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On the rationality of algebraic monodromy groups of compatible systems

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Abstract. Let E be a number field and X a smooth geometrically connected variety defined over a characteristic p finite field. Given an n-dimensional pure E-compatible system of semisimple λ-adic representations of the étale fundamental group of X with connected algebraic monodromy groups $\mathbf{G}_\lambda$, we construct a common E-form $\mathbf{G}$ of all the groups $\mathbf{G}_\lambda$ and in the absolutely irreducible case, a common E-form $\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}$ of all the tautological representations $\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}$. Analogous rationality results in characteristic p assuming the existence of crystalline companions in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \otimes E_v$ for all $v \mid p$ and in characteristic zero assuming ordinariness are also obtained. Applications include the construction of a $\mathbf{G}$-compatible system from some GL_n-compatible system and some results predicted by the Mumford–Tate conjecture.

Keywords: algebraic monodromy groups, Galois representations, compatible system.

1. Introduction

1.1. The Mumford–Tate conjecture

Let A be an abelian variety defined over a number field $K \subset \mathbb{C}$, $V_\ell := H^1(A_{\bar{K}}, \mathbb{Q}_\ell)$ the étale cohomology groups for all primes ℓ, and $V_\infty = H^1(A(\mathbb{C}), \mathbb{Q})$ the singular cohomology group. The famous Mumford–Tate conjecture [47, Section 4] asserts that the ℓ-adic Galois representations $\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V_\ell)$ are independent of ℓ, in the sense that if $\mathbf{G}_\ell$ denotes the *algebraic monodromy group* of ρ_ℓ (the Zariski closure of the image of ρ_ℓ in GL_{V_ℓ}) and $\mathbf{G}_{\mathrm{MT}}$ denotes the *Mumford–Tate group* of the pure Hodge structure of V_∞, then via the comparison isomorphisms $V_\ell \cong V_\infty \otimes \mathbb{Q}_\ell$ one has

$$(\mathbf{G}_{\mathrm{MT}} \hookrightarrow \mathrm{GL}_{V_\infty}) \times_{\mathbb{Q}} \mathbb{Q}_\ell \cong (\mathbf{G}_\ell^\circ \hookrightarrow \mathrm{GL}_{V_\ell}) \quad \text{for all } \ell. \tag{1}$$

In particular, the representations ρ_ℓ are semisimple and the identity components $\mathbf{G}_\ell^\circ$ are reductive with the same absolute root datum. This conjectural ℓ-independence of different algebraic monodromy representations can be formulated almost identically for projective

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smooth varieties Y defined over K, and more generally, for pure motives over K by the universal cohomology theory envisaged by Grothendieck and some deep conjectures in algebraic and arithmetic geometry (see [59, Section 3]).

The same Mumford–Tate type question can also be asked for projective smooth varieties Y defined over a global field K of characteristic $p > 0$. Since $V_\ell = H^w(Y_{\bar{K}}, \mathbb{Q}_\ell)$ are *Weil cohomology theories* for $Y_{\bar{K}}$ only when $\ell \neq p$,[1] one may ask if the algebraic monodromy representations $\mathbf{G}_\ell^\circ \hookrightarrow \mathrm{GL}_{V_\ell}$ of the Galois representations V_ℓ are independent of ℓ for all $\ell \neq p$. This expectation is supported by the philosophy of motives (see [14, Section E]). On the other hand, one can always exploit the fact that the system $\{V_\ell = H^w(Y_{\bar{K}}, \mathbb{Q}_\ell)\}_\ell$ of ℓ-adic Galois representations is a $\mathbb{Q}$-*compatible system* (in the sense of Serre [61, Chapter I-11, Definition]) that is *pure of weight* w (proven by Deligne [11,12]) to directly argue ℓ-independence of the algebraic monodromy representations $\mathbf{G}_\ell^\circ \hookrightarrow \mathrm{GL}_{V_\ell}$. This approach holds regardless of the characteristic of the global field K. By utilizing the compatibility and weight conditions of the compatible system, Serre developed the method of *Frobenius tori* [58] to prove the ℓ-independence result below (Theorem A).

Let us define some notation first. If L is a subfield of $\bar{\mathbb{Q}}$, then denote by $\mathcal{P}_L$ the set of places of L. Denote by $\mathcal{P}_{L,f}$ (resp. $\mathcal{P}_{L,\infty}$) the set of finite (resp. infinite) places of L. Then $\mathcal{P}_L = \mathcal{P}_{L,f} \cup \mathcal{P}_{L,\infty}$. Denote by $\mathcal{P}_{L,f}^{(p)}$ the set of elements of $\mathcal{P}_{L,f}$ not extending p. The residue characteristic of the finite place $v \in \mathcal{P}_{L,f}$ is denoted by p_v. Let V and W be free modules of finite rank over a ring R. Let $\mathbf{G}_m \subset \cdots \subset \mathbf{G}_1 \subset \mathrm{GL}_V$ and $\mathbf{H}_m \subset \cdots \subset \mathbf{H}_1 \subset \mathrm{GL}_W$ be two chains of closed algebraic subgroups over R. We say that the two chain representations (or simply representations if it is clear that they are chains of subgroups of some GL_n) are isomorphic if there is an R-module isomorphism $V \cong W$ such that the induced isomorphism $\mathrm{GL}_V \cong \mathrm{GL}_W$ maps $\mathbf{G}_i$ isomorphically onto $\mathbf{H}_i$ for $1 \le i \le m$.

Theorem A (Serre [58]; see also [44]). (i) (Component groups) *The quotient groups* $\mathbf{G}_\ell/\mathbf{G}_\ell^\circ$ *for all* ℓ *are isomorphic.*

(ii) (Common $\mathbb{Q}$-form of formal characters) *For all* v *in a positive Dirichlet density subset of* $\mathcal{P}_{K,f}$*, there exist a subtorus* $\mathbf{T} := \mathbf{T}_v$ *of* $\mathrm{GL}_{n,\mathbb{Q}}$ *such that for all* $\ell \neq p_v$*, the representation* $(\mathbf{T} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}) \times_{\mathbb{Q}} \mathbb{Q}_\ell$ *is isomorphic to the representation* $\mathbf{T}_\ell \hookrightarrow \mathrm{GL}_{V_\ell}$ *for some maximal torus* $\mathbf{T}_\ell$ *of* $\mathbf{G}_\ell$.

It follows immediately that the connectedness and the absolute rank of $\mathbf{G}_\ell$ are both independent of ℓ. Later, Larsen–Pink obtained some ℓ-independence results for abstract semisimple compatible systems on a Dirichlet density 1 set of primes ℓ [42] and for the geometric monodromy of $\{V_\ell\}_\ell$ if $\mathrm{Char}(K) > 0$ [43]. When $\mathrm{Char}(K) = 0$, the present author proved that the formal bi-character (Definition 2.2 (ii)) of $\mathbf{G}_\ell^\circ \hookrightarrow \mathrm{GL}_{V_\ell}$ is independent of ℓ and obtained ℓ-independence of $\mathbf{G}_\ell^\circ$ under a type A hypothesis [26,28]. The next result is by far the best result in positive characteristic, in a setting more general than the above étale cohomology case.

[1] When $\ell = p$, one has to consider crystalline cohomology group of Y.

Let X be a smooth geometrically connected variety defined over a finite field $\mathbb{F}_q$ of characteristic p. Let E be a number field. For any $\lambda \in \mathcal{P}_E$, denote by E_λ the λ-adic completion of E. Let

$$\rho_\bullet := \{\rho_\lambda : \pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{GL}(V_\lambda)\}_{\lambda \in \mathcal{P}_{E,f}^{(p)}} \tag{2}$$

be an E-compatible system of n-dimensional semisimple λ-adic representations of the *étale fundamental group* $\pi_1^{\text{ét}}(X, \overline{x})$ of X (with base point $\overline{x}$) that is pure of integral weight w. Denote by $\mathbf{G}_\lambda \subset \mathrm{GL}_{V_\lambda}$ the algebraic monodromy group of the representation V_λ. For simplicity, set

$$\pi_1(X) = \pi_1^{\text{ét}}(X, \overline{x})$$

and for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$, choose coordinates for V_λ so that $\mathbf{G}_\lambda$ is identified as a subgroup of $\mathrm{GL}_{n,E_\lambda}$. The following theorem was obtained by Chin when X is a curve [8][2] and is true in general by reducing to the curve case by finding a suitable curve S in some covering X' of X ([2, Section 3.3]; see also [10, Section 4.3]).

Theorem B. *Let $\rho_\bullet$ be an E-compatible system of n-dimensional λ-adic semisimple representations of $\pi_1(X)$ that is pure of integer weight w. The following assertions hold in some coordinates of V_λ:*

(i) (Common E-form of formal characters) *There exists a subtorus $\mathbf{T}$ of $\mathrm{GL}_{n,E}$ such that for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$, $\mathbf{T}_\lambda := \mathbf{T} \times_E E_\lambda$ is a maximal torus of $\mathbf{G}_\lambda$.*

(ii) (λ-independence over an extension) *There exist a finite extension F of E and a chain of subgroups $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,F}$ such that $\mathbf{G}^{\mathrm{sp}}$ is connected split reductive, $\mathbf{T}^{\mathrm{sp}}$ is a split maximal torus of $\mathbf{G}^{\mathrm{sp}}$, and for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$, if F_λ is a completion of F extending λ on E, then there exists an isomorphism of chain representations*

$$f_{F_\lambda} : (\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,F}) \times_F F_\lambda \xrightarrow{\cong} (\mathbf{T}_\lambda \subset \mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda.$$

(iii) (Rigidity) *The isomorphisms f_{F_λ} in* (ii) *can be chosen such that the restriction isomorphisms $f_{F_\lambda} : \mathbf{T}^{\mathrm{sp}} \times_F F_\lambda \to \mathbf{T}_\lambda \times_{E_\lambda} F_\lambda$ admit a common F-form $f_F : \mathbf{T}^{\mathrm{sp}} \to \mathbf{T} \times_E F$ for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$ and F_λ.*

1.2. The results of the paper

1.2.1. Characteristic p.

1.2.1.1. Theorem B (ii) asserts that the algebraic monodromy representations $\mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ have a common (split) F-model after finite extensions F_λ of E_λ. The main theme of this article is to remove these extensions. Based on Theorem B and some ideas seeded in [28], we prove the following E-rationality result (Theorem 1.1). In case the repre-

[2] [8] used pivotally Serre's Frobenius tori and Lafforgue's work [38] on Langlands' conjectures. In case X is a curve, Theorem B (i)–(iii) follow, respectively, from Lemma 6.4, Theorem 1.4, Theorem 6.8 and Corollary 6.9 of [8].

sentations V_λ are absolutely irreducible,[3] it answers the Mumford–Tate type question in positive characteristic.

Theorem 1.1. *Let $\rho_\bullet := \{\rho_\lambda : \pi_1(X) \to \mathrm{GL}(V_\lambda)\}_{\lambda\in\mathcal{P}_{E,f}^{(p)}}$ be an E-compatible system of n-dimensional λ-adic semisimple representations of $\pi_1(X)$ that is pure of integer weight w. Then the following assertions hold:*

(i) *There exists a connected reductive group $\mathbf{G}$ defined over E such that $\mathbf{G}\times_E E_\lambda \cong \mathbf{G}_\lambda^\circ$ for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$.*

(ii) *If moreover $\mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{V_\lambda}$ is absolutely irreducible for some λ, then there exists a connected reductive subgroup $\mathbf{G}$ of $\mathrm{GL}_{n,E}$ such that for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$, the following representations are isomorphic:*

$$(\mathbf{G}\hookrightarrow \mathrm{GL}_{n,E})\times_E E_\lambda \cong (\mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{V_\lambda}).$$

1.2.1.2. Let $\mathcal{O}_\lambda$ be the ring of integers of E_λ, $\mathcal{O}_E$ be the ring of integers of E, $\mathcal{O}_{E,S}$ be the localization for some finite subset $S \subset \mathcal{P}_{E,f}$, and $\mathbb{A}_E^{(p)}$ be the adele ring of E without factors above p. We construct an adelic representation $\rho_{\mathbb{A}}^{\mathbf{G}}$ in Corollary 1.2, and in the absolutely irreducible case, a common model $\mathcal{G} \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}$ of the group schemes $\mathcal{G}_\lambda \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_\lambda}$ (with respect to some $\mathcal{O}_\lambda$-lattice in V_λ) for all but finitely many λ, in Corollary 1.3.

Corollary 1.2. *Let $\rho_\bullet$ be a λ-adic compatible system of $\pi_1(X)$ as above. Suppose $\mathbf{G}_\lambda$ is connected for all λ. Then the following assertions hold:*

(i) *There exist a connected reductive group $\mathbf{G}$ defined over E and an isomorphism $\mathbf{G}\times_E E_\lambda \xrightarrow{\phi_\lambda} \mathbf{G}_\lambda$ for each $\lambda \in \mathcal{P}_{E,f}^{(p)}$ such that the direct product representation*

$$\prod_{\lambda\in\mathcal{P}_{E,f}^{(p)}} \rho_\lambda : \pi_1(X) \to \prod_{\lambda\in\mathcal{P}_{E,f}^{(p)}} \mathbf{G}_\lambda(E_\lambda)$$

factors (via ϕ_λ) through a $\mathbf{G}$-valued adelic representation

$$\rho_{\mathbb{A}}^{\mathbf{G}} : \pi_1(X) \to \mathbf{G}(\mathbb{A}_E^{(p)}).$$

(ii) *If the representations V_λ are absolutely irreducible, then there exist a connected reductive subgroup $\mathbf{G}$ of $\mathrm{GL}_{n,E}$ and an isomorphism of representations $(\mathbf{G}\hookrightarrow \mathrm{GL}_{n,E})\times_E E_\lambda \xrightarrow{\phi_\lambda} (\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{V_\lambda})$ for each $\lambda \in \mathcal{P}_{E,f}^{(p)}$ such that the direct product representation*

$$\prod_{\lambda\in\mathcal{P}_{E,f}^{(p)}} \rho_\lambda : \pi_1(X) \to \prod_{\lambda\in\mathcal{P}_{E,f}^{(p)}} \mathbf{G}_\lambda(E_\lambda) \subset \prod_{\lambda\in\mathcal{P}_{E,f}^{(p)}} \mathrm{GL}_n(E_\lambda)$$

factors (via ϕ_λ) through a $\mathbf{G}$-valued adelic representation

$$\rho_{\mathbb{A}}^{\mathbf{G}} : \pi_1(X) \to \mathbf{G}(\mathbb{A}_E^{(p)}) \subset \mathrm{GL}_{n,E}(\mathbb{A}_E^{(p)}).$$

[3] In general, we expect that there exists a common E-form of the faithful representations $\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{V_\lambda}$ for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$.

Corollary 1.3. *Let $\rho_\bullet$ be a λ-adic compatible system of $\pi_1(X)$ as above. Suppose V_λ is absolutely irreducible and $\mathbf{G}_\lambda$ is connected for all λ. Then there exists a smooth reductive group scheme $\mathcal{G} \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}$ defined over $\mathcal{O}_{E,S}$ (for some finite S) whose generic fiber is $\mathbf{G} \subset \mathrm{GL}_{n,E}$ such that for all $\lambda \in \mathcal{P}_{E,f}^{(p)} \setminus S$, the representations $(\mathcal{G} \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_{E,S}}) \times \mathcal{O}_\lambda$ and $\mathcal{G}_\lambda \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_\lambda}$ are isomorphic, where $\mathcal{G}_\lambda$ is the Zariski closure of $\rho_\lambda(\pi_1(X))$ in $\mathrm{GL}_{n,\mathcal{O}_\lambda}$ after some choice of $\mathcal{O}_\lambda$-lattice in V_λ.*

For almost all λ, $\mathbf{G}(\mathcal{O}_\lambda)$ is a *hyperspecial maximal compact subgroup* of $\mathbf{G}(E_\lambda)$ [67, Section 3.9.1]. Hence, Corollary 1.2 (i) implies that for almost all λ, the image $\rho_\lambda(\pi_1(X))$ is contained in a hyperspecial maximal compact subgroup of $\mathbf{G}_\lambda(E_\lambda)$ (see Proposition 3.6). The next corollary is about the $\mathbf{G}$*-valued compatibility* of the system, motivated by the papers [3, 14] on Langlands conjectures. As shown in [3, Section 6], the results in [10, Section 4] ([8, Section 6] when X is a curve) imply that the E-compatible system $\rho_\bullet$ (assume connectedness of $\mathbf{G}_\lambda$), after some finite extension F/E, factors through an F-compatible system $\rho_\bullet^{\mathbf{G}^{\mathrm{sp}}}$ of $\mathbf{G}^{\mathrm{sp}}$-representations for some connected split reductive group $\mathbf{G}^{\mathrm{sp}}$ defined over F. In some situation, we prove that the extension F/E can be omitted. This gives evidence to the *motivic hope* in [14, Section E] that the Tannakian categories $\mathcal{T}_\lambda(X)$ of semisimple (weight 0) E_λ-representations of $\pi_1(X)$, at least for all λ not extending p, should come from a canonical category $\mathcal{T}(X)$ over E:

$$\mathcal{T}(X) \otimes_E E_\lambda \xrightarrow{\approx} \mathcal{T}_\lambda(X)$$

in a compatible way (see [14, Theorem 1.4.1]). The definition of an E-compatible system of $\mathbf{G}$-representations will be recalled in Section 3.2. Let $\pi_\lambda : \mathbb{A}_E^{(p)} \to E_\lambda$ be the natural surjection to the λ-component.

Corollary 1.4. *Let $\rho_\bullet$ be a λ-adic compatible system of $\pi_1(X)$ as above. Suppose V_λ is absolutely irreducible and $\mathbf{G}_\lambda$ is connected for all λ. Let $\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}$ be the E-form and $\rho_\lambda^{\mathbf{G}}$ be the adelic representation in Corollary 1.2 (ii). Let $N_{\mathrm{GL}_{n,E}}\mathbf{G}$ be the normalizer of $\mathbf{G}$ in $\mathrm{GL}_{n,E}$. Then for each $\lambda \in \mathcal{P}_{E,f}^{(p)}$, there exists (a change of coordinates) $\beta_\lambda \in (N_{\mathrm{GL}_{n,E}}\mathbf{G})(E_\lambda)$ such that the system*

$$\rho_\bullet^{\mathbf{G}} := \{\rho_\lambda^{\mathbf{G}} : \pi_1(X) \xrightarrow{\rho_{\mathbb{A}}^{\mathbf{G}}} \mathbf{G}(\mathbb{A}_E^{(p)}) \xrightarrow{\pi_\lambda} \mathbf{G}(E_\lambda) \xrightarrow{\beta_\lambda} \mathbf{G}(E_\lambda)\}_{\lambda \in \mathcal{P}_{E,f}^{(p)}}$$

is an E-compatible system of $\mathbf{G}$-representations when one of the following holds:

(i) *The group $\mathbf{G}_\lambda$ is split for all λ.*

(ii) *The outer automorphism group of the derived group $\mathbf{G}^{\mathrm{der}} \times_E \overline{E}$ is trivial ($\beta_\lambda = \mathrm{id}$).*

Hence, for any E-representation $\alpha : \mathbf{G} \to \mathrm{GL}_{m,E}$, the system $\{\alpha \circ \rho_\lambda^{\mathbf{G}}\}_{\lambda \in \mathcal{P}_{E,f}^{(p)}}$ of m-dimensional λ-adic semisimple representations is also E-compatible.

1.2.1.3. Denote by $\mathcal{P}_{E,p}$ the set of finite places of E extending p. Let $\mathbb{Q}_{p^k}$ be a degree k unramified extension of $\mathbb{Q}_p$, $v \in \mathcal{P}_{E,p}$, and E_{v,p^k} the composite fields $E_v \cdot \mathbb{Q}_{p^k}$. Let $\rho_\bullet$ be as in Theorem 1.1. The *semisimple crystalline companion object* of $\rho_\bullet$ at v (whose existence[4] is conjectured by Deligne [12, Conjecture 1.2.10]) is an object M_v in the Tannakian category $\mathbf{F\text{-}Isoc}^\dagger(X) \otimes E_{v,q}$ of *overconvergent F-isocrystals of X with coefficients in $E_{v,q}$* (see [35, Section 2] for definition). Any $t \in X(\mathbb{F}_{q^k})$ induces a fiber functor to the category of vector spaces over E_{v,q^k} given by the composition

$$w_t : \mathbf{F\text{-}Isoc}^\dagger(X) \otimes E_{v,q} \to \mathbf{F\text{-}Isoc}^\dagger(x) \otimes E_{v,q^k} \to \mathbf{Vec}_{E_{v,q^k}},$$

where the first one is via the pull-back of $i : t \to X$ and the second one is the forgetful functor. The image $V_{t,v} := w_t(M_v)$ is an n-dimensional vector space. The Tannakian group of the subcategory generated by M_v with respect to w_t can be identified as a reductive subgroup $\mathbf{G}_{t,v} \subset \mathrm{GL}_{V_{t,v}} \cong \mathrm{GL}_{n,E_{v,q^k}}$ and is called the *algebraic monodromy group* of (M_v, w_t). For different closed points t and t' in $X(\mathbb{F}_{q^k})$, $\mathbf{G}_{t,v}$ and $\mathbf{G}_{t',v}$ differ by an inner twist [13, Theorem 3.2]. Let λ be a finite place of E not extending p. The absolute root data of $\mathbf{G}_{t,v}^\circ$ and $\mathbf{G}_\lambda^\circ$ (resp. the component groups of $\mathbf{G}_{t,v}$ and $\mathbf{G}_\lambda$) have been proven to be isomorphic independently by Pal [52] and D'Addezio [10] (relying on [1, 38]). Moreover, given the closed point t one can define the *Frobenius tori* $\mathbf{T}_{t,v}$ in $\mathbf{G}_{t,v}$ (see [10, Section 4.2]) and $\mathbf{T}_{\bar{t},\lambda}$ in $\mathbf{G}_\lambda$ (up to conjugation, see Section 3.3). Assuming the crystalline companions of $\rho_\bullet$ exist for all $v \in \mathcal{P}_{E,p}$ and certain conditions, we prove an E-rationality result (existence + uniqueness) for the above algebraic monodromy groups at all finite places of E.

Theorem 1.5. *Let $\rho_\bullet := \{\rho_\lambda : \pi_1(X) \to \mathrm{GL}(V_\lambda)\}_{\lambda \in \mathcal{P}_{E,f}^{(p)}}$ be an E-compatible system of n-dimensional λ-adic semisimple representations of $\pi_1(X)$ that is pure of integer weight w and $t \in X(\mathbb{F}_{q^k})$ a closed point of X. Suppose the semisimple crystalline companion object M_v of $\rho_\bullet$ exists in $\mathbf{F\text{-}Isoc}^\dagger(X) \otimes E_{v,q}$ for each $v \in \mathcal{P}_{E,p}$ and the following conditions hold:*

(a) *The Frobenius torus $\mathbf{T}_{\bar{t},\lambda}$ is a maximal torus of $\mathbf{G}_\lambda$ for some λ.*

(b) *For all $v \in \mathcal{P}_{E,p}$, the field $\mathbb{Q}_{q^k}$ is contained in E_v.*

(c) *The number field E has at least one real place.*[5]

Then the following assertions hold:

(i) *There exists a chain (of a connected reductive group together with a maximal torus) $\mathbf{T} \subset \mathbf{G}$ defined over E that is the unique common E-form of the chains $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{G}_\lambda^\circ$ for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$ and the chains $\mathbf{T}_{t,v} \subset \mathbf{G}_{t,v}^\circ$ for all $v \in \mathcal{P}_{E,p}$.*

(ii) *If moreover $\mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{V_\lambda}$ is absolutely irreducible for some λ, then there exist an inner form $\mathrm{GL}_{m,D}$ (for some division algebra D over E) of $\mathrm{GL}_{n,E}$ over E containing a chain of subgroups $\mathbf{T} \subset \mathbf{G}$ such that $\mathbf{T} \subset \mathbf{G} \hookrightarrow \mathrm{GL}_{m,D}$ is the unique common E-form of the chain representations $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{V_\lambda}$ for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$ and the*

[4] Recent works of Kedlaya [33, 34] establish the existence of crystalline companion when X is smooth.

[5] This condition is needed to ensure that the E-torus in Main Theorem II (d) is anisotropic at some place v of E.

chain representations $\mathbf{T}_{t,v} \subset \mathbf{G}_{t,v} \hookrightarrow \mathrm{GL}_{V_{t,v}}$ *for all* $v \in \mathcal{P}_{E,p}$. *When* E *has exactly one real place, we have* $\mathrm{GL}_{m,D} \cong \mathrm{GL}_{n,E}$.

1.2.2. Characteristic zero.

1.2.2.1. It turns out that the strategy for proving Theorem 1.1 remains the same in characteristic zero if ordinary representations enter the picture. This part is influenced by the work of Pink [53]. To keep things simple, we only consider the $\mathbb{Q}$-compatible system (with exceptional set S) of n-dimensional ℓ-adic Galois representations $V_\ell := H^w(Y_{\bar{K}}, \mathbb{Q}_\ell)$:

$$\rho_\bullet := \{\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V_\ell)\}_{\ell \in \mathcal{P}_{\mathbb{Q},f}}, \tag{3}$$

arising from a smooth projective variety Y defined over a number field K. The set S consists of the finite places of K at which Y does not have good reduction. Let $\mathbf{G}_\ell$ be the algebraic monodromy group at ℓ. The Grothendieck–Serre semisimplicity conjecture asserts that the representation ρ_ℓ is semisimple (see [65]), which is equivalent to the algebraic group $\mathbf{G}_\ell^\circ$ being reductive. Choose coordinates for V_ℓ and identify $\mathbf{G}_\ell$ as a subgroup of $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ for all ℓ. Embed $\mathbb{Q}_\ell$ into $\mathbb{C}$ for all ℓ.

Let $v \in \mathcal{P}_{K,f} \setminus S$ with $p := p_v$. Let K_v be the completion of K at v, $\mathcal{O}_v$ the ring of integers, and Y_v the special fiber of a smooth model of Y over $\mathcal{O}_v$. The local representation $V_p = H^w(Y_{\bar{K}}, \mathbb{Q}_p)$ of $\mathrm{Gal}(\bar{K}_v/K_v)$ is *crystalline* and corresponds, via a mysterious functor of Fontaine [19–21], to the crystalline cohomology $M_v := H^w(Y_v/\mathcal{O}_v) \otimes_{\mathcal{O}_v} K_v$ [17,22]. The local representation V_p is said to be *ordinary* if the Newton and Hodge polygons of M_v coincide [45]. This notion originates from ordinary abelian varieties defined over finite fields. It has been conjectured by Serre that if K is large enough, then the set of places v in $\mathcal{P}_{K,f}$ for which the local representations V_p are ordinary is of Dirichlet density 1, for abelian varieties of low dimensions, see Serre [61], Ogus [51], Noot [48,49], Tankeev [64]; for abelian varieties in general, see Pink [53]; and for K3 surfaces, see Bogomolov–Zarhin [4].

Theorem 1.6. *Let* $\rho_\bullet$ *be the* $\mathbb{Q}$*-compatible system* (3) *arising from the* ℓ*-adic cohomology* (*of degree* w) *of a smooth projective variety* Y *defined over a number field* K. *Suppose* $\mathbf{G}_\ell$ *is connected for all* ℓ *and the following conditions hold:*

(a) (Ordinariness) *The set of places* v *in* $\mathcal{P}_{K,f}$ *for which the local representations* V_p *of* $\mathrm{Gal}(\bar{K}_v/K_v)$ *are ordinary is of positive Dirichlet density.*

(b) (Absolute ℓ-independence) *There exists a connected reductive subgroup* $\mathbf{G}_\mathbb{C}$ *of* $\mathrm{GL}_{n,\mathbb{C}}$ *such that the representations* $\mathbf{G}_\mathbb{C} \hookrightarrow \mathrm{GL}_{n,\mathbb{C}}$ *and* $(\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}) \times_{\mathbb{Q}_\ell} \mathbb{C}$ *are isomorphic for all* ℓ.

(c) (Invariance of roots) *Let* $\mathbf{T}_\mathbb{C}^{\mathrm{ss}}$ *be a maximal torus of the derived group* $\mathbf{G}_\mathbb{C}^{\mathrm{der}}$. *Then the normalizer* $N_{\mathrm{GL}_{n,\mathbb{C}}}(\mathbf{T}_\mathbb{C}^{\mathrm{ss}})$ *is invariant on the roots of* $\mathbf{G}_\mathbb{C}^{\mathrm{der}}$ *with respect to* $\mathbf{T}_\mathbb{C}^{\mathrm{ss}}$.

Then the following assertions hold:

(i) *There exists a connected reductive group* $\mathbf{G}$ *defined over* $\mathbb{Q}$ *such that* $\mathbf{G} \times_\mathbb{Q} \mathbb{Q}_\ell \cong \mathbf{G}_\ell$ *for all* ℓ. *In particular,* $\mathbf{G}_\ell$ *is unramified for* $\ell \gg 0$.

(ii) *If moreover* $\mathbf{G}_{\mathbb{C}}$ *is irreducible on* $\mathbb{C}^n$*, then there exists a connected reductive subgroup* $\mathbf{G}$ *of* $\mathrm{GL}_{n,\mathbb{Q}}$ *such that* $\mathbf{G} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$ *is a common* $\mathbb{Q}$*-form of the representations* $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ *for all* ℓ*.*

Remark 1.7. Conditions (a)–(c) of Theorem 1.6 are to be compared with (i)–(iii) of Theorem B. Since Theorem A (ii) only gives a common $\mathbb{Q}$-form of formal characters for all but one ℓ, condition (a) is needed if one aims at a $\mathbb{Q}$-common form for all ℓ. Assuming 1.6 (a) and B (i), conditions 1.6 (b) and B (ii) are easily seen to be equivalent ($E = \mathbb{Q}$). The rigidity assertion B (iii) is not known to hold in characteristic zero, and is now replaced with the invariance-of-roots condition 1.6 (c), which holds if the root system of $\mathbf{G}_{\mathbb{C}}^{\mathrm{der}}$ is of a certain type [29, Theorems A1, A2].

Remark 1.8. If ρ_ℓ is abelian at one ℓ, then the rationality of $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$ for all ℓ has been obtained by Serre via the Serre group $\mathbf{S}_{\mathfrak{m}}$ [61].

1.2.2.2. Suppose $\mathbf{G}_\ell$ is connected reductive for all $\ell \in \mathcal{P}_{\mathbb{Q},f}$.

Hypothesis H. For $\ell \gg 0$, the image of ρ_ℓ is contained in a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell(\mathbb{Q}_\ell)$.

This hypothesis follows from a Galois maximality conjecture of Larsen [39] (see Theorem 3.9), which has been established for type A representations [30], abelian varieties and hyper-Kähler varieties (degree $w = 2$) [31]. Further assuming the hypothesis, we obtain the following corollaries which are analogous to Corollaries 1.2 and 1.3.

Corollary 1.9. *Let* $\rho_\bullet$ *be an* ℓ*-adic compatible system of* $\mathrm{Gal}(\bar{K}/K)$ *as above. Suppose* $\mathbf{G}_\ell$ *is connected for all* ℓ *and Hypothesis H holds. Then the following assertions hold:*

(i) *There exist a connected reductive group* $\mathbf{G}$ *defined over* $\mathbb{Q}$ *and an isomorphism* $\mathbf{G} \times_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\phi_\ell} \mathbf{G}_\ell$ *for each* $\ell \in \mathcal{P}_{\mathbb{Q},f}$ *such that the direct product representation*

$$\prod_{\ell \in \mathcal{P}_{\mathbb{Q},f}} \rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \prod_{\ell \in \mathcal{P}_{\mathbb{Q},f}} \mathbf{G}_\ell(\mathbb{Q}_\ell)$$

factors (via ϕ_ℓ*) through a* $\mathbf{G}$*-valued adelic representation*

$$\rho_{\mathbb{A}}^{\mathbf{G}} : \mathrm{Gal}(\bar{K}/K) \to \mathbf{G}(\mathbb{A}_{\mathbb{Q}}).$$

(ii) *If the representations* V_ℓ *are absolutely irreducible, then there exist a connected reductive subgroup* $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{Q}}$ *and an isomorphism of representations* $(\mathbf{G} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}) \times_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\phi_\ell} (\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{V_\ell})$ *for each* $\ell \in \mathcal{P}_{\mathbb{Q},f}$ *such that the direct product representation*

$$\prod_{\ell \in \mathcal{P}_{\mathbb{Q},f}} \rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \prod_{\ell \in \mathcal{P}_{\mathbb{Q},f}} \mathbf{G}_\ell(\mathbb{Q}_\ell) \subset \prod_{\ell \in \mathcal{P}_{\mathbb{Q},f}} \mathrm{GL}_n(\mathbb{Q}_\ell)$$

factors (via ϕ_ℓ*) through a* $\mathbf{G}$*-valued adelic representation*

$$\rho_{\mathbb{A}}^{\mathbf{G}} : \mathrm{Gal}(\bar{K}/K) \to \mathbf{G}(\mathbb{A}_{\mathbb{Q}}) \subset \mathrm{GL}_{n,\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}}).$$

Corollary 1.10. *Let $\rho_\bullet$ be an ℓ-adic compatible system of* $\mathrm{Gal}(\overline{K}/K)$ *as above. Suppose V_ℓ is absolutely irreducible, $\mathbf{G}_\ell$ is connected for all ℓ, and Hypothesis H holds. Then there exists a smooth reductive group scheme $\mathcal{G} \subset \mathrm{GL}_{n,\mathbb{Z}_S}$ defined over $\mathbb{Z}_S$ (for some finite $S \subset \mathcal{P}_{\mathbb{Q},f}$) whose generic fiber is $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{Q}}$ such that for all $\ell \in \mathcal{P}_{\mathbb{Q},f} \setminus S$, the representations $(\mathcal{G} \hookrightarrow \mathrm{GL}_{n,\mathbb{Z}_S}) \times \mathbb{Z}_\ell$ and $\mathcal{G}_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Z}_\ell}$ are isomorphic, where $\mathcal{G}_\ell$ is the Zariski closure of $\rho_\ell(\mathrm{Gal}(\overline{K}/K))$ in $\mathrm{GL}_{n,\mathbb{Z}_\ell}$ after some choice of $\mathbb{Z}_\ell$-lattice in V_ℓ.*

1.2.2.3. Suppose $Y = A$ is an abelian variety defined over K of dimension g, and let $w = 1$. We say that *A has ordinary reduction at v* if the local representation V_p of $\mathrm{Gal}(\overline{K}_v/K_v)$ is ordinary. The following results are due to Pink.

Theorem C ([53, Theorems 5.13 (a, c, d), 7.1]). *Let A be an abelian variety defined over a number field K with $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$ and suppose $\mathbf{G}_\ell$ is connected for all ℓ. There exists a connected reductive subgroup $\mathbf{G}$ of $\mathrm{GL}_{2g,\mathbb{Q}}$ such that the following assertions hold.*

- (i) *$(\mathbf{G} \hookrightarrow \mathrm{GL}_{2g,\mathbb{Q}}) \times \mathbb{Q}_\ell$ is isomorphic to $\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{V_\ell}$ for all ℓ in set $\mathcal{L}$ of primes of Dirichlet density* 1.
- (ii) *The derived group $\mathbf{G}^{\mathrm{der}}$ is $\mathbb{Q}$-simple.*
- (iii) *If the root system of $\mathbf{G}$ is determined uniquely by its formal character, i.e., if $\mathbf{G}$ does not have an ambiguous factor (in Theorem E below), then we can take $\mathcal{L}$ in* (i) *to contain all but finitely many primes.*
- (iv) *If $\mathbf{G} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ does not have any type C_r simple factors with $r \geq 3$, then the abelian variety A has ordinary reduction at a Dirichlet density* 1 *set of places v of K.*

By the Tate conjecture for abelian varieties proven by Faltings [16] and $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$, the representations V_ℓ are absolutely irreducible. The $\mathbb{Q}_\ell$-representation $V_\ell = H^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ has a natural $\mathbb{Z}_\ell$-model $H^1(A_{\overline{K}}, \mathbb{Z}_\ell)$. Consider the representation

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(H^1(A_{\overline{K}}, \mathbb{Z}_\ell))$$

and let $\mathcal{G}_\ell$ be the Zariski closure of the image in $\mathrm{GL}_{H^1(A_{\overline{K}},\mathbb{Z}_\ell)}$. Combining the previous results, we obtain Theorem 1.11 below which extends Theorem C (iii) to all ℓ assuming ordinariness.

Theorem 1.11. *Let A be an abelian variety defined over a number field K with* $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$ *and suppose $\mathbf{G}_\ell$ is connected for all ℓ and the following conditions hold:*

- (a) *The set of places v in $\mathcal{P}_{K,f}$ for which the local representations V_p of* $\mathrm{Gal}(\overline{K}_v/K_v)$ *are ordinary is of positive Dirichlet density.*
- (b) *The root system of $\mathbf{G}_\ell$ is uniquely determined by its formal character.*

Then there exists a smooth group subscheme $\mathcal{G} \subset \mathrm{GL}_{2g,\mathbb{Z}_S}$ over $\mathbb{Z}_S$ (for some finite $S \subset \mathcal{P}_{\mathbb{Q},f}$) with generic fiber $\mathbf{G} \subset \mathrm{GL}_{2g,\mathbb{Q}}$ and an isomorphism of representations $(\mathbf{G} \hookrightarrow \mathrm{GL}_{2g,\mathbb{Q}}) \times_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\phi_\ell} (\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{V_\ell})$ for each $\ell \in \mathcal{P}_{\mathbb{Q},f}$ such that the direct product *representation*

$$\prod_{\ell\in\mathcal{P}_{\mathbb{Q},f}}\rho_\ell:\mathrm{Gal}(\bar{K}/K)\to\prod_{\ell\in\mathcal{P}_{\mathbb{Q},f}}\mathbf{G}_\ell(\mathbb{Q}_\ell)\subset\prod_{\ell\in\mathcal{P}_{\mathbb{Q},f}}\mathrm{GL}_{2g}(\mathbb{Q}_\ell)$$

factors (via ϕ_ℓ) through a **G***-valued adelic representation*

$$\rho_{\mathbb{A}}^{\mathbf{G}}:\mathrm{Gal}(\bar{K}/K)\to\mathbf{G}(\mathbb{A}_{\mathbb{Q}})\subset\mathrm{GL}_{2g,\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}}).$$

Moreover, for $\ell\gg 0$, the representations $(\mathcal{G}\hookrightarrow\mathrm{GL}_{2g,\mathbb{Z}_S})\times\mathbb{Z}_\ell$ and $\mathcal{G}_\ell\hookrightarrow\mathrm{GL}_{H^1(A_{\bar{K}},\mathbb{Z}_\ell)}$ are isomorphic.

Remark 1.12. By Theorem C (iv), Theorem E, and the fact that for every ℓ, every simple factor of $\mathbf{G}_\ell\times_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$ is of type A, B, C, or D [53, Corollary 5.11], conditions (a) and (b) of Theorem 1.11 hold if for some prime ℓ', every simple factor of $\mathbf{G}_{\ell'}\times_{\mathbb{Q}_{\ell'}}\overline{\mathbb{Q}}_{\ell'}$ is of type A_r with $r>1$.

1.3. The structure of the paper

The paper is structured around the purely algebraic Main Theorems I and II in the next section. Roughly speaking, they state that if a family of connected reductive algebraic subgroups $\mathbf{G}_\lambda\hookrightarrow\mathrm{GL}_{n,E_\lambda}$ indexed by $\lambda\in\mathcal{P}_{E,f}^{(p)}$ (resp. $\mathcal{P}_{E,f}$) satisfies some conditions, then there exist a common E-form of the family of subgroups (resp. representations). The results in Section 1.2 are established in two big steps. Firstly, we prove the Main Theorems in Section 2 which require different techniques from representation theory and Galois cohomology. The notation and diagrams we develop in Section 2 are very much influenced by [28]. A crucial step towards the existence of a common E-form in the Main Theorems is based on the local-global aspects of Galois cohomology in Section 2.5. Secondly, we prove Theorems 1.1, 1.5, and 1.6 in Section 3 by checking that the conditions of the Main Theorems are satisfied for the corresponding family of algebraic monodromy groups of the E-compatible systems and applying the Main Theorems. For the characteristic p case, to deduce Theorem 1.1 (resp. Theorem 1.5) from Main Theorem I (resp. II), the required conditions are ensured by Theorem B (resp. recent work [10], see Theorem B′). The characteristic zero case is more involved. It requires the results on formal bi-characters (Section 2.2) and invariance of roots to compensate for the lack of the rigidity condition of Theorem B (iii). The information at the real place (Proposition 3.3) and a finite place (ordinary representation V_p) is also needed. The other results in Section 1.2 will also be established in Section 3. The statements we quote are named using letters (e.g., Theorem A) and the statements we prove are named using numbers (e.g., Theorem 1.1).

2. Main theorems

2.1. Statements

Main Theorem I. *Suppose a connected reductive subgroup $\mathbf{G}_\lambda\subset\mathrm{GL}_{n,E_\lambda}$ is given for each $\lambda\in\mathcal{P}_{E,f}^{(p)}$ such that the following conditions hold:*

(a) (Common E-form of formal characters) *There exists a subtorus* $\mathbf{T}$ *of* $\mathrm{GL}_{n,E}$ *such that for all* $\lambda \in \mathcal{P}_{E,f}^{(p)}$, $\mathbf{T}_\lambda := \mathbf{T} \times_E E_\lambda$ *is a maximal torus of* $\mathbf{G}_\lambda$.

(b) (Absolute λ-independence) *There exists a chain of subgroups* $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,E}$ *such that* $\mathbf{G}^{\mathrm{sp}}$ *is connected split reductive,* $\mathbf{T}^{\mathrm{sp}}$ *is a split maximal torus of* $\mathbf{G}^{\mathrm{sp}}$, *and for all* $\lambda \in \mathcal{P}_{E,f}^{(p)}$, *if* $\bar{E}_\lambda$ *is a completion of* $\bar{E}$ *extending* λ *on* E, *then there exists an isomorphism of chain representations*

$$f_{\bar{E}_\lambda} : (\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,E}) \times_E \bar{E}_\lambda \xrightarrow{\cong} (\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} \bar{E}_\lambda.$$

(c) (Rigidity) *The isomorphisms* $f_{\bar{E}_\lambda}$ *in* (b) *can be chosen such that the restriction isomorphisms* $f_{\bar{E}_\lambda} : \mathbf{T}^{\mathrm{sp}} \times_E \bar{E}_\lambda \to \mathbf{T}_\lambda \times_{E_\lambda} \bar{E}_\lambda$ *admit a common* $\bar{E}$*-form* $f_{\bar{E}} : \mathbf{T}^{\mathrm{sp}} \times_E \bar{E} \to \mathbf{T} \times_E \bar{E}$ *for all* $\lambda \in \mathcal{P}_{E,f}^{(p)}$ *and* $\bar{E}_\lambda$.

(d) (Quasi-split) *The groups* $\mathbf{G}_\lambda$ *are quasi-split for all but finitely many* $\lambda \in \mathcal{P}_{E,f}^{(p)}$.

Then the following assertions hold:

(i) *There exists a connected reductive group* $\mathbf{G}$ *defined over* E *such that* $\mathbf{G} \times_E E_\lambda \cong \mathbf{G}_\lambda$ *for all* $\lambda \in \mathcal{P}_{E,f}^{(p)}$. *In particular,* $\mathbf{G}_\lambda$ *is unramified for all but finitely many* λ.

(ii) *If moreover* $\mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,E}$ *is irreducible, then there exists a connected reductive subgroup* $\mathbf{G}$ *of* $\mathrm{GL}_{n,E}$ *such that* $\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}$ *is a common* E*-form of the representations* $\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ *for all* $\lambda \in \mathcal{P}_{E,f}^{(p)}$.

For any E-algebra B, define $\mathrm{GL}_{m,B}$ to be the affine algebraic group over E such that for any E-algebra C the group of C-points is $\mathrm{GL}_m(B \otimes_E C)$.

Main Theorem II. *Suppose a connected reductive subgroup* $\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}$ *is given for each* $\lambda \in \mathcal{P}_{E,f}$ *such that the following conditions hold:*

(a) (Common E-form of formal characters) *There exists a subtorus* $\mathbf{T}$ *of* $\mathrm{GL}_{n,E}$ *such that for all* $\lambda \in \mathcal{P}_{E,f}$, $\mathbf{T}_\lambda := \mathbf{T} \times_E E_\lambda$ *is a maximal torus of* $\mathbf{G}_\lambda$.

(b) (Absolute λ-independence) *There exists a chain of subgroups* $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,E}$ *such that* $\mathbf{G}^{\mathrm{sp}}$ *is connected split reductive,* $\mathbf{T}^{\mathrm{sp}}$ *is a split maximal torus of* $\mathbf{G}^{\mathrm{sp}}$, *and for all* $\lambda \in \mathcal{P}_{E,f}$, *if* $\bar{E}_\lambda$ *is a completion of* $\bar{E}$ *extending* λ *on* E, *then there exists an isomorphism of chain representations*

$$f_{\bar{E}_\lambda} : (\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,E}) \times_E \bar{E}_\lambda \xrightarrow{\cong} (\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} \bar{E}_\lambda.$$

(c) (Rigidity) *The isomorphisms* $f_{\bar{E}_\lambda}$ *in* (b) *can be chosen such that the restriction isomorphisms* $f_{\bar{E}_\lambda} : \mathbf{T}^{\mathrm{sp}} \times_E \bar{E}_\lambda \to \mathbf{T}_\lambda \times_{E_\lambda} \bar{E}_\lambda$ *admit a common* $\bar{E}$*-form* $f_{\bar{E}} : \mathbf{T}^{\mathrm{sp}} \times_E \bar{E} \to \mathbf{T} \times_E \bar{E}$ *for all* $\lambda \in \mathcal{P}_{E,f}$ *and* $\bar{E}_\lambda$.

(d) (Anisotropic torus) *The twisted* E*-torus* ${}_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C})$ *is anisotropic at some place of* E *and all real places of* E, *where* $\mathbf{C}$ *is the center of* $\mathbf{G}^{\mathrm{sp}}$ *and* $\mu \in Z^1(E, \mathrm{Aut}_{\bar{E}}\, \mathbf{T}^{\mathrm{sp}})$ *the cocycle defined by* $f_{\bar{E}}$ *in* (c).

Then the following assertions hold:

(i) *There exists a unique connected reductive group* $\mathbf{G}$ *defined over* E *containing* $\mathbf{T}$ *such that* $(\mathbf{T} \subset \mathbf{G}) \times_E E_\lambda \cong (\mathbf{T}_\lambda \subset \mathbf{G}_\lambda)$ *for all* $\lambda \in \mathcal{P}_{E,f}$. *In particular,* $\mathbf{G}_\lambda$ *is unramified for all but finitely many* λ.

(ii) *If moreover* $\mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,E}$ *is irreducible, then there exist an inner form* $\mathrm{GL}_{m,D}$ (*for some division algebra* D *over* E) *of* $\mathrm{GL}_{n,E}$ *over* E *containing a chain of subgroups* $\mathbf{T} \subset \mathbf{G}$ *such that* $\mathbf{T} \subset \mathbf{G} \hookrightarrow \mathrm{GL}_{m,D}$ *is a common* E*-form of the chain representations* $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ *for all* $\lambda \in \mathcal{P}_{E,f}$. *Such a chain of* E*-groups is unique.*

Remark 2.1. There are similarities and differences between the two Main Theorems:

(1) The index set for Main Theorem I is $\mathcal{P}^{(p)}_{E,f}$, and for Main Theorem II it is $\mathcal{P}_{E,f}$.

(2) Conditions (a)–(c) of the two Main Theorems are identical except for the index sets.

(3) If we embed $\overline{E}_\lambda$ into $\mathbb{C}$ for all λ, then condition (b) is equivalent to asking that the $\mathbb{C}$-representation $(\mathbf{G}_\lambda \to \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} \mathbb{C}$ is independent of λ.

(4) The rigidity condition (c) rigidifies the isomorphisms $f_{\overline{E}_\lambda}$ in (b) by requiring them to be extensions of an $\overline{E}$-isomorphism $\mathbf{T}^{\mathrm{sp}} \times_E \overline{E} \to \mathbf{T} \times_E \overline{E}$ where $\mathbf{T}^{\mathrm{sp}}$ (resp. $\mathbf{T}$) is the torus in (b) (resp. (a)).

(5) An F-torus $\mathbf{T}$ is said to be *anisotropic* if it has no non-trivial F-character. If F is a number field, $\mathbf{T}$ is said to be *anisotropic at a place* λ of F if it is anisotropic over F_λ. The twisted E-torus ${}_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C})$ in Main Theorem II (d) will be defined in Section 2.6.1.

(6) The conclusion of Main Theorem II is stronger than that of Main Theorem I as the E-torus $\mathbf{T}$ in condition (a) can be found in the common E-form $\mathbf{G}$ in Main Theorem II. Moreover, if E has only one real place, then the inner form $\mathrm{GL}_{m,D}$ in Main Theorem II is equal to $\mathrm{GL}_{n,E}$ by class field theory.

2.2. *The rigidity condition*

The rigidity condition (c) is important for the construction of the E-form $\mathbf{G}$ in the main theorems. It does not come for free. In this section, we would like to prove that the rigidity condition follows from conditions (a), (b) and (c′) below.

(c′) Both the following hold:

- (c′-bi) (Common E-form of formal bi-characters) There exists a subtorus $\mathbf{T}^{\mathrm{ss}}$ of $\mathbf{T}$ such that $\mathbf{T}^{\mathrm{ss}} \times_E E_\lambda$ is a maximal torus of the derived group $\mathbf{G}^{\mathrm{der}}_\lambda$ of $\mathbf{G}_\lambda$ for all $\lambda \in \mathcal{P}_{E,f}$;
- (c′-inv) (Invariance of roots) The normalizer $N_{\mathrm{GL}_{n,E}}(\mathbf{T}^{\mathrm{ssp}})$ is invariant on the roots of the derived group $(\mathbf{G}^{\mathrm{sp}})^{\mathrm{der}}$ of $\mathbf{G}^{\mathrm{sp}}$ with respect to the maximal torus $\mathbf{T}^{\mathrm{ssp}} := \mathbf{T}^{\mathrm{sp}} \cap (\mathbf{G}^{\mathrm{sp}})^{\mathrm{der}}$.

2.2.1. Formal character and bi-character. Let F be a field and $\mathbf{G}$ a connected reductive subgroup of $\mathrm{GL}_{n,F}$. If $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, then $\mathbf{T}^{\mathrm{ss}} := \mathbf{T} \cap \mathbf{G}^{\mathrm{der}}$ is a maximal torus of the derived group $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$.

Definition 2.2 ([28, Definitions 2.2, 2.3]). (i) The inclusion $\mathbf{T} \subset \mathrm{GL}_{n,F}$ is said to be a *formal character* of $\mathbf{G} \subset \mathrm{GL}_{n,F}$.

(ii) The chain $\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,F}$ is said to be a *formal bi-character* of $\mathbf{G} \subset \mathrm{GL}_{n,F}$.

Remark 2.3. A chain of subtori $\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,F}$ is a formal bi-character of $\mathbf{G} \subset \mathrm{GL}_{n,F}$ if and only if $\mathbf{T} \subset \mathrm{GL}_{n,F}$ is a formal character of $\mathbf{G} \subset \mathrm{GL}_{n,F}$ and $\mathbf{T}^{\mathrm{ss}} \subset \mathrm{GL}_{n,F}$ is a formal character of $\mathbf{G}^{\mathrm{der}} \subset \mathrm{GL}_{n,F}$. It is clear that (c′-bi) together with (a) in the Main Theorems means that there exists a chain of subtori, denoted $\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,E}$, such that

$$(\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,E}) \times_E E_\lambda$$

is a formal bi-character of $\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}$ for all λ.

Proposition 2.4. *If conditions* (a) *and* (b) *in the Main Theorems hold and* $\mathbf{G}^{\mathrm{sp}}$ *is irreducible on* E^n*, then* (c′-bi) *holds.*

Proof. Let $\mathbf{T} \subset \mathrm{GL}_{n,E}$ be in (a) and let $\mathbf{T}^{\mathrm{ss}}$ be the identity component of the kernel of the determinant map $\mathbf{T} \to \mathrm{GL}_{n,E} \xrightarrow{\det} \mathbb{G}_{\mathrm{m}}$. Since $\mathbf{G}_\lambda$ is connected and the representation $\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}$ is absolutely irreducible for all λ by the assumptions, $\mathbf{G}_\lambda$ is either $\mathbf{G}_\lambda^{\mathrm{der}}$ or $\mathbf{G}_\lambda^{\mathrm{der}} \cdot \mathbb{G}_{\mathrm{m}}$ by Schur's lemma. Hence by counting dimensions, $\mathbf{T}^{\mathrm{ss}} \times_E E_\lambda$ is a maximal torus of $\mathbf{G}_\lambda^{\mathrm{der}}$ for all λ. ∎

2.2.2. Invariance of roots. Let F be a field of characteristic zero and $\mathbf{G}$ a connected split semisimple subgroup of $\mathrm{GL}_{n,F}$. Fix a split maximal torus $\mathbf{T}$ of $\mathbf{G}$ and denote by $\mathbb{X}$ the character group of $\mathbf{T}$. Let $R \subset \mathbb{X}$ be the set of *roots* of $\mathbf{G}$ with respect to $\mathbf{T}$. Let $\mathbf{N} := N_{\mathrm{GL}_{n,F}}(\mathbf{T})$ be the normalizer of $\mathbf{T}$ in $\mathrm{GL}_{n,F}$. Since $\mathbf{N}$ acts on $\mathbf{T}$, it also acts on $\mathbb{X}$. We would like to know when R is invariant under $\mathbf{N}$. It is easy to see that this invariance-of-roots condition (i.e., $\mathbf{N} \cdot R = R$) is independent of the choice of the maximal torus $\mathbf{T}$ and is invariant under field extension. So, we take $F = \mathbb{C}$ for simplicity. If $\mathbf{H}$ is an almost simple factor of $\mathbf{G}$, then by the Cartan–Killing classification the root system of $\mathbf{H}$ is one of the following: A_r $(r \geq 1)$, B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 4)$, E_6, E_7, E_8, F_4, G_2. We also use the convention that $C_2 = B_2$, $D_2 = A_1^2$, and $D_3 = A_3$.

2.2.2.1. Here are some examples for the invariance-of-roots condition.

Theorem D ([28, Theorem 3.10], [29, Theorem A2]). *The following* $\mathbb{C}$*-connected semisimple groups* $\mathbf{G}$ *satisfy the invariance-of-roots condition for all representations* $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{C}}$*:*

(a) (Hypothesis A) $\mathbf{G}$ *has at most one* A_4 *almost simple factor, and if* $\mathbf{H}$ *is an almost simple factor of* $\mathbf{G}$*, then* $\mathbf{H}$ *is of type* A_r *for some* $r \in \mathbb{N} \setminus \{1, 2, 3, 5, 7, 8\}$.

(b) (Almost simple) $\mathbf{G}$ *is almost simple of type different from* $\{A_7, A_8, B_4, D_8\}$.

Suppose $\mathbf{G}$ is irreducible on the ambient space $\mathbb{C}^n$. If $\mathbf{G}_1$ is a connected normal subgroup of $\mathbf{G}$, then there exists a unique complementary connected normal subgroup $\mathbf{G}_2$ of $\mathbf{G}$ such that the natural map $\mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}$ is an isogeny of semisimple groups. Moreover, there exist unique irreducible representations V_1 and V_2 of respectively $\mathbf{G}_1$ and $\mathbf{G}_2$

such that the composition representation $\mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G} \to \mathrm{GL}_{n,\mathbb{C}}$ is equal to the tensor product representation $(\mathbf{G}_1 \times \mathbf{G}_2, V_1 \otimes V_2)$ (see [23]). We say that the representation $(\mathbf{G}_1, V_1)$ is a *factor* of the representation $(\mathbf{G}, \mathbb{C}^n)$.

Theorem E (by [41, Theorem 4]). *If $\mathbf{G}, \mathbf{G}' \subset \mathrm{GL}_{n,\mathbb{C}}$ are two connected semisimple subgroups with the same formal character $\mathbf{T} \subset \mathrm{GL}_{n,\mathbb{C}}$ and are both irreducible on the ambient space $\mathbb{C}^n$, then the roots R and R' of respectively $\mathbf{G}$ and $\mathbf{G}'$ (with respect to $\mathbf{T}$) are identical in $\mathbb{X}$ and the two representations are isomorphic unless one of the following conditions holds:*

(a) *For $r_1, \dots, r_m, r \in \mathbb{N}$ such that $r_1 + \cdots + r_m = r$, the spin representation of B_r is a factor of $(\mathbf{G}, \mathbb{C}^n)$ and the tensor product of the spin representations of B_{r_j} for all $1 \le j \le m$ is a factor of $(\mathbf{G}', \mathbb{C}^n)$.*

(b) *For $1 \le k \le r-1$ and $r \ge 2$, the representation of C_r (resp. D_r) with highest weight $(k, k-1, \dots, 2, 1, 0, \dots, 0)$ is a factor of $(\mathbf{G}, \mathbb{C}^n)$ (resp. $(\mathbf{G}', \mathbb{C}^n)$).*

(c) *The unique dimension 27 irreducible representation of A_2 (resp. G_2) is a factor of $(\mathbf{G}, \mathbb{C}^n)$ (resp. $(\mathbf{G}', \mathbb{C}^n)$).*

(d) *Pick two out of the three unique dimension $4096 = 2^{12}$ irreducible representations of C_4, D_4, and F_4. Then one is a factor of $(\mathbf{G}, \mathbb{C}^n)$ and the other one is a factor of $(\mathbf{G}', \mathbb{C}^n)$.*

The following corollary follows directly by taking $\mathbf{G}' = g\mathbf{G}g^{-1}$, where $g \in \mathbf{N}$.

Corollary 2.5. *If $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{C}}$ is a connected semisimple subgroup that is irreducible on the ambient space $\mathbb{C}^n$, then the invariance-of-roots condition holds if the following conditions are satisfied:*

(a) *For $r_1, \dots, r_m, r \in \mathbb{N}$ such that $r_1 + \cdots + r_m = r$, the spin representation of B_r and the tensor product of the spin representations of B_{r_j} for all $1 \le j \le m$ are not both factors of $(\mathbf{G}, \mathbb{C}^n)$.*

(b) *For $1 \le k \le r-1$ and $r \ge 2$, the representations of C_r and D_r with highest weight $(k, k-1, \dots, 2, 1, 0, \dots, 0)$ are not both factors of $(\mathbf{G}, \mathbb{C}^n)$.*

(c) *The unique dimension 27 irreducible representations of A_2 and G_2 are not both factors of $(\mathbf{G}, \mathbb{C}^n)$.*

(d) *Any two of the unique dimension 4096 irreducible representations of C_4, D_4, and F_4 are not both factors of $(\mathbf{G}, \mathbb{C}^n)$.*

2.2.2.2. Inspired by Theorem E, we give more examples for the invariance-of-roots condition.

Theorem 2.6. *Suppose $\mathbf{G} \subset \mathrm{GL}_{n,\mathbb{C}}$ is a connected adjoint semisimple subgroup that satisfies the following Lie type assumptions:*

(a) *$\mathbf{G}$ does not have a factor of type B_r $(r \ge 2)$.*

(b) *If $\mathbf{G}$ has a factor of type C_3, then it has no factor of type A_3.*

(c) *If $\mathbf{G}$ has a factor of type C_r, then it has no factor of type D_r $(r \ge 4)$.*

(d) *If* $\mathbf{G}$ *has a factor of type* F_4*, then it has no factor of type* D_4*.*

(e) *If* $\mathbf{G}$ *has a factor of type* G_2*, then it has no factor of type* A_2*.*

Then the invariance-of-roots condition holds.

Proof. Let $\mathbf{G}_1,\ldots,\mathbf{G}_k$ be the almost simple factors of $\mathbf{G}$. Then $\mathbf{T}_i=\mathbf{G}_i\cap\mathbf{T}$ is a maximal torus of $\mathbf{G}_i$ for all i. Let $\mathbb{X}_i$ be the character group of $\mathbf{T}_i$ and R_i the roots of $\mathbf{G}_i$ with respect to $\mathbf{T}_i$. Let $\Phi\subset\mathbb{X}$ (resp. $\Phi_i\subset\mathbb{X}_i$) be the subgroup (root lattice) generated by R (resp. R_i). One can impose a metric on the real vector space $\mathbb{X}_{\mathbb{R}}:=\mathbb{X}\otimes_{\mathbb{Z}}\mathbb{R}$ such that $(R,\mathbb{X}_{\mathbb{R}})$ is a root system, the normalizer $\mathbf{N}$ is isometric on $\mathbb{X}_{\mathbb{R}}$, and the decomposition

$$R=\coprod_{i=1}^{k}R_i\subset\bigoplus_{i=1}^{k}\Phi_i\otimes\mathbb{R}=\bigoplus_{i=1}^{k}\mathbb{X}_i\otimes\mathbb{R}=\mathbb{X}_{\mathbb{R}}\tag{4}$$

is orthogonal (see, e.g., [29, Appendix A]). The root subsystem $(R_i,\mathbb{X}_{i,\mathbb{R}}:=\mathbb{X}_i\otimes\mathbb{R})$ is irreducible for all i. The lemma below is needed.

Lemma 2.7. *Suppose* $\mathbf{G}\subset\mathrm{GL}_{n,\mathbb{C}}$ *is a connected semisimple subgroup that satisfies the assumptions of Theorem 2.6. The following assertions are equivalent:*

(i) R *is invariant under* $\mathbf{N}$*.*

(ii) *If* $g\in\mathbf{N}$*, then* $g\cdot R\subset\Phi$*.*

(iii) *If* $g\in\mathbf{N}$*, then* g *induces an automorphism of* Φ*.*

Proof. (i)⇒(ii): trivial.

(ii)⇒(iii): (ii) is equivalent to $\mathbf{N}\cdot\Phi\subset\Phi$. Since g induces an automorphism of $\mathbb{X}$ and $\mathbb{X}/\Phi$ is finite, $g\cdot\Phi\subset\Phi$ implies that $g\cdot\Phi=\Phi$.

(iii)⇒(i): The set of non-zero elements of Φ_i with the smallest length is equal to the set of short roots R_i° of R_i [41, Section 4, Lemma], which also spans $\mathbb{X}_i\otimes\mathbb{R}$. The decomposition in (4) is orthogonal and $\Phi=\bigoplus_{i=1}^k\Phi_i$ in $\mathbb{X}_{\mathbb{R}}$. Since g is isometric on $X_{\mathbb{R}}$ and induces an automorphism of Φ by (iii), g permutes the union $R_1^\circ\cup\cdots\cup R_m^\circ$. Note that $R_i^\circ=R_i$ if R_i is of type A, D, E, and the following conventions [41, p. 395]:

$$B_r^\circ=A_1^r\ (r\geq 2),\quad C_3^\circ=A_3,\quad C_r^\circ=D_r\ (r\geq 4),\quad F_4^\circ=D_4,\quad G_2^\circ=A_2.$$

These facts and assumption (a) imply that R_i° remains irreducible for all i. Then the orthogonality of the decomposition (4) and the fact that g is isometric on $X_{\mathbb{R}}$ imply that g permutes the set $\{R_1^\circ,\ldots,R_m^\circ\}$. Since g is isometric on $X_{\mathbb{R}}$, the Lie type assumptions (a)–(e) and the above facts about short roots imply that R_i and R_j $(1\leq i,j\leq m)$ are of the same type if $g\cdot R_i^\circ=R_j^\circ$. By observing how the R_i° generate R_i [24, Table 1], we obtain $g\cdot R_i=R_j$. Hence, g actually permutes the union $R_1\cup\cdots\cup R_m$. By the orthogonality of the decomposition (4), the fact that g is isometric on $X_{\mathbb{R}}$, and induction, we conclude that g permutes R. ∎

Back to the theorem, we have $\Phi=\mathbb{X}$ because $\mathbf{G}$ is adjoint. Since $\mathbb{X}$ is invariant under $\mathbf{N}$ by definition, Φ is invariant under $\mathbf{N}$. Therefore, R is invariant under $\mathbf{N}$ by the lemma. ∎

2.2.3. Conditions for rigidity.

Proposition 2.8. *If conditions* (a), (b) *in the Main Theorem(s) and* (c′) *hold, then condition* (c) *in the Main Theorem(s) also holds.*

Proof. By (a) and (c′-bi), we have a chain of subtori $\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,E}$ such that for all λ,

$$\mathbf{T}_\lambda^{\mathrm{ss}} \subset \mathbf{T}_\lambda \subset \mathrm{GL}_{n,E_\lambda} := (\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \subset \mathrm{GL}_{n,E}) \times_E E_\lambda$$

is a formal bi-character of $\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}$. By (b), we have the field extension diagram

$$\begin{tikzcd}
& \bar{E}_\lambda \arrow[dl, dash] \arrow[dr, dash] & \\
\bar{E} \arrow[dr, dash] & & E_\lambda \arrow[dl, dash] \\
& E &
\end{tikzcd}$$

and a chain $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}}$ (over E) such that for all λ, there exists an $\bar{E}_\lambda$-isomorphism of representations $f_{\bar{E}_\lambda}$ taking $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}}$ to $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda$ (omitting the extension field for simplicity). This implies that $f_{\bar{E}_\lambda}$ maps $\mathbf{T}^{\mathrm{ssp}} := \mathbf{T}^{\mathrm{sp}} \cap (\mathbf{G}^{\mathrm{sp}})^{\mathrm{der}}$ to $\mathbf{T}_\lambda^{\mathrm{ss}} = \mathbf{T}_\lambda \cap \mathbf{G}_\lambda^{\mathrm{der}}$ for all λ. Hence, we conclude that for all λ, the two chains

$$\mathbf{T}^{\mathrm{ssp}} \subset \mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \quad \text{and} \quad \mathbf{T}_\lambda^{\mathrm{ss}} \subset \mathbf{T}_\lambda \subset \mathbf{G}_\lambda \tag{5}$$

are conjugate in $\mathrm{GL}_n(\bar{E}_\lambda)$. In particular, the two E-chains

$$\mathbf{T}^{\mathrm{ssp}} \subset \mathbf{T}^{\mathrm{sp}} \quad \text{and} \quad \mathbf{T}^{\mathrm{ss}} \subset \mathbf{T} \tag{6}$$

are conjugate in $\mathrm{GL}_n(\bar{E})$. So we choose $M \in \mathrm{GL}_n(\bar{E})$ such that

$$\mathbf{T}^{\mathrm{ssp}} \subset \mathbf{T}^{\mathrm{sp}} = M(\mathbf{T}^{\mathrm{ss}} \subset \mathbf{T})M^{-1}. \tag{7}$$

To finish the proof, it suffices to find, for all λ, a matrix $B_\lambda \in \mathrm{GL}_n(\bar{E}_\lambda)$ such that conjugation by B_λ takes $M\mathbf{G}_\lambda M^{-1}$ to $\mathbf{G}^{\mathrm{sp}}$ and is the identity on $\mathbf{T}^{\mathrm{sp}} = M\mathbf{T}M^{-1}$. Such a B_λ exists. Indeed, there exists $A_\lambda \in \mathrm{GL}_n(\bar{E}_\lambda)$ such that

$$\mathbf{T}^{\mathrm{ssp}} \subset \mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} = A_\lambda M(\mathbf{T}_\lambda^{\mathrm{ss}} \subset \mathbf{T}_\lambda \subset \mathbf{G}_\lambda)M^{-1}A_\lambda^{-1} \tag{8}$$

because the chains in (5) are conjugate in $\mathrm{GL}_n(\bar{E}_\lambda)$. Then (7) and (8) imply that $A_\lambda \in N_{\mathrm{GL}_n}(\mathbf{T}^{\mathrm{ssp}})$ and conjugation by A_λ takes the roots of $M\mathbf{G}_\lambda^{\mathrm{der}}M^{-1}$ to the roots of $(\mathbf{G}^{\mathrm{sp}})^{\mathrm{der}}$. By (c′-inv), the roots of the two semisimple (derived) groups are identical (in the character group of $\mathbf{T}^{\mathrm{ssp}}$). Hence, [28, Theorem 3.8] implies that the absolute root data of $M\mathbf{G}_\lambda M^{-1}$ and $\mathbf{G}^{\mathrm{sp}}$ are identical with respect to the common maximal torus $M\mathbf{T}_\lambda M^{-1} = \mathbf{T}^{\mathrm{sp}}$. By [62, Theorem 16.3.2], there exists an $\bar{E}_\lambda$-isomorphism b_λ taking the pair $(M\mathbf{G}_\lambda M^{-1}, M\mathbf{T}_\lambda M^{-1})$ to the pair $(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})$ inducing the identity map between their root data. Let i_1 and i_2 be the tautological representations of $M\mathbf{G}_\lambda M^{-1}$ and $\mathbf{G}^{\mathrm{sp}}$ into GL_n. Then the two representations i_1 and $i_2 \circ b_\lambda$ are isomorphic. Therefore, b_λ is just conjugation by a matrix $B_\lambda \in \mathrm{GL}_n(\bar{E}_\lambda)$ that is the identity on $M\mathbf{T}_\lambda M^{-1} = \mathbf{T}^{\mathrm{sp}}$. ■

2.3. *Forms of reductive chains*

This section is foundational for the proofs of the Main Theorems and is developed from [28, Section 4].

2.3.1. Galois cohomology. Let F be a field of characteristic zero, and $\mathbf{G}_1$ and $\mathbf{G}_1'$ be linear algebraic groups defined over F. The Galois group $\mathrm{Gal}(\overline{F}/F)$ acts (on the left) on the set of $\overline{F}$-homomorphisms $\phi : \mathbf{G}_1 \times_F \overline{F} \to \mathbf{G}_1' \times_F \overline{F}$ as follows: if $\sigma \in \mathrm{Gal}(\overline{F}/F)$, then ${}^{\sigma}\phi$ is the homomorphism such that

$$ {}^{\sigma}\phi(x) = \sigma(\phi(\sigma^{-1}x)) \quad \forall x \in \mathbf{G}_1(\overline{F}). $$

Let $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ be a chain of linear algebraic groups defined over F. An *F-form* of the chain $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ is a chain of reductive groups $\mathbf{G}_k' \subset \cdots \subset \mathbf{G}_1'$ defined over F that is isomorphic to $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ over $\overline{F}$, i.e., there exists an $\overline{F}$-homomorphism $\phi : \mathbf{G}_1 \times_F \overline{F} \to \mathbf{G}_1' \times_F \overline{F}$ such that $\phi(\mathbf{G}_i \times_F \overline{F}) \subset \mathbf{G}_i' \times_F \overline{F}$ and the restriction $\phi|_{\mathbf{G}_i \times_F \overline{F}}$ is an isomorphism for all $1 \leq i \leq k$. Since the groups are defined over F, the $\overline{F}$-homomorphism ${}^{\sigma}\phi$ is also an $\overline{F}$-isomorphism between the two chains. In particular, the *automorphism group* $\mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k)$ *of the chain* (i.e., the subgroup of the automorphism group $\mathrm{Aut}_{\overline{F}}\, \mathbf{G}_1$ of $\mathbf{G}_1 \times_F \overline{F}$ preserving the chain $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$) is a $\mathrm{Gal}(\overline{F}/F)$-group. Let $\phi : \mathbf{G}_1 \times_F \overline{F} \to \mathbf{G}_1' \times_F \overline{F}$ be an $\overline{F}$-isomorphism from $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ to $\mathbf{G}_k' \subset \cdots \subset \mathbf{G}_1'$. Then the assignment

$$ \sigma \mapsto a_\sigma := \phi^{-1} \circ {}^{\sigma}\phi \in \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) \tag{9} $$

for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ satisfies the 1-cocycle condition

$$ a_{\sigma\sigma'} = a_\sigma {}^{\sigma}a_{\sigma'}, $$

producing a bijective correspondence (see [60, Chapter 3.1, Proposition 5 and its proof]) between the set of isomorphism classes of F-forms of the chain $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ and the Galois cohomology pointed set $H^1(F, \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k))$ in which the neutral element is the trivial class $[a_\sigma = \mathrm{id}]$ corresponding to the F-isomorphism class of $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$.

Let $\mathrm{Inn}_{\overline{F}}\, \mathbf{G}_1$ be the inner automorphism group of $\mathbf{G}_1 \times_F \overline{F}$. It is a ($\mathrm{Gal}(\overline{F}/F)$-) normal subgroup of $\mathrm{Aut}_{\overline{F}}\, \mathbf{G}_1$. Denote the *inner automorphism group* of the chain by

$$ \mathrm{Inn}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) := \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) \cap \mathrm{Inn}_{\overline{F}}\, \mathbf{G}_1. $$

and the *outer automorphism group* of the chain by

$$ \mathrm{Out}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) := \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k)/\mathrm{Inn}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k). $$

Then we obtain a short exact sequence of $\mathrm{Gal}(\overline{F}/F)$-groups

$$ 1 \to \mathrm{Inn}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) \to \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) \to \mathrm{Out}_{\overline{F}}(\mathbf{G}_1, \ldots, \mathbf{G}_k) \to 1 \tag{10} $$

and an exact sequence of pointed sets [60, Chapter 1.5.5, Proposition 38]

$$H^1(F, \mathrm{Inn}_{\overline{F}}(\mathbf{G}_1, \dots, \mathbf{G}_k)) \xrightarrow{i} H^1(F, \mathrm{Aut}_{\overline{F}}(\mathbf{G}_1, \dots, \mathbf{G}_k)) \\ \xrightarrow{\pi} H^1(F, \mathrm{Out}_{\overline{F}}(\mathbf{G}_1, \dots, \mathbf{G}_k)). \tag{11}$$

The exactness means that the preimage $\pi^{-1}([\mathrm{id}])$ is equal to the image $\mathrm{Im}\,(i)$.

An F-form $\mathbf{G}'_k \subset \cdots \subset \mathbf{G}'_1$ of $\mathbf{G}_k \subset \cdots \subset \mathbf{G}_1$ is called an *inner F-form* (or *inner form*) if there exists an $\overline{F}$-isomorphism ϕ such that in (9), the element a_σ belongs to $\mathrm{Inn}_{\overline{F}}(\mathbf{G}_1, \dots, \mathbf{G}_k)$ for all σ. In general, the isomorphism classes of inner F-forms do not form a subset of the isomorphism classes of F-forms since the map i in (11) is not injective. However, the sequence (11) is a short exact sequence of pointed sets (and thus i is injective) if (10) splits. We will see in later sections that the splitting of (10) holds for some chains (e.g., $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}}$). The following simple lemma is useful to study the conjugacy class of a subgroup in $\mathrm{GL}_{n,F}$.

Lemma 2.9. *Let D be a central division algebra over F. Let $\mathbf{U} = \mathrm{GL}_{m,D}$ be an F-inner form of $\mathrm{GL}_{n,F}$, and $\mathbf{T} \subset \mathbf{G} \subset \mathrm{GL}_{n,F}$ and $\mathbf{T}' \subset \mathbf{G}' \subset \mathbf{U}$ be two chains. If the two chains of $\overline{F}$-representations $(\mathbf{T} \subset \mathbf{G} \hookrightarrow \mathrm{GL}_{n,F}) \times_F \overline{F}$ and $(\mathbf{T}' \subset \mathbf{G}' \hookrightarrow \mathbf{U}) \times_F \overline{F}$ are isomorphic, then the following hold:*

(i) *The chain $\mathbf{T}' \subset \mathbf{G}' \subset \mathbf{U}$ is an inner form of $\mathbf{T} \subset \mathbf{G} \subset \mathrm{GL}_{n,F}$.*

(ii) *If the cohomology class $[\mathbf{T}' \subset \mathbf{G}' \subset \mathbf{U}] \in H^1(F, \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}, \mathbf{T}))$ is the neutral class, then $D = F$ and the two F-representations $\mathbf{T} \subset \mathbf{G} \hookrightarrow \mathrm{GL}_{n,F}$ and $\mathbf{T}' \subset \mathbf{G}' \hookrightarrow \mathbf{U} = \mathrm{GL}_{n,F}$ are isomorphic.*

Proof. Identify $\mathbf{U} \times_F \overline{F}$ with $\mathrm{GL}_{n,\overline{F}}$. The condition implies that there exists an $\overline{F}$-inner automorphism ψ of $\mathrm{GL}_{n,\overline{F}}$ such that $\psi(\mathbf{G} \times_F \overline{F}) = \mathbf{G}' \times_F \overline{F}$ and $\psi(\mathbf{T} \times_F \overline{F}) = \mathbf{T}' \times_F \overline{F}$. This defines a 1-cocycle

$$\sigma \mapsto a_\sigma := \psi^{-1} \circ {}^\sigma\psi \in \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}, \mathbf{T}),$$

which proves (i). If the cocycle is neutral, then there exists $\gamma \in \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}, \mathbf{T}) \subset \mathrm{PGL}_n(\overline{F})$ such that $a_\sigma = \gamma^{-1} \circ {}^\sigma\gamma$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$. This is equivalent to

$$\psi \circ \gamma^{-1} = {}^\sigma(\psi \circ \gamma^{-1}) \quad \forall \sigma \in \mathrm{Gal}(\overline{F}/F).$$

Hence, $\psi \circ \gamma^{-1} \in \mathrm{PGL}_n(F)$ and $\mathrm{GL}_{n,F}$ and $\mathrm{GL}_{m,D}$ are F-isomorphic. Therefore, $D = F$, $\mathbf{U} = \mathrm{GL}_{n,F}$, and $\psi \circ \gamma^{-1}$ is an F-inner automorphism of $\mathrm{GL}_{n,F}$ taking $\mathbf{G}$ to $\mathbf{G}'$ as well as $\mathbf{T}$ to $\mathbf{T}'$, which proves (ii). ∎

2.3.2. Some diagrams. In this section, some diagrams of groups and Galois cohomology will be presented. Let F be a field. Denote by

- $\mathbf{G}^{\mathrm{sp}}$ a connected split reductive group defined over F,
- $\mathbf{T}^{\mathrm{sp}}$ a split maximal torus of $\mathbf{G}^{\mathrm{sp}}$,

- $\mathbf{N}$ the normalizer of $\mathbf{T}^{\mathrm{sp}}$ in $\mathbf{G}^{\mathrm{sp}}$,
- $W := \mathbf{N}/\mathbf{T}^{\mathrm{sp}}$ the Weyl group,
- $\mathbf{B}$ a Borel subgroup of $\mathbf{G}^{\mathrm{sp}}$ containing $\mathbf{T}^{\mathrm{sp}}$,
- $\mathbf{C}$ the center of $\mathbf{G}^{\mathrm{sp}}$,
- $(\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}} := \mathbf{G}^{\mathrm{sp}}/\mathbf{C}$ the adjoint quotient of $\mathbf{G}^{\mathrm{sp}}$,
- $\Theta_{\overline{F}} := \mathrm{Out}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}}$ the outer automorphism group of $\mathbf{G}^{\mathrm{sp}}$,
- $Z^k(F, \mathbf{M}) := Z^k(F, \mathbf{M}(\overline{F}))$ the cocycles if $\mathbf{M}$ is a linear algebraic group defined over F,
- $H^k(F, \mathbf{M}) := H^k(F, \mathbf{M}(\overline{F}))$ the cohomology if $\mathbf{M}$ is a linear algebraic group defined over F.

2.3.2.1. Consider the following diagram of $\mathrm{Gal}(\overline{F}/F)$-groups:

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{N}/\mathbf{C}(\overline{F}) \arrow[r, "i"] \arrow[d, hook] & \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, "\pi"] \arrow[d, hook, "\mathrm{Res}_{\mathbf{G}^{\mathrm{sp}}}"'] & \Theta_{\overline{F}} \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}(\overline{F}) \arrow[r, "i"] & \mathrm{Aut}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}} \arrow[r, "\pi"] & \Theta_{\overline{F}} \arrow[r] & 1
\end{tikzcd} \tag{12}$$

where the top (resp. bottom) row is (10) for $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}}$ by [28, Proposition 4.3] (resp. $\mathbf{G}^{\mathrm{sp}}$) and the vertical arrows are all natural inclusions induced by restricting automorphisms to $\mathbf{G}^{\mathrm{sp}}$:

$$\mathrm{Res}_{\mathbf{G}^{\mathrm{sp}}} : \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \to \mathrm{Aut}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}}. \tag{13}$$

Since $\mathbf{G}^{\mathrm{sp}}$ is split, the Galois group $\mathrm{Gal}(\overline{F}/F)$ acts trivially on the outer automorphism group $\Theta_{\overline{F}}$. The proposition below is well-known.

Proposition F (see, e.g., [28, Proposition 4.1]). *The automorphism group* $\mathrm{Aut}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}}$ *contains a* $\mathrm{Gal}(\overline{F}/F)$*-invariant subgroup that preserves* $\mathbf{T}^{\mathrm{sp}}$ *and* $\mathbf{B}$ *and is mapped isomorphically onto* $\mathrm{Out}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}}$. *Hence, the top (resp. bottom) row in (12) is a split short exact sequence of* $\mathrm{Gal}(\overline{F}/F)$*-groups*

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{N}/\mathbf{C}(\overline{F}) \arrow[r, "i"] & \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, "\pi"] & \Theta_{\overline{F}} \arrow[r] \arrow[l, bend right, "j"'] & 1.
\end{tikzcd} \tag{14}$$

Denote

$$\Omega_{\overline{F}} := \mathrm{Im}(\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}),$$

where $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}$ restricts automorphisms to $\mathbf{T}^{\mathrm{sp}}$:

$$\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}} : \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \to \mathrm{Aut}_{\overline{F}}\, \mathbf{T}^{\mathrm{sp}}. \tag{15}$$

Then the first row in (12) also fits into the following diagram of $\mathrm{Gal}(\overline{F}/F)$-groups with exact rows and columns by [28, Proposition 4.3] and j denotes a splitting induced by (14).

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbf{T}^{\mathrm{sp}}/\mathbf{C}(\bar{F}) \arrow[r] \arrow[d] & \mathbf{T}^{\mathrm{sp}}/\mathbf{C}(\bar{F}) \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{N}/\mathbf{C}(\bar{F}) \arrow[r, "i"] \arrow[d] & \mathrm{Aut}_{\bar{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}"'] & \Theta_{\bar{F}} \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 1 \\
1 \arrow[r] & W \arrow[r, "i"] \arrow[d] & \Omega_{\bar{F}} \arrow[r, "\pi"] \arrow[d] & \Theta_{\bar{F}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{16}$$

2.3.2.2. Suppose we are given a faithful (absolutely) irreducible representation $\mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,F}$. Then we have the chain $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,F}$. The irreducibility condition implies that $\mathbf{C}$ is contained in the subgroup of scalars in $\mathrm{GL}_{n,F}$ and the following inclusions hold:

$$\begin{tikzcd}
\mathbf{N}/\mathbf{C}(\bar{F}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, hook] \arrow[d, hook, "\mathrm{Res}_{(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}})}"'] & \mathrm{Aut}_{\bar{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[d, hook, "\mathrm{Res}_{\mathbf{G}^{\mathrm{sp}}}"] \\
(\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}(\bar{F}) \arrow[r, hook] & \mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}) \arrow[r, hook] & \mathrm{Aut}_{\bar{F}}\, \mathbf{G}^{\mathrm{sp}}
\end{tikzcd} \tag{17}$$

In diagram (16), denote

$$\theta_{\bar{F}} := \pi(\mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})) \in \Theta_{\bar{F}},$$
$$\omega_{\bar{F}} := \mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}(\mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})) \in \Omega_{\bar{F}}.$$

By diagrams (12), (16), (17) and the fact that the squares in (17) are Cartesian, we obtain the following two diagrams with exact rows and columns. Moreover, (18) injects naturally into (12), (19) injects naturally into (16), and j denotes the splitting induced by (14).

$$\begin{tikzcd}
1 \arrow[r] & \mathbf{N}/\mathbf{C}(\bar{F}) \arrow[r, "i"] \arrow[d, hook] & \mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, "\pi"] \arrow[d, hook, "\mathrm{Res}_{(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}})}"'] & \theta_{\bar{F}} \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 1 \\
1 \arrow[r] & (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}(\bar{F}) \arrow[r, "i"] & \mathrm{Inn}_{\bar{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}) \arrow[r, "\pi"] & \theta_{\bar{F}} \arrow[r] \arrow[l, bend right, "j"'] & 1
\end{tikzcd} \tag{18}$$

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathbf{T}^{\mathrm{sp}}/\mathbf{C}(\overline{F}) \arrow[r] \arrow[d] & \mathbf{T}^{\mathrm{sp}}/\mathbf{C}(\overline{F}) \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbf{N}/\mathbf{C}(\overline{F}) \arrow[r, "i"] \arrow[d] & \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}}) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}"'] & \theta_{\overline{F}} \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 1 \\
1 \arrow[r] & W \arrow[r, "i"] \arrow[d] & \omega_{\overline{F}} \arrow[r, "\pi"] \arrow[d] & \theta_{\overline{F}} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
\tag{19}
$$

2.3.2.3. By taking Galois cohomology of diagrams (12), (16), (18), (19), the splitting j, and Hilbert's Theorem 90: $H^1(F, \mathbf{T}^{\mathrm{sp}}/\mathbf{C}) = H^1(F, \mathbb{G}_{\mathrm{m}})^{\oplus k} = 0$, we obtain the following diagrams of pointed sets such that the rows and columns are all exact. Moreover, there are natural maps from (22) to (20), (23) to (21), and j again denotes the splitting.

$$
\begin{tikzcd}
0 \arrow[r] & H^1(F, \mathbf{N}/\mathbf{C}) \arrow[r, "i"] \arrow[d] & H^1(F, \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{\mathbf{G}^{\mathrm{sp}}}"'] & H^1(F, \Theta_{\overline{F}}) \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 0 \\
0 \arrow[r] & H^1(F, (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}) \arrow[r, "i"] & H^1(F, \mathrm{Aut}_{\overline{F}}\, \mathbf{G}^{\mathrm{sp}}) \arrow[r, "\pi"] & H^1(F, \Theta_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j"'] & 0
\end{tikzcd}
\tag{20}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^1(F, \mathbf{N}/\mathbf{C}) \arrow[r, "i"] \arrow[d] & H^1(F, \mathrm{Aut}_{\overline{F}}(\mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}"'] & H^1(F, \Theta_{\overline{F}}) \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 0 \\
0 \arrow[r] & H^1(F, W) \arrow[r, "i"] & H^1(F, \Omega_{\overline{F}}) \arrow[r, "\pi"] & H^1(F, \Theta_{\overline{F}}) \arrow[r] & 0
\end{tikzcd}
\tag{21}
$$

$$
\begin{tikzcd}
0 \arrow[r] & H^1(F, \mathbf{N}/\mathbf{C}) \arrow[r, "i"] \arrow[d] & H^1(F, \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}}, \mathbf{T}^{\mathrm{sp}})) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}})}"'] & H^1(F, \theta_{\overline{F}}) \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 0 \\
0 \arrow[r] & H^1(F, (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}) \arrow[r, "i"] & H^1(F, \mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F}, \mathbf{G}^{\mathrm{sp}})) \arrow[r, "\pi"] & H^1(F, \theta_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j"'] & 0
\end{tikzcd}
\tag{22}
$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^1(F,\mathbf{N}/\mathbf{C}) \arrow[r, "i"] \arrow[d] & H^1(F,\mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F},\mathbf{G}^{\mathrm{sp}},\mathbf{T}^{\mathrm{sp}})) \arrow[r, "\pi"] \arrow[d, "\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}"'] & H^1(F,\theta_{\overline{F}}) \arrow[r] \arrow[d, "="] \arrow[l, bend right, "j"'] & 0 \\
0 \arrow[r] & H^1(F,W) \arrow[r, "i"] & H^1(F,\omega_{\overline{F}}) \arrow[r, "\pi"] & H^1(F,\theta_{\overline{F}}) \arrow[r] & 0
\end{tikzcd} \tag{23}$$

2.4. *Twisting*

Let G be a profinite group and A be a G-group (a discrete group on which G acts continuously). The Galois cohomology $H^1(G,A)$ is a pointed set with neutral element given by the trivial class $[\mathrm{id}_A]$. Let $1 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 1$ be a short exact sequence of G-groups. Then one obtains an exact sequence of pointed sets

$$H^1(G,A) \xrightarrow{i} H^1(G,B) \xrightarrow{\pi} H^1(G,C),$$

meaning that the image of i is equal to $\pi^{-1}([\mathrm{id}_C]) = \pi^{-1}(\pi([\mathrm{id}_B]))$, the fiber of $\pi([\mathrm{id}_B])$. Let $[\beta] \in H^1(G,B)$ be a cohomology class. To study the image of π as well as the fiber of $\pi([\beta])$, that is, the set $\pi^{-1}(\pi([\beta]))$, one uses the method of *twisting* in [60, Chapters 1.5.3–1.5.7]. This technique will be applied to some short exact sequences in Section 2.4.2.

2.4.1. Definition. Let G be a group, M a (left) G-group, and A (resp. B) an M-group on which G acts compatibly on the left, i.e., $g(m(a)) = g(m)(g(a))$ for $g \in G$, $m \in M$, and $a \in A$. Suppose $\mu := (m_g) \in Z^1(G,M)$ is a 1-cocycle. Then one can define a G-group ${}_\mu A$ *twisted by* μ, which can be viewed as A with a new G-action: as a group ${}_\mu A = A$ and the G-action is defined by

$$G \times {}_\mu \to {}_\mu A, \quad (g,a) \mapsto m_g(g(a)). \tag{24}$$

As M acts on itself by inner automorphism (conjugation): $(-) \mapsto m(-)m^{-1}$, denote by ${}_\mu M$ the twisted G-group. Then ${}_\mu A$ is a ${}_\mu M$-group under the identification

$$\begin{tikzcd}
{}_\mu M \times {}_\mu A \arrow[r] \arrow[d, "="'] & {}_\mu A \arrow[d, "="] \\
M \times A \arrow[r] & A
\end{tikzcd} \tag{25}$$

on which G acts compatibly on the left. If $\mu, \mu' \in Z^1(G,M)$ are cohomologous, then ${}_\mu A$ and ${}_{\mu'} A$ are isomorphic. The assignment $A \mapsto {}_\mu A$ is functorial: if $f : A \to B$ is a G-, M-group homomorphism, then ${}_\mu f : {}_\mu A \to {}_\mu B$ is a G-, ${}_\mu M$-group homomorphism [60, Chapter 1.5.3]. Since A acts on itself by inner automorphisms $A \to \mathrm{Inn}(A)$, it acts on B via the map $A \to B \to \mathrm{Inn}(B)$ such that $A \to B$ is an A-group homomorphism. The following correspondences are crucial.

Proposition G ([60, Chapter 1.5.3, Proposition 35 bis]). *Let $f : A \to B$ be a G-group homomorphism, $\alpha = (a_g) \in Z^1(G, A)$ be a cocycle, and $\beta = (b_g) \in Z^1(G, B)$ the image of α. Write $A' = {}_\alpha A$, $B' = {}_\beta B$, and $f' : A' \to B'$ the map. To each cocycle $(a'_g) \in Z^1(G, A')$ (resp. $(b'_g) \in Z^1(G, B')$), associate the cocycle $(a'_g a_g) \in Z^1(G, A)$ (resp. $(b'_g b_g) \in Z^1(G, B)$). This induces the following commutative diagrams whose vertical arrows are bijective correspondences taking neutral cocycles (resp. classes) to α, β (resp. $[\alpha], [\beta]$):*

$$\begin{tikzcd} Z^1(G, A') \arrow[r, "f'"] \arrow[d, "t_\alpha"'] & Z^1(G, B') \arrow[d, "t_\beta"] \\ Z^1(G, A) \arrow[r, "f"] & Z^1(G, B) \end{tikzcd} \qquad \begin{tikzcd} H^1(G, A') \arrow[r, "f'"] \arrow[d, "\tau_\alpha"'] & H^1(G, B') \arrow[d, "\tau_\beta"] \\ H^1(G, A) \arrow[r, "f"] & H^1(G, B) \end{tikzcd} \tag{26}$$

Therefore, $\tau_\alpha : (f')^{-1}(f'([\mathrm{id}_{A'}])) \to f^{-1}(f([\alpha]))$ is a bijective correspondence between the fibers of classes.

2.4.2. Fibers of π. Given a split short exact sequence of G-groups

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "\pi"] & C \arrow[l, bend right, "j"'] \arrow[r] & 1, \end{tikzcd} \tag{27}$$

we obtain a split short exact sequence of pointed sets

$$\begin{tikzcd} 0 \arrow[r] & H^1(G, A) \arrow[r, "i"] & H^1(G, B) \arrow[r, "\pi"] & H^1(G, C) \arrow[l, bend right, "j"'] \arrow[r] & 0. \end{tikzcd} \tag{28}$$

Since C acts on itself by inner automorphism, it also acts on B and A by the splitting j. Let $\chi \in Z^1(G, C)$ be a cocycle. It can also be seen as a cocycle in B via j. Hence, we let

$$\begin{tikzcd} 1 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "\pi'"] & C' \arrow[l, bend right, "j'"'] \arrow[r] & 1 \end{tikzcd} \tag{29}$$

be the split short exact sequence of G-groups constructed by twisting (27) by χ. We obtain the corollary below by Proposition G.

Corollary 2.10. *In the diagram below, the rows are split short exact sequence of pointed sets and the vertical arrows are bijective with $\tau_{j(\chi)}([\mathrm{id}_{B'}]) = [j(\chi)]$, $\tau_\chi([\mathrm{id}_{C'}]) = [\chi]$, and $\tau_\chi \circ \pi' = \pi \circ \tau_{j(\chi)}$:*

$$\begin{tikzcd} 0 \arrow[r] & H^1(G, A') \arrow[r, "i'"] & H^1(G, B') \arrow[r, "\pi'"] \arrow[d, "\tau_{j(\chi)}"'] & H^1(G, C') \arrow[l, bend right, "j'"'] \arrow[r] \arrow[d, "\tau_\chi"] & 0 \\ 0 \arrow[r] & H^1(G, A) \arrow[r, "i"] & H^1(G, B) \arrow[r, "\pi"] & H^1(G, C) \arrow[l, bend right, "j"'] \arrow[r] & 0 \end{tikzcd} \tag{30}$$

2.4.2.1. Let $\mathbf{G}^{\mathrm{sp}}$ be a connected split reductive group defined over F. By Proposition F, there is a split short exact sequence of $\mathrm{Gal}(\overline{F}/F)$-groups

$$\begin{tikzcd} 0 \arrow[r] & (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}(\overline{F}) \arrow[r, "i"] & \mathrm{Aut}_{\overline{F}}\,\mathbf{G}^{\mathrm{sp}} \arrow[r, "\pi"] & \Theta_{\overline{F}} \arrow[r] \arrow[l, bend right, "j"'] & 0, \end{tikzcd}$$

inducing a split short exact sequence of pointed sets

$$\begin{tikzcd} 0 \arrow[r] & H^1(F, (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}) \arrow[r, "i"] & H^1(F, \mathrm{Aut}_{\overline{F}}\,\mathbf{G}^{\mathrm{sp}}) \arrow[r, "\pi"] & H^1(F, \Theta_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j"'] & 0. \end{tikzcd} \tag{31}$$

A reductive group $\mathbf{G}/F$ is said to be *quasi-split* if $\mathbf{G}$ has a Borel subgroup defined over F. The group $\Theta_{\overline{F}}$ via j is a group of F-automorphisms of $\mathbf{G}^{\mathrm{sp}}/F$. The image of j in (31) can be characterized.

Theorem I (see, e.g., [28, Theorem 4.2] and its proof). *The set $j(H^1(F, \Theta_{\overline{F}}))$ in (31) is equal to the set of isomorphism classes of quasi-split F-forms of $\mathbf{G}^{\mathrm{sp}}$. Moreover, if $\chi \in Z^1(F, \Theta_{\overline{F}})$, then the* $\mathrm{Gal}(\overline{F}/F)$*-group ${}_\chi\mathbf{G}^{\mathrm{sp}}(\overline{F})$ is the $\overline{F}$-points of a quasi-split connected reductive group $\mathbf{G}'$ over F corresponding to the $\overline{F}$-isomorphism class* $[\mathbf{G}'] = j([\chi])$.

Since the twisted automorphism group ${}_\chi\mathrm{Aut}_{\overline{F}}\,\mathbf{G}^{\mathrm{sp}}$ acts on ${}_\chi\mathbf{G}^{\mathrm{sp}}(\overline{F}) = \mathbf{G}'(\overline{F})$ by Theorem I, the twisted group ${}_\chi\mathrm{Aut}_{\overline{F}}\,\mathbf{G}^{\mathrm{sp}}$ is naturally isomorphic to $\mathrm{Aut}_{\overline{F}}\,\mathbf{G}'$. Denote by $\mathbf{G}'^{\mathrm{ad}}$ the adjoint quotient of $\mathbf{G}'$. By Corollary 2.10, the following diagram has split short exact rows of pointed sets and the vertical arrows are bijective with $\tau_{j(\chi)}([\mathrm{id}]) = [\mathbf{G}']$ and $\tau_\chi \circ \pi' = \pi \circ \tau_{j(\chi)}$:

$$\begin{tikzcd} 0 \arrow[r] & H^1(F, \mathbf{G}'^{\mathrm{ad}}) \arrow[r, "i'"] & H^1(F, \mathrm{Aut}_{\overline{F}}\,\mathbf{G}') \arrow[r, "\pi'"] \arrow[d, "\tau_{j(\chi)}"'] & H^1(F, \Theta'_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j'"'] \arrow[d, "\tau_\chi"] & 0 \\ 0 \arrow[r] & H^1(F, (\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}) \arrow[r, "i"] & H^1(F, \mathrm{Aut}_{\overline{F}}\,\mathbf{G}^{\mathrm{sp}}) \arrow[r, "\pi"] & H^1(F, \Theta_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j"'] & 0 \end{tikzcd} \tag{32}$$

Remark 2.11. (1) The middle vertical correspondence $\tau_{j(\chi)}$ in (32) is the identity map if we identify the set of isomorphism classes of F-forms of $\mathbf{G}'$ with that of $\mathbf{G}^{\mathrm{sp}}$ in a natural way.

(2) The twisted group $\Theta'_{\overline{F}}$ is naturally isomorphic to $\mathrm{Out}_{\overline{F}}\,\mathbf{G}'$ and corresponds via j' to the set of isomorphism classes of quasi-split F-forms of $\mathbf{G}'$.

(3) Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two F-forms of $\mathbf{G}^{\mathrm{sp}}$. The form $\mathbf{G}_1$ is said to be an *inner form* of $\mathbf{G}_2$ if $\pi([\mathbf{G}_1]) = \pi([\mathbf{G}_2])$. By Theorem I, any F-form $\mathbf{G}_1$ is an inner form of a unique quasi-split F-form $\mathbf{G}'$.

2.4.2.2. Similarly, let $\chi \in Z^1(F, \theta_{\overline{F}})$ and twist the second row of (18) by χ. Then we obtain an F-form $\mathbf{G}' \subset \mathrm{GL}_{m',D'}$ of the chain $\mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,F}$, where $\mathbf{G}'$ is a quasi-split F-

form of $\mathbf{G}^{\mathrm{sp}}$ and $\mathrm{GL}_{m',D'}$ is an inner form of $\mathrm{GL}_{n,F}$ (for some central division algebra D' over F). Since $\mathbf{G}'$ is quasi-split and the tautological representation is absolutely irreducible, it follows that $\mathrm{GL}_{m',D'} = \mathrm{GL}_{n,F}$ [66, Theorem 3.3] and the F-form is

$$\mathbf{G}' \subset \mathrm{GL}_{n,F} \tag{33}$$

such that the following diagram has split short exact rows of pointed sets and the vertical arrows are bijective with $\tau_{j(\chi)}([\mathrm{id}]) = [\mathbf{G}' \subset \mathrm{GL}_{n,F}]$ and $\tau_\chi \circ \pi' = \pi \circ \tau_{j(\chi)}$:

$$\begin{tikzcd}
0 \arrow[r] & H^1(F,\mathbf{G}'^{\mathrm{ad}}) \arrow[r, "i'"] & H^1(F,\mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F},\mathbf{G}')) \arrow[r, "\pi'"] \arrow[d, "\tau_{j(\chi)}"'] & H^1(F,\theta'_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j'"'] \arrow[d, "\tau_\chi"] & 0 \\
0 \arrow[r] & H^1(F,(\mathbf{G}^{\mathrm{sp}})^{\mathrm{ad}}) \arrow[r, "i"] & H^1(F,\mathrm{Inn}_{\overline{F}}(\mathrm{GL}_{n,F},\mathbf{G}^{\mathrm{sp}})) \arrow[r, "\pi"] & H^1(F,\theta_{\overline{F}}) \arrow[r] \arrow[l, bend right, "j"'] & 0
\end{tikzcd} \tag{34}$$

Corollary 2.12. *The fiber* $\pi^{-1}([\chi])$ *in* (32) (*resp.* (34)) *can be identified with* $H^1(F,\mathbf{G}'^{\mathrm{ad}})$.

2.4.3. Image of π. Given a short exact sequence of G-groups with A abelian

$$1 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 1, \tag{35}$$

C acts on A naturally and there is the twisted group ${}_\chi A$ for every $\chi \in Z^1(G,C)$. One associates to χ a cohomology class $\Delta(\chi) \in H^2(G, {}_\chi A)$ as follows. Lift χ to a continuous map $g \mapsto b_g$ of G into B and define

$$a_{g,g'} = b_g g(b_{g'}) b_{gg'}^{-1}, \tag{36}$$

which is a 2-cocycle with values in ${}_\chi A$ [60, Chapter 1.5.6].

Proposition J ([60, Chapter 1.5.6, Proposition 41]). *The cohomology class* $[\chi]$ *belongs to the image of* $\pi : H^1(G,B) \to H^1(G,C)$ *if and only if* $\Delta(\chi)$ *vanishes in* $H^2(G, {}_\chi A)$.

Since the middle columns of (16) and (19) are short exact sequences of $\mathrm{Gal}(\overline{F}/F)$-groups with $\mathbf{T}^{\mathrm{sp}}/\mathbf{C}$ abelian, we obtain the following.

Corollary 2.13. *Let* $\mu \in Z^1(F,\Omega_{\overline{F}})$ (*resp.* $\mu \in Z^1(F,\omega_{\overline{F}})$). *The cohomology class* $[\mu]$ *belongs to the image of* $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}$ *in* (21) (*resp.* (23)) *if and only if* $\Delta(\mu)$ *vanishes in* $H^2(F, {}_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C}))$.

2.5. Local-global aspects

2.5.1. The localization map. Let E be a number field and $\mathscr{P}_E$ be the set of places of E. Let $\mathbf{G}$ be a linear algebraic group (or more generally an automorphism group of a reductive chain in Section 2.3.1) defined over E . For any $\lambda \in \mathscr{P}_E$, denote by E_λ the completion of E with respect to λ and by $i_\lambda : E \to E_\lambda$ the embedding. Let $i_{\bar\lambda} : \overline{E} \to \overline{E}_\lambda$ be an embedding extending i_λ. Then it induces homomorphisms $\mathrm{Gal}(\bar{E}_\lambda/E_\lambda) \to \mathrm{Gal}(\bar{E}/E)$ and $\mathbf{G}(\bar{E}) \to \mathbf{G}(\bar{E}_\lambda)$ for which the $\mathrm{Gal}(\bar{E}_\lambda/E_\lambda)$-module $\mathbf{G}(\bar{E}_\lambda)$ and $\mathrm{Gal}(\bar{E}/E)$-module $\mathbf{G}(\bar{E})$ are compatible. We obtain a map of cocycles ($k = 0, 1$ if $\mathbf{G}$ non-abelian)

$$\mathrm{loc}_{\bar{\lambda}} : Z^k(E, \mathbf{G}) \to Z^k(E_\lambda, \mathbf{G}). \tag{37}$$

The associated map of Galois cohomology

$$\mathrm{loc}_\lambda : H^k(E, \mathbf{G}) \to H^k(E_\lambda, \mathbf{G}) \tag{38}$$

is called the *localization map at* λ. It is functorial and does not depend on $i_{\bar{\lambda}}$ [60, Chapter 2.1.1].

2.5.2. Some results. We would like to present some results for the map

$$\prod_{\lambda\in\mathscr{P}_E} \mathrm{loc}_\lambda : H^k(E, \mathbf{G}) \to \prod_{\lambda\in\mathscr{P}_E} H^k(E_\lambda, \mathbf{G}) \tag{39}$$

when $\mathbf{G}$ is connected reductive and $k = 1$ and when $\mathbf{G}$ is a torus and $k = 2$. For simplicity, we use the notation and formulation of [5] although the results were obtained earlier by Harder [25], Kneser [36], Sansuc [54], Kottwitz [37]. Let $\text{Ш}^k(E, \mathbf{G})$ be the kernel of the map (39). The reductive group $\mathbf{G}$ is said to satisfy the *Hasse principle* if the *Shafarevich–Tate group* $\text{Ш}^1(E, \mathbf{G})$ of $\mathbf{G}$ vanishes.

Denote by $\bar{\mathbf{G}}$ the group $\mathbf{G} \times_E \bar{E}$, by $\bar{\mathbf{G}}^{\mathrm{der}}$ the derived group of $\bar{\mathbf{G}}$, by $\bar{\mathbf{G}}^{\mathrm{sc}}$ the simply-connected cover of $\bar{\mathbf{G}}^{\mathrm{der}}$, by $\rho : \bar{\mathbf{G}}^{\mathrm{sc}} \to \bar{\mathbf{G}}$ the natural map, by $\bar{\mathbf{T}}$ a maximal torus of $\bar{\mathbf{G}}$, and by $\mathbb{X}_*$ the cocharacter functor for a torus. The *algebraic fundamental group* of $\bar{\mathbf{G}}$ [5, Definition 1.3] is a $\mathrm{Gal}(\bar{E}/E)$-module defined as

$$M := \mathbb{X}_*(\bar{\mathbf{T}})/\rho_*(\mathbb{X}_*(\rho^{-1}(\bar{\mathbf{T}}))).$$

For each $\lambda \in \mathscr{P}_E$, one has a map [5, Section 5.15]

$$\mu_\lambda : H^1(E_\lambda, \mathbf{G}) \xrightarrow{\mathrm{ab}^1} H^1_{\mathrm{ab}}(E_\lambda, \mathbf{G}) = \mathcal{T}_\lambda^{-1}(M) \xrightarrow{\mathrm{cor}_\lambda^{-1}} \mathcal{T}^{-1}(M) = (M_{\mathrm{Gal}(\bar{E}/E)})_{\mathrm{tor}}, \tag{40}$$

where $H^1_{\mathrm{ab}}(E_\lambda, \mathbf{G})$ is the *first abelian Galois cohomology group of* $\mathbf{G}$ [5, Definition 2.2] and $(M_{\mathrm{Gal}(\bar{E}/E)})_{\mathrm{tor}}$ denotes the torsion subgroup of the Galois coinvariants of M. The surjectivity of the *abelianization map* ab^1 is by [5, Theorem 5.4]. If E_λ is non-Archimedean, then $\mathcal{T}_\lambda^{-1}(M) = (M_{\mathrm{Gal}(\bar{E}_\lambda/E_\lambda)})_{\mathrm{tor}}$ [5, Propositions 2.8 and 4.1 (i)] and $\mathrm{cor}_\lambda^{-1}$ is the natural map [5, Section 4.7].

Theorem K ([5, Theorem 5.16]). *When $k = 1$, the map in (39) factors through $\bigoplus_{\lambda\in\mathscr{P}_E} H^1(E_\lambda, \mathbf{G})$ and*

$$0 \to \text{Ш}^1(E, \mathbf{G}) \to H^1(E, \mathbf{G}) \to \bigoplus_\lambda H^1(E_\lambda, \mathbf{G}) \xrightarrow{\oplus\mu_\lambda} (M_{\mathrm{Gal}(\bar{E}/E)})_{\mathrm{tor}}$$

is exact.

As M is finite for semisimple $\mathbf{G}$, we obtain the following.

Proposition 2.14. *If* $\mathbf{G}$ *is semisimple and* E_λ *is non-Archimedean, then* μ_λ *in* (40) *is surjective.*

We have the following result for the torus $\mathbf{G} = \mathbf{T}$ by class field theory and [5, Lemma 5.6.2].

Proposition L. *Suppose* $\mathbf{T}$ *is a direct product of a split torus* $\mathbf{T}^{\mathrm{sp}}$ *and a torus* $\mathbf{T}'$ *such that* $\mathbf{T}'$ *is anisotropic over* E_λ *for some place* λ *of* E. *Then* $\text{Ш}^2(E,\mathbf{T}) = \text{Ш}^2(E,\mathbf{T}^{\mathrm{sp}}) \oplus \text{Ш}^2(E,\mathbf{T}') = 0$.

2.6. *Proofs of Main Theorems*

2.6.1. The 1*-cocycles* μ *and* χ. According to conditions (a)–(c) of the Main Theorem(s), we have a chain $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,E}$, a chain $\mathbf{T} \subset \mathrm{GL}_{n,E}$, and an $\bar{E}$-isomorphism of representations

$$f_{\bar{E}} : (\mathbf{T}^{\mathrm{sp}} \times_E \bar{E} \hookrightarrow \mathrm{GL}_{n,\bar{E}}) \xrightarrow{\cong} (\mathbf{T} \times_E \bar{E} \hookrightarrow \mathrm{GL}_{n,\bar{E}}).$$

This produces a 1-cocycle (as well as a Galois representation since $\mathrm{Gal}(\bar{E}/E)$ acts trivially on $\mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}$):

$$\mu = (\mu_\sigma) := (f_{\bar{E}}^{-1} \circ {}^\sigma f_{\bar{E}}) \in Z^1(E, \mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}) = \mathrm{Hom}(\mathrm{Gal}(\bar{E}/E), \mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}). \quad (41)$$

As $\Omega_{\bar{E}}$ (resp. $\omega_{\bar{E}}$) is a subgroup of $\mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}$ (Section 2.3.2), we first show the following.

Proposition 2.15. *The image of the Galois representation* $\mu : \mathrm{Gal}(\bar{E}/E) \to \mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}$ *is contained in* $\Omega_{\bar{E}}$ *(resp.* $\omega_{\bar{E}}$ *if* $\mathbf{G}^{\mathrm{sp}}$ *is irreducible on* E^n*). Thus, it defines a class* μ *in* $Z^1(E,\Omega_{\bar{E}})$ *(resp.* $Z^1(E,\omega_{\bar{E}})$*).*

Proof. For every $i_{\bar{\lambda}} : \bar{E} \to \bar{E}_\lambda$ with $\lambda \in \mathcal{P}_{E,f}$,

$$\begin{aligned} \mathrm{loc}_{\bar{\lambda}}(\mu) &= \mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}} \circ \mathrm{loc}_{\bar{\lambda}}((f_{\bar{E}_\lambda}^{-1} \circ {}^\sigma f_{\bar{E}_\lambda})) \\ &\in \mathrm{Hom}(\mathrm{Gal}(\bar{E}_\lambda/E_\lambda), \Omega_{\bar{E}_\lambda}) = \mathrm{Hom}(\mathrm{Gal}(\bar{E}_\lambda/E_\lambda), \Omega_{\bar{E}}) \end{aligned}$$

(resp. $\mathrm{Hom}(\mathrm{Gal}(\bar{E}_\lambda/E_\lambda), \omega_{\bar{E}})$) by (37), condition (b), and diagram (16) (resp. diagrams (17) and (19)) for $F = E_\lambda$. Hence, all the local representations land on $\Omega_{\bar{E}}$ (resp. $\omega_{\bar{E}}$). Since $\mathrm{Aut}_{\bar{E}}\,\mathbf{T}^{\mathrm{sp}}$ is discrete, the image of μ is finite. We are done by the Chebotarev density theorem. ∎

So it makes sense to define by diagram (16) (resp. (19)) for $F = E$ the twisted torus

$$_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C}) \quad (42)$$

for Main Theorem II (d) and the $\Theta_{\bar{E}}$-valued (resp. $\theta_{\bar{E}}$-valued) 1-cocycle

$$\chi := \pi(\mu). \quad (43)$$

2.6.2. Proof of Main Theorem I (i). By condition (b) and diagram (21) for $F = E_\lambda$, in $H^1(E_\lambda, \Theta_{\bar{E}_\lambda})$ the cohomology class $\pi([\mathbf{T}_\lambda \subset \mathbf{G}_\lambda])$ is equal to $\mathrm{loc}_\lambda[\chi]$. Then by applying $\mathrm{Res}_{\mathbf{G}^{\mathrm{sp}}}$ in diagram (20), we have $\pi([\mathbf{G}_\lambda]) = \mathrm{loc}_\lambda[\chi]$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$. By Theorem I for $F = E$, we obtain a quasi-split connected reductive group $\mathbf{G}'$ over E such that $[\mathbf{G}'] = j[\chi]$ in (31). On the one hand, for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$, $[\mathbf{G}' \times_E E_\lambda]$ and $[\mathbf{G}_\lambda]$ belong to the same fiber of π in (31) for $F = E_\lambda$. On the other hand, for almost all $\lambda \in \mathscr{P}_{E,f}^{(p)}$,

$$[\mathbf{G}' \times_E E_\lambda] = j(\mathrm{loc}_\lambda[\chi]) = [\mathbf{G}_\lambda] \tag{44}$$

by Theorem I for $F = E_\lambda$ and condition (d). Hence, by Corollary 2.12 for $F = E_\lambda$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$ to identify $[\mathbf{G}_\lambda]$ as an element in $H^1(E_\lambda, \mathbf{G}'^{\mathrm{ad}} \times_E E_\lambda)$, we obtain $[\mathbf{G}_\lambda] = 0$ for almost all $\lambda \in \mathscr{P}_{E,f}^{(p)}$. Let λ' be a place of E extending p. Then $\lambda' \notin \mathscr{P}_{E,f}^{(p)}$. Since $\mathbf{G}'^{\mathrm{ad}}$ is semisimple, there exists an element $[\mathbf{G}] \in H^1(E, \mathbf{G}'^{\mathrm{ad}})$ such that $\mathrm{loc}_\lambda[\mathbf{G}] = [\mathbf{G}_\lambda]$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$ by Theorem K and Proposition 2.14. Here $\mathbf{G}$ is an inner form of $\mathbf{G}'^{\mathrm{ad}}$ (Remark 2.11 (3)). Therefore, we conclude that $\mathbf{G} \times_E E_\lambda \cong \mathbf{G}_\lambda$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$ and $\mathbf{G}_\lambda$ is unramified for all but finitely many λ. ∎

Remark 2.16. Besides $\mathrm{loc}_\lambda[\mathbf{G}] = [\mathbf{G}_\lambda]$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$, we can impose conditions at other places of E except λ'. For example, we can require that $\mathrm{loc}_\lambda[\mathbf{G}] = [\mathbf{G}' \times_E E_\lambda]$ for all $\lambda \in \mathscr{P}_E \setminus (\mathscr{P}_{E,f}^{(p)} \cup \{\lambda'\})$.

2.6.3. Proof of Main Theorem I (ii). By condition (b) and diagram (23) for $F = E_\lambda$, the cohomology class $\pi([\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}])$ is equal to $\mathrm{loc}_\lambda[\chi]$ in $H^1(E, \theta_{\bar{E}_\lambda})$. Then by applying $\mathrm{Res}_{(\mathrm{GL}_{n,E_\lambda}, \mathbf{G}^{\mathrm{sp}})}$ in diagram (22), we have

$$\pi([\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}]) = \mathrm{loc}_\lambda[\chi] \quad \text{for all } \lambda \in \mathscr{P}_{E,f}^{(p)}.$$

By (33) for $F = E$, we obtain an E-form $\mathbf{G}' \subset \mathrm{GL}_{n,E}$ of $\mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,E}$ where $\mathbf{G}'$ is quasi-split such that $[\mathbf{G}' \subset \mathrm{GL}_{n,E}] = j[\chi]$ in (34). On the one hand, for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$, $[(\mathbf{G}' \subset \mathrm{GL}_{n,E}) \times_E E_\lambda]$ and $[\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}]$ belong to the same fiber of π in (34) for $F = E_\lambda$. On the other hand, for almost all $\lambda \in \mathscr{P}_{E,f}^{(p)}$,

$$[(\mathbf{G}' \subset \mathrm{GL}_{n,E}) \times_E E_\lambda] = j(\mathrm{loc}_\lambda[\chi]) = [\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}] \tag{45}$$

by Theorem I for $F = E_\lambda$, condition (d), and the proposition below.

Proposition 2.17 ([66, Lemma 3.2, Theorem 3.3]). *Let F be a field of characteristic zero and D_i $(i = 1, 2)$ be central simple algebras over F. Let $\mathbf{H}$ be a connected reductive group over F and $\rho_i : \mathbf{H} \to \mathrm{GL}_{m_i, D_i}$ $(i = 1, 2)$ be two F-representations that are absolutely irreducible. If $\rho_1 \times_F \bar{F} \cong \rho_2 \times \bar{F}$, then $\rho_1 \cong \rho_2$.*

Hence, by Corollary 2.12 for $F = E_\lambda$ for all $\lambda \in \mathscr{P}_{E,f}^{(p)}$ to identify $[\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}]$ as an element in $H^1(E_\lambda, \mathbf{G}'^{\mathrm{ad}} \times_E E_\lambda)$, we find that $[\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}] = 0$ for almost all $\lambda \in \mathscr{P}_{E,f}^{(p)}$. Let λ' be a place of E extending p. Then $\lambda' \notin \mathscr{P}_{E,f}^{(p)}$. Since $\mathbf{G}'^{\mathrm{ad}}$ is semisimple, there exists an element $[\mathbf{G} \subset \mathrm{GL}_{m,D}] \in H^1(E, \mathbf{G}'^{\mathrm{ad}})$ such that

$$\begin{aligned} \mathrm{loc}_\lambda[\mathbf{G} \subset \mathrm{GL}_{m,D}] &= [\mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}], && \forall \lambda \in \mathcal{P}_{E,f}^{(p)}, \\ \mathrm{loc}_\lambda[\mathbf{G} \subset \mathrm{GL}_{m,D}] &= [(\mathbf{G}' \subset \mathrm{GL}_{n,E}) \times E_\lambda], && \forall \lambda \in \mathcal{P}_E \setminus (\mathcal{P}_{E,f}^{(p)} \cup \{\lambda'\}), \end{aligned} \tag{46}$$

by Theorem K and Proposition 2.14. Here $\mathbf{G}$ (resp. $\mathrm{GL}_{m,D}$) is an inner form of $\mathbf{G}'^{\mathrm{ad}}$ (resp. $\mathrm{GL}_{n,E}$) and $\mathrm{GL}_{m,D} = \mathrm{GL}_{n,E}$ by (46) and class field theory. By Lemma 2.9, we conclude that $(\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}) \times_E E_\lambda \cong (\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda})$ as representations for all $\lambda \in \mathcal{P}_{E,f}^{(p)}$ and $\mathbf{G}_\lambda$ is unramified for all but finitely many λ. ∎

2.6.4. Proof of Main Theorem II. Consider the cocycle μ in $Z^1(E, \Omega_{\bar{E}})$ (resp. $Z^1(E, \omega_{\bar{E}})$). By condition (b), $\mathrm{loc}_\lambda[\mu]$ equals $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}[\mathbf{T}_\lambda \subset \mathbf{G}_\lambda]$ (resp. $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}[\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \subset \mathrm{GL}_{n,E_\lambda}]$) for all $\lambda \in \mathcal{P}_{E,f}$. It suffices to show that $[\mu]$ belongs to the image of the injection $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}$ (ensuring uniqueness) in diagram (21) (resp. (23)) for $F = E$. By Corollary 2.13, this is equivalent to $\Delta(\mu) = 0$ in $H^2(E, {}_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C}))$. By condition (d) and Proposition L, it remains to prove that $\mathrm{loc}_\lambda(\Delta(\mu)) = 0$ for all places λ of E. For a finite place λ, this is true by the fact that the image of $\mathrm{Res}_{\mathbf{T}^{\mathrm{sp}}}$ in (21) (resp. (23)) contains $\mathrm{loc}_\lambda[\mu]$ and Corollary 2.13 for $F = E_\lambda$. For a real place, this is true by (d) and $H^2(\mathbb{R}, \mathbf{S}_{\mathbb{R}}) = 0$ if $\mathbf{S}_{\mathbb{R}}$ is an $\mathbb{R}$-anisotropic torus (see [37, Lemma 10.4]). Therefore, we obtain a common E-form $\mathbf{T} \subset \mathbf{G}$ (resp. $\mathbf{T} \subset \mathbf{G} \hookrightarrow \mathrm{GL}_{m,D}$ by Lemma 2.9) of the chain $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda$ (resp. the chain representation $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}$) for all finite places λ of E. ∎

3. Rationality of algebraic monodromy groups

This section is devoted to the proofs of the statements in Section 1.2. Fix a number field E and denote by p_λ the residue characteristic of the finite place $\lambda \in \mathcal{P}_{E,f}$.

3.1. *Profinite group* Π *and Frobenius elements* Fr

Consider two cases.

3.1.1. Characteristic zero. In this case, Π denotes the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$ of a number field K and $\mathcal{P} := \mathcal{P}_{E,f}$. Equip Π with a subset $\mathrm{Fr} \subset \Pi$ of *Frobenius elements* as follows.

For all $v \in \mathcal{P}_{K,f}$, let q_v be the size of the residue field $\mathbb{F}_{q_v}$ of K_v and consider the natural surjection

$$\pi_v : \mathrm{Gal}(\bar{K}_v/K_v) \to \mathrm{Gal}(\bar{\mathbb{F}}_{q_v}/\mathbb{F}_{q_v}).$$

For each v, fix a lift $\phi_v \in \pi_v^{-1}(\mathrm{Fr}_{q_v}^{-1})$, where $\mathrm{Fr}_{q_v}^{-1} \in \mathrm{Gal}(\bar{\mathbb{F}}_{q_v}/\mathbb{F}_{q_v})$ is the geometric Frobenius. Each $\bar{v} \in \mathcal{P}_{\bar{K},f}$ determines an embedding $\iota_{\bar{v}} : \mathrm{Gal}(\bar{K}_v/K_v) \to \mathrm{Gal}(\bar{K}/K)$. For $\bar{v} \in \mathcal{P}_{\bar{K},f}$, define $\mathrm{Fr}_{\bar{v}}$ to be $\iota_{\bar{v}}(\phi_v)$ where v is the restriction of $\bar{v}$ to K. Define

$$\mathrm{Fr}_v := \bigcup_{\bar{v}|v} [\mathrm{Fr}_{\bar{v}}] \quad \text{and} \quad \mathrm{Fr} := \bigcup_{v \in \mathcal{P}_{K,f}} \mathrm{Fr}_v.$$

For any Galois extension L/K that is unramified except at finitely many $v \in \mathcal{P}_{K,f}$ and any finite subset $S \subset \mathcal{P}_{K,f}$, the image of $\bigcup_{v\in\mathcal{P}_{K,f}\backslash S} \mathrm{Fr}_v$ in $\mathrm{Gal}(L/K)$ is dense [61, Chapter I, Section 2.2, Corollary 2]. Assign the number q_v to the elements in Fr_v.

3.1.2. Characteristic p. In this case, Π denotes the étale fundamental group $\pi_1^{\text{ét}}(X,\bar{x})$ (with some base point $\bar{x}$) of a smooth geometrically connected variety $X/\mathbb{F}_q$ in characteristic p and $\mathcal{P} := \mathcal{P}_{E,f}^{(p)}$. Equip Π with a subset $\mathrm{Fr} \subset \Pi$ of *Frobenius elements* as follows.

Let X^{cl} be the set of closed points of X. For any geometric point $\overline{x}'$ over $x' \in X^{\mathrm{cl}}$, let $\mathrm{Fr}_{\overline{x}'}$ be the image of the geometric Frobenius $\mathrm{Fr}_{q_{x'}}^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_{q_{x'}}/\mathbb{F}_{q_{x'}}) = \pi_1(x',\overline{x}')$ under the natural map

$$\pi_1(x',\overline{x}') \to \pi_1(X,\overline{x}') \xrightarrow{\sigma_{xx'}} \pi_1(X,\overline{x}),$$

where $q_{x'}$ is the size of the residue field of x'. Note that the change of base point isomorphism $\sigma_{xx'}$ is unique up to an inner automorphism of $\pi_1(X,\overline{x})$. Since the conjugacy class $[\mathrm{Fr}_{\overline{x}'}]$ depends only on x', write $\mathrm{Fr}_{x'} := [\mathrm{Fr}_{\overline{x}'}]$ and define

$$\mathrm{Fr} := \bigcup_{x'\in X^{\mathrm{cl}}} \mathrm{Fr}_{x'}.$$

The subset Fr is dense in Π [56]. Assign the number $q_{x'}$ to the elements in $\mathrm{Fr}_{x'}$.

3.2. *E-compatible systems*

Let $(\Pi, \mathrm{Fr}, \mathcal{P})$ be one of the two cases in Section 3.1. In characteristic zero, denote by S a finite subset of $\mathcal{P}_{K,f}$. Otherwise, S is the empty set.

3.2.1. GL_n*-valued compatible systems.* A system of n-dimensional λ-adic (continuous) representations

$$\rho_\bullet := \{\rho_\lambda : \Pi \to \mathrm{GL}_n(E_\lambda)\}_{\lambda\in\mathcal{P}}$$

of Π is said to be *semisimple* (resp. *irreducible, absolutely irreducible*) if for all $\lambda \in \mathcal{P}$, ρ_λ is semisimple (resp. irreducible, absolutely irreducible). The system $\rho_\bullet$ is said to be *E-compatible* (with exceptional set S) if

- in characteristic zero, ρ_λ is unramified outside $S \cup \{t \in \mathcal{P}_{K,f} : p_\lambda \,|\, q_t\}$ for each $\lambda \in \mathcal{P}$;
- for each Frobenius element $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$ satisfying $t \notin S$ and for each λ satisfying $p_\lambda \nmid q_t$, the characteristic polynomial

$$P_t(T) := \det(\rho_\lambda(\mathrm{Fr}_{\bar{t}}) - T \cdot I_n) \in E_\lambda[T] \tag{47}$$

  has coefficients in E and depends only on t (independent of $\lambda \in \mathcal{P}$).

The compatible system $\rho_\bullet$ is said to be *pure of weight* $w \in \mathbb{R}$ (resp. *of mixed weights*) if for each $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$ with $t \notin S$ and each root $\alpha \in \overline{E}$ of $P_t(T)$, the absolute value $|i(\alpha)|$ is equal to $q_t^{w/2}$ for every complex embedding $i : \overline{E} \to \mathbb{C}$ (resp. is independent of the complex embedding $i : \overline{E} \to \mathbb{C}$).

3.2.2. Coefficient extension and the Weil restriction. Let $\rho_\bullet$ be an n-dimensional (semisimple) E-compatible system of Π that is pure of weight w (resp. of mixed weights). For a number field E', denote $\mathcal{P}' = \mathcal{P}_{E',f}$ in characteristic zero and $\mathcal{P}' = \mathcal{P}^{(p)}_{E',f}$ in characteristic p.

If E' is an extension of E, then we obtain by *coefficient extension* a (semisimple) system $\rho_\bullet \otimes_E E'$ of n-dimensional λ'-adic representations

$$(\rho_\bullet \otimes_E E')_{\lambda'} := (\Pi \xrightarrow{\rho_\lambda} \mathrm{GL}_n(E_\lambda) \subset \mathrm{GL}_n(E'_{\lambda'})), \tag{48}$$

where λ is the restriction of λ' to E. The system is E'-compatible (with exceptional set S), pure of weight w (resp. of mixed weights), and called the *coefficient extension of* $\rho_\bullet$ *to* E' (see [2, Definition 3.2]).

If E' is a subfield of E, then we obtain by the *Weil restriction of scalars* a (semisimple) system $\mathrm{Res}_{E/E'}\rho_\bullet$ of $n[E:E']$-dimensional λ'-adic representations

$$\begin{aligned}(\mathrm{Res}_{E/E'}\rho_\bullet)_{\lambda'} := \bigoplus_{\lambda|\lambda'} \rho_\lambda : \Pi \to \prod_{\lambda|\lambda'} \mathrm{GL}_n(E_\lambda)\\ = (\mathrm{Res}_{E/E'}\mathrm{GL}_{n,E})(E'_{\lambda'}) \subset \mathrm{GL}_{n[E:E']}(E'_{\lambda'}).\end{aligned} \tag{49}$$

The system is E'-compatible (with exceptional set S), pure of weight w (resp. of mixed weights), and called the *Weil restriction of* $\rho_\bullet$ (see [2, Definition 3.4]).

3.2.3. $\mathbf{G}$*-valued compatible systems.* Let $\mathbf{G}$ be a linear algebraic group defined over E with affine coordinate ring R. Since $\mathbf{G}$ acts on itself by conjugation, $\mathbf{G}$ acts on R. The subring of invariant functions is denoted by $R^{\mathbf{G}}$. For all $g \in \mathbf{G}$, let g_s be the semisimple part of g. If g is defined over a field extension F/E, then g_s is also defined over F. A system $\{\rho_\lambda : \Pi \to \mathbf{G}(E_\lambda)\}_{\lambda\in\mathcal{P}}$ of λ-adic $\mathbf{G}$-representations of Π is said to be *E-compatible* (with exceptional set S) if

- in characteristic zero, ρ_λ is unramified outside $S \cup \{t \in \mathcal{P}_{K,f} : p_\lambda \mid q_t\}$ for each $\lambda \in \mathcal{P}$;
- for each Frobenius element $\mathrm{Fr}_{\bar t} \in \mathrm{Fr}$ satisfying $t \notin S$, each λ satisfying $p_\lambda \nmid q_t$, and each $f \in R^{\mathbf{G}}$ the number
$$f(\rho_\lambda(\mathrm{Fr}_{\bar t})_s) \in E_\lambda \tag{50}$$
belongs to E and depends only on t and f [61, Chapter I, Section 2.4] (independent of $\lambda \in \mathcal{P}$).[6]

It follows that an n-dimensional E-compatible system is the same as an E-compatible system of $\mathrm{GL}_{n,E}$-representations.

3.2.4. Algebraic monodromy groups and connectedness. For all $\lambda \in \mathcal{P}$, the *algebraic monodromy group* of ρ_λ, i.e., the Zariski closure of the image of ρ_λ in $\mathrm{GL}_{n,E_\lambda}$, is denoted by $\mathbf{G}_\lambda$. It is a closed subgroup of $\mathrm{GL}_{n,E_\lambda}$. The image $\rho_\lambda(\Pi)$ is a compact subgroup of the

[6] This is equivalent to the conjugacy class of $\rho_\lambda(\mathrm{Fr}_{\bar t})_s$ in $\mathbf{G}$ being defined over E and depend only on $t \notin S$ (independent of λ).

λ-adic Lie group $\mathbf{G}_\lambda(E_\lambda)$. The following result is well-known by using the compatibility condition [42, Proposition 6.14].

Proposition M. *The component groups $\mathbf{G}_\lambda/\mathbf{G}_\lambda^\circ$ are isomorphic for all $\lambda \in \mathcal{P}$. In particular, the connectedness of $\mathbf{G}_\lambda$ is independent of λ.*

3.2.5. Group schemes. Suppose the algebraic monodromy group $\mathbf{G}_\lambda$ is connected reductive for all λ. Let $\mathcal{O}_\lambda$ be the ring of integers of E_λ with residue field k_λ of characteristic p_λ. Let Λ_λ be an $\mathcal{O}_\lambda$-lattice of E_λ^n that is invariant under the image $\rho_\lambda(\Pi)$. Let $\mathcal{G}_\lambda$ be the Zariski closure of $\rho_\lambda(\Pi)$ in $\mathrm{GL}_{\Lambda_\lambda} \cong \mathrm{GL}_{n,\mathcal{O}_\lambda}$, endowed with the unique structure of reduced closed subscheme. The generic fiber of $\mathcal{G}_\lambda$ is $\mathbf{G}_\lambda$. The special fiber, denoted by $\mathcal{G}_{k_\lambda}$, is identified as a subgroup of $\mathrm{GL}_{n,k_\lambda}$. When $p_\lambda \geq n$, the subgroup $\mathcal{G}_{k_\lambda} \subset \mathrm{GL}_{n,k_\lambda}$ is said to be *saturated* if for any unipotent element $u \in \mathcal{G}_{k_\lambda}(\bar{k}_\lambda)$, the one-parameter subgroup $\{u^a : a \in \bar{k}_\lambda\} \subset \mathrm{GL}_n(\bar{k}_\lambda)$ belongs to $\mathcal{G}_{k_\lambda}(\bar{k}_\lambda)$ [59, Section 4.2].

Proposition N ([43, Proposition 1.3], [2, Proposition 5.51, Theorem 5.52]). *For all but finitely many $\lambda \in \mathcal{P}$, the following assertions hold:*

(i) *The group scheme $\mathcal{G}_\lambda$ is smooth with constant absolute rank over $\mathcal{O}_\lambda$.*

(ii) *The identity component of the special fiber $\mathcal{G}_{k_\lambda} \subset \mathrm{GL}_{n,k_\lambda}$ is saturated.*

3.3. Frobenius torus

3.3.1. Frobenius torus and maximal torus. For all $\lambda \in \mathcal{P}$, let $\mathbf{G}_\lambda$ be the algebraic monodromy group of ρ_λ. The identity component of $\mathbf{G}_\lambda$ is reductive since ρ_λ is semisimple. Let ρ_λ be a member of the system and $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$ be a Frobenius element with $t \notin S$. If $p_\lambda \nmid q_t$, then the *Frobenius torus* $\mathbf{T}_{\bar{t},\lambda}$ of $\mathrm{Fr}_{\bar{t}}$ is defined to be the identity component of the smallest (diagonalizable) algebraic subgroup $\mathbf{S}_{\bar{t},\lambda}$ in $\mathrm{GL}_{n,E_\lambda}$ containing the semisimple part of $\rho_\lambda(\mathrm{Fr}_{\bar{t}})$. It follows that $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{S}_{\bar{t},\lambda} \subset \mathbf{G}_\lambda$. The following theorem is due to Serre.

Theorem O (see [44, Theorem 1.2 and its proof], [8, Theorem 5.7], [28, Theorem 2.6]). *Suppose the algebraic monodromy group $\mathbf{G}_{\lambda'}$ is connected for some $\lambda' \in \mathcal{P}$. Suppose there exists a finite subset $Q \subset \mathbb{Q}$ such that for all $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$ with $t \notin S$, the following conditions are satisfied for every root α of the characteristic polynomial $P_t(T)$ in (47):*

(a) *the absolute values of α in all complex embeddings are equal;*

(b) *α is a unit at any finite place not extending p_t;*

(c) *for any finite place w of $\bar{\mathbb{Q}}$ such that $w(p_t) > 0$, the ratio $w(\alpha)/w(q_t)$ belongs to Q.*

Then there exists a proper closed subvariety $\mathbf{Y}_{\lambda'}$ of $\mathbf{G}_{\lambda'}$ such that $\mathbf{T}_{\bar{t},\lambda'}$ is a maximal torus of $\mathbf{G}_{\lambda'}$ whenever $\rho_{\lambda'}(\mathrm{Fr}_{\bar{t}}) \in \mathbf{G}_{\lambda'} \setminus \mathbf{Y}_{\lambda'}$.

Remark 3.1. (1) If $\mathbf{G}_\lambda$ is connected and the Frobenius torus $\mathbf{T}_{\bar{t},\lambda}$ is maximal, then $\mathbf{T}_{\bar{t},\lambda} = \mathbf{S}_{\bar{t},\lambda}$.

(2) Conditions O (a, b) hold for our mixed compatible system $\rho_\bullet$.

(3) Condition O (c) holds in characteristic p by replacing X with a non-empty open subset U [15, Theorem 1.3.3 (i), Remark 1.3.5].

(4) Condition O (c) holds in characteristic zero if we assume the system is $\{H^w(Y_{\bar{K}}, \mathbb{Q}_\ell)\}_{\ell\in\mathscr{P}_{\mathbb{Q},f}}$ for some smooth projective variety Y/K [44, Theorem 1.1].

(5) In characteristic p, the subset of elements $\mathrm{Fr}_{\bar{t}}$ of Fr whose Frobenius tori $\mathbf{T}_{\bar{t},\lambda'}$ are maximal in $\mathbf{G}_{\lambda'}$ is dense in Π.

(6) In characteristic zero, the subset of places $v \in \mathscr{P}_{K,f}$ such that $\mathbf{T}_{\bar{v},\lambda'}$ is a maximal torus of $\mathbf{G}_{\lambda'}$ is of Dirichlet density 1 (see [28, Corollary 2.7]).

Let $\mathrm{Fr}_{\bar{t}}$ be a Frobenius element. There is a semisimple matrix $M_t \in \mathrm{GL}_n(E)$ with $P_t(T)$ (47) as characteristic polynomial. For all $\lambda \in \mathscr{P}$ with $p_\lambda \nmid q_t$, M_t is conjugate to the semisimple part $\rho_\lambda(\mathrm{Fr}_{\bar{t}})_s$ in $\mathrm{GL}_n(E_\lambda)$ by E-compatibility. Hence, if we let $\mathbf{S}_t$ be the smallest algebraic subgroup of $\mathrm{GL}_{n,E}$ containing M_t, and $\mathbf{T}_t$ be the identity component of $\mathbf{S}_t$, then the chain representations $(\mathbf{T}_t \subset \mathbf{S}_t \hookrightarrow \mathrm{GL}_{n,E}) \times_E E_\lambda$ and $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{S}_{\bar{t},\lambda} \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ are isomorphic for all $\lambda \in \mathscr{P}$ with $p_\lambda \nmid q_t$.

Corollary 3.2. *Under the conditions of Theorem O, the following assertions hold.*

(i) (Common E-form of formal characters) *If the Frobenius torus $\mathbf{T}_{\bar{t},\lambda'}$ is a maximal torus of $\mathbf{G}_{\lambda'}$, then the Frobenius torus $\mathbf{T}_{\bar{t},\lambda}$ is also a maximal torus of $\mathbf{G}_\lambda$ for all $\lambda \in \mathscr{P}$ with $p_\lambda \nmid q_t$. Moreover, the representation $(\mathbf{T}_t \hookrightarrow \mathrm{GL}_{n,E}) \times_E E_\lambda$ is isomorphic to $\mathbf{T}_{\bar{t},\lambda} \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ for all $\lambda \in \mathscr{P}$ with $p_\lambda \nmid q_t$.*

(ii) (Absolute rank) *The absolute rank of $\mathbf{G}_\lambda$ is independent of λ.*

Proof. Assertion (i) is straightforward by Theorem O and the above construction of $\mathbf{T}_t$. Assertion (ii) is obvious by (i) in characteristic p, and follows from (i) and Remark 3.1 (6) in characteristic zero. ∎

3.3.2. Anisotropic subtorus. In this subsection, $\mathbf{G}_\lambda$ is connected for all $\lambda \in \mathscr{P}$. The subtorus $\mathbf{T}_t \subset \mathrm{GL}_{n,E}$ in Corollary 3.2 (i) is studied under the following hypothesis. Let k be the order of $\mathbf{S}_t/\mathbf{T}_t$. Then the Zariski closure of $M_t^{k\mathbb{Z}}$ in $\mathrm{GL}_{n,E}$ is $\mathbf{T}_t$.

Hypothesis R. Assume that for each real embedding $E \to \mathbb{R}$, the set of powers $\det(M_t)^{\mathbb{Z}} \subset \mathbb{R}$ contains some non-zero integral power of the absolute value $|i(\alpha)|$ for every root α of $P_t(T)$ and every complex embedding $i : \bar{E} \to \mathbb{C}$ extending $E \to \mathbb{R}$.

Proposition 3.3. *If Hypothesis R holds, then the subtorus $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ of $\mathbf{T}_t$ is anisotropic at all real places of E.*

Proof. Embed E into $\mathbb{R}$ and let $\mathbf{T}_{t,\mathbb{R}} \subset \mathbf{S}_{t,\mathbb{R}} \subset \mathrm{GL}_{n,\mathbb{R}}$ be the base change to $\mathbb{R}$. If $\chi : \mathbf{T}_{t,\mathbb{R}} \to \mathbb{G}_{\mathrm{m},\mathbb{R}}$ is an $\mathbb{R}$-character, then $\chi(M_t^k) \in \mathbb{G}_{\mathrm{m}}(\mathbb{R}) = \mathbb{R}^*$ Let $i : \bar{E} \to \mathbb{C}$ be an embedding extending $E \to \mathbb{R}$. Then $\chi(M_t^k)$ is the product of some integral powers of the roots $i(\alpha)$ of the polynomial $i(P_t(T)) \in \mathbb{R}[T]$. Hence, there exist integers $h \neq 0$ and m such that

$$\chi^{2h}(M_t^k) = \det(M_t)^{2m} \in \mathbb{R}^*_{>0}$$

by Hypothesis R. This implies

$$\chi^{2hk} = \det{}^{2m} \quad \text{on } \mathbf{T}_{t,\mathbb{R}}$$

since $(M_t^k)^{\mathbb{Z}}$ is Zariski dense in $\mathbf{T}_{t,\mathbb{R}}$. Consequently, χ^{2hk} is trivial on the subtorus $(\mathbf{T}_{t,\mathbb{R}} \cap \mathrm{SL}_{n,\mathbb{R}})^\circ$ for some $2hk \neq 0$. We conclude that the torus $(\mathbf{T}_{t,\mathbb{R}} \cap \mathrm{SL}_{n,\mathbb{R}})^\circ$ is anisotropic. ■

Corollary 3.4. *If Hypothesis R holds and E has a real place, then the subtorus $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ of $\mathbf{T}_t$ is anisotropic on a positive Dirichlet density subset $\mathcal{P}'$ of $\mathcal{P}_{E,f}$.*

Proof. Let r be the absolute rank of the E-torus $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$. Then it is an E-form of the split torus $\mathbb{G}_{\mathrm{m},E}^r$ with automorphism group $\mathrm{GL}_r(\mathbb{Z})$. The isomorphism class of $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ is represented by an element of $H^1(E, \mathrm{GL}_r(\mathbb{Z}))$, which is a continuous group homomorphism $\phi : \mathrm{Gal}(\bar{E}/E) \to \mathrm{GL}_r(\mathbb{Z})$ up to conjugation. Let $c \in \mathrm{Gal}(\bar{E}/E)$ be complex conjugation corresponding to a real place of E. Since $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ is anisotropic over $\mathbb{R}$ by Proposition 3.3 and c is of order 2, it follows that $\phi(c) = -I_r$. Since the image of ϕ is finite, there is a positive Dirichlet density set $\mathcal{P}'$ of finite places λ of E such that $\phi(\mathrm{Fr}_\lambda) = -I_r$ by the Chebotarev density theorem. Therefore, $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ is anisotropic over E_λ for all $\lambda \in \mathcal{P}'$. ■

Remark 3.5. (1) Hypothesis R holds for every $P_t(T)$ if the E-compatible system is pure.

(2) If $\lambda \in \mathcal{P}'$ in Corollary 3.4, then the E_λ-subtorus $(\mathbf{T}_{\bar{t},\lambda} \cap \mathbf{G}_\lambda^{\mathrm{der}})^\circ$ of $(\mathbf{T}_{\bar{t},\lambda} \cap \mathrm{SL}_{n,E_\lambda})^\circ \cong (\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ \times_E E_\lambda$ is also anisotropic. If $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{G}_\lambda$ is a maximal torus, then $\mathbf{T}_{\bar{t},\lambda} \cap \mathbf{G}_\lambda^{\mathrm{der}} \subset \mathbf{G}_\lambda^{\mathrm{der}}$ is also a maximal torus.

(3) Corollary 3.4 is not true for general E since $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ can be a non-trivial split torus over E. This is seen by taking a finite extension E'/E such that $P_t[T]$ splits and replacing the E-compatible system $\rho_\bullet$ with its coefficient extension $\rho_\bullet \otimes_E E'$ (Section 3.2.2).

Let $\mathbf{G}$ be a connected reductive group defined over a field F. A torus $\mathbf{T} \subset \mathbf{G}$ is said to be *fundamental* if it is a maximal torus with minimal F-rank. In characteristic zero, let $\mathcal{S}$ be the subset of elements $v \in \mathcal{P}_{K,f}$ such that for some $\lambda \in \mathcal{P}_{E,f}$, the Frobenius torus $\mathbf{T}_{\bar{v},\lambda} \cong \mathbf{T}_v \times_E E_\lambda$ is a fundamental torus of $\mathbf{G}_\lambda$. A Frobenius torus $\mathbf{T}_{\bar{v},\lambda} \subset \mathbf{G}_\lambda$ being fundamental is equivalent to $\mathbf{T}_{\bar{v},\lambda}$ being a maximal torus and $\mathbf{T}_{\bar{v},\lambda} \cap \mathbf{G}_\lambda^{\mathrm{der}}$ being anisotropic [5, Proposition 5.3.2]. When Hypothesis R holds and E has a real place, Remark 3.1 (6), Corollary 3.4, and Remark 3.5 (2) imply that $\mathcal{S}$ is of Dirichlet density 1.

Question Q. Suppose Hypothesis R holds; what is the Dirichlet density of $\mathcal{S}$ in $\mathcal{P}_{K,f}$ when E is totally imaginary?

We do not know the answer; we even do not know if $\mathcal{S}$ is non-empty. If we want to apply Main Theorem II to the algebraic monodromy representations $\{\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}\}_{\lambda \in \mathcal{P}_{E,f}}$ when E is totally imaginary, then positive Dirichlet density of $\mathcal{S}$ is necessary.

3.4. Proofs of characteristic p results

Let $\mathscr{P}$ be $\mathscr{P}^{(p)}_{E,f}$.

3.4.1. Proof of Theorem 1.1. By Proposition M and taking a finite Galois covering of X, we assume that $\mathbf{G}_\lambda$ is connected for all $\lambda \in \mathscr{P}$. It suffices to check conditions (a)–(d) of Main Theorem I for the system of algebraic monodromy representations

$$\{\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}\}_{\lambda\in\mathscr{P}}.$$

Conditions (a)–(c) follow directly from assertions (i)–(iii) of Theorem B. Condition (d) holds by [2, Corollary 7.9], or by Proposition 3.6 below, for almost all λ, the existence of a hyperspecial maximal compact subgroup of $\mathbf{G}_\lambda(E_\lambda)$ implies that $\mathbf{G}_\lambda$ is unramified [46, Section 1]. We are done by Main Theorem I. ∎

Proposition 3.6. *If $\mathbf{G}_\lambda$ is connected for all $\lambda \in \mathscr{P}$, then the image of ρ_λ is contained in a hyperspecial maximal compact subgroup H_λ of $\mathbf{G}_\lambda(E_\lambda)$ for almost all λ.*

Proof. Since $\pi_1(X)$ is compact, we may assume $\rho_\lambda(\pi_1(X)) \subset \mathrm{GL}_n(\mathcal{O}_\lambda)$ after some change of coordinates $V_\lambda \cong E_\lambda^n$ for all λ. The geometric étale fundamental group $\pi_1^{\mathrm{geo}}(X)$ of X fits into the short exact sequence

$$1 \to \pi_1^{\mathrm{geo}}(X) \to \pi_1(X) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

Denote the Zariski closure of $\rho_\lambda(\pi_1^{\mathrm{geo}}(X))$ in $\mathrm{GL}_{n,E_\lambda}$ by $\mathbf{G}_\lambda^{\mathrm{geo}}$. Let $\mathcal{G}_\lambda$ (resp. $\mathcal{G}_\lambda^{\mathrm{geo}}$) be the Zariski closure of $\mathbf{G}_\lambda$ (resp. the identity component of $\mathbf{G}_\lambda^{\mathrm{geo}}$) in $\mathrm{GL}_{n,\mathcal{O}_\lambda}$ with special fiber $\mathcal{G}_{k_\lambda}$ (resp. $\mathcal{G}_{k_\lambda}^{\mathrm{geo}}$). It suffices to prove that for almost all λ, $H_\lambda := \mathcal{G}_\lambda(\mathcal{O}_\lambda)$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\lambda(E_\lambda)$. By Bruhat–Tits theory, this condition follows if we show that the $\mathcal{O}_\lambda$-group scheme $\mathcal{G}_\lambda$ is reductive [67, Section 3.8.1].

By [2, Theorem 7.3], the $\mathcal{O}_\lambda$-group scheme $\mathcal{G}_\lambda^{\mathrm{geo}}$ is semisimple for almost all λ. Let k_λ be the residue field of E_λ. Since the $\mathcal{O}_\lambda$-group scheme $\mathcal{G}_\lambda$ is smooth with constant absolute rank for almost all λ by Proposition N (i), and contains $\mathcal{G}_\lambda^{\mathrm{geo}}$ as a closed normal subgroup scheme, the inequalities

$$\begin{aligned}\dim \mathbf{G}_\lambda &= \dim(\mathcal{G}_{k_\lambda}) = \dim(\mathcal{G}_{k_\lambda}^{\mathrm{geo}}) + \dim(\mathcal{G}_{k_\lambda}/\mathcal{G}_{k_\lambda}^{\mathrm{geo}}) \\ &\geq \dim(\mathcal{G}_{k_\lambda}^{\mathrm{geo}}) + \mathrm{rk}(\mathcal{G}_{k_\lambda}/\mathcal{G}_{k_\lambda}^{\mathrm{geo}}) = \dim \mathbf{G}_\lambda^{\mathrm{geo}} + \mathrm{rk}(\mathbf{G}_\lambda/\mathbf{G}_\lambda^{\mathrm{geo}}) = \dim \mathbf{G}_\lambda\end{aligned}$$

imply that the special fiber $\mathcal{G}_{k_\lambda}$ has trivial unipotent radical for almost all λ. Therefore, the smooth group scheme $\mathcal{G}_\lambda$ is reductive over $\mathcal{O}_\lambda$ for almost all λ. ∎

3.4.2. Proof of Corollary 1.2. By Theorem 1.1 (i), there is a connected reductive group $\mathbf{G}$ defined over E and an isomorphism $\phi_\lambda : \mathbf{G} \times_E E_\lambda \to \mathbf{G}_\lambda$ for each $\lambda \in \mathscr{P}$. For almost all λ, the $\mathcal{O}_\lambda$-points $\mathbf{G}(\mathcal{O}_\lambda)$ is well-defined (by finding some integral model $\mathcal{G}$ of $\mathbf{G}$) and is a hyperspecial maximal compact subgroup of the E_λ-points $\mathbf{G}(E_\lambda)$ [67, Section 3.8.1]. Let $\mathbf{G}_\lambda^{\mathrm{ad}}$ be the adjoint group of $\mathbf{G}_\lambda$. The subgroup $\mathbf{G}_\lambda^{\mathrm{ad}}(E_\lambda)$ of $\mathrm{Aut}_{\overline{E}_\lambda}\, \mathbf{G}_\lambda(E_\lambda)$ is transitive on the set of hyperspecial maximal compact subgroups of $\mathbf{G}_\lambda(E_\lambda)$ [67, Section 2.5].

Hence, by Proposition 3.6 and adjusting ϕ_λ for almost all λ, we assume $\phi_\lambda(\mathbf{G}(\mathcal{O}_\lambda)) = H_\lambda \subset \mathbf{G}_\lambda(E_\lambda)$ for almost all λ. Then the image of the map $\prod_{\lambda\in\mathcal{P}} \phi_\lambda^{-1} \circ \rho_\lambda$ is contained in the adelic points $\mathbf{G}(\mathbb{A}_E^{(p)})$, which defines the desired $\mathbf{G}$-valued adelic representation $\rho_{\mathbb{A}}^{\mathbf{G}}$. This proves assertion (i).

The proof of (ii) is exactly the same except we want to adjust the isomorphism of representations

$$\phi_\lambda : (\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}) \times_E E_\lambda \to (\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda})$$

in order to have

$$\phi_\lambda^{-1}(H_\lambda) = \mathbf{G}(\mathcal{O}_\lambda) \subset \mathrm{GL}_{n,E}(\mathcal{O}_\lambda) \tag{51}$$

for almost λ (the inclusion is defined by finding some integral model $\mathcal{G} \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}$ of $\mathbf{G} \subset \mathrm{GL}_{n,E}$). This can be achieved since $\mathbf{G}_\lambda^{\mathrm{ad}}(E_\lambda)$ is a subgroup of the group $\mathrm{Inn}_{\bar{E}_\lambda}(\mathrm{GL}_{n,E_\lambda}, \mathbf{G}_\lambda)(E_\lambda)$ (see Section 2.3.1) as the representation ρ_λ is absolutely irreducible. ∎

3.4.3. Proof of Corollary 1.3. Find a smooth $\mathcal{O}_{E,S}$-model $\mathcal{G} \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}$ of $\mathbf{G} \subset \mathrm{GL}_{n,E}$ for some finite $S \subset \mathcal{P}_{E,f}$. Then by enlarging S we find that the group scheme $\mathrm{GL}_{n,\mathcal{O}_{E,S}} \times \mathcal{O}_\lambda$ (resp. $\mathcal{G} \times \mathcal{O}_\lambda$) is the group scheme associated to the hyperspecial maximal compact subgroup $\mathrm{GL}_{n,\mathcal{O}_{E,S}}(\mathcal{O}_\lambda)$ of $\mathrm{GL}_{n,\mathcal{O}_{E,S}}(E_\lambda) = \mathrm{GL}_n(E_\lambda)$ (resp. $\mathcal{G}(\mathcal{O}_\lambda)$ of $\mathbf{G}(E_\lambda)$) for all $\lambda \in \mathcal{P} \setminus S$ [67, Section 3.9.1]. We may assume that for all $\lambda \in \mathcal{P} \setminus S$, the inclusion

$$\mathcal{G}(\mathcal{O}_\lambda) \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}(\mathcal{O}_\lambda) = \mathrm{GL}_n(\mathcal{O}_\lambda)$$

gives the construction $\mathbf{G}(\mathcal{O}_\lambda) \subset \mathrm{GL}_{n,E}(\mathcal{O}_\lambda)$ in (51). Since the λ-component

$$(\rho_{\mathbb{A}}^{\mathbf{G}})_\lambda : \pi_1(X) \to \mathcal{G}(\mathcal{O}_\lambda) \subset \mathrm{GL}_{n,\mathcal{O}_{E,S}}(\mathcal{O}_\lambda) = \mathrm{GL}_n(\mathcal{O}_\lambda) \subset \mathrm{GL}_n(E_\lambda)$$

of the adelic representation $\rho_{\mathbb{A}}^{\mathbf{G}}$ is isomorphic to ρ_λ by Corollary 1.2 (ii), the representation $(\mathcal{G} \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_{E,S}}) \times \mathcal{O}_\lambda$ is isomorphic to $\mathcal{G}_\lambda \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_\lambda}$, where $\mathcal{G}_\lambda$ is the Zariski closure of $\rho_\lambda(\pi_1(X))$ in $\mathrm{GL}_{n,\mathcal{O}_\lambda}$ after some choice of $\mathcal{O}_\lambda$-lattice in V_λ. ∎

Remark 3.7. The proofs of Corollaries 1.2 and 1.3 are standard in the sense that they only require the common E-forms $\mathbf{G}$ and $\mathbf{G} \subset \mathrm{GL}_{n,E}$ in Theorem 1.1, Proposition 3.6, and Bruhat–Tits theory [67].

3.4.4. Proof of Corollary 1.4. By Corollary 1.2 (ii), there exists a common E-form $\iota : \mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}$. For each $\lambda \in \mathcal{P}$, choose an embedding $\bar{E} \to \bar{E}_\lambda$. We claim that the conjugacy class of the semisimple part $\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s \in \mathbf{G}(E_\lambda)$ is defined over $\bar{E}$ for all Frobenius elements $\mathrm{Fr}_{\bar{t}}$ and all $\lambda \in \mathcal{P}$. Indeed, by field extension, we obtain

$$\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s \in (\mathbf{G} \times \bar{E})(\bar{E}_\lambda) \overset{\iota_{\bar{E}}}{\hookrightarrow} \mathrm{GL}_{n,\bar{E}}(\bar{E}_\lambda).$$

It suffices to show that for any irreducible representation ψ of $\mathbf{G} \times \bar{E}$, the trace of $\psi(\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s)$ is in $\bar{E}$. This is true because the roots α of the characteristic polynomial

$P_t(T)$ of $\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}}) \in \mathrm{GL}_{n,\bar{E}}(\bar{E}_\lambda)$ belong to $\bar{E}$ by E-compatibility and ψ is a subrepresentation of $\bigotimes^r \iota_{\bar{E}} \otimes \bigotimes^s \iota_{\bar{E}}^*$ for some $r, s \in \mathbb{Z}_{\geq 0}$.

The next step is to show that for a fixed Frobenius element $\mathrm{Fr}_{\bar{t}}$, the conjugacy class of $\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s$ in $\mathbf{G}$ is independent of λ. By [10, Theorem 4.3.2] ([8, Theorem 6.8, Corollary 6.9] when X is a curve), there is a finite extension F of E and a connected reductive subgroup $\mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,F}$ such that for all $\lambda \in \mathscr{P}$, if F_λ is a completion of F extending λ on E, then there exists an isomorphism of representations

$$f_{F_\lambda} : (\mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,F}) \times_F F_\lambda \xrightarrow{\cong} (\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda. \tag{52}$$

Moreover by [10, proof of Theorem 4.3.2] ([8, Theorem 6.12] when X is a curve), the representations

$$\rho_\lambda^{\mathbf{G}^{\mathrm{sp}}} : \pi_1(X) \xrightarrow{\rho_\lambda} (\mathbf{G}_\lambda \times F_\lambda)(F_\lambda) \xrightarrow{f_{F_\lambda}^{-1}} \mathbf{G}^{\mathrm{sp}}(F_\lambda) \tag{53}$$

for all λ form an F-compatible system of $\mathbf{G}^{\mathrm{sp}}$-representations. Hence, the conjugacy class $[\rho_\lambda^{\mathbf{G}^{\mathrm{sp}}}(\mathrm{Fr}_{\bar{t}})_s]$ in $\mathbf{G}^{\mathrm{sp}}$ is independent of λ. If $\beta_\lambda \in N_{\mathrm{GL}_{n,E}}(\mathbf{G})(E_\lambda)$, we obtain the isomorphisms

$$\begin{aligned}
&(\mathbf{G} \times_E E_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda \xrightarrow{\beta_\lambda^{-1} \times F_\lambda} (\mathbf{G} \times_E E_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda \\
&= (\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}) \times_E F_\lambda \xrightarrow{\phi_\lambda \times F_\lambda} (\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda \xrightarrow{f_{F_\lambda}^{-1}} (\mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,F}) \times_F F_\lambda
\end{aligned} \tag{54}$$

by Corollary 1.2 (ii) and (52). Fix $\lambda' \in \mathscr{P}$, define $\beta_{\lambda'} = \mathrm{id}$, and embed F_λ into $\mathbb{C}$ for all $\lambda \in \mathscr{P}$. It suffices to find β_λ for all $\lambda \in \mathscr{P} \setminus \{\lambda'\}$ such that

$$\Phi_\lambda := [(\beta_\lambda \times F_\lambda) \circ (\phi_\lambda \times F_\lambda)^{-1} \circ f_{F_\lambda} \circ f_{F_{\lambda'}}^{-1} \circ (\phi_{\lambda'} \times F_{\lambda'})] \times \mathbb{C} \in \mathrm{Inn}_{\mathbb{C}}(\mathbf{G} \times \mathbb{C}). \tag{55}$$

Then $\Phi_\lambda([\rho_{\lambda'}^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s]) = [\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s]$ is an equality of conjugacy classes in $\mathbf{G}$ for all $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$.

For (i), since $\mathbf{G}_\lambda$ is split and is irreducible on the ambient space, $N_{\mathrm{GL}_{n,E}}(\mathbf{G})(E_\lambda)$ surjects onto $\theta_{\bar{E}_\lambda}$ in (18). Thus, there is $\beta_\lambda \in N_{\mathrm{GL}_{n,E}}(\mathbf{G})(E_\lambda)$ such that $\Phi_\lambda \in \mathrm{Inn}_{\mathbb{C}}(\mathbf{G} \times \mathbb{C}) = \mathbf{G}^{\mathrm{ad}}(\mathbb{C})$. For (ii), take $\beta_\lambda = \mathrm{id}$ for all λ. Since the outer automorphism group $\mathrm{Out}(\mathbf{G}^{\mathrm{der}} \times \mathbb{C})$ is trivial and $\mathbf{G} \times \mathbb{C} \hookrightarrow \mathrm{GL}_{n,\mathbb{C}}$ is irreducible, the image of Φ_λ in $\mathrm{Out}(\mathbf{G} \times \mathbb{C})$ is also trivial. Hence, in both cases (i) and (ii), Φ_λ is inner and $[\rho_{\lambda'}(\mathrm{Fr}_{\bar{t}})_s] = [\rho_\lambda(\mathrm{Fr}_{\bar{t}})_s]$ for all $\mathrm{Fr}_{\bar{t}}$. For $\mathrm{Fr}_{\bar{t}} \in \mathrm{Fr}$, it follows that the $\bar{E}$-conjugacy class $[\rho_\lambda(\mathrm{Fr}_{\bar{t}})_s]$ is independent of $\lambda \in \mathscr{P}$.

Let R be the affine coordinate ring of $\mathbf{G}$. For any $f \in R^{\mathbf{G}}$, $f_t := f([\rho_\lambda^{\mathbf{G}}(\mathrm{Fr}_{\bar{t}})_s]) \in \bar{E}$ is independent of λ and also belongs to E_λ for all $\lambda \in \mathscr{P}$. Therefore, $f_t \in E$ and we conclude that $\{\rho_\lambda^{\mathbf{G}}\}_{\lambda \in \mathscr{P}}$ is an E-compatible system of $\mathbf{G}$-representations. The last claim of the corollary is immediate. ∎

Remark 3.8. In general, if for each λ there exists $\beta_\lambda \in \mathrm{Inn}_{\bar{E}_\lambda}(\mathrm{GL}_{n,E_\lambda}, \mathbf{G}_\lambda)(E_\lambda)$ such that Φ_λ (defined in (55)) belongs to $\mathrm{Inn}_{\mathbb{C}}(\mathbf{G} \times \mathbb{C})$, then the conclusion of the corollary also follows.

3.4.5. Proof of Theorem 1.5. For each $\lambda \in \mathcal{P}_{E,f}^{(p)}$, we have the chain $\mathbf{T}_{\bar{t},\lambda} \subset \mathbf{G}_\lambda^\circ \subset \mathrm{GL}_{V_\lambda} \cong \mathrm{GL}_{n,E_\lambda}$. For each $v \in \mathcal{P}_{E,p}$, we have the chain $\mathbf{T}_{t,v} \subset \mathbf{G}_{t,v}^\circ \subset \mathrm{GL}_{V_{t,v}} \cong \mathrm{GL}_{n,E_v}$ by condition (b) of Theorem 1.5. By identifying the algebraic monodromy groups as subgroups of GL_n, we obtain a chain $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda^\circ \subset \mathrm{GL}_{n,E_\lambda}$ for each finite place λ of E. Here we simplify our notation by representing places of E extending p also as λ. To prove the theorem, it suffices to find a torus $\mathbf{T} \subset \mathrm{GL}_{n,E}$ and a chain $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,E}$ such that conditions (a)–(d) of Main Theorem II for the system

$$\{\mathbf{T}_\lambda \subset \mathbf{G}_\lambda^\circ \subset \mathrm{GL}_{n,E_\lambda}\}_{\lambda \in \mathcal{P}_{E,f}}$$

hold. Note that the last sentence of Theorem 1.5 (ii) follows from Remark 2.1 (6). The verifications rely on the following result of D'Addezio (enhancing Theorem B) and the fact that $\mathbf{T}_\lambda$ is a maximal torus of $\mathbf{G}_\lambda$ for all $\lambda \in \mathcal{P}_{E,f}$ by condition (a) of Theorem 1.5.

Theorem B′ ([10, Construction 4.2.1 (Frobenius tori), Theorem 4.3.2 and its proof]). *Let $\rho_\bullet$ be the E-compatible system in Theorem 1.5 and $\mathbf{T}_\lambda \subset \mathbf{G}_\lambda^\circ \subset \mathrm{GL}_{n,E_\lambda}$ be the chain defined above for each $\lambda \in \mathcal{P}_{E,f}$. Then the following assertions hold:*

(i) (Common E-form of formal characters) *There exists a subtorus $\mathbf{T} := \mathbf{T}_t$ of $\mathrm{GL}_{n,E}$ such that for all $\lambda \in \mathcal{P}_{E,f}$, $\mathbf{T}_\lambda := \mathbf{T} \times_E E_\lambda$ is a maximal torus of $\mathbf{G}_\lambda^\circ$.*

(ii) (λ-independence over an extension) *There exist a finite extension F of E and a chain of subgroups $\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \subset \mathrm{GL}_{n,F}$ such that $\mathbf{G}^{\mathrm{sp}}$ is connected split reductive, $\mathbf{T}^{\mathrm{sp}}$ is a split maximal torus of $\mathbf{G}^{\mathrm{sp}}$, and for all $\lambda \in \mathcal{P}_{E,f}$, if F_λ is a completion of F extending λ on E, then there exists an isomorphism of chain representations*

$$f_{F_\lambda} : (\mathbf{T}^{\mathrm{sp}} \subset \mathbf{G}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,F}) \times_F F_\lambda \xrightarrow{\cong} (\mathbf{T}_\lambda \subset \mathbf{G}_\lambda^\circ \hookrightarrow \mathrm{GL}_{n,E_\lambda}) \times_{E_\lambda} F_\lambda.$$

(iii) (Rigidity) *The isomorphisms f_{F_λ} in* (ii) *can be chosen such that the restriction isomorphisms $f_{F_\lambda} : \mathbf{T}^{\mathrm{sp}} \times_F F_\lambda \to \mathbf{T}_\lambda \times_{E_\lambda} F_\lambda$ admit a common F-form $f_F : \mathbf{T}^{\mathrm{sp}} \to \mathbf{T} \times_E F$ for all $\lambda \in \mathcal{P}_{E,f}$ and F_λ.*

Then conditions II (a)–(c) are just Theorem B′ (i)–(iii). For condition II (d), let $\mathbf{T}_t \subset \mathrm{GL}_{n,E}$ be the E-form in Theorem B′ (i). By Section 2.6.1 and conditions II (a)–(c), there exists an isomorphism of representations

$$f_{\bar{E}} : (\mathbf{T}^{\mathrm{sp}} \hookrightarrow \mathrm{GL}_{n,E}) \times_E \bar{E} \xrightarrow{\cong} (\mathbf{T}_t \hookrightarrow \mathrm{GL}_{n,E}) \times_E \bar{E}$$

which produces the cocycle μ as in (41). Consider the short exact sequence of E-groups

$$1 \to \mathbf{C} \to \mathbf{T}^{\mathrm{sp}} \to \mathbf{T}^{\mathrm{sp}}/\mathbf{C} \to 1. \tag{56}$$

By Proposition 2.15, μ as Galois representation acts on $\mathbf{C}$ and hence (56) in an equivariant way, inducing a short exact sequence of E-groups by twisting (Section 2.4.1):

$$\begin{tikzcd} & & \mathbf{T}_t \arrow[d, "="] & & \\ 1 \arrow[r] & {}_\mu\mathbf{C} \arrow[r] & {}_\mu\mathbf{T}^{\mathrm{sp}} \arrow[r] & {}_\mu(\mathbf{T}^{\mathrm{sp}}/\mathbf{C}) \arrow[r] & 1 \end{tikzcd} \tag{57}$$

Since μ is constructed from $f_{\overline{E}}$, it has values in $\mathrm{Inn}_{\overline{E}}(\mathrm{GL}_{n,E}, \mathbf{T}^{\mathrm{sp}})$. It follows that μ as Galois representation acts on the surjection of E-groups

$$(\mathbf{T}^{\mathrm{sp}} \cap \mathrm{SL}_{n,E})^\circ \twoheadrightarrow \mathbf{T}^{\mathrm{sp}}/\mathbf{C} \tag{58}$$

in an equivariant way. Hence, we obtain a surjection of E-groups

$$(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ = {}_{\mu}(\mathbf{T}^{\mathrm{sp}} \cap \mathrm{SL}_{n,E})^\circ \twoheadrightarrow {}_{\mu}(\mathbf{T}^{\mathrm{sp}}/\mathbf{C}). \tag{59}$$

By condition (c) of Theorem 1.5, E has a real place. Since $(\mathbf{T}_t \cap \mathrm{SL}_{n,E})^\circ$ is anisotropic over each real place of E by Proposition 3.3, Remark 3.5 (1), and the fact that $\rho_\bullet$ is pure of weight w, it follows that the twisted torus ${}_{\mu}(\mathbf{T}^{\mathrm{sp}}/\mathbf{C})$ is also anisotropic over each real place of E by the surjection (59). ∎

3.5. *Proofs of characteristic zero results*

Let $\mathcal{P}$ be $\mathcal{P}_{\mathbb{Q},f}$.

3.5.1. Proof of Theorem 1.6. It suffices to check conditions (a)–(d) of Main Theorem II for the system of algebraic monodromy representations

$$\{\mathbf{G}_\ell \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}\}_{\ell\in\mathcal{P}}$$

and invoke Remark 2.1 (6). Since the conditions remain the same after taking any finite extension F of K, we are free to do so.

Condition II (a): By condition 1.6 (a) and Remark 3.1 (6), there is a place $v \in \mathcal{P}_{K,f} \setminus S$ such that the Frobenius torus $\mathbf{T}_{\overline{v},\ell}$ is a maximal torus of $\mathbf{G}_\ell$ for all $\ell \neq p := p_v$, and the local representation V_p of $\mathrm{Gal}(\overline{K}_v/K_v)$ is ordinary. It remains to check the condition for the places over p.

Let Y_v be the special fiber of a smooth model of Y over $\mathcal{O}_v$ and let $M_v := H^w(Y_v/\mathcal{O}_v) \otimes_{\mathcal{O}_v} K_v$ be the crystalline cohomology group, which belongs to the category $\mathbf{MF}^f_{K_v}$ of *weakly admissible filtered modules* over K_v. There are algebraic subgroups $(\mathbf{H}_{V_p} \subset \mathrm{GL}_{V_p}) \times_{\mathbb{Q}_p} K_v$ and $\mathbf{H}_{M_v} \subset \mathrm{GL}_{M_v}$ such that their tautological representations (via the *mysterious functor* of Fontaine) are inner forms of each other, in particular isomorphic over $\overline{\mathbb{Q}}_p$,

$$(\mathbf{H}_{V_p} \hookrightarrow \mathrm{GL}_{V_p}) \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p \xrightarrow{\iota \cong} (\mathbf{H}_{M_v} \hookrightarrow \mathrm{GL}_{M_v}) \times_{K_v} \overline{\mathbb{Q}}_p,$$

where $\mathbf{H}_{V_p}$ is the algebraic monodromy group of the local crystalline representation $\rho_p : \mathrm{Gal}(\overline{K}_v/K_v) \to \mathrm{GL}(V_p)$ and $\mathbf{H}_{M_v}$ is the automorphism group of the fiber functor on the full Tannakian subcategory of $\mathbf{MF}^f_{K_v}$ generated by M_v that assigns to a filtered K-module the underlying K-vector space (see [53, Section 2]). Let m_v be the degree $[K_v : \mathbb{Q}_p]$ and f_{M_v} the *crystalline Frobenius*. By Katz–Messing [32] (see [53, Theorem 3.10]), $f_{M_v}^{m_v}$ is an element of $H_{M_v}(K_v) \subset \mathrm{GL}(M_v)$ with characteristic polynomial equal to $P_v(T)$, the characteristic polynomial of $\rho_\ell(\mathrm{Fr}_v)$ ($\ell \neq p$).

Let Φ_{V_p} be the element in $\mathbf{H}_{V_p}(\overline{\mathbb{Q}}_p)$ corresponding to $f_{M_v}^{m_v} \in H_{M_v}(\overline{\mathbb{Q}}_p)$ via ι. The group $\mathbf{H}_{V_p}$ is generated by cocharacters (connected) and the smallest algebraic subgroup containing Φ_{V_p} [53, Proposition 2.6]. It is connected because the characteristic polynomial of $\Phi_{V_p} \in \mathrm{GL}_{V_p}$ is $P_v(T)$ and the (maximal) Frobenius torus $\mathbf{T}_{\overline{v},\ell}$ is equal to $\mathbf{S}_{\overline{v},\ell}$ by Remark 3.1(1). Let V_p^{ss} be the semisimplification of the representation $\mathbf{H}_{V_p} \hookrightarrow \mathrm{GL}_{V_p}$. Since the local representation V_p is ordinary, $\mathbf{H}_{V_p}$ is solvable [53, Proposition 2.9] and its image $\mathbf{H}_{V_p}^{\mathrm{red}}$ in $\mathrm{GL}_{V_p^{\mathrm{ss}}}$ is a torus. Since the conjugacy class of Φ_{V_p} in $\mathbf{H}_{V_p}$ is defined over $\mathbb{Q}_p$ [53, Proposition 2.2] and $\mathbf{H}_{V_p}^{\mathrm{red}}$ is abelian, the image of Φ_{V_p} in $\mathbf{H}_{V_p}^{\mathrm{red}}$, denoted by $\Phi_{V_p}^{\mathrm{red}}$, belongs to $\mathbf{H}_{V_p}^{\mathrm{red}}(\mathbb{Q}_p)$. By the splitting of the surjection $\mathbf{H}_{V_p} \twoheadrightarrow \mathbf{H}_{V_p}^{\mathrm{red}}$, there is a semisimple element $\Phi_{\overline{v}} \in \mathbf{H}_{V_p}(\mathbb{Q}_p) \subset \mathrm{GL}(V_p)$ with characteristic polynomial $P_v(T)$. The smallest algebraic subgroup of $\mathbf{H}_{V_p} \subset \mathbf{G}_p$ containing $\Phi_{\overline{v}}$ is a $\mathbb{Q}_p$-maximal torus $\mathbf{T}_{\overline{v},p}$ of $\mathbf{G}_p$ because the absolute rank of $\mathbf{G}_\ell$ is independent of ℓ by Corollary 3.2 (ii) and $\mathbf{T}_{\overline{v},\ell}$ is a maximal Frobenius torus. Since the characteristic polynomials of $\Phi_{\overline{v}}$ and $\rho_\ell(\mathrm{Fr}_{\overline{v}})_s$ for all $\ell \neq p$ are equal to $P_v(T)$, the tori representations $\mathbf{T}_{\overline{v},\ell} \hookrightarrow \mathrm{GL}_{V_\ell}$ for all ℓ admit a common $\mathbb{Q}$-form $\mathbf{T}_v \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$. ∎

Condition II (b): This is just condition 1.6 (b). ∎

Condition II (c): By Proposition 2.8 and condition 1.6 (c), it suffices to check condition (c′-bi) of Section 2.2. Identify GL_{V_ℓ} as $\mathrm{GL}_{n,\mathbb{Q}_\ell}$ for all ℓ. We employ the technique in [26, Proposition 3.18, Theorem 3.19]. Let $\{\psi_\ell\}_{\ell\in\mathscr{P}}$ be an r-dimensional semisimple $\mathbb{Q}$-compatible system of abelian ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$. Let $\mathbf{S}_\ell \subset \mathrm{GL}_{r,\mathbb{Q}_\ell}$ be the algebraic monodromy group of ψ_ℓ and assume $\mathbf{S}_\ell$ is torus and with the largest possible dimension d_K [26, Theorem 3.8] for all ℓ. Consider the semisimple $\mathbb{Q}$-compatible system $\{\rho_\ell \oplus \psi_\ell\}_{\ell\in\mathscr{P}}$ and let $\mathbf{G}_\ell' \subset \mathrm{GL}_{n,\mathbb{Q}_\ell} \times \mathrm{GL}_{r,\mathbb{Q}_\ell}$ be the algebraic monodromy group at ℓ. Let

$$p_{i,\ell}' : \mathbf{G}_\ell' \to \mathrm{GL}_{n,\mathbb{Q}_\ell} \times \mathrm{GL}_{r,\mathbb{Q}_\ell} \tag{60}$$

be the projection to the ith factor, $i = 1, 2$. By considering $p_{1,\ell}'$, there is a diagonalizable subgroup $\mathbf{D}_\ell$ of $\mathbf{S}_\ell$ with a short exact sequence

$$1 \to \mathbf{D}_\ell \to \mathbf{G}_\ell' \to \mathbf{G}_\ell \to 1. \tag{61}$$

Let k be the number of components of $\mathbf{D}_{\ell'}$ for some prime ℓ'. By replacing $\{\rho_\ell \oplus \psi_\ell\}_{\ell\in\mathscr{P}}$ with $\{\rho_\ell \oplus \psi_\ell^k\}_{\ell\in\mathscr{P}}$, we assume that $\mathbf{D}_{\ell'}$ is connected. Since $\mathbf{G}_{\ell'}$ is connected, $\mathbf{G}_{\ell'}'$ is connected by (61). Hence, $\mathbf{G}_\ell'$ is connected for all ℓ by Proposition M. Since the dimension of the center of $\mathbf{G}_\ell'$ is $d_K = \dim \mathbf{S}_\ell$ for all ℓ [26, Proposition 3.8, Theorem 3.19], it follows that for all ℓ,

$$\ker(p_{2,\ell}')^\circ = (\mathbf{G}_\ell')^{\mathrm{der}} = \mathbf{G}_\ell^{\mathrm{der}}. \tag{62}$$

Proposition P ([16], [61, Chapter II], [55, Chapter 1, Theorem 4.1]). *Fix a prime ℓ''. There exist a finite extension F of K and an abelian variety A over F that is a direct product of CM abelian varieties with the following properties. Let*

$$\{\epsilon_\ell : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(W_\ell)\}_{\ell\in\mathscr{P}}$$

be the semisimple compatible system of Galois representations with $W_\ell := H^1(A_{\overline{F}}, \mathbb{Q}_\ell)$. *Let* $\mathbf{M}_\ell$ *and* $\mathbf{G}''_\ell$ *be respectively the algebraic monodromy groups of the Galois representations* ϵ_ℓ *and* $\rho_\ell \oplus \epsilon_\ell$ *of* $\mathrm{Gal}(\overline{F}/F)$. *Then the following assertions hold:*

(i) *For all* ℓ, $\mathbf{G}''_\ell$ *is connected and* $\mathbf{M}_\ell$ *is a torus with dimension independent of* ℓ.

(ii) *The restriction map* $\psi_{\ell''} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_{\ell''})$ *factors through a morphism* $\mathbf{M}_{\ell''} \times \overline{\mathbb{Q}}_{\ell''} \to \mathrm{GL}_{r,\overline{\mathbb{Q}}_{\ell''}}$.

Since $\mathbf{G}'_\ell$ is connected for all ℓ, it is again the algebraic monodromy group of the restriction of $\rho_\ell \oplus \psi_\ell$ to $\mathrm{Gal}(\overline{F}/F)$. Again, let $p''_{i,\ell} : \mathbf{G}''_\ell \to \mathrm{GL}_{n,\mathbb{Q}_\ell} \times \mathbf{M}_\ell$ be the projection to the ith factor, $i = 1, 2$. Since there exists a surjective map $\mathbf{G}''_{\ell''} \to \mathbf{G}'_{\ell''}$ by Proposition P (ii), it follows from (62) and the connectedness of $\mathbf{G}''_{\ell''}$ (Proposition P (i)) that

$$\ker(p''_{2,\ell''})^\circ = \mathbf{G}^{\mathrm{der}}_{\ell''} = (\mathbf{G}''_{\ell''})^{\mathrm{der}} \tag{63}$$

is the semisimple part of $\mathbf{G}''_{\ell''}$. Since $\{\rho_\ell \oplus \epsilon_\ell\}_{\ell \in \mathcal{P}}$ is a compatible system of representations of $\mathrm{Gal}(\overline{F}/F)$, the semisimple rank and the dimension of the center of $\mathbf{G}''_\ell$ are independent of ℓ [26, Theorem 3.19]. This, together with (63) and the ℓ-independence of $\dim \mathbf{M}_\ell$ (Proposition P (i)), implies that

$$\ker(p''_{2,\ell})^\circ = (\mathbf{G}''_\ell)^{\mathrm{der}} \tag{64}$$

for all ℓ. Hence, if $\mathbf{T}''_\ell$ is a maximal torus of $\mathbf{G}''_\ell$, then

$$\ker(p''_{2,\ell} : \mathbf{T}''_\ell \to \mathbf{M}_\ell)^\circ \subset p''_{1,\ell}(\mathbf{T}''_\ell) \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell}$$

is a formal bi-character of $\mathbf{G}_\ell$.

Finally, we follow the strategy in condition II (a). If v is a finite place of F such that $Y \times_K F$ and A have good reduction, then write $p := p_v$ and the Frobenius element $\mathrm{Fr}_{\overline{v}}$ have characteristic polynomials $P_v(T) \in \mathbb{Q}[T]$ on V_ℓ and $Q_v(T) \in \mathbb{Q}[T]$ on W_ℓ for all $\ell \neq p$. By condition 1.6 (a), there exists $v \in \mathcal{P}_{F,f}$ such that the Frobenius torus $\mathbf{T}''_{\overline{v},\ell} \subset \mathbf{G}''_\ell$ is maximal for all $\ell \neq p$ and the local representation V_p of $\mathrm{Gal}(\overline{F}_v/F_v)$ is ordinary. Then we let $\mathbf{H}_{V_p \oplus W_p} \subset \mathrm{GL}_{V_p} \times \mathrm{GL}_{W_p}$ be the algebraic monodromy group of the local crystalline representation

$$\rho_p \oplus \epsilon_p : \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{GL}(V_p) \times \mathrm{GL}(W_p)$$

and $\mathbf{H}^{\mathrm{red}}_{V_p \oplus W_p}$ its image (semisimplification) in the (abelian) diagonalizable subgroup $\mathbf{H}^{\mathrm{red}}_{V_p} \times \mathbf{M}_p \subset \mathrm{GL}_{V_p} \times \mathrm{GL}_{W_p}$, where $\mathbf{H}^{\mathrm{red}}_{V_p}$ is defined in condition II (a). Since the local representation $V_p \oplus W_p$ is crystalline, we conclude by repeating the arguments in the second and third paragraphs of the proof of condition II (a) that there exists an element in $\mathbf{H}^{\mathrm{red},\circ}_{V_p \oplus W_p}(\mathbb{Q}_p)$ lifting to a semisimple element $\Phi''_{\overline{v}} \in \mathbf{H}^\circ_{V_p \oplus W_p}(\mathbb{Q}_p) \in \mathbf{G}''_p(\mathbb{Q}_p)$ with characteristic polynomials $P_v(T)$ on V_p and $Q_v(T)$ on W_p. The smallest algebraic subgroup $\mathbf{T}''_{\overline{v},p}$ of $\mathbf{G}''_p$ containing $\Phi''_{\overline{v}}$ is also a maximal torus because the absolute rank of $\mathbf{G}''_\ell$ is independent of ℓ. By using the polynomials $P_v(T), Q_v(T) \in \mathbb{Q}[T]$, we construct a common $\mathbb{Q}$-form $\mathbf{T}''_v \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}} \times \mathrm{GL}_{2\dim A,\mathbb{Q}}$ of the formal characters $\mathbf{T}''_{\overline{v},\ell} \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}_\ell} \times \mathrm{GL}_{W_\ell}$ of

$\mathbf{G}''_\ell \subset \mathrm{GL}_{n,\mathbb{Q}_\ell} \times \mathrm{GL}_{W_\ell}$ for all ℓ such that

$$\ker(p_2 : \mathbf{T}''_v \to \mathrm{GL}_{2\dim A,\mathbb{Q}})^\circ \subset p_1(\mathbf{T}''_v) \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}} \tag{65}$$

is a common $\mathbb{Q}$-form of formal bi-characters of $\mathbf{G}_\ell \subset \mathrm{GL}_{n,\mathbb{Q}_\ell}$ for all ℓ, where p_1, p_2 are the obvious projections. We may replace $\mathbf{T}_v \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$ constructed in the proof of condition II (a) with $p_1(\mathbf{T}''_v) \hookrightarrow \mathrm{GL}_{n,\mathbb{Q}}$ in (65). ∎

Condition II (d): Let $\mathbf{T}_v \subset \mathrm{GL}_{n,\mathbb{Q}}$ be the $\mathbb{Q}$-form we found in the proof of condition II (a). This part is exactly the same as the verification of condition II (d) for Theorem 1.5 once we replace the field E by $\mathbb{Q}$ and the E-torus $\mathbf{T}_t$ by the $\mathbb{Q}$-torus $\mathbf{T}_v$. ∎

3.5.2. Proofs of Corollaries 1.9 *and* 1.10. Since Corollaries 1.9 and 1.10 (of Theorem 1.6) assume Hypothesis H, their proofs follow the lines of the proofs of Corollaries 1.2 and 1.3 by Remark 3.7. ∎

3.5.3. Galois maximality and Hypothesis H. Let K be a number field and $\{\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_{\ell\in\mathcal{P}}$ be a $\mathbb{Q}$-compatible system of ℓ-adic representations. Let Γ_ℓ be the image of ρ_ℓ and $\mathbf{G}_\ell$ be the algebraic monodromy group of ρ_ℓ. Then Γ_ℓ is a compact subgroup of $\mathbf{G}_\ell(\mathbb{Q}_\ell)$. Suppose for simplicity that $\mathbf{G}_\ell$ is connected for all ℓ. Denote by $\mathbf{G}_\ell^{\mathrm{ss}}$ the quotient of $\mathbf{G}_\ell$ by its radical and by $\mathbf{G}_\ell^{\mathrm{sc}}$ the simply-connected covering of $\mathbf{G}_\ell^{\mathrm{ss}}$. Denote by $\Gamma_\ell^{\mathrm{ss}}$ the image of Γ_ℓ in $\mathbf{G}_\ell^{\mathrm{ss}}(\mathbb{Q}_\ell)$ and by $\Gamma_\ell^{\mathrm{sc}}$ the inverse image of $\Gamma_\ell^{\mathrm{ss}}$ in $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$. When $\ell \gg 0$ compared to the absolute rank of $\mathbf{G}_\ell^{\mathrm{sc}}$, a compact subgroup H_ℓ of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is hyperspecial maximal compact if the "mod ℓ reduction" of H_ℓ is "of the same Lie type" as the semisimple group $\mathbf{G}_\ell^{\mathrm{sc}}$ (see [30]). In [39], Larsen proved that the set of primes ℓ for which $\Gamma_\ell^{\mathrm{sc}} \subset \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is hyperspecial maximal compact is of Dirichlet density 1 and conjectured the following.

Conjecture S. *For all $\ell \gg 0$, $\Gamma_\ell^{\mathrm{sc}}$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$.*

This conjecture is also related to the conjectures of Serre on maximal motives [59, Sections 11.4, 11.8]. Suppose the ℓ-adic compatible system is $\{H^w(Y_{\bar{K}}, \mathbb{Q}_\ell)\}_{\ell\in\mathcal{P}}$, where Y is a smooth projective variety defined over a number field K. When Y is an elliptic curve without complex multiplication and $w = 1$, a well-known theorem of Serre states that for $\ell \gg 0$, $\Gamma_\ell \cong \mathrm{GL}_2(\mathbb{Z}_\ell)$ is maximal compact in $\mathrm{GL}(V_\ell)$ [57]. In general, by studying the *mod ℓ compatible system* $\{H^w(Y_{\bar{K}}, \mathbb{F}_\ell)\}_{\ell\gg 0}$, we proved that $\Gamma_\ell \subset \mathbf{G}_\ell(\mathbb{Q}_\ell)$ is large in the sense that its mod ℓ reduction has "the same semisimple rank" as the algebraic group $\mathbf{G}_\ell$ for $\ell \gg 0$ [27, Theorem A]. This result is crucial to the following.

Theorem T ([30, 31]). *Let $\rho_\bullet$ be the $\mathbb{Q}$-compatible system* (3) *arising from a smooth projective variety Y defined over K. Conjecture* S *holds in the following cases:*

(i) *For $\ell \gg 0$, $\mathbf{G}_\ell^{\mathrm{sc}}$ is of type A, i.e., isomorphic to $\prod_i \mathrm{SL}_{n_i}$ over $\bar{\mathbb{Q}}_\ell$.*

(ii) *Y is an abelian variety.*

(iii) *Y is a hyper-Kähler variety and $w = 2$.*

Let Λ_ℓ be a $\mathbb{Z}_\ell$-lattice of $\mathbb{Q}_\ell^n$ that is invariant under Γ_ℓ and let $\mathscr{G}_\ell$ (resp. $\mathscr{G}_\ell^{\mathrm{der}}$) be the Zariski closure of Γ_ℓ (resp. the derived group $[\Gamma_\ell,\Gamma_\ell]$) in $\mathrm{GL}_{n,\mathbb{Z}_\ell}$, endowed with the unique reduced closed subscheme structure. Write $\mathscr{G}_{\mathbb{F}_\ell}$ (resp. $\mathscr{G}_{\mathbb{F}_\ell}^{\mathrm{der}}$) for the special fiber.

Theorem 3.9. *Suppose $\mathbf{G}_\ell$ is connected reductive for all ℓ. Then Conjecture S implies that $\mathscr{G}_\ell$ is a reductive group scheme over $\mathbb{Z}_\ell$ for $\ell \gg 0$ and Hypothesis H.*

Proof. Let $\pi_\ell : \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell) \to \mathbf{G}_\ell^{\mathrm{der}}(\mathbb{Q}_\ell) \to \mathbf{G}_\ell(\mathbb{Q}_\ell)$ be the natural morphism. Consider the following natural commutative diagram where each vertical map is the commutator map:

$$\begin{tikzcd}
\mathbf{G}_\ell^{\mathrm{sc}} \times \mathbf{G}_\ell^{\mathrm{sc}} \arrow[r] \arrow[d] & \mathbf{G}_\ell \times \mathbf{G}_\ell \arrow[r] \arrow[d] & \mathbf{G}_\ell^{\mathrm{ss}} \times \mathbf{G}_\ell^{\mathrm{ss}} \arrow[d] \\
\mathbf{G}_\ell^{\mathrm{sc}} \arrow[r, "\pi_\ell"] & \mathbf{G}_\ell \arrow[r] & \mathbf{G}_\ell^{\mathrm{ss}}
\end{tikzcd}$$

Then by the definition of $\Gamma_\ell^{\mathrm{sc}}$, the inclusion $\pi_\ell([\Gamma_\ell^{\mathrm{sc}},\Gamma_\ell^{\mathrm{sc}}]) \subset [\Gamma_\ell,\Gamma_\ell]$ holds. Suppose Conjecture S holds. Then the hyperspecial maximal compact $\Gamma_\ell^{\mathrm{sc}}$ is perfect for $\ell \gg 0$ (see, e.g., [30, proof of Corollary 11]). Thus, for $\ell \gg 0$, it follows that $\pi_\ell(\Gamma_\ell^{\mathrm{sc}}) \subset [\Gamma_\ell,\Gamma_\ell]$. The closed subscheme $\mathscr{G}_\ell \subset \mathrm{GL}_{n,\mathbb{Z}_\ell}$ is smooth by Proposition N for $\ell \gg 0$. Also, the subscheme $\mathscr{G}_\ell^{\mathrm{der}}$ is smooth for $\ell \gg 0$ (see, e.g., [7, Theorem 9.1.1, Section 9.2.1], note that $\mathbf{G}_\ell^{\mathrm{der}}$ is connected). Then for $\ell \gg 0$,

$$\pi_\ell(\Gamma_\ell^{\mathrm{sc}}) \subset [\Gamma_\ell,\Gamma_\ell] \subset \mathscr{G}_\ell^{\mathrm{der}}(\mathbb{Z}_\ell) \subset \mathscr{G}_\ell(\mathbb{Z}_\ell) \subset \mathrm{GL}_n(\mathbb{Z}_\ell). \tag{66}$$

If we can prove that $\mathscr{G}_\ell$ is a reductive group scheme over $\mathbb{Z}_\ell$, then $\mathscr{G}_\ell(\mathbb{Z}_\ell) \subset \mathbf{G}_\ell(\mathbb{Q}_\ell)$ is hyperspecial maximal compact by Bruhat–Tits theory. So it remains to prove that the special fiber $\mathscr{G}_{\mathbb{F}_\ell}$ is reductive.

Taking mod ℓ reduction of (66), we see by Hensel's lemma that for $\ell \gg 0$,

$$\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})} \subset \overline{\mathscr{G}_\ell^{\mathrm{der}}(\mathbb{Z}_\ell)} = \mathscr{G}_{\mathbb{F}_\ell}^{\mathrm{der}}(\mathbb{F}_\ell) \subset \mathrm{GL}_n(\mathbb{F}_\ell). \tag{67}$$

For $\ell \gg n$, let $\mathbf{S}_\ell \subset \mathrm{GL}_{n,\mathbb{F}_\ell}$ be the *Nori envelope* [50] of the finite subgroup $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})} \subset \mathrm{GL}_n(\mathbb{F}_\ell)$. It is the connected algebraic subgroup of $\mathrm{GL}_{n,\mathbb{F}_\ell}$ generated by the one-parameter unipotent subgroups $\{u^t : t \in \overline{\mathbb{F}}_\ell\}$ for all order ℓ elements of $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}$. It is semisimple by unipotent. Let $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^+$ be the (normal) subgroup of $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}$ generated by the order ℓ elements. Then $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^+$ is a subgroup of $\mathbf{S}_\ell(\mathbb{F}_\ell)$ and $[\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})} : \overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^+]$ is prime to ℓ. The Nori envelope $\mathbf{S}_\ell$ approximates the finite subgroup $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})} \subset \mathrm{GL}_n(\mathbb{F}_\ell)$ in the sense that the index $[\mathbf{S}_\ell(\mathbb{F}_\ell) : \overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^+]$ is bounded by a constant depending only on n for all prime ℓ large enough compared to n [50, Theorems B (1), 3.6 (v)].

Proposition 3.10. *For $\ell \gg 0$, the smooth group scheme $\mathscr{G}_\ell^{\mathrm{der}}$ is reductive.*

Proof. Suppose $\ell \geq n$. Since $\Gamma_\ell^{\mathrm{sc}}$ is maximal compact in $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ for $\ell \gg 0$, the equality $\pi_\ell^{-1}(\mathscr{G}_\ell^{\mathrm{der}}(\mathbb{Z}_\ell)) = \Gamma_\ell^{\mathrm{sc}}$ holds for $\ell \gg 0$. Thus, there is a constant c such that

$$[\mathscr{G}_\ell^{\mathrm{der}}(\mathbb{Z}_\ell) : \pi_\ell(\Gamma_\ell^{\mathrm{sc}})] \leq c \tag{68}$$

for all $\ell \gg 0$ [31, Corollary 2.5]. Hence, after reduction we also have

$$[\mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{der}}(\mathbb{F}_\ell) : \overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}] \leq c. \tag{69}$$

If the proposition is false, then the unipotent radical of the special fiber $\mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{der}}$ is non-trivial for infinitely many primes ℓ. Thus, (69) implies that $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}$ contains a non-trivial normal unipotent subgroup U_ℓ (consisting of order ℓ elements) for infinitely many primes ℓ. Let $\mathbf{S}'_\ell$ be the Nori envelope of the semisimplification of $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})} \hookrightarrow \mathrm{GL}_n(\mathbb{F}_\ell)$ (with image $\overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^{\mathrm{red}}$) for $\ell \gg 0$. By the definition of Nori envelope [50, Sections 1, 3], for all $\ell \gg 0$ we have a short exact sequence

$$1 \to \mathbf{U}_\ell \to \mathbf{S}_\ell \xrightarrow{\pi} \mathbf{S}'_\ell \to 1 \tag{70}$$

where π is induced by semisimplification. For infinitely many primes ℓ, we have $\dim \mathbf{U}_\ell \geq 1$ since $\mathbf{U}_\ell$ contains a one-parameter subgroup $t \mapsto u^t := \exp(t\log(u))$ [50] for some non-identity element $u \in U_\ell$.

Since $\mathbf{S}'_\ell$ is semisimple, [30, Proposition 4 (iii)] asserts that $\dim \mathbf{S}'_\ell = \dim_\ell \mathbf{S}'_\ell(\mathbb{F}_\ell)$ (the *ℓ-dimension* [30, Section 2]). Since $\Gamma_\ell^{\mathrm{sc}}$ is hyperspecial maximal compact in $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$, there is a reductive group scheme $\mathcal{H}_\ell$ over $\mathbb{Z}_\ell$ such that the generic fiber is $\mathbf{G}_\ell^{\mathrm{sc}}$ and $\mathcal{H}_\ell(\mathbb{Z}_\ell) = \Gamma_\ell^{\mathrm{sc}}$. By the definition of ℓ-dimension and [30, Proposition 4 (iii)] again, we obtain

$$\begin{aligned} \dim_\ell \mathbf{S}'_\ell(\mathbb{F}_\ell) &= \dim_\ell \mathbf{S}'_\ell(\mathbb{F}_\ell)^+ = \dim_\ell \overline{\pi_\ell(\Gamma_\ell^{\mathrm{sc}})}^{\mathrm{red}} = \dim_\ell \Gamma_\ell^{\mathrm{sc}} = \dim_\ell \mathcal{H}_\ell(\mathbb{F}_\ell) \\ &= \dim \mathbf{G}_\ell^{\mathrm{sc}}. \end{aligned} \tag{71}$$

It follows from (70) that $\dim \mathbf{S}_\ell > \dim \mathbf{G}_\ell^{\mathrm{der}}$ for infinitely many ℓ, but this contradicts [40, Theorem 7]. ■

Let $\mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{red}}$ be the quotient of $\mathcal{G}_{\mathbb{F}_\ell}^{\circ}$ by its unipotent radical. For $\ell \gg 0$, the special fiber $\mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{der}}$ (of $\mathcal{G}_\ell^{\mathrm{der}}$) is a normal connected semsimple subgroup of $\mathcal{G}_{\mathbb{F}_\ell}^{\circ}$ (Proposition 3.10), which injects into $\mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{red}}$. It follows that

$$\begin{aligned} \dim \mathcal{G}_{\mathbb{F}_\ell} \geq \dim \mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{red}} &\geq \dim \mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{der}} + \mathrm{rk}\, \mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{red}} - \mathrm{rk}\, \mathcal{G}_{\mathbb{F}_\ell}^{\mathrm{der}} \\ &= \dim \mathbf{G}_\ell^{\mathrm{der}} + \mathrm{rk}\, \mathbf{G}_\ell - \mathrm{rk}\, \mathbf{G}_\ell^{\mathrm{der}} = \dim \mathbf{G}_\ell. \end{aligned} \tag{72}$$

Therefore, (72) is an equality and the special fiber $\mathcal{G}_{\mathbb{F}_\ell}$ is reductive for $\ell \gg 0$. ■

Remark 3.11. Let F be a finitely generated field of characteristic p and Y be a smooth projective variety defined over F. Conjecture S holds for the $\mathbb{Q}$-compatible system $\{H^w(Y_{\bar{F}}, \mathbb{Q}_\ell)\}_{\ell \neq p}$ [7, Theorem 1.2].

3.5.4. Proof of Theorem 1.11. Embed $\mathbb{Q}_\ell$ into $\mathbb{C}$ for all ℓ. Since $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$, the representations ρ_ℓ are all absolutely irreducible by the Tate conjecture proven by Faltings [16]. Since the formal bi-character of $(\mathbf{G}_\ell \hookrightarrow \mathrm{GL}(V_\ell)) \times \mathbb{C}$ is independent of ℓ [26, Theorem 3.19], condition 1.11 (b) and Theorem E imply that the tautological representation $(\mathbf{G}_\ell \to \mathrm{GL}(V_\ell)) \times \mathbb{C}$ is independent of ℓ. Since condition 1.11 (b) and Theorem C (ii) hold, the simple factors of $\mathbf{G}_\ell \times_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ are of the same type and the invariance-of-roots condition holds by Corollary 2.5. We conclude that conditions 1.6 (a)–(c) hold. Hence, Theorems 1.6 (ii) T (ii), and 3.9, and Corollaries 1.9 and 1.10, give Theorem 1.11 except the last assertion. It suffices to show that for $\ell \gg 0$, the two $\mathbb{Z}_\ell$-representations

$$\begin{aligned} V_{\mathbb{Z}_\ell} &: \mathrm{Gal}(\overline{K}/K) \to \mathbf{G}(\mathbb{Z}_\ell) = \mathscr{G}(\mathbb{Z}_\ell) \to \mathrm{GL}_{2g}(\mathbb{Z}_\ell), \\ H^1(A_{\overline{K}}, \mathbb{Z}_\ell) &: \mathrm{Gal}(\overline{K}/K) \to \mathscr{G}_\ell(\mathbb{Z}_\ell) \to \mathrm{GL}(H^1(A_{\overline{K}}, \mathbb{Z}_\ell)) \end{aligned} \tag{73}$$

are isomorphic.

Since $V_{\mathbb{Z}_\ell} \otimes \mathbb{Q}_\ell \cong H^1(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H^1(A_{\overline{K}}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$, there is an element Φ_ℓ in the free $\mathbb{Z}_\ell$-module $\mathrm{Hom}_{\mathrm{Gal}(\overline{K}/K)}(V_{\mathbb{Z}_\ell}, H^1(A_{\overline{K}}, \mathbb{Z}_\ell))$ that is non-zero after mod ℓ reduction. Since $\mathrm{End}_{\overline{K}}(A) = \mathbb{Z}$, the representation $H^1(A_{\overline{K}}, \mathbb{F}_\ell)$ is absolutely irreducible for $\ell \gg 0$ [18, Theorem 4.2]. Thus, the non-zero $\mathrm{Gal}(\overline{K}/K)$-equivariant map

$$\Phi_\ell \times \mathbb{F}_\ell : V_{\mathbb{Z}_\ell} \otimes \mathbb{F}_\ell \to H^1(A_{\overline{K}}, \mathbb{F}_\ell)$$

is surjective for $\ell \gg 0$. By Nakayama's lemma, Φ_ℓ is surjective for $\ell \gg 0$. Therefore, Φ_ℓ is bijective and induces an isomorphism of the Galois representations $V_{\mathbb{Z}_\ell}$ and $H^1(A_{\overline{K}}, \mathbb{Z}_\ell)$ for $\ell \gg 0$. ∎

Remark 3.12. Embed $\mathbb{Q}_\ell$ into $\mathbb{C}$. Let $\{(\mathbf{H}_i, V_i) : 1 \le i \le k\}$ be the irreducible factors of the irreducible representation $(\mathbf{G}_\ell^{\mathrm{der}} \to \mathrm{GL}(V_\ell)) \times \mathbb{C}$, i.e., $\mathbf{H}_i$ is almost simple and V_i is irreducible (Section 2.2.2.1). By [53, Theorem 3.18], the irreducible representation $(\mathbf{G}_\ell \to \mathrm{GL}(V_\ell)) \times \mathbb{C}$ is a strong Mumford–Tate pair of weight $\{0, 1\}$. Then [53, Proposition 4.5] and [53, Table 4.6] imply that k is odd and for the representations $(\mathbf{H}_i, V_i)$ we have the following possibilities:

A_r: $\bigwedge^r$(standard), $r \equiv 1 \bmod 4$, $r \ge 1$.

B_r: Spin, $r \equiv 1, 2 \bmod 4$, $r \ge 2$.

C_r: Standard, $r \ge 3$.

D_r: Spin^+, $r \equiv 2 \bmod 4$, $r \ge 6$.

One observes that each simple Lie algebra has at most one possible representation.

3.6. *Final remarks*

(1) We construct a common E-form $\mathbf{G} \hookrightarrow \mathrm{GL}_{n,E}$ of the algebraic monodromy representations $\mathbf{G}_\lambda \hookrightarrow \mathrm{GL}_{n,E_\lambda}$ of the system (2) in case it is absolutely irreducible and $\mathbf{G}_\lambda$ is connected (for all λ) in Theorem 1.1 (ii). The non-absolutely-irreducible case and the non-connected case remain open.

(2) Let $\rho_\bullet$ be the system in Theorem 1.6 and assume Conjecture S. Then Corollary 1.9 (i) produces an adelic representation $\rho_{\mathbb{A}}^{\mathbf{G}} : \mathrm{Gal}(\overline{K}/K) \to \mathbf{G}(\mathbb{A}_{\mathbb{Q}})$. Let $\rho_{\mathbb{F}_\ell}^{\mathbf{G}}$ be the mod ℓ reduction of the ℓ-component $\rho_\ell^{\mathbf{G}}$ of $\rho_{\mathbb{A}}^{\mathbf{G}}$ for $\ell \gg 0$. One can deduce by [27, Theorem A, Corollary B] that there is a constant $C > 0$ such that the index satisfies

$$[\mathbf{G}(\mathbb{F}_\ell) : \rho_{\mathbb{F}_\ell}^{\mathbf{G}}(\mathrm{Gal}(\overline{K}/K))] \le C, \quad \forall \ell \gg 0.$$

Thus, the composition factors of Lie type in characteristic ℓ of $\rho^{\mathbf{G}}_{\mathbb{F}_\ell}(\mathrm{Gal}(\bar{K}/K))$ can be described when $\ell \gg 0$; see a similar result [28, Corollary 1.5] for certain type A compatible systems.

(3) The smooth subgroup scheme $\mathcal{G}_\lambda \subset \mathrm{GL}_{n,\mathcal{O}_\lambda}$ in Corollary 1.3 depends on the choice of an $\mathcal{O}_\lambda$-lattice of V_λ. It is shown in [6] that for almost all λ, the subscheme $\mathcal{G}_\lambda \subset \mathrm{GL}_{n,\mathcal{O}_\lambda}$ is unique up to isomorphism.

(4) The E-forms $\mathbf{G}$ and $\mathbf{G} \subset \mathrm{GL}_{n,E}$ we constructed in Theorem 1.1 are not unique for the simple reason that $Ш^1(E, \mathbf{G}^{\mathrm{ad}})$ in Theorem K may not be trivial, where $\mathbf{G}^{\mathrm{ad}}$ denotes the adjoint quotient of $\mathbf{G}$.

(5) Let S' be a non-empty finite subset of $\mathcal{P}_{E,f}$. Actually, by examining the proof, Main Theorem I holds if we replace $\mathcal{P}^{(p)}_{E,f}$ with $\mathcal{P}_{E,f} \setminus S'$.

(6) In characteristic zero, Question Q in Section 3.3.2 should be addressed if one wants to apply Main Theorem II to an E-compatible system when E is totally imaginary. However, one can always use Main Theorem I by omitting a finite place of E if one knows that $\mathbf{G}_\lambda$ is quasi-split for almost all λ, or, one can take the Weil restriction $\mathrm{Res}_{E/\mathbb{Q}}$ (Section 3.2.2) to obtain a $\mathbb{Q}$-compatible system and see if Main Theorem II can be applied.

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