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Kazhdan constants for Chevalley groups over the integers

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Abstract. We compute lower bounds for Kazhdan constants of Chevalley groups over the integers, endowed with the standard Steinberg generators. For types other than $\mathtt{A_n}$, these are the first explicit asymptotically sharp such bounds. The method relies on establishing a new connection between the structure of a root system grading a family of groups and the behaviour of the square of the Laplace operator in the family.

1. Introduction

In his seminal paper [19], Kazhdan introduced property T, an important rigidity property for locally compact groups. For such a group G, the definition stipulates the existence of a compact subset $S \subset G$ and a constant κ, such that for every unitary representation π of G, either every element of S moves all unit vectors by at least κ, or π contains a non-zero G-invariant vector. When G is discrete, the set S can be easily seen to be necessarily generating, and the maximal $\kappa = \kappa(G, S)$ as above is known as the *Kazhdan constant*.

In the paper, Kazhdan proved that all simple algebraic groups over local fields of rank at least 3, and most such groups of rank 2, have property T; his result also covered lattices in such groups. This was later extended by Vaserstein [30] to cover all groups of rank at least 2. In particular, using the theorem of Borel–Harish-Chandra [4], we see that the result applies to Chevalley groups over the integers associated to any irreducible root system of rank at least 2.

It turned out that for many applications, not limited to the estimates of the expansion constants, one needs the quantitative knowledge of the associated Kazhdan constants (see, e.g., the introduction of [2] for an overview). Unfortunately, neither of the results quoted above sheds light on the possible values of such constants.

The computation of these values is a problem with a long and distinguished history, discussed for example by de la Harpe and Valette in Chapter 1 of [8]. In particular, they restate the question of Serre about the Kazhdan constant for $\mathrm{SL}_3(\mathbb{Z})$ endowed with the standard generating set.

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Finding Kazhdan constants, somewhat surprisingly, seems much harder than establishing property T. For finite groups, Kazhdan constants can be computed, since the representation theory of such groups is completely understood, see for example [1]. To the authors' knowledge, the only cases of *infinite* groups where the exact values are known are groups acting transitively on $\widetilde{A}_2$ buildings, as computed by Cartwright–Młotkowski–Steger [7]. However, the problem of establishing a lower bound on the constants seems slightly more accessible and has been an area of very active research recently.

There are several methods that can be used to bound Kazhdan constants from below. Broadly speaking, one can: analyse angles between subspaces in unitary representations [5, 9]; use the property of bounded generation [18, 22, 28]; establish large spectral gap of the graph Laplacian of a link in a Cayley graph [9, 31]; present the group as a graph of groups with sufficient co-distance at vertices [10, 11]; or use sums-of-squares decompositions in the group ring [25]. We are going to focus on this last method. A different, geometric method of partially defined cocycles (see Section 4 in [31]) seems to be underexplored so far from the theoretical point of view (cf. [23, 26]).

The mentioned method of Ozawa focuses on finding a decomposition of the squared group Laplacian into sum of squares. While the advantage of this formulation is its simplicity, the decomposition must be repeated for every group, separately, unless the family of groups which we are interested in has some internal structure. For instance, [15] leverages the structure of $\{\mathrm{SL}_n(\mathbb{Z})\}_{n\geqslant 3}$ to exhibit a *single computation* that proves property T for *all* groups in the family. The external similarities between the groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{Aut}(F_n)$ (for which the method of [15] also works) lead the authors to believe that these results are a specialization of a more general pattern which is independent of the nature of the particular groups and depends on the internal structure of the family instead.

The central aim of this paper is to generalize the method of [15] beyond the family of special automorphisms groups of $\mathbb{Z}^n$ (or F_n), and establish a new general method for computing lower bounds for Kazhdan constants across families of groups. In the process, the authors found that the method is particularly well tailored to the families of Steinberg groups, which is exemplified in the following sections by executing the envisioned programme for the universal Chevalley groups over the integers.

The key example of our settings is the family of special linear groups $\mathrm{SL}_n(\mathbb{Z})$ (for $n \geqslant 3$), generated by the set S_n of elementary matrices that differ from the identity by a single ± 1 in an entry away from the diagonal. With these generating sets one can think of $\mathrm{SL}_n(\mathbb{Z})$ as the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{A}_{\mathtt{n}-1}$.

Kazhdan constants for this family have been particularly well studied. The first (partial) lower bounds of $\mathrm{SL}_3(\mathbb{Z})$ was produced by Burger [5], and eight years later, Shalom [28] gave lower bounds for all groups $\mathrm{SL}_n(\mathbb{Z})$ using the concept of bounded generation. Subsequently, Kassabov [18] obtained a much better estimate which we know to be asymptotically tight, as witnessed by the upper bound $\sqrt{2/n}$ by Żuk. However, even for small n these lower bounds differ by two orders of magnitude from the upper bound.

A completely new method of establishing lower bounds for Kazhdan constants of finitely generated groups emerged from the work of Ozawa [25]. Using a quantitative version due to Netzer–Thom [21], one can utilise a computer to calculate such lower bounds for specific groups. This was done by Netzer–Thom and Fujiwara–Kabaya [12], who obtained vastly improved bounds for $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{SL}_4(\mathbb{Z})$. This very direct approach cul-

		lower bound for κ		
type	n	$R=2$	$R=3$	comments
$\mathtt{A}_2$		0.21618	0.30069	direct computations [17]
$\mathtt{A}_3$		0.33122		
$\mathtt{A}_4$		0.3668		
$\mathtt{A}_n$	$n \geqslant 5$	$\sqrt{\frac{0.5(n-1)}{n(n+1)}}$		see Section 5.1 of [15]
$\mathtt{D}_n$	$n \geqslant 4$		$\sqrt{\frac{0.273954(n-2)}{n(n-1)}}$	see Theorem 3.9
$\mathtt{E}_6$			0.19506	
$\mathtt{E}_7$			0.18651	
$\mathtt{E}_8$			0.17877	
$\mathtt{B}_2 = \mathtt{C}_2$		0.3315	0.42014	direct computations
$\mathtt{B}_n, \mathtt{C}_n$	$n \geqslant 3$	$\sqrt{\frac{1.2086965}{n^2}}$	$\sqrt{\frac{0.1224675(n-1)}{n^2}}$	see Theorems 3.16 and 3.11
$\mathtt{F}_4$			0.80804	
$\mathtt{G}_2$		0.28397		see Theorem 3.17

Table 1. The best known lower bounds for Kazhdan constants. Note: for type $\mathtt{C}_n$ the estimate with $R = 2$ (via Theorem 3.16) is better than with $R = 3$ (via Theorem 3.11) for $n \leqslant 9$.

minated with the paper of the first author with Nowak [16], in which the best known lower bounds for $\mathrm{SL}_n(\mathbb{Z})$ with $n \in \{3, 4, 5\}$ were computed. Finally, using a derived computer-assisted calculation and a form of induction, the authors, together with Nowak [15], computed the best known bounds for $\mathrm{SL}_{n+1}(\mathbb{Z})$ with $n \geqslant 5$. We list them in Table 1. What is worth pointing out is that these results show that the actual value of Kazhdan constants for special linear groups seems to be in the upper half of the upper bound.

In this work, we show that the somewhat ad hoc constructions of [15] does in fact follow from a much more general construction and applies to groups graded by root systems, as defined below. We explore, in a quantitative and systematic way, the connection of various group ring elements defined by the root systems of the underlying Chevalley groups and we uncover a general method of proving property T of which [15] is a special case. As an outcome, we are able to give the very first explicit asymptotically tight lower bounds for universal Chevalley groups over $\mathbb{Z}$ corresponding to all irreducible root systems of rank at least 2. In particular, we give concrete lower bounds for Kazhdan constants of symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$, with respect to the usual (Steinberg) generators. This was previously done by Neuhauser [22], but his bounds are asymptotic to $1/n$, rather than the optimal $1/\sqrt{n}$. This result complements therefore Theorem 7.12 in [11], which gives the asymptotics of such lower bounds without providing concrete values.

Theorem A. *Let G be an elementary Chevalley group of type Ω over the integers, and let S denote the set of its Steinberg generators. When Ω is irreducible and of rank at least 2, then the pair (G, S) has property T with Kazhdan constant bounded below by the number indicated in Table 1.*

The parameter R of Table 1 is the radius of a ball in the Cayley graph supporting the elements in the group ring that are used in a sum-of-Hermitian-squares decomposition, see Definition 2.1.

It is very curious that the elements in the group ring mentioned above turn out to satisfy a form of spectral gap in all the cases considered above, but we are not able to prove it without a computer even in the simplest case of $\Omega = \mathtt{A}_2$. The only computer-free result of relevance here is contained in a recent article of Ozawa [26].

Part of the motivation for looking into the symplectic groups was the rough analogy between special linear groups over $\mathbb{Z}$ and automorphism groups of free groups on one side, and symplectic groups over $\mathbb{Z}$ and mapping class groups of surfaces on the other. To have a hope of employing a similar line of attack on the question of property T for mapping class groups, one needs to first find appropriate "Steinberg" generators. The authors had constructed such a generating set, but were unable to prove the requred positivity statements, and therefore our attempts eventually failed.

Related work

A very basic idea behind this work lies in encoding the combinatorics of Steinberg generators in a root system. In this spirit, Ershov–Kassabov–Jaikin-Zapirain [11] introduced groups graded by root systems by imposing a condition binding the group structure with the additive structure of the root system. The gradings we introduce are somewhat weaker, even though (in case of arithmetic groups investigated here) they do implicitly satisfy the conditions of [11].

On the other hand, the induced "non-commutative" grading for $\mathrm{SAut}(F_n)$ does not fall into the framework of [11], and yet our definition is sufficient to successfully apply our results (Theorem 2.6) to the group. Explicitly, $\mathrm{SAut}(F_n)$ is generated by the set of *Nielsen transformations*, or *transvections*. These automorphisms multiply one of the generators by another one, either on the right or on the left. Their image in $\mathrm{SL}_n(\mathbb{Z})$ is an elementary matrix, one of the Chevalley generators, and such matrices are naturally assigned to roots. This way we obtain an assignment sending transvections to roots – this is the "non-commutative grading" alluded to above. If we pick a root, say α, then the matrices in $\mathrm{SL}_n(\mathbb{Z})$ assigned to $\pm\alpha$ generate a copy of $\mathrm{SL}_2(\mathbb{Z})$, but this is no longer true for transvections associated to $\pm\alpha$, as they generate a copy of $\mathrm{SAut}(F_2)$. Moreover, consider $F_3 = F(a,b,c)$, and the transvections

$$\rho_{12} : \begin{cases} a & \mapsto & ab, \\ b & & b, \\ c & & c, \end{cases} \qquad \rho_{23} : \begin{cases} a & \mapsto & a, \\ b & & bc, \\ c & & c. \end{cases}$$

Their commutator $[\rho_{12}, \rho_{23}] = \rho_{12}\rho_{23}{\rho_{12}}^{-1}{\rho_{23}}^{-1}$ does not lie in the subgroup generated by transvections of the free factor $F(a,c)$. This violates the definition of a grading as introduced in [11], and the problem persists for all $n > 2$. Nevertheless, $\mathrm{SAut}(F_n)$ for $n > 3$ does have property T, and for $n > 5$, it is proved using a technique that can be expressed in the same terms as in the proof of Theorem 3.16: in [15], we introduced elements Sq_n, Adj_n, and Op_n lying in the group ring of $\mathrm{SAut}(F_n)$; these elements correspond to Lev_2^n, Lev_3^n, and Lev_4^n, respectively.

We use gradings to establish property T in a way that is very different to that used by Ershov–Kassabov–Jaikin-Zapirain. They require that the generators corresponding to orthogonal roots commute, in order to ensure that certain subspaces of a Hilbert space on which they act are orthogonal as well. For us, orthogonality considerations are replaced by ensuring that products of positive operators related to subspaces are positive.

2. Roots and sums of Hermitian squares

2.1. Ozawa's theorem

Let G be a discrete group generated by a symmetric generating set S. The map

$$G \to G, \quad g \mapsto g^{-1}$$

extends linearly to a map $\mathbb{R}G \to \mathbb{R}G$ that we will denote by $x \mapsto x^*$. Elements with $x^* = x$ will be called *self adjoint*.

Definition 2.1. We say that an element $x \in \mathbb{R}G$ is a *sum of Hermitian squares* if and only if there exist $\xi_1, \ldots, \xi_n \in \mathbb{R}G$ with

$$x = \sum_{i=1}^{n} \xi_i{}^* \xi_i .$$

If additionally all the elements ξ_i are supported in the ball of radius R with respect to the word metric on G coming from S, we will say that the sum of squares decomposition is (*witnessed*) *on radius* R, and we will write $x \geqslant_R 0$.

Theorem 2.2 ([25]). *Let G be a group with a finite symmetric generating set S, and let*

$$\Delta = |S| - \sum_{s \in S} s \in \mathbb{Z}G$$

be the Laplacian. The group G has property T if and only if there exists $\lambda > 0$ such that

$$\Delta^2 - \lambda\Delta$$

admits a decomposition into a sum of Hermitian squares.

If the decomposition is witnessed on radius R, i.e., $\Delta^2 - \lambda\Delta \geqslant_R 0$, we will say that the sum of Hermitian squares is a *witness* of property T *of type* (λ, R). We remark that the number λ gives a lower bound of $\sqrt{2\lambda/|S|}$ for the Kazhdan constant of (G, S) – see the proof of Proposition 5.4.5 in [2].

2.2. Root systems

Definition 2.3. A subset Ω of a finite-dimensional real vector space V endowed with an inner product $\langle \cdot, \cdot \rangle$ is a *root system* if and only if for every $\alpha, \beta \in \Omega$, all of the following hold:

(1) $0 \notin \Omega$,

(2) $\Omega \cap \{\lambda\alpha : \lambda \in \mathbb{R}\} = \{\pm\alpha\}$,

(3) $\beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in \Omega$,

(4) $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Roots $\alpha, \beta \in \Omega$ are called *proportional* (written $\alpha \sim \beta$) if and only if they are related by homothety. If α and β are not proportional, we call them *non-proportional* and write $\alpha \not\sim \beta$. The *rank* of Ω denotes the dimension of its linear span. The root system is *reducible* if and only if it is the disjoint union of two root systems that are orthogonal with respect to the inner product on V; otherwise, the system is *irreducible*.

The irreducible root systems are classified by Dynkin diagrams, and hence come in the following types: $\mathtt{A_n}$ with $n \geqslant 1$; $\mathtt{B_n}$ with $n \geqslant 2$; $\mathtt{C_n}$ with $n \geqslant 3$; $\mathtt{D_n}$ with $n \geqslant 4$; $\mathtt{E_n}$ with $n \in \{6, 7, 8\}$; $\mathtt{F_4}$; and $\mathtt{G_2}$. Types $\mathtt{A_n}, \mathtt{D_n}$ and $\mathtt{E_n}$ are called *simply laced*.

Since we insist on item (2) above, we are only considering reduced root systems.

Every root system Ω has the associated Weyl group W_Ω acting on the roots. When the system is irreducible, the action is transitive on all roots of the same length. In particular, in the simply laced case, the Weyl group acts transitively.

2.3. Decomposing the Laplacian

Let G be a finitely generated group, and Ω a root system. To every root $\alpha \in \Omega$ we associate a finite symmetric subset $S_\alpha \subseteq G$ such that G is generated by the finite set $S = \bigsqcup_{\alpha \in \Omega} S_\alpha$, the disjoint union of the sets S_α.

We do not impose any further conditions on the subsets S_α and their interactions, and so the observations we will make in this section will be very general. It is however worth keeping in mind that in the subsequent sections we will look at groups where the subsets S_α and the subgroups they generate have specific properties. We will be mainly interested in the case where $\langle S_\alpha \rangle \cong \mathrm{SL}_2(\mathbb{Z})$ for every α, and where the elements of S_α and S_β commute when α and β are orthogonal.

Definition 2.4 (Laplacians for subgroups). For every $\alpha \in \Omega$, we set

$$\Delta_\alpha = |S_\alpha| - \sum_{s \in S_\alpha} s \in \mathbb{Z}G$$

and call it a *root Laplacian*. For every subspace $W \leqslant V$, we set $\Omega_W = \Omega \cap W$, and

$$\Delta_W = \sum_{\alpha \in \Omega_W} \Delta_\alpha.$$

In particular, Δ_V is the usual group Laplacian Δ associated to G with respect to S. Since the sets S_α are assumed to be symmetric, the element Δ_α is self adjoint for every $\alpha \in \Omega$, and hence every Δ_W is also self adjoint.

Definition 2.5. Let W be a linear subspace of V. We set G_W to be the subgroup of G generated by $S_W = \bigsqcup_{\alpha \in \Omega_W} S_\alpha$. We define the *adjacency element* in $\mathbb{Z}G_W$ as follows:

$$\mathrm{Adj}_W = \sum_{\alpha \in \Omega_W} \Delta_\alpha \Big(\sum_{\alpha \not\sim \beta \in \Omega_W} \Delta_\beta \Big),$$

with the convention that the empty sum is 0.

We will say that W is

- *irreducible* when $\Omega_W = \Omega \cap W$ is irreducible as a root system;
- *admissible* when $\dim W = 2$ and Ω_W contains at least two roots which are non-proportional (and so, in particular, non-opposite).

The set of all admissible subspaces will be denoted by $\mathcal{A}$, and the set of irreducible and admissible subspaces will be denoted by $\mathcal{A}_{irr}$. Note that $\mathcal{A}$ is finite, since there are only finitely many roots, and every two non-proportional roots span a 2-dimensional subspace.

Theorem 2.6. *Let G be a group generated by a finite set S decomposing as a disjoint union $\bigsqcup_{\alpha\in\Omega} S_\alpha = S$, where Ω is a root system. Suppose that $\dim V \geqslant 2$, that V is the span of subspaces in $\mathcal{A}$, and that for every $W \in \mathcal{A}$,*

$$\mathrm{Adj}_W - \lambda_W \Delta_W \geqslant_{R_W} 0$$

for some $\lambda_W \geqslant 0$ and $R_W \geqslant 1$. Suppose additionally that for every $\alpha \in \Omega$, there exists at least one $W \in \mathcal{A}$ such that $\alpha \in W$ and $\lambda_W > 0$. Then (G, S) has property T with a witness of type (λ, R), where

$$\lambda = \min\Big\{ \sum_{\alpha\,\in\, W\in\mathcal{A}} \lambda_W : \alpha \in \Omega \Big\} > 0 \quad \textit{and} \quad R = \max_{W\in\mathcal{A}} R_W.$$

Proof. Let $\Delta = \Delta_V$ be the Laplacian of G with respect to S. We have

$$\Delta^2 = \Big(\sum_{\alpha\in\Omega} \Delta_\alpha\Big)^2 = \mathrm{Sq}_V + \mathrm{Adj}_V,$$

where $\mathrm{Sq}_V = \sum_{\alpha\in\Omega} {\Delta_\alpha}^2$. Since every Δ_α is self adjoint, $\mathrm{Sq}_V \geqslant_1 0$ is already a sum of squares witnessed on radius 1. Thus, it is sufficient to show that

$$\mathrm{Adj}_V - \lambda\Delta \geqslant_R 0.$$

Since V is spanned by admissible subspaces, we note that Adj_V is a sum of Adj_W taken over admissible subspaces, and $\Delta = \sum_\alpha \Delta_\alpha$ is a sum of root Laplacians. Explicitly, we have for Adj_V,

$$\begin{aligned}\sum_{W\in\mathcal{A}} \mathrm{Adj}_W &= \sum_{W\in\mathcal{A}} \Big(\sum_{\alpha\in\Omega_W} \Delta_\alpha\Big(\sum_{\alpha\not\sim\beta\,\in\,\Omega_W} \Delta_\beta\Big)\Big) = \sum_{\alpha\in\Omega} \Delta_\alpha\Big(\sum_{\alpha\in W\in\mathcal{A}}\ \sum_{\alpha\not\sim\beta\,\in\,\Omega_W} \Delta_\beta\Big)\\ &= \sum_{\alpha\in\Omega} \Delta_\alpha\Big(\sum_{\alpha\not\sim\beta\in\Omega} \Delta_\beta\Big) = \mathrm{Adj}_V,\end{aligned}$$

and for $\Delta = \Delta_V$,

$$\sum_{W\in\mathcal{A}} \lambda_W \Delta_W = \sum_{W\in\mathcal{A}} \Big(\lambda_W \sum_{\alpha\in\Omega_W} \Delta_\alpha\Big) = \sum_{\alpha\in\Omega} \Delta_\alpha\Big(\sum_{\alpha\in W\in\mathcal{A}} \lambda_W\Big) = \sum_{\alpha\in\Omega} \lambda_\alpha \Delta_\alpha,$$

where

$$\lambda_\alpha = \sum_{\alpha\in W\in\mathcal{A}} \lambda_W > 0 \quad \text{for every } \alpha.$$

Taking

$$\lambda = \min_{\alpha \in \Omega} \lambda_\alpha$$

and combining these two equalities, we conclude that

$$\operatorname{Adj}_V - \lambda \Delta = \underbrace{\sum_{W \in \mathcal{A}} (\operatorname{Adj}_W - \lambda_W \Delta_W)}_{\geqslant_{R_W} 0} + \underbrace{\sum_{\alpha \in \Omega} (\lambda_\alpha - \lambda) \Delta_\alpha}_{\geqslant_1 0},$$

as required. We deduce that G has property T from Theorem 2.2. ■

The above theorem shows that, in order to prove property T for G, it is enough to understand Adj_W for admissible subspaces W.

3. Chevalley groups

In this section, we look at Chevalley groups, as defined by Steinberg in Section 3 of [29]. Since we will not need the precise definition, let us only sketch it here. Let $\mathcal{L}$ denote a semi-simple Lie algebra over $\mathbb{C}$ associated to a root system Ω. For every $\alpha \in \Omega$, one can choose a distinguished element $X_\alpha \in \mathcal{L}$. The choice of the elements X_α is unique up to signs and automorphisms of $\mathcal{L}$.

Now take a faithful finite-dimensional $\mathbb{C}$-linear representation of $\mathcal{L}$ on a complex vector space U. We define an automorphism $x_\alpha(t) = \exp(tX_\alpha)$ of U. A subgroup of $\mathrm{GL}(U)$ generated by all the elements $x_\alpha(t)$ with $\alpha \in \Omega$ and $t \in \mathbb{C}$ is called a *Chevalley group* of type Ω.

In general, for a single Ω, one gets more than one Chevalley group, depending on the representation U. There are however only finitely many Chevalley groups of type Ω, and among them, there exists a *universal* one that maps onto all the others, via a map that restricts to the identity on the Steinberg generators $x_\alpha(t)$.

Definition 3.1 (Elementary Chevalley group over $\mathbb{Z}$). In the above setup, we will refer to a subgroup of $\mathrm{GL}(U)$ generated by

$$S = \{x_\alpha(\pm 1) : \alpha \in \Omega\}$$

as an *elementary Chevalley group* over $\mathbb{Z}$ of type Ω. As above, we also have a *universal* such group, with the same property. The set S is the set of *Steinberg generators*.

In this definition, S comes with a natural map $S \to \Omega$, $x_\alpha(\pm 1) \mapsto \alpha$, which we call a *grading* of $(\langle S \rangle, S)$ by Ω.

In the universal case, the groups above are isomorphic to the universal Chevalley groups over $\mathbb{Z}$ that one obtains from the Chevalley–Demazure group scheme.

Given a Chevalley group over $\mathbb{Z}$, we set

$$S_\alpha = \{x_\alpha(\pm 1)\}.$$

This allows us to use the notation and results of the previous section.

Note that an embedding of a root system Ω into a root system Ω' gives an embedding of the associated Lie algebras $\mathcal{L}_\Omega$ and $\mathcal{L}_{\Omega'}$ – this can be seen by considering presentations of the Lie algebras, as given by Steinberg. The tricky aspect is that the presentations involve *structure constants* $N_{\alpha,\beta}$, where α and β range over all roots, and these structure constants cannot be read of the root system. Starting from a presentation for $\mathcal{L}_{\Omega'}$, we obtain structure constants $N'_{\alpha,\beta}$. We can now remember these structure constants for all *extraspecial pairs* of roots in Ω, and obtain structure constants $N_{\alpha,\beta}$ for all pairs of roots in Ω from them, see Proposition 4.2.2 in [6]. In principle, these structure constants $N_{\alpha,\beta}$ may not agree with the constants $N'_{\alpha,\beta}$. The discussion after Proposition 4.2.2 in [6] tells us, however, that the constants $N_{\alpha,\beta}$ are uniquely determined by, and can be computed using a number of rules from the constants for extraspecial pairs. The constants $N'_{\alpha,\beta}$ also satisfy these rules, since they satisfy them over Ω'. Hence, there exists a choice of a Chevalley basis for $\mathcal{L}_\Omega$ such that the map between the Chevalley bases of $\mathcal{L}_\Omega$ and $\mathcal{L}_{\Omega'}$ induces a map between Lie algebras. It is easy to see that this map is injective as a map of vector spaces.

The upshot is that the representation U used to define Chevalley groups of type Ω' works also for Ω, and therefore a subgroup of a Chevalley group of type Ω' generated by $x_\alpha(t)$ with $\alpha \in \Omega$ is itself a Chevalley group of type Ω. The same argument works for elementary Chevalley groups over $\mathbb{Z}$.

Example 3.2. There are two explicit examples of immediate interest.

(1) The universal Chevalley group of type $\mathtt{A_n}$ over $\mathbb{C}$ is $\mathrm{SL}_{n+1}(\mathbb{C})$. The corresponding Chevalley group over $\mathbb{Z}$ is $\mathrm{SL}_{n+1}(\mathbb{Z})$. We may realise the system $\mathtt{A_n}$ inside $\mathbb{R}^{n+1}$ endowed with the standard basis $\{e_i : 1 \leqslant i \leqslant n+1\}$ as the set

$$\{\pm(e_i - e_j) : 1 \leqslant i < j \leqslant n+1\}.$$

We then have $x_{\pm(e_i-e_j)}$ equal to $\mathrm{I} \pm \delta_{i,j}$, where $\delta_{i,j}$ differs from the zero matrix by a single 1 in position (i, j).

(2) Similarly, the universal Chevalley group of type $\mathtt{C_n}$ is the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, with the corresponding Chevalley group over $\mathbb{Z}$ being $\mathrm{Sp}_{2n}(\mathbb{Z})$. Let us now describe the matrices x_α in detail.

The root system $\mathtt{C_n}$ can be realised inside $\mathbb{R}^n$ as the set

$$\{\pm e_i \pm e_j : 1 \leqslant i, j \leqslant n\} \smallsetminus \{0\}.$$

We thus have long roots $\pm 2e_i$, with

$$x_{2e_i} = \mathrm{I} + \delta_{i,i+n} \quad \text{and} \quad x_{-2e_i} = \mathrm{I} + \delta_{i+n,i},$$

short roots of the form $e_i - e_j$ with $i \neq j$, corresponding to

$$x_{e_i-e_j} = \mathrm{I} + \delta_{i,j} - \delta_{n+j,n+i},$$

and finally short roots of the form $\pm(e_i + e_j)$ with $i \neq j$, corresponding to

$$x_{e_i+e_j} = \mathrm{I} + \delta_{i,j+n} + \delta_{j,i+n} \quad \text{and} \quad x_{-e_i-e_j} = \mathrm{I} + \delta_{i+n,j} + \delta_{j+n,i}.$$

3.1. Simply laced types

Let us introduce the following number related to a root system.

Definition 3.3. The number $\gamma(\Omega)$ for a root system Ω is defined as

$$\gamma(\Omega) = \min_{\alpha\in\Omega} |\{\mathrm{Span}(\alpha, \beta) : \alpha \not\sim \beta \in \Omega \text{ and } \langle\alpha, \beta\rangle \neq 0\}| ,$$

where Span denotes the $\mathbb{R}$-linear span.

Equivalently, we may set

$$\gamma(\Omega) = \min_{\alpha\in\Omega} |\{W \in \mathcal{A}_{irr} : \alpha \in W\}| .$$

If $\gamma(\Omega) = 0$, then Ω contains a root α that is orthogonal to all other roots but $-\alpha$, and therefore $\Omega = \{\pm\alpha\} \sqcup \Omega'$ for some root system Ω', i.e., Ω is reducible.

Example 3.4. Let us compute $\gamma(\Omega)$ when Ω is of type $\mathtt{A_n}$. Since the Weyl group acts transitively on Ω, we have

$$\gamma(\Omega) = |\{W \in \mathcal{A}_{irr} : \alpha \in W\}|$$

for every $\alpha \in \Omega$. Fixing $\alpha = e_1 - e_2$, we see that it is precisely the roots of the form $\pm(e_1 - e_j)$ and $\pm(e_2 - e_j)$ with $j > 2$ that are not proportional and not orthogonal to α. There are in total $4(n-1)$ such roots. Now, each of these roots lies in a unique two-dimensional subspace containing α, and each such subspace intersects Ω in a system of type $\mathtt{A_2}$. Such systems have precisely 6 roots, two of which are proportional to α. This means that the number of such spaces is $4(n-1)/(6-2) = n-1$.

The values of γ, listed in Table 2, will be used when computing witnesses of property T for groups of simply laced type in Theorem 3.9.

Now we may state an easy strengthening of Theorem 2.6.

Corollary 3.5. *Let Ω be an irreducible root system of rank at least 2, and suppose that for every irreducible and admissible subspace $W \in \mathcal{A}_{irr}$,*

$$\mathrm{Adj}_W - \lambda_W \Delta_W \geqslant_{R_W} 0$$

for some $\lambda_W > 0$ and $R_W \geqslant 1$. Then (G, S) has property T witnessed by a sum of Hermitian squares decomposition of type (λ, R), where

$$\lambda = \gamma(\Omega) \min\{\lambda_W : W \in \mathcal{A}_{irr}\} \quad \textit{and} \quad R = \max\{R_W : W \in \mathcal{A}_{irr}\}.$$

Proof. We need to prove that for every $\alpha \in \Omega$, there exists $W \in \mathcal{A}_{irr}$ containing it, and therefore that V is spanned by all subspaces from $\mathcal{A}_{irr}$. The remaining conclusions will follow directly from Theorem 2.6.

Since Ω is of rank at least 2, there always exists $\beta \in \Omega$ such that $W_\beta = \mathrm{Span}(\alpha, \beta)$ is admissible. If no such W_β is irreducible, then α must be orthogonal to all roots in Ω (except $\pm\alpha$). This contradicts the irreducibility of Ω, hence at least one of those spaces W_β must be irreducible. ∎

type	$\lvert S\rvert$	γ	lower bound for κ
$\mathtt{A_n}$	$2n(n+1)$	$(n-1)$	$\sqrt{\frac{(n-1)\,\lambda_{\mathtt{A}_2}}{n(n+1)}}$
$\mathtt{D_n}$	$4n(n-1)$	$2(n-2)$	$\sqrt{\frac{(n-2)\,\lambda_{\mathtt{A}_2}}{n(n-1)}}$
$\mathtt{E}_6$	144	10	0.19506
$\mathtt{E}_7$	252	16	0.18651
$\mathtt{E}_8$	480	28	0.17877

Table 2. Explicit values for lower bounds for Kazhdan constants according to Theorem 3.9. Note that the values for the exceptional root systems are based on using $(\lambda_{\mathtt{A}_2}, R) = (0.273954, 3)$. For the type $\mathtt{A_n}$, we obtain here the same bound as in [15].

Remark 3.6. Note that the irreducibility condition on Ω_W above is equivalent to $|\Omega_W| \geqslant 5$, which might be easier to compute. Indeed, admissible subspaces are spanned by at least two non-collinear roots and if Ω_W contains exactly 4 roots it is necessarily reducible.

Theorem 3.7 ([15]). *Let $G = \mathrm{SL}_3(\mathbb{Z})$ be the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{A}_2$ endowed with the set of Steinberg generators S. Let V denote the ambient vector space of the root system. Then*

$$\mathrm{Adj}_V - \lambda \Delta_V \geqslant_R 0$$

whenever $(\lambda, R) \in \{(0.158606, 2), (0.273954, 3)\}$.

Remark 3.8. The constant given in [15] differs slightly from what is above. However, one can obtain these constants using the same methods, the same code and patience.

Theorem 3.9. *Let G be an elementary Chevalley group over $\mathbb{Z}$ associated to a root system Ω of type $\mathtt{A_n}$ with $n \geqslant 2$, or of type $\mathtt{D_n}$ with $n \geqslant 4$, or of types $\mathtt{E}_6$, $\mathtt{E}_7$, or $\mathtt{E}_8$. Then (G, S) has property T with a witness of type $(\gamma(\Omega)\lambda, R)$, where*

$$(\lambda, R) \in \{(0.158606, 2), (0.273954, 3)\}.$$

Proof. We will apply Corollary 3.5. First note that all of the root systems above are irreducible. Secondly, since all of these types are simply laced, the only two possible cardinalities of admissible Ω_W are 4 and 6. The statement of Corollary 3.5 tells us that we need only worry about the latter possibility, so let us fix an admissible subspace W of V with $|\Omega_W| = 6$. This means precisely that Ω_W is itself isomorphic to the root system $\mathtt{A}_2$, and the group G_W is an elementary Chevalley group over $\mathbb{Z}$ of type $\mathtt{A}_2$, and hence a quotient of $\mathrm{SL}_3(\mathbb{Z})$. By Theorem 3.7,

$$\mathrm{Adj}_W - \lambda \Delta_W \geqslant_R 0$$

is a sum of Hermitian squares and the conclusion follows from Corollary 3.5. ∎

The explicit lower bounds for the Kazhdan constants that can be extracted from the above result are listed in Table 2.

3.2. Types with double bonds

In this section, we concentrate on the universal Chevalley groups over $\mathbb{Z}$ of type $\mathtt{C_n}$, that is, symplectic groups $\mathrm{Sp}_{2n}(\mathbb{Z})$. The discussion will also give a new proof of property T for the elementary Chevalley groups over $\mathbb{Z}$ of types $\mathtt{B_n}$ and $\mathtt{F_4}$. Since the focus here is really on symplectic groups, we will often write $\mathtt{C_2}$ instead of $\mathtt{B_2}$.

The first two results are analogous to Theorems 3.7 and 3.9, and the first one is proved by a computer-assisted calculation as well.

Theorem 3.10. *Let $G = \mathrm{Sp}_4(\mathbb{Z})$ be the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{B_2} = \mathtt{C_2}$, endowed with the set of Steinberg generators S. Let V denote the ambient vector space of the root system. Then*

$$\mathrm{Adj}_V - \lambda\Delta_V \geqslant_R 0$$

for $(\lambda, R) = (0.244935, 3)$.

Theorem 3.11. *Let G be an elementary Chevalley group over $\mathbb{Z}$ associated to a root system Ω of type $\mathtt{B_n}$ or $\mathtt{C_n}$ with $n \geqslant 2$, or of type $\mathtt{F_4}$. Then (G, S) has property T with a witness of type $(\lambda, 3)$, where $\lambda = (n-1)\lambda_{\mathtt{C_2}}$ for types $\mathtt{B_n}$ and $\mathtt{C_n}$, and $\lambda = 3\lambda_{\mathtt{C_2}} + 4\lambda_{\mathtt{A_2}}$ for type $\mathtt{F_4}$, with $\lambda_{\mathtt{A_2}} = 0.273954$ and $\lambda_{\mathtt{C_2}} = 0.244935$.*

Proof. First let us give a quick argument for why, in the cases under consideration, there is a witness of type $(\lambda, 3)$. For admissible subspaces W, only three different types of root systems appear as Ω_W: $\mathtt{A_1} \times \mathtt{A_1}$, $\mathtt{A_2}$, and $\mathtt{C_2}$. By Corollary 3.5, we need to only worry about the last two types.

We take $\lambda_W = \lambda_{\mathtt{A_2}} = 0.273954$ for the second, and $\lambda_W = \lambda_{\mathtt{C_2}} = 0.244935$ for the third. We now apply Theorems 3.7 and 3.10 and conclude that (G, S) has property T with a witness of type $(\lambda, 3)$.

In order to compute λ, we will use Theorem 2.6, again disregarding type $\mathtt{A_1} \times \mathtt{A_1}$, since it does not contribute.

We will focus on $\mathtt{C_n}$, since $\mathtt{B_n}$ is its dual, and so the computation is literally the same. When Ω is of type $\mathtt{C_n}$ and α is a long root, then we know that every $W \in \mathcal{A}_{irr}$ containing α is of type $\mathtt{C_2}$, and thus the corresponding λ is $(n-1)\lambda_{\mathtt{C_2}}$, where the coefficient $n-1$ comes from counting $W \in \mathcal{A}_{irr}$ of type $\mathtt{C_2}$ containing α. When α is a short root, there is a single $W \in \mathcal{A}_{irr}$ of type $\mathtt{C_2}$ that contains α, and there are $2(n-2)$ such subspaces of type $\mathtt{A_2}$; hence we obtain λ equal to $\lambda_{\mathtt{C_2}} + 2(n-2)\lambda_{\mathtt{A_2}}$. The minimum of these two is $(n-1)\lambda_{\mathtt{C_2}}$.

Finally, consider Ω of type $\mathtt{F_4}$. Since this type is self-dual, we may look at short roots without losing any generality, and since all roots of the same length form an orbit under the action of the Weyl group, we need to only carry out our computation for a single root of our choosing. By counting, we find that for every short root $\alpha \in \mathtt{F_4}$ there exist 18 other, non-proportional roots β such that (α, β) spans an admissible subspace of type $\mathtt{C_2}$. Since the same subspace will be spanned for 6 different choices of β, we see that there are precisely 3 admissible subspaces containing β. By similar considerations, we arrive at 16 roots, which give rise to 4 admissible subspaces V such that $\Omega_V \cong \mathtt{A_2}$. Hence we conclude that $\lambda = 3\lambda_{\mathtt{C_2}} + 4\lambda_{\mathtt{A_2}}$ for type $\mathtt{F_4}$. ∎

For symplectic groups, with a more involved argument, we will now find a witness for property T on radius 2. Our considerations will apply to $\mathrm{Sp}_{2n}(\mathbb{Z})$ for $n \geqslant 3$; for $n = 2$, we offer the following direct computation.

Theorem 3.12. *Let $G = \mathrm{Sp}_4(\mathbb{Z})$ be the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{B}_2 = \mathtt{C}_2$ endowed with the set of Steinberg generators S. The pair (G, S) has property T with a witness of type $(\lambda, R) \in \{(0.879159, 2), (1.412187, 3)\}$.*

Now let Ω denote the root system of type $\mathtt{C_n}$ embedded in a vector space V, and let G be the associated universal Chevalley group over $\mathbb{Z}$, that is $\mathrm{Sp}_{2n}(\mathbb{Z})$, endowed with the Steinberg generators. Let $\Delta = \Delta_V$. We have

$$\Delta^2 = \sum_{\alpha,\beta \in \Omega} \Delta_\alpha \Delta_\beta = \sum_{\alpha \in \Omega} {\Delta_\alpha}^2 + \sum_{W \in \mathcal{A}} \mathrm{Adj}_W .$$

Let Ω_{long} denote the set of long roots in Ω, and let Ω_{short} denote the set of short roots. We let

$$\mathrm{Sq}_{\mathrm{long}} = \sum_{\alpha \in \Omega_{\mathrm{long}}} {\Delta_\alpha}^2 \quad \text{and} \quad \mathrm{Sq}_{\mathrm{short}} = \sum_{\alpha \in \Omega_{\mathrm{short}}} {\Delta_\alpha}^2.$$

For a two-dimensional admissible subspace W, the root system Ω_W can be of types $\mathtt{A}_1 \times \mathtt{A}_1$, $\mathtt{A}_1 \times \mathtt{C}_1$, $\mathtt{A}_2$, or $\mathtt{C}_2$, where the distinction between types $\mathtt{A}_1$ and $\mathtt{C}_1$ is that we consider the former to consist of short roots in Ω, and the latter to consist of long roots. Note that type $\mathtt{C}_1 \times \mathtt{C}_1$ does not appear, since any plane containing two long roots will intersect Ω in a system of type $\mathtt{C}_2$.

For $\mathtt{Z}$ being one of the above types, we let $\mathrm{Adj}_{\mathtt{Z}}$ denote the sum of the elements Adj_W with Ω_W of type $\mathtt{Z}$.

We now have

$$\Delta^2 = \mathrm{Sq}_{\mathrm{long}} + \mathrm{Sq}_{\mathrm{short}} + \mathrm{Adj}_{\mathtt{C}_2} + \mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{C}_1} + \mathrm{Adj}_{\mathtt{A}_2} + \mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{A}_1}.$$

In the above equation, all the elements depend on n. We have been suppressing this dependence so far, but we will now make it explicit.

Definition 3.13 (Levels). For every $n \geqslant 2$, we define four elements of $\mathbb{Z}\,\mathrm{Sp}_{2n}(\mathbb{Z})$ as follows:

$$\mathrm{Lev}_1^n = \mathrm{Sq}_{\mathrm{long}}, \qquad \mathrm{Lev}_2^n = \mathrm{Sq}_{\mathrm{short}} + \mathrm{Adj}_{\mathtt{C}_2},$$
$$\mathrm{Lev}_3^n = \mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{C}_1} + \mathrm{Adj}_{\mathtt{A}_2}, \qquad \mathrm{Lev}_4^n = \mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{A}_1}.$$

The reason for calling these elements levels comes from the way they behave under the action of the Weyl group. More specifically, consider the standard embedding of $\mathrm{Sp}_{2n}(\mathbb{Z})$ into $\mathrm{Sp}_{2m}(\mathbb{Z})$ for $m > n$. Let $W_{\mathtt{C_m}}$ denote the Weyl group of type $\mathtt{C_m}$, and denote by x^w the action of $w \in W_{\mathtt{C_m}}$ on an element x from the corresponding group (ring). It is clear from the definitions that if we take $\mathrm{Adj}_{\mathtt{Z}} \in \mathbb{R}\,\mathrm{Sp}_{2n}(\mathbb{Z})$, then the sum

$$\sum_{w \in W_{\mathtt{C_m}}} {\mathrm{Adj}_{\mathtt{Z}}}^w$$

is equal to $\mathrm{Adj}_{\mathtt{Z}}$ formed in $\mathbb{R}\,\mathrm{Sp}_{2m}(\mathbb{Z})$, up to multiplication by a constant depending on n, m and $\mathtt{Z}$, and similarly for $\mathrm{Sq}_{\mathrm{long}}$ and $\mathrm{Sq}_{\mathrm{short}}$. We group these elements into the levels, precisely depending on these constants.

Lemma 3.14. *Take $n \geqslant m \geqslant i$ and $i \in \{1,2,3,4\}$. We have*

$$\sum_{w \in W_{\mathtt{C}_\mathtt{n}}} (\mathrm{Lev}_i^m)^w = \frac{2^n \cdot m! \cdot (n-i)!}{(m-i)!} \mathrm{Lev}_i^n .$$

A direct computation now gives us the following.

Theorem 3.15. *Let $G = \mathrm{Sp}_6(\mathbb{Z})$ be the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{C}_3$ endowed with the set of Steinberg generators S. Let V denote the ambient vector space of the root system. Then for $\lambda = 2.417393$, we have*

$$\mathrm{Lev}_2^3 + \mathrm{Lev}_3^3 - \lambda \Delta_V \geqslant_2 0.$$

Theorem 3.16. *Let $G = \mathrm{Sp}_{2n}(\mathbb{Z})$ be the universal Chevalley group over $\mathbb{Z}$ of type $\mathtt{C}_\mathtt{n}$ for $n \geqslant 3$, endowed with the set of Steinberg generators S. The pair (G, S) has property T with a witness of type $(2.417393, 2)$.*

Proof. Let Ω denote the root system of type $\mathtt{C}_\mathtt{n}$ embedded in a vector space V. Let V_3 be a 3-dimensional subspace of V such that Ω_{V_3} is a root system of type $\mathtt{C}_3$; this way we have a fixed copy of $\mathrm{Sp}_6(\mathbb{Z})$ embedded in G. By Theorem 3.15,

$$\mathrm{Lev}_2^3 + \mathrm{Lev}_3^3 - \lambda \Delta_{V_3} \geqslant_2 0$$

for $\lambda = 2.417393$. We see that

$$\begin{aligned} 0 &\leqslant_2 \frac{1}{3 \cdot 2^n \cdot (n-3)!} \sum_{w \in W_{\mathtt{C}_\mathtt{n}}} (\mathrm{Lev}_2^3 + \mathrm{Lev}_3^3 - \lambda \Delta_{V_3})^w \\ &= 2(n-2)\,\mathrm{Lev}_2^n + 2\,\mathrm{Lev}_3^n - \lambda(n-2)\Big((n-1) \sum_{\alpha \in \Omega_{\mathrm{long}}} \Delta_\alpha + 2 \sum_{\alpha \in \Omega_{\mathrm{short}}} \Delta_\alpha\Big). \end{aligned}$$

Since every $\Delta_\alpha \geqslant_1 0$, we conclude that

$$(\star) \qquad 2(n-2)\,\mathrm{Lev}_2^n + 2\,\mathrm{Lev}_3^n - 2\lambda(n-2)\Delta \geqslant_2 0,$$

where $\Delta = \Delta_V$.

Let V_2 be a two-dimensional subspace of V such that Ω_{V_2} is a root system of type $\mathtt{A}_2$. By Theorem 3.9, the element Adj_{V_2} is a sum of Hermitian squares. This implies that so is the element $\mathrm{Adj}_{\mathtt{A}_2}$ in the definition of Lev_3^3. By our definition of the grading, generating sets S_α and S_β assigned to orthogonal roots commute with each other and therefore $\mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{C}_1}$ is a sum of Hermitian squares – the details of this argument can be seen in the proof of Lemma 3.6 in [15]. We conclude that Lev_3^3, and hence Lev_3^n, are sums of Hermitian squares. Adding a positive multiple of Lev_3^n to $(\star)$ and dividing by a positive constant shows that

$$\mathrm{Lev}_2^n + \mathrm{Lev}_3^n - \lambda \Delta \geqslant_2 0.$$

Since Lev_1^n is blatantly a sum of Hermitian squares, and since the argument we just used for $\mathrm{Adj}_{\mathtt{A}_1 \times \mathtt{C}_1}$ applies also to Lev_4^n, we finish by observing that

$$\Delta^2 - \lambda \Delta = \sum_{i=1}^{4} \mathrm{Lev}_i^n - \lambda \Delta \geqslant_2 0. \qquad \blacksquare$$

3.3. The triple bond

A direct computer calculation yields the following result.

Theorem 3.17. *Let G be the Chevalley group over $\mathbb{Z}$ of type $\mathtt{G}_2$, and let S be the set of its Steinberg generators. The pair (G, S) has property T with a witness of type* $(0.967685, 2)$.

Let us also mention the following result, also obtained by a direct calculation. No result of this paper makes use of it, but for the sake of completeness (i.e., just because we can), we show that the element Adj of the last remaining type behaves in a similar manner to all the previous ones.

Theorem 3.18. *Let G be the Chevalley group over $\mathbb{Z}$ of type $\mathtt{G}_2$, and let S be the set of its Steinberg generators. Let V denote the ambient vector space of the root system. Then*

$$\mathrm{Adj}_V - \lambda \Delta_V \geqslant_R 0$$

for $(\lambda, R) = (1.56799, 3)$.

4. Replication

The code used to perform the computations has been stored in the Zenodo repository [14]. A fair share of commentaries (both about computations and the certification) is included in notebooks contained therein.

For all the gory details, we direct the interested reader to study the code used for replication of [16, 17].

To pay tribute to the authors, below we list a few significant pieces of software that are used for our computations:

- the code is written in julia programming language [3];
- authors' PropertyT.jl package is used to formulate the optimization problems through the JuMP.jl [20] package for mathematical programming;
- Splitting conic solver (scs) [24] and Conic operator splitting method (COSMO) [13] solvers are used to solve the problems;
- IntervalArithmetic.jl package [27] is used to certify the soundness of computations in floating point arithmetic.

Finally, the authors' own packages Groups.jl, StarAlgebras.jl, and SymbolicWedderburn.jl provide the modelling of the algebraic structures. Thanks especially to the last package, all computational statements with $R = 2$ can be reproven on an average desktop computer within minutes. Same computations with $R = 3$ though require substantial computational resources and patience.

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