Rev. Mat. Iberoam. 41 (2025), no. 4, 1489–1512
DOI 10.4171/RMI/1547

On strictly elliptic K3 surfaces and del Pezzo surfaces

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Abstract. This article primarily aims at classifying, on certain K3 surfaces, the elliptic fibrations induced by conic bundles on smooth del Pezzo surfaces. The key geometric tool employed is the Alexeev–Nikulin correspondence between del Pezzo surfaces with log-terminal singularities of Gorenstein index two and K3 surfaces with non-symplectic involutions of elliptic type: the latter surfaces are realized as appropriate double covers obtained from the former ones. The main application of this correspondence is in the study of linear systems that induce elliptic fibrations on K3 surfaces admitting a strictly elliptic non-symplectic involution, i.e., whose fixed locus consists of a single curve of genus $g \geq 2$. The obtained results are similar to those achieved by Garbagnati and Salgado for jacobian elliptic fibrations.

1. Introduction

We work over the field of complex numbers $\mathbf{C}$.

One of the principal algebraic invariants of a projective algebraic variety X is its group of biregular automorphisms, denoted as $\mathrm{Aut}(X)$. In many cases, this group can be used to discover geometric properties of the underlying algebraic variety and its projective models. Furthermore, as we will explore, the mere existence of suitable involutions has non-trivial consequences regarding special linear systems on the corresponding variety.

One of the main successful approaches to study the automorphism group of a smooth projective variety X is to consider the natural induced action on cohomology lattices. Indeed, a classical result by Lieberman [26] states that the neutral component $\mathrm{Aut}^\circ(X)$ has finite index in the kernel of the linear representation $\mathrm{Aut}(X) \to \mathrm{GL}(\mathrm{H}^2(X, \mathbf{Z}))$, i.e., $\mathrm{Aut}(X)$ splits into its neutral component $\mathrm{Aut}^\circ(X)$ and its discrete image in $\mathrm{GL}(\mathrm{H}^2(X, \mathbf{Z}))$ (see, e.g., [9] for further details).

The aforementioned strategy has been classically employed to investigate the automorphism groups of smooth *del Pezzo surfaces*, which are projective algebraic surfaces Z such that the anti-canonical divisor $-K_Z$ is ample. In cases where these groups are finite, they are realized as subgroups of the Weyl group of an appropriate lattice (see Chapters 8-9 of [17] for details and a historical account). Another remarkable class of algebraic varieties for which this method is commonly used are *projective K3 surfaces*, i.e., simply connected

Mathematics Subject Classification 2020: 14J28 (primary); 14J26, 14J27 (secondary).
Keywords: K3 surfaces, del Pezzo surfaces, conic bundles, elliptic fibrations.

smooth projective surfaces X with trivial canonical bundle $K_X \sim 0$. Indeed, for these surfaces the relevance of the study of automorphisms is a consequence of the celebrated Torelli theorem due to Pyatetskii-Shapiro and Shafarevich [35], and general lattice theoretic results by Nikulin [31, 32].

Remarkably, these two classes of surfaces are naturally related thanks to the work of Alexeev and Nikulin [2] (see also [30, 43]) on the classification of del Pezzo surfaces Z of Gorenstein index 2, i.e., the Weil divisor $-2K_Z$ is an ample Cartier divisor. More precisely, let us recall that in any K3 surface X, there is a holomorphic and non-vanishing (i.e., symplectic) 2-form ω_X such that $\mathrm{H}^0(X, \Omega_X^2) = \mathbf{C} \cdot \omega_X$, and thus we say that $\sigma \in \mathrm{Aut}(X)$ is *symplectic* if $\sigma^*(\omega_X) = \omega_X$, and that is *non-symplectic* otherwise. In this terms, the correspondence [2] between K3 surfaces X and (possibly) singular del Pezzo surfaces Z of Gorenstein index 2 goes as follows: the so-called smooth divisor theorem (see Theorem 1.5 in [2]) ensures that, given such a del Pezzo surface, there is a smooth irreducible curve C in the linear system $|-2K_Z|$. Subsequently, a double cover $W \to Z$ can be constructed, branched over both C and $\mathrm{Sing}(Z)$. The crucial observation lies in the fact that the minimal resolution $X \to W$ is a K3 surface, and the involution $\iota\colon X \to X$ associated with the covering is such that $\iota^*\omega_X = -\omega_X$, i.e., ι is a *non-symplectic involution*. Through an analysis of X and the quotient surface $X/\langle\iota\rangle$, the authors in [2] reduce the classification problem for Z to the study of K3 surfaces with non-symplectic involutions.

It is worth mentioning that the above construction is a vast generalization of the fact that the double cover of $\mathbf{P}^2$ branched over a *smooth* sextic curve C is a K3 surface with a non-symplectic involution. This classical construction (revisited by Dolgachev [16] and Reid [36] using modern methods) was considered by Enriques and Campedelli (see [8]), who investigated double coverings of $\mathbf{P}^2$ that are birational to a K3 surface. Another remarkable recent construction in birational geometry, closely related to our context, is the work of Peters and Sterk [34] where they consider nodal Enriques surfaces constructed from a K3 surface obtained as a double cover of a smooth del Pezzo surface of degree 6.

Even in the simplest case where the quotient del Pezzo surfaces are smooth (which will be our case of interest), the correspondence by Alexeev and Nikulin enables a connection between the geometric properties of the del Pezzo surface $Z = X/\langle\iota\rangle$ and the lattice-theoretic properties of the invariant lattice $\mathrm{NS}(X)^{\iota^*}$ associated with the non-symplectic involution ι on the K3 surface (see Section 2.2 for details). For instance, it follows from results by Nikulin in [32] that the topology of the fixed locus X^ι is determined by suitable discrete invariants (r, a, δ) of the lattice $\mathrm{NS}(X)^{\iota^*}$ which, in turn, can be interpreted through Z (e.g., the genus of the fixed curve $X^\iota = C_g$ will be given by $g = K_Z^2 + 1$).

After understanding the topology of the fixed locus, a natural next step is to describe the possible linear systems on these K3 surfaces, following the work initiated by Saint-Donat in [38]. Notably, linear systems inducing elliptic fibrations (see Definition 2.11, and note that we do not require the existence of a section) are of special interest, as they can be characterized numerically thanks to the results in Section 3 of [35] and since they have important arithmetic applications (see, e.g., [40]). Significant progress has been made in this direction, particularly in the case where the K3 surface is *generic* among those admitting a non-symplectic automorphism with a given fixed locus (see Convention 2.15). More precisely, in [13, 19, 20, 23, 32, 33], the authors classify elliptic fibrations on K3 surfaces with a non-symplectic involution in many cases, using the fact that the fixed locus of the involution is either empty, the disjoint union of two elliptic curves or contains at least a rational curve (we refer the reader to Remark 3.4 for further details). In order to complete the classification of elliptic fibrations on such generic surfaces, it remains to study the case when $X^\iota = C_g$ consists of a single smooth irreducible curve of genus $g \geq 2$. In regard of the previous discussion,

> the main purpose of this article is to address, through the Alexeev–Nikulin correspondence, the remaining case in the classification of (not necessarily jacobian) elliptic fibrations on generic K3 surfaces (in the sense of Convention 2.15) that admit a non-symplectic involution $\iota\colon X \to X$.

To achieve this, we will restrict ourselves to the case where $X^\iota = C_g$ is a smooth irreducible curve of genus $g \geq 2$, and we will say that $\iota\colon X \to X$ is a *strictly elliptic involution* (see Definition 3.1). As we will observe in Proposition 3.5, the main feature of the pair (X, ι) is that the Alexeev–Nikulin correspondence results in a *smooth* del Pezzo surface $Z = X/\langle\iota\rangle$. In this context, our main result (see Theorem 3.10) establishes that the quotient projection $\pi\colon X \to Z = X/\langle\iota\rangle$ induces a correspondence between elliptic fibrations $\mathcal{E}\colon X \to \mathbf{P}^1$ and conic bundles (see Definition 2.4) $f\colon Z \to \mathbf{P}^1$, and moreover, $\mathcal{E} = f \circ \pi$. It is worth noting that analogous results have been obtained for other K3 surfaces with non-symplectic involutions, provided the fixed locus X^ι contains rational curves (see, e.g., Section 5 of [19], Section 7.1 of [20] and Section 5.10 of [13]). In contrast, the main advantage of our approach using del Pezzo surfaces, as opposed to previous works relying on lattice-theoretic methods (where the existence of a section of the elliptic fibration or the presence of a rational curve in the fixed locus X^ι is important), is its compatibility with standard tools from Mori theory (see, e.g., Chapter 6 of [14]). Notably, in our case, we can classify the effective classes of curves inducing conic bundles on the surface Z (see Proposition 2.9), following a similar approach as in the case of (-1)-curves in Section 26 of [27], and use them to describe all admissible singular fibers of the induced elliptic fibrations on the corresponding K3 surface X (see Section 5).

Finally, it is noteworthy that, as a consequence of recent work [12] by Clingher and Malmendier, the considered elliptic fibrations are not jacobian, i.e., they do not admit sections. However, it is not difficult to observe (see Section 5) that they admit bisections which can be induced by the (-1)-curves in the associated del Pezzo surface (see Example 3.11). Despite the absence of jacobian elliptic fibrations, and consequently the inability to consider Weierstrass models, these K3 surfaces are quite special due to the fact that by Section 2.8 in [2], they have *finite automorphism groups*. Remarkably, they fall within the recent work by Roulleau [37], where some explicit projective models are studied and where the full lattice $\mathrm{NS}(X)$ is described (see Section 4).

2. Background and preliminaries

2.1. Conic bundles and smooth del Pezzo surfaces

Let us recall that a smooth projective surface Z is called a del Pezzo surface if the anticanonical divisor $-K_Z$ is ample. The positive integer $d(Z) = (-K_Z)^2$ is called the degree of Z: it is the main invariant that allows for their classification. More precisely, we have the following classical result (see, e.g., Section 24 of [27] and Proposition 8.1.25 in [17]).

Theorem 2.1. *Let Z be a smooth del Pezzo surface of degree d. Then, $1 \leq d \leq 9$ and we have that*:

(i) *If $d = 9$, then $Z \simeq \mathbf{P}^2$.*

(ii) *If $d = 8$, then Z is isomorphic to either $\mathbf{P}^1 \times \mathbf{P}^1$ or to the blow-up of $\mathbf{P}^2$ at one point* (*i.e., the Hirzebruch surface $\mathbf{F}_1$*)*.*

(iii) *If $1 \leq d \leq 7$, then Z is isomorphic to the blow-up of $\mathbf{P}^2$ at $9 - d$ points in general position.*

Here, we say that the points are in general position *if the following hold*:

(1) *no three points are on a line*;

(2) *no six points are on a conic*;

(3) *no nodal or cuspidal cubic passes through eight points with one of them being the singular point.*

Conversely, any blow-up of i points in general position, $i = 1, \dots, 8$, is a smooth del Pezzo surface.

Convention 2.2. We will denote by Z an arbitrary smooth del Pezzo surface. Additionally, we will denote by Z_d the del Pezzo surface of degree $d \in \{1, \dots, 8\}$ obtained as the blow-up of $9 - d$ points in general position in $\mathbf{P}^2$, and by $E_1, \dots, E_{9-d}$ the corresponding exceptional divisors.

It is worth mentioning that the geometry of exceptional curves on del Pezzo surfaces is completely understood. For instance, it is known that every irreducible curve with negative self-intersection on a smooth del Pezzo surface Z is a (-1)-curve (see, e.g., Theorem 24.3 in [27]). More precisely, we have the following result (see Section 26 of [27] and Section 8.2.6 of [17]).

Theorem 2.3. *Let Z_d be a smooth del Pezzo surface of degree $1 \leq d \leq 8$ and let $\varepsilon\colon Z_d \to \mathbf{P}^2$ be its representation as the blow-up of $9 - d$ points $p_i \in \mathbf{P}^2$, $i \in \{1, \dots, 9 - d\}$, in general position. Let $\Gamma \subseteq Z_d$ be a (-1)-curve. Then the image $\varepsilon(\Gamma) \subseteq \mathbf{P}^2$ is of one of the following types*:

(1) *one of the points p_i*;

(2) *a line passing through two of the points p_i*;

(3) *a conic passing through five of the points p_i*;

(4) *a cubic passing through seven of the points p_i such that one of them is a double point*;

(5) *a quartic passing through eight of the points p_i such that three of them are double points*;

(6) *a quintic passing through eight of the points p_i such that six of them are double points*;

(7) *a sextic passing through eight of the points p_i such that seven of them are double points and one is a triple point.*

Moreover, the number n of (-1)-curves on Z_d is given by the following table:

d	8	7	6	5	4	3	2	1
n	1	3	6	10	16	27	56	240

Following the same line of ideas that allow the classification of the images of (-1)-curves in the above result, we can describe the possible conic bundles on smooth del Pezzo surfaces. For the reader's benefit, we recall the relevant notions about conic bundles below.

Definition 2.4 (Section 1 of [39]). A conic bundle on a smooth projective surface Z is a surjective morphism onto a smooth curve $f: Z \to C$ whose general fiber is a smooth, irreducible curve of genus 0.

Remark 2.5. According to Lüroth's theorem, if Z is a rational surface (e.g., a smooth del Pezzo surface), then the curve C must be isomorphic to $\mathbf{P}^1$.

Definition 2.6. Let Z be a smooth projective surface. We say that an element $[D] \in \mathrm{NS}(Z)$ is a conic class if D is nef, $D^2 = 0$, and $D \cdot K_Z = -2$.

Lemma 2.7. *Let Z be smooth rational surface. Then there exists a correspondence between the set of conic bundles on Z and the set of conic classes of* $\mathrm{NS}(Z)$.

Proof. First, let $[D] \in \mathrm{NS}(Z)$ be a conic class. Then, by the Riemman–Roch theorem and Theorem III.1.(a) in [21], the associated map $\phi_{|D|}: Z \to \mathbf{P}^1$ is a well-defined morphism, and since every fiber is linearly equivalent to D it is a conic bundle on Z. On the other hand, let $f: Z \to \mathbf{P}^1$ be a conic bundle, then it is clear that the class of any fiber of f, say F, is a conic class. ∎

Remark 2.8. The number of singular fibers of a conic bundle is a numerical invariant. For instance, for a conic bundle $f: Z_d \to \mathbf{P}^1$ on the del Pezzo surface Z_d, it is well known that the number of singular fibers is $8 - K_{Z_d}^2 = 8 - d$, as seen in Section 1 of [24].

Proposition 2.9. *Let Z_d be a del Pezzo surface of degree $d \leq 8$ obtained as the blow-up of $9 - d$ points in general position in $\mathbf{P}^2$. Then the conic classes are listed in Table 1. Moreover, the number N of conic bundles on Z_d is given by the following table*:

d	8	7	6	5	4	3	2	1
N	1	2	3	5	10	27	126	2160

Proof. This is a classical fact that can be found in Section 2 of [15] (see also Table 2 in [41]). For the reader's convenience, we give a self-contained proof.

Lemma 2.7 allows us to classify for each Z_d, with $1 \leq d \leq 8$, the conic classes $[D] \in \mathrm{NS}(Z_d)$ that produce conic bundles. Indeed, since Z_d is the blow-up of $\mathbf{P}^2$ in $9 - d$ points in general position, then

$$\mathrm{Pic}(Z_d) = \mathbf{Z}L \oplus \bigoplus_{i=1}^{9-d} \mathbf{Z}E_i,$$

where L is the class of pull-back of a line and each E_i is an exceptional divisor. Thus, $K_{Z_d} = -3L + E_1 + \cdots + E_{9-d}$ and $D = \ell L + a_1 E_1 + \cdots + a_{9-d} E_{9-d}$, so the numerical conditions that impose Lemma 2.7 are

$$\begin{cases} \ell^2 = a_1^2 + \cdots + a_{9-d}^2, \\ -3\ell + 2 = a_1 + \cdots + a_{9-d}. \end{cases}$$

For each d, one gets the possibilities for D of Table 1 solving the Diophantine equation system. A straightforward combinatorial computation gives the total number of conic bundles N for each d depending on n. ∎

Remark 2.10. If $Z \simeq \mathbf{P}^1 \times \mathbf{P}^1$, the only conic bundles are the two different projections onto $\mathbf{P}^1$.

In what follows, we will relate the presence of conic bundles and elliptic fibration, therefore we recall the definition of the latter.

Definition 2.11. An elliptic fibration on a smooth projective surface X is a surjective morphism $\pi\colon X \to T$, where T is a smooth algebraic curve, the fiber $\pi^{-1}(t)$ is a curve of genus 1 for all but finitely many $t \in T$, and π is relatively minimal.

Remark 2.12. Observe that following Definition I.3.1 in [28], and contrary to other works, e.g., [20], we do not ask for the presence of a section in the definition of elliptic fibration. When a section exists, we will refer to the elliptic fibration as *jacobian*.

2.2. Non-symplectic involutions on K3 surfaces

We refer the reader to [22,25] for the notation and preliminaries on K3 surfaces and elliptic fibrations (e.g., we consider the ADE lattices to be definite negative).

Let X be a projective K3 surface and let $\varphi\colon X \to X$ be an automorphism of finite order n. Let us recall that φ is symplectic (respectively, purely non-symplectic) if $\varphi^*\omega = \omega$ (respectively, $\varphi^*\omega = \lambda\omega$ for some λ a primitive n-rooth of the unity), where ω is a non-degenerate holomorphic 2-form on X (i.e., $\mathrm{H}^0(X, \Omega_X^2) = \mathbf{C}\cdot\omega$). Given a K3 surface X and a non-symplectic involution ι on X, we denote X^ι the fixed locus of the involution and $\mathrm{NS}(X)^{\iota^*}$ the invariant lattice, i.e.,

$$\mathrm{NS}(X)^{\iota^*} = \{x \in \mathrm{NS}(X) : \iota^* x = x\}.$$

Classical results on non-symplectic involutions on K3 surfaces allow us to classify their fixed loci and invariant lattices. It is an elementary but very useful observation that $\mathrm{H}^2(X, \mathbf{Z})^{\iota^*} \subset \mathrm{NS}(X)$ as lattices for any non-symplectic involution ι.

Proposition 2.13 (Theorem 4.2.2 in [32]). *Let X be a K3 surface and let ι be a non-symplectic involution with a non-empty fixed locus. The fixed locus of ι can be either the disjoint union of two elliptic curves, $X^\iota = E_1 \sqcup E_2$, or the disjoint union*

$$X^\iota = C_g \sqcup R_1 \sqcup \cdots \sqcup R_k,$$

where C_g is a smooth curve of genus g and the R_i's are rational curves. Moreover, $g = (22 - r - a)/2$ and $k = (r - a)/2$, where $r = \operatorname{rk}\mathrm{NS}(X)^{\iota^*}$ and a is the length of $\mathrm{NS}(X)^{\iota^*}$, i.e., $2^a = |\,\mathrm{NS}(X)^{\iota^*}|$.

Remark 2.14. In the case $X^\iota = \emptyset$, it follows from Theorem 4.2.2 in [32] that necessarily $(r, a, \delta) = (10, 10, 0)$. Moreover, in that case, Theorem 4.2.4 and Proposition 4.2.5 in [32] imply that $\mathrm{NS}(X)^{\iota^*} \simeq U(2) \oplus E_8(2)$ and the group $\mathrm{Aut}(X)$ is infinite. Since $X^\iota = \emptyset$, we have that $X/\langle\iota\rangle$ is an Enriques surface.

$d = 8$	$D = L - E_1$
$d = 7$	$D = L - E_i, i = 1, 2$
$d = 6$	$D = L - E_i, i = 1, 2, 3$
$d = 5$	$D = L - E_i, i = 1, 2, 3, 4$
	$D = 2L - \sum_{k=1}^{4} E_k$
$d = 4$	$D = L - E_i, i = 1, 2, 3, 4, 5$
	$D = 2L - \sum_{j \in J} E_j, J \subset \{1, \ldots, 5\}, \|J\| = 4$
$d = 3$	$D = L - E_i, i = 1, 2, 3, 4, 5, 6.$
	$D = 2L - \sum_{j \in J} E_j, J \subset \{1, \ldots, 6\}, \|J\| = 4$
	$D = 3L - 2E_i - \sum_{j \in J} E_j, i \in \{1, \ldots, 6\}, J = \{1, \ldots, 6\} \setminus \{i\}$
$d = 2$	$D = L - E_i, i = 1, \ldots, 7$
	$D = 2L - \sum_{j \in J} E_j, J \subset \{1, \ldots, 7\}, \|J\| = 4$
	$D = 3L - 2E_i - \sum_{j \in J} E_j, i \in \{1, \ldots, 7\}, J = \{1, \ldots, 7\} \setminus \{i\}, \|J\| = 5$
	$D = 4L - 2\sum_{j \in J} E_j - \sum_{k \in K} E_k, J \subset \{1, \ldots, 7\}, \|J\| = 4, K = \{1, \ldots, 7\} \setminus J$
	$D = 5L - E_i - 2\sum_{j \in J} E_j, i \in \{1, \ldots, 7\}, J = \{1, \ldots, 7\} \setminus \{i\}$
$d = 1$	$D = L - E_i, i = 1, \ldots, 8$
	$D = 2L - \sum_{j \in J} E_j, J \subset \{1, \ldots, 8\}, \|J\| = 4$
	$D = 3L - 2E_i - \sum_{j \in J} E_j, i \in \{1, \ldots, 8\}, J = \{1, \ldots, 8\} \setminus \{i\}, \|J\| = 5$
	$D = 4L - 2\sum_{j \in J} E_j - \sum_{k \in K} E_k, J, K \subset \{1, \ldots, 8\}, \|J\| = 4, \|K\| = 3, J \cap K = \emptyset$
	$D = 4L - 3E_i - \sum_{j \in J} E_j,\ i \in \{1, \ldots, 8\},\ J = \{1, \ldots, 8\} \setminus \{i\}$
	$D = 5L - 3E_i - 2\sum_{j \in J} E_j - \sum_{k \in K} E_{i_k}, i \in \{1, \ldots, 8\},$ $J, K \subset \{1, \ldots, 8\} \setminus \{i\}, \|J\| = 3, \|K\| = 4, J \cap K = \emptyset$
	$D = 5L - E_i - 2\sum_{j \in J} E_j,\ i \in \{1, \ldots, 8\},\ J \subset \{1, \ldots, 8\} \setminus \{i\},\ \|J\| = 6$
	$D = 6L - 3\sum_{j \in J} E_j - 2\sum_{k \in K} E_k - \sum_{r \in R} E_r,$ $J, K, R \subset \{1, \ldots, 8\}, \|J\| = 2, \|K\| = 4, \|R\| = 2, J \cap K = J \cap R = K \cap R = \emptyset$
	$D = 7L - 4E_i - 3E_j - 2\sum_{k \in K} E_k,\ i \in \{1, \ldots, 8\},\ j \in \{1, \ldots, 8\} \setminus \{i\},$ $K = \{1, \ldots, 8\} \setminus \{i, j\}$
	$D = 7L - E_i - 2\sum_{j \in J} E_j - 3\sum_{k \in K} E_k,\ i \in \{1, \ldots, 8\},\ J, K \subset \{1, \ldots, 8\} \setminus \{i\},$ $\|J\| = 3,\ \|K\| = 4,\ J \cap K = \emptyset$
	$D = 8L - 4E_i - 2\sum_{j \in J} E_j - 3\sum_{k \in K} E_k,\ i \in \{1, \ldots, 8\},\ J, K \subset \{1, \ldots, 8\} \setminus \{i\},$ $\|J\| = 3,\ \|K\| = 4,\ J \cap K = \emptyset$
	$D = 8L - E_i - 3\sum_{j \in J} E_j,\ i \in \{1, \ldots, 8\},\ J = \{1, \ldots, 8\} \setminus \{i\}$
	$D = 9L - 2E_i - 4\sum_{j \in J} E_j - 3\sum_{k \in K} E_k,\ i \in \{1, \ldots, 8\},\ J, K \subset \{1, \ldots, 8\} \setminus \{i\},$ $\|J\| = 2,\ \|K\| = 5,\ J \cap K = \emptyset$
	$D = 10L - 3\sum_{i \in I} E_i - 4\sum_{j \in J},\ I, J \subset \{1, \ldots, 8\},\ \|J\| = \|K\| = 4,\ J \cap K = \emptyset$
	$D = 11L - 3E_i - 4\sum_{j \in J} E_j,\ i \in \{1, \ldots, 8\},\ J = \{1, \ldots, 8\} \setminus \{i\}$

Table 1. Conic classes on the del Pezzo surface Z_d.

Convention 2.15. In what follows, we will assume that X is *generic* among the K3 surfaces admitting a non-symplectic involution with a given fixed locus. This is equivalent to the condition the action of ι^* is trivial on $\mathrm{NS}(X)$.

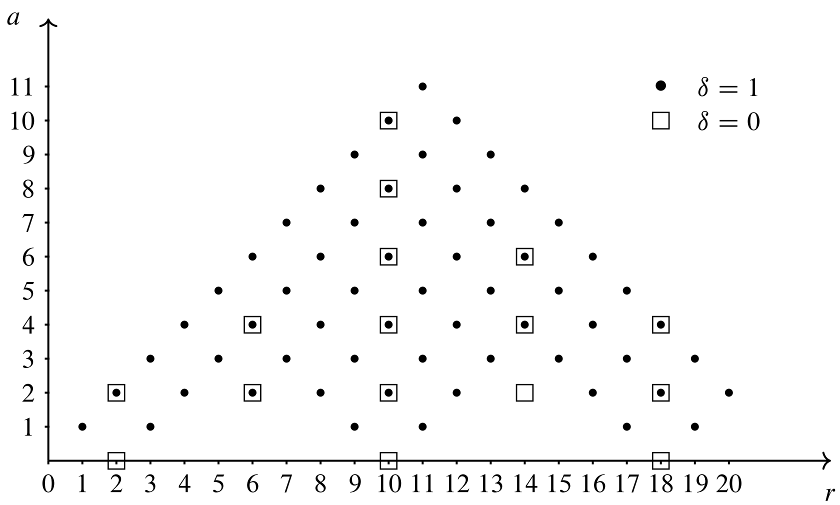

Figure 1. All possible invariants (r, a, δ).

As observed in [32], the invariants (r, a) allow to recover the topological invariants (g, k) of the fixed locus X^ι. Viceversa, if the pair (g, k) is known, a third invariant δ is needed to identify uniquely $\mathrm{NS}(X)^{\iota^*}$: $\delta = 0$ if and only if $X^\iota \sim 0 \bmod 2$ in $H_2(X, \mathbf{Z})$, otherwise $\delta = 1$. The picture of all possible invariant (r, a, δ) is presented in Figure 1 of [1]. For completeness, we include it in Figure 1.

The following result characterizes the quotient of a K3 surfaces by a non-symplectic automorphism.

Proposition 2.16. *Let X be a projective K3 surface and let φ be a non-symplectic automorphism of finite order. Then,*

(i) *the quotient $X/\langle\varphi\rangle$ is a rational surface or birational to an Enriques surface*;

(ii) *if φ is an involution, then the fixed locus of φ is empty if and only if $X/\langle\varphi\rangle$ is an Enriques surface.*

Proof. See Lemma 4.8 in [22] for (i). For (ii), the “only if” part is proven in Lemma 1.2 of [43]. Suppose that φ is an involution. We know that if the fixed locus is non-empty, then $X^\varphi = D$ has codimension one and it is a disjoint union of smooth curves. Furthermore, the canonical divisor of cyclic coverings implies that $D \sim 2K_{X/\langle\varphi\rangle}$ and so the quotient is not an Enriques surface. Suppose that the fixed locus X^φ is empty. Then,

$$2\pi^* K_{X/\langle\varphi\rangle} \sim K_X \sim 0,$$

implying that $X/\langle\varphi\rangle$ is an Enriques surface. ∎

Remark 2.17. It is a classical fact that K3 surfaces admitting a non-symplectic automorphism of finite order are projective and the Néron–Severi lattice is hyperbolic. See Corollary 1.10 in Chapter 15 of [22] and Theorem 3.1 in [31], for instance.

3. Strictly elliptic involutions

Let X be a K3 surface and ι a non-symplectic involution on X. According to the notation in Section 2.8 of [2], it is natural to classify ι into three categories: ι is of *elliptic* type if X^ι contains a curve of genus $g \geq 2$. This is equivalent to $r + a \leq 18$, $(r, a, \delta) \neq (10, 8, 0)$. The involution ι is of *parabolic* type if X^ι contains a genus 1 curve, which is equivalent to $r + a = 20$ or $(r, a, \delta) = (10, 8, 0)$. Finally, ι is of *hyperbolic* type if $X^\iota = \emptyset$ or if X^ι only contains rational curves, which is equivalent to $r + a = 22$ or $(r, a, \delta) = (10, 10, 0)$.

In this work, we specifically focus on the case of elliptic type, and in particular, we consider K3 surfaces that admit a non-symplectic involution with no rational curves in the fixed locus.

Definition 3.1. Let X be a K3 surface and let ι be a non-symplectic involution on X. We say that ι is of strictly elliptic type if its fixed locus is given by $X^\iota = C_g$, where C_g is a smooth irreducible curve of genus $g \geq 2$.

Remark 3.2. Looking at Figure 1, one can observe that the Néron–Severi groups of the strictly elliptic type K3 surfaces correspond to points on the line $r = a$ with $r \leq 9$, and they all have $\delta = 1$ except for the case $r = 2$, where both $\delta = 0$ and $\delta = 1$ are possible. As a consequence of [32], possibilities for $\mathrm{NS}(X)$ are as follows: if $r = 2, \delta = 0$, then $\mathrm{NS}(X) \cong U(2)$; otherwise, if $\delta = 1$, one has

$$\mathrm{NS}(X) \cong \begin{cases} \langle 2 \rangle, & \text{if } r = 1, \\ \langle 2 \rangle \oplus A_1, & \text{if } r = 2, \\ U(2) \oplus A_1^{\oplus r-2}, & \text{if } r \geq 3. \end{cases}$$

Observe that by Section 2.8 of [2], the automorphism group of a K3 surface with a non-symplectic involution of elliptic type is finite. Moreover, in Section 2 of [20], the authors consider a non-symplectic involutions $\iota\colon X \to X$ such that X satisfies the Convention 2.15 and divide the possible elliptic fibrations $\mathcal{E}\colon X \to \mathbf{P}^1$ into two types:

- Type 1: ι maps each fiber of $\mathcal{E}$ to itself.
- Type 2: ι maps at least one fiber of $\mathcal{E}$ to another fiber of $\mathcal{E}$.

The next statement follows from the proof of Proposition 2.5 in [20].

Lemma 3.3 (Garbagnati–Salgado). *Let X be a K3 surface and let $\iota\colon X \to X$ be a non-symplectic involution of strictly elliptic type satisfying Convention* 2.15*. Then, X only admits elliptic fibrations of Type* 1 *with respect to ι.*

Remark 3.4. As mentioned in the introduction, our methods rely on the correspondence between strictly elliptic K3 surfaces and smooth del Pezzo surfaces, as established in [2]. This correspondence allows us to analize the case where the fixed locus corresponds to a single smooth irreducible curve C_g of genus $g \geq 2$. Additionally, prior works [13,19,20,23,33] are devoted to describe all (jacobian) elliptic fibrations in Figure 1 except for the cases $(r, a, \delta) = (10, 10, 0)$ and $(r, a, \delta) = (10, 10, 1)$. The former case corresponds precisely to the situation $X^\iota = \emptyset$ (see Remark 2.14) and thus $X/\langle\iota\rangle$ is an Enriques surface, while the latter case arises when the fixed locus is given by a single smooth elliptic curve E.

It is noteworthy that the case $(r, a, \delta) = (10, 10, 1)$ falls beyond our analysis, and it should be noted that, by Lemma 3.3, the nature of the corresponding elliptic fibrations is necessarily different from the strictly elliptic case.

On the other hand, in the case $(r, a, \delta) = (10, 10, 1)$ we have that the quotient $Z = X/\langle\iota\rangle$ is a smooth rational surface with $K_Z^2 = 0$ and $-2K_Z \sim E$, where $X^\iota \simeq E$ is an elliptic curve (cf. the proof of Proposition 3.5 below). In particular, it follows from Proposition 2.2 in [10] that there exist an irreducible pencil of sextics in $\mathbf{P}^2$ with 9 nodes $p_1, \dots, p_9$ as base points such that $Z \cong \mathrm{Bl}_{p_1,\dots,p_9}(\mathbf{P}^2)$, and that the elliptic fibration $\phi_{|E|}: Z \to \mathbf{P}^1$ is the proper transform of this pencil.

Finally, observe that by Corollary 3.3 in [12], the case $(r, a, \delta) = (6, 4, 0)$ does not admit jacobian elliptic fibrations and therefore it is not considered in the analysis of [20] despite the presence of a rational curve in X^ι. On the other hand, by Table 9 in [30], $Z = \operatorname{Proj} \oplus_{m\geq 0} \mathrm{H}^0(X, \mathcal{O}_X(mC_g))^\iota$ is a *singular* del Pezzo surface of Gorenstein index two that can be realized via a Sarkisov link starting from $\mathbf{P}(1, 1, 4)$ or as a hypersurface of degree 5 in $\mathbf{P}(1, 1, 1, 4)$ (see Propositions 7.4 and 7.11 in [30]).

Proposition 3.5. *Let X be a K3 surface admitting an involution of strictly elliptic type. Suppose that C_g is the smooth irreducible curve of genus $2 \leq g \leq 10$ fixed by ι.*

(1) *If $\delta = 0$, then the K3 surface X can be realized as the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ over a smooth irreducible curve $B \in |\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(4, 4)|$ of genus 9.*

(2) *If $\delta = 1$, then the K3 surface X can be realized as the double cover of a del Pezzo surface Z_d of degree $d = g - 1$ ramified over a smooth irreducible curve B belonging to the linear system $|-2K_{Z_d}|$.*

Moreover, in the case $\delta = 1$, the image of B in $\mathbf{P}^2$ is a nodal sextic curve Γ_d that passes through $9 - d$ points corresponding to the points where the blow-up of $\mathbf{P}^2$ is done to obtain Z_d.

Proof. By Proposition 2.16 and the fact that X^ι is of pure codimension 1 (see Proposition 2.13), the quotient $X/\langle\iota\rangle$ is a smooth rational surface. Furthermore, since the fixed locus of ι is an irreducible curve C_g of genus $g \geq 2$, the quotient $X/\langle\iota\rangle$ is a del Pezzo surface of degree $d = K^2_{X/\langle\iota\rangle} = g - 1$. Denote by $Z = X/\langle\iota\rangle$ and by B the image under the quotient map of the curve C_g. We have that $B \in |-2K_Z|$, and by the genus formula, $g(B) = g(C_g) = K_Z^2 + 1 = d + 1$. According to the classification of strictly involutions of elliptic type, we know that g is an integer number at most 10, hence the degree of Z_d satisfies $1 \leq d \leq 9$. Consequently, either $Z \simeq \mathbf{P}^1 \times \mathbf{P}^1$ (and $d = 8$) or $Z \simeq Z_d$ is obtained as the blow-up of $\mathbf{P}^2$ at some generic points $p_1, \dots, p_{9-d}$. In the latter case, let β be the blow-up map and $E_i's$ be the corresponding exceptional divisors, then $B \in |-2K_{Z_d}| = |6L - 2\sum_{i=1}^{9-d} E_i|$, where L is the pull-back of a line in $\mathbf{P}^2$ via β. This implies that B can be considered as the strict transform of an irreducible curve $\Gamma_d \subseteq \mathbf{P}^2$ of degree 6 passing with multiplicity two at each point $p_1, \dots, p_{9-d}$. It is worth noting that curves with these properties can always be found; see, e.g., Theorem 1 in Section 5.2, Chapter 5, of [18].

Finally, the fact that $\delta = 0$ corresponds precisely to double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ follows from Proposition 2.13, Table 1, and the topological condition (see formula (38) on page 32 of [2]) imposing that $\frac{1}{4}[B]$ is an integral homology class in $H_2(X/\langle\iota\rangle, \mathbf{Z})$. ∎

Remark 3.6. It is worth noticing that the case where X^ι consist of a curve of genus 10 fits in the framework of the previous proposition when considering $d = 9$, i.e., the rational surface is $Z_9 \simeq \mathbf{P}^2$. In this case, as we will see in Theorem 3.10, there are no elliptic fibrations on X, therefore our analysis will focus on $d \leq 8$.

Following the construction presented in Proposition 3.5, if we take the double cover of the plane $\mathbf{P}^2$ branched at a nodal sextic curve Γ_d we get a singular surface Y, with ADE singularities, such that its minimal resolution X is a K3 surface, and the strict transform of Γ_d is a smooth curve C_g on X. So, we have a diagram as below.

$$\begin{tikzcd}
C_g \subset \arrow[d, mapsto, "1:1"] & X \arrow[r, "\text{min. res}"] \arrow[d, "2:1"'] & Y \arrow[d, "2:1"] & \\
B \subset & Z_d \arrow[r, "\beta"] & \mathbf{P}^2 & \supset \Gamma_d
\end{tikzcd}$$

Figure 2. Strictly elliptic K3 surfaces and smooth del Pezzo surfaces as in Proposition 3.5 (2).

Remark 3.7. It is worth mentioning that in Section 7 of [20], the authors consider a K3 surface X together with a non-symplectic involution $\iota\colon X \to X$ as in Convention 2.15 and such that $X^\iota = C_g \sqcup R_1 \sqcup \cdots \sqcup R_k$ consists of a smooth irreducible curve C_g of genus $g \geq 2$ and $k \geq 1$ rational curves. In this setting, some of these K3 surfaces are such that $X/\langle\iota\rangle \simeq \mathbf{P}^2$, and it is observed that in such instances, X can be realized as the double covering of $\mathbf{P}^2$ along a *reducible* sextic curve. We refer the reader to the Table in Section 7.1 of [20] for further details (see also Section 5 of [19] and Section 5.10 of [13]).

Lemma 3.8. *Let Z be a del Pezzo surface of degree d and let $f\colon Z \to \mathbf{P}^1$ be a conic bundle on Z with F_x the fiber over a point $x \in \mathbf{P}^1$. If E is an irreducible curve such that $E \subseteq \operatorname{Supp}(F_x)$, then $E^2 \in \{-1, 0\}$ and $g(E) = 0$. Let*[1] *$B \in |-2K_Z|$ be a smooth irreducible curve, then*

$$E^2 = \begin{cases} -1 & \text{if } E \cdot B = 2, \\ 0 & \text{if } E \cdot B = 4. \end{cases}$$

Proof. By the correspondence in Lemma 2.7, to any conic bundle $f\colon Z \to \mathbf{P}^1$, we can associate a conic class, i.e., an effective class $[D] \in \operatorname{NS}(Z)$ such that $D^2 = 0$, $K_Z \cdot D = -2$, and $f = \phi_{|D|}$ is corresponding the induced map. Furthermore, if E is an irreducible component of $\operatorname{Supp}(F_x)$, we have that $E \cdot F_x = E \cdot D = 0$. Suppose that $E^2 > 0$. By the Hodge index theorem, D is numerically trivial. However, this contradicts $K_Z \cdot D = -2$. Thus, $E^2 \leq 0$.

By the genus formula, we have that

$$2p_a(E) - 2 = E^2 + K_Z \cdot E \leq 0,$$

[1]The existence of such a curve can be deduced from the smooth divisor theorem (see Theorem 1.5 in [2]), or simply from the classification of smooth del Pezzo surfaces. Indeed, the divisor $-2K_{Z_d}$ is very ample for $d \geq 2$ and it defines a double cover $\phi_{|-2K_Z|}\colon Z \to Q$ for $d = 1$, where $Q \subseteq \mathbf{P}^3$ is a quadric cone. See Chapter IV of [6] for details.

and thus

$$2p_a(E) - 2 < 0,$$

since $E^2 \leq 0$ and $-K_Z \cdot E > 0$ by the Nakai–Moishezon ampleness criterion. Hence, $p_a(E) = 0$, and thus $E \simeq \mathbf{P}^1$. Finally, for a smooth irreducible curve $B \subseteq Z$ such that $-2K_Z \sim B$, we have that $E \cdot B = 2E^2 + 4$, from which the last statement of the lemma follows thoroughly. ∎

Remark 3.9. Note that the smooth fibers of any such conic bundle over del Pezzo surfaces are rational curves intersecting the curve $B \in |-2K_Z|$ at four points, taking multiplicities into account. Additionally, the singular fibers consist of two rational curves with self-intersection -1 that intersect at a single point and each component intersects B at two points.

We can now state our main theorem concerning the correspondence between conic bundles on Z and elliptic fibrations on X.

Theorem 3.10. *Let X be a K3 surface and $\iota\colon X \to X$ a non-symplectic involution of strictly elliptic type. Let $Z = X/\langle\iota\rangle$ be the quotient smooth del Pezzo surface as in Proposition 3.5, and let $\pi\colon X \to Z$ be the quotient map.*

If $d = 9$ and $g = 10$, the K3 surface X does not admit any elliptic fibration. If $d \leq 8$, there is a correspondence

$$\left\{ \begin{array}{c} \text{Conic bundles} \\ f : Z \to \mathbf{P}^1 \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Elliptic fibrations} \\ \mathcal{E} : X \to \mathbf{P}^1 \end{array} \right\}, \quad f \longmapsto f \circ \pi.$$

Moreover, the fixed curve $X^\iota = C$ is a bisection of $\mathcal{E}$, i.e., $E \cdot C = 2$ for the general fiber E of $\mathcal{E}\colon X \to \mathbf{P}^1$.

Proof. The case $Z \simeq \mathbf{P}^1 \times \mathbf{P}^1$ is treated in Proposition 2.4 of [16] and in Section 3 of [36]: observe that the composition of the projection π with the projection p_1 on $\mathbf{P}^1 \times \mathbf{P}^1$ (and similarly for p_2) defines an elliptic fibration on X, as a consequence of the genus formula for the double cover. Hence, X admits at least two distinct elliptic fibrations. Now assume $\mathcal{E}$ is an elliptic fibration on X; this induces a pencil on $\mathbf{P}^1 \times \mathbf{P}^1$, but the only conic bundles are the projections (see Remark 2.10). Therefore, the elliptic fibration $\mathcal{E}$ must be of the form $\pi \circ f$, where f is either p_1 or p_2.

In what follows, we will assume that $Z \simeq Z_d$ is the blow-up of $9 - d$ points in $\mathbf{P}^2$ in general position (cf. Theorem 3.2 in [36], where the case of $Z_8 \simeq \mathbf{F}_1$ is also considered).

We first observe that when $g = 10$, the K3 surface X does not admit elliptic fibrations. Indeed, as observed in Remark 3.2, in this case $\mathrm{NS}(X) \cong \langle 2\rangle$, i.e., it is generated by a class of square 2. Thus there are no classes $D \not\sim 0$ with $D^2 = 0$, and therefore no elliptic fibrations.

First, let us consider $\mathcal{E}\colon X \to \mathbf{P}^1$ an arbitrary elliptic fibration. Since ι is strictly elliptic, we know that ι must map each fiber of $\mathcal{E}$ to itself (see Lemma 3.3), and thus $\mathcal{E}$ factors through a fibration $f\colon X/\langle\iota\rangle \simeq Z_d \to \mathbf{P}^1$. Since Z_d is a smooth del Pezzo surface and f has connected fibers, it follows that f is a K_{Z_d}-negative contraction with one-dimensional fibers and thus is a conic bundle by Theorem 3.1 (ii) in [3].

$$\begin{tikzcd} X \arrow[r, "\mathcal{E}"] \arrow[d, "\pi"'] & \mathbf{P}^1 \\ Z \arrow[ur, "\exists f"'] & \end{tikzcd}$$

Figure 3. Elliptic fibrations on strictly elliptic K3 surfaces and conic bundles on del Pezzo surfaces.

Conversely, consider a conic bundle $f\colon Z_d \to \mathbf{P}^1$ and define $\mathcal{E}\colon= f \circ \pi\colon X \to \mathbf{P}^1$. Here, we have that $B := \pi(C) \in |-2K_{Z_d}|$. In particular, if we denote by F the general fiber of $f\colon Z_d \to \mathbf{P}^1$, we have that $B \cdot F = 4$ and then it follows from the Riemann–Hurwitz formula that $\mathcal{E}\colon X \to \mathbf{P}^1$ is an elliptic fibration. The induced conic bundle is precisely $f\colon Z_d \to \mathbf{P}^1$ and thus we get the desired correspondence. Finally, the fact that C is a bisection of $\mathcal{E}\colon X \to \mathbf{P}^1$ follows directly from the projection formula. ■

Example 3.11. Let Z_d be a del Pezzo surface of degree $d \leq 8$ obtained as the blow-up of $9-d$ points in general position in $\mathbf{P}^2$. Now fix a conic bundle $f\colon Z_d \to \mathbf{P}^1$ and let $\mathcal{E}\colon X \to \mathbf{P}^1$ be the corresponding elliptic fibration.

Let F be the general fiber of $\mathcal{E}\colon X \to \mathbf{P}^1$, let Γ be the general fiber of $f\colon Z_d \to \mathbf{P}^1$, and let $E \subseteq Z_d$ be a (-1)-curve. By the projection formula,

$$F \cdot \pi^*(E) = \deg(\pi)(\Gamma \cdot E) = \deg(\pi) = 2 \quad \text{as long as } \Gamma \cdot E = 1,$$

and hence these (-1)-curves on Z_d induce bisections of $\mathcal{E}\colon X \to \mathbf{P}^1$. The condition $\Gamma \cdot E = 1$ can be explicitly verified by means of Theorem 2.3 and Proposition 2.9. For instance, following Convention 2.2, in Z_4 we can consider

$$E = 2L - E_1 - E_2 - E_3 - E_4 - E_5$$

and we can check that

- if $\Gamma = L - E_1$, then $E \cdot \Gamma = 1$;
- if $\Gamma = 2L - E_1 - E_2 - E_3 - E_4$, then $E \cdot \Gamma = 0$.

4. Examples and Néron–Severi lattices

We will now present some examples.

Example 4.1. Let X be the smooth quartic surface given by the equation

$$\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6(x_0^2x_1^2 - x_1^2x_2^2 + x_1^2x_3^2 + x_0^2x_2^2 - x_0^2x_3^2 + x_2^2x_3^2) = 0\} \subseteq \mathbf{P}^3.$$

We consider the automorphism

$$\iota : X \to X,\ [x_0 : x_1 : x_2 : x_3] \mapsto [x_0 : x_1 : x_2 : -x_3],$$

which is a non-symplectic involution on X, and the fixed locus of ι is the smooth quartic curve

$$C = \{x_0^4 + x_1^4 + x_2^4 + 6(x_0^2x_1^2 - x_1^2x_2^2 + x_0^2x_2^2) = 0\} \subseteq \{x_0 = 0\} \cong \mathbf{P}^2$$

of genus $g(C) = 3$.

The quotient $X/\langle\iota\rangle = Z$ is a smooth rational surface, and by the formula of the canonical divisor of double coverings (see Section 22 in Chapter V of [7]), we have that $B := \pi(C) \sim -2K_Z$. Therefore, the bi-anticanonical divisor $-2K_Z$ is ample and $K_Z^2 = 2$, i.e., $Z \simeq Z_2$ a del Pezzo surface of degree two. Conversely, a del Pezzo surface of degree 2 is isomorphic to a double cover of $\mathbf{P}^2$ branched at a smooth quartic curve (see, e.g., Chapter IV of [6]).

More generally, Section 5.3 of [11] gives a correspondence between plane sextic curves with 7 nodes in general position and smooth planar quartic curves. In particular, the smooth quartic curve $C \subseteq \mathbf{P}^2$ corresponds a sextic curve $\Gamma \subseteq \mathbf{P}^2$ with 7 nodes in general position. By considering the blow-up of $\mathbf{P}^2$ at the 7 nodes of Γ, followed by taking the double cover ramified at the strict transform of Γ, we obtain the K3 surface X.

The surface X has Néron–Severi lattice $U(2) \oplus A_1^{\oplus 6}$, as outlined in Section 8.4 of [37]. It is noteworthy, in accordance with Theorem 8.2 in [5], that X is projectively equivalent to Burnside's quartic surface. Specifically, its group of projective automorphisms is $\mathbf{Z}_2^4 \cdot \mathfrak{S}_5$.

As we observed, K3 surfaces of strictly elliptic type have finite automorphism group (see Section 2.8 of [2]) and thus they are related with the works of Roulleau, Artebani and Correa Deisler [4, 37].

For instance, if $d = 7$, the corresponding K3 surface has Picard rank 3 and is obtained as double cover of $\mathbf{P}^2$ branched over a sextic curve with two nodes p_1 and p_2. By Proposition 3.4 in [37], the surface admits three (-2)-curves. Two of them are contracted to p_1 and p_2, while the image of the third is the line through the two nodal points.

Similar descriptions and properties regarding the configuration of the (-2)-curves for $1 \leq d \leq 6$ are provided in [4, 37]. We have summarized the references for each case in Table 2, where $r = \operatorname{rk} \operatorname{NS}(X)$.

r	d	$\operatorname{NS}(X)$
3	7	$S_{1,1,2} \simeq U(2) \oplus A_1$, Section 3.4 of [37]
4	6	$U(2) \oplus A_1^{\oplus 2}$, Proposition 2.11 in [4]
5	5	$U(2) \oplus A_1^{\oplus 3}$, Section 5.2 of [37]
6	4	$U(2) \oplus A_1^{\oplus 4}$, Section 6.7 of [37]
7	3	$U(2) \oplus A_1^{\oplus 5}$, Section 7.3 of [37]
8	2	$U(2) \oplus A_1^{\oplus 6}$, Section 8.4 of [37]
9	1	$U(2) \oplus A_1^{\oplus 7}$, Section 9.4 of [37]

Table 2. Néron–Severi group of strictly elliptic K3 surfaces with $r \geq 3$.

In the case $d = 8$ (and subsequently $r = 2$), we distinguish between the cases $\delta = 0$ and $\delta = 1$, where $Z \simeq \mathbf{P}^1 \times \mathbf{P}^1$ and $Z \simeq \mathbf{F}_1$ respectively. In the $\delta = 0$ case, the Néron–Severi lattice $\operatorname{NS}(X)$ is generated by the pull-back via $\pi\colon X \to Z$ of the two fibers of the canonical projections, and thus $\operatorname{NS}(X) \simeq U(2)$. In the $\delta = 1$ case, $\operatorname{NS}(X)$ is generated by the pull-back via $\pi\colon X \to Z$ of the unique (-1)-curve in $\mathbf{F}_1$ and the class in $\mathbf{F}_1$ obtained by the pull-back of a line in $\mathbf{P}^2$ via the contraction $\mathbf{F}_1 \to \mathbf{P}^2$, yielding $\operatorname{NS}(X) \simeq \langle 2\rangle \oplus A_1$ (cf. Remark 3.2).

Example 4.2. Let X be the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched at a smooth bi-quartic curve $B = \{f_{4,4}([x,y],[s,t]) = 0\}$, i.e.,

$$X := \{w^2 = f_{4,4}([x,y],[s,t])\}.$$

We will exhibit an elliptic fibration $\mathcal{E}\colon X \to \mathbf{P}^1$ that admits fibers of types I_0, I_1, I_2, II and III. Let B be the smooth bi-quartic curve in $\mathbf{P}^1 \times \mathbf{P}^1$ given by the equation

$$\begin{aligned} B = \{&x^4s^4 - \tfrac{1}{2}x^3ys^3t + \tfrac{1}{2}x^3ys^2t^2 - x^2y^2s^4 - \tfrac{1}{2}x^2y^2s^3t + \tfrac{3}{2}x^2y^2s^2t^2 \\ &- x^2y^2t^4 + \tfrac{1}{2}xy^3s^2t^2 + y^4z^4 - 2y^4s^2t^2 + y^4t^4 = 0\}. \end{aligned}$$

We consider the elliptic fibration $\mathcal{E}\colon X \to \mathbf{P}^1$ obtained composing with the first projection onto $\mathbf{P}^1$. In the affine chart $t = 1$, the fiber over a generic point $[a:b] \in \mathbf{P}^1$ is given by

$$\begin{aligned} w^2 = {}&(a^4 - a^2b^2 + b^4)s^4 + \frac{1}{2}(-a^3b + a^2b^2)s^3 + \frac{1}{2}(a^3b + 3a^2b^2 + ab^3 - 4b^4)s^2 \\ &+ (-a^2b^2 + b^4). \end{aligned}$$

Using classical Weierstrass' methods (see, e.g., Theorem 2 in Chapter 10 of [29]), we can see that the generic fiber can be written as $w^2 = f_3(z,a,b)$, where $f_3(z)$ is a polynomial of degree three in z such that for generic $[a:b] \in \mathbf{P}^1$ the corresponding curve is irreducible.

If we denote by $L_{[a:b]}$ the line $\{[a:b]\} \times \mathbf{P}^1$, we have the following:

(1) $B \cap L_{[1:0]} = \{s^4 = 0\}$, so the fiber over $[1:0]$ is $w^2 = s^4$, of type III.

(2) $B \cap L_{[1:1]} = \{s^4 - s^3t = 0\}$, so the fiber over $[1:1]$ is $w^2 = s^3(s-t)$, of type II.

(3) $B \cap L_{[0:1]} = \{s^4 - 2s^2t^2 + t^4 = 0\}$, so the fiber over $[0:1]$ is $w^2 = (s-t)^2(s+t)^2$, of type I_2.

(4) $B \cap L_{[1:-1]} = \{s^4 - s^2t^2 = 0\}$, so the fiber over $[1:-1]$ is $w^2 = s^2(s^2 - t^2)$, of type I_1.

As mentioned in the introduction, the analysis of double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over curves of bi-degree $(4,4)$ was undertaken in [16, 36], where the authors already observed that the corresponding K3 surface X admits an elliptic fibration (see Theorem 4.4 in [16] and Section 3 of [36]). This study builds upon classical works by Enriques and Campedelli.

5. Singular fibers and bisections

Let (X,ι) and let $\pi\colon X \to Z = X/\langle\iota\rangle$ be as in Theorem 3.10. It is a well-known fact that if there exists a primitive embedding of the lattice $U \hookrightarrow \mathrm{NS}(X)$, then the K3 surface X admits a jacobian elliptic fibration (see, e.g., Remark 1.4 in Section 11 of [22]). However, according to Corollary 3.3 in [12], K3 surfaces of strictly elliptic type do not possess such embeddings, and consequently, they do not admit jacobian elliptic fibrations. On the other hand, since there is an embedding of the lattice $U(2)$ in $\mathrm{NS}(X)$, these surfaces admit bisections. More precisely, one can choose a class L in $U(2)$ with $L^2 = 0$ and assume that L is nef modulo the action of the Weyl group. By Remark 2.13 in Section 8 of [22], X admits an elliptic fibration. Furthermore, since the divisibility of L is either 1 or 2, the elliptic fibration admits either sections or bisections. In our specific context, where elliptic fibrations do not admit sections, we can therefore conclude the existence of bisections.

Proposition 5.1. *Let $f: Z \to \mathbf{P}^1$ be a conic bundle on the smooth del Pezzo surface Z and let $\mathcal{E}: X \to \mathbf{P}^1$ be the induced elliptic fibration on X. If $B \in |-2K_Z|$ is the branching locus of $\pi: X \to Z$, then the following statements hold:*

(1) *If F is a smooth fiber of f, then the corresponding fiber of $\mathcal{E}$ can be of one of the following types* (*see Table* 3):
- I_0, *if B meets F in four distinct points*;
- I_1, *if B meets F in two simple points and a double point*;
- I_2, *if B meets F in two double points*;
- II, *if B meets F in two points*: *a simple one and a point with multiplicity* 3;
- III, *if B meets F in a single point with multiplicity* 4.

(2) *If $F = F_1 + F_2$ is a singular fiber of f, let $P = F_1 \cap F_2$. Then the corresponding fiber of $\mathcal{E}$ can be of one of the following types* (*see Table* 4):
- I_2, *if B meets each F_i in two simple points, distinct from P*;
- I_3, *if B meets F_2 in two simple points, and B meets F_1 in a double point, distinct from P*;
- I_4, *if B meets each F_i in a double point, distinct from P*;
- III, *if $P \in B$ and B meets each $F_i = 2$ in two simple points*;
- IV, *if B meets F_1 in two simple points, while B meets F_2 in P with multiplicity* 2.

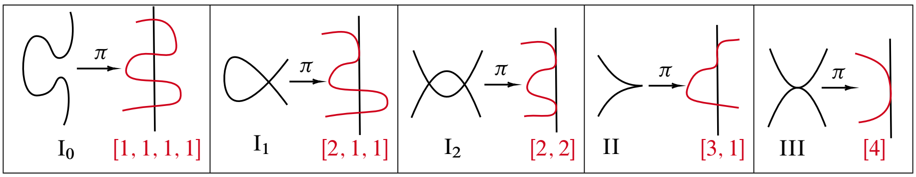

Table 3. Singular fibers of the elliptic fibration induced by smooth fibers of the conic bundle.

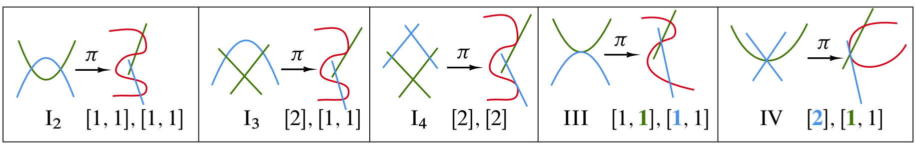

Table 4. Singular fibers of the elliptic fibration induced by singular fibers of the conic bundle.

Proof. Let F be a smooth fiber of f. By the previous construction, the branching locus of $\pi\colon X \to Z$ is a smooth irreducible curve B (in red in Table 3) and by Remark 3.9, F meets B in 4 points (with multiplicity). We study each case separately.

If B meets F in four distinct point, then the corresponding fiber of $\mathcal{E}$ is a double cover of $\mathbf{P}^1$ with four ramification points. In Table 3, we show the possible multiplicities of the points. Then by Riemann–Hurwitz formula, the fiber of $\mathcal{E}$ is a smooth curve of genus 1, i.e., a curve of type I_0. If B meets F in two simple points and a double point p, the preimage of p is a nodal point of the fiber, thus the fiber is of type I_1. Similarly, one obtains a fiber of type I_2 when $B \cap F$ consists of two double points. If B meets F in a triple point and a simple one, then the fiber of $\mathcal{E}$ has a singular point which is a cusp: the triple point in the double cover gives a singular point of the fiber where the equation is locally given by $y^2 = x^3$, thus a cusp. Similarly, if B meets F in a single point with multiplicity 4, the induced fiber of $\mathcal{E}$ is of type III.

Now let F be a singular fiber of f. According to Remark 3.9, singular fibers of a conic bundle on Z are union of two (-1)-curves F_1 and F_2 that intersect at one point $P = F_1 \cap F_2$. Furthermore, the branch curve B of the 2-cover $X \to Z$ (in red in Table 4) intersects each rational curve F_i at two points (with multiplicity). In Table 4, we show the possible multiplicities of the points. If B meets each F_i in two simple points, distinct from P, this defines a fiber of type I_2 of $\mathcal{E}$, since the preimage of each F_1 defines a rational curve and $\pi^{-1}(P)$ consists of two points. If B meets F_2 in two simple points and F_1 in a double point, distinct from P, this defines a singular fiber of $\mathcal{E}$ of type I_3: the double cover of F_1 gives two components of the fiber I_3, while F_2 contributes with one component and $\pi^{-1}(P)$ consist of two points. Similarly, if B meets each F_i in a double point, distinct from P, the fiber is of type I_4. The last case to study is when B meets F_1 in two simple points F_2 in P with multiplicity 2. In this case, the fiber $\pi^{-1}(F)$ has three components and they all meet in $\pi^{-1}(P)$. Three concurrent rational curves form a fiber of type IV. ■

Now we want to classify which types of fibers are compatible in each case with the classification given in Proposition 2.9. Note that in the case $Z \simeq \mathbf{P}^1 \times \mathbf{P}^1$ all types of singular fibers in Proposition 5.1 (1) can be realized, as Example 4.2 shows. Now, given a del Pezzo surface Z_d (see Convention 2.2), Theorem 3.10 establishes a correspondence between conic bundles on Z_d and elliptic fibrations on the K3 surface X. Proposition 2.9 then classifies conic bundles on Z_d. Given the potential fibers outlined in Proposition 5.1, our goal is to determine which ones are admissible for each Z_d and to establish their connection with the geometry of the sextic curve Γ_d introduced in Proposition 3.5.

Proposition 5.2. *Let (X, ι) be a pair of strictly elliptic type and let $Z_d = X/\langle\iota\rangle$ be the quotient smooth del Pezzo surface as in Proposition 3.5 (2). If $d \leq 5$, all types of singular fibers in Proposition 5.1 are admissible. If $d = 6, 7$ (respectively, 8), then the fiber* I_4 *and* IV *(respectively,* I_3, I_4 *and* IV*) are not admissible.*

In other words, the admissible fibers for the elliptic fibration $\mathcal{E}\colon X \to \mathbf{P}^1$ are described in Table 5.

Proof. Let $d = 8$. By Proposition 2.9, the only conic class on Z_8 is $D = L - E_1$, where E_1 is the exceptional divisor of the blow-up. In this case the conic bundle has no

d	Singular fibers
8	$\mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2, \mathrm{II}, \mathrm{III}$
7, 6	$\mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{II}, \mathrm{III}$
≤ 5	$\mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4, \mathrm{II}, \mathrm{III}, \mathrm{IV}$

Table 5. Admissible singular fibers of the elliptic fibration $\mathcal{E}: X \to \mathbf{P}^1$ induced by a conic bundle $f: Z_d \to \mathbf{P}^1$.

singular fibers, thus Γ_d meets the fiber in the nodal point and four more points. According to possibilities given in Table 3, one can have singular fiber of the elliptic fibrations of types I_0, I_1, I_2, II or III.

If $d = 7$, the curve Γ_d is a sextic curve with 2 nodes and by Proposition 2.9, conic classes on Z_7 are $D = L - E_i$, where E_i is the exceptional divisor over one of the two nodal points p_1, p_2. We take $i = 1$ without loss of generality. If the conic bundle has no singular fibers, then as before the possible fibers of the elliptic fibration are of types I_0, I_1, I_2, II or III. If $F = F_1 + F_2$ reducible, which corresponds to the case when L passes through the other nodal point p_2, we observe that each component F_1 and F_2 meets B, the strict transform of Γ_d, in 2 points. If they are distinct for both F_1, F_2, one has again a fiber of type I_2, while if one of the two components meets B with multiplicity 2, one obtains a fiber of type I_3 (see case [1, 1], [1, 1] and case [2], [1, 1] of Table 4). Observe that the strict transform always meets one of the two components in two distinct points (coming from the blow-up of the nodal points). Thus the only cases of Table 4 that appear are [1, 1], [1, 1] and [2], [1, 1], giving raise to fibers I_2 and I_3. One concludes that fibers of type I_4 and IV are not possible, since otherwise the sextic curve would have singularities worse than nodes (cf. Table 2). The case $d = 6$ is analogous.

When $d = 5$, according to Proposition 2.9, one can have the conic class $D = 2L - \sum_{i=1}^{4} E_i$. In this case, we distinguish if conics of the bundle are irreducible or reducible. The first case will give fibers of type $\mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2$, II and III as in Table 3. In the case of conics reduced as the union of two lines $L_1 \cup L_2$, one observes that this allows the cases of Table 4: each line passes through 2 of the 5 nodal points of the sextic Γ_5, thus meeting Γ_5 in two more points (with multiplicity). According to the distribution of these points, one obtain all cases of Table 4.

The remaining cases with $d \leq 4$ can be treated in the same way, and thus admit all types of fibers listed in Proposition 5.1. ∎

Remark 5.3. The proof of the previous result, along with Table 1, not only allows for the determination of admissible singular fibers in the induced elliptic fibration but also the complete configuration of these fibers. In fact, given a del Pezzo surface of degree d and a conic bundle, Table 5 tells which are possible types of singular fibers on the elliptic fibration induced on the K3 surface X. Moreover, they can be computed explicitly, as we show in the following example.

Example 5.4 (Wiman sextic). We describe the configuration of singular fibers of the strictly elliptic K3 surface associated to the classical Wiman sextic [42]. More precisely, we consider

$$B_W := \{[x:y:z] \in \mathbf{P}^2,\ W(x,y,z) = 0\},$$

where

$$W(x,y,z) = x^6 + y^6 + z^6 + (x^2+y^2+z^2)(x^4+y^4+z^4) - 12x^2y^2z^2.$$

Wiman observed that $B_W \subseteq \mathbf{P}^2$ has exactly four nodes $p_1, p_2, p_3, p_4 \in B_W$ as singularities, with

$$p_1 = [1:1:1], \quad p_2 = [1:-1:1], \quad p_3 = [-1:1:1] \quad \text{and} \quad p_4 = [1:1:-1].$$

The blow-up $\varepsilon\colon Z_W := \mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbf{P}^2) \to \mathbf{P}^2$ produces a smooth quintic del Pezzo surface, with exactly ten exceptional (-1)-curves. Explicitly, these curves are given by the exceptional divisors E_1, E_2, E_3, E_4 and the strict transforms $\overline{L}_{ij} \subseteq Z_W$ of the lines $L_{ij} = \overline{p_i p_j} \subseteq \mathbf{P}^2$, where

$$L_{12} = V(x-z), \quad L_{13} = V(y-z), \quad L_{14} = V(y-x), \quad L_{23} = V(x+y),$$
$$L_{24} = V(y+z) \quad \text{and} \quad L_{34} = V(x+z)$$

(see Figure 4). In particular, we observe that each intersection $B_W \cap L_{ij}$ is given by p_i and p_j as double points and by two other simple points[2] q_{ij} and $\bar{q}_{ij}$ (not depicted in Figure 4).

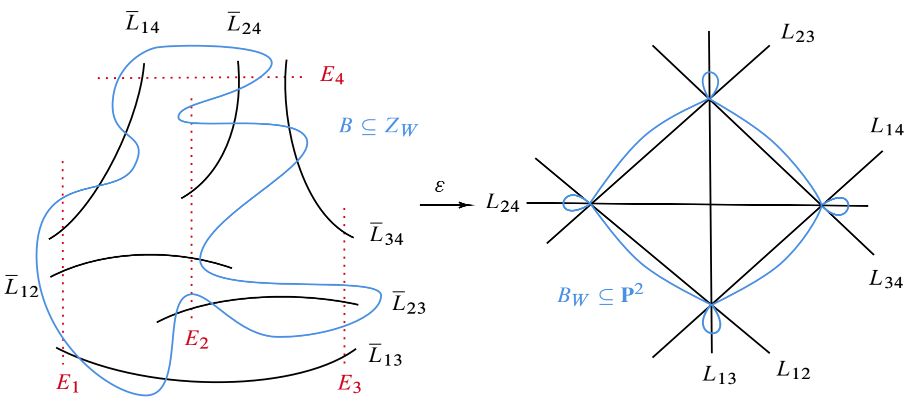

Figure 4. Conic bundle associated to the Wiman sextic.

Following Table 1, we study the conic bundle induced by the linear system

$$|2L - E_1 - E_2 - E_3 - E_4| \cong \mathbf{P}^1_{[\lambda:\mu]},$$

[2]Explicitly, $B_W \cap L_{ij} = \{p_i, p_j, q_{ij}, \bar{q}_{ij}\}$ where $q_{12} = [1, i\sqrt{3}, 1]$, $q_{13} = [i\sqrt{3}, 1, 1]$, $q_{14} = [i, i, -\sqrt{3}]$, $q_{23} = [i, -i, -\sqrt{3}]$, $q_{24} = [-\sqrt{3}, i, -i]$, $q_{34} = [i, \sqrt{3}, i]$, and where $\bar{q}_{ij}$ is obtained by applying to each coordinate of $q_{ij} \in \mathbf{P}^2(\mathbf{Q}(i,\sqrt{3}))$ the automorphism $\tau \in \mathrm{Gal}(\mathbf{Q}(i,\sqrt{3})/\mathbf{Q})$ given by $\tau(i) = i$ and $\tau(\sqrt{3}) = -\sqrt{3}$ (e.g., $\bar{q}_{13} = [-i\sqrt{3}, 1, 1]$).

i.e., induced by the pencil of conics in $\mathbf{P}^2$ passing through the points $p_1, p_2, p_3, p_4 \in B_W$. Explicitly, this is given by the conics

$$\mathcal{C}_{[\lambda:\mu]} := \{[x:y:z] \in \mathbf{P}^2 \mid \lambda(y^2-x^2)+\mu(z^2-x^2)=0\},\ [\lambda:\mu] \in \mathbf{P}^1.$$

By Theorem 3.10, the fibers of the induced elliptic fibration $\mathcal{E}_W\colon X_W \to \mathbf{P}^1$ come from fibers of the conic bundle. We first observe that for a general point $[\lambda:\mu] \in \mathbf{P}^1$, the conic $\mathcal{C}_{[\lambda:\mu]}$ meets the sextic B_W in p_1, p_2, p_3 and p_4, and in four more simple points by Bezout's theorem. Therefore the general fiber of the induced fibrations $\mathcal{E}_W$ is of type I_0.

Denote by x_F the number of singular fibers of type F, which is a non-negative integer. Observe that, since $\mathcal{E}_W\colon X_W \to \mathbf{P}^1$ is an elliptic fibration on a K3 surface, we expect that $\sum_F \chi(F) = 24$ (see Remark 11.1.12 in [22]) and then

$$x_{\mathrm{I}_1} + 2x_{\mathrm{I}_2} + 3x_{\mathrm{I}_3} + 4x_{\mathrm{I}_4} + 2x_{\mathrm{II}} + 3x_{\mathrm{III}} + 4x_{\mathrm{IV}} = 24. \tag{5.1}$$

Singular fibers of $\mathcal{E}_W$ are obtained from

(1) either singular fiber of the conic bundle, i.e., reduced conics of the pencil, or

(2) conics $\mathcal{C}_{[\lambda:\mu]}$ meeting B_W in less than 4 points.

We now study the two cases separately.

Case (1). The pencil of conics $\mathcal{C}_{[\lambda:\mu]}$ has exactly three reducible conics: $V(y^2-x^2)$, $V(y^2-z^2)$, and $V(z^2-x^2)$, corresponding to the points $[\lambda,\mu]$ given by $[1:0]$, $[1:-1]$, and $[0:1]$, respectively. Consequently, the induced conic bundle $f\colon Z \to \mathbf{P}^1$ has three singular fibers:

$$F_1 := \overline{L}_{14} \cup \overline{L}_{23}, \quad F_2 := \overline{L}_{13} \cup \overline{L}_{24}, \quad \text{and} \quad F_3 := \overline{L}_{12} \cup \overline{L}_{34}.$$

Each $\overline{L}_{ij}$ intersects the strict transform $B \subseteq Z_W$ of the Wiman sextic $B_W \subseteq \mathbf{P}^2$ in precisely two simple points, and hence the corresponding three singular fibers of the associated elliptic fibration $\mathcal{E}_W\colon X_W \to \mathbf{P}^1$ are of type I_2.

Case (2). Now consider smooth conics of the pencil such that the intersection with the sextic is not four simple points (out of nodes). According to Table 3, the multiplicities of the intersection points will determine the type of fibers of $\mathcal{E}_W$. Since the case $\lambda = 0$ corresponds to a reducible conic, we can assume $\lambda = 1$ and thus conics the pencil have equation

$$y^2 = x^2 - \mu(z^2-x^2), \quad \mu \in \mathbf{C} \setminus \{0,-1\}.$$

As observed in Proposition 5.1, the singular fibers of the elliptic fibration associated with smooth fibers of the conic bundle $\mathcal{C}_{[\lambda:\mu]}$ are induced by tangency conditions between the strict transform of conics of the pencil and the strict transform $B \subseteq Z_W$ of the Wiman sextic $B_W \subseteq \mathbf{P}^2$. Computation shows that there are nine values of μ such that the conic and the sextic meet in the four double points p_1, p_2, p_3, p_4 and in two additional points q_1, q_2, in each of them with multiplicity 2. Thus this corresponds to case I_2 of Table 3. The nine values of $[\lambda:\mu]$, as well as the equations of the conics, are shown in Table 6. Observe that everything is invariant applying a permutation of coordinates: this is due to the symmetric form of the sextic polynomial W.

Finally, one gets from Table 6 that $x_{\mathrm{I}_2} = 12$ and $x_F = 0$ for other types of fibers. therefore (5.1) is satisfied.

$[\lambda : \mu]$	conic	fiber	type
$[0:1]$	$(z-x)(z+x)=0$	reducible, $\bar{L}_{12} \cup \bar{L}_{34}$	I_2
$[1:0]$	$(y-x)(y+x)=0$	reducible, $\bar{L}_{14} \cup \bar{L}_{23}$	I_2
$[1:-1]$	$(y+z)(y-z)=0$	reducible, $\bar{L}_{13} \cup \bar{L}_{24}$	I_2
$[1:-2]$	$x^2+y^2-2z^2=0$	smooth, bitangent	I_2
$[1:1]$	$y^2+z^2-2x^2=0$	smooth, bitangent	I_2
$[2:-1]$	$z^2+x^2-2y^2=0$	smooth, bitangent	I_2
$[4:-1+i\sqrt{15}]$	$x^2(\frac{3}{4}+\frac{i\sqrt{15}}{4})-y^2+z^2(+\frac{1}{4}-\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2
$[6:-3+i\sqrt{15}]$	$y^2(\frac{3}{4}+\frac{i\sqrt{15}}{4})-z^2+x^2(+\frac{1}{4}-\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2
$[4:-3+i\sqrt{15}]$	$z^2(\frac{3}{4}+\frac{i\sqrt{15}}{4})-x^2+y^2(+\frac{1}{4}-\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2
$[4:-1-i\sqrt{15}]$	$x^2(\frac{3}{4}-\frac{i\sqrt{15}}{4})-y^2+z^2(+\frac{1}{4}+\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2
$[6:-3-i\sqrt{15}]$	$y^2(\frac{3}{4}-\frac{i\sqrt{15}}{4})-z^2+x^2(+\frac{1}{4}+\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2
$[4:-3-i\sqrt{15}]$	$z^2(\frac{3}{4}-\frac{i\sqrt{15}}{4})-x^2+y^2(+\frac{1}{4}+\frac{i\sqrt{15}}{4})=0$	smooth, bitangent	I_2

Table 6. Singular fibers of $\mathcal{E}_W$.

Acknowledgements. The authors would like to sincerely thank Michela Artebani, Giacomo Mezzedemi, Cecília Salgado, and Giancarlo Urzúa for various fruitful discussions regarding the constructions used in this article. Special thanks go to Cinzia Casagrande, who informed us about reference [15] and allowed us to correct errors in the counting of conic classes at low degree. The authors express their gratitude to Cecília Salgado for her generous hospitality extended to S. Troncoso during his visit to the Bernoulli Institute at the University of Groningen. This project began while the author Y. Prieto was visiting ICTP, and she would like to thank the institute for its warm hospitality and stimulating environment. P. Comparin is member of INdAM-GNSAGA.

Funding. P. Comparin has been partially funded by Universidad de La Frontera, Proyecto DIM23-0001 and Fondecyt ANID Project 1240360. P. Montero has been partially funded by Fondecyt ANID Projects 1231214 and 1240101. S. Troncoso was partially supported by Fondecyt ANID Project 3210518 and PRIN project "Multilinear Algebraic Geometry" of MUR (2022ZRRL4C).

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Received April 12, 2024; revised November 18, 2024.

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