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Extrinsic GJMS operators for submanifolds

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Abstract. We derive extrinsic GJMS operators and Q-curvatures associated to a submanifold of a conformal manifold. The operators are conformally covariant scalar differential operators on the submanifold with leading part a power of the Laplacian in the induced metric. Upon realizing the conformal manifold as the conformal infinity of an asymptotically Poincaré–Einstein space and the submanifold as the boundary of an asymptotically minimal submanifold thereof, these operators arise as obstructions to smooth extension as eigenfunctions of the Laplacian of the induced metric on the minimal submanifold. We derive explicit formulas for the operators of orders 2 and 4. We prove factorization formulas when the original submanifold is a minimal submanifold of an Einstein manifold. We also show how to reformulate the construction in terms of the ambient metric for the conformal manifold, and use this to prove that the operators defined by the factorization formulas are conformally invariant for all orders in all dimensions.

1. Introduction

The GJMS operators [28] are a family of conformally covariant natural differential operators on a Riemannian manifold with principal part a power of the Laplacian. They are basic objects in conformal geometry which arise in many contexts. In this paper, we construct analogs of the GJMS operators associated to a submanifold of a conformal manifold. These are differential operators on the submanifold which depend on its extrinsic geometry in the background space.

The original construction of operators in [28] used the ambient metric of [15]. Their derivation was reformulated in [33] in terms of a Poincaré metric. There they arise as obstructions to smoothly extending functions on the conformal manifold as eigenfunctions of the Laplacian of the Poincaré metric with prescribed leading order asymptotics. In [33], it was pointed out that the same construction can be carried out upon replacing the Poincaré metric by any asymptotically hyperbolic metric. In the general case, the operators still act on functions on the boundary at infinity and satisfy the same conformal transformation law with respect to rescaling the boundary metric. However, they now depend on

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the Taylor coefficients at the boundary of a compactification of the asymptotically hyperbolic metric, which in general need not have any relation to the intrinsic geometry of the boundary itself.

In order to construct operators associated to a submanifold of a conformal manifold, we apply the latter construction to an asymptotically hyperbolic metric determined asymptotically by the extrinsic geometry of the submanifold. Let Σ be a k-dimensional submanifold of an n-dimensional Riemannian manifold (M, g). Let $X = M \times [0, \varepsilon_0)$ for some small $\varepsilon_0 > 0$. We realize the conformal class $(M, [g])$ as the conformal infinity of a Poincaré metric g_+ on $\mathring{X}$, i.e., a smooth, even asymptotically hyperbolic approximate solution of the Einstein equation $\mathrm{Ric}(g_+) = -ng_+$ (see [15]). In turn, we realize Σ as the boundary of a smooth submanifold $Y \subset X$ which is asymptotically minimal with respect to g_+ and even in a suitable sense. We equip $\mathring{Y}$ with the metric h_+ induced by g_+, which is also asymptotically hyperbolic. We then derive our operators by applying the usual construction of [33] on the space $(\mathring{Y}, h_+)$. We call the resulting operators (*minimal submanifold*) *extrinsic GJMS operators.*

Branson's (critical) Q-curvature [6] is another fundamental object in conformal geometry. It is a natural scalar defined in even dimensions whose conformal transformation law is linear in the log of the conformal factor. Branson defined it from the zeroth order terms of the GJMS operators via analytic continuation. Just as for the operators, the same construction can be carried out on a general asymptotically hyperbolic manifold. Applying the construction on $(\mathring{Y}, h_+)$, we obtain for k even a Q-curvature associated to the submanifold $\Sigma \subset (M, g)$.

Our main existence result is the following.

Theorem 1.1. *Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ a submanifold, with $n \geq 3$ and $1 \leq k \leq n-1$.*

(1) *For the following values of ℓ, there is a minimal submanifold extrinsic GJMS operator $P_{2\ell}\colon C^\infty(\Sigma) \to C^\infty(\Sigma)$ of order 2ℓ:*

(a) $1 \leq \ell < \infty$ *if n and k are both odd,*

(b) $1 \leq \ell < n/2$ *if n is even and k is odd,*

(c) $1 \leq \ell \leq k/2 + 1$ *if k is even. (If $\ell = k/2 + 1$ and n is even, we also assume $n > k + 2$.)*

The operator $P_{2\ell}$ is formally self-adjoint, has leading term $(-\Delta_h)^\ell$, where h is the metric induced on Σ by g, is natural (as defined in Section 2), and satisfies the transformation law

$$\widehat{P}_{2\ell} = e^{(-k/2-\ell)\,\omega|_\Sigma} \circ P_{2\ell} \circ e^{(k/2-\ell)\,\omega|_\Sigma}, \tag{1.1}$$

under a conformal change $\widehat{g} = e^{2\omega} g$ with $\omega \in C^\infty(M)$.

(2) *If k is even, there is a minimal submanifold extrinsic Q-curvature Q_k, which is a natural scalar on Σ. Under conformal change $\widehat{g} = e^{2\omega} g$, it satisfies*

$$e^{k\,\omega|_\Sigma}\, \widehat{Q}_k = Q_k + P_k(\omega|_\Sigma). \tag{1.2}$$

Moreover, $P_k 1 = 0$.

Equation (1.2), the formal self-adjointness of P_k, and the fact that $P_k 1 = 0$ imply that if Σ is compact, then $\int_\Sigma Q_k \, da$ is a conformal invariant, where da denotes the area density on Σ determined by g. This invariant is a multiple of the higher-dimensional Willmore energy of Σ studied in [31, 37, 40]. See Remark 4.3.

We give explicit formulas for P_2 and P_4 in Theorems 5.2 and 5.3. These are discussed in more detail in the last paragraph of this introduction.

The transformation law (1.1) can be interpreted as saying that for fixed ℓ, the g-dependent family of operators $P_{2\ell}$ defines a single invariant operator on conformal densities on Σ. The zeroth order terms of the operators $P_{2\ell}$ for $2\ell \neq k$ define non-critical curvatures $Q_{2\ell}$ which we also study.

In this paper, by a submanifold we mean an embedded submanifold. But since the construction is local and an immersed submanifold is locally embedded, Theorem 1.1 also holds on an immersed submanifold $i: \Sigma \to M$. Likewise, Theorems 1.2 and 1.3 below apply to immersed submanifolds.

Note that the transformation laws (1.1) and (1.2) only involve $\omega|_\Sigma$. A consequence is that the $P_{2\ell}$ and Q_k only depend on the conformal class $[g]$ on M near Σ and the representative h on Σ. That is, they are independent of the way that the representative h is extended off of Σ to a metric g in the conformal class on M. This observation is pertinent to the application of our results to immersed submanifolds in the follow-up paper [9] described below. When we view the $P_{2\ell}$ and Q_k as determined by $(M, [g])$ and a choice of representative h on Σ, in the immersed case h can be any representative, allowing rescalings by any positive function in $C^\infty(\Sigma)$, not just the pullback of a representative g on M. In this setting, (1.1) and (1.2) hold for any $\omega \in C^\infty(\Sigma)$, again not just the pullback of a function on M.

We usually write as if we are working in Riemannian signature. But everything in this paper is formal, so is valid for metrics g of general signature. If g has mixed signature, it is assumed that the submanifold Σ is such that $g|_{T\Sigma}$ is everywhere nondegenerate. A nondegenerate submanifold is called minimal if its mean curvature vector vanishes.

The restrictions in Theorem 1.1 when n or k is even arise from the fact that the extension problems for Poincaré metrics and minimal submanifolds are obstructed at finite order in these cases. The original GJMS construction is similarly obstructed in even dimensions. The instance $\ell = k/2 + 1$ of case (c) is subtle. Unlike the other cases, the ambiguity in the minimal submanifold expansion does enter into the induced metric at the order which could affect P_{k+2}. However, this ambiguity does not contribute in the derivation of P_{k+2}; see Section 5 below for details.

There are other constructions of operators and Q-curvatures on submanifolds of conformal manifolds.

The simplest is just to forget about the extrinsic geometry. That is, consider the induced metric h on Σ and take the usual GJMS operators and Q-curvatures determined by h. We call these the *intrinsic* operators and Q-curvatures. Unfortunately, in general these do not satisfy the factorization identities given in Theorem 1.2 below, which are crucial for the application we have in mind. However, as we show in Section 4, our operators coincide with the intrinsic operators for umbilic submanifolds of locally conformally flat spaces.

In [22, 24], a calculus on conformally compact manifolds is developed, including a construction of families of operators and Q-curvatures on the conformal infinity as well as generalizations thereof to operators acting on sections of more general vector bundles.

In the setting of a hypersurface $\Sigma \subset (M,[g])$, the papers [4, 23, 25, 26] apply this general construction to a distinguished asymptotically hyperbolic representative of the conformal class $(M,[g])$ to derive and study natural extrinsic operators and Q-curvatures on Σ analogous to those in Theorem 1.1. This distinguished representative is an asymptotic solution of the singular Yamabe problem, i.e., a metric which is asymptotically hyperbolic along Σ with asymptotically constant scalar curvature $-n(n+1)$. It follows from the explicit formulas for our operators P_2 and P_4 derived in Theorems 5.2 and 5.3 that our operators in the case of hypersurfaces are different from the Gover–Waldron singular Yamabe extrinsic operators. In particular, the singular Yamabe extrinsic operators also do not satisfy the factorization identities. Unlike our construction via minimal submanifold extension, the construction via the singular Yamabe problem also produces operators of odd order. Further studies of the singular Yamabe extrinsic operators and Q-curvature can be found in [11, 35, 36]. The paper [3] uses minimal extension into a singular Yamabe space to construct operators, Q-curvatures, and associated boundary transgression curvatures when the hypersurface $\Sigma \subset M$ itself has a boundary.

The singular Yamabe problem and therefore the construction of operators using it are obstructed at finite order in all dimensions. But in [26], a tractor construction is applied to the highest order operator produced by the singular Yamabe construction to produce operators of all even orders in all dimensions when Σ is a hypersurface. This is in contrast to the situation for the original GJMS operators, where it is known that operators of higher orders do not exist [20, 27]. It would be interesting to determine whether such operators exist for all ℓ in higher codimension when k and/or n is even.

Suppose now that g is Einstein. In this case, a Poincaré metric can be written explicitly. If $\mathrm{Ric}(g) = \lambda(n-1)g$, then the Schouten tensor is given by $\mathsf{P}_{ij} = \frac{\lambda}{2} g_{ij}$, and a Poincaré metric for g is

$$g_+ = r^{-2}(dr^2 + (1 - \tfrac{1}{4}\lambda r^2)^2 g) \tag{1.3}$$

(see [15]). We call this the *canonical Poincaré metric* associated to the Einstein metric g. This explicit identification of g_+ leads to a factorization formula for the GJMS operators for an Einstein metric as a product of second order operators of the form $\Delta + c$. It also leads to the conclusion that for conformal classes containing an Einstein metric, the operators defined by the factorization formula are invariantly associated to the conformal class for all $\ell \geq 1$ in all dimensions. See [15]. Another treatment of these results is contained in [19].

A minimal extension Y can also be written explicitly if $\Sigma \subset M$ is a minimal submanifold with respect to the Einstein metric g. Namely, it was observed in the proof of Proposition 4.5 of [31] that $Y = \Sigma \times [0, \varepsilon_0) \subset X$ is a minimal extension of Σ with respect to g_+ (we recall this argument in Section 4). We call $\Sigma \times [0, \varepsilon_0)$ the *canonical minimal extension* of the minimal submanifold Σ of the Einstein manifold (M, g). The next theorem establishes that the same factorization formula holds for the minimal submanifold extrinsic GJMS operators. We regard this as a fundamental feature of these operators.

Theorem 1.2. *Suppose that* $\mathrm{Ric}(g) = \lambda(n-1)g$ *and* Σ *is minimal in* (M, g). *Let* g_+ *be the canonical Poincaré metric* (1.3) *and let* $Y = \Sigma \times [0, \varepsilon_0)$ *be the canonical minimal extension of* Σ. *Denote by* h *the metric on* Σ *induced by* g, *and by* h_+ *the metric on* $\mathring{Y}$ *induced by* g_+. *Let* $\ell \in \mathbb{N}$. *The operators produced by the GJMS construction for* $(\mathring{Y}, h_+)$

are given by

$$(1.4)\qquad P_{2\ell}=\prod_{j=1}^{\ell}(-\Delta_h+\lambda c_j),\quad \textit{where}\quad c_j=(k/2+j-1)(k/2-j).$$

If k is even, then

$$Q_k=\lambda^{k/2}(k-1)!.$$

In the case that g is Einstein and $\Sigma\subset(M,g)$ is minimal, (1.4) gives a formula for the operator $P_{2\ell}$ for ℓ in the ranges stated in Theorem 1.1. The next theorem shows that for all $\ell\geq 1$ in all dimensions, the operators (1.4) satisfy (1.1) under conformal change to another Einstein metric for which Σ is also minimal (if there is another such Einstein metric). This result can be used to define consistently operators via (1.1) for non-Einstein metrics in the same conformal class.

Theorem 1.3. *Let (M,g) be Einstein with $\operatorname{Ric}(g)=\lambda(n-1)g$ and let $\widehat{g}=e^{2\omega}g$ be a conformally related Einstein metric with $\operatorname{Ric}(\widehat{g})=\widehat{\lambda}(n-1)\widehat{g}$. Suppose $\Sigma\subset M$ is minimal relative to both g and $\widehat{g}$. Define $P_{2\ell}$ by (1.4). Let $\widehat{h}$ denote the metric induced on Σ by $\widehat{g}$ and define $\widehat{P}_{2\ell}$ by (1.4) with h replaced by $\widehat{h}$ and λ replaced by $\widehat{\lambda}$. Then (1.1) is valid for all $\ell\geq 1$.*

In this paper, asymptotically minimal submanifolds for Poincaré metrics are a tool used to derive and study the minimal submanifold extrinsic GJMS operators $P_{2\ell}$. The paper [18] is in the same spirit. In the case $k=1$, it uses the even asymptotically minimal extension to study canonical parametrizations of the curve Σ and to characterize when it is a conformal geodesic.

Asymptotically hyperbolic metrics that are exact solutions of $\operatorname{Ric}(g_+)=-ng_+$ are known as *Poincaré–Einstein metrics*. Actual minimal submanifolds of Poincaré–Einstein spaces, not just asymptotic ones, have been and continue to be an object of intense study themselves, motivated partially by physical considerations. In a follow-up paper [9], Theorem 1.2 and a construction based on scattering theory are applied to derive a formula of Gauss–Bonnet type for the renormalized area of such an even-dimensional minimal submanifold of a Poincaré–Einstein space, assuming a submanifold version of a result of Alexakis [1] establishing a decomposition of integrands of conformally invariant integrals. This application was the genesis of our project: we were led to search for extrinsic GJMS operators satisfying a factorization of the form (1.4) in order to derive such a formula. The formula takes the form

$$\mathcal{A}=a_k\chi(Y)+\int_Y \mathcal{W}_k\,dv_{h_+},$$

where k is even, Y is a minimal submanifold of dimension k of a Poincaré–Einstein space, h_+ is the induced metric on Y, $\chi(Y)$ denotes the Euler characteristic and $\mathcal{A}$ the renormalized area of Y, $\mathcal{W}_k$ is a pointwise conformal submanifold invariant, and $a_k\in\mathbb{R}$. A formula of this type was derived for $k=2$ in [2], and a formula in the same spirit for $k=4$ in the case of hypersurfaces in [39]. The derivation in [9] follows the same outline as the proof in [12] of an analogous formula for the renormalized volume of even-dimensional Poincaré–Einstein manifolds.

The paper [33] showed that the usual GJMS operators can be embedded in a continuous family of fractional order scattering operators. In general, such a family depends on a choice of an exact or asymptotic Poincaré–Einstein manifold with prescribed conformal infinity. Such fractional order operators have been of great interest in recent years, motivated in part by the connection to the Caffarelli–Silvestre extension [7, 8, 10, 13]. As indicated in [33], the scattering construction can be carried out for any asymptotically hyperbolic metric. Consequently, our minimal submanifold extrinsic GJMS operators can also be embedded in families of fractional order minimal submanifold extrinsic scattering operators upon choosing an extension of Σ as an exact or asymptotically minimal submanifold of an exact or asymptotically Poincaré–Einstein space. It would be interesting to explore possible uses of such operators. The scattering operators for general asymptotically hyperbolic metrics and nonzero potentials were studied also in [34], where moreover inverse scattering results were obtained.

A summary of the paper is as follows. In Section 2, we describe our notation and conventions and formulate the notion of naturality in the submanifold setting. In Section 3, we review the formal asymptotics of smooth, even Poincaré metrics, minimal submanifolds, and their induced metrics, mostly following [15] and [31]. In Section 4, we review the GJMS construction in the setting of a general asymptotically hyperbolic metric and prove Theorem 1.1 except for case (c) with $\ell = k/2 + 1$. We discuss minimal submanifolds of Einstein manifolds and prove Theorem 1.2. We then formulate Theorem 4.10, which asserts the infinite order diffeomorphism invariance of the canonical minimal extension, and prove Theorem 1.3 assuming Theorem 4.10. We close Section 4 by showing that if $(M, [g])$ is locally conformally flat and Σ is umbilic, then the extrinsic operators equal the intrinsic operators. In Section 5, we derive two versions of formulas for P_2, Q_2 and P_4, Q_4 for general g, Σ, k and n. One version is Theorem 5.2, in which the operators are expressed in terms of the second fundamental form and curvature of the background metric g. This version is well-suited to seeing the factorization formula (1.4) when g is Einstein and Σ is minimal. The second one is Theorem 5.3, in which P_2 and P_4 are written as the GJMS operators $\overline{P}_2$ and $\overline{P}_4$ intrinsic to Σ plus additional terms involving extrinsic quantities. The derivation of both is based on a fourth-order passage to normal form of the induced metric h_+ on Y. We conclude Section 5 by proving Theorem 1.1 in the remaining case (c) for $\ell = k/2 + 1$. The main step is Lemma 5.5, which uses a calculation of a higher order passage to normal form to identify explicitly the contribution to the induced metric of the ambiguity in the minimal submanifold expansion. Finally, in Section 6 we show how to reformulate the whole construction in terms of the ambient metric. We use this to prove Theorem 4.10, thereby completing the proof of Theorem 1.3. The proof of Theorem 4.10 is modeled on the proof of the analogous result in [15] asserting the diffeomorphism invariance of the canonical Poincaré metric associated to an Einstein metric.

2. Notation and conventions

For a Riemannian manifold (M^n, g), we denote the Levi-Civita connection by ${}^g\nabla$, the curvature tensor by R_{ijkl}, the Ricci tensor by $\mathrm{Ric}(g)$ or $R_{ij} = R^k{}_{ikj}$, and the scalar curvature by $R = R^i{}_i$. Our sign convention for R_{ijkl} is such that spheres have positive scalar curvature.

The Schouten tensor of (M, g) is

$$\mathsf{P}_{ij} = \frac{1}{n-2}\Big(R_{ij} - \frac{R}{2(n-1)}\, g_{ij}\Big)$$

and the Weyl tensor is defined by the decomposition

$$R_{ijkl} = W_{ijkl} + \mathsf{P}_{ik}\, g_{jl} - \mathsf{P}_{jk}\, g_{il} - \mathsf{P}_{il}\, g_{jk} + \mathsf{P}_{jl}\, g_{ik}.$$

The Cotton and Bach tensors are

$$C_{ijk} = {}^{g}\nabla_k \mathsf{P}_{ij} - {}^{g}\nabla_j \mathsf{P}_{ik}$$

and

$$B_{ij} = {}^{g}\nabla^k C_{ijk} - \mathsf{P}^{kl} W_{kijl}.$$

Latin indices i, j, k run between 1 and n in local coordinates, or can be interpreted as labels for TM or its dual in invariant expressions such as those above (Penrose abstract index notation).

We will denote by Σ a submanifold of (M, g) of dimension k, $1 \le k \le n-1$. All considerations in this paper are local, so all submanifolds are assumed to be embedded. We use α, β, γ as index labels for $T\Sigma$, and α', β', γ' for the normal bundle $N\Sigma$. A Latin index i thus specializes either to an α or an α'. So, for instance, when restricted to Σ, the Schouten tensor P_{ij} splits into its tangential $\mathsf{P}_{\alpha\beta}$, mixed $\mathsf{P}_{\alpha\alpha'}$, and normal $\mathsf{P}_{\alpha'\beta'}$ pieces. Likewise, the restriction of the metric g_{ij} to Σ can be identified with the metric $g_{\alpha\beta}$ induced on Σ together with the bundle metric $g_{\alpha'\beta'}$ induced on $N\Sigma$. We use $g_{\alpha\beta}$ and $g_{\alpha'\beta'}$ and their inverses to lower and raise unprimed and primed indices.

The *second fundamental form* $L\colon S^2T\Sigma \to N\Sigma$ is defined by $L(X, Y) = ({}^{g}\nabla_X Y)^{\perp}$. We typically write it as $L_{\alpha\beta}^{\alpha'}$, or perhaps as $L_{\alpha\beta\alpha'}$ or $L_{\alpha}{}^{\beta}{}_{\alpha'}$ upon lowering and/or raising indices. Since L has only one primed index and is symmetric in $\alpha\beta$, it is not necessary to pay attention to the order of the three indices. The mean curvature vector is $H = \frac{1}{k}\operatorname{tr} L$, i.e., the section of $N\Sigma$ given by $H^{\alpha'} = \frac{1}{k} g^{\alpha\beta} L_{\alpha\beta}^{\alpha'} = \frac{1}{k} L_{\alpha}{}^{\alpha\alpha'}$.

The Levi-Civita connection of g induces connections on $T\Sigma$ and $N\Sigma$ together with their duals and tensor products, all of which we denote ∇. So, for instance, we can form the covariant derivative $\nabla_\alpha H^{\alpha'}$, which is a section of $T^*\Sigma \otimes N\Sigma$.

When working in coordinates, we always use a local coordinate system

$$z^i = (x^\alpha, u^{\alpha'}), \quad 1 \le \alpha \le k, \quad k+1 \le \alpha' \le n$$

for M near Σ, with the properties that $\Sigma = \{u^{\alpha'} = 0\}$ and $\partial_\alpha \perp \partial_{\alpha'}$ on Σ. We call such a coordinate system *adapted*. The coordinates x^α restrict to a coordinate system on Σ. On Σ, the vectors ∂_α span $T\Sigma$, the $\partial_{\alpha'}$ span $N\Sigma$, and the mixed metric components $g_{\alpha\alpha'}$ vanish. This use of indices for coordinates is consistent with the abstract interpretation described above. Partial derivatives in local coordinates are expressed using either of the two notations $\partial_\alpha u_\beta = u_{\beta,\alpha}$. In Section 6, indices preceded by a semicolon, such as $u_{\beta;\alpha}$, are used to denote covariant differentiation.

By a (scalar, linear) *natural differential operator* on k-dimensional submanifolds of n-dimensional Riemannian manifolds, we will mean an assignment to each $\Sigma^k \subset (M^n, g)$ of a differential operator P on Σ, such that the following two conditions hold:

(1) If $\Sigma' \subset (M', g')$ and $\varphi\colon (M, g) \to (M', g')$ is an isometry for which $\varphi(\Sigma) = \Sigma'$, then $\varphi^* P' = P$.

(2) There are $m \in \mathbb{N} \cup \{0\}$ and universal polynomials $q_{\mathcal{J}}$ such that in any adapted local coordinate system $z = (x, u)$, P has the form

$$Pf(x) = \sum_{|\mathcal{J}| \leq m} q_{\mathcal{J}}(g^{\alpha\beta}, g^{\alpha'\beta'}, \partial_z^J g_{ij})\, \partial_x^{\mathcal{J}} f(x), \quad f \in C^\infty(\Sigma). \tag{2.1}$$

Here J is an n-multiindex and $\mathcal{J}$ is a k-multiindex. The argument $\partial_z^J g_{ij}$ denotes all derivatives of all g_{ij} of orders up to N, for some N, except that the variables $\partial_x^{\mathcal{J}} g_{\alpha\alpha'}$ for k-multiindices $\mathcal{J}$ do not appear (since these vanish in adapted coordinates). The $g^{\alpha\beta}$, $g^{\alpha'\beta'}$ and $\partial_z^J g_{ij}$ are evaluated at $z = (x, 0)$.

To clarify, $q_{\mathcal{J}}$ is a polynomial function on the vector space in which the inverse metric and the metric and its derivatives take values in local coordinates, taking into account the symmetry in the metric and partial derivative indices. For instance, for $N = 1$, the arguments are

$$(g^{\alpha\beta}, g^{\alpha'\beta'}, g_{\alpha\beta}, g_{\alpha'\beta'}, g_{\alpha\beta,i}, g_{\alpha\alpha',\gamma'}, g_{\alpha'\beta',i}),$$

so each $q_{\mathcal{J}}$ is a universal polynomial function on the vector space

$$S^2\mathbb{R}^k \oplus S^2\mathbb{R}^{n-k} \oplus S^2\mathbb{R}^{k^*} \oplus S^2\mathbb{R}^{n-k^*} \oplus (S^2\mathbb{R}^{k^*} \otimes \mathbb{R}^{n*}) \oplus (\mathbb{R}^{k^*} \otimes (\otimes^2\mathbb{R}^{n-k^*})) \oplus (S^2\mathbb{R}^{n-k^*} \otimes \mathbb{R}^{n*}).$$

The special case $m = 0$ serves to define natural scalars of k-dimensional submanifolds of n-dimensional Riemannian manifolds. In a follow-up paper [29], it is shown that any natural differential operator as above can be expressed as a linear combination of contractions of covariant derivatives of the curvature tensor of g, covariant derivatives of the second fundamental form, and covariant derivatives of f.

Our sign convention for Laplacians is that $\Delta = \sum \partial_i^2$ on Euclidean space. Norms are always taken with respect to the metric on tensor products induced by the metric on the underlying bundle.

3. Background: Smooth even formal asymptotics

In this section, we review the formal asymptotics of Poincaré metrics and minimal submanifolds thereof. We restrict consideration here to smooth even expansions, which we use in Section 4 to derive extrinsic GJMS operators. We largely follow [15] for Poincaré metrics and [31] for minimal submanifold asymptotics. The asymptotics of minimal submanifolds of Poincaré–Einstein spaces have also been studied in [32, 38, 40].

Let $(M^n, [g])$ be a conformal manifold, $n \geq 2$, and g a chosen metric in the conformal class. Set $X = M \times [0, \varepsilon_0)_r$, $\mathring{X} = M \times (0, \varepsilon_0)_r$, and identify M with $M \times \{0\} \subset X$. By an *even Poincaré metric* in normal form relative to g, we will mean a metric g_+ on $\mathring{X}$, for some $\varepsilon_0 > 0$, of the form

$$g_+ = \frac{dr^2 + g_r}{r^2}, \tag{3.1}$$

where g_r is a smooth 1-parameter family of metrics on M for which $g_0 = g$, such that the Taylor expansion of g_r at $r = 0$ is even, and satisfying the following:

(1) If n is odd, then $\mathrm{Ric}(g_+) + ng_+$ vanishes to infinite order at $r = 0$.

(2) If n is even, then $|\,\mathrm{Ric}(g_+) + ng_+|_{g_+} = O(r^n)$.

(For n even, the definition in [15] includes an additional trace condition that will not be relevant here.) An even Poincaré metric in normal form relative to g exists and g_r is unique, to infinite order if n is odd, and modulo $O(r^n)$ if n is even. Even Poincaré metrics in normal form relative to conformally related metrics are related, to infinite order if n is odd, and modulo $O(r^n)$ if n is even, by an even diffeomorphism between neighborhoods of M in X which restricts to the identity on M. See [15]. Set $\bar{g} = r^2 g_+ = dr^2 + g_r$. We view $M = \partial X$ as the boundary at infinity relative to g_+.

Let $\Sigma \subset M$ be a smooth embedded submanifold of dimension k, $1 \le k \le n-1$. Let $Y^{k+1} \subset X$ be a smooth submanifold which is transverse to M and satisfies $Y \cap M = \Sigma$. We describe Y near Σ in terms of a 1-parameter family of sections of the g-normal bundle $N\Sigma$ of Σ in M as follows: the normal exponential map of Σ with respect to g, denoted $\exp_\Sigma$, defines a diffeomorphism from a neighborhood of the zero section in $N\Sigma$ to a neighborhood of Σ in M. For $r \ge 0$ small, let $Y_r \subset M$ denote the slice of Y at height r, defined by $Y \cap (M \times \{r\}) = Y_r \times \{r\}$. Then Y_r is a smooth submanifold of M of dimension k and $Y_0 = \Sigma$. For each r, there is a unique section $U_r \in \Gamma(N\Sigma)$ so that $\exp_\Sigma\{U_r(p) : p \in \Sigma\} = Y_r$. This defines a smooth 1-parameter family U_r of sections of $N\Sigma$ for which, near Σ, we have

$$Y = \big\{(\exp_\Sigma U_r(p), r) : p \in \Sigma, r \ge 0\big\}. \tag{3.2}$$

In particular, $U_0 = 0$. The submanifolds $Y \subset X$ that we consider will all be orthogonal to M along Σ with respect to $\bar{g}$. Thus the tangent bundle to Y along Σ is $T\Sigma \oplus \operatorname{span} \partial_r$, and the normal bundle to Y along Σ can be identified with $N\Sigma$. Orthogonality of Y to M along Σ is equivalent to the condition $\partial_r U_r|_{r=0} = 0$, i.e., $U_r = O(r^2)$.

The inverse normal exponential map determines a boundary identification diffeomorphism ψ from a neighborhood of Σ in Y to a neighborhood of Σ in $\Sigma \times [0, \varepsilon_0)$ by

$$\psi(q, r) = (\pi((\exp_\Sigma)^{-1} q), r),$$

where $(q, r) \in Y \subset M \times [0, \varepsilon_0)$ and $\pi : N\Sigma \to \Sigma$ is the projection onto the base. It is easily seen that ψ is indeed a diffeomorphism if Y is transverse to M.

It is useful to realize ψ explicitly in terms of geodesic normal coordinates. Choose a local coordinate system $\{x^\alpha : 1 \le \alpha \le k\}$ for an open subset $\mathcal{V} \subset \Sigma$ and a local frame $\{e_{\alpha'}(x) : k+1 \le \alpha' \le n\}$ for $N\Sigma|_{\mathcal{V}}$. Let $\{u^{\alpha'} : k+1 \le \alpha' \le n\}$ denote the corresponding linear coordinates on the fibers of $N\Sigma|_{\mathcal{V}}$. The map $\exp_\Sigma(u^{\alpha'} e_{\alpha'}(x)) \mapsto (x, u)$ defines a geodesic normal coordinate system $(x^\alpha, u^{\alpha'})$ in a neighborhood $\mathcal{W}$ of $\mathcal{V}$ in M, with respect to which Σ is given by $u^{\alpha'} = 0$. For each (x, u), the curve $t \mapsto (x, tu)$ is a geodesic for g normal to Σ. In particular, in these coordinates the mixed metric components $g_{\alpha\alpha'}$ vanish on $\mathcal{V}$, so that (x, u) is an adapted coordinate system as defined in §2. Extend the coordinates (x, u) to $\mathcal{W} \times [0, \varepsilon_0) \subset X$ to be constant in r. In these coordinates, the diffeomorphism ψ is given by $\psi(x, u, r) = (x, r)$ for $(x, u, r) \in Y$. The coordinates (x, r) restrict to a coordinate system on Y. If U_r is a 1-parameter family of sections of $N\Sigma$ and we define $u^{\alpha'}(x,r)$ by $U_r(x) = u^{\alpha'}(x,r)e_{\alpha'}(x)$, then the description (3.2) of Y is the same as saying that, in the coordinates (x,u,r) on X, Y is the graph $u^{\alpha'} = u^{\alpha'}(x,r)$. The notation $u^{\alpha'}(x,r)$ can therefore be interpreted as the components of U_r in the frame $e_{\alpha'}(x) = \partial_{\alpha'}$, or equivalently as the graphing function $u^{\alpha'} = u^{\alpha'}(x,r)$ for Y in these coordinates. When we write $U_r^{\alpha'}$, the index is interpreted as an abstract index indicating that U_r is a section of $N\Sigma$.

We now impose the condition that Y is asymptotically minimal with respect to the metric g_+. This becomes a system of partial differential equations on the normal vector fields U_r. Recall that minimality of Y is equivalent to the statement that the mean curvature vector field of Y with respect to g_+ obeys $H_Y = 0$.

Proposition 3.1. *Let g_+ be an even Poincaré metric in normal form and Σ a submanifold of M as above.*

(1) *If k is odd, then there exists U_r whose Taylor expansion in r at $r = 0$ is even and for which H_Y vanishes to infinite order. Such U_r is unique to infinite order. If n is even, the Taylor expansion of U_r modulo $O(r^{n+2})$ is independent of the $O(r^n)$ ambiguity in g_r.*

(2) *If k is even, then there exists U_r so that $|H_Y|_{\bar{g}} = O(r^{k+2})$. The Taylor expansion of U_r modulo $O(r^{k+2})$ is uniquely determined (and is independent of the $O(r^n)$ ambiguity in g_r if n is even), and is even modulo $O(r^{k+2})$.*

Proposition 3.1 is proved in Theorem 3.1 of [31] for k even. It is straightforward to verify that the same sort of analysis can be used to prove Proposition 3.1 for k odd. One expresses Y as a graph in local coordinates and then carries out a perturbative analysis of the minimal submanifold equation order by order. The main point is that the equation respects parity and has indicial roots of 0 and $k+2$. The freedom at the indicial root of 0 corresponds to the freedom to prescribe Σ arbitrarily. When k is odd, the freedom at the indicial root of $k+2$ is fixed by requiring the expansion of U_r to be even. When k is even, the indicial root of $k+2$ generates an obstruction to existence of a smooth solution. Note that $U_r = O(r^2)$; a minimal submanifold is orthogonal to M along Σ.

If φ is an even diffeomorphism that restricts to the identity on M and pulls back g_+ to another even Poincaré metric $\widetilde{g}_+$ in normal form relative to a conformally related metric, then φ pulls back the minimal extension Y for g_+ to that for $\widetilde{g}_+$, to infinite order if k is odd, and modulo $O(r^{k+2})$ if k is even. This follows from the isometry invariance of the minimality condition, the parity preservation of φ, and the uniqueness of Y. In this sense Y is conformally invariant, to infinite order if k and n are odd, to order $O(r^{n+2})$ if k is odd and n is even, and to order $O(r^{k+2})$ if k is even.

In case (2), the condition $|H_Y|_{\bar{g}} = O(r^{k+2})$ only determines the expansion of U_r modulo $O(r^{k+2})$. Here and in Section 4, we will take the expansion to be even to infinite order, so that the full Taylor expansion of U_r is even in all cases. We write the expansion of U_r in the form

$$U_r = U_{(2)}r^2 + U_{(4)}r^4 + \cdots, \tag{3.3}$$

where the $U_{(2j)}$ are globally and invariantly defined sections of $N\Sigma$ determined by the choice of metric g in the conformal class, up to the order specified by Proposition 3.1.

The first coefficient is given by $U_{(2)} = \frac{1}{2}H$, where H is the mean curvature vector of $\Sigma \subset M$ with respect to g; see (5.1) of [31].

Let h_+ denote the metric on Y induced by g_+. Since g_+ and Y are invariant up to diffeomorphism to the orders stated above under conformal change of g, it follows that h_+ is likewise invariant up to diffeomorphism. Set $\bar{h} = r^2 h_+$, so that $\bar{h}$ is the metric induced by $\bar{g} = dr^2 + g_r$. In terms of the coordinates (x^α, r) on Y introduced above, $\bar{h}$ is given by

$$
\begin{aligned}
\bar{h}_{\alpha\beta} &= g_{\alpha\beta} + 2g_{\alpha'(\alpha} u^{\alpha'}{}_{,\beta)} + g_{\alpha'\beta'} u^{\alpha'}{}_{,\alpha} u^{\beta'}{}_{,\beta},\\
\bar{h}_{\alpha 0} &= g_{\alpha\alpha'} u^{\alpha'}{}_{,r} + g_{\alpha'\beta'} u^{\alpha'}{}_{,\alpha} u^{\beta'}{}_{,r},\\
\bar{h}_{00} &= 1 + g_{\alpha'\beta'} u^{\alpha'}{}_{,r} u^{\beta'}{}_{,r}.
\end{aligned}
\tag{3.4}
$$

We use a “0” index for the r-direction. The components of $\bar{h}$ and the derivatives of u are evaluated at (x, r). The above formulas for components of $\bar{h}$ were obtained from the pullback of $\bar{g}$ upon writing

$$
g_r = g_{\alpha\beta}(x,u,r)\, dx^\alpha dx^\beta + 2g_{\alpha\alpha'}(x,u,r)\, dx^\alpha du^{\alpha'} + g_{\alpha'\beta'}(x,u,r)\, du^{\alpha'} du^{\beta'}.
$$

In (3.4), all g_{ij} are understood to be evaluated at $(x, u(x,r), r)$. Since the expansions of g_r and $u(x,r)$ are even in r, it follows upon inspection of (3.4) that the Taylor expansions of $\bar{h}_{\alpha\beta}$ and $\bar{h}_{00}$ in r at $r = 0$ are even and the Taylor expansion of $\bar{h}_{\alpha 0}$ is odd. The following proposition is easily verified from (3.4), Proposition 3.1, and the formal determination of the Poincaré metric (3.1).

Proposition 3.2.

(1) *If n and k are both odd, then the infinite order Taylor expansions of $\bar{h}_{\alpha\beta}$, $\bar{h}_{\alpha 0}$, and $\bar{h}_{00}$ are uniquely determined by Σ and g.*

(2) *If n is even and k is odd, then the Taylor expansions of $\bar{h}_{\alpha\beta}$ and $\bar{h}_{00}$* mod $O(r^n)$ *and of $\bar{h}_{\alpha 0}$* mod $O(r^{n+1})$ *are independent of the $O(r^n)$ ambiguity in g_r, and therefore are uniquely determined by Σ and g.*

(3) *If k is even, then the Taylor expansions of $\bar{h}_{\alpha\beta}$ and $\bar{h}_{00}$* mod $O(r^{k+2})$ *and of $\bar{h}_{\alpha 0}$* mod $O(r^{k+3})$ *are independent of the $O(r^{k+2})$ ambiguity in U_r (and independent of the $O(r^n)$ ambiguity in g_r if n is even), and therefore are uniquely determined by Σ and g.*

Since $g_{\alpha\alpha'} = O(r^2)$ and $u^{\alpha'} = O(r^2)$, it is evident from (3.4) that $\bar{h}_{\alpha 0} = O(r^3)$ and $\bar{h}_{00} = 1 + O(r^2)$. In particular, h_+ is asymptotically hyperbolic since $|dr|^2_{\bar{h}} = 1$ at $r = 0$.

4. Extrinsic GJMS operators

As described in the introduction, it was noted in [33] that the GJMS construction can be carried out for general asymptotically hyperbolic metrics. (In this paper, asymptotically hyperbolic metrics have smooth compactifications.) The conclusions of this general GJMS/Q-curvature construction are summarized in the following proposition. We formulate the characterization of the operators as obstructions to the existence of smooth expansions for eigenfunctions of the Laplacian of the asymptotically hyperbolic metric, rather than by the equivalent characterization as log coefficients in the expansions of non-smooth solutions. Our choice of notation is governed by our intended application to extrinsic GJMS operators.

Proposition 4.1. *Let Y^{k+1} be a manifold with boundary Σ^k, $k \geq 1$. Let h_+ be an asymptotically hyperbolic metric on $\mathring{Y}$. Let h be a representative of the conformal infinity of h_+, and let r be a defining function for Σ satisfying $r^2 h_+|_{T\Sigma} = h$. Let $\ell \in \mathbb{N}$.*

(1) *Given $f \in C^\infty(\Sigma)$, there exists $F \in C^\infty(Y)$, uniquely determined modulo $O(r^{2\ell})$, so that $F|_\Sigma = f$ and $u := r^{k/2-\ell} F$ satisfies*

$$\left(\Delta_{h_+} + ((k/2)^2 - \ell^2)\right)u = O(r^{k/2+\ell}).$$

The function

$$\left(r^{-k/2-\ell}(\Delta_{h_+} + ((k/2)^2 - \ell^2))\, u\right)\big|_\Sigma \tag{4.1}$$

is independent of the $O(r^{2\ell})$ ambiguity in F, independent of the choice of r, and can be written as $a_\ell P_{2\ell} f$, where $a_\ell^{-1} = (-1)^\ell 2^{2(\ell-1)}(\ell-1)!^2$ and $P_{2\ell}$ is a formally self-adjoint differential operator on Σ with leading term $(-\Delta_h)^\ell$. If $\widehat{h} = e^{2\omega} h$ for $\omega \in C^\infty(\Sigma)$, then

$$\widehat{P}_{2\ell} = e^{(-k/2-\ell)\omega} \circ P_{2\ell} \circ e^{(k/2-\ell)\omega}. \tag{4.2}$$

(2) *There exists a function $Q_{2\ell}$, which depends polynomially on k, so that $P_{2\ell} 1 = (k/2-\ell) Q_{2\ell}$. For k even, if $\widehat{h} = e^{2\omega} h$ for $\omega \in C^\infty(\Sigma)$, then*

$$e^{k\omega} \widehat{Q}_k = Q_k + P_k \omega.$$

Remark 4.2. As written, the definition of $Q_{2\ell}$ in (2) fails in the critical case $2\ell = k$, where the factor $k/2 - \ell$ vanishes. Branson's original definition was by analytic continuation in k. There are now other constructions avoiding this analytic continuation [5, 14, 16, 21, 22, 33].

Remark 4.3. If k is even, the properties stated in Proposition 4.1 imply that $\int_\Sigma Q_k \, dv_h$ is conformally invariant. This invariant can be identified: the volume expansion for h_+ reads

$$\mathrm{vol}_{h_+}\{r > \varepsilon\} = c_0 \varepsilon^{-k} + c_1 \varepsilon^{1-k} + \cdots + c_{k-1}\varepsilon^{-1} + L \log \tfrac{1}{\varepsilon} + O(1),$$

where r is the geodesic defining function determined by h. Then

$$\int_\Sigma Q_k \, dv_h = b_k L, \quad \text{where} \quad b_k = (-1)^{k/2} 2^{k-1} (k/2)!\,(k/2-1)!.$$

This is proved for asymptotically Poincaré–Einstein metrics in [33] and [14], and both proofs are valid for general asymptotically hyperbolic metrics. The invariant L was studied in [31,37,40] and interpreted as a conformally invariant generalization of the Willmore energy of Σ in the case that h_+ is the induced metric on an asymptotically minimal submanifold of an asymptotically Poincaré–Einstein space.

Remark 4.4. The noncritical Q-curvatures $Q_{2\ell}$ for $\ell \neq k/2$ satisfy the transformation law that follows from (4.2); namely,

$$e^{2\ell\omega}\,\widehat{Q}_{2\ell} = Q_{2\ell} + (k/2-\ell)^{-1} e^{(\ell-k/2)\omega}\,\mathring{P}_{2\ell}\,(e^{(k/2-\ell)\omega}),$$

where $\mathring{P}_{2\ell} = P_{2\ell} - (k/2-\ell)Q_{2\ell}$.

Remark 4.5. In the setting of Proposition 4.1, there are nonzero obstruction operators for generic h_+ also for $\ell \in 1/2 + \mathbb{N}$. These vanish by parity considerations for the h_+ induced by even Poincaré metrics on submanifolds defined by even U_r.

Remark 4.6. For $k = 2$, one has $P_2 = -\Delta$ for any asymptotically hyperbolic metric.

In Proposition 4.1, it is clear that if $\widetilde{Y}$ is a second manifold with boundary Σ and if $\varphi\colon \widetilde{Y} \to Y$ is a diffeomorphism that restricts to the identity on Σ, then for each representative h, the operators $P_{2\ell}$ generated by $\varphi^* h_+$ are the same as those generated by h_+. In particular, one can take h_+ to be in normal form relative to h. In the next lemma, we calculate explicitly the operators P_2 and P_4 for a general asymptotically hyperbolic metric h_+ that is even and in normal form.

Lemma 4.7. *Let*

$$h_+ = r^{-2}(dr^2 + h_r)$$

be an asymptotically hyperbolic metric in normal form on $\Sigma \times (0,\varepsilon_0)$, where Σ has dimension k. Suppose h_r has the form

$$h_r = h + h_2 r^2 + h_4 r^4 + \cdots .$$

Then

$$P_2 = -\Delta + \frac{k-2}{2}\,Q_2, \qquad P_4 = \Delta^2 + \nabla^\alpha(T_{\alpha\beta}\nabla^\beta) + \frac{k-4}{2}\,Q_4, \tag{4.3}$$

where

$$Q_2 = -\operatorname{tr} h_2, \qquad T = -4h_2 + (k-2)(\operatorname{tr} h_2)h, \qquad Q_4 = 8\operatorname{tr} h_4 + \Delta(\operatorname{tr} h_2) - 4|h_2|^2 + \frac{k}{2}(\operatorname{tr} h_2)^2. \tag{4.4}$$

Proof. It is useful to introduce $\rho = r^2$ since h_+ is even. Then $h_+ = d\rho^2/(4\rho^2) + \mathrm{h}_\rho/\rho$, where

$$\mathrm{h}_\rho := h_r = h_{\sqrt{\rho}}.$$

At $\rho = 0$, we have

$$\mathrm{h}' = h_2 \quad \text{and} \quad \mathrm{h}'' = 2h_4, \tag{4.5}$$

where $' = \partial_\rho$.

The operator Δ_{h_+} takes the form

$$\Delta_{h_+} = 4\rho^2\,\partial_\rho^2 + 2(2-k)\,\rho\,\partial_\rho + 2\rho^2\,\mathrm{h}^{\alpha\beta}\,\mathrm{h}'_{\alpha\beta}\,\partial_\rho + \rho\,\Delta_{\mathrm{h}_\rho}.$$

Straightforward calculation shows that

$$\begin{aligned} &r^{\ell-k/2-2}\circ(\Delta_{h_+} + ((k/2)^2-\ell^2))\circ r^{k/2-\ell} \\ &\qquad = 4\rho\,\partial_\rho^2 + 4\big(1-\ell+\tfrac{1}{2}\rho\,\mathrm{h}^{\alpha\beta}\,\mathrm{h}'_{\alpha\beta}\big)\,\partial_\rho + \Delta_{\mathrm{h}_\rho} + (k/2-\ell)\,\mathrm{h}^{\alpha\beta}\,\mathrm{h}'_{\alpha\beta}. \end{aligned} \tag{4.6}$$

Setting $\ell = 1$ and $\rho = 0$ and recalling the definition of P_2 in Proposition 4.1 give $P_2 = -\Delta - (k/2-1)\operatorname{tr} h_2$, as claimed.

When $\ell = 2$, evaluating the equation

$$r^{\ell-k/2-2}(\Delta_{h_+} + ((k/2)^2-\ell^2))\,(r^{k/2-\ell}F) = 0$$

at $\rho = 0$ gives

$$4F' = \Big(\Delta + \frac{k-4}{2}\operatorname{tr} h_2\Big)F. \tag{4.7}$$

Upon differentiating at $\rho = 0$, the $\partial_\rho^2 F$ terms drop out and one sees that (4.1) equals

$$\Big(\Delta + \frac{k}{2}\operatorname{tr}\mathrm{h}'\Big)F' + \Big((\Delta_{\mathrm{h}_\rho})' + \frac{k-4}{2}\,(\mathrm{h}^{\alpha\beta}\,\mathrm{h}'_{\alpha\beta})'\Big)F.$$

Now

$$\begin{aligned} (\Delta_{\mathrm{h}_\rho})' &= -(\mathrm{h}')^{\alpha\beta}\,\nabla^2_{\alpha\beta} - \big(\nabla_\beta(\mathrm{h}')^{\alpha\beta} - \tfrac{1}{2}\nabla^\alpha(\mathrm{h}')^\beta{}_\beta\big)\,\nabla_\alpha, \\ (\mathrm{h}^{\alpha\beta}\,\mathrm{h}'_{\alpha\beta})' &= \operatorname{tr}\mathrm{h}'' - |\mathrm{h}'|^2. \end{aligned} \tag{4.8}$$

Substituting (4.5), (4.7), (4.8) and multiplying by $a_2^{-1} = 4$ give

$$\begin{aligned} P_4 = {}&\Big(\Delta + \frac{k}{2}\operatorname{tr} h_2\Big)\Big(\Delta + \frac{k-4}{2}\operatorname{tr} h_2\Big) \\ &- 4\,(h_2)^{\alpha\beta}\,\nabla^2_{\alpha\beta} - 4\big(\nabla_\beta(h_2)^{\alpha\beta} - \tfrac{1}{2}\nabla^\alpha(h_2)^\beta{}_\beta\big)\nabla_\alpha + 2(k-4)\,(2\operatorname{tr} h_4 - |h_2|^2). \end{aligned}$$

Elementary manipulations reduce this to the expression for P_4 written in (4.3). ∎

Later we will need the following lemma, which identifies the contribution of $h_{2\ell}$ to $P_{2\ell}$.

Lemma 4.8. *Suppose we are in the setting of Lemma 4.7. Let $\ell \geq 1$. Write $P_{2\ell} = \mathring{P}_{2\ell} + (k/2-\ell)\,Q_{2\ell}$. Then $\mathring{P}_{2\ell}$ depends only on h_{2j} for $j < \ell$, and*

$$Q_{2\ell} = \ell\, a_\ell^{-1}\operatorname{tr} h_{2\ell} + \mathring{Q}_{2\ell}, \tag{4.9}$$

where $\mathring{Q}_{2\ell}$ also depends only on h_{2j} for $j < \ell$.

Proof. Let f, F and u be as in the statement of Proposition 4.1. Introduce $\rho = r^2$ as in the proof of Lemma 4.7. Equation (4.6) implies that

$$\big(4\rho\partial_\rho^2 + 4\big(1-\ell+\tfrac{1}{2}\rho \mathrm{h}^{\alpha\beta}\mathrm{h}'_{\alpha\beta}\big)\partial_\rho + \Delta_{\mathrm{h}_\rho} + (k/2-\ell)\mathrm{h}^{\alpha\beta}\mathrm{h}'_{\alpha\beta}\big) F = \rho^{\ell-1}G,$$

where $G|_{\rho=0} = a_\ell P_{2\ell} f$. Differentiate $\ell-1$ times with respect to ρ and set $\rho=0$. The right-hand side becomes $(\ell-1)!a_\ell P_{2\ell} f$. The left-hand side depends only on $\partial_\rho^j F|_{\rho=0}$ for $j<\ell$ and $\partial_\rho^j \mathrm{h}_\rho|_{\rho=0}$ for $j\le\ell$. All the derivatives $\partial_\rho^j F|_{\rho=0}$ for $j<\ell$ are determined and depend only on h_{2j} for $j<\ell$. The only term which involves $\partial_\rho^\ell \mathrm{h}_\rho|_{\rho=0}$ arises when all the derivatives hit $\mathrm{h}'_{\alpha\beta}$ in the zeroth order term $(k/2-\ell)\mathrm{h}^{\alpha\beta}\mathrm{h}'_{\alpha\beta}F$. It follows that $\mathring{P}_{2\ell}$ depends only on h_{2j} for $j<\ell$. Equation (4.9) for $Q_{2\ell}$ follows upon equating the zeroth order term on both sides and using that $\partial_\rho^\ell \mathrm{h}_\rho|_{\rho=0} = \ell!\,h_{2\ell}$. ∎

The original GJMS operators were defined by taking h_+ to be a Poincaré metric for h. For a Poincaré metric, the coefficients are given by

$$\begin{aligned}(h_2)_{\alpha\beta} &= -\mathsf{P}_{\alpha\beta},\\ (h_4)_{\alpha\beta} &= \frac{1}{4}\Big(-\frac{B_{\alpha\beta}}{k-4} + \mathsf{P}_\alpha{}^\gamma \mathsf{P}_{\gamma\beta}\Big),\end{aligned}$$

where the Schouten and Bach tensors refer to the metric h on Σ. In this case, (4.3) reduces to the formulas for the usual Yamabe and Paneitz operators.

Now return to the setting of Section 3. So Σ^k is a submanifold of (M^n, g) with induced metric h, g_+ is an even Poincaré metric in normal form relative to (M,g) on $\mathring{X} = M\times(0,\varepsilon_0)$, Y^{k+1} is an asymptotically minimal extension of Σ to X, and h_+ is the metric induced on $\mathring{Y}$ by g_+. Since g_+ was chosen to be in normal form relative to the chosen metric g, the defining function r on X is the geodesic defining function for g_+ determined by g, and we denote also by r its restriction to Y. The compactification $\bar{h} = r^2 h_+$ is uniquely determined to the orders stated in Proposition 3.2.

Proof of Theorem 1.1 (*except case* (c) *for* $\ell = k/2+1$). We prove here cases (a) and (b), and case (c) with $\ell\le k/2$. The proof for case (c) with $\ell = k/2+1$ will be given after Lemma 5.5 below.

Since h_+ is asymptotically hyperbolic, we can construct the operators $P_{2\ell}$ and associated Q-curvatures according to Proposition 4.1. These will depend only on the geometry of $\Sigma\subset(M,g)$ so long as the numbers of derivatives applied to the components of $\bar{h}$ are constrained as in the statement of Proposition 3.2. In this case, invariance of h_+ up to diffeomorphism under conformal change of g as discussed in Section 3 implies that the resulting operators satisfy (1.1).

Next we show that for the stated ranges of ℓ, the operators $P_{2\ell}$ of Proposition 4.1 associated to h_+ are uniquely determined independently of the ambiguities in g_r and U_r. If n and k are both odd, there are no ambiguities, so the operators are well-defined for all ℓ. This proves case (a).

Since $h_+ = r^{-2}\bar{h}$, we have

$$\Delta_{h_+} = r^2\Delta_{\bar{h}} + (1-k)r\bar{h}^{ij}\partial_j r\,\partial_i = r^2\bar{h}^{ij}(\partial^2_{ij} - \bar{\Gamma}^m_{ij}\partial_m) + (1-k)r\bar{h}^{ij}\partial_j r\,\partial_i. \quad (4.10)$$

It follows that the differential operator

$$r^{-k/2+\ell} \circ \big(\Delta_{h_+} + ((k/2)^2 - \ell^2)\big) \circ r^{k/2-\ell} \tag{4.11}$$

has smooth coefficients up to $r = 0$ which involve $\bar{h}$ only through $\bar{h}^{ij}$ and the Christoffel symbols $\bar{\Gamma}^m_{ij}$.

The construction of the obstruction operator $P_{2\ell}$ involves differentiating the operator (4.11) up to 2ℓ times at $r = 0$. By hypothesis, $2\ell < n$ if n is even and $2\ell < k + 2$ if k is even. So according to Proposition 3.2, all derivatives of all components of $\bar{h}$ of orders at most 2ℓ are determined independently of the ambiguities in U_r and g_r. Thus all derivatives of the $\bar{h}^{ij}$ terms which can enter are independent of these ambiguities.

The Christoffel symbols involve an extra derivative of $\bar{h}$. The Christoffel symbol term in (4.10) is $-r^2 \bar{h}^{ij} \bar{\Gamma}^m_{ij} \partial_m$. Since this includes the factor r^2 and only the first order differentiation ∂_m, it follows that each occurrence of $\bar{\Gamma}^m_{ij}$ in the operator (4.11) is multiplied by a factor of either r or r^2. So if (4.11) is differentiated at most 2ℓ times at $r = 0$, at most $2\ell - 1$ of the differentiations can hit $\bar{\Gamma}^m_{ij}$. Thus again the maximum number of differentiations of a component of $\bar{h}$ is 2ℓ.

We now explain why the operators $P_{2\ell}$ are natural. The isometry invariance is a consequence of the uniqueness and invariance of the Poincaré metric and the minimal submanifold extension, and of the GJMS construction as formulated in Proposition 4.1. The polynomial dependence on the inverse and the derivatives of the metric can be seen by following through each step of the construction as outlined below.

Our data is the metric g, written in terms of an adapted coordinate system (x, u) for Σ. First consider the Poincaré metric for g. Proposition 3.5 of [15] asserts that each determined derivative $\partial_r^j g_r|_{r=0}$ of g_r in (3.1) can be written as a linear combination of contractions of Ricci curvature and covariant derivatives of Ricci curvature of the initial metric g. Of course, this step is independent of Σ.

Next consider the minimal submanifold Y. If (x, u) is an adapted coordinate system, then Y is written as a graph $u = u(x, r)$ in (x, u, r) space. The graphing function $u(x, r)$ is determined by the minimal submanifold equation, which is written explicitly in any adapted coordinate system in (2.11) of [32] in terms of the metric g_r and the induced metric $\bar{h}$ given by (3.4). It is easily verified from (3.4) that the derivatives $\partial_r^j \bar{h}|_{r=0}$ of components of $\bar{h}$ can be expressed as universal polynomials in derivatives of components of g_r and its inverse and derivatives of components of $u(x, r)$. The Taylor expansion of $u(x, r)$ is derived inductively from the minimal submanifold equation in terms of derivatives of $\bar{h}$ and g_r and previously determined Taylor coefficients of $u(x, r)$, beginning with $u(x, 0) = 0$. It follows that each component of each determined derivative $\partial_r^j u|_{r=0}$ can be expressed as a universal polynomial in derivatives of the initial metric g and its inverse. Hence the same is true for the components of the determined derivatives $\partial_r^j \bar{h}|_{r=0}$.

Next consider the GJMS algorithm applied to the induced metric $h_+ = r^{-2}\bar{h}$. This is perhaps best analyzed by first putting h_+ into normal form relative to the metric h on Σ induced by g. The passage to normal form is affected (see Section 5 of [30]) by solving the eikonal equation $|d\tilde{r}|^2_{\tilde{r}^2 h_+} = 1$ for the geodesic defining function $\tilde{r}$ and then straightening the flow of the vector field $\mathrm{grad}_{\tilde{r}^2 h_+} \tilde{r}$ by solving ordinary differential equations. The

Taylor expansions of the solutions for both steps can be inductively calculated from the equations in terms of the expansions of the components of $\bar{h}$. It therefore follows that when written in normal form,

$$h_+ = \tilde{r}^{-2}\,(d\tilde{r}^2 + \tilde{h}_{\tilde{r}}), \tag{4.12}$$

the components of each determined derivative $\partial_{\tilde{r}}^j \tilde{h}_{\tilde{r}}|_{\tilde{r}=0}$ can be written as universal polynomials in derivatives of the initial metric g and its inverse. See Lemmas 5.1 and 5.5 below for some explicit special cases.

Finally, when applied to a metric in normal form (4.12), the GJMS algorithm produces operators whose coefficients can be written polynomially in terms of derivatives of the Taylor coefficients of $\tilde{h}$, as in Lemmas 4.7 and 4.8. ■

As described in the introduction, if g is Einstein, then a Poincaré metric is given by (1.3). If in addition $\Sigma \subset (M, g)$ is minimal, then $Y = \Sigma \times [0, \varepsilon_0)$ is a minimal extension. We recall the argument that Y is minimal for completeness. Upon setting $s = -\log r$, the metric g_+ takes the form

$$g_+ = ds^2 + (e^s - \tfrac{1}{4}\lambda e^{-s})^2 g.$$

But it is a general fact that if Σ is a minimal submanifold of a Riemannian manifold (M, g), then $\Sigma \times \mathbb{R}$ is a minimal submanifold of $M \times \mathbb{R}$ with respect to any warped product metric g_+ of the form $g_+ = ds^2 + A(s)g$, where s denotes the variable in $\mathbb{R}$ and $A(s)$ is a positive function.

Proof of Theorem 1.2. Recall that h_+ is defined to be the pull back of g_+ given by (1.3) to $Y = \Sigma \times [0, \varepsilon_0)$. Clearly, this gives

$$h_+ = r^{-2}\big(dr^2 + (1 - \tfrac{1}{4}\lambda r^2)^2 h\big). \tag{4.13}$$

If g is an Einstein metric on M^n satisfying $\mathrm{Ric}(g) = \lambda(n-1)g$, then its usual GJMS operators are given by the same formula (1.4) with Δ_h replaced by Δ_g and k replaced by n. There are now several proofs of this fact. In Chapter 7 of [15], it is shown that in the Einstein case, the recursion for the GJMS operators reduces to a recursion for a sequence of polynomials of the one variable Δ_g/λ. The last proof presented there solves this recursion explicitly to obtain the factorization. This proof carries over verbatim for the operators determined by the asymptotically hyperbolic metric h_+ given by (4.13). The relevant point is that in (1.3), the metric g is independent of r. Since h_+ in (4.13) is given by the same formula with h independent of r, the same proof applies. This proves (1.4). The formula for Q follows upon inspecting the constant term in (1.4). ■

Remark 4.9. Inspection of the constant term in the factorization (1.4) shows that the $Q_{2\ell}$ for general $\ell \geq 1$ are given by

$$Q_{2\ell} = \lambda^\ell \prod_{j=1}^{2\ell-1} \Big(\frac{k}{2} - \ell + j\Big).$$

If n is odd, then (1.3) is the unique even Poincaré metric in normal form for g to infinite order. But if n is even, there are others, differing at order n. Proposition 7.5 of [15] shows that the canonical Poincaré metric g_+ defined by (1.3) is conformally invariant in the sense that if $\widehat{g} = e^{2\omega} g$ is a conformally related Einstein metric with $\mathrm{Ric}(\widehat{g}) = \widehat{\lambda}(n-1)\widehat{g}$, then g_+ and $\widehat{g}_+ = r^{-2}(dr^2 + (1 - \frac{1}{4}\widehat{\lambda}r^2)^2 \widehat{g})$ are diffeomorphic to infinite order by a diffeomorphism that restricts to the identity on M. The diffeomorphism is unique to infinite order, since the diffeomorphism putting any asymptotically hyperbolic metric into normal form relative to a given representative for the conformal infinity is unique.

If k is odd, then $Y = \Sigma \times [0, \varepsilon_0)$ is to infinite order the unique even minimal extension. If k is even, the fact that Y is smooth implies that the obstruction to existence of a smooth minimal extension of Σ vanishes, see Proposition 4.5 in [31]. But there are other infinite order minimal extensions, corresponding to different choices of the freedom in U_r at order $k+2$. The following theorem shows that the canonical minimal extension is conformally invariant.

Theorem 4.10. *Let (M, g) be Einstein and let $\widehat{g} = e^{2\omega} g$ be a conformally related Einstein metric. Suppose $\Sigma \subset M$ is minimal relative to both g and $\widehat{g}$. Then the diffeomorphism that pulls back $\widehat{g}_+$ to g_+ preserves $\Sigma \times [0, \varepsilon_0)$ to infinite order.*

For k odd, this follows from the uniqueness to infinite order of the even minimal extension. We will prove Theorem 4.10 for k even in Section 6 using the realization of the minimal extension in the ambient space. So for minimal submanifolds Σ of Einstein manifolds (M, g), for all k and n the canonical choices of both g_+ and Y are invariant to infinite order up to diffeomorphism.

Proof of Theorem 1.3. Theorem 4.10 implies that the metrics h_+ and $\widehat{h}_+$ induced on $\Sigma \times [0, \varepsilon_0)$ by g_+ and $\widehat{g}_+$ are diffeomorphic to infinite order. The diffeomorphism invariance in Proposition 4.1 shows that the operators $P_{2\ell}$ and $\widehat{P}_{2\ell}$ arising from the GJMS construction for h_+ and $\widehat{h}_+$ satisfy (1.1). Theorem 1.2 shows that these operators are given by the factorization formula (1.4). ∎

We close this section by identifying the extrinsic operators in another special case: when g is locally conformally flat and Σ is umbilic. Any locally conformally flat manifold $(M, [g])$, of any dimension $n \geq 3$, is the conformal infinity of a hyperbolic metric g_+ in a deleted neighborhood of M in $X = M \times [0, \varepsilon)$; see Proposition 7.2 and Theorem 7.4 in [15]. The hyperbolic metric g_+ is uniquely determined by $(M, [g])$ up to a diffeomorphism restricting to the identity on M. There is an explicit formula for g_+ when written in normal form relative to any representative g, but we will not need this formula.

Suppose $\Sigma \subset (M, [g])$ is umbilic of dimension k, $2 \leq k \leq n-1$. (We rule out the case $k = 1$ because the umbilic condition is vacuous in this case.) We claim that there is an extension Y in a neighborhood of Σ in X which is totally geodesic with respect to g_+, and such a Y is unique. To see this, choose locally a representative $g_0 \in [g]$ of constant sectional curvature one on an open set $\mathcal{U} \subset M$ and an isometric embedding of $(\mathcal{U}, g_0)$ into the round n-sphere. Use this to regard $\mathcal{U} \subset S^n = \partial B^{n+1}$. Now X can be realized locally as a neighborhood of $\mathcal{U}$ in B^{n+1} and g_+ as the standard hyperbolic metric $\frac{4}{(1-|x|^2)^2}|dx|^2$. A connected component of $\Sigma \cap \mathcal{U}$ is a piece of a sphere $S^k \subset \mathcal{U} \subset S^n$. The sphere S^k extends into B^{n+1} as a spherical cap intersecting the boundary orthogonally. This extension is the unique totally geodesic extension into the hyperbolic ball.

Since the hyperbolic metric g_+ and the totally geodesic extension Y are uniquely determined by $(M,[g])$ and Σ up to diffeomorphism, the induced metric h_+ on Y is too. So the GJMS construction on (Y,h_+) produces extrinsic operators $P_{2\ell}$ satisfying the conformal covariance relation (1.1) for all $\ell \geq 1$ and for all $n \geq 3$ and $k \geq 2$.

Consider now the intrinsic GJMS operators for the induced conformal class on Σ, which we denote by $\overline{P}_{2\ell}$. We have to rule out $k = 1$ here too since intrinsic operators do not exist on 1-manifolds. For $k = 2$, the only intrinsic operator is $\overline{P}_2 = -\Delta = P_2$ (recall Remark 4.6). So suppose $k \geq 3$. If as above we choose locally a round representative g_0, then since $\Sigma \cap \mathcal{U}$ is a piece of a sphere, the metric induced by g_0 on $\Sigma \cap \mathcal{U}$ is Einstein. So the induced conformal class contains (local) Einstein representatives. As mentioned in the introduction and discussed in Chapter 7 of [15], for such a conformal class, operators can be defined for all $\ell \geq 1$ for all representatives so that the conformal covariance relation holds. For a local Einstein representative, the operator $\overline{P}_{2\ell}$ is given by the factorization formula (1.4) with $P_{2\ell}$ on the left-hand side replaced by $\overline{P}_{2\ell}$.

Proposition 4.11. *Let g be locally conformally flat and let $\Sigma \subset (M,g)$ be umbilic of dimension $k \geq 3$. For any $\ell \geq 1$, the minimal submanifold extrinsic operator $P_{2\ell}$ equals the intrinsic operator $\overline{P}_{2\ell}$.*

Proof. Since both operators satisfy the same covariance relation, it suffices to prove the equality for a single representative, and it suffices to prove it locally. Take g_0 as above. The extrinsic operator $P_{2\ell}$ is constructed via the GJMS algorithm on (Y,h_+). Since Y is a piece of a totally geodesic sphere, the induced metric h_+ is hyperbolic and thus is the canonical Poincaré metric associated to the Einstein metric induced by g_0 on Σ. So $P_{2\ell}$ and $\overline{P}_{2\ell}$ are determined by the same construction. ∎

5. The operators P_2 and P_4

We now turn to the derivation of formulas for the extrinsic GJMS operators P_2 and P_4. We intend to use Lemma 4.7 for this purpose. But h_+ is not in normal form in the boundary identification $\psi\colon Y \to \Sigma \times [0,\varepsilon_0)$ introduced in Section 3. This is clear from (3.4): the equations $\bar{h}_{\alpha 0} = 0$, $\bar{h}_{00} = 1$ need not hold. The defining function r on Y is the restriction of a geodesic defining function for g_+ on X but need not be a geodesic defining function for h_+. We must first rewrite h_+ in normal form in order to apply Lemma 4.7.

By Proposition 3.2 and the statement immediately thereafter, the beginning terms of $\bar{h}$ in (3.4) can be written in the following form:

$$\begin{aligned}
\bar{h}_{\alpha\beta} &= h_{\alpha\beta} + D_{\alpha\beta} r^2 + K_{\alpha\beta} r^4 + o(r^4),\\
\bar{h}_{\alpha 0} &= A_\alpha r^3 + o(r^3),\\
\bar{h}_{00} &= 1 + E r^2 + F r^4 + o(r^4),
\end{aligned} \tag{5.1}$$

relative to the boundary identification ψ, for some coefficients $D_{\alpha\beta}$, $K_{\alpha\beta}$, A_α, E, and F defined on Σ.

Lemma 5.1. *Let $h_+ = r^{-2}\bar{h}$ be an asymptotically hyperbolic metric on $\Sigma \times (0, \varepsilon_0)_r$ with Taylor coefficients given by (5.1). When written in normal form (4.12) with respect to the same representative h on Σ, the Taylor expansion of $\tilde{h}_{\tilde{r}}$ has the form*

$$\tilde{h}_{\tilde{r}} = h + \tilde{h}_2 \tilde{r}^2 + \tilde{h}_4 \tilde{r}^4 + \cdots, \tag{5.2}$$

with

$$\begin{aligned} (\tilde{h}_2)_{\alpha\beta} &= D_{\alpha\beta} + \tfrac{1}{2} E\, h_{\alpha\beta}, \\ (\tilde{h}_4)_{\alpha\beta} &= K_{\alpha\beta} - \tfrac{1}{2}\nabla_{(\alpha} A_{\beta)} + \tfrac{1}{8}\nabla^2_{\alpha\beta} E + \big(\tfrac{1}{4} F - \tfrac{3}{16} E^2\big) h_{\alpha\beta}. \end{aligned} \tag{5.3}$$

Proof. The existence result for the normal form states that there exists a unique change of variables $x = \tilde{x}(x, r)$, $r = \tilde{r}(x, r)$ with $\tilde{x}(x, 0) = x$, $\tilde{r} = r + O(r^2)$, such that h_+ is in normal form (4.12) relative to $(\tilde{x}, \tilde{r})$. It is tedious but straightforward to verify that h_+ takes the claimed form modulo higher order terms under the change

$$\begin{aligned} x^\alpha &= \tilde{x}^\alpha + \tfrac{1}{4}\big(-A^\alpha + \tfrac{1}{4}\nabla^\alpha E\big)\tilde{r}^4, \\ r &= \tilde{r} - \tfrac{1}{4} E\, \tilde{r}^3 + \big(-\tfrac{1}{8} F + \tfrac{3}{16} E^2\big)\tilde{r}^5. \end{aligned}$$

In carrying out this verification, note that (5.1) refers to the coefficients in

$$\bar{h} = \bar{h}_{\alpha\beta}(x, r)\, dx^\alpha dx^\beta + 2\bar{h}_{\alpha 0}(x, r)\, dx^\alpha dr + \bar{h}_{00}(x, r)\, dr^2,$$

i.e., $h_{\alpha\beta}$ and the coefficients of the powers of r in (5.1) are evaluated at x. Similarly, in (5.2) one has

$$\tilde{h}_{\tilde{r}} = \big(h_{\alpha\beta}(\tilde{x}) + (\tilde{h}_2)_{\alpha\beta}(\tilde{x})\tilde{r}^2 + (\tilde{h}_4)_{\alpha\beta}(\tilde{x})\tilde{r}^4 + \cdots\big)\, d\tilde{x}^\alpha d\tilde{x}^\beta,$$

i.e., h in (5.2) and the coefficients in (5.3) are evaluated at $\tilde{x}$. ∎

Observe that h_4 enters in (4.4) only via its trace. Equation (5.3) gives

$$\operatorname{tr} \tilde{h}_4 = K_\alpha{}^\alpha - \tfrac{1}{2}\nabla^\alpha A_\alpha + \tfrac{1}{8}\Delta E + \tfrac{k}{4}\big(F - \tfrac{3}{4} E^2\big). \tag{5.4}$$

The coefficients $D_{\alpha\beta}$, $K_{\alpha\beta}$, E, and F in (5.1) were identified in (5.15) and (5.16) of [31] for the metrics h_+ that are induced on minimal submanifolds Y by Poincaré metrics g_+. (In [31], $K_{\alpha\beta}$ was called $Q_{\alpha\beta}$, and the convention $H^{\alpha'} = L_\alpha{}^{\alpha\alpha'}$ was used.) These coefficients were calculated in [31] by first evaluating the coefficients $U_{(2)}$ and $U_{(4)}$ in the expansion (3.3) of the normal graph U_r, then Taylor expanding (3.4) and inserting the expansion of U_r and the Poincaré metric expansion of g_r. The same process gives A_α; only the leading term for $\bar{h}_{\alpha 0}$ in (3.4) is needed for this. The formulas are:

$$\begin{aligned} D_{\alpha\beta} &= -H^{\alpha'} L_{\alpha\beta\alpha'} - \mathsf{P}_{\alpha\beta}, \\ K_{\alpha\beta} &= -2L_{\alpha\beta\alpha'} U^{\alpha'}_{(4)} + \tfrac{1}{4} R_{\alpha'\alpha\beta\beta'} H^{\alpha'} H^{\beta'} + \tfrac{1}{4} L^{\alpha'}_{\alpha\gamma} L_\beta{}^\gamma{}_{\beta'} H_{\alpha'} H^{\beta'} - \tfrac{1}{2}{}^g\nabla_{\alpha'} \mathsf{P}_{\alpha\beta} H^{\alpha'} \\ &\quad + L^{\alpha'}_{\gamma(\alpha} \mathsf{P}_{\beta)}{}^\gamma H_{\alpha'} + \tfrac{1}{4(4-n)} B_{\alpha\beta} + \tfrac{1}{4}\mathsf{P}_\alpha{}^i \mathsf{P}_{i\beta} - \mathsf{P}_{\alpha'(\alpha}\nabla_{\beta)} H^{\alpha'} + \tfrac{1}{4}\nabla_\alpha H^{\alpha'} \nabla_\beta H_{\alpha'}, \\ A_\alpha &= -\mathsf{P}_{\alpha\alpha'} H^{\alpha'} + \tfrac{1}{2} H_{\alpha'} \nabla_\alpha H^{\alpha'}, \\ E &= |H|^2, \\ F &= -\mathsf{P}_{\alpha'\beta'} H^{\alpha'} H^{\beta'} + 8 H_{\alpha'} U^{\alpha'}_{(4)}. \end{aligned} \tag{5.5}$$

The $U_{(4)}$ expression appearing in $K_{\alpha\beta}$ and F is defined in (3.3). Its explicit form is given in Proposition 5.5 of [31], but will not be needed here, since the occurrences in $K_\alpha{}^\alpha$ and F cancel in (5.4).

Theorem 5.2. *The first two extrinsic GJMS operators P_2, P_4 are given by*

$$P_2 = -\Delta + \frac{k-2}{2} Q_2, \qquad P_4 = \Delta^2 + \nabla^\alpha (T_{\alpha\beta}\nabla^\beta) + \frac{k-4}{2} Q_4, \tag{5.6}$$

where

$$\begin{aligned} Q_2 &= \mathsf{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2, \\ T_{\alpha\beta} &= 4\mathsf{P}_{\alpha\beta} + 4H^{\alpha'}L_{\alpha\beta\alpha'} - \Big[(k-2)\mathsf{P}_\gamma{}^\gamma + \frac{1}{2}(k^2-2k+4)|H|^2\Big]h_{\alpha\beta}, \\ Q_4 &= -\Delta\Big(\mathsf{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2\Big) - 2\Big|\mathsf{P}_{\alpha\beta} + H^{\alpha'}L_{\alpha\beta\alpha'} - \frac{1}{2}|H|^2 h_{\alpha\beta}\Big|^2 \\ &\quad + 2|\mathsf{P}_{\alpha\alpha'} - \nabla_\alpha H_{\alpha'}|^2 + \frac{k}{2}\Big(\mathsf{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2\Big)^2 \\ &\quad - 2W^\alpha{}_{\alpha'\alpha\beta'}H^{\alpha'}H^{\beta'} - 4C^\alpha{}_{\alpha\alpha'}H^{\alpha'} - \frac{2}{n-4}B^\alpha{}_\alpha. \end{aligned} \tag{5.7}$$

In these formulas, L and H denote the second fundamental form and the mean curvature vector of $\Sigma \subset M$, all curvature quantities are of the background metric g, ∇_α denotes the induced connection on $T\Sigma$ and $N\Sigma$, and $\Delta = \nabla^\alpha\nabla_\alpha$.

Proof. Substitution of (5.5) into (5.3) gives

$$(\tilde{h}_2)_{\alpha\beta} = -\Big(\mathsf{P}_{\alpha\beta} + H^{\alpha'}L_{\alpha\beta\alpha'} - \frac{1}{2}|H|^2 h_{\alpha\beta}\Big). \tag{5.8}$$

Taking the trace gives

$$\operatorname{tr}\tilde{h}_2 = -\Big(\mathsf{P}_\alpha{}^\alpha + \frac{k}{2}|H|^2\Big). \tag{5.9}$$

Substituting these into the first two formulas of (4.4) with h_2 replaced by $\tilde{h}_2$ gives immediately the formulas for Q_2 and $T_{\alpha\beta}$ in (5.7).

We claim next that

$$\begin{aligned} 8\operatorname{tr}\tilde{h}_4 &= 2|\mathsf{P}_{\alpha\alpha'} - \nabla_\alpha H_{\alpha'}|^2 + 2|\tilde{h}_2|^2 - 2W^\alpha{}_{\alpha'\alpha\beta'}H^{\alpha'}H^{\beta'} \\ &\quad - 4C^\alpha{}_{\alpha\alpha'}H^{\alpha'} - \frac{2}{n-4}B^\alpha{}_\alpha. \end{aligned} \tag{5.10}$$

This can be verified by direct computation upon substituting (5.5) into (5.4). All of the computations are straightforward except for the following point. Note from (5.4) that $\operatorname{tr}\tilde{h}_4$ includes the term $-\frac{1}{2}\nabla^\alpha A_\alpha$. Clearly,

$$\nabla^\alpha A_\alpha = -H^{\alpha'}\nabla^\alpha \mathsf{P}_{\alpha\alpha'} - \mathsf{P}_{\alpha\alpha'}\nabla^\alpha H^{\alpha'} + \tfrac{1}{2}|\nabla H|^2 + \tfrac{1}{2}H_{\alpha'}\Delta H^{\alpha'}.$$

In the first term, the ∇^α denotes the induced connection on the bundle $T\Sigma \otimes N\Sigma$. One rewrites this in terms of the Levi-Civita connection ${}^g\nabla$ using the relation

$$\nabla_\alpha \mathsf{P}_{\beta\alpha'} = {}^g\nabla_\alpha \mathsf{P}_{\beta\alpha'} + L_{\alpha\beta}^{\beta'} \mathsf{P}_{\beta'\alpha'} - L_{\alpha\alpha'}^{\gamma} \mathsf{P}_{\beta\gamma},$$

where the first term on the right-hand side is interpreted as the $\alpha\beta\alpha'$ component of ${}^g\nabla\mathsf{P}$. This relation is a consequence of the definitions of the second fundamental form and the induced connection. See (2.2) of [31] and the discussion there for elaboration of this point.

Now substituting (5.8), (5.9), and (5.10) into (4.4) gives the formula for Q_4 written in (5.7). ■

The derivation of Theorem 5.2 automatically produces formulas (5.6) for P_2 and P_4 written in terms of curvature of the background metric g. This is advantageous for verifying the factorization formula of Theorem 1.2 when g is Einstein and Σ is minimal, since in that case all the terms involving the second fundamental form drop out, the Schouten tensor is constant, and the Bach tensor vanishes. This approach also avoids application of the Gauss–Codazzi equations. However, it can also be useful to express the operators in terms of intrinsic geometry, for instance to compare with the intrinsic operators on Σ and thereby to verify directly the conformal transformation law (1.1). This cannot be done in all cases, though. Intrinsic operators do not exist when $k = 1$, and intrinsic P_4 does not exist when $k = 2$.

In the rest of this section, we will denote intrinsic quantities for the induced metric h on Σ with an overline. For instance, $\bar{\mathsf{P}}_{\alpha\beta}$ is the Schouten tensor and $\overline{R}$ the scalar curvature of h. Set

$$\bar{\mathsf{J}} := \frac{\overline{R}}{2(k-1)},$$

so that $\bar{\mathsf{J}} = \bar{\mathsf{P}}_\alpha{}^\alpha$ if $k > 2$. Recall that our convention is that ∇_α denotes the induced connection on $T\Sigma$ or $N\Sigma$, so we do not need the overline on $\Delta = \nabla^\alpha\nabla_\alpha$. Thus the intrinsic Yamabe and Paneitz operators on Σ are written

$$\begin{aligned}
\overline{P}_2 &= -\Delta + \frac{k-2}{2}\bar{\mathsf{J}}, \\
\overline{P}_4 &= \Delta^2 + \nabla^\alpha\big(\overline{T}_{\alpha\beta}\nabla^\beta\big) + \frac{k-4}{2}\overline{Q}_4,
\end{aligned}$$

where

$$\begin{aligned}
\overline{T}_{\alpha\beta} &= 4\bar{\mathsf{P}}_{\alpha\beta} - (k-2)\bar{\mathsf{J}}h_{\alpha\beta}, \\
\overline{Q}_4 &= -\Delta\bar{\mathsf{J}} - 2|\bar{\mathsf{P}}|^2 + \frac{k}{2}\bar{\mathsf{J}}^2.
\end{aligned}$$

Extrinsic geometry enters via the following three objects. The (manifestly conformally invariant) Fialkow tensor [17] is

$$\mathsf{F}_{\alpha\beta} := \frac{1}{k-2}\big(\mathring{L}_{\alpha\gamma\alpha'}\mathring{L}_\beta{}^{\gamma\alpha'} - W_{\alpha\gamma\beta}{}^\gamma - \mathsf{G}h_{\alpha\beta}\big),$$

where

$$\mathsf{G} := \frac{1}{2(k-1)}\,(|\mathring{L}|^2 - W_{\alpha\beta}{}^{\alpha\beta}) = \mathsf{F}_\alpha{}^\alpha.$$

Set

$$\mathsf{D}_{\alpha\alpha'} := \frac{1}{1-k}\,(\nabla^\beta \mathring{L}_{\alpha\beta\alpha'} + W_{\alpha\beta\alpha'}{}^\beta).$$

Theorem 5.3. *The extrinsic operators P_2 and P_4 of Theorem 5.2 can be written*

$$P_2 = \overline{P}_2 + \frac{k-2}{2}\,\mathsf{G}, \qquad \textit{for } k > 1, \tag{5.11}$$

$$P_4 = \overline{P}_4 + \nabla^\alpha(\widetilde{T}_{\alpha\beta}\nabla^\beta) + \frac{k-4}{2}\,\widetilde{Q}_4, \quad \textit{for } k > 2, \tag{5.12}$$

where

$$\begin{aligned}\widetilde{T}_{\alpha\beta} &= 4\mathsf{F}_{\alpha\beta} - (k-2)\,\mathsf{G}\,h_{\alpha\beta},\\ \widetilde{Q}_4 &= -\Delta\mathsf{G} - 2|\mathsf{F}|^2 + \frac{k}{2}\,\mathsf{G}^2 - 4\mathsf{F}_{\alpha\beta}\,\bar{\mathsf{P}}^{\alpha\beta} + k\,\mathsf{G}\,\bar{\mathsf{J}} + 2|\mathsf{D}|^2\\ &\quad - 2H^{\alpha'}H^{\beta'}W^\alpha{}_{\alpha'\alpha\beta'} - 4H^{\alpha'}C^\alpha{}_{\alpha\alpha'} - \frac{2}{n-4}\,B_\alpha{}^\alpha.\end{aligned} \tag{5.13}$$

In particular,

$$\begin{aligned}Q_2 &= \bar{\mathsf{J}} + \mathsf{G},\\ T_{\alpha\beta} &= \overline{T}_{\alpha\beta} + \widetilde{T}_{\alpha\beta},\\ Q_4 &= \overline{Q}_4 + \widetilde{Q}_4.\end{aligned} \tag{5.14}$$

Proof. The Gauss–Codazzi equations imply the following relations (see [17]):

$$\mathsf{P}_{\alpha\beta} + H^{\alpha'}L_{\alpha\beta\alpha'} - \frac{1}{2}\,|H|^2 h_{\alpha\beta} = \bar{\mathsf{P}}_{\alpha\beta} + \mathsf{F}_{\alpha\beta}, \tag{5.15}$$

$$\mathsf{P}_{\alpha\alpha'} - \nabla_\alpha H_{\alpha'} = \mathsf{D}_{\alpha\alpha'}. \tag{5.16}$$

The trace of the first equation gives

$$\mathsf{P}_\alpha{}^\alpha + \frac{k}{2}\,|H|^2 = \bar{\mathsf{J}} + \mathsf{G}. \tag{5.17}$$

Combining (5.17) with (5.6) and (5.7) shows that

$$P_2 = -\Delta + \frac{k-2}{2}\,Q_2 = -\Delta + \frac{k-2}{2}\,(\bar{\mathsf{J}} + \mathsf{G}) = \overline{P}_2 + \frac{k-2}{2}\,\mathsf{G}.$$

This proves (5.11).

For (5.12), it suffices to show that $T_{\alpha\beta}$ and Q_4 in (5.7) satisfy (5.14). Using (5.15) and (5.17) in (5.7) gives

$$\begin{aligned}T_{\alpha\beta} &= 4\left(\bar{\mathsf{P}}_{\alpha\beta} + \mathsf{F}_{\alpha\beta} + \frac{1}{2}|H|^2 h_{\alpha\beta}\right)\\ &\qquad - \left((k-2)\left(\bar{\mathsf{J}} + \mathsf{G} - \frac{k}{2}|H|^2\right) + \frac{1}{2}\,(k^2 - 2k + 4)\,|H|^2\right)h_{\alpha\beta}\\ &= \overline{T}_{\alpha\beta} + \widetilde{T}_{\alpha\beta}.\end{aligned}$$

Finally, substituting (5.15), (5.16), and (5.17) in (5.7) gives

$$Q_4 = -\Delta(\mathsf{\bar{J}} + \mathsf{G}) - 2|\mathsf{\bar{P}} + \mathsf{F}|^2 + 2|\mathsf{D}|^2 + \frac{k}{2}(\mathsf{\bar{J}} + \mathsf{G})^2$$
$$- 2W^{\alpha}{}_{\alpha'\alpha\beta'}H^{\alpha'}H^{\beta'} - 4C^{\alpha}{}_{\alpha\alpha'}H^{\alpha'} - \frac{2}{n-4}B^{\alpha}{}_{\alpha}.$$

Expanding and collecting terms, one sees that this equals $\overline{Q}_4 + \widetilde{Q}_4$. ■

Remark 5.4. If we set

$$\widetilde{P}_4 = P_4 - \overline{P}_4 = \nabla^{\alpha}(\widetilde{T}_{\alpha\beta}\nabla^{\beta}) + \frac{k-4}{2}\widetilde{Q}_4,$$

it follows from the conformal covariance of P_4 and $\overline{P}_4$ that under a conformal change $\hat{g} = e^{2\omega}g$ on M, we have

(5.18)
$$\hat{\widetilde{P}}_4 = e^{(-k/2-2)\,\omega|_{\Sigma}} \circ \widetilde{P}_4 \circ e^{(k/2-2)\,\omega|_{\Sigma}}, \quad \text{for } k \geq 3,$$
$$e^{4\omega|_{\Sigma}}\hat{\widetilde{Q}}_4 = \widetilde{Q}_4 + \widetilde{P}_4(\omega|_{\Sigma}), \quad \text{for } k = 4.$$

Reversing the logic, these conformal transformation laws can be checked directly and thereby used to verify by direct calculation the conformal covariance of P_4.

Note that the second and third terms $-2|\mathsf{F}|^2 + \frac{k}{2}\mathsf{G}^2$ on the first line of formula (5.13) for $\widetilde{Q}_4$ are conformally invariant. So (5.18) still holds if they are omitted. However, they are needed to obtain the factorization in the case of minimal submanifolds of Einstein manifolds.

We next prove Theorem 1.1 in the remaining case (c) for $\ell = k/2 + 1$. The following lemma is the key. It generalizes to higher order the cancellation of the occurrences of $U_{(4)}$ in $K_{\alpha}{}^{\alpha}$ and F in the formula (5.4) for $\operatorname{tr}\tilde{h}_4$.

Lemma 5.5. *Let Y be an extension of Σ described as a normal graph* (3.2), (3.3)*, with $U_{(2)} = \frac{1}{2}H$. Let $m \geq 2$. When the induced metric h_+ on Y is written in the normal form* (4.12), (5.2)*, the contribution of $U_{(2m)}$ to $\tilde{h}_{2m}$ is $-2\mathring{L}_{\alpha\beta\alpha'}U^{\alpha'}_{(2m)}$.*

Proof. First consider the Taylor expansions of $\bar{h}_{\alpha\beta}$, $\bar{h}_{\alpha 0}$, and $\bar{h}_{00}$ given by (3.4). The coefficients of orders up to r^{2m} in $\bar{h}_{\alpha\beta}$ and $\bar{h}_{00}$ and up to r^{2m-1} in $\bar{h}_{\alpha 0}$ are the only ones that can affect $\tilde{h}_{2m}$. Staring at (3.4) and recalling that the g_{ij} are evaluated at $(x, u(x,r), r)$ and that $g_{\alpha\alpha'} = O(r^2)$ and $u^{\alpha'} = O(r^2)$, one sees that for $j < 2m$, $U_{(2m)}$ cannot enter into the coefficient of r^j in any of $\bar{h}_{\alpha\beta}$, $\bar{h}_{\alpha 0}$, or $\bar{h}_{00}$. So we are left with determining the contribution of $U_{(2m)}$ to the r^{2m} coefficient in $\bar{h}_{\alpha\beta}$ and $\bar{h}_{00}$.

It is clear that $U_{(2m)}$ cannot contribute to the r^{2m} coefficient of the second or third terms on the right-hand side in the formula for $\bar{h}_{\alpha\beta}$ in (3.4). To obtain the expansion in r of the first term $g_{\alpha\beta}(x, u(x,r), r)$, first Taylor expand $g_{\alpha\beta}(x, u, r)$ in r and then expand in u the resulting coefficients and substitute for each u the expansion of $u(x,r)$ in r. It is clear that the r^{2m} coefficient in $u(x,r)$ can only contribute to the r^{2m} coefficient in the resulting expansion when $u(x,r)$ is substituted into the first term $g_{\alpha\beta}(x,u,0)$. And since on Σ we have $g_{\alpha\beta,\alpha'} = -2L_{\alpha\beta\alpha'}$, it follows that

$$\begin{aligned} g_{\alpha\beta}(x,u(x,r),0) &= g_{\alpha\beta}(x,0,0) + g_{\alpha\beta,\alpha'}(x,0,0)\,u^{\alpha'}(x,r) + O(u^2) \\ &= h_{\alpha\beta}(x) - 2\,L_{\alpha\beta\alpha'}(x)\,u^{\alpha'}(x,r) + O(u^2). \end{aligned}$$

So the contribution of $U_{(2m)}$ to the r^{2m} coefficient in $\bar{h}_{\alpha\beta}$ is $-2L_{\alpha\beta\alpha'}U^{\alpha'}_{(2m)}$. For $\bar{h}_{00}$, substitute

$$u^{\alpha'}{}_{,r} = H^{\alpha'}r + \cdots + 2m\,u^{\alpha'}_{(2m)}\,r^{2m-1}$$

into (3.4) to see that the contribution of $U_{(2m)}$ to the r^{2m} coefficient of $\bar{h}_{00}$ is $4mH_{\alpha'}U^{\alpha'}_{(2m)}$. Set

$$\lambda = 4m\,H_{\alpha'}\,U^{\alpha'}_{(2m)}.$$

Now we have to transform h_+ to normal form as in Lemma 5.1. Let $h_+^{(0)}$ be the metric obtained by truncating the expansion of U_r at order $2m-2$, i.e., by using

$$U_r = U_{(2)}\,r^2 + \cdots + U_{(2m-2)}\,r^{(2m-2)}.$$

The above identification of the contribution of $U_{(2m)}$ shows that $h_+ = h_+^{(0)} + r^{-2}\,\bar{k}$, where

$$\begin{aligned} \bar{k}_{\alpha\beta} &= -2\,L_{\alpha\beta\alpha'}\,U^{\alpha'}_{(2m)}\,r^{2m} + O(r^{2m+2}), \\ \bar{k}_{\alpha 0} &= O(r^{2m+1}), \\ \bar{k}_{00} &= \lambda\,r^{2m} + O(r^{2m+2}). \end{aligned}$$

There exists a diffeomorphism φ which restricts to the identity on Σ, which satisfies $\varphi^* r = r + O(r^2)$, and for which $h_+^{(1)} := \varphi^* h_+^{(0)}$ is in normal form. It follows that

$$\varphi^* h_+ = h_+^{(1)} + \varphi^*(r^{-2}\,\bar{k}) = r^{-2}\,\bar{h}_\varphi,$$

with $\bar{h}_\varphi$ of the form

$$\begin{aligned} (\bar{h}_\varphi)_{\alpha\beta} &= \bar{h}^{(1)}_{\alpha\beta} - 2\,L_{\alpha\beta\alpha'}\,U^{\alpha'}_{(2m)}\,r^{2m} + O(r^{2m+2}), \\ (\bar{h}_\varphi)_{\alpha 0} &= O(r^{2m+1}), \\ (\bar{h}_\varphi)_{00} &= 1 + \lambda\,r^{2m} + O(r^{2m+2}). \end{aligned}$$

It is easily verified that the coordinate change

$$x^\alpha = \tilde{x}^\alpha, \quad r = \tilde{r} - \frac{\lambda}{4m}\,\tilde{r}^{2m+1}$$

transforms $\varphi^* h_+$ to normal form to the same order, with

$$\tilde{h}_{\tilde{r}} = \left(\bar{h}^{(1)}_{\alpha\beta} - 2\,L_{\alpha\beta\alpha'}\,U^{\alpha'}_{(2m)}\,\tilde{r}^{2m} + \frac{\lambda}{2m}\,h_{\alpha\beta}\,\tilde{r}^{2m} + O(\tilde{r}^{2m+2})\right) d\tilde{x}^\alpha\,d\tilde{x}^\beta.$$

Since $\frac{\lambda}{2m} = 2H_{\alpha'}U^{\alpha'}_{(2m)}$, it follows that the contribution of $U_{(2m)}$ to $\tilde{h}_{2m}$ is

$$-2\,L_{\alpha\beta\alpha'}\,U^{\alpha'}_{(2m)} + 2\,H_{\alpha'}\,U^{\alpha'}_{(2m)}h_{\alpha\beta} = -2\,\mathring{L}_{\alpha\beta\alpha'}\,U^{\alpha'}_{(2m)}. \qquad \blacksquare$$

Remark 5.6. In the more general context in which odd powers and/or log terms are allowed in the expansion of U_r, the argument of the proof of Lemma 5.5 also identifies the contribution of these terms to the expansion of h_+ in normal form.

Proof of Theorem 1.1 in case (c) for $\ell = k/2+1$. As in the other cases, it suffices to show that for k even (and $n > k+2$ if n is even), the operator P_{k+2} depends only on the determined coefficients $U_{(2j)}$ for $2j \le k$, and not on the undetermined coefficient $U_{(k+2)}$.

Let (5.2) be the expansion when h_+ is written in normal form (4.12). Lemma 4.8 shows that P_{k+2} depends on $\tilde{h}_{k+2}$ only through $\operatorname{tr}\tilde{h}_{k+2}$. And Lemma 5.5 shows that U_{k+2} drops out when calculating $\operatorname{tr}\tilde{h}_{k+2}$, so that $\operatorname{tr}\tilde{h}_{k+2}$ depends only on the $U_{(2j)}$ for $2j \le k$, as desired. ∎

6. Ambient realization

In this section, we show how to reformulate the minimal extension Y in terms of the ambient space and use this to prove Theorem 4.10 for k even.

We first recall the ambient space its relationship with Poincaré metrics. See [15] for details. The metric bundle of the conformal manifold $(M,[g])$ is

$$\mathcal{G} = \{(p, t^2 g(p)) : p \in M, t > 0\} \subset S^2T^*M$$

and the ambient space is

$$\widetilde{\mathcal{G}} := \mathcal{G} \times (-\varepsilon,\varepsilon)_\rho \quad \text{for } \varepsilon > 0 \text{ small.}$$

The choice of representative metric g induces an identification of $\widetilde{\mathcal{G}}$ with $\mathbb{R}_+ \times M \times (-\varepsilon,\varepsilon)$, points of which we denote (t,p,ρ). The ambient space admits dilations $\delta_\lambda : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ for $\lambda > 0$ defined by $\delta_\lambda(t,p,\rho) = (\lambda t, p, \rho)$. A straight ambient metric in normal form relative to g is a Lorentzian metric on $\widetilde{\mathcal{G}}$ of the form

$$\tilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_\rho \tag{6.1}$$

that asymptotically solves $\operatorname{Ric}(\tilde{g}) = 0$ to an appropriate order which depends on the dimension. Here g_ρ is a smooth one-parameter family of metrics on M with $g_0 = g$. The metric $\tilde{g}$ is homogeneous of degree 2 with respect to the dilations δ_λ: $\delta_\lambda^*\tilde{g} = \lambda^2\tilde{g}$.

Let $T = \frac{d}{d\lambda}\delta_\lambda|_{\lambda=1}$ denote the infinitesimal generator of the dilations, so that $T = t\partial_t$ in the identification $\widetilde{\mathcal{G}} = \mathbb{R}_+ \times M \times (-\varepsilon,\varepsilon)$. Note that $\|T\|^2_{\tilde{g}} = 2\rho t^2$. Introduce a new variable $s > 0$ on $\{\rho < 0\}$ by $s^2 = -\|T\|^2_{\tilde{g}}$, so that $s = rt$ where $r = \sqrt{-2\rho}$. In the new variables (s,p,r), $\tilde{g}$ becomes

$$\tilde{g} = s^2 g_+ - ds^2. \tag{6.2}$$

One consequence of this relation is that the asymptotic conditions $\operatorname{Ric}(\tilde{g}) = 0$ and $\operatorname{Ric}(g_+) = -ng_+$ are equivalent.

Define the projection $\pi_X : \widetilde{\mathcal{G}} \to X = M \times [0,\varepsilon_0)_r$ by

$$\pi_X(t,p,\rho) = (p,r), \quad \text{with} \quad r = \sqrt{-2\rho}.$$

Define the hypersurface

$$\mathcal{H} := \{\|T\|^2_{\tilde{g}} = -1\} = \{s = 1\} \subset \tilde{\mathcal{G}},$$

and denote by $i\colon \mathcal{H} \to \tilde{\mathcal{G}}$ the inclusion. If we identify $\mathcal{H}$ with $\mathring{X}$ via π_X, then clearly (6.2) implies $g_+ = i^*\tilde{g}$.

Let Y be a submanifold of X with $1 \le \dim Y \le \dim X - 1$. Define the *homogeneous lift* $\overline{Y}$ of Y by $\overline{Y} = \pi_X^*(Y)$, which we can identify with $\mathbb{R}_+ \times Y$ under our product decomposition $\mathbb{R}_+ \times M \times (-\varepsilon, \varepsilon)$ and the change of variable $r = \sqrt{-2\rho}$. Clearly $\overline{Y}$ is invariant under the dilations δ_λ. Observe from (6.2) that $\tilde{g}|_{T\overline{Y}}$ has Lorentzian signature.

Proposition 6.1. *A submanifold $Y \subset \mathring{X} = M \times (0, \varepsilon_0)$ is minimal with respect to g_+ if and only if its homogeneous lift $\overline{Y}$ is minimal with respect to $\tilde{g}$.*

Proof. Observe that (6.2) exhibits $\tilde{g}$ as a warped product of the form $A(s)g_+ - ds^2$. Thus the conclusion follows from the same observation that we made in Section 4 to identify the minimal extension of a minimal submanifold of an Einstein manifold: Y is minimal in $\mathring{X}$ with respect to g_+ if and only if $\overline{Y} = \mathbb{R}_+ \times Y$ is minimal in $\tilde{\mathcal{G}}$ with respect to any warped product of this form. ∎

We mentioned in the introduction that in the case $k = 1$, Fine and Herfray in [18] use the even minimal extension Y of a curve $\Sigma \subset M$ to study conformal geodesics and conformal canonical parametrizations of Σ. In order to relate the construction to tractors, they also analyze the unique lift of $\mathring{Y}$ to a surface contained in the hypersurface $\mathcal{H} \subset \tilde{\mathcal{G}}$. But they do not consider the homogeneous lift of Y.

Proposition 6.1 shows that the problem of extending Σ to a minimal submanifold Y is equivalent to the problem of extending its homogeneous lift $\overline{\Sigma}$ to a homogeneous minimal submanifold $\overline{Y}$ of $\tilde{\mathcal{G}}$. Therefore Proposition 3.1 gives asymptotic solutions of the latter problem.

Remark 6.2. As mentioned in the introduction, the GJMS operators were originally constructed in [28] in terms of the ambient metric, and it was shown in [33] that the construction is equivalent to the one used here in terms of the Poincaré metric. The equivalence argument in [33] applies in the general setting of an asymptotically hyperbolic metric h_+ and the ambient-like metric it determines upon replacing g_+ by h_+ in (6.2). Consequently, our extrinsic minimal submanifold GJMS operators can also be obtained from either of the two constructions given in [28] applied to the metric on the homogeneous minimal extension $\overline{Y}$ that is induced from the ambient metric $\tilde{g}$ associated to (M, g).

Now we turn to the proof of Theorem 4.10. The proof follows the same general outline as the proof in Chapter 7 of [15] of the diffeomorphism invariance of the canonical ambient metric associated to an Einstein metric. We have to develop a number of ingredients before we can present the proof at the end of the section.

Let $\Sigma^k \subset (M^n, g)$ and let $(x^\alpha, u^{\alpha'})$ be adapted coordinates near Σ defined via the normal exponential map, as constructed in Section 3. We will need the following lemma, which makes explicit certain consequences of the fact that Σ is minimal for two conformally related Einstein metrics.

Lemma 6.3. *Suppose g and $\widehat{g} = e^{2\omega} g$ are conformally related Einstein metrics on M. Suppose $\Sigma \subset M$ is minimal for both g and $\widehat{g}$. For $s \geq 0$, let $\nabla^s L$ be the section of $\otimes^{s+2} T^*\Sigma \otimes N\Sigma$ which is the s-th covariant derivative of the second fundamental form of Σ with respect to g. Then $(\operatorname{grad}_b \omega \lrcorner)^{s+1}(\nabla^s L) = 0$, where $(\operatorname{grad}_b \omega \lrcorner)^{s+1}(\nabla^s L)$ denotes the contraction of the tangential gradient of $\omega|_\Sigma$ into any $s+1$ of the $s+2$ covariant indices of $\nabla^s L$.*

Proof. Recall that the mean curvature vector transforms by

$$e^{2\omega}\, \widehat{H}^{\alpha'} = H^{\alpha'} - \omega^{\alpha'}.$$

Since Σ is minimal for both g and $\widehat{g}$, it follows that $\omega^{\alpha'} = 0$ on Σ.

The conformal transformation law for the Schouten tensor reads

$$\widehat{\mathsf{P}}_{ij} = \mathsf{P}_{ij} - \omega_{ij} + \omega_i \omega_j - \tfrac{1}{2}\omega_k \omega^k g_{ij}, \tag{6.3}$$

where $\omega_{ij} = {}^g\nabla^2_{ij}\omega$. Take the $\alpha\alpha'$ component of (6.3) on Σ. Since $\widehat{\mathsf{P}}_{ij}$ and P_{ij} are both multiples of g_{ij} and $\omega_{\alpha'} = 0$, this component reduces to simply $\omega_{\alpha\alpha'} = 0$. But $\omega_{\alpha\alpha'} = \partial_\alpha \omega_{\alpha'} - \Gamma^i_{\alpha\alpha'}\omega_i$. Now $\partial_\alpha\omega_{\alpha'} = 0$ since $\omega_{\alpha'} = 0$ on Σ, and $\Gamma^\beta_{\alpha\alpha'} = \frac{1}{2} g^{\beta\gamma}\partial_{\alpha'} g_{\alpha\gamma} = -L^\beta_{\alpha\alpha'}$. Therefore $L^\beta_{\alpha\alpha'}\omega_\beta = 0$, which is equivalent to $L^{\alpha'}_{\alpha\beta}\omega^\beta = 0$. This proves the case $s = 0$.

We proceed by induction on s. First consider a case with $s = 1$, which is indicative of the general argument. Write

$$L^{\alpha'}_{\alpha\beta;\gamma}\,\omega^\alpha\omega^\gamma = \left(L^{\alpha'}_{\alpha\beta}\,\omega^\alpha\omega^\gamma\right)_{;\gamma} - L^{\alpha'}_{\alpha\beta}\,\omega^\alpha{}_\gamma\omega^\gamma - L^{\alpha'}_{\alpha\beta}\,\omega^\alpha\omega^\gamma{}_\gamma. \tag{6.4}$$

The first and last terms on the right-hand side vanish by the case $s = 0$. For the middle term, take the $\alpha\gamma$ component of (6.3) and use as above that P_{ij} and $\widehat{\mathsf{P}}_{ij}$ are multiples of g_{ij} to obtain $\omega_{\alpha\gamma} = \omega_\alpha\omega_\gamma + f g_{\alpha\gamma}$ for some function f on Σ. In this equation, $\omega_{\alpha\gamma} = {}^g\nabla^2_{\alpha\gamma}\omega$, whereas in (6.4), $\omega^\alpha{}_\gamma$ refers to the covariant derivative of $\operatorname{grad}_b \omega$ with respect to the induced connection. But these agree since $\omega_{\alpha'} = 0$ on Σ. Substituting $\omega^\alpha{}_\gamma = \omega^\alpha\omega_\gamma + f\delta^\alpha{}_\gamma$ into (6.4) shows that the right-hand side of (6.4) vanishes by the case $s = 0$ as desired. The argument for the other $s = 1$ case, where ω^γ is replaced by ω^β in the left-hand side of (6.4), is easier: after factoring out the covariant derivative index γ, all terms on the right-hand side vanish by the $s = 0$ case.

The general inductive step going from s to $s+1$ is similar. One factors out the last covariant derivative index on $\nabla^{s+1}L$ at the expense of terms involving $\nabla^2\omega$. If the last covariant derivative index is the free index, the induction hypothesis immediately implies the result. If the last covariant derivative index is one of the contracted indices, substitute the tangential component of (6.3) for all the second derivative terms. It is easily seen that all terms vanish by the induction hypothesis. ∎

If g is Einstein with $\operatorname{Ric}(g) = \lambda(n-1)g$, then the canonical ambient metric is given by (6.1), with $g_\rho = (1 + \frac{1}{2}\lambda\rho)^2 g$. If Σ is minimal, then $\overline{Y} = \mathbb{R}_+ \times \Sigma \times (-\varepsilon, 0]$ is a homogeneous minimal extension of $\mathbb{R}_+ \times \Sigma$, which we call the *canonical extension*. More general homogeneous extensions are written as a graph

$$\overline{Y} = \{(t, x^\alpha, u^{\alpha'}, \rho) : t \in \mathbb{R}_+, u^{\alpha'} = u^{\alpha'}(x, \rho)\};$$

the canonical extension is given by $u^{\alpha'} = 0$. For comparison, the graphing function in the Poincaré metric picture is $u^{\alpha'} = u^{\alpha'}(x, -r^2/2)$. In particular, the indeterminacy in $u^{\alpha'}$ appears at order $\rho^{k/2+1}$. Let

$$\tilde{h} = \iota^* \tilde{g}$$

be the induced metric, where $\iota\colon \overline{Y} \to \widetilde{\mathcal{G}}$. We observed above that $\tilde{h}$ has Lorentzian signature for ρ small. This will be explicit in (6.5) below. Write

$$g = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2 g_{\alpha\alpha'}\, dx^\alpha\, du^{\alpha'} + g_{\alpha'\beta'}\, du^{\alpha'} du^{\beta'},$$

where all g_{ij} are functions of (x, u) and $g_{\alpha\alpha'} = 0$ at $u = 0$. Since g is Einstein,

$$\mathrm{g}_\rho = \left(1 + \tfrac{1}{2}\lambda\rho\right)^2 g.$$

Now

$$\iota^* \mathrm{g}_\rho = k_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2 k_{\alpha\infty}\, dx^\alpha\, d\rho + k_{\infty\infty}\, d\rho^2,$$

where

$$\begin{aligned}
k_{\alpha\beta} &= \left(1 + \tfrac{1}{2}\lambda\rho\right)^2 \left(g_{\alpha\beta} + 2 g_{\alpha'(\alpha} u^{\alpha'}{}_{,\beta)} + g_{\alpha'\beta'} u^{\alpha'}{}_{,\alpha} u^{\beta'}{}_{,\beta}\right),\\
k_{\alpha\infty} &= \left(1 + \tfrac{1}{2}\lambda\rho\right)^2 \left(g_{\alpha\alpha'} u^{\alpha'}{}_{,\rho} + g_{\alpha'\beta'} u^{\alpha'}{}_{,\alpha} u^{\beta'}{}_{,\rho}\right),\\
k_{\infty\infty} &= \left(1 + \tfrac{1}{2}\lambda\rho\right)^2 g_{\alpha'\beta'} u^{\alpha'}{}_{,\rho} u^{\beta'}{}_{,\rho}.
\end{aligned}$$

The coefficients $k_{\alpha\beta}$, $k_{\alpha\infty}$, and $k_{\infty\infty}$ are functions of (x, ρ), and the g_{ij} are evaluated at $(x, u(x, \rho))$. It follows that

$$(\tilde{h}_{\mathcal{I}\mathcal{J}}) = \begin{pmatrix} 2\rho & 0 & t \\ 0 & t^2 k_{\alpha\beta} & t^2 k_{\alpha\infty} \\ t & t^2 k_{\alpha\infty} & t^2 k_{\infty\infty} \end{pmatrix} \tag{6.5}$$

in (t, x, ρ) coordinates. We use 0 for the t direction and ∞ for the ρ direction. Indices $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$ run over $0, \alpha, \infty$, i.e., $0, 1, \dots, k, \infty$, and in the following, indices I, J, K run over $0, i, \infty$, i.e., $0, 1, \dots, n, \infty$.

Set

$$e_{\mathcal{I}} = \partial_{\mathcal{I}} + u^{\alpha'}{}_{,\mathcal{I}}\, \partial_{\alpha'},$$

so that $\{e_{\mathcal{I}}\}$ is a basis for $T\overline{Y}$. Let $^\top$ and $^\perp$ denote, respectively, the orthogonal projections with respect to $\tilde{g}$ of $T\widetilde{\mathcal{G}}|_{\overline{Y}}$ onto $T\overline{Y}$ and $N\overline{Y}$. Set

$$e_{\alpha'} = \partial_{\alpha'}^\perp.$$

Let $\overline{\nabla}$ denote the induced connections on $T\overline{Y}$ and $N\overline{Y}$, and let $\overline{\Gamma}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}$ and $\overline{\Gamma}^{\alpha'}_{\mathcal{I}\beta'}$ denote the Christoffel symbols of $\overline{\nabla}$ on $T\overline{Y}$ and $N\overline{Y}$, respectively. Let $\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}}$ denote the components of the second fundamental form of $\overline{Y}$ with respect to the frames $e_{\mathcal{I}}$ and $e_{\alpha'}$. The defining relations for these quantities are

$$(\widetilde{\nabla}_{e_{\mathcal{I}}} e_{\mathcal{J}})^\top = \overline{\Gamma}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}\, e_{\mathcal{K}}, \quad (\widetilde{\nabla}_{e_{\mathcal{I}}} e_{\mathcal{J}})^\perp = \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}}\, e_{\alpha'}, \quad \text{and} \quad (\widetilde{\nabla}_{e_{\mathcal{I}}} e_{\beta'})^\perp = \overline{\Gamma}^{\alpha'}_{\mathcal{I}\beta'}\, e_{\alpha'}.$$

The Christoffel symbols $\widetilde{\Gamma}^K_{IJ}$ of $\widetilde{g}$ with respect to a coordinate frame $\{\partial_t, \partial_i, \partial_\rho\}$ are given by (3.16) of [15]. In the case that g is Einstein and $g_\rho = (1 + \frac{1}{2}\lambda\rho)^2 g$, these become

$$\widetilde{\Gamma}^0_{IJ} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}t\lambda(1+\frac{1}{2}\lambda\rho)g_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\Gamma}^k_{IJ} = \begin{pmatrix} 0 & t^{-1}\delta^k_j & 0 \\ t^{-1}\delta^k_i & \Gamma^k_{ij} & \frac{\lambda}{2}(1+\frac{1}{2}\lambda\rho)^{-1}\delta^k_i \\ 0 & \frac{\lambda}{2}(1+\frac{1}{2}\lambda\rho)^{-1}\delta^k_j & 0 \end{pmatrix}, \tag{6.6}$$

$$\widetilde{\Gamma}^\infty_{IJ} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & (-1+\frac{1}{4}\lambda^2\rho^2)g_{ij} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}.$$

In the middle equation, Γ^k_{ij} denotes the Christoffel symbol of g on M.

If $u^{\alpha'} = 0$, then $e_{\mathcal{I}} = \partial_{\mathcal{I}}$ and $e_{\alpha'} = \partial_{\alpha'}$, so

$$\overline{\Gamma}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}} = \widetilde{\Gamma}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}, \quad \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}} = \widetilde{\Gamma}^{\alpha'}_{\mathcal{I}\mathcal{J}}, \quad \text{and} \quad \overline{\Gamma}^{\beta'}_{\mathcal{I}\alpha'} = \widetilde{\Gamma}^{\beta'}_{\mathcal{I}\alpha'}.$$

Equation (6.6) gives

$$\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Gamma^{\alpha'}_{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L^{\alpha'}_{\alpha\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.7}$$

where $\Gamma^{\alpha'}_{\alpha\beta}$ are the Christoffel symbols for the metric g on M, which is independent of ρ (and t), and $L^{\alpha'}_{\alpha\beta} = \Gamma^{\alpha'}_{\alpha\beta}$ is the second fundamental form of $\Sigma \subset M$.

Equation (6.7) evaluates $\overline{L}$ everywhere on $\overline{Y}$ for the canonical extension $u^{\alpha'} = 0$. Next we derive the form of the covariant derivatives of $\overline{L}$. We begin with components involving a 0 index. These relations depend only on the homogeneity, so they hold for any homogeneous extension $\overline{Y}$ of Σ. Recall that $T = d/d\lambda|_{\lambda=1}\delta_\lambda$. This is a vector field on $\widetilde{\mathcal{G}}$ which is tangent to $\overline{Y}$. The components of T are given by $T^I = t\delta^I_0$. It is shown in Proposition 3.4 of [15] that $\widetilde{\nabla}T = \mathrm{Id}$ is the identity endomorphism of $T\widetilde{\mathcal{G}}$ for any ambient metric of the form (6.1), with g_ρ an arbitrary 1-parameter family of metrics on M.

Proposition 6.4. *The covariant derivatives of the second fundamental form of any homogeneous $\overline{Y}$ satisfy*

$$T^{\mathcal{I}}\,\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_r} = -\sum_{s=1}^{r}\overline{L}^{\alpha'}_{\mathcal{J}\mathcal{K}_s;\mathcal{K}_1\cdots\widehat{\mathcal{K}_s}\cdots\mathcal{K}_r}, \tag{6.8}$$

$$T^{\mathcal{L}}\,\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_s\mathcal{L}\mathcal{K}_{s+1}\cdots\mathcal{K}_r} = -(s+1)\,\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_r} - \sum_{t=s+1}^{r}\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_s\mathcal{K}_t\mathcal{K}_{s+1}\cdots\widehat{\mathcal{K}_t}\cdots\mathcal{K}_r}. \tag{6.9}$$

On the right-hand side of (6.8) and (6.9), the index with the hat is omitted.

Condition (6.8) in the case $r = 0$ is interpreted as the statement $T^{\mathcal{I}} \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}} = 0$, or equivalently, $\overline{L}^{\alpha'}_{\mathcal{J}0} = 0$. Note that the case $s = r$ in (6.9) reduces to

$$T^{\mathcal{L}} \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_r\mathcal{L}} = -(r+1)\, \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_r}. \tag{6.10}$$

Proof. If V is a vector field tangent to $\overline{Y}$, then $\widetilde{\nabla}_V T = V$ is also tangent to $\overline{Y}$. So $\overline{L}(T, V) = 0$. In terms of indices, this is written $T^{\mathcal{I}} \overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}} = 0$. Now differentiate successively using $T^{\mathcal{I}}{}_{;\mathcal{J}} = \delta^{\mathcal{I}}{}_{\mathcal{J}}$ to obtain (6.8).

For (6.9), we first prove the special case (6.10). Recall the identity

$$\nabla_X U = \mathcal{L}_X U + (\nabla X).U \tag{6.11}$$

for a torsion free connection ∇ on a manifold, where X is a vector field and U is a tensor field, $\mathcal{L}_X$ denotes the Lie derivative, and $(\nabla X).U$ denotes the algebraic action of the endomorphism ∇X on U. Apply this on $\widetilde{\mathcal{G}}$, taking $\nabla = \widetilde{\nabla}$, $X = T$, and U equal to an extension of $\overline{\nabla}^r \overline{L}$ to a tensor field on $\widetilde{\mathcal{G}}$ near $\overline{Y}$. Since $\overline{L}$ is homogeneous of degree 0 with respect to the dilations δ_s, we can choose U to have the same homogeneity so that $\mathcal{L}_T U = 0$. The identity endomorphism acts on a tensor covariant in ℓ indices and contravariant in m indices by multiplication by $m - \ell$. So (6.11) becomes $\widetilde{\nabla}_T U = -(r+1)U$. This implies that the restriction of $\widetilde{\nabla}_T U$ to $\overline{Y}$ is also a section of $\otimes^{r+2} T^*\overline{Y} \otimes N\overline{Y}$, so that $\widetilde{\nabla}_T U|_{\overline{Y}} = \overline{\nabla}_T (\overline{\nabla}^r \overline{L})$. This proves (6.10).

To obtain (6.9), replace now r by s in (6.10) and differentiate $r - s$ more times using again $T^{\mathcal{I}}{}_{;\mathcal{J}} = \delta^{\mathcal{I}}{}_{\mathcal{J}}$. ∎

Now we restrict to the case $\overline{Y} = \mathbb{R}_+ \times \Sigma \times (-\varepsilon, 0]$, which is the canonical extension when g is Einstein and Σ is minimal. At each point $p \in \overline{Y}$, we can identify $N_p\overline{Y}$ with $N_{\pi_\Sigma(p)}\Sigma$, where $\pi_\Sigma\colon \overline{Y} \to \Sigma$ is the projection induced by this product decomposition. Construct tensors on Σ from the covariant derivatives of $\overline{L}$ as follows. Choose an order $r \geq 0$ of covariant differentiation. Divide the set of symbols $\mathcal{I}\mathcal{J}\mathcal{K}_1\cdots\mathcal{K}_r$ into three disjoint subsets $\mathcal{S}_0$, $\mathcal{S}_\Sigma$ and $\mathcal{S}_\infty$ of cardinalities s_0, s_Σ and s_∞, respectively. Set the indices in $\mathcal{S}_0$ equal to 0, those in $\mathcal{S}_\infty$ equal to ∞, and let those in $\mathcal{S}_\Sigma$ correspond to Σ in the decomposition $\overline{Y} = \mathbb{R}_+ \times \Sigma \times (-\varepsilon, 0]$. (In local coordinates, the indices in $\mathcal{S}_\Sigma$ vary between 1 and k.) Evaluate the resulting component $\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_r}$ at $\rho = 0$ and $t = 1$. This defines a tensor on Σ which is a section of $\otimes^{s_\Sigma} T^*\Sigma \otimes N\Sigma$, which we denote by $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}$.

Proposition 6.5. *The tensor $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}$ is a linear combination of tensor products of an iterated covariant derivative of the second fundamental form L of Σ with some power of the induced metric $g|_{T\Sigma}$, with some ordering of the covariant indices.*

We provide a clarification of the statement. For any order $s \geq 0$ of covariant differentiation and any $m \geq 0$, and any ordering of the indices, we can form a tensor on Σ of the form $\nabla^s L \otimes (\otimes^m g|_{T\Sigma})$. The statement is that for any $\mathcal{S}_0$, $\mathcal{S}_\Sigma$, $\mathcal{S}_\infty$, the tensor $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}$ is a linear combination of tensors of this form. Since the ranks must agree, only terms for which $s + 2 + 2m = s_\Sigma$ can appear in the linear combination.

Corollary 6.6. $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty} = 0$ *if* $s_\Sigma = 0$ *or* 1.

Proof. $s_\Sigma = s + 2 + 2m \geq 2$. ∎

Proof of Proposition 6.5. We shall prove a stronger statement: denote by $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}(\rho)$ the 1-parameter family of tensors on Σ constructed exactly the same way, but not restricting to $\rho = 0$. So $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty} = \overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}(0)$. We prove by induction on $r \geq 0$ that for each choice of $\mathcal{S}_0$, $\mathcal{S}_\Sigma$, $\mathcal{S}_\infty$ with $s_0 + s_\Sigma + s_\infty = r + 2$, the tensor $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}(\rho)$ is a linear combination of tensors of the form $f(\rho)\nabla^s L \otimes (\otimes^m g|_{T\Sigma})$ with some ordering of the indices of each term, where $f(\rho)$ is a smooth function near $\rho = 0$ which can vary from term to term. The desired statement follows upon setting $\rho = 0$.

The case $r = 0$ is clear from (6.7): the only nonzero component of $\overline{L}$ is $\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J}} = L^{\alpha'}_{\mathcal{I}\mathcal{J}}$. For this component, $s = m = 0$ and $f(\rho) = 1$.

Assume the induction statement is true for r. Consider a tensor $\overline{L}^{(r+1)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}(\rho)$, whose components we write $\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}\mathcal{I}}$. Proposition 6.4 shows that a zero index can be removed at the expense of commuting the remaining indices. So we can assume that $\mathcal{S}_0 = \emptyset$. Write

$$\begin{aligned}\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}\mathcal{I}} = \partial_{\mathcal{I}}\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}} - \overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_1}\overline{L}^{\alpha'}_{\mathcal{J}\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}} - \overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_2}\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{J};\mathcal{K}_3\cdots\mathcal{K}_{r+2}} \\ - \cdots - \overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_{r+2}}\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+1}\mathcal{J}} + \overline{\Gamma}^{\alpha'}_{\mathcal{I}\beta'}\overline{L}^{\beta'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}}.\end{aligned}$$

As noted above, for the Christoffel symbols we have

$$\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}} = \widetilde{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}} \quad \text{and} \quad \overline{\Gamma}^{\alpha'}_{\mathcal{I}\beta'} = \widetilde{\Gamma}^{\alpha'}_{\mathcal{I}\beta'},$$

and the $\widetilde{\Gamma}^K_{IJ}$ are given by (6.6).

First consider the case $\mathcal{I} = \infty$, so $\partial_{\mathcal{I}} = \partial_\rho$. For the first term on the right-hand side, apply the induction hypothesis to $\overline{L}^{\alpha'}_{\mathcal{K}_1\mathcal{K}_2;\mathcal{K}_3\cdots\mathcal{K}_{r+2}}$. The ∂_ρ just hits the function $f(\rho)$ in each term in the linear combination, so the first term on the right-hand side has the desired form. For the last term, substitute

$$\overline{\Gamma}^{\alpha'}_{\infty\beta'} = \tfrac{\lambda}{2}(1 + \tfrac{1}{2}\lambda\rho)^{-1}\delta^{\alpha'}_{\beta'}$$

and apply the induction hypothesis to see that it also has the desired form. Since $\widetilde{\Gamma}^{\mathcal{J}}_{\infty\infty} = 0$ for all $\mathcal{J}$, the terms with a factor $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_i}$ for which $\mathcal{K}_i = \infty$ all vanish. The only other terms involve $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_i}$ with $\mathcal{K}_i \in \mathcal{S}_\Sigma$. For these terms, $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_i} = 0$ unless $1 \leq \mathcal{J} \leq k$, in which case

$$\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\mathcal{K}_i} = \tfrac{\lambda}{2}(1 + \tfrac{1}{2}\lambda\rho)^{-1}\delta^{\mathcal{J}}_{\mathcal{K}_i}.$$

So again, the induction hypothesis implies these terms have the desired form.

Finally, consider the case $\mathcal{I} \in \mathcal{S}_\Sigma$. Recall that $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\infty} = 0$ unless $1 \leq \mathcal{J} \leq k$, in which case

$$\overline{\Gamma}^{\mathcal{J}}_{\mathcal{I}\infty} = \tfrac{\lambda}{2}(1 + \tfrac{1}{2}\lambda\rho)^{-1}\delta^{\mathcal{J}}_{\mathcal{I}}.$$

So the terms on the right-hand side with a factor $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{J}\mathcal{K}_i}$ with $\mathcal{K}_i = \infty$ all are of the desired form. If $\mathcal{K}_i \in \mathcal{S}_\Sigma$, then $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{J}\mathcal{K}_i}$ is nonvanishing for all $\mathcal{J}$. Since

$$\overline{\Gamma}^{0}_{\mathcal{J}\mathcal{K}} = -\tfrac{1}{2}t\lambda(1+\tfrac{1}{2}\lambda\rho)g_{\mathcal{J}\mathcal{K}},$$

we can apply the induction hypothesis to conclude that the terms with $\mathcal{J} = 0$ are of the desired form (recall that $t = 1$ in the definition of $\overline{L}^{(r+1)}_{\mathcal{S}_0,\mathcal{S}_\Sigma;\mathcal{S}_\infty}(\rho)$). Since

$$\overline{\Gamma}^{\infty}_{\mathcal{J}\mathcal{K}} = (-1+\tfrac{1}{4}\lambda^2\rho^2)g_{\mathcal{J}\mathcal{K}},$$

we can likewise apply the induction hypothesis to obtain the desired form for these terms. This leaves the terms with $\mathcal{K}_i \in \mathcal{S}_\Sigma$ and $1 \le \mathcal{J} \le k$. In this case, $\overline{\Gamma}^{\mathcal{J}}_{\mathcal{J}\mathcal{K}} = \Gamma^{\mathcal{J}}_{\mathcal{J}\mathcal{K}}$ is the Christoffel symbol for g on M. Applying the induction hypothesis again, all of these terms combine with the first and last term to produce the covariant derivative on M applied to each of the terms $\nabla^s L \otimes (\otimes^m g|_{T\Sigma})$ in the linear combination. This gives further terms of the desired form. ∎

Next consider extensions $\overline{Y}$ defined by nonzero $u^{\alpha'}$.

Proposition 6.7. *Let $k \ge 2$ be even. Let g be Einstein and let $\widetilde{g}$ be given by (6.1) with $g_\rho = (1+\frac{1}{2}\lambda\rho)^2 g$. If $\Sigma \subset M$ is minimal, then $\overline{Y} = \mathbb{R}_+ \times \Sigma \times (-\varepsilon, 0]$ is to infinite order the unique homogeneous minimal extension of Σ satisfying*

$$\overline{L}^{\alpha'}_{\infty\infty;\underbrace{\infty\cdots\infty}_{k/2-1}}\Big|_{\rho=0} = 0. \tag{6.12}$$

Proof. First calculate the leading term in $\overline{L}^{\alpha'}_{\infty\infty}$, which recall is defined by

$$(\widetilde{\nabla}_{e_\infty}e_\infty)^\perp = \overline{L}^{\alpha'}_{\infty\infty}\,e_{\alpha'}.$$

Recalling from (6.6) that $\widetilde{\nabla}_{\partial_\rho}\partial_\rho = 0$, we have

$$\begin{aligned}
\widetilde{\nabla}_{e_\infty}e_\infty &= \widetilde{\nabla}_{\partial_\rho + u^{\alpha'}{}_{,\rho}\partial_{\alpha'}}e_\infty = \widetilde{\nabla}_{\partial_\rho}e_\infty + u^{\alpha'}{}_{,\rho}\widetilde{\nabla}_{\partial_{\alpha'}}e_\infty \\
&= \widetilde{\nabla}_{\partial_\rho}(\partial_\rho + u^{\alpha'}{}_{,\rho}\partial_{\alpha'}) + u^{\alpha'}{}_{,\rho}\widetilde{\nabla}_{\partial_{\alpha'}}(\partial_\rho + u^{\beta'}{}_{,\rho}\partial_{\beta'}) \\
&= (\partial_\rho^2 u^{\alpha'})\partial_{\alpha'} + 2u^{\alpha'}{}_{,\rho}\widetilde{\nabla}_{\partial_\rho}\partial_{\alpha'} + u^{\alpha'}{}_{,\rho}u^{\beta'}{}_{,\rho\alpha'}\partial_{\beta'} + u^{\alpha'}{}_{,\rho}u^{\beta'}{}_{,\rho}\widetilde{\nabla}_{\partial_{\alpha'}}\partial_{\beta'}.
\end{aligned}$$

Since $e_{\alpha'} = (\partial_{\alpha'})^\perp$, we obtain

$$\overline{L}^{\alpha'}_{\infty\infty} = \partial_\rho^2 u^{\alpha'} + \text{LOTS},$$

where LOTS consists of terms involving fewer ρ-derivatives of $u^{\alpha'}$.

Now successive differentiation shows that

$$\overline{L}^{\alpha'}_{\infty\infty;\underbrace{\infty\cdots\infty}_{m}} = \partial_\rho^{m+2}u^{\alpha'} + \cdots$$

for $m \ge 0$.

In particular, this shows that $\partial_\rho^{k/2+1} u^{\alpha'}|_{\rho=0}$ is uniquely determined by

$$\overline{L}^{\alpha'}_{\infty\infty;\underbrace{\infty\cdots\infty}_{k/2-1}}|_{\rho=0}$$

and the values of $\partial_\rho^j u^{\alpha'}\big|_{\rho=0}$ for $j \leq k/2$. We know that any minimal extension of Σ satisfies $u^{\alpha'} = O(\rho^{k/2+1})$, and $u^{\alpha'}$ vanishes to infinite order if and only if $u^{\alpha'} = O(\rho^{k/2+2})$. Corollary 6.6 shows that the choice $u^{\alpha'} = 0$ makes

$$\overline{L}^{\alpha'}_{\infty\infty;\underbrace{\infty\cdots\infty}_{k/2-1}}|_{\rho=0} = 0. \qquad \blacksquare$$

Proof of Theorem 4.10. We have already observed that this follows from the uniqueness of the even minimal extension when k is odd. So we can assume that $k \geq 2$ is even.

We prove the analogous statement for the diffeomorphism relating the ambient metrics of g and $\widehat{g}$.

Let $\widehat{\widetilde{g}}$ be the ambient metric on $\widehat{\widetilde{\mathcal{G}}} = \mathbb{R}_+ \times M \times (-\varepsilon, \varepsilon)$ obtained by replacing $\widetilde{g}$ by $\widehat{\widetilde{g}}$ and g_ρ by

$$\widehat{\mathrm{g}}_\rho = (1 + \tfrac{1}{2}\widehat{\lambda}\rho)^2 \widehat{g}$$

in (6.1). Let χ be a homogeneous diffeomorphism mapping $\widehat{\widetilde{\mathcal{G}}}$ to $\widetilde{\mathcal{G}}$ which pulls back $\widetilde{g}$ to $\widehat{\widetilde{g}}$ to infinite order at $\rho = 0$ and restricts to the identity on $\mathcal{G}$. Then χ is uniquely determined to infinite order. Let $\overline{Y} \subset \widetilde{\mathcal{G}}$ and $\widehat{\overline{Y}} \subset \widehat{\widetilde{\mathcal{G}}}$ be the canonical extensions of Σ with respect to g and $\widehat{g}$, respectively. Proposition 6.7 implies that $\chi^{-1}(\overline{Y}) = \widehat{\overline{Y}}$ to infinite order if and only if $\chi^{-1}(\overline{Y})$ satisfies (6.12). The condition (6.12) transforms tensorially under the Jacobian of χ. This Jacobian is identified at $\rho = 0$ in terms of ω in (6.8)–(6.10) of [15]. It follows that $\chi^{-1}(\overline{Y})$ satisfies (6.12) if and only if

$$\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_{k/2-1}}\, p^{\mathcal{I}}{}_\infty\, p^{\mathcal{J}}{}_\infty\, p^{\mathcal{K}_1}{}_\infty \cdots p^{\mathcal{K}_{k/2-1}}{}_\infty = 0 \tag{6.13}$$

at $\rho = 0$, where

$$p^I{}_J = \begin{pmatrix} 1 & \omega_j & -\frac{1}{2}\omega_k\omega^k \\ 0 & \delta^i{}_j & -\omega^i \\ 0 & 0 & 1 \end{pmatrix}$$

and $\overline{L}^{\alpha'}_{\mathcal{I}\mathcal{J};\mathcal{K}_1\cdots\mathcal{K}_{k/2-1}}$ refers to the covariant derivative of $\overline{L}$ on $\overline{Y}$. (See the proof of Proposition 6.5 of [15], and recall that $\omega^{\alpha'} = 0$ on Σ.)

Expanding out (6.13), one obtains a linear combination with smooth coefficients of contractions of $\operatorname{grad}_b \omega$ with tensors $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}$ in which each index in $\mathcal{S}_\Sigma$ is contracted against a factor of $\operatorname{grad}_b \omega$. Write the tensor $\overline{L}^{(r)}_{\mathcal{S}_0,\mathcal{S}_\Sigma,\mathcal{S}_\infty}$ as a linear combination of tensors of the form $\nabla^s L \otimes (\otimes^m g|_{T\Sigma})$ as in Proposition 6.5. Each index of each tensor $\nabla^s L$ which appears is contracted against a factor of $\operatorname{grad}_b \omega$. Lemma 6.3 implies that all the terms vanish. $\blacksquare$

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