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Large cliques in extremal incidence configurations

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Abstract. Let $P \subset \mathbb{R}^2$ be a Katz–Tao (δ, s)-set, and let $\mathcal{L}$ be a Katz–Tao (δ, t)-set of lines in $\mathbb{R}^2$. A recent result of Fu and Ren gives a sharp upper bound for the δ-covering number of the set of incidences $\mathcal{I}(P, \mathcal{L}) = \{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}$. In fact, for $s, t \in (0, 1]$,

$$|\mathcal{I}(P, \mathcal{L})|_\delta \lesssim_\varepsilon \delta^{-\varepsilon - f(s,t)}, \quad \varepsilon > 0,$$

where $f(s,t) = (s^2 + st + t^2)/(s + t)$. For $s, t \in (0, 1]$, we characterise the near-extremal configurations $P \times \mathcal{L}$ of this inequality: we show that if $|\mathcal{I}(P, \mathcal{L})|_\delta \approx \delta^{-f(s,t)}$, then $P \times \mathcal{L}$ contains "cliques" $P' \times \mathcal{L}'$ satisfying $|\mathcal{I}(P', \mathcal{L}')|_\delta \approx |P'|_\delta |\mathcal{L}'|_\delta$,

$$|P'|_\delta \approx \delta^{-s^2/(s+t)} \quad \text{and} \quad |\mathcal{L}'|_\delta \approx \delta^{-t^2/(s+t)}.$$

1. Introduction

This paper studies the δ-covering number of incidences between sets of points and lines in $\mathbb{R}^2$. Let $P \subset \mathbb{R}^2$ and $\mathcal{L} \subset \mathcal{A}(2)$, where $\mathcal{A}(2)$ is the space of all (affine) lines in $\mathbb{R}^2$. The *incidences* between P and $\mathcal{L}$ are the pairs

$$\mathcal{I}(P, \mathcal{L}) = \{(p, \ell) \in P \times \mathcal{L} : p \in \ell\}.$$

We equip $\mathbb{R}^2$ with the Euclidean norm $|\cdot|$, and $\mathcal{A}(2)$ with the metric

$$d_{\mathcal{A}(2)}(\ell_1, \ell_2) = \|\pi_{L_1} - \pi_{L_2}\| + |a_1 - a_2|,$$

whenever $\ell_j = L_j + a_j$, and L_j is the 1-dimensional subspace parallel to ℓ_j. If $P \subset \mathbb{R}^2$, $\mathcal{L} \subset \mathcal{A}(2)$, and $\delta > 0$, the notations $|P|_\delta$ and $|\mathcal{L}|_\delta$ refer to the δ-covering numbers relative to the Euclidean and $d_{\mathcal{A}(2)}$-metrics, respectively. For $\mathcal{I} \subset \mathbb{R}^2 \times \mathcal{A}(2)$, the notation $|\mathcal{I}|_\delta$ refers to the δ-covering number in the metric

$$d((x, \ell), (x', \ell')) = \max\{|x - x'|, d_{\mathcal{A}(2)}(\ell, \ell')\}.$$

The following notion is central to this paper.

Definition 1.1 ((δ, θ)-clique). For $\delta \in 2^{-\mathbb{N}}$ and $\theta \in [0, 1]$, a (δ, θ)-*clique* is a pair $P \times \mathcal{L} \subset \mathbb{R}^2 \times \mathcal{A}(2)$ with $|\mathcal{I}(P, \mathcal{L})|_\delta \geq \theta |P|_\delta |\mathcal{L}|_\delta$.

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The main purpose will be, roughly speaking, to show that if $|\mathcal{I}(P,\mathcal{L})|_\delta$ is "extremal", then $P \times \mathcal{L}$ needs to contain large (δ,θ)-sub-cliques with $\theta \approx 1$. What is meant by "extremal"? A typical result in (δ-discretised) incidence geometry gives an upper bound for $|\mathcal{I}(P,\mathcal{L})|_\delta$, provided that P and $\mathcal{L}$ satisfy some non-concentration conditions. Then, an extremal configuration is a pair $P \times \mathcal{L}$ which satisfies these non-concentration conditions, and such that $|\mathcal{I}(P,\mathcal{L})|_\delta$ (nearly) realises the upper bound. In particular, the definition of "extremal" depends on the choice of non-concentration conditions.

We focus on the following definition, originally introduced by Katz and Tao [9].

Definition 1.2 (Katz–Tao (δ,s,C)-set). Let $P \subset \mathbb{R}^d$ be a bounded set, $d \geq 2$. Let $\delta \in (0,1]$, $0 \leq s \leq d$ and $C > 0$. We say that P is a *Katz–Tao* (δ,s,C)*-set* if

$$|P \cap B(x,r)|_\delta \leq C\left(\frac{r}{\delta}\right)^s, \quad x \in \mathbb{R}^2, \delta \leq r \leq 1. \tag{1.1}$$

If $\mathcal{P} \subset \mathcal{D}_\delta(\mathbb{R}^2)$, we say that $\mathcal{P}$ is a Katz–Tao (δ,s,C)-set if $P := \bigcup \mathcal{P}$ satisfies (1.1).

A line family $\mathcal{L} \subset \mathcal{A}(2)$ is called a *Katz–Tao* (δ,s,C)*-set* if

$$|\mathcal{L} \cap B_{\mathcal{A}(2)}(\ell,r)|_\delta \leq C\left(\frac{r}{\delta}\right)^s, \quad \ell \in \mathcal{A}(2), \delta \leq r \leq 1,$$

Here, $B_{\mathcal{A}(2)}(\ell,r)$ refers to a ball in the metric $d_{\mathcal{A}(2)}$.

A (δ,s,C)-set (of points or lines) is called a (δ,s)-set if the value of the constant $C > 0$ is irrelevant.

Remark 1.3. Note that a Katz–Tao (δ,s)-set of points or lines may well be infinite. A reasonable intuition is that P is a finite union of δ-discs or δ-squares, whereas $\mathcal{L}$ is the collection of lines foliating a finite union of δ-tubes or *dyadic* δ*-tubes* (see Definition 2.4).

If $P \subset \mathbb{R}^2$ is a Katz–Tao (δ,s)-set, and $\mathcal{L} \subset \mathcal{A}(2)$ is a Katz–Tao (δ,t)-set of lines, the sharp upper bound for $|\mathcal{I}(P,\mathcal{L})|_\delta$ was recently established by Fu and Ren [5]:

Theorem 1.4. *Let $s,t \in (0,1]$ and $K_P, K_{\mathcal{L}} \geq 1$. For every $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, K_P, K_{\mathcal{L}})$ such that the following holds. Assume $P \subset [0,1]^2$ is a Katz–Tao (δ,s,K_P)-set and $\mathcal{L} \subset \mathcal{A}(2)$ is a Katz–Tao $(\delta,t,K_{\mathcal{L}})$- set. Then*

$$|\mathcal{I}(P,\mathcal{L})|_\delta \leq C\delta^{-\varepsilon - f(s,t)},$$

where $f(s,t) = (s^2 + st + t^2)/(s+t)$. Moreover, this bound is sharp up to $C\delta^{-\varepsilon}$.

In fact, Fu and Ren established a sharp bound for all $s \in (0,2]$ and $t \in (0,2]$ (the definition of f is then piece-wise, depending on the range of s and t). In this paper, we restrict attention to the cases $s,t \in (0,1]$.

Remark 1.5. The result of Fu and Ren was originally stated slightly differently. In [5], the set $P = \bigcup \mathcal{B}$ is a finite union of δ-discs and $\mathcal{L} = \bigcup \mathcal{T}$ is a finite union of δ-tubes. The incidences are defined in [5] as $\bar{\mathcal{I}}(\mathcal{B},\mathcal{T}) = \{(B,T) \in \mathcal{B} \times \mathcal{T} : B \cap T \neq \emptyset\}$. Under the hypotheses of Theorem 1.4, the authors established the inequality $|\bar{\mathcal{I}}(\mathcal{B},\mathcal{T})| \lesssim_\varepsilon \delta^{-\varepsilon - f(s,t)}$. We will check in Remark 2.7 that Theorem 1.4 follows, as stated, from its original version in [5].

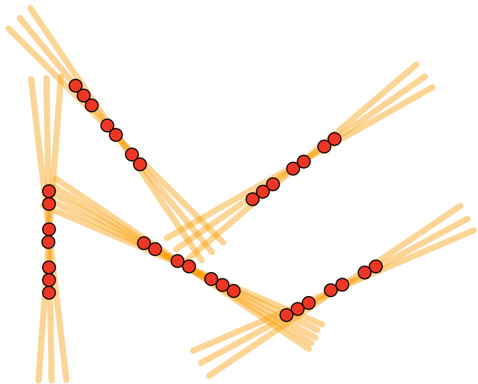

Figure 1. A pair $P \cap \mathcal{L}$ admitting a decomposition into large δ-sub-cliques.

Remark 1.6. It took a few attempts before we ended up studying the quantity $|\mathcal{I}(P, \mathcal{L})|_\delta$. Other, perhaps more obvious, alternatives would have been to consider incidences between δ-balls and ordinary tubes (as in [5]), or dyadic δ-squares and dyadic δ-tubes (as in, e.g., [11]). However, if our main result, Theorem 1.7, was stated for any one of these "standard" choices, it seemed hard to deduce the other "standard" choices as corollaries.

We will not explain the problem in detail, but it has to do with the following phenomenon. Assume that $\mathcal{B}$ and $\mathcal{T}$ are δ-neighbourhoods of points and lines in some metric of $\mathbb{R}^2$. Then, the cardinality $|\bar{\mathcal{I}}(\mathcal{B}, \mathcal{T})| = |\{(B, T) \in \mathcal{B} \times \mathcal{T} : B \cap T \neq \emptyset\}|$ is far from being (roughly) invariant under bi-Lipschitz transformations of that metric. For example, there are configurations where a large family of Euclidean δ-tubes narrowly avoids a large family of Euclidean δ-balls, but the collinear (2δ)-tubes already hit all the concentric (2δ)-balls. In contrast, the δ-covering number $|\mathcal{I}(P, \mathcal{L})|_\delta$ only changes by a constant if the metrics of $\mathbb{R}^2$ and $\mathcal{A}(2)$ are replaced by bi-Lipschitz equivalent ones.

Theorem 1.7 is formulated in terms of $|\mathcal{I}(P, \mathcal{L})|_\delta$ to make it more robust. Now it actually implies other (possibly more) "standard" versions as corollaries. We mention one concrete example in Remark 1.9, but omit the straightforward details.

The bound in Theorem 1.4 is sharp, but weaker than the *Szemerédi–Trotter* bound on incidences between families of points and lines. If $P \subset \mathbb{R}^2$ is a finite set, and $\mathcal{L}$ is a finite set of lines, Szemerédi and Trotter [20] established in 1983 the following:

$$|\mathcal{I}(P, \mathcal{L})| \lesssim |P|^{2/3}|\mathcal{L}|^{2/3} + |P| + |\mathcal{L}|. \tag{1.2}$$

For example, if $s = t \in (0, 1]$, then Theorem 1.4 gives $|\mathcal{I}(P, \mathcal{L})|_\delta \lesssim_\varepsilon \delta^{-3s/2-\varepsilon}$, whereas a formal application of (1.2) would predict that $|\mathcal{I}(P, \mathcal{L})|_\delta \lesssim \delta^{-4s/3}$.

In the sharp examples provided by Fu and Ren to Theorem 1.4, the large number of incidences is due to many large $(\delta, 1)$-cliques. In fact, in these examples both P and $\mathcal{L}$ are partitioned as $P = P_1 \cup \dots \cup P_N$ and $\mathcal{L} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_N$ in such a way that $P_j \times \mathcal{L}_j$ is a $(\delta, 1)$-clique for all $1 \le j \le N$, see Figure 1 for an illustration.

The Katz–Tao conditions impose the following restriction on $(\delta, 1)$-cliques: if $P \times \mathcal{L}$ is a $(\delta, 1)$-clique, where P is a Katz–Tao (δ, s)-set and $\mathcal{L} \subset \mathcal{A}(2)$ is a Katz–Tao (δ, t)-set, then $|P|_\delta^t |\mathcal{L}|_\delta^s \lesssim \delta^{-st}$. This follows from Proposition 1.11 below. Optimising under this constraint, one finds that the most δ-incidences are generated by a $(\delta, 1)$-clique decomposition where

$$|P_j|_\delta \equiv \delta^{-s^2/(s+t)} \quad \text{and} \quad |\mathcal{L}_j|_\delta \equiv \delta^{-t^2/(s+t)}. \tag{1.3}$$

Indeed, the number of incidences in such a configuration matches the upper bound in Theorem 1.4, up to the constant $C\delta^{-\varepsilon}$.

Our main result shows that any (near-)extremal configuration for Theorem 1.4 must contain cliques of (nearly) the size (1.3):

Theorem 1.7. *For every $u \in (0, 1]$ and $s, t \in (0, 1]$, there exist $\delta_0 = \delta_0(s, t, u) > 0$ and $\varepsilon = \varepsilon(s,t,u) > 0$ such that the following holds for any $\delta \in (0, \delta_0]$. Write $f(s,t) = \frac{s^2+st+t^2}{s+t}$. Let $P \subset [0,1]^2$ be a Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set, and let $\mathcal{L} \subset \mathcal{A}(2)$ be a Katz–Tao $(\delta, t, \delta^{-\varepsilon})$-set. If*

$$|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - f(s,t)}, \tag{1.4}$$

then there exists a (δ, δ^u)-clique $P' \times \mathcal{L}' \subset P \times \mathcal{L}$ with

$$|P'|_\delta \geq \delta^{u-s^2/(s+t)} \quad \textit{and} \quad |\mathcal{L}'|_\delta \geq \delta^{u-t^2/(s+t)}. \tag{1.5}$$

Remark 1.8. The proof of Theorem 1.7 (see (3.22)) yields a (δ, δ^u)-clique $P' \times \mathcal{L}'$ of the form $P' = P \cap Q$ and $\mathcal{L}' = \mathcal{L} \cap \mathcal{T}'$, where Q is a dyadic square of some side-length $\Delta \in [\delta, 1]$, and $\mathcal{T}' \subset \mathcal{T}^\delta$ is a family of dyadic δ-tubes (see Definition 2.4). Here $\mathcal{L} \cap \mathcal{T}'$ refers to the set of lines in $\mathcal{L}$ contained in at least one element of $\mathcal{T}'$.

Remark 1.9. Let $s, t \in (0, 1]$ and $u > 0$. In the notation of Remark 1.5, assume that $\mathcal{B}$ is a Katz–Tao (δ, s)-set of δ-discs, and $\mathcal{T}$ is a Katz–Tao (δ, t)-set of δ-tubes satisfying $|\bar{\mathcal{I}}(\mathcal{B}, \mathcal{L})| \geq \delta^{\varepsilon - f(s,t)}$. Then, if $\delta > 0$ is sufficiently small in terms of s, t and u, there exist $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{T}' \subset \mathcal{T}$ such that $|\bar{\mathcal{I}}(\mathcal{B}', \mathcal{T}')| \geq \delta^u |\mathcal{B}'||\mathcal{T}'|$, and $\mathcal{B}'$ and $\mathcal{T}'$ satisfy (1.5). This follows easily from Theorem 1.7 applied to $P = \bigcup \mathcal{B}$ and $\mathcal{L} = \{\ell \in \mathcal{A}(2) : \ell \subset T \text{ for some } T \in \mathcal{T}\}$.

While Theorem 1.7 only states the existence of a single (δ, δ^u)-clique, a formal "exhaustion argument" shows that there are many (δ, δ^u)-cliques: they are indeed responsible for a major part of the incidences.

Corollary 1.10. *Under the hypotheses of Theorem 1.7, there exists a list*

$$(P_1 \times \mathcal{L}_1), \ldots, (P_n \times \mathcal{L}_n) \subset P \times \mathcal{L}$$

of (δ, δ^u)-cliques satisfying condition (1.5), with the sets $\mathcal{D}_\delta(P_j)$ disjoint, and such that $\sum_j |\mathcal{I}(P_j, \mathcal{L}_j)|_\delta \geq \delta^{u - f(s,t)}$.

Does Corollary 1.10 imply that the only configurations $P \times \mathcal{L}$ satisfying (1.4) must contain a sub-configuration as in Figure 1? In other words, do (δ, δ^u)-cliques resemble the "sheaves" from Figure 1? The answer is affirmative, up to passing to further subsets. This follows from the next proposition, combined with the subsequent remark.

Proposition 1.11. *There exists an absolute constant $C \geq 1$ such that the following holds. Let $P \times \mathcal{L}$ be a (δ, θ)-clique. Then, there exists a rectangle $R \subset \mathbb{R}^2$ of dimensions $C(\delta \times \Delta)$, where $\Delta \in [\delta, 2]$, such that*

$$|P \cap R|_\delta \gtrsim \theta^2 |P|_\delta \quad \textit{and} \quad |\{\ell \in \mathcal{L} : R \subset [\ell]_{C\delta}\}|_\delta \gtrapprox \theta^4 |\mathcal{L}|_\delta. \tag{1.6}$$

Here, $[\ell]_{C\delta}$ is the $C\delta$-neighbourhood of ℓ. In particular, if P is a Katz–Tao (δ, s)-set and $\mathcal{L}$ is a Katz–Tao (δ, t)-set, then $|P|_\delta^t |\mathcal{L}|_\delta^s \lessapprox \theta^{-6} \delta^{-st}$.

In Proposition 1.11, the notation $A \lessapprox B$ means that $A \leq C(\log(1/\delta))^C B$ for some absolute constant $C \geq 1$. This notation will serve various purposes in the paper, and we will always define it separately.

Remark 1.12. When Proposition 1.11 is applied to the (δ, δ^u)-clique $P' \times \mathcal{L}'$ in Theorem 1.7, the diameter Δ of the rectangle R is (almost) uniquely determined. In fact,

$$\operatorname{diam}(R) \sim \Delta \approx \delta^{t/(s+t)}. \tag{1.7}$$

The "$\approx$" and "$\lessapprox$" notations in this remark are allowed to hide factors of the form $\delta^{-C\varepsilon}$ and δ^{-Cu}. To verify (1.7), we first deduce from the lower bound $|P'|_\delta \gtrapprox \delta^{-s^2/(s+t)}$, combined with the Katz–Tao (δ, s)-set condition of P, that

$$(\Delta/\delta)^s \gtrapprox |P' \cap R|_\delta \gtrapprox |P'|_\delta \gtrapprox \delta^{-s^2/(s+t)} \implies \Delta \gtrapprox \delta^{1-s/(s+t)} = \delta^{t/(s+t)}.$$

Second, all the lines $\ell \in \mathcal{L}'$ with $R \subset [\ell]_{C\delta}$ are themselves contained in a $d_{\mathcal{A}(2)}$-ball of radius $\sim (\delta/\Delta)$. Consequently, now using the lower bound $|\mathcal{L}'|_\delta \gtrapprox \delta^{-t^2/(s+t)}$ and the Katz–Tao (δ, t)-set condition of $\mathcal{L}' \subset \mathcal{L}$,

$$\delta^{-t^2/(s+t)} \lessapprox |\mathcal{L}'|_\delta \lessapprox |\{\ell \in \mathcal{L}' : R \subset [\ell]_{C\delta}\}|_\delta \lessapprox \Delta^{-t} \implies \Delta \lessapprox \delta^{t/(s+t)}.$$

Combining these inequalities gives (1.7). Therefore, combined with Proposition 1.11, Theorem 1.7 provides the following geometric information: there exists a rectangle $R \subset \mathbb{R}^2$ of dimensions $\approx (\delta \times \delta^{t/(s+t)})$ such that

$$|P \cap R|_\delta \gtrapprox \delta^{-s^2/(s+t)} \quad \text{and} \quad |\{\ell \in \mathcal{L} : R \subset [\ell]_{C\delta}\}|_\delta \gtrapprox \delta^{-t^2/(s+t)}.$$

The δ-neighbourhoods of the sets $P \cap R$ and $\{\ell : R \subset [\ell]_{C\delta}\}$ are the "sheaves" in Figure 1.

1.1. Related work and further problems

Theorem 1.7 and Corollary 1.10 give a characterisation of the extremal configurations in Fu and Ren's Theorem 1.4 when $s, t \in (0, 1]$. It is a natural – and difficult – open problem to study the structure of extremal configurations in the original Szemerédi–Trotter incidence bound (1.2). Of course, any answers (and methods) in this problem will be completely different from the one provided by Theorem 1.7: for example, if $P \times \mathcal{L}$ is a $(0, 1)$-clique of points and lines, then $\min\{|\mathcal{L}|, |P|\} = 1$. For recent work on this discrete variant of the problem, see the papers of Solymosi [19], Sheffer–Silier [16], Katz–Silier [8], and Currier–Solymosi–Yu [3].

In the δ-discretised setting, we are not aware of previous structural results analogous to Theorem 1.7. On the other hand, Theorem 1.7 is far from exhaustive. For example, it only covers the range $s, t \in (0, 1]$ of the Fu–Ren incidence theorem. The reason is that the known sharpness examples in other ranges of s and t have rather different structure than the "unions of cliques" shown in Figure 1. We are not even sure what to expect if $\max\{s, t\} > 1$, and certainly the required proof techniques would be different from ours.

Another further direction is to relax or change the non-concentration conditions we impose in Theorem 1.7. This will typically change the sharp upper bounds for $|\mathcal{I}(P, \mathcal{L})|_\delta$, and therefore the problem of characterising the extremal configurations. However, this is not always the case. For example, if $P \subset [0, 1]^2$ is a Katz–Tao $(\delta, 1)$-set, and $\mathcal{L} \subset \mathcal{A}(2)$ is *any* set of lines with $|\mathcal{L}|_\delta \le \delta^{-1}$, then $|\mathcal{I}(P, \mathcal{L})|_\delta \lessapprox \delta^{-3/2}$. This folklore result (see, e.g., Proposition 2.13 in [11]) matches Fu and Ren's bound in the case $s = t = 1$, and the Katz–Tao $(\delta, 1)$-set condition on $\mathcal{L}$ is not needed. So, the following question makes sense: assume that $P \subset [0, 1]^2$ is a Katz–Tao $(\delta, 1)$-set, and $\mathcal{L} \subset \mathcal{A}(2)$ satisfies $|\mathcal{L}|_\delta \le \delta^{-1}$. If $|\mathcal{I}(P, \mathcal{L})|_\delta \gtrapprox \delta^{-3/2}$, does the conclusion of Theorem 1.7 (in the case $s = t = 1$) continue to hold? Our proof heavily relies on the Katz–Tao $(\delta, 1)$-set properties of both P and $\mathcal{L}$.

Finally, we refer the reader to further recent advances in the active area of estimating δ-discretised incidences between points and lines in $\mathbb{R}^d$: [2, 4, 6, 7, 10, 13–15, 18, 21, 22].

1.2. Proof ideas

We explain here a few key points of the proof of Theorem 1.7 in the case $s = 1 = t$. We warn the reader that some statements in this subsection are inaccurate.

Let $P \subset [0, 1]^2$ be a Katz–Tao $(\delta, 1)$-set, and let $\mathcal{L} \subset \mathcal{A}(2)$ be a Katz–Tao $(\delta, 1)$-set of lines. According to Theorem 1.4, in this case

$$|\mathcal{I}(P, \mathcal{L})|_\delta \lessapprox \delta^{-3/2}, \tag{1.8}$$

where the $\lessapprox$ notation in this section hides factors of the order $\delta^{-\varepsilon}$. Assume that $(P, \mathcal{L})$ is a pair of almost extremisers for (1.8). One can infer – after pigeonholing and refining – that typically $|P \cap T| \approx \delta^{-1/2}$ for $T \in \mathcal{T} := \{T = [\ell]_\delta : \ell \in \mathcal{L}\}$ (the δ-tubes around the lines in $\mathcal{L}$). A key challenge is that we know a priori nothing about the distribution of the set $P \cap T$. In fact, the main goal is to show that $P \cap T$ is maximally concentrated:

$$|P \cap T \cap B(x_0, \delta^{1/2})| \approx \delta^{-1/2} \tag{1.9}$$

for some $x_0 \in \mathbb{R}^2$. Proving this contains the main work in finding the clique predicted by Theorem 1.7.

Suppose (1.9) fails. Another "extreme" possibility is where $P \cap T$ is a (δ, η)-set for some $\eta > 0$ (see Definition 2.1 – this is different from a Katz–Tao (δ, η)-set). If so, a recent result of Shmerkin and Wang (see Corollary 1.7 in [18]) implies that $|\mathcal{T}| \ge \delta^{-\rho-1}$ for some $\rho = \rho(\eta) > 0$. On the other hand, $|\mathcal{T}| \lessapprox \delta^{-1}$ by the Katz–Tao $(\delta, 1)$-set property of $\mathcal{L}$. So, $P \cap T$ cannot satisfy a (δ, η)-condition for any definite $\eta > 0$. This conclusion is in the direction of (1.9), but weaker.

To proceed, we make the following central observation: it is always possible to restrict $P \cap T$ to some appropriately chosen square $Q \subset [0, 1]^2$ of side $\Delta \ge \delta$ such that $P \cap T \cap Q$ looks η-dimensional for some $\eta > 0$. Formally, the rescaled set $S_Q(P \cap T \cap Q)$ is a

$((\delta/\Delta),\eta)$-set, up to passing to a refinement. Moreover, this can be done in such a way that $|P\cap T\cap Q|\geq \delta^{c\eta}|P\cap T|$ for an arbitrarily small constant $c>0$ of our choosing. The catch is that the smaller we pick $c>0$, the smaller also $\eta>0$ needs to be. The technical statement is Proposition 2.17.

After plenty of pigeonholing, and using the extremal property of P and $\mathcal{L}$, we can use this observation to find a single square $Q\subset[0,1]^2$ such that

(a) there are "many" tubes $T\in\mathcal{T}$ intersecting Q,

(b) each intersection $P\cap T\cap Q$ is a (rescaled) $((\delta/\Delta),\eta)$-set with

$$|P\cap T\cap Q|\geq \delta^{c\eta}|P\cap T|\gtrapprox \delta^{c\eta-1/2}.$$

We can also ensure that the family $\mathcal{T}_Q:=\{T\in\mathcal{T}:T\cap Q\neq\emptyset\}$ satisfies a "single-scale" non-concentration condition whenever $\Delta\gg\delta^{1/2}$ (see (3.21)). Since we want to prove that $\Delta\approx\delta^{1/2}$ – this amounts to (1.9) –, it suffices to find a contradiction in the case $\Delta\gg\delta^{1/2}$.

The single-scale non-concentration condition is weaker than the Katz–Tao $(\delta,1)$-set property of $\mathcal{L}$, but nonetheless strong enough to apply the theorem of Shmerkin and Wang (Corollary 1.7 in [18]) to $P\cap Q$ and $\mathcal{T}_Q$. A big issue is, however, that the constant $\rho=\rho(\eta)>0$ in Corollary 1.7 of [18] is non-explicit. It turns out that we could reach a contradiction provided that $\rho\geq c_0\eta$ for some $c_0>0$ (which may also depend on the "single scale" non-concentration exponent of $\mathcal{T}_Q$). Then, a suitable choice $c\ll c_0$ (as in (b)) would yield a contradiction.

Fortunately, the linear dependence of ρ on η has been established recently in Theorem 5.61 of [12], or see Theorem 2.8. Applying this "black box" allows us to reach a contradiction in the case $\Delta\gg\delta^{1/2}$, and finally conclude (1.9).

Remark 1.13. Some form of non-trivial Furstenberg set estimate is necessary to prove Theorem 1.7, since Theorem 1.7 itself could be – with some effort – used to deduce a non-trivial Furstenberg set estimate. We give a very rough sketch. It is well known (see Theorem 1.16 in [9]) that non-trivial Furstenberg set estimates are equivalent to a non-trivial sum-product estimate of the following form: if $A\subset[1,2]$ is a Katz–Tao $(\delta,1/2)$-set with $|A|=\delta^{-1/2}$, then either $|A+A|_\delta\gg|A|$ or $|A\cdot A|_\delta\gg|A|$. So, we only need to verify this sum-product estimate, starting from Theorem 1.7.

If both $|A+A|_\delta\approx|A|$ and $|A\cdot A|_\delta\approx|A|$, then Proposition 6.6 in [1] shows that also $|A'+A'A'|_\delta\approx|A|$ for some refinement $A'\subset A$ with $|A'|\approx|A|$. This is best interpreted by saying that the Katz–Tao $(\delta,1)$-set $P:=A'\times A'$ has $\frac{1}{2}$-dimensional projections in a Katz–Tao $(\delta,1/2)$-set of directions determined by A'. This eventually implies that there exists a $(\delta,1)$-set of lines $\mathcal{L}$ such that $|\mathcal{I}(P,\mathcal{L})|_\delta\approx\delta^{-3/2}$. However, one can check that the $\frac{1}{2}$-dimensional Katz–Tao Property of A' ensures that there can exist no cliques $(P'\times A')\subset P\times\mathcal{L}$ as large as the ones predicted by Theorem 1.7 in the case $s=1=t$.

1.3. Outline of the paper

In Section 2, we gather preliminary results required to prove Theorem 1.7. The main technical result in that section is Proposition 2.17, which may have some independent interest to experts.

The proof of Theorem 1.7 occupies Section 3. There is a substantial difference between the complexity of the proofs when $s=t$ (harder) and $s\neq t$ (easier). For the case

$s = t$, we need the non-trivial Theorem 5.61 in [12], repeated here as Theorem 2.8. This is a quantitative *Furstenberg set estimate*, although not the sharp one from [15]. This auxiliary result is not required in the case $s \neq t$. It might have been possible to combine the cases $s = t$ and $s \neq t$, but we decided to separate them for clarity. Where the details are very similar, we give all of them in the harder case $s = t$, and a sketch when $s \neq t$. Regarding the cases $s \neq t$, we only give a (fairly) detailed argument for $s < t$, and then infer the case $s > t$ by point-line duality (see Section 3.3 for the details).

Finally, Section 4 contains the proofs of Corollary 1.10 and Proposition 1.11.

2. Preliminaries

2.1. Notations and (δ, s)-sets

We adopt the standard notations $\lesssim, \gtrsim, \sim$. For example, $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$, while $A \lesssim_r B$ stands for $A \leq C(r)B$ for a positive function $C(r)$. We will denote $A \lessapprox_\delta B$, $A \gtrapprox_\delta B$, $A \approx_\delta B$ or $A \approx B$ to hide slowly growing functions of δ such as $\log(1/\delta)$ and $\delta^{-\varepsilon}$. The precise meaning of the $\lessapprox$ notation will always be explained separately.

For $\delta \in 2^{-\mathbb{N}}$, dyadic δ-cubes in $\mathbb{R}^d$ are denoted $\mathcal{D}_\delta(\mathbb{R}^d)$. Elements of $\mathcal{D}_\delta(\mathbb{R}^d)$ are typically denoted with letters p, q. For $P \subset \mathbb{R}^d$, we write $\mathcal{D}_\delta(P) := \{p \in \mathcal{D}_\delta(\mathbb{R}^d) : P \cap p \neq \emptyset\}$.

In addition to the Katz–Tao (δ, s)-set condition (Definition 1.2), also the following slightly different non-concentration property will be needed in the paper:

Definition 2.1 ((δ, s, C)-set). For $\delta \in (0, 1]$, $s \in [0, d]$ and $C > 0$, a nonempty bounded set $P \subset \mathbb{R}^d$ is called a (δ, s, C)-set if

$$|P \cap B(x, r)|_\delta \leq C r^s |P|_\delta, \quad x \in \mathbb{R}^d, \ r \in [\delta, 1]. \tag{2.1}$$

A family $\mathcal{P} \subset \mathcal{D}_\delta(\mathbb{R}^d)$ is called a (δ, s, C)-set if $\cup\mathcal{P} \subset \mathbb{R}^d$ is a (δ, s, C)-set.

Since both Definitions 1.2 and 2.1 will be used in the paper, we will always be careful and explicit in either including the words "Katz–Tao", or omitting them.

2.2. Point-line duality and dyadic tubes

Definition 2.2. Let $D \colon \mathbb{R}^2 \to \mathcal{A}(2)$ be the *point-line duality map* sending (a, b) to a corresponding line in $\mathbb{R}^2$, defined by

$$D(a, b) := \ell_{a,b} := \{(x, y) \in \mathbb{R}^2 : y = ax + b\} \in \mathcal{A}(2).$$

The following useful lemma follows by chasing the definitions:

Lemma 2.3. *The map $D \colon ([-1, 1] \times \mathbb{R}, |\cdot|) \to (\mathcal{A}(2), d_{\mathcal{A}(2)})$ is bi-Lipschitz.*

Definition 2.4 (Dyadic δ-tubes). Let $\delta \in 2^{-\mathbb{N}}$ and

$$Q = [a_0, a_0 + \delta) \times [b_0, b_0 + \delta) \in \mathcal{D}_\delta([-1, 1] \times \mathbb{R}).$$

The union of lines $T := \cup\{D(a, b) : (a, b) \in Q\} \subset \mathbb{R}^2$ is called a *dyadic δ-tube*. The *slope of T* is defined to be $\sigma(T) := a_0$. The family of dyadic δ-tubes in $\mathbb{R}^2$ is denoted $\mathcal{T}^\delta$.

If $\mathcal{L} \subset \mathcal{A}(2)$, we denote $\mathcal{T}^\delta(\mathcal{L})$ the family of dyadic tubes which contain at least one line from $\mathcal{L}$. Whenever $\mathcal{L} \subset D([-1,1) \times \mathbb{R})$, the family $\mathcal{T}^\delta(\mathcal{L})$ is a cover of $\mathcal{L}$.

By an abuse of notation and terminology, we sometimes view dyadic δ-tubes as subsets of $\mathcal{A}(2)$. In fact, we already did so in the last sentence of Definition 2.4.

We introduce notation for "dyadic covers" of sets $\mathcal{I} \subset \mathbb{R}^2 \times \mathcal{A}(2)$:

$$\mathcal{D}_\delta(\mathcal{I}) := \{(p, T) \in \mathcal{D}_\delta \times \mathcal{T}^\delta : x \in p \text{ and } \ell \subset T \text{ for some } (x, \ell) \in \mathcal{I}\}.$$

To be accurate, the elements of $\mathcal{D}_\delta(\mathcal{I})$ only cover $\mathcal{I}$ when the $\mathcal{A}(2)$-component of $\mathcal{I}$ consists of lines with slops in $[-1,1]$. We will only use this notation when $\mathcal{I} \subset \mathbb{R}^2 \times D([-1,1) \times \mathbb{R})$.

Lemma 2.5. *Let $P \subset \mathbb{R}^2$, and let $\mathcal{L} \subset D([-1,1) \times \mathbb{R}) \subset \mathcal{A}(2)$. Then,*

$$|\mathcal{I}(P, \mathcal{L})|_\delta \sim |\mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))|. \tag{2.2}$$

Proof. We start with the inequality "$\lesssim$". Let $(x_1, \ell_1), \ldots, (x_n, \ell_n) \in \mathcal{I}(P, \mathcal{L})$ be a maximal (11δ)-separated set. For each $1 \leq j \leq n$, pick $(p_j, T_j) \in \mathcal{D}_\delta \times \mathcal{T}^\delta$ with $x_j \in p_j$ and $\ell_j \subset T_j$. Then $(p_j, T_j) \in \mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))$, since $(x_j, \ell_j) \in \mathcal{I}(P, \mathcal{L})$. Furthermore, the map $(x_j, \ell_j) \mapsto (p_j, T_j)$ is injective, because if $(p, T) \in \mathcal{D}_\delta \times \mathcal{T}^\delta$ is fixed, then the set $\{(x, \ell) : x \in p \text{ and } \ell \subset T\}$ is contained in a $d_{\mathcal{A}(2)}$-ball of radius 5δ. Therefore, $|\mathcal{I}(P, \mathcal{L})|_\delta \sim n \leq |\mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))|$.

We then prove the inequality "$\gtrsim$". Write $\mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L})) = \{(p_1, T_1), \ldots, (p_n, T_n)\}$, and $T_j = \bigcup D(q_j)$, where $q_j \in \mathcal{D}_\delta(\mathbb{R}^2)$. We say that (p_i, T_i) and (p_j, T_j) are *neighbours* if $\operatorname{dist}(p_i, p_j) \leq C\delta$ and $\operatorname{dist}(q_i, q_j) \leq C\delta$ for a suitable absolute constant $C \geq 1$. Pick any maximal neighbour-free subset $\mathcal{J} \subset \mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))$. It is easy to check that $|\mathcal{J}| \sim_C n$.

We claim that $|\mathcal{I}(P, \mathcal{L})|_\delta \geq |\mathcal{J}|$, which will complete the proof. To see this, pick $(p, T) \in \mathcal{J} \subset \mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))$. Then, by definition there exist $x_p \in p$ and $\ell_T \subset T$ such that $(x_p, \ell_T) \in \mathcal{I}(P, \mathcal{L})$, and in particular, $\ell_T \in \mathcal{L} \subset D([-1,1) \times \mathbb{R})$.

Now, it suffices to note that the pairs $(x_p, \ell_T) \in \mathcal{I}(P, \mathcal{L})$ obtained this way are δ-separated. If $(p, T), (p', T') \in \mathcal{J}$ are distinct, then either $\operatorname{dist}(p, p') \geq C\delta$ or $\operatorname{dist}(q, q') \geq C\delta$. In the former case, $|x_p - x_{p'}| \geq 10\delta$. In the latter case, $d_{\mathcal{A}(2)}(\ell_T, \ell_{T'}) \gtrsim \operatorname{dist}(q, q') \geq C\delta$ by the bi-Lipschitz property of D. Therefore, $d_{\mathcal{A}(2)}(\ell_T, \ell_{T'}) \geq \delta$ if $C \geq 1$ is large enough. ■

2.3. Incidence bounds

The following result is a version of Fu and Ren's Theorem 1.4 where the dependence on the non-concentration constants has been quantified. It is also due to Fu–Ren, see Theorems 3.1 and 3.2 in [5].

Theorem 2.6. *Let $0 \leq s, t \leq 1$ and $K_P, K_{\mathcal{L}} \geq 1$. Assume $P \subset [0,1]^2$ is a Katz–Tao (δ, s, K_P)-set and $\mathcal{L} \subset \mathcal{A}(2)$ is a Katz–Tao $(\delta, t, K_{\mathcal{L}})$-set. Then,*

$$|\mathcal{I}(P, \mathcal{L})|_\delta^{s+t} \lesssim_\varepsilon \delta^{-st(1+\varepsilon)}\, K_P^t\, K_{\mathcal{L}}^s\, |P|_\delta^s\, |\mathcal{L}|_\delta^t, \quad \varepsilon > 0. \tag{2.3}$$

Remark 2.7. The original formulation (Theorems 3.1 and 3.2 in [5]) of Theorem 2.6 concerned incidences of the form $\bar{\mathcal{I}}(\mathcal{B}, \mathcal{T}) = \{(B, T) \in \mathcal{B} \times \mathcal{T} : B \cap T \neq \emptyset\}$, where $\mathcal{B}$ is a family of δ-discs and $\mathcal{T}$ is a family of δ-tubes. Let us clarify why the original formulation implies Theorem 2.6 as stated. (We give the full details to make sure that the original dependence on the constants K_P and $K_{\mathcal{L}}$ can be maintained.)

Let P and $\mathcal{L}$ be defined as in Theorem 2.6, and pick a maximal (3δ)-separated set $(x_1,\ell_1),\ldots,(x_n,\ell_n)\in\mathcal{I}(P,\mathcal{L})$ in the d-metric of $\mathbb{R}^2\times\mathcal{A}(2)$. Thus, $|\mathcal{I}(P,\mathcal{L})|_\delta\sim n$, and $x_j\in\ell_j$ for all $1\le j\le n$. Let $P'\subset\{x_1,\ldots,x_n\}$ and $\mathcal{L}'\subset\{\ell_1,\ldots,\ell_n\}$ be maximal δ-separated sets, and consider the families of (10δ)-balls and (10δ)-tubes

$$\mathcal{B}:=\{B(x',10\delta):x'\in P'\}\quad\text{and}\quad\mathcal{T}:=\{[\ell]_{10\delta}:\ell\in\mathcal{L}'\},$$

where $[\ell]_r$ is the r-neighbourhood of ℓ. Then $\mathcal{B}$ is a Katz–Tao $(10\delta,s,O(K_P))$-set and $\mathcal{T}$ is a Katz–Tao $(10\delta,t,O(K_{\mathcal{L}}))$-set in the terminology of [5]. Further, we claim that $n\le|\bar{\mathcal{I}}(\mathcal{B},\mathcal{T})|$. To see this, fix $1\le j\le n$. By the definitions of P' and $\mathcal{L}'$, there exist $x'\in P'$ and $\ell'\in\mathcal{L}'$ such that $|x_j-x'|\le\delta$ and $|\ell_j-\ell'|\le\delta$. Since $x_j\in\ell_j$,

$$B(x',10\delta)\cap[\ell']_{10\delta}\neq\emptyset.$$

Moreover, the map $(x_j,\ell_j)\mapsto(x',\ell')$ is injective: two pairs (x_i,ℓ_i) and (x_j,ℓ_j) corresponding to the same pair (x',ℓ') would satisfy $|x_i-x_j|\le2\delta$ and $|\ell_i-\ell_j|\le2\delta$, and therefore $d((x_i,\ell_i),(x_j,\ell_j))\le2\delta$, contrary to the (3δ)-separation. This proves the inequality $n\le|\bar{\mathcal{I}}(\mathcal{B},\mathcal{T})|$, and finally (2.3) follows from the original formulation of [5].

Besides Theorem 2.6, a main tool in the proof of Theorem 1.7 is Theorem 5.61 in [12], stated below as Theorem 2.8. To be accurate, the statement below is the "dual" version of Theorem 5.61 in [12], which is more convenient for our application. Another small difference is that Theorem 2.8 is stated for ("ordinary") δ-tubes, whereas Theorem 5.61 in [12] is formulated in terms of dyadic δ-tubes. The introduction of dyadic δ-tubes in [12] brings technical convenience in the proof, but the two versions are a posteriori easily seen to be equivalent. In the statement, a δ-tube is any rectangle of dimensions $\delta\times1$, and two δ-tubes T,T' are called *distinct* if $\mathrm{Leb}(T\cap T')\le\frac{1}{2}\mathrm{Leb}(T)$.

Theorem 2.8. *Fix $\eta\in(0,1]$, $t\in(0,2)$, $u\in(0,\min\{t,2-t\}]$, and $0\le\alpha<\eta u/4$. There exist $\varepsilon=\varepsilon(\eta,t,u)>0$ and $\delta_0=\delta_0(\alpha,\eta,t,u)>0$ such that the following holds for all $\delta\in(0,\delta_0]$.*

Let $\mathcal{T}$ be a family of distinct δ-tubes with $|\mathcal{T}|=\delta^{-t}$ and satisfying the following non-concentration condition at the single scale $\rho:=\delta|\mathcal{T}|^{1/2}$:

$$|\{T\in\mathcal{T}:T\subset\mathbf{T}\}|\le\delta^u|\mathcal{T}|, \tag{2.4}$$

where $\mathbf{T}\subset\mathbb{R}^2$ is an arbitrary $(\rho\times2)$-rectangle. Let $N\ge1$. For every $T\in\mathcal{T}$, assume that there exists a $(\delta,\eta,\delta^{-\varepsilon})$-set $\mathcal{P}_T\subset\mathcal{D}_\delta([0,1)^2)$ satisfying $|\mathcal{P}_T|\ge N$, and with the property that every square in $\mathcal{P}_T$ intersects T. Then,

$$\Big|\bigcup_{T\in\mathcal{T}}\mathcal{P}_T\Big|\ge N\cdot|\mathcal{T}|^{1/2}\cdot\delta^{-\alpha}.$$

While Theorem 2.8 is an improvement (enabled by (2.4)) over the classical "2-ends" incidence bound, we will also employ the classical bound, recorded below:

Proposition 2.9. *Let $\mathcal{T}$ be a family of dyadic δ-tubes or distinct (ordinary) δ-tubes. Let $\mathfrak{N}\ge1$ and $r>0$. For every $T\in\mathcal{T}$, assume that there exists a set $\mathcal{P}_T\subset\mathcal{D}_\delta([0,1)^2)$ with*

$|\mathcal{P}_T| = \mathfrak{N}$, *with the property that every square in* $\mathcal{P}_T$ *intersects* T, *and* $\mathcal{P}_T$ *satisfies the following* 2-ends condition*:*

$$|\mathcal{P}_T \cap B(x,r)| \leq \frac{1}{3}\mathfrak{N}, \quad x \in \mathbb{R}^2. \tag{2.5}$$

Then,

$$\Big|\bigcup_{T\in\mathcal{T}} \mathcal{P}_T\Big| \gtrsim |\mathcal{T}|^{1/2} \cdot \mathfrak{N} \cdot r^{1/2}. \tag{2.6}$$

Proof. According to (2.5), we may for every $T \in \mathcal{T}$ find two subsets $\mathcal{P}_T^1, \mathcal{P}_T^2 \subset \mathcal{P}_T$ such that $|\mathcal{P}_T^j| \sim \mathfrak{N}$ for $j \in \{1,2\}$, and $\operatorname{dist}(p,q) \gtrsim r$ for all $(p,q) \in \mathcal{P}_T^1 \times \mathcal{P}_T^2$. Consequently,

$$\sum_{T\in\mathcal{T}} |\mathcal{P}_T^1 \times \mathcal{P}_T^2| \gtrsim |\mathcal{T}| \cdot \mathfrak{N}^2.$$

On the other hand, denoting by "$\mathcal{P}$" the set appearing in (2.6), we have

$$\sum_{T\in\mathcal{T}} |\mathcal{P}_T^1 \times \mathcal{P}_T^2| \leq \sum_{\substack{(\mathbf{p},\mathbf{q})\in\mathcal{P}^2 \\ \operatorname{dist}(\mathbf{p},\mathbf{q})\gtrsim r}} |\{T \in \mathcal{T} : \mathbf{p} \cap T \neq \emptyset \neq \mathbf{q} \cap T\}| \lesssim |\mathcal{P}|^2/r.$$

Combining these estimates gives (2.6). ■

2.4. Uniform sets

The items in this section are repeated from Section 2.3 in [12].

Definition 2.10. Let $n \geq 1$, and let

$$\delta = \Delta_n < \Delta_{n-1} < \cdots < \Delta_1 \leq \Delta_0 = 1$$

be a sequence of dyadic scales. We say that a set $P \subset [0,1)^2$ is $\{\Delta_j\}_{j=1}^n$*-uniform* if there is a sequence $\{N_j\}_{j=1}^n$ such that $N_j \in 2^{\mathbb{N}}$ and $|P \cap Q|_{\Delta_j} = N_j$ for all $j \in \{1,\ldots,n\}$ and all $Q \in \mathcal{D}_{\Delta_{j-1}}(P)$. A family of dyadic cubes $\mathcal{P} \subset \mathcal{D}_\delta([0,1)^d)$ is called $\{\Delta_j\}_{j=1}^n$-uniform if the set $P = \bigcup \mathcal{P}$ is $\{\Delta_j\}_{j=1}^n$-uniform.

The following simple but key lemma asserts that one can always find "dense uniform subsets". See, e.g., Lemma 3.6 in [17] for the short proof.

Lemma 2.11. *Let* $P \subset [0,1)^d$, $m, H \in \mathbb{N}$, *and* $\delta := 2^{-mH}$. *Let also* $\Delta_j := 2^{-jH}$ *for* $0 \leq j \leq m$, *so in particular,* $\delta = \Delta_m$. *Then, there is a* $\{\Delta_j\}_{j=1}^m$*-uniform set* $P' \subset P$ *such that*

$$|P'|_\delta \geq (2H)^{-m}|P|_\delta.$$

In particular, if $\varepsilon > 0$ *and* $H^{-1}\log(2H) \leq \varepsilon$, *then* $|P'|_\delta \geq \delta^\varepsilon|P|_\delta$.

Lemma 2.11 also holds for families $\mathcal{P} \subset \mathcal{D}_\delta([0,1)^d)$ (as can be seen by applying the lemma to $P = \bigcup \mathcal{P}$). The lemma has the following superficially stronger corollary, which we will also need. The details can be found in Corollary 7.9 of [12].

Corollary 2.12. *For every $\varepsilon > 0$, there exists $H_0 = H_0(\varepsilon) \geq 1$ such that the following holds for all $\delta = 2^{-mH}$ with $m \geq 1$ and $H \geq H_0$. Let $\mathcal{P} \subset \mathcal{D}_\delta$. Then, there exist disjoint $\{2^{-jH}\}_{j=1}^m$-uniform subsets $\mathcal{P}_1, \dots, \mathcal{P}_N \subset \mathcal{P}$ with the properties*

- $|\mathcal{P}_j| \geq \delta^{2\varepsilon}|\mathcal{P}|$ *for all* $1 \leq j \leq N$,
- $|\mathcal{P} \setminus (\mathcal{P}_1 \cup \dots \cup \mathcal{P}_N)| \leq \delta^{\varepsilon}|\mathcal{P}|$.

Definition 2.13 (Branching function). Let $H \in \mathbb{N}$, let $P \subset [0,1)^d$ be a $\{\Delta_j\}_{j=1}^m$-uniform set, with $\Delta_j := 2^{-jH}$, and let $\{N_j\}_{j=1}^m \subset \{1, \dots, 2^{dH}\}^m$ be the associated sequence. We define the *branching function* $\beta\colon [0,m] \to [0,dm]$ by setting $\beta(0) = 0$, and

$$\beta(j) := \frac{\log |P|_{2^{-jH}}}{H} = \frac{1}{H}\sum_{i=1}^{j} \log N_i, \quad i \in \{1, \dots, m\},$$

and then interpolating linearly.

Definition 2.14 (ε-linear and superlinear functions). Given a function $f\colon [a,b] \to \mathbb{R}$ and numbers $\varepsilon, \sigma \geq 0$, we say that (f,a,b) is (σ,ε)-*superlinear* if

$$f(x) \geq f(a) + \sigma(x-a) - \varepsilon(b-a), \quad x \in [a,b].$$

If $\varepsilon = 0$, we simply say that (f,a,b) is σ-superlinear.

Definition 2.15 (Homothety map). Let $r \in 2^{-\mathbb{N}}$ and let $Q \in \mathcal{D}_r([0,1)^d)$. Define $S_Q\colon Q \to [0,1)^2$ to be the affine homothety with $S_Q(Q) = [0,1)^d$.

The following lemma is Lemma 8.3 in [11], but we give the proof to record the dependence on the constant Δ more explicitly.

Lemma 2.16. *Let $P \subset [0,1)^d$ be $\{\Delta^j\}_{j=1}^m$-uniform with branching function β and let $\delta = \Delta^m$. If (β,a,b) is s-superlinear for any integers $0 \leq a < b \leq m$ and $s > 0$, then for any $Q \in \mathcal{D}_{\Delta^a}(P)$, the rescaled set $S_Q(P \cap Q)$ is a $(\Delta^{b-a}, s, C\Delta^{-s})$-set for some $C = C(d) > 0$.*

Proof. By Lemma 2.24 in [12], $S_Q(P)$ is $\{\Delta^j\}_{j=1}^{m-a}$-uniform for any $Q \in \mathcal{D}_{\Delta^a}(P)$, and the corresponding branching function β_Q satisfies

$$\beta_Q(x) = \beta(x+a) - \beta(a), \quad x \in [0, m-a],$$

and $(\beta_Q, 0, b-a)$ is s-superlinear. Let $\{N_j\}_{j=1}^m$ be the corresponding sequence defined as in Definition 2.10. For any $q \in \mathcal{D}_{\Delta^i}(P)$, $0 \leq i \leq b-a$, we have

$$\begin{aligned} |S_Q(P) \cap q|_{\Delta^{b-a}} &= N_{i+1}N_{i+2}\cdots N_{m-a} = \frac{|S_Q(P)|_{\Delta^{b-a}}}{N_1N_2\cdots N_i} \\ &= |S_Q(P)|_{\Delta^{b-a}}\, 2^{-\log(\Delta^{-1})\beta_Q(i)} \leq |S_Q(P)|_{\Delta^{b-a}}\, 2^{-\log(\Delta^{-1})is} \\ &= \Delta^{is}|S_Q(P)|_{\Delta^{b-a}}. \end{aligned}$$

In general, for any $r \in [\Delta^{b-a}, 1]$, there exists i such that $r \in [\Delta^{i+1}, \Delta^i)$ and thus $\Delta^i \leq \Delta^{-1}r$. For any $q \in \mathcal{D}_r(S_Q(P))$, choose any $q_1 \in \mathcal{D}_{\Delta^i}(P)$, we simply get

$$|S_Q(P) \cap q|_{\Delta^{b-a}} \leq |S_Q(P) \cap q_1|_{\Delta^{b-a}} \leq \Delta^{is}|S_Q(P)|_{\Delta^{b-a}} \leq \Delta^{-s}r^s|S_Q(P)|_{\Delta^{b-a}},$$

as required. By uniformity of P, we deduce that $S_Q(P \cap Q)$ is a $(\Delta^{b-a}, s, C\Delta^{-s})$-set. ∎

2.5. Finding non-concentrated subsets

One further key tool in the proof of Theorem 1.7 is the next proposition, which allows us to find reasonably large reasonably non-concentrated subsets within arbitrary families of δ-cubes.

Proposition 2.17. *For every $d \in \mathbb{N}$, $C \geq 1$, $\delta \in 2^{-\mathbb{N}}$, $\emptyset \neq \mathcal{P} \subset \mathcal{D}_\delta([0,1)^d)$ and $\eta_0 > 0$, there exist a scale $\Delta \in 2^{-\mathbb{N}} \cap [\delta, 1]$, a number $\eta \in [\eta_0, \eta_0 e^{2C}]$, a cube $Q \in \mathcal{D}_\Delta(\mathcal{P})$, and a subset $\mathcal{P}_Q \subset \mathcal{P} \cap Q$ with the following properties*:

(1) $|\mathcal{P}_Q| \geq \delta^\eta |\mathcal{P}|$, *and*

(2) $S_Q(\mathcal{P}_Q)$ *is a* $((\delta/\Delta), C\eta, O_{d,\eta_0}(1))$*-set.*

Here S_Q is the affine homothety mapping in Definition 2.15.

Remark 2.18. The main point of the proposition is the distinction between passing to a subset $\mathcal{P}_Q$ of cardinality $\geq \delta^\eta |\mathcal{P}|$, and gaining the $(C\eta)$-dimensional non-concentration condition for $S_Q(\mathcal{P}_Q)$ – for any prescribed $C \geq 1$. We also note that the non-concentration condition in (2) refers to Definition 2.1, and not to the Katz–Tao condition.

Proof of Proposition 2.17. Fix $C \geq 1$ and $\eta_0 > 0$ as in the statement. Applying initially Lemma 2.11 with "η_0" in place of "ε", we may find a $\{2^{-jH}\}_{j=1}^m$-uniform subset $\mathcal{P}' \subset \mathcal{P}$, where $H^{-1}\log(2H) \leq \eta_0$, and $|\mathcal{P}'| \geq \delta^{\eta_0/2}|\mathcal{P}|$. After this initial step, our efforts will be directed towards finding the numbers $\eta \in [\eta_0, \eta_0 e^C]$ and the subset $\mathcal{P}_Q$ inside $\mathcal{P}'$ instead of $\mathcal{P}$, satisfying $|\mathcal{P}_Q| \geq \delta^{\eta/2}|\mathcal{P}'|$. Therefore finally $|\mathcal{P}_Q| \geq \delta^{\eta/2+\eta_0/2}|\mathcal{P}| \geq \delta^\eta|\mathcal{P}|$. To simplify notation, we will continue denoting $\mathcal{P}'$ by $\mathcal{P}$ – or in other words, we assume without loss of generality that $\mathcal{P}$ is $\{2^{-jH}\}_{j=1}^m$-uniform to start with.

Let $\bar{\beta}\colon [0,m] \to [0,dm]$ be the branching function of $\mathcal{P}$, and consider also the normalised version defined by

$$\beta(x) := \frac{1}{m}\bar{\beta}(mx), \quad x \in [0,1].$$

We first dispose of a special case where $\beta(1) \leq (\eta_0/2)e^{2C}$. In this case, we set

$$\eta := \max\{\eta_0, 2\beta(1)\} \in [\eta_0, \eta_0 e^{2C}].$$

We set $\Delta := \delta$, pick $Q \in \mathcal{P} = \mathcal{D}_\Delta(\mathcal{P})$ arbitrarily, and define $\mathcal{P}_Q := \{Q\}$. Then,

$$|\mathcal{P}_Q| = 1 = \delta^{\beta(1)}|\mathcal{P}| \geq \delta^{\eta/2}|\mathcal{P}|.$$

Furthermore, $S_Q(\mathcal{P}_Q) = \{[0,1)^d\}$ and $\delta/\Delta = 1$, so $S_Q(\mathcal{P}_Q)$ is vacuously a $((\delta/\Delta), C\eta, 1)$-set, because indeed $\{[0,1)^d\}$ is an $(1,s,1)$-set for every $s > 0$ (using $1^s \equiv 1$).

In the sequel, we may assume that $\beta(1) > (\eta_0/2)e^{2C}$. On the other hand, always $\beta(0) = 0 < (\eta_0/2)e^{2C\cdot 0}$. Let x_0 be the largest value of $x \in \{0, 1/m, 2/m, \ldots, 1\}$ satisfying

$$\beta(x) \leq \frac{\eta_0}{2} \cdot e^{2Cx}. \tag{2.7}$$

Write

$$\eta := \eta_0 e^{2Cx_0} \in [\eta_0, \eta_0 e^{2C}]. \tag{2.8}$$

By the choice of x_0, the converse inequality $\beta(y) > (\eta_0/2)e^{2Cy}$ holds for all $y \in (x_0, 1] \cap \frac{1}{m}\mathbb{Z}$. Therefore, for $y \in [x_0, 1] \cap \frac{1}{m}\mathbb{Z}$,

$$\begin{aligned}\beta(y) - \beta(x_0) &\geq \frac{\eta_0}{2}(e^{2Cy} - e^{2Cx_0}) = C\eta_0 \int_{x_0}^{y} e^{2C\xi}\, d\xi \\ &\geq C\eta_0 e^{2Cx_0}(y - x_0) = C\eta(y - x_0).\end{aligned}$$

(From this inequality, we deduce in particular that $C\eta \leq (\beta(y) - \beta(x))/(y - x_0) \leq d$, since β is d-Lipschitz.) In terms of the original branching function $\bar{\beta}$, we find

$$\bar{\beta}(y) - \bar{\beta}(mx_0) \geq C\eta(y - mx_0), \quad y \in [mx_0, m] \cap \mathbb{Z}.$$

In other words, $(\bar{\beta}, mx_0, m)$ is $C\eta$-superlinear in the sense of Definition 2.14. Recall that $\mathcal{P}'$ is $\{2^{-jH}\}_{j=1}^{m}$-uniform with $H \sim_{\eta_0} 1$, and write

$$\Delta := 2^{-(mx_0)H}.$$

Let $Q \in \mathcal{D}_\Delta(\mathcal{P}')$ be arbitrary. Lemma 2.16 (applied with $a = mx_0$ and $b = m$ and 2^{-H} taking the role of Δ) implies that $S_Q(\mathcal{P} \cap Q)$ is a $((\delta/\Delta), C\eta, O(2^{CH\eta}))$-set. Furthermore, $O(2^{CH\eta}) = O_{d,\eta_0}(1)$, using $C\eta \leq d$. This completes the proof of part (2) of Proposition 2.17. (Note that $\mathcal{P} \cap Q$ here is in fact $\mathcal{P}_Q := \mathcal{P}' \cap Q \subset \mathcal{P} \cap Q$ in the original notation.)

It remains to check part (1); more precisely, $|\mathcal{P}_Q| \geq \delta^{\eta/2}|\mathcal{P}|$. By the uniformity of $\mathcal{P}$,

$$|\mathcal{P}_Q| = \frac{|\mathcal{P}|}{|\mathcal{P}|_\Delta}.$$

Here, by the definitions of Δ, and the branching functions $\bar{\beta}, \beta$, and recalling $\delta = 2^{-Hm}$,

$$|\mathcal{P}|_\Delta = |\mathcal{P}|_{2^{-(mx_0)H}} = 2^{H\bar{\beta}(mx_0)} = 2^{Hm\beta(x_0)} = \delta^{-\beta(x_0)} \leq \delta^{-\eta/2},$$

using (2.7)–(2.8) in the final inequality. Therefore $|\mathcal{P}_Q| \geq \delta^{\eta/2}|\mathcal{P}|$, as desired. ∎

3. Proof of Theorem 1.7

3.1. Case $s = t$

We restate Theorem 1.7 in the case $s = t$.

Theorem 3.1. *For every $s, u \in (0, 1]$, there exist $\delta_0 = \delta_0(s, u) > 0$ and $\varepsilon = \varepsilon(s, u) > 0$ such that the following holds for $\delta \in (0, \delta_0]$. Let $P \subset [0, 1]^2$ and $\mathcal{L} \subset \mathcal{A}(2)$ be Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-sets. If*

$$|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - 3s/2}, \tag{3.1}$$

then there exists a (δ, δ^u)-clique $P' \times \mathcal{L}' \subset P \times \mathcal{L}$ with

$$|P'|_\delta \geq \delta^{u - s/2} \quad \textit{and} \quad |\mathcal{L}'|_\delta \geq \delta^{u - s/2}.$$

From now on, we fix the parameters $s, u \in (0,1]$, as in the statement of Theorem 3.1. The parameter $\varepsilon > 0$ will be determined in the proof, see (3.5). We record that in the case $s = t$, the incidence inequality of Fu–Ren in Theorem 2.6 simplifies to

$$|\mathcal{I}(P,\mathcal{L})|_\delta \le C_\varepsilon \delta^{-\varepsilon}\sqrt{\delta^{-s} K_{\mathcal{P}} K_{\mathcal{T}} |P|_\delta |\mathcal{L}|_\delta}, \quad \varepsilon > 0. \tag{3.2}$$

We will use $A \lessapprox B$ to signify that there exists a constant $C > 0$, depending only on s and u, such that $A \le C\delta^{-C\varepsilon}B$. The two-sided inequality $A \lessapprox B \lessapprox A$ is abbreviated to $A \approx B$.

Proof of Theorem 3.1. After an initial reduction performed right away, the proof will be divided into *Steps* 1–3. The initial reduction is this: we may assume that the lines $\ell = \ell_{a,b} \in \mathcal{L}$ have slopes $a \in [-1,1]$. In fact, there always exists a subset $\mathcal{L}' \subset \mathcal{L}$ such that (a) every pair of lines from $\mathcal{L}'$ forms an angle $\le 1/10$, and (b) $|\mathcal{I}(P,\mathcal{L}')|_\delta \sim |\mathcal{I}(P,\mathcal{L}')|_\delta$. We may then rotate both P and $\mathcal{L}'$ such that the $[-1,1]$-slope condition is satisfied, and afterwards we proceed to find a (δ,δ^u)-clique inside $P \times \mathcal{L}'$.

The $[-1,1]$-slope hypothesis is equivalent to $\mathcal{L} \subset D([-1,1]\times\mathbb{R})$, so Lemma 2.5 will now allow us to express the δ-covering number $|\mathcal{I}(P',\mathcal{L}')|_\delta$, for $P' \subset P$ and $\mathcal{L}' \subset \mathcal{L}$, in the convenient dyadic form $|\mathcal{D}_\delta(\mathcal{I}(P',\mathcal{L}'))|$.

Step 1. *Reduction to the case where $\mathcal{D}_\delta(P)$ is uniform.*

We first apply Corollary 2.12 with parameter $\sqrt{\varepsilon}$ to the family $\mathcal{P} := \mathcal{D}_\delta(P)$. This produces a constant $H \sim_\varepsilon 1$ and a list of disjoint $\{2^{-jH}\}_{j=1}^m$-uniform subsets $\mathcal{P}_1,\dots,\mathcal{P}_{N_0} \subset \mathcal{P}$ such that $|\mathcal{P}_j| \ge \delta^{2\sqrt{\varepsilon}}|\mathcal{P}|$, and the "remainder set"

$$\mathcal{R} := \mathcal{P} \setminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{N_0})$$

satisfies $|\mathcal{R}| \le \delta^{\sqrt{\varepsilon}}|\mathcal{P}| \le \delta^{\sqrt{\varepsilon}-\varepsilon-s}$. In particular, $N_0 \le \delta^{-2\sqrt{\varepsilon}}$. Since $\mathcal{R}$ is also a Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set, using (3.2) and the upper bound of $|\mathcal{R}|$ gives

$$|\mathcal{I}(\mathcal{R},\mathcal{L})|_\delta \lesssim_\varepsilon \delta^{-\varepsilon}\sqrt{\delta^{-s-2\varepsilon}|\mathcal{R}|\,|\mathcal{L}|_\delta} \le \delta^{\sqrt{\varepsilon}/2-3\varepsilon}\,\delta^{-3s/2} \le \frac{1}{2}|\mathcal{I}(P,\mathcal{L})|_\delta,$$

provided that $\varepsilon < 1/64$ and $\delta > 0$ is small enough. Since P is contained in the union of $\mathcal{R}$ and $\{\mathcal{P}_j\}$, there exists at least one index $j \in \{1,\dots,N_0\}$ such that

$$|\mathcal{I}(P \cap \mathcal{P}_j,\mathcal{L})|_\delta \ge \frac{1}{2}N_0^{-1}|\mathcal{I}(P,\mathcal{L})|_\delta \gtrsim \delta^{3\sqrt{\varepsilon}-3s/2}.$$

Here $\mathcal{P}_j = \mathcal{D}_\delta(P \cap \mathcal{P}_j)$ is $\{2^{-jH}\}_{j=1}^m$-uniform. Thus, at the cost of replacing "ε" by "$3\sqrt{\varepsilon}$" in the hypothesis (3.1), we can assume that $\mathcal{P}$ is $\{2^{-jH}\}_{j=1}^m$-uniform for some $H \sim_\varepsilon 1$.

Write $\mathcal{T} := \mathcal{T}^\delta(\mathcal{L})$. We claim that $|\mathcal{P}| \ge \delta^{6\varepsilon-s}$ and $|\mathcal{T}| \ge \delta^{6\varepsilon-s}$, thus $|\mathcal{P}| \approx \delta^{-s} \approx |\mathcal{T}|$. Indeed, otherwise one can check that (3.2) already gives an upper bound smaller than (3.1). Moreover, we claim that there exists a subset $\overline{\mathcal{T}} \subset \mathcal{T}$ with $|\overline{\mathcal{T}}| \approx |\mathcal{T}| \approx \delta^{-s}$ such that

$$|\{p \in \mathcal{P} : x \in \ell \text{ for some } x \in P \cap p, \ell \in \mathcal{L} \cap T\}| \sim |\mathcal{I}(P,\mathcal{L} \cap T)|_\delta \approx \delta^{-s/2}, \tag{3.3}$$

for $T \in \overline{\mathcal{T}}$. (The middle "$\sim$" follows from Lemma 2.5 applied to P and $\mathcal{L} \cap T = \{\ell \in \mathcal{L} : \ell \subset T\}$.) To see (3.3), we pigeonhole a subset $\overline{\mathcal{T}} \subset \mathcal{T}$ with the properties that $|\mathcal{I}(P,\mathcal{L} \cap T)|_\delta$ is roughly constant for any $T \in \overline{\mathcal{T}}$, say $|\mathcal{I}(P,\mathcal{L} \cap T)|_\delta \sim M_0$, and moreover,

$$|\mathcal{I}(P,\mathcal{L} \cap \overline{\mathcal{T}})|_\delta \approx |\mathcal{I}(P,\mathcal{L})|_\delta \approx \delta^{-3s/2}.$$

By (3.2), we have $|\mathcal{I}(P, \mathcal{L} \cap \overline{\mathcal{T}})| \lessapprox \delta^{-s}|\overline{\mathcal{T}}|^{1/2}$. Thus $|\overline{\mathcal{T}}| \approx \delta^{-s}$, and then $M_0 \cdot |\overline{\mathcal{T}}| \approx \delta^{-3s/2}$ implies that $M_0 \approx \delta^{-s/2}$. In the sequel, we simplify notation by dropping the "bar" and denoting $\overline{\mathcal{T}}$ still by $\mathcal{T}$. (So, formally, the proof will finally produce a (δ, δ^u)-clique inside $(P \cap \mathcal{P}_j) \times (\mathcal{L} \cap \overline{\mathcal{T}})$.) We have now reduced matters to a situation where $\mathcal{P} = \mathcal{D}_\delta(P)$ is $\{2^{-jH}\}_{j=1}^m$-uniform with $H \sim_\varepsilon 1$, and $\mathcal{T} = \mathcal{T}^\delta(\mathcal{L})$ satisfies (3.3).

Step 2. *Finding the dyadic scale* $\Delta \in [\delta, 1]$.

Fix $T \in \mathcal{T}$. We want to show, roughly speaking, that most of the squares in the set

$$\overline{\mathcal{P}}_T := \{p \in \mathcal{P} : x \in \ell \text{ for some } x \in P \cap p, \ell \in \mathcal{L} \cap T\},$$

familiar with (3.3), lie inside a single rectangle of dimensions roughly $\delta \times \sqrt{\delta}$. A priori, we have no information about the distribution of $\overline{\mathcal{P}}_T$, but at least Proposition 2.17 applied to each individual $\overline{\mathcal{P}}_T$, $T \in \mathcal{T}$, allows us to find a few useful objects.

Namely, we apply Proposition 2.17 with $d = 2$, $\mathbf{C} := 400/(su)$, and

$$\eta_0 := \frac{e^{-2\mathbf{C}}su}{200} = \frac{2e^{-2\mathbf{C}}}{\mathbf{C}}. \tag{3.4}$$

With this notation established, we can state the sufficient condition for the constant "ε" in Theorem 3.1:

$$0 < \varepsilon \leq c\eta_0, \tag{3.5}$$

where $c = c(s, u) > 0$ is a small constant depending only on s, u, determined later.

The conclusion of Proposition 2.17 is that for every $T \in \mathcal{T}$, the following objects exist:

(i) a number $\eta_T \in [\eta_0, \eta_0 e^{2\mathbf{C}}] = [\eta_0, su/200]$ and a scale $\Delta_T \in [\delta, 1]$;

(ii) a square $Q_T \in \mathcal{D}_{\Delta_T}(\overline{\mathcal{P}}_T)$ and a subset $\mathcal{P}_T \subset \overline{\mathcal{P}}_T \cap Q_T$.

These objects satisfy

(P1) $|\mathcal{P}_T| \geq \delta^{\eta_T}|\overline{\mathcal{P}}_T| \gtrapprox \delta^{\eta_T - s/2}$ (using (3.3)).

(P2) $S_{Q_T}(\mathcal{P}_T)$ is a $((\delta/\Delta_T), \mathbf{C}\eta_T, O_{\mathbf{C}}(1))$-set.

The parameters Δ_T and η_T initially depend on "T", but this can be fixed by another pigeonholing. Indeed, there exist a subset $\mathcal{T}' \subset \mathcal{T}$ of cardinality $|\mathcal{T}'| \gtrsim (\log \frac{1}{\delta})^{-1}|\mathcal{T}|$, and fixed numbers $\eta \in [\eta_0, 2/\mathbf{C}]$ and $\Delta \in [\delta, 1] \cap \{2^{-jH}\}_{j=1}^m$ such that

$$\eta_T \in [\eta, 2\eta] \quad \text{and} \quad \Delta_T \in [\Delta, 2^H \Delta], \quad T \in \mathcal{T}'.$$

Since there will be no difference between $\mathcal{T}'$ and $\mathcal{T}$ for the remainder of the argument, we simplify notation by denoting $\mathcal{T}'$ again by $\mathcal{T}$. In other words, we assume that $\eta_T \sim \eta$ and $\Delta_T \sim_\varepsilon \Delta$ for all $T \in \mathcal{T}$.

Now we arrive at a key claim in the proof: assuming (3.4), (3.5) and $\delta > 0$ sufficiently small in terms of s and u, we have

$$\delta^{1/2 + 2\eta/s} \lessapprox \Delta \leq \delta^{1/2 - u/3}. \tag{3.6}$$

This will be the key estimates to establish the existence of a (δ, δ^u)-clique in *Step* 3. The rest of *Step* 2 is devoted to proving (3.6).

By the Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set property of P (hence $\mathcal{P}$) and the bound (P1) for $|\mathcal{P}_T|$,

$$(\Delta/\delta)^s \gtrapprox |\mathcal{P} \cap Q_T| \geq |\mathcal{P}_T| \gtrapprox \delta^{2\eta - s/2}.$$

This yields the lower bound in (3.6) for Δ.

By using the lower bound, we briefly record that the sets $S_{Q_T}(\mathcal{P}_T)$ satisfy a 2-ends condition, more precisely (3.7) below. Indeed, by property (P2) and using $\eta_T \geq \eta$, for every $y \in \mathbb{R}^2$, $r \in [\delta/\Delta_T, 1]$ and $T \in \mathcal{T}$, it holds

$$|S_{Q_T}(\mathcal{P}_T) \cap B(y,r)| \leq O_{\mathrm{C}}(1) r^{\mathrm{C}\eta_T} |S_{Q_T}(\mathcal{P}_T)| \leq O_{\mathrm{C}}(1) r^{\mathrm{C}\eta} |S_{Q_T}(\mathcal{P}_T)|.$$

Let $r_0 := (3 O_{\mathrm{C}}(1))^{-1/(\mathrm{C}\eta)}$, a constant depending only on s, u. Note that $r_0 \geq \delta/\Delta \geq \delta/\Delta_T$ by the lower bound in (3.6), provided $\delta > 0$ is sufficiently small. Then,

$$|S_{Q_T}(\mathcal{P}_T) \cap B(y, r_0)| \leq \tfrac{1}{3} |S_{Q_T}(\mathcal{P}_T)|, \quad \forall y \in \mathbb{R}^2, T \in \mathcal{T}. \tag{3.7}$$

The estimate (3.7) will eventually allow us to use Proposition 2.9.

Now we begin to establish the upper bound in (3.6). As an intermediate goal, we want to show that

$$|\mathcal{P}|_\Delta \lessapprox \delta^{-4\eta} \Delta^{-s}. \tag{3.8}$$

Since $\mathcal{P}$ is $\{2^{-jH}\}_{j=1}^m$-uniform, and $\Delta \in \{2^{-jH}\}_{j=1}^m$, there exists a constant $M \lessapprox (\Delta/\delta)^s$ such that $|\mathcal{P} \cap Q| = M$ for all $Q \in \mathcal{D}_\Delta(\mathcal{P})$, and in fact $M = |\mathcal{P}|/|\mathcal{P}|_\Delta$. We claim that $M \gtrapprox \delta^{4\eta}(\Delta/\delta)^s$, which will give (3.8) because $|\mathcal{P}|_\Delta \approx \delta^{-s}/M$.

Recall the squares $Q_T \in \mathcal{D}_\Delta(\mathcal{P})$, $T \in \mathcal{T}$, in (ii). By the pigeonhole principle, there exists at least one $Q_0 \in \mathcal{D}_\Delta(\mathcal{P})$ such that

$$|\{T \in \mathcal{T} : Q_T = Q_0\}| \geq \frac{|\mathcal{T}|}{|\mathcal{P}|_\Delta} \approx \frac{|\mathcal{P}|}{|\mathcal{P}|_\Delta} = M. \tag{3.9}$$

The square Q_0 will be fixed for the rest of the proof, and we write

$$\mathcal{T}_0 := \{T \in \mathcal{T} : Q_T = Q_0\}. \tag{3.10}$$

For $T_1, T_2 \in \mathcal{T}_0$, we say that two intersections $T_1 \cap Q_0$ and $T_2 \cap Q_0$ are *comparable* if there exists a rectangle R of dimensions $\sim (\Delta \times \delta)$ containing both $T_j \cap Q_0$, $j \in \{1,2\}$. (The exact requirement for the dimensions of R is determined by the following: if $T_1 \cap Q_0$ and $T_2 \cap Q_0$ are incomparable, then the rescaled sets $S_{Q_0}(T_j \cap Q_0)$ are contained in distinct ordinary $C(\delta/\Delta)$-tubes. This will be used when we soon apply Proposition 2.9.)

We claim that the family

$$\mathcal{T}_0 \cap Q_0 := \{T \cap Q_0 : T \in \mathcal{T}_0\}$$

contains $\gtrapprox \Delta^s M$ incomparable intersections. This is based on the Katz–Tao (δ, s)-condition of $\mathcal{L}$ (hence $\mathcal{T}$). Assume that $\{T_1, \ldots, T_k\}$ is a family of dyadic δ-tubes such that every intersection $\{T_i \cap Q_0\}$ is comparable to one fixed intersection $\{T_{i_0} \cap Q_0\}$. Let l_i be some line contained in T_i. Then the angle between any two lines of $\{l_1, \ldots, l_k\}$ is $\lesssim \delta/\Delta$, thus there exists a $(2 \times (C\delta/\Delta))$-rectangle $\mathbf{T}$ such that $[0,1]^2 \cap T_i \subset \mathbf{T}$ for $1 \le i \le k$, where $C \ge 1$ is absolute. Since $\mathcal{T}$ is a Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set, it follows

$$k \lessapprox \Delta^{-s}. \tag{3.11}$$

From this and (3.9), we deduce that the family $\mathcal{T}_0 \cap Q_0$ has $\gtrapprox \Delta^s M$ incomparable intersections, as desired. We denote this in a slightly ad hoc manner as

$$|\mathcal{T}_0 \cap Q_0|_{\Delta \times \delta} \gtrapprox \Delta^s M.$$

This enables us to use Proposition 2.9 after rescaling. Indeed, if $T_1 \cap Q_0$ and $T_2 \cap Q_0$ are incomparable, then the rescaled sets $S_{Q_0}(T_1 \cap Q_0)$ and $S_{Q_0}(T_2 \cap Q_0)$ are distinct (δ/Δ)-tubes. Since $S_{Q_T}(\mathcal{P}_T)$ satisfies the 2-ends condition by (3.7), we infer from Proposition 2.9 applied at scale δ/Δ that

$$\begin{aligned} M &= |\mathcal{P} \cap Q_0| = |S_{Q_0}(\mathcal{P} \cap Q_0)| \\ &\gtrapprox \left|\mathcal{T} \cap Q_0\right|_{\Delta \times \delta}^{1/2} \cdot \delta^{2\eta - s/2} \gtrapprox \delta^{2\eta} \left(\frac{\Delta}{\delta}\right)^{s/2} M^{1/2}. \end{aligned} \tag{3.12}$$

Rearranging this inequality leads to $M \gtrapprox \delta^{4\eta}(\Delta/\delta)^s$. This finally proves (3.8). From (3.9), we also obtain

$$|\mathcal{T}_0| \gtrapprox \delta^{4\eta} \left(\frac{\Delta}{\delta}\right)^s. \tag{3.13}$$

Next, define a subset $\Xi \subset \mathcal{T}_0$ to be a *tube packet* if Ξ has the form

$$\Xi = \{T \in \mathcal{T}_0 : T \cap Q_0 \subset R\}, \tag{3.14}$$

where R is a rectangle of dimensions $\sim (\delta \times \Delta)$. Thus, the intersections $T \cap Q_0$, $T \in \Xi$, are pairwise comparable. In (3.11), we showed that the cardinality of every tube packet Ξ satisfies $|\Xi| \lessapprox \Delta^{-s}$. By the pigeonhole principle, there exists a value $n \in \{1, \ldots, \lessapprox \Delta^{-s}\}$ such that $\approx |\mathcal{T}_0|$ tubes of $\mathcal{T}_0$ are contained in tube packets $\Xi_1, \ldots, \Xi_L$ with $|\Xi_j| \sim n$. Since $n \lessapprox \Delta^{-s}$, we have the lower bound

$$L \approx |\mathcal{T}_0|/n \overset{(3.13)}{\gtrapprox} \delta^{4\eta}(\Delta/\delta)^s \cdot \Delta^s = \delta^{4\eta}(\Delta^2/\delta)^s. \tag{3.15}$$

On the other hand, we can match this with an upper bound by repeating the argument of (3.12) (and using $|\mathcal{T} \cap Q_0|_{\Delta \times \delta} \ge L$):

$$\left(\frac{\Delta}{\delta}\right)^s \gtrapprox M = |\mathcal{P} \cap Q_0| \gtrapprox L^{1/2} \cdot \delta^{2\eta - s/2}. \tag{3.16}$$

Rearranging this gives $L \lessapprox \delta^{-4\eta}(\Delta^2/\delta)^s$. Hence we get the following useful bounds for L:

$$\delta^{4\eta}(\Delta^2/\delta)^s \lessapprox L \lessapprox \delta^{-4\eta}(\Delta^2/\delta)^s. \tag{3.17}$$

It will also be useful to record that

$$n \approx \frac{|\mathcal{T}_0|}{L} \overset{(3.13)}{\gtrapprox} \frac{\delta^{4\eta}(\Delta/\delta)^s}{(\Delta^2/\delta)^s \cdot \delta^{-4\eta}} = \delta^{8\eta} \cdot \Delta^{-s}. \tag{3.18}$$

To proceed, for each tube packet Ξ_j, choose one representative $T_j \in \Xi_j$; we may assume that the intersections $T_j \cap Q_0$ are incomparable for different indices "j" (there exist $\sim L$ packets such that their representatives have this property, and we restrict attention to those packets without changing notation). Let $S_{Q_0}\colon Q_0 \to [0,1)^2$ be the rescaling map, and define the following rescaled sets:

- $\mathbb{T} := \{S_{Q_0}(T_j \cap Q_0) : 1 \le j \le L\}$;
- $\mathbb{P} := \{S_{Q_0}(p) : p \in \mathcal{P} \cap Q_0\} \subset \mathcal{D}_{\delta/\Delta}$;
- $\mathbb{P}_T := S_{Q_0}(\mathcal{P}_T)$, where $\mathcal{P}_T \subset \mathcal{P} \cap T \cap Q_T$ is the subset obtained in (ii).

Here $\mathbb{T}$ is a collection of distinct (δ/Δ)-tubes with $|\mathbb{T}| = L$. We also recall from (P1) that $|\mathbb{P}_T| \gtrapprox \delta^{2\eta - s/2}$.

We are about to apply Theorem 2.8 to the objects $\mathbb{T}$ and $\mathbb{P}_T$. We will leave to the reader the small technical point that the sets in $\mathbb{T}$ are not exactly (ordinary) (δ/Δ)-tubes. Each element of $\mathbb{T}$ is, however, contained in some $C(\delta/\Delta)$-tube. Theorem 2.8 can then be applied to a maximal distinct subset in the ensuing family of $C(\delta/\Delta)$-tubes.

The main challenge in applying Theorem 2.8 is to verify the non-concentration condition (2.4) for the collection of (δ/Δ)-tubes $\mathbb{T}$. This amounts to checking the following. Let

$$\rho := \Big(\frac{\delta}{\Delta}\Big)|\mathbb{T}|^{1/2} \overset{(3.17)}{\approx} \Big(\frac{\delta}{\Delta}\Big) \cdot \Big(\frac{\Delta^2}{\delta}\Big)^{s/2} \cdot \delta^{\pm 4\eta} = \Delta^{s-1}\delta^{1-s/2\pm 4\eta}. \tag{3.19}$$

Let T_ρ be an arbitrary $(2 \times \rho)$-rectangle, and consider the quantity

$$X := |\{\mathbf{T} \in \mathbb{T} : \mathbf{T} \subset T_\rho\}|. \tag{3.20}$$

After rescaling back to Q_0, there exist X indices "j" such that $T_j \cap Q_0 \subset \bar{T}_\rho$, where $\bar{T}_\rho := S_{Q_0}^{-1}(T_\rho)$ is now a rectangle of dimensions $(2\Delta \times \rho\Delta)$. A little trigonometry shows that whenever $T_j \cap Q_0 \subset \bar{T}_\rho$, then there exists an $(A \times A\rho)$-rectangle $T_{A\rho}$ such that all the tubes $T \in \Xi_j$ in the packet represented by T_j satisfy

$$T \cap [0,1]^2 \subset T_{A\rho},$$

where $A \ge 1$ is an absolute constant. Now, recalling from (3.18) that $|\Xi_j| \sim n \gtrapprox \delta^{8\eta}\Delta^{-s}$, we infer from the Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set condition of $\mathcal{T}$ that

$$X \cdot \Delta^{-s} \cdot \delta^{8\eta} \lessapprox |\{T \in \mathcal{T} : T \subset T_{A\rho}\}| \lessapprox \Big(\frac{\rho}{\delta}\Big)^s \overset{(3.19)}{\lessapprox} (\Delta^{s-1}\delta^{-s/2-4\eta})^s.$$

This implies $X \lessapprox \delta^{-8\eta-4s\eta}(\Delta^s \cdot \delta^{-s/2})^s$. Since $|\mathbb{T}| = L \gtrapprox \delta^{4\eta}(\Delta^2/\delta)^s$ by (3.15), we see that

$$|\{\mathbf{T} \in \mathbb{T} : \mathbf{T} \subset T_\rho\}| = X \lessapprox \delta^{-16\eta}\Big(\frac{\delta}{\Delta^2}\Big)^{s(1-s/2)}|\mathbb{T}|. \tag{3.21}$$

We claim that this implies the upper bound $\Delta \le \delta^{1/2-u/3}$ asserted in (3.6).

Assume that this fails: thus $\Delta > \delta^{1/2-u/3}$. Then $\delta^{-16\eta}(\delta/\Delta^2)^{s(1-s/2)} < \delta^{su/5}$ thanks to $\eta \le su/200$. Therefore, the non-concentration condition of (2.4) is satisfied with exponent "$su/5$" in place of "u". Recalling that the sets $\mathbb{P}_T$ are $(\delta/\Delta, \mathbf{C}\eta, O_{\mathbf{C},s}(1))$-sets with $|\mathbb{P}_T| \gtrapprox \delta^{-s/2+2\eta} =: N$, Theorem 2.8 (with $\alpha = \frac{1}{5} \cdot (\mathbf{C}\eta)su/8$) implies the following improvement over the 2-ends bound in (3.12):

$$\left(\frac{\Delta}{\delta}\right)^s \gtrapprox |\mathbb{P}| \geq N \cdot |\mathbb{T}|^{1/2} \cdot (\delta/\Delta)^{-\mathbf{C}\eta \cdot su/40}$$
$$\overset{(3.15)}{\gtrapprox} \delta^{4\eta - s/2} \cdot \left(\frac{\Delta^2}{\delta}\right)^{s/2} \cdot (\delta/\Delta)^{-\mathbf{C}\eta \cdot su/40}.$$

Rearranging the inequality, using $\Delta > \delta^{1/2-u/3} \geq \delta^{1/2}$, and recalling from above (3.4) that $\mathbf{C} = 400/(su)$, we obtain

$$1 \geq C\delta^{-\mathbf{C}\eta \cdot su/80 + 4\eta + C\varepsilon} = C\delta^{-\eta + C\varepsilon}.$$

Here $C = C(s,u) > 0$ is a constant depending only on s and u. Since $\varepsilon \leq c\eta_0 \leq \eta/(2C)$ by (3.5) and (i), we get a contradiction for all $\delta > 0$ sufficiently small. This concludes the proof of the upper bound in (3.6).

Step 3. *Finding a* (δ, δ^u)*-clique.*

We now use the information of tube packets (introduced in (3.14)) to define a clique and then apply (3.6) to verify the properties claimed in Theorem 3.1. Recalling (3.18), note that there exists at least one tube packet, denoted $\Xi_0 = \{T \in \mathcal{T}_0 : T \cap Q_0 \subset R_\Delta\}$, such that

$$|\Xi_0| \sim n \gtrapprox \delta^{8\eta} \cdot \Delta^{-s} \geq \delta^{-s/2+su/3+8\eta}.$$

Here R_Δ is a rectangle of dimensions $\sim (\delta \times \Delta)$. Since $\mathcal{P}_T \subset \overline{\mathcal{P}}_T \cap Q_T$ (recall the definition of $\overline{\mathcal{P}}_T$ at the beginning of *Step* 2), whenever $Q_T = Q_0$, then $\cup\mathcal{P}_T \subset Q_0$, and

$$|\{p \in \mathcal{P} \cap Q_0 : x \in \ell \text{ for some } x \in P \cap p, \ell \in \mathcal{L} \cap T\}| \geq |\mathcal{P}_T| \gtrapprox \delta^{2\eta - s/2}.$$

Since $\Xi_0 \subset \mathcal{T}_0$ (recall the definition of $\mathcal{T}_0$ from (3.10)),

$$|\{p \in \mathcal{P} \cap Q_0 : x \in \ell \text{ for some } p \in P \cap p, \ell \in \mathcal{L} \cap T\}| \gtrapprox \delta^{-s/2+2\eta}, \quad T \in \Xi_0.$$

We are ready to define the clique:

$$P' := P \cap Q_0 \quad \text{and} \quad \mathcal{L}' := \mathcal{L} \cap \Xi_0 := \{\ell \in \mathcal{L} : \ell \subset T \text{ for some } T \in \Xi_0\}. \tag{3.22}$$

Then $|\mathcal{I}(P', \mathcal{L}')|_\delta \gtrapprox |\Xi_0|\delta^{-s/2+2\eta} \gtrapprox \delta^{-s+su/3+10\eta}$. On the other hand, using the Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-property of P, the upper bound $|\mathcal{L}'|_\delta = |\Xi_0| \lessapprox \Delta^{-s}$ valid for all tube packets, and both inequalities in (3.6), we get

$$\delta^{-s/2+3\eta} \leq |P'|_\delta \leq \delta^{-s/2-su/3-\eta} \quad \text{and} \quad \delta^{-s/2+su/3+10\eta} \leq |\mathcal{L}'|_\delta \leq \delta^{-s/2-3\eta}.$$

By assumption, $\eta \leq us/200$. It follows from the numerology above that

$$|\mathcal{I}(P', \mathcal{L}')|_\delta \geq \delta^u |P'|_\delta |\mathcal{L}'|_\delta, \quad \text{with} \quad |P'|_\delta \geq \delta^{u-s/2} \text{ and } |\mathcal{L}'|_\delta \geq \delta^{u-s/2}.$$

Thus, $P' \times \mathcal{L}'$ is a (δ, δ^u)-clique satisfying the claims of Theorem 3.1. ■

3.2. Case $s < t$

The proof is similar to the case $s = t$, except that the argument does not rely on Theorem 2.8: in the variant of *Step* 2 below, a completely elementary argument gives the desired upper bound for Δ. Where the proof is virtually the same as in the case $s = t$, we will omit some repeated details.

From now on, we fix the parameters $u \in (0, 1]$ and $s, t \in (0, 1]$ with $s < t$, as in the statement of Theorem 1.7. Recall that $f(s,t) = (s^2 + st + t^2)/(s + t)$.

Proof of Theorem 1.7 in the case $s < t$. We use $A \lessapprox B$ to signify that there exists a constant $C > 0$, depending only on s, t and u, such that $A \leq C\delta^{-C\varepsilon}B$. Here $\varepsilon > 0$ is the constant from the main hypothesis (1.4). The constant $\varepsilon > 0$ will be specified at (3.25). Just like in the case $s = t$, it is easy to reduce matters to the situation where the slopes of the lines in $\mathcal{L}$ lie in $[-1, 1]$. This makes Lemma 2.5 applicable.

Step 1. *Reduction to the case where $\mathcal{D}_\delta(P)$ is uniform.*

We also denote $\mathcal{P} := \mathcal{D}_\delta(P)$ and $\mathcal{T} := \mathcal{T}^\delta(\mathcal{L})$. Then it suffices to prove Theorem 1.7 under the following additional hypotheses:

(i) $\mathcal{P}$ is $\{2^{-jH}\}_{j=1}^m$-uniform for some $H \sim_\varepsilon 1$.

(ii) $|\mathcal{P}| \approx \delta^{-s}$ and $|\mathcal{T}| \approx \delta^{-t}$.

(iii) $|\mathcal{I}(P, \mathcal{L} \cap T)|_\delta \approx \delta^{-s^2/(s+t)}$, $T \in \mathcal{T}$, where by Lemma 2.5,

$$|\mathcal{I}(P, \mathcal{L} \cap T)|_\delta \sim |\{p \in \mathcal{P} : x \in \ell \text{ for some } x \in P \cap p, \ell \in \mathcal{L} \cap T\}|.$$

This reduction was carried out in detail in *Step* 1 of the case $s = t$, and the arguments are exactly the same, up to changing the numerology, and applying the case $s < t$ of Fu and Ren's Theorem 2.6. Morally, (ii) follows from (1.4), because if either $|\mathcal{P}| \ll \delta^{-s}$ or $|\mathcal{T}| \ll \delta^{-t}$, then Theorem 2.6 already gives an improvement over (1.4). Eventually, (iii) follows from (ii) and (1.4) after another pigeonholing argument: morally but inaccurately, this is the computation $|\mathcal{I}(P, \mathcal{L} \cap T)|_\delta \approx |\mathcal{T}|^{-1}|\mathcal{I}(P, \mathcal{L})|_\delta \approx \delta^{t-f(s,t)} = \delta^{-s^2/(s+t)}$.

Step 2. *Finding the dyadic scale $\Delta \in [\delta, 1]$.*

The argument in this step will initially resemble the case $s = t$ closely, but eventually Theorem 2.8 will not be needed. For each $T \in \mathcal{T}$, we write

$$\overline{\mathcal{P}}_T := \{p \in \mathcal{P} : x \in \ell \text{ for some } x \in P \cap p, \ell \in \mathcal{L} \cap T\}. \tag{3.23}$$

By property (iii) in *Step* 1, $|\overline{\mathcal{P}}_T| \approx \delta^{-s^2/(s+t)}$. We apply Proposition 2.17 for each $\overline{\mathcal{P}}_T$ with parameters $d = 2$, $\mathbf{C} := 160/(su(t - s))$ and

$$\eta_0 := \frac{e^{-2\mathbf{C}}su(t - s)}{80} = \frac{2e^{-2\mathbf{C}}}{\mathbf{C}}. \tag{3.24}$$

We claim that the following bound suffices for the parameter "ε" in (1.4):

$$0 < \varepsilon \leq c\eta_0. \tag{3.25}$$

Here $c = c(s, t, u) > 0$ is a small constant depending only on s, t, u, determined later.

The conclusion of Proposition 2.17 is that for every $T \in \mathcal{T}$, the following objects exist:

(a) a number $\eta_T \in [\eta_0, \eta_0 e^{2\mathbf{C}}] = [\eta_0, su(t-s)/80]$ and a scale $\Delta_T \in [\delta, 1]$;

(b) a square $Q_T \in \mathcal{D}_{\Delta_T}(\overline{\mathcal{P}}_T)$ and a subset $\mathcal{P}_T \subset \overline{\mathcal{P}}_T \cap Q_T$;

which satisfy

(L1) $|\mathcal{P}_T| \geq \delta^{\eta_T} |\overline{\mathcal{P}}_T| \gtrapprox \delta^{\eta_T - s^2/(s+t)}$;

(L2) $S_{Q_T}(\mathcal{P}_T)$ is a $((\delta/\Delta_T), \mathbf{C}\eta_T, O_{\mathbf{C}}(1))$-set.

To remove the dependence on T for η_T and Δ_T, pigeonhole a subset $\mathcal{T}' \subset \mathcal{T}$ of cardinality $|\mathcal{T}'| \gtrsim (\log \frac{1}{\delta})^{-1} |\mathcal{T}|$, and fixed numbers $\eta \in [\eta_0, 2/\mathbf{C}]$ and $\Delta \in [\delta, 1] \cap \{2^{-jH}\}_{j=1}^m$ such that

$$\eta_T \in [\eta, 2\eta] \quad \text{and} \quad \Delta_T \in [\Delta, 2^H \Delta], \quad T \in \mathcal{T}'.$$

In the following, we simplify notation by denoting $\mathcal{T}'$ again by $\mathcal{T}$.

The rest of *Step* 2 is devoted to proving the claim: under our choices for $\mathbf{C}$ and ε, if $\delta > 0$ is sufficiently small in terms of s, t and u, then

$$\delta^{t/(s+t)+2\eta/s} \lessapprox \Delta \leq \delta^{t/(s+t)-u/8}. \tag{3.26}$$

By the Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set property of P and the lower bound on $|\mathcal{P}_T|$, we deduce

$$\left(\frac{\Delta}{\delta}\right)^s \gtrapprox |\mathcal{P} \cap Q_T| \geq |\mathcal{P}_T| \gtrapprox \delta^{2\eta - s^2/(s+t)},$$

which yields the lower bound in (3.26).

By the lower bound of Δ, we can also verify that the sets $S_{Q_T}(\mathcal{P}_T)$ satisfy a 2-ends condition. Indeed, for any $x \in \mathbb{R}^2$ and $r \in [\delta/\Delta_T, 1]$, we have from (L2) and $\eta_T \geq \eta$ that

$$|S_{Q_T}(\mathcal{P}_T) \cap B(x,r)| \leq O_{\mathbf{C}}(1)\, r^{\mathbf{C}\eta_T} |S_{Q_T}(\mathcal{P}_T)| \leq O_{\mathbf{C}}(1)\, r^{\mathbf{C}\eta} |S_{Q_T}(\mathcal{P}_T)|.$$

Choose $r_0 := (3 O_{\mathbf{C}}(1))^{-1/(\mathbf{C}\eta)}$, which is a constant depending only on s, t and u, and by the lower bound of Δ we have $r_0 \geq \delta/\Delta \geq \delta/\Delta_T$ provided that δ is small enough. Then

$$|S_{Q_T}(\mathcal{P}_T) \cap B(x, r_0)| \leq \frac{1}{3} |S_{Q_T}(\mathcal{P}_T)|, \quad x \in \mathbb{R}^2. \tag{3.27}$$

In particular, a dependence on "r_0" is allowed in the $\lessapprox$ notation below.

We then proceed to prove the upper bound in (3.26). For any $Q \in \mathcal{D}_\Delta(\mathcal{P})$, we get from the $(\delta, s, \delta^{-\varepsilon})$-set condition and uniformity of $\mathcal{P}$ that

$$M := \frac{|\mathcal{P}|}{|\mathcal{P}|_\Delta} = |\mathcal{P} \cap Q| \lessapprox \left(\frac{\Delta}{\delta}\right)^s.$$

Recall $Q_T \in \mathcal{D}_\Delta(\overline{\mathcal{P}}_T)$ in (b). By the pigeonhole principle, there exists $Q_0 \in \mathcal{D}_\Delta(\mathcal{P})$ such that

$$|\{T \in \mathcal{T} : Q_T = Q_0\}| \geq \frac{|\mathcal{T}|}{|\mathcal{P}|_\Delta} \approx M \delta^{s-t}. \tag{3.28}$$

The last equation follows from (ii) in *Step* 1. We also write

$$\mathcal{T}_0 := \{T \in \mathcal{T} : Q_T = Q_0\}. \tag{3.29}$$

As in the case $s = t$, we need to find a lower bound on the number of incomparable intersections in the family

$$\mathcal{T}_0 \cap Q_0 := \{T \cap Q_0 : T \in \mathcal{T}_0\}.$$

Recall that $T_1 \cap Q_0$ and $T_2 \cap Q_0$ are comparable if there exists a rectangle of dimensions $\sim (\delta \times \Delta)$ containing both $T_j \cap Q_0$, $j \in \{1, 2\}$. Assume $\{T_j\}_{j=1}^n \subset \mathcal{T}$ is a family (a *tube packet*) such that every intersection $\{T_j \cap Q_0\}$ is comparable to one fixed intersection $\{T_{j_0} \cap Q_0\}$. Then $n \lessapprox \Delta^{-t}$ by the $(\delta, t, \delta^{-\varepsilon})$-set condition of $\mathcal{T}$, see the proof of (3.11). From this and (3.28), we deduce that the family $\mathcal{T}_0 \cap Q_0$ has $L \gtrapprox \Delta^t M \delta^{s-t}$ incomparable elements. By applying Proposition 2.9 after rescaling by S_{Q_0}, and the 2-ends condition we established in (3.27),

$$M = |\mathcal{P} \cap Q_0| \gtrsim L^{1/2} \cdot \delta^{2\eta - s^2/(s+t)} \gtrapprox (\Delta^t M \delta^{s-t})^{1/2} \cdot \delta^{2\eta - s^2/(s+t)}, \tag{3.30}$$

which implies $M \gtrapprox \delta^{4\eta - 2s^2/(s+t) + s - t} \Delta^t$, and consequently,

$$\delta^{4\eta - (s^2+t^2)/(s+t)} \Delta^t = \delta^{4\eta - 2s^2/(s+t) + s - t} \Delta^t \lessapprox M \lessapprox \left(\frac{\Delta}{\delta}\right)^s. \tag{3.31}$$

Since $s < t$, we may infer that

$$\Delta \lessapprox \delta^{t/(s+t) - 4\eta/(t-s)}. \tag{3.32}$$

Finally, recall from (3.24) and (a) that $\max\{\eta, \eta_0\} \leq su(t - s)/80$. Recall also that the "$\lessapprox$" notation hides a constant of the form $C\delta^{-C\varepsilon}$, where $C = C(s, t, u) > 0$. Recalling from (3.25) that $\varepsilon \leq c\eta_0$, and finally taking $c := C(s, t, u)^{-1}$, we may deduce from (3.32) that $\Delta \leq \delta^{t/(s+t) - u/8}$, provided that $\delta > 0$ is sufficiently small in terms of s, t, u. This completes the proof of (3.26).

Step 3. *Finding a* (δ, δ^u)*-clique.*

Recall definition (3.29), and from a combination of (3.28) and (3.31) we deduce $|\mathcal{T}_0| \gtrapprox M\delta^{s-t} \gtrapprox \delta^{4\eta - 2s^2/(s+t) + 2s - 2t} \Delta^t$. A subset $\Xi \subset \mathcal{T}_0$ is a *tube packet* if Ξ has the form

$$\Xi = \{T \in \mathcal{T}_0 : T \cap Q_0 \subset R\},$$

where R is a rectangle of dimensions $\sim (\delta \times \Delta)$. Thus, the intersections $T \cap Q_0$, for $T \in \Xi$, are pairwise comparable. We claim that there exists a tube packet Ξ_0 with $|\Xi_0| \geq \delta^{-t^2/(s+t) + u/2}$ (this is roughly the extremal cardinality of a tube packet allowed by the Katz–Tao (δ, t)-set property of $\mathcal{T} \subset \mathcal{T}^\delta(\mathcal{L})$).

In *Step* 2, we already showed that every tube packet $\Xi \subset \mathcal{T}_0$ satisfies $|\Xi| \lessapprox \Delta^{-t}$. By the pigeonhole principle, there exists a value $n \in \{1, \ldots, \lessapprox \Delta^{-t}\}$ such that $\approx |\mathcal{T}_0|$ tubes of $\mathcal{T}_0$ are contained in tube packets $\Xi_1, \ldots, \Xi_L$ with $|\Xi_j| \sim n$. To get an upper bound for L, we recall the ("2-ends") lower bound (3.30):

$$\left(\frac{\Delta}{\delta}\right)^s \gtrapprox |\mathcal{P} \cap Q_0| \gtrapprox L^{1/2} \cdot \delta^{2\eta - s^2/(s+t)}.$$

This implies $L \lessapprox \delta^{-4\eta} \Delta^{2s} \delta^{-2st/(s+t)}$, so

$$n \approx \frac{|\mathcal{T}_0|}{L} \gtrapprox \frac{\delta^{4\eta - 2s^2/(s+t) + 2s - 2t} \Delta^t}{\delta^{-4\eta} \Delta^{2s} \delta^{-2st/(s+t)}} = \delta^{8\eta} \cdot \delta^{(2st - 2t^2)/(s+t)} \Delta^{t-2s}.$$

Using finally $\delta^{t/(s+t)+2\eta/s} \lessapprox \Delta \leq \delta^{t/(s+t)-u/8}$ (by (3.26)), $\eta \leq su(t-s)/80$ (by (a)) and $\varepsilon \leq c\eta_0 \leq c\eta$ (by (3.25) and (a)), we obtain after a little algebra the desired inequality $n \geq \delta^{-t^2/(s+t)+u/2}$. In particular, there exists a tube packet $\Xi_0 = \{T \in \mathcal{T}_0 : T \cap Q_0 \subset R_\Delta\}$ such that $|\Xi_0| \sim n \geq \delta^{-t^2/(s+t)+u/2}$. Here R_Δ is a rectangle of dimensions $\sim (\delta \times \Delta)$.

As in the case $s = t$, we now define

$$P' := P \cap Q_0 \quad \text{and} \quad \mathcal{L}' := \mathcal{L} \cap \Xi_0 := \{\ell \in \mathcal{L} : \ell \subset T \text{ for some } T \in \Xi_0\}.$$

Note that whenever $T \in \mathcal{T}_0$, then $Q_T = Q_0$ by definition, and (b) and (L1) imply

$$|\overline{\mathcal{P}}_T \cap Q_0| \geq |\mathcal{P}_T| \gtrapprox \delta^{2\eta - s^2/(s+t)}, \quad T \in \mathcal{T}_0. \tag{3.33}$$

Recalling the definition of $\overline{\mathcal{P}}_T$ from (3.23), this implies

$$|\mathcal{I}(P', \mathcal{L}')|_\delta \gtrapprox |\Xi_0| \delta^{2\eta - s^2/(s+t)} \gtrapprox \delta^{2\eta + u/2 - (s^2+t^2)/(s+t)}.$$

It further follows from the Katz–Tao conditions of $P' \subset P$ and $\mathcal{L}' \subset \mathcal{L}$ that

$$\delta^{-s^2/(s+t)+2\eta} \leq |P'|_\delta \leq \delta^{-s^2/(s+t)-u/6-\eta},$$
$$\delta^{-t^2/(s+t)+u/2} \leq |\mathcal{L}'|_\delta \leq \delta^{-t^2/(s+t)-u/40-2\eta}.$$

Recalling that $\eta \leq su(t-s)/80$, we easily conclude

$$|\mathcal{I}(P', \mathcal{L}')|_\delta \sim |\mathcal{D}_\delta(\mathcal{I}(P', \mathcal{L}'))| \geq \delta^u |P'|_\delta |\mathcal{L}'|_\delta,$$

where

$$|P'|_\delta \geq \delta^{u - s^2/(s+t)} \quad \text{and} \quad |\mathcal{L}'|_\delta \geq \delta^{u - t^2/(s+t)}.$$

This means that $P' \times \mathcal{L}'$ satisfies the claims in Theorem 1.7. ■

3.3. Case $s > t$

This is a standard duality argument, but we record the details. Our assumptions are the following: $P \subset [0,1]^2$ is a Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set, $\mathcal{L} \subset \mathcal{A}(2)$ is a Katz–Tao $(\delta, t, \delta^{-\varepsilon})$-set, $|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - f(s,t)}$, and $s > t$.

We may assume that the slopes of the lines in $\mathcal{L}$ lie in $[-1, 1]$, equivalently, $\mathcal{L} \subset D([-1,1] \times \mathbb{R})$. This is the same argument we already described at the beginning of Section 3.1. Assuming this, we infer from Lemma 2.5 that

$$|\mathcal{D}_\delta(\mathcal{I}(P, \mathcal{L}))| \gtrapprox \delta^{\varepsilon - f(s,t)}. \tag{3.34}$$

As a second initial reduction, we may assume that all the lines in $\mathcal{L}$ cross the y-axis in $[-2, 2]$. Indeed, other lines (with slopes in $[-1, 1]$) do not contain points of $P \subset [0,1]^2$.

For any line $l_{a,b} := \{(x, y) \in \mathbb{R}^2 : y = ax + b\}$, define the map $D^\star$ by $D^\star(l_{a,b}) = (-a, b)$. Then $D^\star$ is bi-Lipschitz on the subset of $\mathcal{A}(2)$ consisting of lines with slopes in $[-1, 1]$, in particular, on $\mathcal{L}$. We write

$$D^\star(T) = \{(-a, b) : (a, b) \in q\}, \quad T = D(q) \in \mathcal{T}^\delta.$$

Now we set

$$P^\star = D^\star(\mathcal{L}) \quad \text{and} \quad \mathcal{L}^\star = D(P).$$

From the bi-Lipschitz properties of D and $D^\star$, it follows that $P^\star$ is a $(\delta, t, \delta^{-\varepsilon})$-set contained in $[-1,1] \times [-2,2]$, and $\mathcal{L}^\star$ is a $(\delta, s, \delta^{-\varepsilon})$-set of lines with slopes in $[-1,1]$ and y-intersects in $\{0\} \times [-1,1]$.

We claim that

$$|\mathcal{I}(P^\star, \mathcal{L}^\star)|_\delta \gtrsim \delta^{\varepsilon - f(s,t)}. \tag{3.35}$$

Indeed, if (p, T) is a pair counted by the left-hand side of (3.34), there exist $(x, y) \in P \cap p$ and a line $l_{a,b} \in T \cap \mathcal{L}$ such that $(x, y) \in l_{a,b}$. Thus $y = ax + b$, then $b = -ax + y$, which means $(-a, b) \in D(x, y)$. Since

$$(-a, b) \in P^\star \cap D^\star(T), \quad D(x, y) \in \mathcal{L}^\star \cap D(p),$$

the pair $(D^\star(T), D(p)) \in \mathcal{D}_\delta(P^\star) \times \mathcal{T}^\delta(\mathcal{L}^\star)$ lies in the set

$$\{(q, T') \in \mathcal{D}_\delta(P^\star) \times \mathcal{T}^\delta(\mathcal{L}^\star) : y \in \ell' \text{ for some } y \in P^\star \cap q, \ell' \in \mathcal{L}^\star \cap T'\}.$$

The map $(T, p) \mapsto (D^\star(T), D(p))$ is injective, so the set above has the same cardinality as the set in (3.34). This proves $|\mathcal{I}(P^\star, \mathcal{L}^\star)|_\delta \gtrsim \delta^{\varepsilon - f(s,t)}$.

Recalling that $s > t$, we have reduced our problem to the case treated in Section 3.2, where the Katz–Tao exponent of $P^\star$ (namely t) is strictly lower than the Katz–Tao exponent of $L^\star$ (namely s), and moreover $|\mathcal{I}(P^\star, \mathcal{L}^\star)|_\delta \gtrsim \delta^{\varepsilon - f(s,t)}$. By the result in Section 3.2, there exists a (δ, δ^u)-clique $(P^\star)' \times (\mathcal{L}^\star)' \subset P^\star \times \mathcal{L}^\star$ such that

$$|\mathcal{I}((P^\star)', (\mathcal{L}^\star)')|_\delta \geq \delta^u \, |(P^\star)'|_\delta \, |(\mathcal{L}^\star)'|_\delta,$$

where

$$|(P^\star)'|_\delta \geq \delta^{u - t^2/(t+s)} \quad \text{and} \quad |(\mathcal{L}^\star)'|_\delta \geq \delta^{u - s^2/(t+s)}.$$

We finally transform $(P^\star)' \times (\mathcal{L}^\star)'$ back to a subset of $P \times \mathcal{L}$ by setting

$$P' := D^{-1}(\mathcal{L}^\star) \quad \text{and} \quad \mathcal{L}' := D^{\star -1}(P^\star).$$

Now $P' \times \mathcal{L}' \subset P \times \mathcal{L}$ is a (δ, δ^{2u})-clique; this uses a similar argument as the proof of (3.35), where we showed that the transformations D and $D^\star$ roughly preserve the δ-covering number of incidences, as well as the δ-covering numbers of the sets $\mathcal{L}^\star$ and $P^\star$.

4. Proofs of Corollary 1.10 and Proposition 1.11

We start by proving Proposition 1.11. Here is the statement again. We have added (4.2) to the statement for future reference.

Proposition 4.1. *There exists an absolute constant $C \geq 1$ such that the following holds. Let $P \times \mathcal{L} \subset [0,1)^2 \times \mathcal{A}(2)$ be a (δ, θ)-clique. Then, there exists a rectangle $R \subset \mathbb{R}^2$ of dimensions $C(\delta \times \Delta)$, where $\Delta \in [\delta, 2]$, such that*

$$|P \cap R|_\delta \gtrsim \theta^2 |P|_\delta \quad \textit{and} \quad |\{\ell \in \mathcal{L} : R \subset [\ell]_{C\delta}\}|_\delta \gtrapprox \theta^4 |\mathcal{L}|_\delta. \tag{4.1}$$

In particular, if P is a Katz–Tao (δ, s, A)-set, and $\mathcal{L}$ is a Katz–Tao (δ, t, B)-set, with $A, B \geq 1$, then $|P|_\delta^t |\mathcal{L}|_\delta^s \lessapprox A^t B^s \theta^{-6} \delta^{-st}$. Finally, we have the individual estimates

$$|P|_\delta \lesssim \theta^{-2} A (\Delta/\delta)^s \quad \textit{and} \quad |\mathcal{L}|_\delta \lesssim \theta^{-4} B \Delta^{-t}. \tag{4.2}$$

Proof. Arguing as in Section 3, we may assume $\mathcal{L} \subset D([-1,1) \times \mathbb{R})$. Denote $\mathcal{P} := \mathcal{D}_\delta(P)$ and $\mathcal{T} := \mathcal{T}^\delta(\mathcal{L})$. Since $(P, \mathcal{L})$ is a (δ, θ)-clique, we deduce, by Lemma 2.5,

$$\theta|\mathcal{P}||\mathcal{T}| \lesssim |\{(p,T) \in \mathcal{P} \times \mathcal{T} : x \in \ell \text{ for some } x \in P \cap p, \ell \in T \cap \mathcal{L}\}|. \tag{4.3}$$

For each $p \in \mathcal{P}$ and $T \in \mathcal{T}$, write

- $\mathcal{P}_T := \{p \in \mathcal{P} : x \in \ell \text{ for some } x \in p \cap P, \ell \in T \cap \mathcal{L}\}$,
- $\mathcal{T}_p := \{T \in \mathcal{T} : x \in \ell \text{ for some } x \in p \cap P, \ell \in T \cap \mathcal{L}\}$.

We start by applying Cauchy–Schwarz:

$$\theta\,|\mathcal{P}||\mathcal{T}| \overset{(4.3)}{\lesssim} \sum_{p \in \mathcal{P}} |\mathcal{T}_p| \leq |\mathcal{P}|^{1/2}\Big(\sum_{T,T'} |\mathcal{P}_T \cap \mathcal{P}_{T'}|\Big)^{1/2} \implies \sum_{T,T'} |\mathcal{P}_T \cap \mathcal{P}_{T'}| \geq \theta^2|\mathcal{P}||\mathcal{T}|^2.$$

Therefore, there exist $\geq \frac{1}{2}\theta^2|\mathcal{T}|^2$ pairs $(T,T') \in \mathcal{T} \times \mathcal{T}$ such that

$$|\mathcal{P}_T \cap \mathcal{P}_{T'}| \geq \frac{1}{2}\,\theta^2\,|\mathcal{P}|.$$

In particular, we may fix $T_0 \in \mathcal{T}$ such that the set

$$\mathcal{T}_0 := \Big\{T \in \mathcal{T} : |\mathcal{P}_T \cap \mathcal{P}_{T_0}| \geq \frac{1}{2}\,\theta^2\,|\mathcal{P}|\Big\}$$

has cardinality $|\mathcal{T}_0| \geq \frac{1}{2}\theta^2|\mathcal{T}|$. Let $\sigma_0 := \sigma(T_0) \in [-1,1]$ be the slope of T_0 (Definition 2.4). We further split $\mathcal{T}_0$ into $\sim \log(1/\delta)$ subsets such that $|\sigma(T) - \sigma_0| \in [\alpha, 2\alpha]$ for all T in a fixed subset. (For $\alpha = \delta$, the defining condition is $|\sigma(T) - \sigma_0| \leq 2\delta$). One subset has cardinality $\gtrapprox \theta^2|\mathcal{T}|$, and we keep denoting this subset $\mathcal{T}_0$. Note that

$$\operatorname{diam}([0,1)^2 \cap T \cap T_0) \leq C(\delta/\alpha), \quad T \in \mathcal{T}_0.$$

Next, let $\mathcal{R} = \mathcal{R}(\alpha)$ be a boundedly overlapping cover of $[0,1]^2 \cap T_0$ by rectangles of dimensions $C'\delta \times C'(\delta/\alpha)$, with the property that if $T \in \mathcal{T}_0$, then some element $R \in \mathcal{R}$ covers the intersection $[0,1]^2 \cap T \cap T_0$ with further requirement

$$\bigcup (\mathcal{P}_T \cap \mathcal{P}_{T_0}) \subset R. \tag{4.4}$$

This can be done if we choose $C' \geq 1$ large enough. We say that a rectangle $R \in \mathcal{R}$ is *good* if $|P \cap R|_\delta \geq \frac{1}{2}\theta^2|\mathcal{P}|$. By the bounded overlap of the family $\mathcal{R}$, there are $\lesssim \theta^{-2}$ good rectangles in $\mathcal{R}$. Moreover, if $R \in \mathcal{R}$ satisfies (4.4) for some $T \in \mathcal{T}_0$, then R is good, because in that case,

$$|P \cap R|_\delta \geq |\mathcal{P}_{T_0} \cap \mathcal{P}_T| \geq \frac{1}{2}\,\theta^2\,|\mathcal{P}|, \quad T \in \mathcal{T}_0.$$

Now, since there are $\lesssim \theta^{-2}$ good rectangles and $\gtrapprox \theta^2|\mathcal{T}|$ possible intersections $T \cap T_0$, there exists a good rectangle $R_0 \in \mathcal{R}$ covering $\gtrapprox \theta^4|\mathcal{T}|$ intersections $T \cap T_0$ (in the sense (4.4)). Whenever R_0 satisfies (4.4), we can choose a suitable constant $C'' \geq 1$ such that $R_0 \subset [\ell]_{C''\delta}$ for any line $\ell \in \mathcal{L}$ contained in T (and there are such lines, since $T \in \mathcal{T} = \mathcal{T}^\delta(\mathcal{L})$).

Consequently,

$$|P \cap R_0|_\delta \geq \frac{1}{2}\,\theta^2\,|\mathcal{P}| \sim \theta^2|P|_\delta \quad \text{and} \quad |\{\ell \in \mathcal{L} : R_0 \subset [\ell]_{C''\delta}\}| \gtrapprox \theta^4|\mathcal{T}| \sim \theta^4|\mathcal{L}|_\delta.$$

This completes the proof of the proposition, except for the final remarks concerning the Katz–Tao sets P and $\mathcal{L}$. The following are consequences of the Katz–Tao hypotheses:

$$|P \cap R|_\delta \lesssim A(\Delta/\delta)^s \quad \text{and} \quad |\{\ell \in \mathcal{L} : R \subset [\ell]_{C\delta}\}|_\delta \lesssim B\Delta^{-t}.$$

Combining these with (4.1), we immediately obtain the estimates (4.2). Finally, also

$$|P|_\delta^t\,|\mathcal{L}|_\delta^s \lessapprox A^t B^s \theta^{-6}(\Delta/\delta)^{st}\Delta^{-st} = A^t B^s \theta^{-6}\delta^{-st},$$

as desired. ∎

Here is again the statement of Corollary 1.10.

Corollary 4.2. *Under the hypotheses of Theorem 1.7, there exists a list*

$$(P_1 \times \mathcal{L}_1), \ldots, (P_n \times \mathcal{L}_n) \subset P \times \mathcal{L}$$

of (δ, δ^u)-cliques satisfying (1.5), with the sets $\mathcal{D}_\delta(P_j)$ disjoint, and $\sum_j |\mathcal{I}(P_j, \mathcal{L}_j)|_\delta \geq \delta^{u-f(s,t)}$.

We will use the following observation.

Lemma 4.3. *Let $P \subset \mathbb{R}^2$ and $\mathcal{L} \subset \mathcal{A}(2)$. Assume that there exists a constant $M \geq 0$ such that*

$$|\{\ell \in \mathcal{L} : x \in \ell \text{ for some } x \in P \cap p\}|_\delta \leq M, \quad p \in \mathcal{D}_\delta(P).$$

Then, $|\mathcal{I}(P, \mathcal{L})|_\delta \lesssim M|P|_\delta$.

Proof. Note that $\mathcal{I}(P, \mathcal{L}) \subset \bigcup_{p \in \mathcal{D}_\delta(P)} \mathcal{I}(P \cap p, \mathcal{L})$, so

$$|\mathcal{I}(P, \mathcal{L})|_\delta \leq \sum_{p \in \mathcal{D}_\delta(P)} |\mathcal{I}(P \cap p, \mathcal{L})|_\delta.$$

Here further $\mathcal{I}(P \cap p, \mathcal{L}) \subset p \times \{\ell \in \mathcal{L} : x \in \ell \text{ for some } x \in P \cap p\}$, so

$$|\mathcal{I}(P \cap p, \mathcal{L})|_\delta \lesssim |\{\ell \in \mathcal{L} : x \in \ell \text{ for some } x \in P \cap p\}|_\delta.$$

Combining these inequalities gives the claim. ∎

Proof of Corollary 4.2. We may assume that every $x \in P$ is contained on at least one line from $\mathcal{L}$. This is because we may remove from P the points for which this fails without affecting the hypothesis $|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - f(s,t)}$.

Fix $u > 0$. The statement of Corollary 1.10 only gets stronger for smaller values of $u > 0$, so we may assume that $0 < u \leq st/(s+t)$.

In this proof, the notation "$\lessapprox$" only hides constants of order $(\log(1/\delta))^C$. Now let $c \in (0, 1/20]$ be an absolute constant to be determined later. Let $\bar{\varepsilon} = \bar{\varepsilon}(s, t, cu) > 0$ be the constant provided by Theorem 1.7 with parameters, s, t and cu, and let

$$\varepsilon := \min\{u/(8C(s,t)), \bar{\varepsilon}/2\}, \tag{4.5}$$

where $C(s,t) := \max\{(s+t)/s, 100\}$. We now claim the conclusion of Corollary 4.2 holds if P is a Katz–Tao $(\delta, s, \delta^{-\varepsilon})$-set, $\mathcal{L}$ is a Katz–Tao $(\delta, t, \delta^{-\varepsilon})$-set, $|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - f(s,t)}$, and finally $\delta > 0$ small enough, depending on s, t and u (although we will not track the necessary smallness of δ explicitly).

We start by finding a subset $\bar{P} \subset P$ such that $|\mathcal{I}(\bar{P}, \mathcal{L})|_\delta \gtrapprox |\mathcal{I}(P, \mathcal{L})|_\delta$, and moreover,

$$|\{\ell \in \mathcal{L} : x \in \ell \text{ for some } x \in \bar{P} \cap p\}|_\delta \leq \delta^{-u/4 - t^2/(s+t)}, \quad p \in \mathcal{D}_\delta(\bar{P}). \tag{4.6}$$

To see how this is done, write $\mathcal{P} := \mathcal{D}_\delta(P)$. For every $p \in \mathcal{P}$, define the quantity

$$M(p) := |\{\ell \in \mathcal{L} : x \in \ell \text{ for some } x \in P \cap p\}|_\delta \in \{1, \ldots, |\mathcal{L}|_\delta\}.$$

Note that $M(p) \geq 1$, since every $x \in p \cap P$ is contained on at least one line in $\mathcal{L}$ by the reduction in the first paragraph. Let

$$P_M := \{x \in P : M(p_x) \in [M, 2M]\},$$

where $p_x \in \mathcal{D}_\delta$ is the dyadic square containing x. Since the condition $M(p_x) \in [M, 2M]$ only depends on the dyadic "δ-parent" of x, one has $P_M = P \cap (\cup \mathcal{D}_\delta(P_M))$. Now, find a number $M \in \{1, \ldots, |\mathcal{L}|_\delta\}$ and a subset $\bar{P} := P_M$ such that $M(p_x) \sim M$ for all $x \in \bar{P}$, and

$$|\mathcal{I}(P, \mathcal{L})|_\delta \approx |\mathcal{I}(\bar{P}, \mathcal{L})|_\delta \gtrapprox M |\bar{P}|_\delta.$$

By Lemma 2.6, and using the Katz–Tao conditions of $\bar{P}$ and $\mathcal{L}$,

$$|\mathcal{I}(\bar{P}, \mathcal{L})| \lesssim_\varepsilon \delta^{-3\varepsilon} \delta^{-st/(s+t)} |\bar{P}|_\delta^{s/(s+t)} |\mathcal{L}|_\delta^{t/(s+t)} \leq \delta^{-4\varepsilon} \delta^{-(st+t^2)/(s+t)} |\bar{P}|_\delta^{s/(s+t)}.$$

Chaining these inequalities, and using that $|\mathcal{I}(P, \mathcal{L})|_\delta \geq \delta^{\varepsilon - f(s,t)}$, we first infer $|\bar{P}|_\delta \gtrapprox \delta^{C(s,t)\varepsilon - s}$, and next,

$$M \lessapprox \frac{|\mathcal{I}(P, \mathcal{L})|_\delta}{\delta^{C(s,t)\varepsilon - s}} \lesssim_\varepsilon \delta^{-C(s,t)\varepsilon} \delta^{s - f(s,t)} = \delta^{-C(s,t)\varepsilon} \delta^{-t^2/(s+t)}.$$

This proves (4.6) by the choice of $\varepsilon > 0$ in (4.5).

Noting that $|\mathcal{I}(\bar{P}, \mathcal{L})|_\delta \geq \delta^{2\varepsilon - f(s,t)} \geq \delta^{\bar{\varepsilon} - f(s,t)}$, we may now apply Theorem 1.7 to find our first (δ, δ^{cu})-clique $\bar{P}_1 \times \mathcal{L}_1 \subset \bar{P} \times \mathcal{L}$ satisfying

$$|\bar{P}_1|_\delta \geq \delta^{cu - s^2/(s+t)} \quad \text{and} \quad |\mathcal{L}_1|_\delta \geq \delta^{cu - t^2/(s+t)}. \tag{4.7}$$

Here the constant $c > 0$ will be chosen later. We note that $\bar{P}_1$ can be selected to be of the form $\bar{P}_1 = \bar{P} \cap (\cup \bar{\mathcal{P}}_1)$, where $\bar{\mathcal{P}}_1 \subset \mathcal{D}_\delta(\bar{P})$, recall Remark 1.8.

We also need a matching upper bound for $|\bar{P}_1|_\delta$. This will follow from the first part of (4.2), where the number "Δ" is comparable to the diameter of the "covering" rectangle R in (4.1). Moreover, as early as in (1.7), we recorded the following estimate for $\operatorname{diam}(R)$:

$$\Delta \sim \operatorname{diam}(R) \lesssim \delta^{-C(\varepsilon + cu) + t/(s+t)}.$$

Here $C > 0$ is an absolute constant. If the absolute constant $c > 0$ (introduced above (4.5)) is chosen small enough relative to "C", and also recall from (4.5) that ε is much smaller than u, we obtain $\Delta \lessapprox \delta^{-u/8 + t/(s+t)}$. Combining this upper bound for Δ with (4.2) yields

$$|\bar{P}_1|_\delta \overset{(4.2)}{\lessapprox} \delta^{-2cu - \varepsilon} (\Delta/\delta)^s \leq \delta^{-u/4} \left(\frac{\delta^{t/(s+t)}}{\delta} \right)^s = \delta^{-u/4 - s^2/(s+t)}.$$

In particular, recalling (4.6), and using Lemma 4.3,

$$|\mathcal{I}(\bar{P}_1, \mathcal{L})|_\delta \lessapprox \delta^{-u/2 - (s^2+t^2)/(s+t)}. \tag{4.8}$$

For $\delta > 0$ small enough, and since $u \leq st/(s+t)$, this upper bound is far smaller than $|\mathcal{I}(\bar{P}, \mathcal{L})| \gtrapprox \delta^{\varepsilon - f(s,t)}$. In particular, we still have $|\mathcal{I}(\bar{P} \setminus \bar{P}_1, \mathcal{L})| \geq \delta^{2\varepsilon - f(s,t)}$. Therefore, the previous reasoning can be repeated to the $(\delta, s, \delta^{-\varepsilon})$-set $\bar{P} \setminus \bar{P}_1$ and $(\delta, t, \delta^{-\varepsilon})$-set $\mathcal{T}$ to produce a second δ^{cu}-clique $\bar{P}_2 \times \mathcal{L}_2 \subset (\bar{P} \setminus \bar{P}_1) \times \mathcal{L}$. Again, by Remark 1.8, the set $\bar{P}_2$ can be selected to be of the form $\bar{P}_2 = (\bar{P} \setminus \bar{P}_1) \cap (\cup \bar{\mathcal{P}}_2)$, where $\bar{\mathcal{P}}_2 \subset \mathcal{D}_\delta(\bar{P} \setminus \bar{P}_1)$. In particular, using the analogous structure of $\bar{P}_1$ discussed above, $\mathcal{D}_\delta(\bar{P}_1) \cap \mathcal{D}_\delta(\bar{P}_2) = \emptyset$.

How many times can this argument be repeated?

For every δ^{cu}-clique $\bar{P}_j \times \mathcal{L}_j \subset \bar{P} \times \mathcal{T}$, the estimate (4.8) holds with "1" replaced by "j" (by the same argument). Therefore,

$$|\mathcal{I}(\bar{P}_1 \cup \cdots \cup \bar{P}_n, \mathcal{L})|_\delta \lesssim n \cdot \delta^{-u/2-(s^2+t^2)/(s+t)}.$$

This upper bound remains smaller than $\frac{1}{2}|\mathcal{I}(\bar{P}, \mathcal{L})| \gtrapprox \delta^{\varepsilon - f(s,t)}$ as long as

$$n \lessapprox \delta^{u/2+2\varepsilon-st/(s+t)}.$$

So, the argument can safely be repeated at least $n := \delta^{3u/4-st/(s+t)}$ times. At this stage, by the δ^{cu}-clique property of $\bar{P}_j \times \mathcal{L}_j$, and the covering number bounds (4.7),

$$\sum_{1 \le j \le n} |\mathcal{I}(\bar{P}_j, \mathcal{L})|_\delta \ge n \cdot \delta^{cu} \cdot |\bar{P}_j|_\delta |\mathcal{L}_j|_\delta = \delta^{3u/4+3cu-f(s,t)}.$$

Since $c \le 1/20$, this completes the proof of the corollary. ■

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