Z. Anal. Anwend. (Online first)
DOI 10.4171/ZAA/1799

On the partial regularity of weak solutions for the magnetohydrodynamics system in $\mathbb{R}^4$

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Abstract. In this paper, the partial regularity of the weak solutions to the magnetohydrodynamics (MHD) system in $\mathbb{R}^4$ is studied. In order to tackle the lack of compactness arising in the spatially high-dimensional setting, inspired by Wu [Arch. Rational Mech. Anal. 239 (2021), 1771–1808], we use the defect measures and prove the existence of partially regular weak solutions (satisfying certain local energy inequality) to the 4-dimensional MHD system. As an application, we obtain that the 2-dimensional Hausdorff dimension of singular sets of these weak solutions is finite.

1. Introduction

In this paper, we study partial regularity results for the 4D incompressible MHD equations, which are given by the following system:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - B \cdot \nabla B + \nabla \Pi = 0, \\ \partial_t B - \Delta B + u \cdot \nabla B - B \cdot \nabla u = 0, \qquad \text{in } \mathbb{R}^4 \times [0, \infty), \\ \operatorname{div} u = \operatorname{div} B = 0. \end{cases} \tag{1.1}$$

Here, the unknown variable u denotes fluid velocity field, B represents the magnetic field, and $\Pi = \pi + \frac{1}{2} B^2$ signifies the total pressure, respectively. The MHD system (1.1) plays a crucial role in the dynamics of electrically conducting fluids, such as plasmas (see [2], for example). Because of the nonlinear interaction between the fluid velocity field and the magnetic field, the MHD system can accommodate much richer phenomena than the Navier–Stokes system. One important example is that the magnetic field can actually stabilize the fluid motion [1]. There have been extensive studies on various topics concerning the MHD system (see, e.g., [4, 6, 9, 12–14, 19, 28] and references therein). The global weak solutions and the local strong solutions to the 3D MHD system were constructed by Duvaut and Lions [9] and Sermange and Teman [24]. Meanwhile, as in the classical incompressible Navier–Stokes equations (see [3,21,22,26,27]), partial regularity of suitable weak solutions to the 3D MHD system was investigated by He and Xin in [13]. Wang and Zhang [28] removed the magnetic field hypothesis of the regularity criteria

Mathematics Subject Classification 2020: 35B65.
Keywords: magnetohydrodynamics equations, partial regularity, weak solutions, defect measures.

for the suitable weak solutions to the 3D MHD system. The results obtained in [13, 28] indicate that the velocity field plays a more dominant role than the magnetic field does on the regularity of solutions to the MHD system. Recently, by introducing the notion of dissipative solutions, Chamorro and He [5] weakened the hypothesis on the pressure and obtained a generalization of C-K-N type theorem for weak solutions of MHD system (1.1).

The partial regularity of weak solutions satisfying the local energy inequality was originated from Scheffer [21–23] for the Navier–Stokes equations. And the author studied the Hausdorff measure of potential singular points set of the solutions to the 3D and 4D Navier–Stokes equations. Later, Caffarelli, Kohn, and Nirenberg [3] proved that the one-dimensional Hausdorff measure of potential singular set of suitable weak solutions to the Navier–Stokes equations is zero. Since then, there have been extensive studies on the partial regularity of solutions to the Navier–Stokes equations, MHD equations, and other hydrodynamic models; see [15–17, 27], for example. For the high-dimensional Navier–Stokes equations, Dong and Du [7] proved that the 2-dimensional Hausdorff measure of the set of singular points at the first blow-up time is zero. As in [3] for the dimensional analysis of the Navier–Stokes equations, time is equivalent to two space dimensions. Later, Dong and Strain [8] proved the partial regularity theory of suitable weak solutions of the 6D stationary Navier–Stokes equations. Wang and Wu [29] gave a unified proof on the results of [3, 7, 8]. Recently, by introducing the defect measure, Wu [30, 31] constructed partially regular weak solutions which satisfy local energy inequalities to the non-stationary incompressible Navier–Stokes equations in $\mathbb{R}^4$ and stationary Navier–Stokes equations in $\mathbb{R}^6$, whose singular sets have a local finite 2-dimensional parabolic Hausdorff measure. In context of MHD equations in high-dimensional space, Han and He [11] studied the 4-dimensional non-stationary MHD equations (1.1) interior partial regularity. Recently, Gu [10] and Liu–Wang [18] also obtained some boundary regularity criteria for the 4D non-stationary MHD equations and 6D stationary MHD equations. Motivated by [3, 30] for Navier–Stokes equations, we obtained the existence of partially regular weak solutions for the 4D MHD equations and proved that the 2-dimensional parabolic Hausdorff measure of singular set is finite. Before presenting our main result, we first give the definition of parabolic Hausdorff dimension.

Definition 1.1. Given a set $D \subset \mathbb{R}^4 \times \mathbb{R}$, for a fixed positive real number s, s-dimensional parabolic Hausdorff measure is defined as

$$\mathcal{H}^s(D) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(D),$$

where

$$\mathcal{H}^s_\delta(D) = \inf\left\{ \sum_{i=1}^{\infty} r_i^s \middle| D \subset \bigcup_{i \in \mathbb{N}^+} Q_{r_i}(z_0),\ 0 < r_i < \delta,\ z_0 = (x_0, t_0) \in \mathbb{R}^4 \times \mathbb{R} \right\}.$$

Here, $Q_r(z_0)$ is centered parabolic cylinder defined by

$$Q_r(z_0) := B_r(x_0) \times \left(t_0 - \frac{r^2}{2}, t_0 + \frac{r^2}{2} \right).$$

The main result of the present paper is the following statement.

Theorem 1.1. *For a fixed time $T > 0$, there exists a weak solution set $(u, B, \Pi, \lambda, \omega)$ for the non-stationary MHD system (1.1) in $\mathbb{R}^4 \times [0, T]$, which satisfies the local energy inequalities (2.11) and (2.12). Moreover, the 2-dimensional Hausdorff dimension of singular set S of weak solutions to (1.1) is finite.*

2. Existence of weak solutions set and local energy inequalities

In order to construct generalized weak solutions to the non-stationary MHD system (1.1), we consider the following regularized MHD system:

$$\begin{cases} \partial_t u_k - \Delta u_k + [(\chi_k * u_k)\cdot\nabla]u_k - [(\chi_k * B_k)\cdot\nabla]B_k + \nabla\Pi = 0, \\ \operatorname{div} u_k = \operatorname{div} B_k = 0, \\ \partial_t B_k - \Delta B_k + [(\chi_k * u_k)\cdot\nabla]B_k - [(\chi_k * B_k)\cdot\nabla]u_k = 0, \\ u_k(\cdot,0) = u_0,\ B_k(\cdot,0) = B_0. \end{cases} \tag{2.1}$$

Here, $\{\chi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^4)$ denotes the standard mollifiers.

Lemma 2.1. *For the regularized non-stationary MHD system (2.1), we have a sequence $\{(u_k, B_k, \Pi_k)\}_{k\in\mathbb{N}} \subset L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^4 \times [0,T]) \times L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^4 \times [0,T]) \times L^{\frac{3}{2}}(\mathbb{R}^4 \times [0,T])$ such that (u_k, B_k, Π_k) is a weak solution to the system (2.1). Moreover,*

(1) *$\{u_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^4 \times [0,T])$;*

(2) *$\{\Pi_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{3}{2}}(\mathbb{R}^4 \times [0,T])$;*

(3) *$\{\partial_t u_k\}_{k\in\mathbb{N}}$ and $\{\partial_t B_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L_t^1 H_{x,\mathrm{loc}}^{-1}(\mathbb{R}^4 \times [0,T])$.*

Therefore, there exists a triple (u, B, Π) such that

$$\begin{aligned} &u_k \to u, B_k \to B \quad && \text{weakly in } L_t^2 H_x^1(\mathbb{R}^4 \times [0,T]), \\ &u_k \to u, B_k \to B \quad && \text{weakly-* in } L_t^\infty L_x^2(\mathbb{R}^4 \times [0,T]), \\ &\Pi_k \to \Pi && \text{weakly in } L^{\frac{3}{2}}(\mathbb{R}^4 \times [0,T]). \end{aligned} \tag{2.2}$$

Additionally, for any bounded smooth function ϕ with bounded derivatives, this sequence satisfies the following local energy inequality:

$$\begin{aligned} &\int_{\mathbb{R}^4} (|u_k(t)|^2 + |B_k(t)|^2)\phi(t)\mathrm{d}x + \int_0^t \int_{\mathbb{R}^4} (|\nabla u_k|^2 + |\nabla B_k|^2)\phi \mathrm{d}x\mathrm{d}s \\ &\quad \le \int_0^t \int_{\mathbb{R}^4} (|u_k|^2 + |B_k|^2)|\partial_t\phi + \Delta\phi|\mathrm{d}x\mathrm{d}s + \int_0^t \int_{\mathbb{R}^4} (|u_k|^2 + |B_k|^2)(\tilde{u}_k\cdot\nabla)\phi \mathrm{d}x\mathrm{d}s \\ &\qquad + 2\int_0^t \int_{\mathbb{R}^4} \Pi_k (u_k\cdot\nabla)\phi \mathrm{d}x\mathrm{d}s - \int_0^t \int_{\mathbb{R}^4} (\widetilde{B}_k\cdot\nabla\phi)(u_k\cdot B_k)\mathrm{d}x\mathrm{d}s \\ &\qquad + \int_{\mathbb{R}^4} (|u_0|^2 + |B_0|^2)\phi(0)\mathrm{d}x, \end{aligned} \tag{2.3}$$

where $\tilde{u}_k := \chi_k * u_k$, $\widetilde{B}_k := \chi_k * B_k$.

Proof. The existence of weak solutions u and B can be obtained by a standard Galerkin method (see [20, Theorems 4.4 and 14.1], for example). The existence of Π can be proved by Calderon–Zygmund theory. It is noteworthy that $u_k\phi$ and $B_k\phi$ are admissible test functions in the regularized system (2.1). Multiplying equation $(2.1)_1$ by $u_k\phi$, integrating the resulting equation over $\mathbb{R}^4$, and using integration by parts and incompressible conditions, we get that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^4}|u_k|^2\phi\mathrm{d}x+\int_{\mathbb{R}^4}|\nabla u_k|^2\phi\mathrm{d}x\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^4}|u_k|^2(\partial_t+\Delta\phi)\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^4}|u_k|^2\tilde{u}_k\cdot\nabla\phi\mathrm{d}x+\int_{\mathbb{R}^4}\Pi u_k\cdot\nabla\phi\mathrm{d}x\\
&\qquad-\int_{\mathbb{R}^4}(\widetilde{B}_k\cdot\nabla)u_k\cdot B_k\phi+(\widetilde{B}_k\cdot\nabla\phi)(u_k\cdot B_k)\mathrm{d}x. \qquad (2.4)
\end{aligned}$$

Next, multiplying equation $(2.1)_3$ by $B_k\phi$, integrating the resulting equation over $\mathbb{R}^4$, and applying integration by parts and incompressible conditions again, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^4}|B_k|^2\phi\mathrm{d}x+\int_{\mathbb{R}^4}|\nabla B_k|^2\phi\mathrm{d}x\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^4}|B_k|^2(\partial_t+\Delta\phi)\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^4}|B_k|^2\tilde{u}_k\cdot\nabla\phi\mathrm{d}x+\int_{\mathbb{R}^4}(\widetilde{B}_k\cdot\nabla)u_k\cdot B_k\phi\mathrm{d}x. \qquad (2.5)
\end{aligned}$$

Hence, by adding (2.4) and (2.5) and after integrating over [0,t], we yield the local energy inequality (2.3). For the uniform boundedness of $\{\partial_t u_k\}_{k\in\mathbb{N}}$ and $\{\partial_t B_k\}_{k\in\mathbb{N}}$, applying duality theory, we know that, for almost every $t\in[0,T]$, the weak formulations (2.1) for u_k and B_k are equivalent to

$$\begin{aligned}
&\langle\partial_t u_k,\xi\rangle_{H_x^{-1}\times H_x^1}\\
&\quad=-\int_{\mathbb{R}^4}\nabla u_k\nabla\xi+(\tilde{u}_k\cdot\nabla)u_k\xi-(\widetilde{B}_k\cdot\nabla)B_k\xi\mathrm{d}x,\quad\forall\xi\in H_x^1\text{ with div }\xi=0
\end{aligned}$$

and

$$\langle\partial_t B_k,\eta\rangle_{H_x^{-1}\times H_x^1}=-\int_{\mathbb{R}^4}\nabla B_k\nabla\eta+(\tilde{u}_k\cdot\nabla)B_k\eta-(\widetilde{B}_k\cdot\nabla)u_k\eta\mathrm{d}x,\quad\forall\eta\in H_x^1.$$

For every $\xi\in C_c^\infty(\Omega)$ with $\Omega\subset\subset\mathbb{R}^4$, the Hölder inequality and the Sobolev embedding inequality imply that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^4}\nabla u_k\nabla\xi+(\tilde{u}_k\cdot\nabla)u_k\xi-(\widetilde{B}_k\cdot\nabla)B_k\xi\mathrm{d}x\right|\\
&\quad\le\|\nabla u_k\|_{L_x^2}\|\nabla\xi\|_{L_x^2}+(\|\nabla u_k\|_{L_x^2}\|u_k\|_{L_x^4}+\|\nabla B_k\|_{L_x^2}\|B_k\|_{L_x^4})\|\xi\|_{L_x^4}\\
&\quad\le C(\|u_k\|_{H_x^1}^2+\|u_k\|_{H_x^1}+\|B_k\|_{H_x^1}^2+\|B_k\|_{H_x^1})\|\xi\|_{H_x^1}.
\end{aligned}$$

Integrating in time yields that $\{\partial_k u_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^1_t H^{-1}_{x,\mathrm{loc}}(\mathbb{R}^4\times[0,T])$. Applying the same procedure, we can obtain the uniform boundedness of $\{\partial_t B_k\}_{k\in\mathbb{N}}$ in $L^1_t H^{-1}_{x,\mathrm{loc}}(\mathbb{R}^4\times[0,T])$. ∎

Next, we prove that certain measures in the limit are relatively compact. We recall that a collection of measures $\{\mu_i\}_{i\in\Lambda}$ on $\mathbb{R}^N$ is called tight if, for any $\varepsilon>0$, there exists a compact set $\Omega_\varepsilon\subset\mathbb{R}^N$ such that $\mu_i(\mathbb{R}^N\backslash\Omega_\varepsilon)<\varepsilon$ for any $i\in\Lambda$.

Lemma 2.2. *Let the assumptions be as in Lemma 2.1; then, $\{(|\nabla u_k|^2+|\nabla B_k|^2)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$, $\{(|u_k|^2+|B_k|^2)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$, and $\{(|u_k|^3+|B_k|^3)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$ are tight in the sense of measures.*

Proof. For fixed $r>0$, we consider a smooth cutoff function $\xi\in C_c^\infty(\mathbb{R}^4)$ such that $0\le\xi\le1$, $\xi|_{B_r}=0$, $\xi|_{\mathbb{R}^4\backslash B_{2r}}=1$, satisfying property $|\nabla\xi|^2+|\nabla^2\xi|\le Cr^{-2}$. Testing the regularity non-stationary MHD system $(2.1)_1$ with $u_k\xi$ and $(2.1)_3$ with $B_k\xi$, respectively, we get

$$\begin{aligned}
&\sup_t\int_{\mathbb{R}^4}(|u_k(t)|^2+|B_k(t)|^2)\xi\mathrm{d}x-\int_{\mathbb{R}^4}(|u_0|^2+|B_0|^2)\xi\mathrm{d}x\\
&\quad+\int_0^T\int_{\mathbb{R}^4}(|\nabla u_k|^2+|\nabla B_k|^2)\xi\mathrm{d}x\mathrm{d}t\\
&\le\int_0^T\int_{\mathbb{R}^4}(|u_k|^2+|B_k|^2)|\Delta\xi|\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^4}(|u_k|^2+|B_k|^2)|\tilde{u}_k||\nabla\xi|\mathrm{d}x\mathrm{d}t\\
&\quad+2\int_0^T\int_{\mathbb{R}^4}|\Pi_k u_k||\nabla\xi|\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^4}|\widetilde{B}_k||\nabla\xi||u_k||B_k|\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

According to the properties of test function and the Hölder inequality, we have

$$\begin{aligned}
&\sup_t\int_{\mathbb{R}^4\backslash B_{2r}}(|u_k(t)|^2+|B_k(t)|^2)\mathrm{d}x-\int_{\mathbb{R}^4\backslash B_r}(|u_0|^2+|B_0|^2)\mathrm{d}x\\
&\quad+\int_0^T\int_{\mathbb{R}^4\backslash B_{2r}}(|\nabla u_k|^2+|\nabla B_k|^2)\mathrm{d}x\mathrm{d}t\\
&\le Cr^{-2}\int_0^T\int_{B_{2r}\backslash B_r}(|u_k|^2+|B_k|^2)\mathrm{d}x\mathrm{d}t\\
&\quad+Cr^{-1}\int_0^T\int_{B_{2r}\backslash B_r}(|u_k|^2+|B_k|^2)|\tilde{u}_k|\mathrm{d}x\mathrm{d}t\\
&\quad+Cr^{-1}\int_0^T\int_{B_{2r}\backslash B_r}|\Pi_k u_k|\mathrm{d}x\mathrm{d}t+Cr^{-1}\int_0^T\int_{B_{2r}\backslash B_r}|\widetilde{B}_k||u_k||B_k|\mathrm{d}x\mathrm{d}t\\
&\le Cr^{-\frac{2}{3}}T^{\frac{1}{3}}+Cr^{-1},
\end{aligned}$$

where we have used the fact that $\{u_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^3(\mathbb{R}^3\times[0,T])$ and $\{\Pi\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{3}{2}}(\mathbb{R}^3\times[0,T])$. Letting r be arbitrarily large implies the tightness of $\{(|\nabla u_k|^2 + |\nabla B_k|^2)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$ and $\{(|u_k(t)|^2 + |B_k(t)|^2)\mathrm{d}x\}_{k\in\mathbb{N}}$ uniformly in t, which implies the tightness of $\{(|u_k|^2 + |B_k|^2)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$. On the other hand, applying Sobolev inequality, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^4} (|u_k\xi|^3 + |B_k\xi|^3)\mathrm{d}x\mathrm{d}t \\
&\le \|u_k\xi\|_{L_t^\infty L_x^2} \int_0^T \int_{\mathbb{R}^4} |\nabla(u_k\xi)|^2 \mathrm{d}x\mathrm{d}t + \|B_k\xi\|_{L_t^\infty L_x^2} \int_0^T \int_{\mathbb{R}^4} |\nabla(B_k\xi)|^2 \mathrm{d}x\mathrm{d}t \\
&\le C\|u_k\xi\|_{L_t^\infty L_x^2} \left(\int_0^T \int_{\mathbb{R}^4} |\nabla u_k|^2 |\xi|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} |u_k|^2 |\nabla\xi|^2 \mathrm{d}x\mathrm{d}t \right) \\
&\quad + C\|B_k\xi\|_{L_t^\infty L_x^2} \left(\int_0^T \int_{\mathbb{R}^4} |\nabla B_k|^2 |\xi|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} |B_k|^2 |\nabla\xi|^2 \mathrm{d}x\mathrm{d}t \right) \\
&\le C(\|u_k\xi\|_{L_t^\infty L_x^2} + \|B_k\xi\|_{L_t^\infty L_x^2}) \int_0^T \int_{\mathbb{R}^4} (|\nabla u_k|^2 + |\nabla B_k|^2)|\xi|^2 \mathrm{d}x\mathrm{d}t \\
&\quad + C(\|u_k\xi\|_{L_t^\infty L_x^2} + \|B_k\xi\|_{L_t^\infty L_x^2}) r^{-\frac{2}{3}} T^{\frac{1}{3}} \left(\int_0^T \int_{\mathbb{R}^4} (|u_k|^3 + |B_k|^3)\mathrm{d}x\mathrm{d}t \right)^{\frac{2}{3}}.
\end{aligned}$$

Combining the tightness of $\{(|\nabla u_k|^2 + |\nabla B_k|^2)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$ with the uniform boundedness of u_k and B_k in the natural energy space implies the tightness of $\{(|u_k|^3+|B_k|^3)\mathrm{d}x\mathrm{d}t\}_{k\in\mathbb{N}}$. ∎

In order to obtain local energy inequalities for the weak limit (u, B, Π), we will pass to the limit $k \to \infty$ in the local energy inequalities (2.3).

Lemma 2.3. *Given a bounded sequence* $\{u_k\}_{k\in\mathbb{N}}$,

$$\{B_k\}_{k\in\mathbb{N}} \subset L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{R}^4 \times [0, T]),$$

let (u, B) *be given by the limit in* (2.2). *Assume that* u_k *converges to* u *and* B_k *converges to* B, *respectively, in* $L^1_{\mathrm{loc}}(\mathbb{R}^4 \times [0, T])$. *Suppose that* $\mu_k = (|\nabla u_k|^2 + |\nabla B_k|^2)\mathrm{d}x\mathrm{d}t \to \mu$, $\nu_k = (|u_k|^3 + |B_k|^3)\mathrm{d}x\mathrm{d}t \to \nu$ *weakly in the sense of measures, where* μ *and* ν *are bounded nonnegative measures on the* $\mathbb{R}^4 \times [0, T]$. *Then, there exist nonnegative finite measures* λ *and* ω *such that, for any* $\eta \in C_c^\infty(\mathbb{R}^4 \times [0, T])$,

$$\int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\mu = \int_0^T \int_{\mathbb{R}^4} \eta(|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\lambda, \tag{2.6}$$

$$\int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\nu = \int_0^T \int_{\mathbb{R}^4} \eta(|u|^3 + |B|^3)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\omega. \tag{2.7}$$

Moreover, $\omega \ll \lambda$, *and for any open subdomain* Ω *of* $\mathbb{R}^4 \times [0, T]$,

$$\int_\Omega \mathrm{d}\omega \le C \liminf_{k\to\infty} (\|u_k - u\|_{L_t^\infty L_x^2(\Omega)} + \|B_k - B\|_{L_t^\infty L_x^2(\Omega)}) \int_\Omega \mathrm{d}\lambda. \tag{2.8}$$

In particular, the Radon–Nikodym derivative satisfies

$$\frac{\mathrm{d}\omega}{\mathrm{d}\lambda} \leq C \lim_{r\to 0} \liminf_{k\to\infty} (\|u_k - u\|_{L_t^\infty L_x^2(Q_r(x_0,t_0))} + \|B_k - B\|_{L_t^\infty L_x^2(Q_r(x_0,t_0))}),$$

where $Q_r(x_0,t_0) := B_r(x_0) \times (t_0 - \frac{r^2}{2}, t_0 + \frac{r^2}{2})$.

Proof. We let $v_k = u_k - u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, $A_k = B_k - B \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, Combining Simon compactness theory (see [25]) with Lemma 2.1 implies that

$$\begin{aligned} &v_k \to 0, A_k \to 0 \quad \text{strongly in } L_{t,x}^2, \text{ locally in space,} \\ &v_k \to 0, A_k \to 0 \quad \text{weakly in } L_t^2 H_x^1, \\ &v_k \to 0, A_k \to 0 \quad \text{weakly-* in } L_t^\infty L_x^2. \end{aligned}$$

Defining $\lambda_k := (|\nabla v_k|^2 + |\nabla A_k|^2)\mathrm{d}x\mathrm{d}t$, we claim that $\{\lambda_k\}_{k\in\mathbb{R}^N}$ is tight. Indeed, for any compactness subset $\Omega \subset \mathbb{R}^4$, let $D^c := (\mathbb{R}^4 \backslash \Omega) \times [0,T]$. Then,

$$\begin{aligned} \|\nabla v_k\|_{L^2(D^c)} &\leq \|\nabla u_k\|_{L^2(D^c)} + \|\nabla u\|_{L^2(D^c)}, \\ \|\nabla A_k\|_{L^2(D^c)} &\leq \|\nabla B_k\|_{L^2(D^c)} + \|\nabla B\|_{L^2(D^c)}. \end{aligned}$$

Due to the weak convergence of $\{\mu_k\}_{k\in\mathbb{N}}$, we know that $\{\mu_k\}_{k\in\mathbb{N}}$ is tight; hence, $\|\nabla v_k\|_{L^2(D^c)}$ and $\|\nabla A_k\|_{L^2(D^c)}$ are arbitrarily small given Ω large enough. Thus, we can extract a weakly convergent subsequence with a limit denoted by λ. For any $\eta \in C_c^\infty(\mathbb{R}^4 \times [0,T])$, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\mu &= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\mu_k \\ &= \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \eta(|\nabla u_k|^2 + |\nabla B_k|^2)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^4} \eta(|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t \\ &\quad + \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \eta(|\nabla v_k|^2 + |\nabla A_k|^2)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^4} \eta(|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\lambda, \end{aligned}$$

where we have used the fact that interaction term vanishes since $u_k \to u$ and $B_k \to B$ weakly in $L_t^2 H_x^1(\mathbb{R}^4 \times [0,T])$. Let

$$\omega_k := (|v_k|^3 + |A_k|^3)\mathrm{d}x\mathrm{d}t \to \omega$$

weakly in the sense of measures. A similar process verifies (2.7). On the other hand, for any $\eta \in C_c^\infty(\mathbb{R}^4 \times [0,T])$, using interpolation inequality and Sobolev embedding inequality, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^4} |\eta|^3 \mathrm{d}\omega \\
&\quad = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} |\eta|^3 \mathrm{d}\omega_k \\
&\quad = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} |v_k\eta|^3 + |A_k\eta|^3 \mathrm{d}x\mathrm{d}t \\
&\quad \le \liminf_{k\to\infty} \sup_{0<t<T} (\|v_k\eta\|_{L^2_x} + \|A_k\eta\|_{L^2_x}) \int_0^T \|v_k\eta\|^2_{L^4_x} + \|A_k\eta\|^2_{L^4_x} \mathrm{d}t \\
&\quad \le C \liminf_{k\to\infty} \sup_{0<t<T} (\|v_k\eta\|_{L^2_x} + \|A_k\eta\|_{L^2_x}) \int_0^T \int_{\mathbb{R}^4} |\nabla(v_k\eta)|^2 + |\nabla(A_k\eta)|^2 \mathrm{d}x\mathrm{d}t \\
&\quad \le C \liminf_{k\to\infty} \sup_{0<t<T} (\|v_k\eta\|_{L^2_x} + \|A_k\eta\|_{L^2_x}) \int_0^T \int_{\mathbb{R}^4} |\eta|^2|\nabla v_k|^2 + |\eta|^2|\nabla A_k|^2 \mathrm{d}x\mathrm{d}t \\
&\quad \le C \liminf_{k\to\infty} \sup_{0<t<T} (\|v_k\eta\|_{L^2_x} + \|A_k\eta\|_{L^2_x}) \int_0^T \int_{\mathbb{R}^4} |\eta|^2 \mathrm{d}\lambda.
\end{aligned}
$$

Notice that, in the third inequality, the terms converge to zero when at least one derivative hits η. Using smooth functions to approximate the indicator function Ω yields the inequality (2.8). Hence, ω is absolutely continuous with respect to λ, and by Radon–Nikodym theorem, we have

$$
\frac{\mathrm{d}\omega}{\mathrm{d}\lambda} \in L^1(\mathbb{R}^4 \times [0,T]; \lambda)
$$

with, for any $(x_0, t_0) \in \mathbb{R}^4 \times (0,T)$,

$$
\frac{\mathrm{d}\omega}{\mathrm{d}\lambda}(x_0,t_0) \le C \lim_{r\to 0} \liminf_{k\to\infty} (\|v_k\|_{L^\infty_t L^2_x(Q_r(x_0,t_0))} + \|B_k\|_{L^\infty_t L^2_x(Q_r(x_0,t_0))}). \qquad \blacksquare
$$

Using the parabolic concentration-compactness framework in Lemma 2.3, we can define the weak solution sets involving concentration measures.

Definition 2.1. (u,B,Π,λ, ω) is a weak solution set of the MHD system (1.1) if

(1) u, B, and Π are obtained as weak limits of the weak solutions $\{(u_k, B_k, \Pi_k)\}_{k\in\mathbb{N}}$ of the regularized MHD system (2.1);

(2) λ and ω are obtained as weak limits of the measures in Lemma 2.3.

In order to obtain local energy inequalities for weak solution sets, we need the following lemma, which shows that the concentration in $|u\Pi|\mathrm{d}x\mathrm{d}t$ is localizable and comparable to the concentration in $|u|^3\mathrm{d}x\mathrm{d}t$.

Lemma 2.4. *Assume that $\{(u_k, B_k, \Pi_k)\}_{k\in\mathbb{N}}$ are the solutions of the regularized system (2.1) and $(u, B, \Pi, \lambda, \omega)$ is the corresponding weak solution set; then,*

$$
\limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u_k(\Pi_k - \gamma) - u(\Pi - \gamma)| \mathrm{d}x\mathrm{d}t \le C \int_0^T \int_{\mathbb{R}^4} \xi \mathrm{d}\omega
$$

for any $\psi \in C_c^\infty(\mathbb{R}^4 \times [0,T])$ and $\gamma \in \mathbb{R}$ with $\psi \ge 0$.

Proof. According to the regularized system (2.1)$_1$, we see that Π satisfies Poisson equation formally:

$$-\Delta\pi_k = \partial_i\partial_j(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j).$$

Then, we localize the above equation with an arbitrary Lipschitz function $\psi \in C^{0,1}(\mathbb{R}^4)$, which yields

$$\begin{aligned}
-\Delta(\Pi_k\psi) &= \psi\partial_i\partial_j(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j) - \operatorname{div}(\Pi_k\nabla\psi) - \nabla\Pi_k\cdot\nabla\psi \\
&= \partial_i\partial_j[\psi(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)] - \partial_i[(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)\partial_j\psi] \\
&\quad - \partial_j(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)\partial_i\psi - \operatorname{div}(\Pi_k\nabla\psi) - \nabla\Pi_k\cdot\nabla\psi \\
&= \partial_i\partial_j[\psi(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)] \\
&\quad - \partial_i[(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)\partial_j\psi + \Pi_k\nabla\psi] \\
&\quad - \partial_j(\tilde{u}_k^i u_k^j - \widetilde{B}_k^i B_k^j)\partial_i\psi - \nabla\Pi_k\cdot\nabla\psi \\
&:= -\Delta\Pi_k^1 - \Delta\Pi_k^2 - \Delta\Pi_k^3.
\end{aligned} \tag{2.9}$$

Similarly, we decompose $\Pi\psi$ in a similar way. From the Calderon–Zygmund theory, we get

$$\begin{aligned}
&\|\Pi_k^1(t) - \Pi^1(t)\|_{L_x^{\frac{3}{2}}} \\
&\quad \le C\|\psi\tilde{u}_k^i(t)u_k^j(t) - \psi\tilde{u}^i(t)u^j(t)\|_{L_x^{\frac{3}{2}}} + C\|\psi\widetilde{B}_k^i(t)B_k^j(t) - \psi\widetilde{B}^i(t)B^j(t)\|_{L_x^{\frac{3}{2}}} \\
&\quad \le C\|\psi(\tilde{u}_k^i(t) - \tilde{u}^i(t))(u_k^j(t) - u^j(t))\|_{L_x^{\frac{3}{2}}} + C\|\psi\tilde{u}^i(t)(u_k^j(t) - u^j(t))\|_{L_x^{\frac{3}{2}}} \\
&\qquad + C\|\psi u^j(t)(\tilde{u}_k^i(t) - \tilde{u}^i(t))\|_{L_x^{\frac{3}{2}}} + \|\psi(\widetilde{B}_k^i(t) - \widetilde{B}^i(t))(B_k^j(t) - B^j(t))\|_{L_x^{\frac{3}{2}}} \\
&\qquad + C\|\psi\widetilde{B}^i(t)(B_k^j(t) - B^j(t))\|_{L_x^{\frac{3}{2}}} + C\|\psi B^j(t)(\widetilde{B}_k^i(t) - \widetilde{B}^i(t))\|_{L_x^{\frac{3}{2}}} \\
&\quad \le C\|\psi^{\frac{1}{2}}(u_k^j(t) - u^j(t))\|_{L_x^3}^2 + C\|\psi\tilde{u}^i(t)(u_k^j(t) - u^j(t))\|_{L_x^{\frac{3}{2}}} \\
&\qquad + C\|\psi u^j(t)(\tilde{u}_k^i(t) - \widetilde{B}^i(t))\|_{L_x^{\frac{3}{2}}} + \|\psi^{\frac{1}{2}}(B_k^j(t) - B^j(t))\|_{L_x^3}^2 \\
&\qquad + C\|\psi\widetilde{B}^i(t)(B_k^j(t) - B^j(t))\|_{L_x^{\frac{3}{2}}} + C\|\psi B^j(t)(\widetilde{B}_k^i(t) - \widetilde{B}^i(t))\|_{L_x^{\frac{3}{2}}}.
\end{aligned}$$

Since $\omega_k \to \omega$ weakly, $|u_k^j - u^j|^{\frac{3}{2}}$ and $|B_k^j - B^j|^{\frac{3}{2}}$ are uniformly integrable with respect to $|\psi|^{\frac{3}{2}}|\tilde{u}^i|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t$ and $|\psi|^{\frac{3}{2}}|\widetilde{B}^i|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t$, respectively, we have

$$\begin{aligned}
&\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|\Pi_k^1 - \Pi^1|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t \\
&\quad \le C\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|\psi|^{\frac{3}{2}}(|u_k - u|^3 + |B_k - B|^3)\mathrm{d}x\mathrm{d}t \\
&\qquad + C\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|\psi|^{\frac{3}{2}}(|\tilde{u}^i(u_k^j - u^j)|^{\frac{3}{2}} + |\widetilde{B}^i(B_k^j - B^j)|^{\frac{3}{2}})\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ C\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|\psi|^{\frac32}(|u^j(\tilde{u}_k^i-\tilde{u}^i)|^{\frac32}+|B^j(\widetilde{B}_k^i-\widetilde{B}^i)|^{\frac32})\mathrm{d}x\mathrm{d}t\\
&\le C\int_0^T\int_{\mathbb{R}^4}|\psi|^{\frac32}\mathrm{d}\omega,
\end{aligned}
$$

where we have used the Vitali's convergence theorem for the second and third lines:

$$
\begin{aligned}
&\|\Pi_k^2-\Pi^2\|_{L^{\frac32}_{t,x}}\\
&\quad=\|(-\Delta)^{-1}[-\operatorname{div}((u_ku_k^j-uu^j)\partial_j\psi-(B_kB_k^j-BB^j)\partial_j\psi+(\Pi_k-\Pi)\nabla\psi)]\|_{L^{\frac32}_{t,x}}\\
&\quad\le C\|(u_ku_k^j-uu^j)\partial_j\psi-(B_kB_k^j-BB^j)\partial_j\psi+(\Pi_k-\Pi)\nabla\psi\|_{L_t^{\frac32}L_x^{\frac{12}{11}}}\\
&\quad\le C\|\nabla\psi\|_{L^\infty}(\|u_ku_k^j-uu^j\|_{L_t^{\frac32}L_x^{\frac{12}{11}}}+\|B_kB_k^j-BB^j\|_{L_t^{\frac32}L_x^{\frac{12}{11}}}+\|\Pi_k-\Pi\|_{L_t^{\frac32}L_x^{\frac{12}{11}}})\\
&\quad\le C\|\nabla\psi\|_{L^\infty}(\|u_k-u\|_{L_t^3L_x^{\frac{24}{11}}}+\|B_k-B\|_{L_t^3L_x^{\frac{24}{11}}}+\|\Pi_k-\Pi\|_{L_t^{\frac32}L_x^{\frac{12}{11}}}).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\|\Pi_k^3-\Pi^3\|_{L^{\frac32}_{t,x}}\\
&\quad\le C\|\nabla\psi\|_{L^\infty}(\|u_k-u\|_{L_t^3L_x^{\frac{24}{11}}}+\|B_k-B\|_{L_t^3L_x^{\frac{24}{11}}}+\|\Pi_k-\Pi\|_{L_t^{\frac32}L_x^{\frac{12}{11}}}).
\end{aligned}
$$

Since for any $f\in L_t^\infty L_x^2\cap L_t^2H_x^1$, with the help of interpolation inequality and Sobolev inequality, we have

$$
\begin{aligned}
\int_0^T\|f(t)\|^3_{L_x^{\frac{24}{11}}}\mathrm{d}t&=\int_0^T\|f(t)\|_{L_x^{\frac{24}{11}}}\cdot\|f(t)\|^2_{L_x^{\frac{24}{11}}}\mathrm{d}t\\
&\le\|f\|_{L_t^{12}L_x^{\frac{24}{11}}}\|f\|^2_{L_{t,x}^{\frac{24}{11}}}\le\|f\|_{L_t^{12}L_x^{\frac{24}{11}}}\|f\|^{\frac32}_{L^2_{t,x}}\|f\|^{\frac12}_{L^3_{t,x}}\\
&\le\|f\|^{\frac32}_{L^2_{t,x}}\|f\|^{\frac12}_{L^3_{t,x}}\left(\int_0^T\|f\|^{10}_{L_x^2}\|f\|^2_{L_x^4}\mathrm{d}t\right)^{\frac{1}{12}}\\
&\le\|f\|^{\frac32}_{L^2_{t,x}}\|f\|^{\frac12}_{L^3_{t,x}}\|f\|^{\frac56}_{L_t^\infty L_x^2}\|\nabla f\|^{\frac16}_{L^2_{t,x}}.
\end{aligned}
$$

Hence, taking $f=u_k-u$ or $f=B_k-B$, we get that

$$
\limsup_{k\to\infty}\|u_k-u\|_{L_t^3L_x^{\frac{24}{11}}}=0,\quad\limsup_{k\to\infty}\|B_k-B\|_{L_t^3L_x^{\frac{24}{11}}}=0.
$$

Combining Calderon–Zygmund theory, this yields

$$
\limsup_{k\to\infty}\|\Pi_k-\Pi\|_{L_t^3L_x^{\frac{12}{11}}}=0. \tag{2.10}
$$

Collecting (2.9)–(2.10), we know that

$$
\limsup_{k\to\infty}\|(\Pi_k-\Pi)\psi\|^{\frac32}_{L^{\frac32}_{t,x}}\le\limsup_{k\to\infty}\sum_{l=1}^3\|\Pi_k^l-\Pi^l\|^{\frac32}_{L^{\frac32}_{t,x}}\le\int_0^T\int_{\mathbb{R}^4}|\psi|^{\frac32}\mathrm{d}\omega.
$$

Thus, applying Vitali's convergence theorem and taking $\xi = \psi^{\frac{2}{3}}$, we get

$$
\begin{aligned}
&\limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u_k(\Pi_k - \gamma) - u(\Pi - \gamma)| \mathrm{d}x\mathrm{d}t \\
&\quad \le \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u_k||\Pi_k - p| \mathrm{d}x\mathrm{d}t + \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u_k - u||\Pi - \gamma| \mathrm{d}x\mathrm{d}t \\
&\quad \le \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u_k - u||\Pi_k - p| \mathrm{d}x\mathrm{d}t + \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^4} \xi |u||\Pi_k - p| \mathrm{d}x\mathrm{d}t \\
&\quad \le C \int_0^T \int_{\mathbb{R}^4} \xi \mathrm{d}\omega.
\end{aligned}
$$

Hence, we complete the proof of Lemma 2.4. ∎

Now, we can prove the following local energy inequalities.

Proposition 2.1. *Let the assumptions be as in Lemma 2.1; then, for any nonnegative cut-off functions $\eta \in C_c^\infty(\mathbb{R}^4 \times [0,T])$ with $\eta(\cdot,0) = 0$ and $\{\eta_i\}_{1\le i\le n} \subset C_c^\infty(\mathbb{R}^4 \times [0,T])$with $\eta = \sum_{i=1}^n \eta_i$, any functions $\{\gamma_i\}_{1\le i\le n} \subset L^{\frac{3}{2}}([0,T],\mathbb{R})$ and $\beta \subset L^3([0,T];\mathbb{R}^4)$, the following local energy inequalities hold:*

$$
\begin{aligned}
&\limsup_{k\to\infty} \sup_t \int_{\mathbb{R}^4} (|u_k(t)|^2 + |B_k(t)|^2)\eta(t)\mathrm{d}x \\
&\quad + \int_0^T \int_{\mathbb{R}^4} (|\nabla u|^2 + |\nabla B|^2)\eta \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\lambda \\
&\le \int_0^T \int_{\mathbb{R}^4} (|u|^2 + |B|^2)|\partial_t \eta + \Delta\eta| \mathrm{d}x\mathrm{d}t + 2\sum_{i=1}^n \int_0^T \int_{\mathbb{R}^4} (|u|^3 + |B|^3)|\nabla \eta_i| \mathrm{d}x\mathrm{d}t \\
&\quad + 3\sum_{i=1}^n \int_0^T \int_{\mathbb{R}^4} |\nabla\eta_i| \mathrm{d}\omega + 2\sum_{i=1}^n \int_0^T \int_{\mathbb{R}^4} |\nabla\eta_i||\Pi - \gamma_i|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t; \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
&\limsup_{k\to\infty} \sup_t \int_{\mathbb{R}^4} (|u_k(t)|^2 + |B_k(t)|^2)\eta(t)\mathrm{d}x \\
&\quad + \int_0^T \int_{\mathbb{R}^4} (|\nabla u|^2 + |\nabla B|^2)\eta \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} \eta \mathrm{d}\lambda \\
&\le \int_0^T \int_{\mathbb{R}^4} (|u|^2 + |B|^2)|\partial_t \eta + \Delta\eta| \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} (|u|^2 + |B|^2)(u\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t \\
&\quad + 2\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3 + |B-\beta|^3)|\nabla\eta| \mathrm{d}x\mathrm{d}t + 3\int_0^T \int_{\mathbb{R}^4} |\nabla\eta| \mathrm{d}\omega \\
&\quad + 2\int_0^T \int_{\mathbb{R}^4} \Pi(u\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t. \qquad (2.12)
\end{aligned}
$$

Proof. Substituting the test function η defined above into the local energy inequality (2.3) yields

$$\begin{aligned}
&\int_{\mathbb{R}^4}(|u_k(t)|^2+|B_k(t)|^2)\eta(t)\mathrm{d}x+\int_0^t\int_{\mathbb{R}^4}(|\nabla u_k|^2+|\nabla B_k|^2)\eta\mathrm{d}x\mathrm{d}t\\
&\quad\le\int_0^t\int_{\mathbb{R}^4}(|u_k|^2+|B_k|^2)|\partial_t\eta+\Delta\eta|\mathrm{d}x\mathrm{d}t+\int_0^t\int_{\mathbb{R}^4}(|u_k|^2+|B_k|^2)(\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t\\
&\qquad+2\int_0^t\int_{\mathbb{R}^4}\Pi_k(u_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t-\int_0^t\int_{\mathbb{R}^4}(\widetilde{B}_k\cdot\nabla\eta)(u_k\cdot B_k)\mathrm{d}x\mathrm{d}t. \qquad (2.13)
\end{aligned}$$

The convergence of the second term can be obtained by Lemma 2.2 and (2.6). Since $u_k\to u$ and $B_k\to B$ in $L^2(\mathbb{R}^4\times[0,T])$, then the convergence of the third term is straightforward. On the other hand, we note that

$$\int_0^T\int_{\mathbb{R}^4}|\tilde{u}_k|^3|\nabla\eta|\mathrm{d}x\mathrm{d}t=\|g_1+g_2\|_{L^3}^3,$$

where

$$\begin{aligned}
g_1(x,t)&=\int_{\mathbb{R}^4}u_k(x-y,t)\chi_k(y)(|\nabla\eta(x,t)|^{\frac13}-|\nabla\eta(x-y,t)|^{\frac13})\mathrm{d}y,\\
g_2(x,t)&=\int_{\mathbb{R}^4}u_k(x-y,t)\chi_k(y)|\nabla\eta(x-y,t)|^{\frac13}\mathrm{d}y.
\end{aligned}$$

For g_1, using Young's inequality for convolution, we have

$$\begin{aligned}
\|g_1\|_{L^3}&\le\left\|\int_{\mathbb{R}^4}u_k(x-y,t)\chi_k(y)\frac{|\nabla\eta(x,t)|^{\frac13}-|\nabla\eta(x-y,t)|^{\frac13}}{|y|^{\frac13}}d_k^{\frac13}\mathrm{d}y\right\|_{L^3}\\
&\le Cd_k^{\frac13}\|\eta\|_{C^2}\|\tilde{u}\|_{L^3}\\
&\le Cd_k^{\frac13}\|\eta\|_{C^2}\|u_k\|_{L^3},
\end{aligned}$$

where $d_k:=\mathrm{diam}(\mathrm{supp}\,\chi_k)\to0$ when $k\to\infty$. For g_2, using Young's inequality for convolution again, we have

$$\|g_2\|_{L^3}=\|(u_k|\nabla\eta|^{\frac13})*\chi_k\|_{L^3}\le\|u_k|\nabla\eta|^{\frac13}\|_{L^3}.$$

Hence, combining the estimates of g_1 and g_2, we have

$$\begin{aligned}
&\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|u_k|^2(\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t\\
&\quad\le C\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|u_k|^3|\nabla\eta|\mathrm{d}x\mathrm{d}t+C\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}|\tilde{u}_k|^3|\nabla\eta|\mathrm{d}x\mathrm{d}t\\
&\quad\le\int_0^T\int_{\mathbb{R}^4}|u|^3|\nabla\eta|\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^4}|\nabla\eta|\mathrm{d}\omega.
\end{aligned}$$

Going through a similar process, we have

$$\begin{aligned}&\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}(\widetilde{B}_k\cdot\nabla\eta)(u_k\cdot B_k)\mathrm{d}x\mathrm{d}t\\&\quad\le C\int_0^T\int_{\mathbb{R}^4}|B|^3|\nabla\eta|\mathrm{d}x\mathrm{d}t+C\int_0^T\int_{\mathbb{R}^4}|\nabla\eta|\mathrm{d}\omega.\end{aligned}$$

For the term involving pressure, with the help of incompressible condition and Lemma 2.4, we have

$$\begin{aligned}\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}\Pi_k(u_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t&=\sum_{i=1}^n\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}\Pi_k(u_k\cdot\nabla)\eta_i\mathrm{d}x\mathrm{d}t\\&=\sum_{i=1}^n\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}\Pi_k(u_k\cdot\nabla)\eta_i\mathrm{d}x\mathrm{d}t\\&\le C\sum_{i=1}^n\int_0^T\int_{\mathbb{R}^4}|u|^3|\nabla\eta_i|\mathrm{d}x\mathrm{d}t\\&\quad+\sum_{i=1}^n\int_0^n\int_{\mathbb{R}^4}|\nabla\eta_i|\mathrm{d}\omega\\&\quad+\sum_{i=1}^n C\int_0^T\int_{\mathbb{R}^4}|\nabla\eta_i||\Pi-\gamma_i|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.14}$$

Combining (2.13)–(2.14), we complete the proof of (2.11). For the local energy inequality (2.12), from incompressible conditions, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^4}(|u_k|^2+|B_k|^2)(\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^4}(\widetilde{B}_k\cdot\nabla\eta)(u_k\cdot B_k)\mathrm{d}x\mathrm{d}t\\&\le C\int_0^T\int_{\mathbb{R}^4}|u_k-\beta+\beta|^2(\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t+C\int_0^T\int_{\mathbb{R}^4}|B_k-\beta+\beta|^2(\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t\\&\le C\int_0^T\int_{\mathbb{R}^4}|u_k-\beta|^2[(\tilde{u}_k-\beta)\cdot\nabla]\eta\mathrm{d}x\mathrm{d}t+C\int_0^T\int_{\mathbb{R}^4}|u_k-\beta|^2(\beta\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t\\&\quad+C\int_0^T\int_{\mathbb{R}^4}[(u_k-\beta)\cdot\beta](\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t+C\int_0^T\int_{\mathbb{R}^4}|B_k-\beta|^2[(\tilde{u}_k-\beta)\cdot\nabla]\eta\mathrm{d}x\mathrm{d}t\\&\quad+C\int_0^T\int_{\mathbb{R}^4}|B_k-\beta|^2(\beta\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t+C\int_0^T\int_{\mathbb{R}^4}[(B_k-\beta)\cdot\beta](\tilde{u}_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{2.15}$$

Since

$$\tilde{u}_k-\beta=(u_k-\beta)*\chi_k\quad\text{and}\quad\widetilde{B}_k-\beta=(u_k-\beta)*\chi_k,$$

applying Young's inequality for convolution yields

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^4} |u_k-\beta|^2[(\tilde{u}_k-\beta)\cdot\nabla]\eta \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^4} |B_k-\beta|^2[(\tilde{u}_k-\beta)\cdot\nabla]\eta \mathrm{d}x\mathrm{d}t \\
&\le \int_0^T \int_{\mathbb{R}^4} (|u_k-u|^3+|B_k-B|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} |u_k-u|^2|u-\beta||\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} |u_k-u||u-\beta|^2|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} |B_k-B|^2|B-\beta||\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} |B_k-B||B-\beta|^2|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^4} (|u_k-u|^3+|B_k-B|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\to C\int_0^T \int_{\mathbb{R}^4} |\nabla\eta|\mathrm{d}\omega + C\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \text{ as } (k\to\infty).
\end{aligned}$$

Now, we can pass $k\to\infty$ in (2.15), which yields

$$\begin{aligned}
&\limsup_{k\to\infty}\int_0^T \int_{\mathbb{R}^4} (|u_k|^2+|B_k|^2)(\tilde{u}_k\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t \\
&\quad + \limsup_{k\to\infty}\int_0^T \int_{\mathbb{R}^4} (\widetilde{B}_k\cdot\nabla\eta)(u_k\cdot B_k)\mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^4} |\nabla\eta|\mathrm{d}\omega + C\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^2+|B-\beta|^2)(\beta\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} [(u-\beta)\cdot\beta+(B-\beta)\cdot\beta](u\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T \int_{\mathbb{R}^4} |\nabla\eta|\mathrm{d}\omega + C\int_0^T \int_{\mathbb{R}^4} (|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t \\
&\quad + C\int_0^T \int_{\mathbb{R}^4} (|u|^2+|B|^2)(u\cdot\nabla)\eta \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^T\int_{\mathbb{R}^4}(|u-\beta|^2+|B-\beta|^2)[(u-\beta)\cdot\eta]\eta\mathrm{d}x\mathrm{d}t\\
&\leq C\int_0^T\int_{\mathbb{R}^4}|\nabla\eta|\mathrm{d}\omega+C\int_0^T\int_{\mathbb{R}^4}(|u-\beta|^3+|B-\beta|^3)|\nabla\eta|\mathrm{d}x\mathrm{d}t\\
&+C\int_0^T\int_{\mathbb{R}^4}(|u|^2+|B|^2)(u\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Finally, using Lemma 2.4 yields

$$
\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^4}\Pi_k(u_k\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t\leq\int_0^T\int_{\mathbb{R}^4}\Pi(u\cdot\nabla)\eta\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^4}|\nabla\eta|\mathrm{d}\omega.
$$

Hence, we complete the proof of Proposition 2.1. ■

3. Partial regularity theory: Part I

This section is devoted to establishing an ε-regularity criterion for weak solution sets $(u, B, \Pi, \lambda, \omega)$ of (1.1) in terms of L^3-norm of (u, B). The argument we will present is based on an iterative method. Firstly, as mentioned in [30], the weak solution sets have the following scaling property.

Lemma 3.1. *If $(u, B, \Pi, \lambda, \omega)$ is a weak solution set of the MHD system (1.1), then, for any $r > 0$, the scaled set $(u_r, B_r, \Pi_r, \lambda_r, \omega_r)$ is also a weak solution set of (1.1), where u_r, B_r, Π_r, λ_r, and ω_r are defined as*

$$
u_r(x,t)=ru(rx,r^2t),\quad B_r(x,t)=rB(rx,r^2t),\quad \Pi_r(x,t)=r^2\Pi(rx,r^2t),
$$
$$
\int_0^T\int_{\mathbb{R}^4}\mathrm{d}\lambda_r:=\frac{1}{r^2}\iint_{\{(rx,r^2t)|(x,t)\in\Omega\}}\mathrm{d}\lambda,\quad \int_0^T\int_{\mathbb{R}^4}\mathrm{d}\omega_r:=\frac{1}{r^3}\iint_{\{(rx,r^2t)|(x,t)\in\Omega\}}\mathrm{d}\omega
$$

for any $\omega\subset\mathbb{R}^4\times\mathbb{R}$.

Proposition 3.1. *There exist an $r > 0$ and an absolute positive constant $\varepsilon > 0$ such that if a weak solution set $(u, B, \Pi, \lambda, \omega)$ of the MHD system (1.1) satisfies*

$$
\frac{1}{r^3}\int_{Q_r(z_0)}|u|^3+|B|^3+|\Pi|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t+\frac{1}{r^3}\int_{Q_r(z_0)}\mathrm{d}\omega\leq\varepsilon,
$$

then $z_0 = (x_0, t_0)$ is a regular point and $Q_r(z_0) := B_r(x_0)\times(t_0-r^2, t_0)$ denote the parabolic cylinder centered at $z_0 = (x_0, t_0)$.

For the proof of Proposition 3.1, we need two auxiliary lemmas, which play an important role in the proof of Proposition 3.1. Before stating lemmas, we introduce the following dimensionless quantities due to the scaling property of system (1.1):

$$
\begin{aligned}
A(r) &= \limsup_{k\to\infty} \sup_{t_0-r^2<t<t_0} \frac{1}{r^2}\int_{B_r(x_0)} |u_k(t)|^2+|B_k(t)|^2 \mathrm{d}x,\\
B(r) &= \frac{1}{r^2}\int_{Q_r(z_0)} |\nabla u|^2+|\nabla B|^2 \mathrm{d}x\mathrm{d}t, \quad \widetilde{B}(r)=\frac{1}{r^2}\int_{Q_r(z_0)} \mathrm{d}\lambda,\\
C(r) &= \frac{1}{r^3}\int_{Q_r(z_0)} |u|^3+|B|^3 \mathrm{d}x\mathrm{d}t, \quad \widetilde{C}(r)=\frac{1}{r^3}\int_{Q_r(z_0)} \mathrm{d}\omega,\\
\overline{C}(r) &= \frac{1}{r^3}\int_{Q_r(z_0)} |u-u_r|^3+|B-B_r|^3 \mathrm{d}x\mathrm{d}t,\\
D(r) &= \frac{1}{r^3}\int_{Q_r(z_0)} |\Pi|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t, \quad \overline{D}(r)=\frac{1}{r^3}\int_{Q-r(z_0)} |\Pi-\Pi_r|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $f_r(t) := \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f(x,t)\mathrm{d}x$ is the integral average of $f(x,t)$ on $B_r(x_0)$. By using iteration method, we can state that u and B are locally bounded if u, B, Π, and concentration measure ω satisfy a local smallness condition.

Lemma 3.2. *Assume that* $(u, B, \Pi, \lambda, \omega)$ *is a weak solution set of the MHD system* (1.1) *in* $Q_r(z_0)$*; then, there exists a universal positive constant* $C > 0$ *such that, for any* $r > 0$,

$$C(r) \le CA^{\frac{3}{2}}(r) + CB^{\frac{3}{2}}(r), \tag{3.1}$$

$$\widetilde{C}(r) \le CA^{\frac{1}{2}}(r)\widetilde{B}(r), \tag{3.2}$$

$$\overline{C}(r) \le CA^{\frac{1}{2}}(r)B(r). \tag{3.3}$$

Proof. By using interpolation inequality and Sobolev inequality, we have

$$
\begin{aligned}
&\|u-u_r\|^3_{L^3(B_r(x_0))} + \|B-B_r\|^3_{L^3(B_r(x_0))}\\
&\le C\|u-u_r\|^2_{L^4(B_r(x_0))}\|u-u_r\|_{L^2(B_r(x_0))}\\
&\quad + C\|B-B_r\|^2_{L^4(B_r(x_0))}\|B-B_r\|_{L^2(B_r(x_0))}\\
&\le \|\nabla u\|^2_{L^2(B_r(x_0))}\|u\|_{L^2(B_r(x_0))} + \|\nabla B\|^2_{L^2(B_r(x_0))}\|B\|_{L^2(B_r(x_0))}.
\end{aligned}
$$

Then, integrating in time, we get

$$\int_{Q_r(z_0)} |u-u_r|^3+|B-B_r|^3 \mathrm{d}x\mathrm{d}t \le Cr^3A^{\frac{1}{2}}(r)B(r),$$

where we have used the lower semi-continuity of weak-$*$ convergence, this is, (3.3). Noting that $\|f\|_{L^4(B_r(x_0))} \le C\|f\|_{L^2(B_r(x_0))} + \|\nabla f\|_{L^2(B_r(x_0))}$, by a direct calculation, we get (3.1). The inequality (3.2) follows directly from Lemma 2.3. ∎

In order to prove Proposition 3.1, we will define a quantity $E(r)$, which is not scale-invariant: $E(r) = \frac{1}{r^{\frac{5}{2}}}\int_{Q_r(z_0)} |\Pi-\Pi_r|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t = r^{\frac{1}{2}}\overline{D}(r)$.

Lemma 3.3. *Suppose that $(u, B, \Pi, \lambda, \omega)$ is a weak solution set of the MHD system (1.1) in $Q_\rho(z_0)$. Then, there exists a universal positive constant $C > 0$ such that, for any $0 < r \le \frac{\rho}{2}$,*

$$\begin{aligned} E(r) \le\, & C\frac{1}{r^{\frac{5}{2}}}\int_{Q_{2r}(z_0)} |u|^3 + |B|^3 \mathrm{d}x\mathrm{d}t \\ & + Cr^5\left(\sup_{t_0-r^2<t<t_0}\int_{2r<|y-x_0|<\rho}\frac{|u|^2+|B|^2}{|y-x_0|^5}\mathrm{d}y\right)^{\frac{3}{2}} \\ & + C\frac{r^3}{\rho^{\frac{11}{2}}}\int_{Q_\rho(z_0)} |u|^3+|B|^3+|\Pi|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.4}$$

Proof. We consider a smooth cutoff function $\phi \in C_c^\infty(\mathbb{R}^4)$ such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_{\frac{3}{4}\rho}(x_0)$, $\phi \equiv 0$ in $\mathbb{R}^4 \backslash B_\rho(x_0)$, satisfying property $|\nabla\phi|^2 + |\nabla^2\phi| \le C\frac{1}{\rho^2}$. Then, similar to (2.9) for the local decomposition for pressure, we have

$$\begin{aligned} \Pi(x,t)\phi(x) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{1}{|x-y|^2}(\eta\partial_i\partial_j(u_iu_j - B_iB_j) - 2\nabla\phi\cdot\nabla\Pi - \Pi\Delta\phi)\mathrm{d}y \\ &= \frac{1}{4\pi^2}\int_{B_{2r}(x_0)}(u_iu_j - B_iB_j)\phi\partial_i\partial_j\left(\frac{1}{|x-y|^2}\right)\mathrm{d}y \\ &\quad + \frac{1}{4\pi^2}\int_{B_\rho(x_0)\backslash B_{2r}(x_0)}(u_iu_j - B_iB_j)\phi\partial_i\partial_j\left(\frac{1}{|x-y|^2}\right)\mathrm{d}y \\ &\quad + \frac{1}{4\pi^2}\int_{B_\rho(x_0)}(u_iu_j - B_iB_j)\left(\frac{\partial_i\partial_j\phi}{|x-y|^2} + \partial_j\phi\frac{4(x_i-y_i)}{|x-y|^4}\right)\mathrm{d}y \\ &\quad + \frac{1}{4\pi^2}\int_{\mathbb{R}^4}\Pi\left(\frac{\Delta\phi}{|x-y|^2} + \frac{4(x-y)\cdot\nabla\phi}{|x-y|^4}\right)\mathrm{d}y \\ &:= \Pi_1(x,t) + \Pi_2(x,t) + \Pi_3(x,t) + \Pi_4(x,t). \end{aligned} \tag{3.5}$$

Hence, the term $E(r)$ can be decomposed into four terms involving $\Pi_1, \Pi_2, \Pi_3, \Pi_4$, respectively:

$$E(r) \le \sum_{l=1}^{4}\frac{1}{r^{\frac{5}{2}}}\int_{Q_r(z_0)}|\Pi_l - \Pi_{l,r}|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t. \tag{3.6}$$

Using the Calderon–Zygmund theory for Π_1, we get

$$\int_{B_r(x_0)}|\Pi_1|^{\frac{3}{2}}\mathrm{d}x \le C\int_{B_{2r}(x_0)}|u|^3 + |B|^3\mathrm{d}x,$$

which implies

$$\int_{Q_r(z_0)}|\Pi_1 - \Pi_{1,r}|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t \le C\int_{Q_{2r}(z_0)}|u|^3 + |B|^3\mathrm{d}x\mathrm{d}t.$$

For Π_2, it is noteworthy that $2|x-y|>|y|$ when $x\in B_r(x_0)$ and $y\in B^c_{2r}(x_0)$; thus, for any $(x,t)\in Q_r(z_0)$,

$$|\nabla\Pi_2(x,t)|\leq C\int_{2r<|y|<\rho}\frac{\phi(|u|^2+|B|^2)}{|x-y|^5}\mathrm{d}x\leq C\int_{2r<|y|<\rho}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}x,$$

by which one has

$$\begin{aligned}\int_{Q_r(z_0)}|\Pi_2-\Pi_{2,r}|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t&\leq Cr^4\int_{t_0-r^2}^{t_0}\|\Pi_2-\Pi_{2,r}\|^{\frac{3}{2}}_{L^\infty(B_r(x_0))}\\&\leq Cr^{\frac{11}{2}}\int_{t_0-r^2}^{t_0}\|\nabla\Pi_2\|^{\frac{3}{2}}_{L^\infty(B_r(x_0))}\mathrm{d}t\\&\leq Cr^{\frac{15}{2}}\left(\sup_{t_0-r^2<t<t_0}\int_{2r<|y|<\rho}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}x\right)^{\frac{3}{2}}.\end{aligned}$$

Similarly, we note that $\nabla\phi=\nabla^2\phi=0$ in $B_{\frac{3}{4}}(x_0)$, and $|x-y|>\frac{\rho}{4}$ when $x\in B_r(x_0),\ y\in B_\rho(x_0)\backslash B_{\frac{3\rho}{4}}(x_0)$. Thus, for any $(x,t)\in Q_r(z_0)$,

$$\begin{aligned}|\nabla\Pi_3|&\leq C\int_{B_\rho(x_0)\backslash B_{\frac{3}{4}}\rho(x_0)}(|u|^2+|B|^2)\left(\frac{|\nabla\phi|}{|x-y|^3}+\frac{|\nabla\phi|}{|x-y|^4}\right)\mathrm{d}y\\&\leq C\frac{1}{\rho^5}\int_{B_\rho(x_0)\backslash B_{\frac{3}{4}}\rho(x_0)}(|u|^2+|B|^2)\mathrm{d}y.\\|\nabla\Pi_4|&\leq C\int_{B_\rho(x_0)\backslash B_{\frac{3}{4}}\rho(x_0)}|\Pi|\left(\frac{|\nabla\phi|}{|x-y|^3}+\frac{|\nabla\phi|}{|x-y|^4}\right)\mathrm{d}y\\&\leq C\frac{1}{\rho^5}\int_{B_\rho(x_0)\backslash B_{\frac{3}{4}}\rho(x_0)}|\Pi|\mathrm{d}y.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{l=3}^4\int_{Q_r(z_0)}|\Pi_l-\Pi_{l,r}|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t&\leq C\sum_{l=3}^4 r^4\int_{t_0-r^2}^{t_0}\|\Pi_l-\Pi_{l,r}\|^{\frac{3}{2}}_{L^\infty(B_r(x_0))}\mathrm{d}t\\&\leq C\sum_{l=3}^4 r^{\frac{11}{2}}\int_{Q_\rho(z_0)}\|\nabla\Pi_l\|^{\frac{3}{2}}_{L^\infty(B_r(x_0))}\mathrm{d}t\\&\leq C(\frac{r}{\rho})^{\frac{11}{2}}\int_{Q_\rho(z_0)}|u|^3+|B|^3+|\Pi|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.7}$$

Combining (3.6) with (3.7) yields the inequality (3.4). ■

Proof of Proposition 3.1. We let $r_n=2^{-n}$ with $n\geq 2$. The method is to iteratively prove the following estimates:

$$C(r_n)+\widetilde{C}(r_n)+E(r_n)\leq\varepsilon^{\frac{2}{3}}r_n^3,\tag{3.8}$$

$$A(r_n)+B(r_n)+\widetilde{B}(r_n)\leq Cr_n^2\varepsilon^{\frac{2}{3}}.\tag{3.9}$$

First, we claim that the inequality $(3.8)_1$ holds. Indeed, the Hölder inequality yields

$$
\begin{aligned}
&C(r_1) + \widetilde{C}(r_1) + E(r_1) \\
&\quad \le 8 \int_{Q_{\frac{1}{2}}(z_0)} |u|^3 + |B|^3 \mathrm{d}x\mathrm{d}t + 8 \int_{Q_{\frac{1}{2}}(z_0)} \mathrm{d}\omega + 16 \int_{Q_{\frac{1}{2}}(z_0)} |\Pi|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Then, we take $\varepsilon \le 2^{-21}$ and apply the smallness assumption, which implies that

$$
C(r_1) + \widetilde{C}(r_1) + E(r_1) \le 16\varepsilon \le \varepsilon^{\frac{2}{3}} r_1^3.
$$

Claim 1: $\{(3.8)_k\}_{2\le k\le n}$ *implies* $(3.9)_{n+1}$. Let ϕ_n be the localized solution of the backward heat equation

$$
\phi_n(x,t) = \frac{\chi(x,t)}{(r_n^2 - t)^2} \exp\left(- \frac{|x|^2}{4(r_n^2 - t)} \right),
$$

where χ is a cutoff function such that $\chi \equiv 1$ in $Q_{\frac{1}{4}}(z_0)$ and $\chi \equiv 0$ in $\mathbb{R}^4 \times (-\infty, 0) \backslash Q_{\frac{1}{3}}(z_0)$. A direct computation yields that ϕ_n satisfies the following properties:

$$
\begin{aligned}
&\partial_t \phi_n + \Delta \phi_n = 0 \text{ in } Q_{\frac{1}{4}}(z_0), \quad |\partial_t \phi_n + \Delta \phi_n| \le C \text{ in } \mathbb{R}^4 \times (-\infty, 0), \\
&C^{-1} r_n^{-4} \le \phi_n \le C r_n^{-4} \quad \text{and} \quad |\nabla \phi_n| \le C r_n^{-5} \text{ in } Q_{r_n}(z_0), \\
&\phi_n \le C r_k^{-4} \quad \text{and} \quad |\nabla \phi_n| \le C r_k^{-5} \text{ in } Q_{r_{k-1}}(z_0) \backslash Q_{r_k}(z_0) \text{ for any } 2 \le k \le n.
\end{aligned}
$$

Next, we define the smooth cutoff function $\{\eta_k\}_{k\in\mathbb{N}}$ such that

$$
\eta_k \equiv 1 \text{ in } Q_{r_{\frac{7}{8}k}}(z_0), \quad \eta_k \equiv 0 \text{ in } \mathbb{R}^4 \times (-\infty, 0) \backslash Q_{r_k}(z_0), \quad |\nabla \eta_k| \le C r_k^{-1}.
$$

Then, we define $\varphi_k := \phi_n(\eta_k - \eta_{k+1})$ for $1 \le k \le n-1$ and $\varphi_n := \phi_n \eta_n$. It is easy to see that $\phi_n = \sum_{k=1}^n \varphi_k$ and

$$
|\nabla \varphi_k| = |\phi_n \nabla \eta_k + \eta_k \nabla \phi_n| \le C r_k^{-5} \quad \text{for any } k \le n.
$$

Substituting ϕ_n into the local energy inequality (2.11) yields

$$
\begin{aligned}
&\limsup_{k\to\infty} \sup_t \int_{B_{\frac{1}{2}}(x_0)} (|u_k(t)|^2 + |B_k(t)|^2) \phi_n \mathrm{d}x \\
&\qquad + \int_{Q_{\frac{1}{2}}(z_0)} (|\nabla u|^2 + |\nabla B|^2) \phi_n \mathrm{d}x\mathrm{d}t + \int_{Q_{\frac{1}{2}}(z_0)} \phi_n \mathrm{d}\lambda \\
&\quad \le \int_{Q_{\frac{1}{2}}(z_0)} (|u|^2 + |B|^2) |\partial_t \phi_n + \Delta \phi_n| \mathrm{d}x\mathrm{d}t \\
&\qquad + 2 \sum_{k=1}^n \int_{Q_{\frac{1}{2}}(z_0)} |\nabla \varphi_k| [(|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega] \\
&\qquad + 2 \sum_{k=1}^n \int_{Q_{\frac{1}{2}}(z_0)} |\nabla \varphi_k| |\Pi - \Pi_{r_k}|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \\
&\quad := I_1 + I_2 + I_3. \qquad (3.10)
\end{aligned}
$$

With the properties of ϕ_n, we deduce that

$$C^{-1} r_{n+1}^{-2} (A(r_{n+1}) + B(r_{n+1}) + \widetilde{B}(r_{n+1})) \le I_1 + I_2 + I_3.$$

For I_1, applying the Hölder inequality and the smallness assumption yields

$$\begin{aligned}
I_1 &\le C \int_{Q_{\frac{1}{2}}(z_0)} |u|^2 + |B|^2 \mathrm{d}x\mathrm{d}t \\
&\le C \left(\int_{Q_{\frac{1}{2}}(z_0)} 1 \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{3}} \left(\int_{Q_{\frac{1}{2}}(z_0)} |u|^3 + |B|^3 \mathrm{d}x\mathrm{d}t \right)^{\frac{2}{3}} \\
&\le C \left(\int_{Q_1(z_0)} |u|^3 + |B|^3 \mathrm{d}x\mathrm{d}t \right)^{\frac{2}{3}} \\
&\le C \varepsilon^{\frac{2}{3}}, \\
I_2 &= \sum_{k=2}^{n} \int_{Q_{r_{k-1}}(z_0) \setminus Q_{r_k}(z_0)} |\nabla \varphi_k| [(|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega] \\
&\quad + \int_{Q_{r_n}(z_0)} |\nabla \varphi_k| [(|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega] \\
&\le C \sum_{k=2}^{n} \frac{1}{r_k^5} \int_{Q_{r_{k-1}}(z_0) \setminus Q_{r_k}(z_0)} (|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega \\
&\quad + C \frac{1}{r_n^5} \int_{Q_{r_n}(z_0)} (|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega \\
&\le C \sum_{k=2}^{n} \frac{1}{r_{k-1}^5} \int_{Q_{r_{k-1}}(z_0)} (|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega \\
&\quad + C \frac{1}{r_n^5} \int_{Q_{r_n}(z_0)} (|u|^3 + |B|^3) \mathrm{d}x\mathrm{d}t + \mathrm{d}\omega \\
&\le C \sum_{k=1}^{n} r_k \varepsilon^{\frac{2}{3}}, \\
I_3 &\le C \sum_{k=2}^{n} \frac{1}{r_k^5} \int_{Q_{r_{k-1}}(z_0) \setminus Q_{r_k}(z_0)} |\Pi - \Pi_{r_k}|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \\
&\quad + C \frac{1}{r_n^5} \int_{Q_{r_n}(z_0)} |\Pi - \Pi_{r_n}|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \\
&\le C \sum_{k=1}^{n} r_k^{\frac{1}{2}} \varepsilon^{\frac{2}{3}}.
\end{aligned}$$

Combining the estimates for I_1, I_2, I_3 and the local energy inequality (3.10), we deduce that $(3.9)_{n+1}$ holds.

Claim 2: $\{(3.9)_k\}_{2\le k\le n}$ *implies* $(3.8)_n$. Lemma 3.2 yields that, for any $2 \le k \le n$,

$$C(r_k) + \widetilde{C}(r_k) \le CA^{\frac{3}{2}}(r_k) + CB^{\frac{3}{2}}(r_k) + C\widetilde{B}(r_k) \le C\varepsilon r_k^3. \tag{3.11}$$

Taking $\rho = r_1$, $r = r_n$ and using Lemmas 3.2 and 3.3, we have

$$\begin{aligned}
E(r) &\le C\frac{1}{r_n^{\frac{5}{2}}}\int_{Q_{r_{n-1}}(z_0)} |u|^3 + |B|^3 \mathrm{d}x\mathrm{d}t \\
&\quad + Cr_n^5\left(\sup_{t_0-r_n^2<t<t_0}\int_{2r_n<|y|<r_1}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}y\right)^{\frac{3}{2}} \\
&\quad + C\frac{r_n^3}{r_1^{\frac{11}{2}}}\int_{Q_{r_1}(z_0)} |u|^3 + |B|^3 + |\Pi|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t \\
&\le Cr_n^{\frac{1}{2}}C(r_{n-1}) + Cr_n^5\left(\sup_{t_0-r_n^2<t<t_0}\int_{2r_n<|y|<r_1}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}y\right)^{\frac{3}{2}} + C\varepsilon r_n^3 \\
&\le C\varepsilon r_n^3 + Cr_n^5\left(\sup_{t_0-r_n^2<t<t_0}\int_{2r_n<|y|<r_1}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}y\right)^{\frac{3}{2}}.
\end{aligned}$$

It is noteworthy that

$$\begin{aligned}
\sup_{t_0-r_n^2<t<t_0}\int_{2r_n<|y|<r_1}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}y &\le \sum_{k=2}^{n-1}\sup_{t_0-r_{k-1}^2<t<t_0}\int_{r_k<|y|<r_{k-1}}\frac{|u|^2+|B|^2}{|y|^5}\mathrm{d}y \\
&\le C\sum_{k=2}^{n-1}\frac{1}{r_k^3}A(r_{k-1}) \le C\varepsilon^{\frac{2}{3}}\sum_{k=2}^{n-1}\frac{1}{r_k} \le C\varepsilon^{\frac{2}{3}}r_n^{-1}.
\end{aligned} \tag{3.12}$$

Hence, from (3.11)–(3.12), we can deduce that

$$C(r_n) + \widetilde{C}(r_n) + E(r_n) \le C\varepsilon r_n^3 \le \varepsilon^{\frac{2}{3}}r_n^3.$$

Now, we can see that $(3.9)_k$ holds for any $k \ge 2$. Hence, using Lebesgue convergence theorem, we get that $z_0 = (x_0, t_0)$ is a regular point. ∎

4. Partial regularity theory: Part II

With the help of Proposition 3.1, we can prove the following partial regularity result, which plays a vital role in the proof of Theorem 1.1.

Proposition 4.1. *Assume that* $(u, B, \Pi, \lambda, \omega)$ *is a weak solution set of the MHD system* (1.1) *in some cylinder* $Q_\rho(z_0)$. *Then, there exists a universal positive constant* $\varepsilon_0 > 0$

such that if

$$\limsup_{r\to 0}\frac{1}{r^2}\int_{Q_r(z_0)}(|\nabla u|^2+|\nabla B|^2)\mathrm{d}x\mathrm{d}t+\mathrm{d}\lambda\le\varepsilon_0,$$

then $\|u\|_{L^\infty(Q_r(z_0))}<Cr_0^{-1}$ *and* $\|B\|_{L^\infty(Q_r(z_0))}<Cr_0^{-1}$ *for some* $0<r_0<\rho$.

For the proof of Proposition 4.1, we also need another decay estimate for the pressure Π.

Lemma 4.1. *Assume that* (u,B,Π,λ,ω) *is a weak solution set of the MHD system* (1.1) *in* $Q_r(z_0)$. *Then, there exists a universal positive constant* $C>0$ *such that, for* $r>0$ *and* $\theta\in(0,\frac{1}{2}]$,

$$D(\theta r)\le C\theta^{-3}A^{\frac{1}{2}}(r)B(r)+C\theta D(r).$$

Proof. We consider a smooth cutoff function $\psi\in C_c^\infty(\mathbb{R}^4)$ such that $0\le\psi\le 1$ and $\psi\equiv 1$ in $B_{\frac{3}{4}}(x_0)$, $\psi\equiv 0$ in $\mathbb{R}^4\backslash B_r(x_0)$, $|\nabla\psi|^2+|\nabla^2\psi|\le C\frac{1}{r^2}$. The pressure equation can be written as

$$-\Delta\Pi=\partial_i\partial_j[(u_i-u_{i,r})(u_j-u_{j,r})-(B_i-B_{i,r})(B_j-B_{j,r})].$$

We can localize this equation like (3.5) to obtain

$$\Pi(x,t)\psi(x)=\widetilde{\Pi_1}(x,t)+\widetilde{\Pi_2}(x,t)+\widetilde{\Pi_3}(x,t),$$

where

$$\begin{aligned}\widetilde{\Pi_1}(x,t)&=\frac{1}{4\pi^2}\int_{B_r(x_0)}[(u_i-u_{i,r})(u_j-u_{j,r})-(B_i-B_{i,r})(B_j-B_{j,r})]\\&\quad\times\psi\partial_i\partial_j\left(\frac{1}{|x-y|^2}\right)\mathrm{d}y,\\\widetilde{\Pi_2}(x,t)&=\frac{1}{4\pi^2}\int_{B_r(x_0)}[(u_i-u_{i,r})(u_j-u_{j,r})-(B_i-B_{i,r})(B_j-B_{j,r})]\\&\quad\times\left(\frac{\partial_i\partial_j\psi}{|x-y|^2}+\partial_j\psi\frac{4(x_i-y_i)}{|x-y|^4}\right)\mathrm{d}y,\\\widetilde{\Pi_3}(x,t)&=\frac{1}{4\pi^2}\int_{B_r(x_0)}\Pi\left(\frac{\Delta\psi}{|x-y|^2}+\frac{4(x-y)\cdot\nabla\psi}{|x-y|^4}\right)\mathrm{d}y.\end{aligned}$$

For $\widetilde{\Pi_1}$, the Calderon–Zygmund theory yields

$$\begin{aligned}\int_{B_{\theta r}(x_0)}|\widetilde{\Pi_1}|^{\frac{3}{2}}\mathrm{d}x\mathrm{d}t&\le C\int_{B_r(x_0)}|(u_i-u_{i,r})(u_j-u_{j,r})|^{\frac{3}{2}}\\&\quad+|(B_i-B_{i,r})(B_j-B_{j,r})|^{\frac{3}{2}}\mathrm{d}x\\&\le C\int_{B_r(x_0)}|u-u_r|^3+|B-B_r|^3\mathrm{d}x.\end{aligned}$$

Integrating in time gives

$$\int_{Q_{\theta r}(z_0)} |\widetilde{\Pi}_1|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \le C \int_{Q_r(z_0)} |u-u_r|^3 + |B-B_r|^3 \mathrm{d}x\mathrm{d}t.$$

For $\widetilde{\Pi}_2$, we note that $\nabla\psi$ is supported in $B_r(x_0)\backslash B_{\frac{3}{4}r}(x_0)$. Then, for $x \in B_{\theta r}(x_0)$ and $|x-y| > \frac{r}{4}$, combining the properties of ψ, we know that

$$|\widetilde{\Pi}_2| \le C\frac{1}{r^4} \int_{B_r(x_0)} |u-u_r|^2 + |B-B_r|^2 \mathrm{d}x,$$

which yields that

$$\begin{aligned}\int_{Q_{\theta r}(z_0)} |\widetilde{\Pi}_2|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t &\le C(\theta r)^4 \int_{t_0-(\theta r)^2}^{t_0} \|\widetilde{\Pi}_2\|_{L^\infty} \mathrm{d}t \\ &\le C\theta^4 \int_{Q_r(z_0)} |u-u_r|^3 + |B-B_r|^3 \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Similarly, for $\widetilde{\Pi}_3$, we have

$$\int_{Q_{\theta r}(z_0)} |\widetilde{\Pi}_3|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t \le C\theta^4 \int_{Q_r(z_0)} |\Pi|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t.$$

Collecting all estimates for $\widetilde{\Pi}_1$, $\widetilde{\Pi}_2$, $\widetilde{\Pi}_3$ and applying Lemma 3.2, we complete the proof of Lemma 4.1. ■

Proof of Proposition 4.1. According to Proposition 3.1, we need only to prove that, for $r_1 > 0$,

$$\frac{1}{r_1^3} \int_{Q_{r_1}(z_0)} |u|^3 + |B|^3 + |\Pi|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t + \frac{1}{r_1^3} \int_{Q_{r_1}(z_0)} \mathrm{d}\omega \le \varepsilon. \tag{4.1}$$

For fixed $r > 0$ and any $0 < \theta \le \frac{1}{2}$, we consider the cutoff function

$$\phi_\theta(x,t) = \frac{1}{[(\theta r)^2 - t]} \exp\left(-\frac{|x|^2}{4[(\theta r)^2 - t]}\right)\chi(\frac{x}{r}, \frac{t}{r^2})(x,t) \in \mathbb{R}^4 \times (-\infty, 0),$$

where $\chi \in C_c^\infty(B_1(x_0) \times (-1,1))$ is a cutoff function such that

$$\chi \equiv 1 \quad \text{in } B_{\frac{1}{2}}(x_0) \times \left(-\frac{1}{4}, \frac{1}{4}\right),$$

satisfying the following properties:

$$\begin{aligned}&C^{-1}(\theta r)^{-4} \le \phi_\theta \le C(\theta r)^{-4} \quad \text{in } Q_{\theta r}(z_0),\\ &\phi_\theta \le C(\theta r)^{-4}, \quad |\nabla\phi_\theta| \le C(\theta r)^{-5}, \quad |\partial_t\phi_\theta + \Delta\phi_\theta| \le Cr^{-6} \quad \text{in } Q_r(z_0).\end{aligned}$$

Substituting ϕ_θ into the local energy inequality (2.12) yields that

$$
\begin{aligned}
&\frac{1}{C(\theta r)^2}\limsup_{k\to\infty}\sup_{t_0-(\theta r)^2<t<t_0}\int_{B_{\theta r}(x_0)}|u_k|^2+|B_k|^2\mathrm{d}x\\
&\quad+\frac{1}{C(\theta r)^2}\int_{Q_{\theta r}(z_0)}(|\nabla u|^2+|\nabla B|^2)\mathrm{d}x\mathrm{d}t+\mathrm{d}\lambda\\
&\leq(\theta r)^2\int_{Q_r(z_0)}(|u|^2+|B|^2)(\partial_t\phi_\theta+\Delta\phi_\theta)\mathrm{d}x\mathrm{d}t\\
&\quad+(\theta r)^2\int_{Q_r(z_0)}(|u|^2+|B|^2)u\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\quad+(\theta r)^2\int_{Q_r(z_0)}|\nabla\phi_\theta|[(|u-u_r|^3+|B-B_r|^3)\mathrm{d}x\mathrm{d}t+\mathrm{d}\omega]\\
&\quad+(\theta r)^2\int_{Q_r(z_0)}\Pi u\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&:=K_1+K_2+K_3+K_4.
\end{aligned}\tag{4.2}
$$

Applying the Hölder inequality and the interpolation inequality in Lemma 3.2, we have

$$
K_1\leq C\theta^2C^{\frac{2}{3}}(r),\quad K_3\leq C\theta^{-3}[\widetilde{C}(r)+\bar{C}(r)],\quad K_4\leq C\theta^{-3}D^{\frac{2}{3}}(r)C^{\frac{1}{3}}.\tag{4.3}
$$

For K_2, using the property of ϕ_θ and integration by parts, we know that

$$
\begin{aligned}
K_2&=(\theta r)^2\int_{Q_r(z_0)}(|u|^2+|B|^2)(u-u_r)\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\quad+(\theta r)^2\int_{Q_r(z_0)}(|u|^2+|B|^2)u_r\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&=(\theta r)^2\int_{Q_r(z_0)}(|u-u_r|^2+|B-B_r|^2)(u-u_r)\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\quad+2(\theta r)^2\int_{Q_r(z_0)}[u\cdot u_r(u-u_r)+B\cdot B_r(u-u_r)]\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\quad-(\theta r)^2\int_{Q_r(z_0)}(|u_r|^2+|B_r|^2)(u-u_r)\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\quad+(\theta r)^2\int_{Q_r(z_0)}(|u|^2+|B|^2)u_r\cdot\nabla\phi_\theta\mathrm{d}x\mathrm{d}t\\
&\leq C(\theta r)^{-3}\int_{Q_r(z_0)}|u-u_r|^3+|B-B_r|^3\mathrm{d}x\mathrm{d}t\\
&\quad-2(\theta r)^2\int_{Q_r(z_0)}\phi_\theta[(u-u_r)\cdot\nabla u\cdot u_r+(u-u_r)\cdot\nabla B\cdot B_r]\mathrm{d}x\mathrm{d}t\\
&\quad-2(\theta r)^2\int_{Q_r(z_0)}\phi_\theta[u\cdot(u_r\cdot\nabla)u+B\cdot(u_r\cdot\nabla)B]\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{4.4}
$$

where we have used the fact that u and $u - u_r$ are divergence-free. Furthermore,

$$\begin{aligned}
&\int_{Q_r(z_0)} \phi_\theta[(u-u_r)\cdot\nabla u\cdot u_r + (u-u_r)\cdot\nabla B\cdot B_r]\mathrm{d}x\mathrm{d}t \\
&\quad\le \frac{C}{(\theta r)^4}\int_{t_0-r^2}^{t_0}\frac{1}{r^4}\int_{B_r(x_0)}|u|+|B|\mathrm{d}x\int_{B_r(x_0)}|u-u_r|(|\nabla u|+|\nabla B|)\mathrm{d}x\mathrm{d}t \\
&\quad\le \frac{C}{r^{\theta^4\frac{16}{3}}}\int_{t_0-r^2}^{t_0}\left(\int_{B_r(x_0)}|u|^3+|B|^3\mathrm{d}x\right)^{\frac{1}{3}}\left(\int_{B_r(x_0)}|u-u_r|^2\mathrm{d}x\right)^{\frac{1}{2}} \\
&\qquad\times\left(\int_{B_r(x_0)}|\nabla u|^2+|\nabla B|^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t \\
&\quad\le \frac{C}{\theta^4 r^5}\left(\int_{Q_r(z_0)}|u|^3+|B|^3\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{3}}\left(\int_{Q_r(z_0)}|\nabla u|^2+|\nabla B|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\times\left(\sup_{t_0-r^2<t<t_0}\int_{B_r(x_0)}|u|^2\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, using the Hölder equality, we have

$$\begin{aligned}
&\int_{Q_r(z_0)} \phi_\theta[u\cdot(u_r\cdot\nabla)u + B\cdot(u_r\cdot\nabla)B]\mathrm{d}x\mathrm{d}t \\
&\quad\le \frac{C}{\theta^4 r^5}\left(\int_{Q_r(z_0)}|u|^3+|B|^3\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{3}}\left(\int_{Q_r(z_0)}|\nabla u|^2+|\nabla B|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\times\left(\sup_{t_0-r^2<t<t_0}\int_{B_r(x_0)}|u|^2\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

Combining (4.4)–(4.5), we get

$$K_2 \le C\theta^{-2}C^{\frac{1}{3}}(r)A^{\frac{1}{2}}(r)B^{\frac{1}{2}}(r) + C\theta^{-3}\overline{C}(r). \tag{4.6}$$

Substituting (4.3) and (4.6) into (4.2) and using Lemma 3.2, we get

$$\begin{aligned}
&A(\theta r) + B(\theta r) + \widetilde{B}(\theta r) \\
&\quad\le C(\theta^2 C^{\frac{2}{3}}(r) + \theta^{-3}D^{\frac{2}{3}}(r)C^{\frac{1}{3}}(r) + \theta^{-3}[\overline{C}(r)+\widetilde{C}(r)] + \theta^{-2}C^{\frac{1}{3}}A^{\frac{1}{2}}(r)B^{\frac{1}{2}}) \\
&\quad\le C[\theta^2 C^{\frac{2}{3}}(r) + \theta^{-8}D^{\frac{4}{3}}(r) + \theta^{-3}A^{\frac{1}{2}}(r)(B(r)+\widetilde{B}(r)) + \theta^{-6}A(r)B(r)] \\
&\quad\le C[\theta^2 A(r) + \theta^2[B(r)+\widetilde{B}(r)] + \theta^{-8}A(r)[B(r)+\widetilde{B}(r)] \\
&\qquad + \theta^{-6}A(r)B(r) + \theta^{-8}D^{\frac{4}{3}}].
\end{aligned} \tag{4.7}$$

On the other hand, from Lemma 4.1, we get

$$\begin{aligned}
D^{\frac{4}{3}}(\theta r) &\le C\theta^{-4}A^{\frac{2}{3}}(r)B^{\frac{4}{3}}(r) + \theta^{\frac{4}{3}}D^{\frac{4}{3}}(r) \\
&\le C\theta^{-12}A(r)B(r) + \theta^{12}B^2(r) + C\theta^{\frac{4}{3}}D^{\frac{4}{3}}(r).
\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$\begin{aligned}
&A^{\theta r} + \theta^{-9} D^{\frac{4}{3}}(\theta r) + B(\theta r) + \widetilde{B}(\theta r) \\
&\quad \leq C[\theta^2 A(r) + \theta^2[B(r) + \widetilde{B}(r)] + \theta^3 B^2(r) \\
&\qquad + (1 + \theta^{-15} + \theta^{-2})\theta^{-6} A(r)[B(r) + \widetilde{B}(r)] + (\theta + \theta^{\frac{4}{3}})\theta^{-9} D^{\frac{4}{3}}(r)]. \qquad (4.9)
\end{aligned}$$

We can fix $\theta \in (0, \frac{1}{2}]$ such that

$$C(\theta + \theta^{\frac{4}{3}} + \theta^2) \leq \frac{1}{4}.$$

According to the smallness assumption, we deduce that there exists an $r' > 0$ such that, for any $0 < r \leq r'$, we have

$$B(r) + \widetilde{B}(r) \leq 2\varepsilon_0.$$

Then, we take ε_0 sufficiently small such that

$$C(1 + \theta^{-10} + \theta^{-2})\theta^{-6}\varepsilon_0 \leq \frac{1}{4}.$$

Letting $G(r) = A(r) + \theta^{-9} D^{\frac{4}{3}}(r) + B(r) + \widetilde{B}(r)$, then, for any $0 < r \leq r'$, from (4.9) we know that

$$G(\theta r) \leq \frac{1}{2} G(r) + \frac{\varepsilon_0}{2}.$$

Iterating this inequality yields

$$A(\theta^k r') + \theta^{-9} D^{\frac{4}{3}}(\theta^k r') + B(\theta^k r') + \widetilde{B}(\theta^k r') = G(\theta^k r') \leq \frac{1}{2} G(r') + \varepsilon_0.$$

Then, there exists some $r_1 > 0$ such that

$$A(r_1) + B(r_1) + \widetilde{B}(r_1) + \theta^{-9} D^{\frac{4}{3}}(r_1) \leq 4\varepsilon_0.$$

Using Lemma 3.2 again, we can bound $C(r_1) + \widetilde{C}(r_1)$ with $A(r_1) + B(r_1) + \widetilde{B}(r_1)$. Then, we can impose additional condition on ε_0 to ensure (4.1). Hence, we complete the proof of Proposition 4.1. ∎

Proof of Theorem 1.1. Without loss of generality, we may suppose the S is bounded. Let V be a parabolic neighborhood of S and fix $\delta > 0$. According to Proposition 4.1, for each $(x, t) \in S$, we can choose $Q_r(x, t) \subset V$ with $r < \delta$ such that

$$\frac{1}{r^2} \int_{Q_r(z_0)} (|\nabla u|^2 + |\nabla B|^2) \mathrm{d}x\mathrm{d}t + \mathrm{d}\lambda > \varepsilon_0.$$

Since S is bounded, we can use Vitali covering lemma to obtain a family of finite disjoint parabolic cylinders $\{Q_{r_i}(x_i, t_i)\}_{i \in \Lambda}$ such that

$$S \subset \bigcup_{i \in \Lambda} Q_{5r_i(x_i, t_i)}.$$

Then,

$$\sum_{i\in\Lambda} r_i^2 \leq \frac{1}{\varepsilon_0}\sum_{i\in\Lambda}\int_{Q_{r_i}(x_i,t_i)} (|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t + \mathrm{d}\lambda$$
$$\leq \frac{1}{\varepsilon_0}\int_V (|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t + \mathrm{d}\lambda.$$

Since δ is arbitrary, we conclude that S has Lebesgue measure zero and

$$\mathcal{H}^2(S) \leq \frac{5}{\varepsilon_0}\int_V (|\nabla u|^2 + |\nabla B|^2)\mathrm{d}x\mathrm{d}t + \mathrm{d}\lambda$$

for every neighborhood V of S. From the fact that $|\nabla u|^2$ and $|\nabla B|^2$ are integrable function, hence, we finish the proof of Theorem 1.1. ∎

Conflicts of interest. The author declares that there are no conflicts of interest related to this work.

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Received 14 October 2023; revised 13 March 2025.

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