

Distortion for multifactor bimodules and representations of multifusion categories

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Abstract. We call a von Neumann algebra with finite-dimensional center a multifactor. We introduce an invariant of bimodules over II_1 multifactors that we call modular distortion, and use it to formulate two classification results.

We first classify finite depth finite index connected hyperfinite II_1 multifactor inclusions $A \subset B$ in terms of the standard invariant (a unitary planar algebra), together with the restriction to A of the unique Markov trace on B . The latter determines the modular distortion of the associated bimodule. Three crucial ingredients are Popa’s uniqueness theorem for such inclusions which are also homogeneous, for which the standard invariant is a complete invariant, a generalized version of the Ocneanu Compactness Theorem, and the notion of Morita equivalence for inclusions.

Second, we classify fully faithful representations of unitary multifusion categories into bimodules over hyperfinite II_1 multifactors in terms of the modular distortion. Every possible distortion arises from a representation, and we characterize the proper subset of distortions that arise from connected II_1 multifactor inclusions.

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1. Introduction

By a deep theorem of Popa [54], a finite depth finite index hyperfinite II_1 subfactor $A \subset B$ is completely determined by its standard invariant. This standard invariant has many equivalent axiomatizations, including Ocneanu’s paragroups/bi-unitary connections [14, 48], Popa’s canonical commuting square [54], Popa’s λ -lattices [56], and Jones’ planar algebras [40]. Popa’s classification theorem can be bootstrapped to show that every unitary fusion category admits an essentially unique fully faithful unitary tensor functor into $\text{Bim}(R)$, where R is the hyperfinite II_1 factor [26, §3.2]; see [31] for the analogous statement for embedding into endomorphisms of the hyperfinite III_1 factor based on [57].

In this article, we extend these results to *multifactor* inclusions and unitary *multifusion* categories. A unitary multifusion category is a semisimple rigid C^* tensor category with finitely many isomorphism classes of simple objects (unlike fusion categories the unit need not be simple). A multifactor is a von Neumann algebra with finite-dimensional center. A (unital) inclusion of finite multifactors $A \subset B$ is called *connected* if $Z(A) \cap Z(B) = C$, *finite index* if the standard bimodule ${}_A L^2 B_B$ is dualizable, and *finite depth* if ${}_A L^2 B_B$ generates a unitary multifusion subcategory of $\text{Bim}(A \oplus B)$.

Our first main theorem gives a complete classification of finite depth finite index connected hyperfinite II_1 multifactor inclusions. In [54], Popa shows that a finite depth finite index hyperfinite II_1 subfactor is completely classified by its standard invariant. This result is no longer true for connected II_1 multifactors; an additional datum is needed. The standard invariant is only a complete invariant up to *Morita equivalence* of inclusions, which we define after Theorem A below.

We now need to recall some terminology before we state the first main theorem. By [20, Thm. 3.7.3], a finite index connected inclusion of finite multifactors $A \subset B$ has a unique *Markov trace* $\text{tr}_B^{\text{Markov}}$, which is characterized by a certain Frobenius–Perron condition (see (2.9) below), and by [20, Thm. 3.6.4 (i)], B_A is finitely generated and projective as a right A -module, so there is a finite Pimsner–Popa basis for B over A [51]. The inclusion $A \subset (B, \text{tr}_B^{\text{Markov}})$ is said to be *strongly Markov* following [41], and the standard invariant $\mathcal{P}_\bullet^{A \subset B}$ is a 2-shaded unitary (i.e., C^* with finite-dimensional box spaces) planar algebra.

Theorem A. *The map which takes $A \subset B$ to the pair $(\mathcal{P}_\bullet^{A \subset B}, \text{tr}_B^{\text{Markov}}|_{Z(A)})$ descends to a bijection*

$$\frac{\left\{ \begin{array}{l} \text{Finite depth finite index con-} \\ \text{nected hyperfinite } \text{II}_1 \text{ multi-} \\ \text{factor inclusions } A \subset B \end{array} \right\}}{\varphi: B_1 \xrightarrow{\sim} B_2 \text{ taking } A_1 \text{ onto } A_2} \cong \frac{\left\{ \begin{array}{l} \text{Pairs } (\mathcal{P}_\bullet, \tau) \text{ with } \mathcal{P}_\bullet \text{ a finite depth} \\ \text{indecomposable unitary 2-shaded planar} \\ \text{algebra and } \tau \text{ a faithful state on } \mathcal{P}_{0,+} \end{array} \right\}}{\varphi_\bullet: \mathcal{P}_\bullet^1 \xrightarrow{\sim} \mathcal{P}_\bullet^2 \text{ such that } \tau^2 \circ \varphi_{0,+} = \tau^1}.$$

The map which takes $A \subset B$ to $\mathcal{P}_\bullet^{A \subset B}$ descends to a bijection

$$\frac{\left\{ \begin{array}{l} \text{Finite depth finite index connected hyperfi-} \\ \text{nite } \text{II}_1 \text{ multifactor inclusions } A \subset B \end{array} \right\}}{\text{Morita equivalence}} \cong \frac{\left\{ \begin{array}{l} \text{Finite depth indecomposable uni-} \\ \text{tary 2-shaded planar algebras } \mathcal{P}_\bullet \end{array} \right\}}{\text{Planar } * \text{-algebra isomorphism}}.$$

Importantly, the standard invariant $\mathcal{P}_\bullet^{A \subset B}$ is only a complete invariant up to *Morita equivalence* and not isomorphism of inclusions. Here, two inclusions $A_1 \subset B_1$ and $A_2 \subset B_2$ are said to be *Morita equivalent* if there is an invertible bimodule $A_2 Y_{A_1}$ and a $*$ -isomorphism

$$\psi: B_2 \rightarrow (B_1^{\text{op}})' \cap B(Y \boxtimes_{A_1} L^2 B_1)$$

that restricts to the identity on A_2 :

$$\begin{array}{ccc} B_2 & \xrightarrow[\cong]{\psi} & (B_1^{\text{op}})' \cap B(Y \boxtimes_{A_1} L^2 B_1). \\ \uparrow & & \uparrow \\ A_2 & \xrightarrow{\text{id}_{A_2}} & A_2 \end{array}$$

We prove Theorem A in two parts: the first part in Theorem 5.13, preceded by the second part in Theorem 5.12. The other main tool we use, besides Morita equivalence, is Popa’s uniqueness theorem for finite depth finite index *homogeneous* connected hyperfinite II_1 multifactor inclusions, i.e., those such inclusions which admit a generating Jones tunnel. See Section 4.3 for other characterizations of homogeneity. For completeness and convenience of the reader, we provide a complete proof of Popa’s theorem in the multifactor setting in Appendix B.

When the inclusion $A \subset B$ is not homogeneous, we may no longer have any Jones downward basic construction, let alone a generating tunnel. An easy example is $A_0 \otimes R \subset B_0 \otimes R$ where $A_0 \subset B_0$ is any finite-dimensional inclusion with Bratteli diagram A_4 [57, Ex. 1.2.8]. We treat this example in detail as Example 4.19 below. Nevertheless, we prove in Theorem 5.6 that any finite depth II_1 inclusion is Morita equivalent to a homogeneous one.

To measure/quantify how an inclusion might fail to be homogeneous, and give further characterizations of homogeneity, we introduce André Henriques’ notion of the *modular distortion* for II_1 multifactor bimodules. Suppose that A and B are hyperfinite II_1 multifactors with minimal central projections p_1, \dots, p_a and q_1, \dots, q_b , respectively, and define $A_i := p_i A$ and $B_j := q_j B$. Given a connected dualizable A - B bimodule X , we write $X_{ij} := p_i X q_j$. The modular distortion of X is the partially defined function

$$\delta = \delta(X): \{1, \dots, a\} \times \{1, \dots, b\} \rightarrow \mathbb{R}_{>0}$$

given by

$$\delta_{ij} := \left(\frac{\text{vNdim}_L(A_i X_{ij})}{\text{vNdim}_R(X_{ij} B_j)} \right)^{1/2} \quad \text{whenever } X_{ij} \neq (0).$$

We analyze the behavior of the distortion under the Jones basic construction and the Jones downward basic construction. This expands on [57, Cor. 1.2.10] to give a quantitative answer to when one can perform a downward basic construction.

Remark. The notion of modular distortion for a II_1 multifactor bimodule is closely related to Izumi’s notion of Connes–Takesaki module for an endomorphism of a properly infinite factor; we explain the connection in Remark 3.10 below.

In Lemma 3.18, we characterize those situations when δ extends (uniquely) to an everywhere defined function $\{1, \dots, a\} \times \{1, \dots, b\} \rightarrow \mathbb{R}_{>0}$ satisfying

$$\delta_{ij}\delta_{i'j'} = \delta_{ij'}\delta_{i'j} \quad \forall i, i' \leq a, j, j' \leq b. \tag{1.1}$$

In those cases, there exist $\eta_i, \xi_j \in \mathbb{R}_{>0}$ (well defined up to global rescaling) such that $\delta_{ij} = \xi_j/\eta_i$. When in addition the *statistical dimension* D_{ij} of each X_{ij} is equal to its *Jones dimension* (the square root of the Jones index), we call the bimodule ${}_A X_B$ *extremal*; these two conditions together are the multifactor analog of the notion of extremality for II_1 factor bimodules from [12, 49]. In Corollary 3.22, we show that any *finite depth* bimodule (i.e., which generates a unitary multifusion category under taking finite direct sums, Connes fusion tensor products, subobjects, and contragredients) is automatically extremal. In Section 3.3, when $A \subset B$ is a finite index connected inclusion of finite multifactors, we connect our definition of extremality for ${}_A L^2 B_B$ with the minimality of the Markov trace-preserving conditional expectation $E : B \rightarrow A$.

Theorem A tells us that not every distortion function satisfying (1.1) can arise from a finite depth inclusion. Indeed, by Corollary 5.9, if $A \subset B$ is a finite depth finite index connected inclusion of finite multifactors, then the distortion is determined by $\text{tr}_A = \text{tr}_B^{\text{Markov}}|_A$, the restriction to A of the unique Markov trace on B :

$$\delta_{ij} = \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right) \sum_{h=1}^a \left(\frac{\text{tr}_A(p_h)}{\alpha_h} \right) D_{hj}. \tag{1.2}$$

Here, $\alpha_1, \dots, \alpha_a$ are the first a coordinates of a Frobenius–Perron eigenvector of $\begin{pmatrix} 0 & D \\ D^T & 0 \end{pmatrix}$. The space of possible distortion functions satisfying (1.1) is parameterized by $\mathbb{R}_{>0}^{a+b-1}$, whereas the space of distortion functions realizable by an inclusion $A \subset B$ is parameterized by the space of faithful traces on A , which is homeomorphic to $\mathbb{R}_{>0}^{a-1}$.

As an application of the modular distortion, we give a complete classification of representations of unitary multifusion categories into bimodules over hyperfinite II_1 multifactors. If \mathcal{C} is an indecomposable unitary multifusion category with $\dim(\text{End}_{\mathcal{C}}(1_{\mathcal{C}})) = n$, then we call \mathcal{C} an $n \times n$ unitary multifusion category. A *representation* of an $n \times n$ unitary multifusion category consists of a hyperfinite II_1 multifactor $A = \bigoplus_{i=1}^n A_i$ (where each A_i is a hyperfinite II_1 factor), together with a fully faithful unitary tensor functor $\alpha : \mathcal{C} \rightarrow \text{Bim}(A)$. The *modular distortion of α* (an idea due to André Henriques) is the matrix $\delta^\alpha \in M_n(\mathbb{R}_{>0})$ given by $\delta_{ij}^\alpha := \delta_{(A_i \alpha(c)_{A_j})}$ for $c \in \mathcal{C}_{ij}$ – this is independent of the choice of object $c \in \mathcal{C}_{ij}$ (see Definition 6.3 below). The modular distortion of α is a groupoid homomorphism $\delta^\alpha : \mathcal{G}_n \rightarrow \mathbb{R}_{>0}$ from the groupoid \mathcal{G}_n with n objects and a unique isomorphism between any two objects to the group(oid) $\mathbb{R}_{>0}$, namely $\delta_{ij}^\alpha \delta_{jk}^\alpha = \delta_{ik}^\alpha$ for all $i, j, k \leq n$.

An *isomorphism* between two representations $\alpha : \mathcal{C} \rightarrow \text{Bim}(A)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(B)$ consists of an invertible bimodule ${}_B \Phi_A$ together with a family of unitary natural isomorphisms

$$\phi = \{ {}_B \Phi \boxtimes_A \alpha(c)_A \rightarrow {}_B \beta(c) \boxtimes_B \Phi_A \}$$

satisfying a certain coherence axiom (6.1). We say that an isomorphism (Φ, ϕ) is *induced* by an algebra isomorphism

$$\varphi: A \rightarrow B \quad \text{if } \Phi = {}_B L^2 B_{\varphi(A)}$$

where the right A -action is transported by φ (and ϕ is arbitrary).

Our second main theorem is as follows.

Theorem B. *Let \mathcal{C} be an $n \times n$ unitary multifusion category. Then the map $\alpha \mapsto \delta^\alpha$ which assigns to a representation α its modular distortion descends to a bijection*

$$\frac{\{\text{Representations } \alpha: \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})\}}{\text{Iso } (\Phi, \phi) \text{ induced by } \varphi \in \text{Aut}(R^{\oplus n})} \cong \{\text{Groupoid homomorphisms } \delta: \mathcal{G}_n \rightarrow \mathbb{R}_{>0}\}.$$

Moreover, there is a unique representation $\mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ up to isomorphism:

$$\frac{\{\text{Representations } \alpha: \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})\}}{\text{Isomorphism } (\Phi, \phi)} \cong \{*\}.$$

We prove the first part of Theorem B in Theorem 6.11, and the second part in Theorem 6.12.

2. Background

2.1. Unitary multitensor categories

We refer the reader to [25, §2.1], [33, §2.1 and §2.2], and [26, §2.1 and §2.2] for a rapid introduction to C^* and W^* tensor categories. More references on $(C^*$ and W^*) tensor categories include [13, 15, 63]. We refer the reader to [17] for a background on C^* and W^* 2-categories. It is well known that 2-categories with exactly one object are monoidal categories [3, Periodic table in §2.1 and §5.6]; a similar statement holds for C^* and W^* 2-categories.

There is a powerful graphical calculus of string diagrams for 2-categories where objects correspond to shaded regions, 1-morphisms correspond to strands, and 2-morphisms correspond to coupons [28, §8.1.2]. As monoidal categories can be viewed as 2-categories with one object, there is only one shading for regions in string diagrams for monoidal categories, where we denote objects by strands and morphisms by coupons [42, 63]. In the graphical calculus, all associator and unitor isomorphisms in our 2-category/monoidal category are left implicit.

Definition 2.1. Let \mathcal{C} be a 2-category and let $a, b \in \mathcal{C}$ be two objects. A 1-morphism $X \in \mathcal{C}(a \rightarrow b)$ is called *dualizable* if there is a *dual 1-morphism* $X^\vee \in \mathcal{C}(b \rightarrow a)$ together with 2-morphisms

$$\text{ev}_X \in \mathcal{C}(X^\vee \otimes X \Rightarrow 1_b), \quad \text{coev}_X \in \mathcal{C}(1_a \Rightarrow X \otimes X^\vee)$$

satisfying the *zig-zag* or *snake equations*:

$$\begin{array}{c}
 \begin{array}{|c|} \hline X \\ \hline \end{array} \begin{array}{|c|} \hline X^\vee \\ \hline \end{array} \begin{array}{|c|} \hline X \\ \hline \end{array} = \begin{array}{|c|} \hline X \\ \hline \end{array} \quad (\text{id}_X \otimes \text{ev}_X) \circ (\text{coev}_X \otimes \text{id}_X) = \text{id}_X, \quad (2.1)
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{|c|} \hline X^\vee \\ \hline \end{array} \begin{array}{|c|} \hline X \\ \hline \end{array} \begin{array}{|c|} \hline X^\vee \\ \hline \end{array} = \begin{array}{|c|} \hline X^\vee \\ \hline \end{array} \quad (\text{ev}_X \otimes \text{id}_{X^\vee}) \circ (\text{id}_{X^\vee} \otimes \text{coev}_X) = \text{id}_{X^\vee}. \quad (2.2)
 \end{array}$$

Here, the shaded regions denote the two objects $\text{○} = a$, $\text{●} = b$, and we represent the evaluation and coevaluation morphisms by a cap and cup, respectively:

$$\begin{array}{|c|} \hline X^\vee \\ \hline \end{array} \begin{array}{|c|} \hline X \\ \hline \end{array} = \text{ev}_X, \quad \begin{array}{|c|} \hline X \\ \hline \end{array} \begin{array}{|c|} \hline X^\vee \\ \hline \end{array} = \text{coev}_X.$$

We also require that the 1-morphism X admits a *predual 1-morphism* X_\vee such that $(X_\vee)^\vee \cong X$.

We call \mathcal{C} *rigid* if every 1-morphism in \mathcal{C} is dualizable. Similarly, we call a monoidal category *rigid* if every object is dualizable.¹

Definition 2.2. A *unitary multitensor category* is a semisimple rigid monoidal C^* category. We call such a category *indecomposable* if it is not the direct sum of two unitary multitensor categories. An $r \times r$ unitary multitensor category is an indecomposable unitary multitensor category such that $\dim(\text{End}_{\mathcal{C}}(1_{\mathcal{C}})) = r$. A *unitary multifusion category* is a unitary multitensor category with finitely many isomorphism classes of simple objects.

In this article, we focus mainly on unitary multifusion categories. The main difference between these and unitary fusion categories is that $1_{\mathcal{C}}$ need not be simple. This leads to a direct sum decomposition of our category as follows.

Notation 2.3. We let $1_{\mathcal{C}} = \bigoplus_{i=1}^r 1_i$ be a decomposition into simples, and write $p_i \in \text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ for the minimal projection onto 1_i .

We write $\mathcal{C}_{ij} := 1_i \otimes \mathcal{C} \otimes 1_j$, so that $\mathcal{C} = \bigoplus \mathcal{C}_{ij}$. We may view \mathcal{C} as a rigid C^*/W^* 2-category with r objects $1_1, \dots, 1_r$, and hom categories $\text{Hom}(1_i \rightarrow 1_j) := \mathcal{C}_{ij}$. In our graphical calculus for \mathcal{C} , we may choose to use different shaded regions to denote various summands of $1_{\mathcal{C}}$.

Suppose \mathcal{C} is a unitary multitensor category. A choice of triple $(c^\vee, \text{ev}_c, \text{coev}_c)$ for every object $c \in \mathcal{C}$ gives rise to a monoidal *dual functor*

$$\vee: \mathcal{C} \rightarrow \mathcal{C}^{\text{mop}}; c \mapsto c^\vee.$$

¹When one views a monoidal category as a 2-category with one object, the objects in the monoidal category are thought of as 1-morphisms in the 2-category, and dualizability is the same notion as in (2.1) and (2.2) above.

Theorem 2.5 ([50, Thm. A]). *Let \mathcal{C} be a unitary multitensor category and let \mathcal{U} be its universal grading groupoid. Then for \vee a unitary dual functor on \mathcal{C} , and $c \in \mathcal{C}$ a non-zero object graded by $g \in \mathcal{U}$, the quantity*

$$\pi(g) := \frac{\dim_{\mathcal{L}}^{\vee}(c)}{\dim_{\mathcal{R}}^{\vee}(c)}$$

is independent of the choice of object c .

The map sending \vee to π establishes a bijection between unitary dual functors on \mathcal{C} up to unitary equivalence and groupoid homomorphisms $\mathcal{U} \rightarrow \mathbb{R}_{>0}$.

If \mathcal{C} is an $r \times r$ unitary multifusion, then (since \mathcal{U} is finite) unitary equivalence classes of unitary dual functors are in canonical bijection with groupoid homomorphisms $\mathcal{G}_r \rightarrow \mathbb{R}_{>0}$.

The inverse map sends $\pi: \mathcal{U} \rightarrow \mathbb{R}_{>0}$ to a unitary dual functor characterized by the so-called ‘ π -balanced’ solutions to the conjugate equations, and the case $\pi = 1$ gives the unique unitary spherical structure on \mathcal{C} . We refer the reader to [50] for more details.

2.1.1. 2-shadings. Of particular importance to this article are *2-shaded* $r \times r$ unitary multitensor categories.

Definition 2.6. A *2-shading* on an $r \times r$ unitary multitensor category \mathcal{C} is an orthogonal decomposition $1_{\mathcal{C}} = 1^+ \oplus 1^-$ of the unit object of \mathcal{C} into two non-zero objects (the objects 1^+ and 1^- are not assumed to be simple).

Let $a := \dim(\text{End}_{\mathcal{C}}(1^+))$ and $b := \dim(\text{End}_{\mathcal{C}}(1^-))$ so that $r = a + b$, let $1^+ = \bigoplus_{i=1}^a 1_i^+$ and $1^- = \bigoplus_{j=1}^b 1_j^-$ be orthogonal decompositions into simples, and let $p_i \in \text{End}_{\mathcal{C}}(1_i^+)$ be the minimal projection onto 1_i^+ and $q_j \in \text{End}_{\mathcal{C}}(1_j^-)$ the minimal projection onto 1_j^- . We denote the objects 1^+ and 1^- by the following two shadings:

$$\text{○} = 1^+, \quad \text{●} = 1^-.$$

An object $X \in \mathcal{C}^{+-} := 1^+ \otimes \mathcal{C} \otimes 1^-$ is said to *generate* \mathcal{C} if \mathcal{C} is Cauchy tensor generated by X and X^\vee (the dual object X^\vee is well defined up to isomorphism), i.e., every simple object of \mathcal{C} is a subobject of an object of the form $Y_1 \otimes \cdots \otimes Y_n$ where each Y_i is either X or X^\vee .² In this setting, we write $X_{ij} := 1_i^+ \otimes X \otimes 1_j^-$ for the homogeneous components of X of degree e_{ij} . In the graphical calculus, we denote X by a strand with the two shaded regions for 1^\pm on either side, and X_{ij} is denoted by tensoring with p_i and q_j on the left and right

$$X = \text{○} \text{---} \text{●}, \quad X_{ij} = \text{○}(p_i) \text{---} \text{●}(q_j).$$

²For this definition of Cauchy tensor generated, we have assumed that \mathcal{C} is *a priori* multitensor, and thus semisimple.

Observe that since X generates \mathcal{C} , and the latter is indecomposable:

- (1) we have inclusions of finite-dimensional von Neumann algebras $\text{End}_{\mathcal{C}}(1^+) \hookrightarrow \text{End}(X)$ and $\text{End}_{\mathcal{C}}(1^-) \hookrightarrow \text{End}_{\mathcal{C}}(X)$ given by

$$\begin{array}{c} \textcircled{p} \\ \textcircled{q} \end{array} \mapsto \begin{array}{c} \textcircled{p} \\ \textcircled{q} \end{array} \Big|, \quad \begin{array}{c} \textcircled{q} \\ \textcircled{p} \end{array} \mapsto \begin{array}{c} \textcircled{q} \\ \textcircled{p} \end{array}.$$

- (2) X is *connected*, that is, the intersection of the images of $\text{End}_{\mathcal{C}}(1^+)$ and $\text{End}_{\mathcal{C}}(1^-)$ in $\text{End}_{\mathcal{C}}(X)$ is Cid_X . Equivalently, the bipartite graph with a even vertices b odd vertices and an edge from i to j whenever $X_{ij} \neq 0$ is connected.

Let $\text{Dim}(X) = \begin{pmatrix} 0 & D_X \\ 0 & 0 \end{pmatrix}$ be the dimension matrix for X with respect to the unique unitary spherical structure on \mathcal{C} . It is block upper triangular, and its upper right corner D_X an $a \times b$ matrix. Let $d_X := \|\text{Dim}(X)\| = \|D_X\|$ be the Frobenius–Perron eigenvalue. By Frobenius–Perron theory applied to the matrix $\begin{pmatrix} 0 & D_X \\ D_X^T & 0 \end{pmatrix}$, there are unique vectors $\alpha \in \mathbb{R}_{>0}^a$ and $\beta \in \mathbb{R}_{>0}^b$ with strictly positive entries satisfying

$$D_X \beta = d_X \alpha, \quad D_X^T \alpha = d_X \beta, \quad \text{and} \quad \|\alpha\|_2 = 1 = \|\beta\|_2.$$

The object X induces a *standard unitary dual functor* on \mathcal{C} , as follows.

Definition 2.7 ([17, Def. 8.29], [50, §4.2]). Let \mathcal{C} be a 2-shaded multitensor category, and let $D_X = (D_{ij})$ and d_X be as above. The standard unitary dual functor associated to X is determined by the following identities:

$$\begin{array}{c} \textcircled{p_i} \\ \textcircled{q_j} \end{array} = \frac{D_{ij} \beta_j}{\alpha_i} \textcircled{p_i}, \quad \begin{array}{c} \textcircled{p_i} \\ \textcircled{q_j} \end{array} = \frac{D_{ij} \alpha_i}{\beta_j} \textcircled{q_j}. \quad (2.3)$$

It satisfies the property that the two loop parameters for X are both equal to the scalar d_X :

$$\begin{array}{c} \textcircled{} \\ \textcircled{} \end{array} := \text{coev}_X^* \circ \text{coev}_X = d_X \text{id}_{1^+}, \quad \begin{array}{c} \textcircled{} \\ \textcircled{} \end{array} := \text{ev}_X \circ \text{ev}_X^* = d_X \text{id}_{1^-}.$$

When \mathcal{C} is in addition multifusion, we will see in Theorem 2.22 below that this property uniquely characterizes the standard unitary dual functor.

Taking the ratio of the scalars for the left and right dimensions in (2.3) above gives the formula for a groupoid homomorphism $\pi: \mathcal{G}_{a+b} \rightarrow \mathbb{R}_{>0}$ describing the standard unitary dual functor associated to X :

$$\pi(e_{ij}) = \frac{D_{ij} \alpha_i / \beta_j}{D_{ij} \beta_j / \alpha_i} = \frac{\alpha_i^2}{\beta_j^2} \quad \forall 1 \leq i \leq a \text{ and } \forall 1 \leq j \leq b. \quad (2.4)$$

This last formula appears in [50, Lem. 4.5 and (27)]. By universality of the grading groupoid \mathcal{U} , we get a groupoid homomorphism $\mathcal{U} \rightarrow \mathcal{G}_{a+b} \rightarrow \mathbb{R}_{>0}$.

2.2. Bimodules over multifactors

In this section, A, B, C, M, N denote von Neumann algebras, and H, K, L denote separable Hilbert spaces. All von Neumann algebras are assumed to have separable preduals.

Definition 2.8. An A - B bimodule is a Hilbert space H together with normal unital $*$ -algebra homomorphisms $\lambda: A \rightarrow B(H)$ and $\rho: B^{\text{op}} \rightarrow B(H)$ such that $[\lambda(a), \rho(b)] = 0$ for all $a \in A$ and $b \in B$. For notational simplicity, we suppress λ, ρ , and write $a\xi b := \lambda(a)\rho(b)\xi$.

The collection vNAlg of von Neumann algebras with separable preduals, bimodules, and intertwiners forms a W^* -2-category, where composition of 1-morphisms is the *Connes fusion relative tensor product* [59], [11, Appendix B.8], [61]. The tensor product $H \boxtimes_B K$ of ${}_A H_B$ and ${}_B K_C$ is the completion of the complex vector space

$$\text{Hom}_{-B}(L^2 B \rightarrow H) \otimes_B L^2 B \otimes_B \text{Hom}_{B-}(L^2 B \rightarrow K),$$

under the sesquilinear form given by (the linear extension of)

$$\langle f_1 \otimes \xi_1 \otimes g_1, f_2 \otimes \xi_2 \otimes g_2 \rangle := \langle (f_2^* \circ f_1)\xi_1(g_2^* \circ g_1), \xi_2 \rangle_{L^2 B}.$$

Here, $f_2^* \circ f_1 \in \text{End}_{-B}(L^2 B) \cong B$ where the identification is via the left action map, and $g_2^* \circ g_1 \in \text{End}_{B-}(L^2 B) \cong B$ where the identification is via the right action map.

Here above, $L^2 B$ is the canonical Haagerup L^2 space [22] which can be defined state-independently [5]. For every faithful normal state φ on B , there is a canonical B - B bimodule unitary isomorphism $L^2(B, \varphi) \cong L^2 B$. We write $\sqrt{\varphi} \in L^2 B$ for the image of the canonical cyclic vector $\Omega_\varphi \in L^2(B, \varphi)$.

Given an A - B bimodule ${}_A H_B$, the *conjugate* bimodule ${}_B \bar{H}_A$ is the complex conjugate Hilbert space \bar{H} of H (whose elements we denote by $\bar{\xi}$ for $\xi \in H$), equipped with the B - A bimodule structure given by $b\bar{\xi}a := \overline{a^* \xi b^*}$. Given an intertwiner $f \in \text{Hom}_{A-B}(H \rightarrow K)$, we define $\bar{f} \in \text{Hom}_{B-A}(\bar{H} \rightarrow \bar{K})$ by $\bar{f}(\bar{\xi}) := \overline{f(\xi)}$. It is straightforward to verify that $\overline{\bar{f}^*} = f^*$ and $\overline{\bar{f} \circ g} = \bar{f} \circ \bar{g}$. Hence, vNAlg is a *bi-involutive* W^* -2-category in the sense of [25, Def. 2.3] and [26, §4.2 and §5.2].

Given a fixed von Neumann algebra A , we denote by $\text{Bim}(A)$ the full bi-involutive W^* -2-subcategory of vNAlg whose only object is A . In other words, $\text{Bim}(A)$ is a bi-involutive W^* tensor category.

Recall that a *factor* is a von Neumann with trivial center. Factors are the fundamental building blocks of the theory of von Neumann algebras. Indeed any von Neumann can be decomposed as a direct integral of factors. In particular, a von Neumann algebra A with finite-dimensional center $Z(A) = \mathbb{C}^k$ decomposes as a finite direct sum of factors $A = \bigoplus_{i=1}^k p_i A p_i$, where p_1, \dots, p_k are the minimal central projections (and $p_i A p_i = p_i A$).

Definition 2.9. A *multifactor* is a finite direct sum of factors. A multifactor is called *finite* if every summand is a finite von Neumann algebra. A multifactor is called a II_1 multifactor if every summand is a II_1 factor.

Example 2.10. Every finite-dimensional von Neumann algebra is a multifactor.

Example 2.11. Consider the group von Neumann algebra LG of an infinite discrete group G , which is infinite-dimensional with faithful normal tracial state $\text{tr}(x) = \langle x\delta_e, \delta_e \rangle$ where $\delta_e \in \ell^2 G$ is the indicator function at the identity. Writing $x = \sum_{g \in G} x_g u_g$ for the Fourier expansion of $x \in LG$ it is easily seen that $x \in Z(LG)$ if and only if x_g depends only on the conjugacy class of g . Since the sequence $(x_g) \in \ell^2 G$, the dimension of the center is the number of finite conjugacy classes. Hence $Z(LG)$ is finite-dimensional if and only if there are finitely many finite conjugacy classes. (When only the trivial conjugacy class is finite, the group is called *ICC* and the group von Neumann algebra is a II_1 factor.) In this case, let $\{z_1, \dots, z_n\}$ denote the minimal central projections of LG . Then each $z_i LG$ is a finite factor. Any type I_n summand of LG would also be a finite-dimensional invariant subspace of $\ell^2 G$ for the left regular action, which can only exist when G is finite. Indeed, the left regular representation $\lambda_G: G \rightarrow B(\ell^2(G))$ of an infinite discrete group G satisfies both $\|\lambda_G(g)v\| = \|v\|$ and $\lim_{g \rightarrow \infty} \langle \lambda_G(g)v, w \rangle = 0$ for all $v, w \in \ell^2(G)$. However, both equations cannot hold for a finite-dimensional representation, since weak and strong convergence coincide on finite-dimensional Hilbert spaces. Explicit examples of such groups include $\text{SL}(2, \mathbb{Z})$ and $G_1 \times G_2$ where G_1 is ICC and G_2 is finite.

Notation 2.12. For the remainder of this article, unless stated otherwise, M and N are finite factors, ${}_M H_N$ is an M - N bimodule, A and B are finite multifactors, and ${}_A X_B$ is an A - B bimodule. We let $\{p_i\}_{1 \leq i \leq a}$ denote the minimal central projections of A , and $\{q_j\}_{1 \leq j \leq b}$ the minimal central projections of B . We define $A_i := p_i A$ and $B_j := q_j B$, and we let $X_{ij} := p_i X q_j$, which is an A_i - B_j bimodule.

From the M - N bimodule ${}_M H_N$, we can define subfactors $M \subset (N^{\text{op}})'$ and $N^{\text{op}} \subset M'$. Let $\text{vNdim}_L(H)$ and $\text{vNdim}_R(H)$ denote the left and right von Neumann dimensions of H .

Definition 2.13. The *Jones dimension* of ${}_M H_N$ is

$$\Delta(H) := \sqrt{\text{vNdim}_L(H) \text{vNdim}_R(H)}.$$

The *Jones dimension matrix* of ${}_A X_B$ is the $a \times b$ matrix $\Delta = \Delta(X)$ whose (i, j) -th entry is given by $\Delta_{ij} := \Delta(X_{ij})$. Note that we always have $\Delta(\bar{H}) = \Delta(H)$ and $\Delta(\bar{X}) = \Delta(X)^T$.

Remark 2.14. When both $\text{vNdim}_L(MH) < \infty$ and $\text{vNdim}_R(HN) < \infty$, recall from [37, (2.1.2) and Prop. 2.1.7] that

$$\begin{aligned} [(N^{\text{op}})' : M] &= \frac{\text{vNdim}_L(MH)}{\text{vNdim}_L((N^{\text{op}})'H)} = \text{vNdim}_L(MH) \text{vNdim}_R(HN) \\ &= \frac{\text{vNdim}_L(N^{\text{op}}H)}{\text{vNdim}_L(M'H)} = [M' : N^{\text{op}}]. \end{aligned}$$

Taking square roots, $\Delta(H) = [(N^{\text{op}})' : M]^{1/2} = [M' : N^{\text{op}}]^{1/2}$.

Definition 2.15. The *statistical dimension* of ${}_M H_N$ is

$$D(H) := \min \{ \sqrt{\text{Ind}(E)} \mid E : (N^{\text{op}})' \rightarrow M \text{ is a conditional expectation} \}$$

where $\text{Ind}(E) \geq 1$ is the Kosaki index of the faithful normal expectation E [44, Def. 2.1]. For subfactors, this number is finite for every E as long as there exists one expectation with finite index. We define the *statistical dimension matrix* of ${}_A X_B$ to be the $a \times b$ matrix $D(X)$ with (i, j) -th entry $D_{ij} := D(X_{ij})$. Observe that $D(\bar{H}) = D(H)$ and $D(\bar{X}) = D(X)^T$ [5, Cor. 5.17], [17, Cor. 6.8].

Note that we always have $D(H) \leq \Delta(H)$, as $\Delta(H)^2$ is the index of the trace-preserving conditional expectation. Moreover, if H is *simple* (i.e., if $\text{End}_{N-M}(H) = \mathbb{C}$), then $D(H) = \Delta(H)$ because an irreducible subfactor admits at most one normal faithful conditional expectation.

Lemma 2.16. For an A - B bimodule ${}_A X_B$, the following are equivalent:

- (1) X is dualizable.
- (2) X_{ij} is dualizable for all i, j .
- (3) Every entry of $D(X)$ is finite.
- (4) Every entry of $\Delta(X)$ is finite.
- (5) X is finitely generated as both a left A -module and a right B -module.

Proof. (1) \Leftrightarrow (2): Immediate from the fact that X is the orthogonal direct sum of the X_{ij} .

(2) \Leftrightarrow (3): By [5, Props. 7.3 and 7.5], X_{ij} is dualizable if and only if there exists a conditional expectation $E : (B_j^{\text{op}})' \rightarrow A_i$ whose index is finite.

(3) \Leftrightarrow (4): By [4, Cor. 3.19], if either $D(X)$ or $\Delta(X)$ is finite, then the relative commutant $A'_i \cap (B_j^{\text{op}})'$ is finite-dimensional. It then follows by [23, Thm. 6.6] that the existence of a conditional expectation $(B_j^{\text{op}})' \rightarrow A_i$ of finite index implies that all conditional expectations have finite index.

(4) \Leftrightarrow (5): This follows by [1, Prop. 9.3.2], as the conditions (i)–(iii) of that proposition are equivalent for multifactors. ■

Remark 2.17. Given a dualizable multifactor bimodule ${}_A X_B$ and any unitary dual functor \vee on the unitary multitensor category $\text{Bim}(A \oplus B)$, there is a canonical unitary isomorphism

$${}_B (X^\vee)_A \cong {}_B \bar{X}_A$$

by [5, Cor. 6.12] and [50, Cor. B].

2.3. Connected multifactor inclusions

We now consider the special case of $X = {}_A L^2 B_B$ where $A \subset B$ is a unital inclusion of finite multifactors. We call $A \subset B$ *finite index* if ${}_A L^2 B_B$ is dualizable (cf. Lemma 2.16), and *connected* if ${}_A L^2 B_B$ is connected as an object $\text{Bim}(A \oplus B)$ equivalently, if $Z(A) \cap Z(B) = \mathbb{C}1$.

Let $A \subset B$ be a finite index connected inclusion of finite multifactors. Let tr_B be a faithful normal trace on B , and let $E_A : B \rightarrow A$ be the unique trace-preserving conditional expectation. By [20, Prop. 3.5.2 (ii)], $\text{tr}_A := \text{tr}_B|_A$ is characterized by the formula

$$\text{tr}_A(p_i) = \sum_{j=1}^b T_{ij} \text{tr}_B(q_j) \quad \forall 1 \leq i \leq a \tag{2.5}$$

where $T \in M_{a \times b}(\mathbb{R}_{>0})$ is the *trace matrix*

$$T_{ij} := \text{tr}_{B_j}(p_i q_j) = \text{vNdim}_R((p_i(L^2 B)q_j)_{B_j}) \tag{2.6}$$

(the second equality in (2.6) follows by [20, Proof of Lem. 3.6.7 and Prop. 3.2.5 (h)]). We call $(\text{tr}_A(p_i))_{i=1}^a$ and $(\text{tr}_B(q_j))_{j=1}^b$ the *trace vectors*.

The *Jones projection* $e_A \in B(L^2(B, \text{tr}_B))$ is the orthogonal projection onto $L^2(A, \text{tr}_A)$. It satisfies $e_A b \Omega = E_A(b) \Omega$, where $\Omega \in L^2(A, \text{tr}_A) \subset L^2(B, \text{tr}_B)$ is the image of $1 \in A \subset B$. The *Jones basic construction* [37, §3] is the von Neumann algebra generated by B and by the Jones projection:

$$\langle B, A \rangle := \langle B, e_A \rangle = JA'J \subset B(L^2(B, \text{tr}_B)).$$

By setting $A_0 := A$, $A_1 := B$, and $e_1 := e_A$, and inductively $A_n := \langle A_{n-1}, e_{n-1} \rangle = JA'_{n-2}J \subset B(L^2 A_{n-1})$, we get the Jones tower

$$A_0 = A \subset B = A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \subset \dots \tag{2.7}$$

Definition 2.18. Let $A \subset B$ be a finite index connected inclusion of finite multifactors. A faithful normal trace tr_B on B is called *Markov* if there exists a number $d > 0$ and an extension $\text{tr}_{\langle B, A \rangle}$ of tr_B to $\langle B, A \rangle$ that satisfies

$$\text{tr}_{\langle B, A \rangle}(x e_A) = d^{-2} \text{tr}_B(x)$$

for all $x \in B$. We call an inclusion $A \subset (B, \text{tr}_B)$ equipped with a Markov trace a *Markov inclusion*.

By [20, §2.7 and §3.7], the Markov trace exists and is unique. The number d^2 is called the *Markov index* of the inclusion. It is equal to the spectral radius of $\tilde{T}T$, where $\tilde{T} \in M_{b \times a}(\mathbb{R}_{\geq 0})$ is given by

$$\tilde{T}_{ji} := \left\{ \begin{array}{ll} \Delta_{ij}^2 / T_{ij} & \text{if } p_i q_j \neq 0 \\ 0 & \text{if } p_i q_j = 0 \end{array} \right\} = \text{vNdim}_L(A_i(p_i L^2 B q_j)). \tag{2.8}$$

By [20, Proof of Thm. 3.7.3], the Markov trace is completely determined by the equation

$$d^2 \text{tr}_B(q_j) = \sum_{i=1}^a \tilde{T}_{ji} \text{tr}_A(p_i). \tag{2.9}$$

By definition, the minimal central projections of $\langle B, A \rangle$ are the $Jp_i J$ for $i = 1, \dots, a$. By [20, (3.7.3.1)], the trace vector $(\text{tr}_{\langle B, A \rangle}(Jp_i J))_{i=1}^a$ is given by

$$\text{tr}_{\langle B, A \rangle}(Jp_i J) = d^{-2} \text{tr}_A(p_i) \sum_{\substack{1 \leq k \leq b \\ p_i q_k \neq 0}} \frac{\Delta_{ik}^2}{T_{ik}} \tag{2.10}$$

and by [20, Prop. 3.6.8], the trace matrix $T^{B \subset \langle B, A \rangle} \in M_{b \times a}(\mathbb{R}_{\geq 0})$ for the inclusion $B \subset \langle B, A \rangle$ is given by

$$T_{ji}^{B \subset \langle B, A \rangle} = \frac{\tilde{T}_{ji}}{\sum_{\substack{1 \leq k \leq b \\ p_i q_k \neq 0}} \frac{\Delta_{ik}^2}{T_{ik}}}. \tag{2.11}$$

From here on, tr_B is the unique Markov trace on B . By [20, Thm. 3.6.4(i)] (see also [51]), there is a finite *Pimsner–Popa basis* $\{b\}$ for B over A satisfying

$$\sum_b bE_A(b^*x) = x \quad \forall x \in B \quad \text{and} \quad \sum_b bb^* = d^2 \in [1, \infty).$$

This means E_A is of *index finite type* in the sense of [64], and the Markov index $d^2 = \sum_b bb^*$ is equal to the *Watatani index*. Hence a Markov connected inclusion $A \subset (B, \text{tr}_B)$ of finite multifactors is *strongly Markov* in the sense of [41, Def. 2.8] (Markov with a finite Pimsner–Popa basis).

Iterating Jones’ basic construction, we get a Jones tower

$$A_0 = A \subset B = A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \subset \dots$$

abbreviated $(A_n, \text{tr}_n, e_{n+1})_{n \geq 0}$, where each inclusion is strongly Markov.

Example 2.19 ([37, §3.2]). An inclusion of finite-dimensional von Neumann algebras $A \subset B$ equipped with a faithful trace tr_B is Markov if and only if the trace vectors $\vec{\lambda}_B$ for B and $\vec{\lambda}_A$ for A (whose k -th entry is the trace of a minimal projection in the k -th summand) satisfy $\Lambda^T \Lambda \vec{\lambda}_B = d^2 \vec{\lambda}_B$ and $\Lambda \Lambda^T \vec{\lambda}_A = d^2 \vec{\lambda}_A$ where $d > 0$ such that $d^2 = \|\Lambda \Lambda^T\| = \|\Lambda^T \Lambda\|$. Here, Λ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion $A \subset B$, whose (i, j) -th entry $\Lambda_{i,j}$ is the number of edges between the i -th even vertex/simple summand of A and j -th odd vertex/simple summand of B .

Facts 2.20. We have the following facts concerning the Jones tower.

- (Multistep basic construction [41, Prop. 2.20]) For every $n \in \mathbb{N}$ and $0 \leq k \leq n$, the inclusion $A_{n-k} \subset (A_n, \text{tr}_n) \subset (A_{n+k}, \text{tr}_{n+k}, f_{n-k}^n)$ is *standard* in the sense of [41, Def. 2.14], i.e., isomorphic to a basic construction. Here, f_{n-k}^n is proportional to the unique word in the Jones projections e_n, \dots, e_{n-k+1} of maximal length [52].
- (Conjugation by J [41, Rem. 2.21]) Under the multistep basic construction isomorphisms above, on $L^2(A_n, \text{tr}_n)$, $(J_n A_{n-k} J_n)' = A_{n+k}$.

- (Centralizer algebras as endomorphisms [41, Prop. 2.20, Rem. 2.26]) Using the multi-step basic construction together with the bimodule isomorphisms $L^2 A_n \cong L^2 B^{\boxtimes A^n}$, we get the following isomorphisms between centralizer and endomorphism algebras:

$$\begin{aligned} A'_0 \cap A_{2n} &\cong \text{End}_{A-A}(L^2 B^{\boxtimes A^n}), & A'_1 \cap A_{2n+1} &\cong \text{End}_{B-B}(L^2 B^{\boxtimes A^{n+1}}), \\ A'_0 \cap A_{2n+1} &\cong \text{End}_{A-B}(L^2 B^{\boxtimes A^{n+1}}), & A'_1 \cap A_{2n+2} &\cong \text{End}_{B-A}(L^2 B^{\boxtimes A^{n+1}}). \end{aligned} \tag{2.12}$$

2.4. Planar algebras

Suppose we have a unitary multitensor category \mathcal{C} together with a chosen unitary dual functor \vee , a 2-shading $1_{\mathcal{C}} = 1^+ \oplus 1^-$, and a generator $X \in \mathcal{C}^{+-}$. Using [12, §5] and [50, §4], we can construct a *unitary 2-shaded planar algebra*, i.e., a 2-shaded C^* -*planar algebra* [40, Def. 1.37] $\mathcal{P}_{\bullet} = \mathcal{P}(\mathcal{C}, X, \vee)_{\bullet}$ with finite-dimensional box spaces

$$\begin{aligned} \mathcal{P}_{2n,+} &:= \text{End}_{\mathcal{C}}((X \otimes \bar{X})^{\otimes n}), & \mathcal{P}_{2n,-} &:= \text{End}_{\mathcal{C}}((\bar{X} \otimes X)^{\otimes n}), \\ \mathcal{P}_{2n+1,+} &:= \text{End}_{\mathcal{C}}((X \otimes \bar{X})^{\otimes n} \otimes X), & \mathcal{P}_{2n+1,-} &:= \text{End}_{\mathcal{C}}((\bar{X} \otimes X)^{\otimes n} \otimes \bar{X}). \end{aligned}$$

Given such a unitary 2-shaded planar algebra \mathcal{P}_{\bullet} , one can recover the tuple (\mathcal{C}, \vee, X) by taking its *category of projections* (see [46, §4.1], [8, §2.3], [50, §4], and [26, §3.3]). Moreover, these two constructions are mutually inverse.

Theorem 2.21. *There is an equivalence of categories³*

$$\left\{ \begin{array}{l} \text{unitary 2-shaded planar} \\ \text{algebras } \mathcal{P}_{\bullet} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Triples } (\mathcal{C}, \vee, X) \text{ with } \mathcal{C} \text{ a unitary multitensor} \\ \text{category, } \vee \text{ a unitary dual functor, } 1_{\mathcal{C}} = 1^+ \oplus 1^- \\ \text{a 2-shading, and a generator } X \in \mathcal{C}^{+-} \end{array} \right\}.$$

Using the language of strongly Markov inclusions of finite-dimensional von Neumann algebras, we now prove that the standard unitary dual functor \vee_{standard} with respect to a chosen generator X as in Definition 2.7 in a 2-shaded unitary multifusion category is the unique unitary dual functor whose loop parameters for X are the same scalar. As a corollary, if we add scalar loop parameter to the left-hand side of the equivalence in Theorem 2.21, we may remove the unitary dual functor from the right-hand side as it is uniquely determined by the other data.

Theorem 2.22. *Suppose \mathcal{C} is a 2-shaded indecomposable unitary multifusion category and $X \in \mathcal{C}^{+-}$ generates \mathcal{C} . If \vee is a unitary dual functor on \mathcal{C} whose loop parameters for X are scalars*

$$\begin{array}{c} \text{⊙} \\ \text{⊙} \end{array} = \text{coev}_X^* \circ \text{coev}_X = \lambda \text{id}_{1^+}, \quad \begin{array}{c} \text{⊙} \\ \text{⊙} \end{array} = \text{ev}_X \circ \text{ev}_X^* = \lambda \text{id}_{1^-},$$

then $\lambda = d_X$, and \vee is unitarily equivalent to the standard unitary dual functor with respect to X .

³The collection of triples (\mathcal{C}, \vee, X) forms a 2-category which is equivalent to a 1-category [27, Lem. 3.5].

Moreover, there exists a trace on C given by

$$\text{tr}_C := \frac{1}{\lambda^{2n+2} \cdot \psi(\text{id}_{1+})} \cdot (\psi \circ \text{tr}_R^\vee).$$

Since $\text{tr}_C|_B = \text{tr}_B$ and $F(e_A) = \lambda^{-1}$, we see that tr_C is a (λ, B) -trace on C [37, Def. on p. 8]. By [37, Thm. 3.3.2], we have that $A \subset (B, \text{tr}_B)$ is a connected Markov inclusion. Thus tr_B is the unique Markov trace for the inclusion $A \subset B$. From the existence of the Giorgetti–Longo unitary dual functor and induced Markov trace from [17], we immediately have that $\lambda = d_X$ and

$$\begin{aligned} \alpha_i^2 &= \text{tr}_A(p_i \otimes \text{id}_{(X \otimes \bar{X})^{\otimes n}}) = \frac{\psi(p_i)}{\psi(\text{id}_{1+})} \quad \forall i = 1, \dots, a, \\ \beta_j^2 &= \text{tr}_B(\text{id}_{(X \otimes \bar{X})^{\otimes n} \otimes X} \otimes q_j) = \frac{\psi(q_j)}{\psi(\text{id}_{1+})}, \quad \forall j = 1, \dots, b. \end{aligned}$$

Finally, since \mathcal{C} is indecomposable multifusion, the relevant classifying grading groupoid for unitary dual functors is the matrix unit groupoid \mathcal{G}_{a+b} . We have that for all i, j , the classifying groupoid homomorphism

$$\pi: \mathcal{G}_{a+b} \rightarrow \mathbb{R}_{>0}$$

is given by

$$\pi(e_{ij}) = \frac{\dim_L^\vee(X_{ij})}{\dim_R^\vee(X_{ij})} = \frac{\psi \left(\begin{array}{|c|c|} \hline \text{○ } p_i \text{ ○} & \text{○ } q_j \text{ ○} \\ \hline \end{array} \right) / \psi(q_j)}{\psi \left(\begin{array}{|c|c|} \hline \text{○ } p_i \text{ ○} & \text{○ } q_j \text{ ○} \\ \hline \end{array} \right) / \psi(p_i)} = \frac{\alpha_i^2}{\beta_j^2}$$

which is exactly the formula (2.4). Thus, \vee is exactly the Giorgetti–Longo standard unitary dual functor by Theorem 2.5. ■

Corollary 2.23. *There is an equivalence of categories (see Footnote 3)*

$$\left\{ \begin{array}{l} \text{Indecomposable finite depth uni-} \\ \text{tary 2-shaded planar algebras } \mathcal{P}_\bullet \\ \text{with equal scalar loop moduli} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Pairs } (\mathcal{C}, X) \text{ with } \mathcal{C} \text{ an indecompos-} \\ \text{able unitary multifusion category, } 1_{\mathcal{C}} = \\ 1^+ \oplus 1^- \text{ a 2-shading, and a generator} \\ X \in \mathcal{C}^{+-} \end{array} \right\}.$$

Warning 2.24. While the unitary 2-shaded planar algebra \mathcal{P}_\bullet has scalar loop moduli which are the same for both shadings, \mathcal{P}_\bullet is not necessarily spherical. Indeed, there can be endomorphisms of X which have distinct left and right traces with respect to the standard unitary dual functor. A typical example of this behavior is the planar algebra of a bipartite graph [38].

2.5. Standard invariants

Definition 2.25. Given multifactors A, B and a dualizable bimodule ${}_A X_B$, the *standard invariant* of ${}_A X_B$ is the abstract 2-shaded unitary multitensor category $\mathcal{C}(X) \subset \text{Bim}(A \oplus B)$ generated by X together with the choice of generating object X and the standard unitary dual functor with respect to X , whose loop moduli are both the scalar d_X as in Definition 2.7. Here, we forget the existence of the forgetful fiber functor $\mathcal{C}(X) \rightarrow \text{Bim}(A \oplus B)$. Observe that \mathcal{C} is 2-shaded with $1^+ = {}_A L^2 A_A$ and $1^- = {}_B L^2 B_B$. We say that ${}_A X_B$ has *finite depth* if $\mathcal{C}(X)$ is a unitary multifusion category.

An equivalence of standard invariants $(F, u): \mathcal{C}(X) \rightarrow \mathcal{D}(Y)$ is a unitary tensor equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$ together with a unitary isomorphism $u \in \mathcal{D}(Y \rightarrow F(X))$.

The *standard invariant* of a finite index connected inclusion $A \subset B$ of finite multifactors is the standard invariant $\mathcal{C}({}_A L^2 B_B)$. We say that the inclusion $A \subset B$ has *finite depth* if ${}_A L^2 B_B$ has finite depth, i.e., $\mathcal{C}({}_A L^2 B_B)$ is multifusion.

By Theorem 2.21, the standard invariant $\mathcal{C}(X)$ of ${}_A X_B$ may also be viewed as a unitary 2-shaded planar algebra $\mathcal{P}(X)_\bullet$ with equal scalar loop moduli.

Suppose now $A \subset (B, \text{tr}_B)$ is a finite index connected inclusion of finite multifactors equipped with the unique Markov trace. In addition to the unitary 2-shaded planar algebra $\mathcal{P}({}_A L^2 B_B)$, one can construct another unitary 2-shaded planar algebra $\mathcal{P}_\bullet^{A \subset B}$ using the construction from [41, §3] whose box spaces are the higher centralizer algebras of the Jones tower:

$$\mathcal{P}_{n,+}^{A \subset B} := A'_0 \cap A_n, \quad \mathcal{P}_{n,-}^{A \subset B} := A'_1 \cap A_{n+1}, \quad \forall n \geq 0.$$

In fact, the two unitary planar algebras $\mathcal{P}({}_A L^2 B_B)_\bullet$ and $\mathcal{P}_\bullet^{A \subset B}$ are $*$ -isomorphic by [12, Proof of Thm. 5.4 and Rem. 5.5], which do not rely in any substantial way on factoriality and can be adapted to our situation using (2.12) from Facts 2.20 in place of the results from [6]. Hence we have a commutative diagram

$$\begin{array}{ccc}
 \left\{ \begin{array}{l} \text{Finite index connected mul-} \\ \text{tifactor inclusions } A \subset B \end{array} \right\} & \xrightarrow{[41]} & \left\{ \begin{array}{l} \text{Indecomposable unitary 2-shaded} \\ \text{planar algebras } \mathcal{P}_\bullet \text{ with equal} \\ \text{scalar loop moduli} \end{array} \right\} \\
 \downarrow & \nearrow [12] & \uparrow \cong \\
 \left\{ \begin{array}{l} \text{Dualizable connected multi-} \\ \text{factor bimodules } {}_A X_B \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{l} \text{Triples } (\mathcal{C}, X, \vee) \text{ with } \mathcal{C} \text{ indecom-} \\ \text{posable such that } X \in \mathcal{C}^{+-} \text{ has} \\ \text{equal scalar loop values} \end{array} \right\}
 \end{array} \tag{2.13}$$

By [56, Thm. 3.1], as explained in [40, Proof of Thm. 4.3.1], given a spherical unitary 2-shaded planar algebra \mathcal{P}_\bullet which is *connected* ($\mathcal{P}_{0,\pm}$ are 1-dimensional), there is a II_1 subfactor $A \subset B$ which is *extremal* (the traces $\text{tr}_{A'}$ and tr_B agree on $A' \cap B$) whose standard invariant is $*$ -isomorphic to \mathcal{P}_\bullet . If moreover \mathcal{P}_\bullet is finite depth, A and B can be taken to be hyperfinite.

Hence in the case of connected spherical unitary 2-shaded planar algebras, the top horizontal map in (2.13) above is surjective. We conjecture that this map is always surjective; we expect the techniques of [21, 35, 58] will be useful towards verification.

Conjecture 2.26. *Given a unitary 2-shaded planar algebra \mathcal{P}_\bullet , there is a finite index homogeneous connected II_1 multifactor inclusion $A \subset B$ whose standard invariant is $*$ -isomorphic to \mathcal{P}_\bullet .*

We discuss *homogeneous* inclusions in Section 4.3 below cf. [57, Def. 1.2.11]. In Section 5.1 below, we prove Conjecture 2.26 for finite depth unitary 2-shaded planar algebras, in which case A, B can be taken to be hyperfinite.

The commutative diagram (2.13) is functorial for isomorphisms; this statement must be interpreted with the subtlety that the top line in (2.13) consists of two 1-groupoids, while the second line consists of two 2-groupoids (the second is 1-truncated and equivalent to a 1-groupoid by Theorem 2.21, see also [27, Lem. 3.5]). We now discuss the functoriality of these arrows in more detail.

An isomorphism of two finite index multifactor inclusions $A \subset B$ and $\tilde{A} \subset \tilde{B}$ is a $*$ -isomorphism $\varphi: B \rightarrow \tilde{B}$ taking A onto \tilde{A} . Given such an isomorphism φ of inclusions, there is a unique extension of φ taking the Jones tower of $A \subset B$ onto the Jones tower of $\tilde{A} \subset \tilde{B}$ preserving the Jones projections, i.e., $e_k^{A \subset B} \mapsto e_k^{\tilde{A} \subset \tilde{B}}$ for all k . This gives us an induced planar algebra isomorphism

$$\mathcal{P}(\varphi)_\bullet: \mathcal{P}_\bullet^{A \subset B} \rightarrow \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}}$$

on the centralizer algebras.

To describe 1-isomorphisms between dualizable bimodules, we must introduce the notion of Morita equivalence.

Definition 2.27. A *Morita equivalence* between von Neumann algebras M and N is an invertible bimodule ${}_M H_N$, i.e., an $M - N$ bimodule H equipped with two unitary bimodule isomorphisms ${}_M H \boxtimes_N \bar{H}_M \cong {}_M L^2 M_M$ and ${}_N \bar{H} \boxtimes_M H_N \cong {}_N L^2 N_N$ which satisfy the zig-zag relations (2.1), (2.2). The unitarity of these cups and caps is exactly the *recabling relations*:

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} \bar{H} \quad H \\ \text{---} \\ \bar{H} \quad H \end{array} & = & \begin{array}{c} \bar{H} \quad H \\ | \quad | \\ \bar{H} \quad H \end{array}, & \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} & = & \begin{array}{c} \text{---} \\ \square \\ \text{---} \end{array} & = & \text{id}_{L^2 N}, \\
 \end{array} \\
 \begin{array}{ccc}
 \begin{array}{c} H \quad \bar{H} \\ \text{---} \\ H \quad \bar{H} \end{array} & = & \begin{array}{c} H \quad \bar{H} \\ | \quad | \\ H \quad \bar{H} \end{array}, & \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} & = & \begin{array}{c} \text{---} \\ \square \\ \text{---} \end{array} & = & \text{id}_{L^2 M}.
 \end{array}
 \tag{2.14}$$

By [60, Prop. 3.1], these unitary solutions to the conjugate equations are unique up to unique $M - N$ bilinear unitary isomorphism of ${}_M H_N$.

As an example, given a von Neumann algebra N , any faithful right N -module H_N is canonically an $N' - N$ Morita equivalence bimodule by [60, Prop. 3.1] where N' is the commutant of the right N -action. When A is a finite multifactor, such faithful right A -modules Y_A are parameterized up to right A -linear unitary isomorphism by $\mathbb{R}_{>0}^a$ via the map $Y_A \mapsto (\text{vNdim}_R(Y p_{iA_i}))_{i=1}^a$.

Suppose now $N \subset M$ is a unital inclusion of von Neumann algebras. A faithful right N -module H_N induces a faithful right M -module $K_M := H \boxtimes_N L^2 M_M$. Setting $N' := (N^{\text{op}})' \cap B(H)$ and $M' := (M^{\text{op}})' \cap B(K)$, the induced left N' -action on K commutes with the right M -action and is thus contained in M' . The inclusion $N' \subset M'$ is called the *Morita equivalent inclusion induced by H_N* . (We warn the reader that the commutants N' and M' are taken in different representations, and thus M' is *not* contained in N' even though $N \subset M$.)

A 1-isomorphism between two dualizable bimodules ${}_A X^1_B$ and ${}_{A'} X^2_{B'}$ consists of a triple $({}_{A'} Y_A, {}_{B'} Z_B, \psi)$ where ${}_{A'} Y_A$ and ${}_{B'} Z_B$ are Morita equivalence bimodules (which always come equipped with two distinguished unitary isomorphisms satisfying the zig-zag axioms) and an $A' - B$ bilinear unitary isomorphism $\psi: {}_{A'} Y \boxtimes_A X^1_B \rightarrow {}_{A'} X^2 \boxtimes_{B'} Z_B$. A 2-isomorphism between triples $({}_{A'} Y^1_A, {}_{B'} Z^1_B, \psi_1)$ and $({}_{A'} Y^2_A, {}_{B'} Z^2_B, \psi_2)$ consists of an $A' - A$ bilinear unitary $u: {}_{A'} Y^1_A \rightarrow {}_{A'} Y^2_A$ and a $B' - B$ bilinear unitary $v: {}_{B'} Z^1_B \rightarrow {}_{B'} Z^2_B$ satisfying the relation

$$\psi_2 \circ (u \boxtimes \text{id}_{X^1}) = (v \boxtimes \text{id}_{X^2}) \circ \psi_1, \tag{2.15}$$

where we denote the four von Neumann algebras A, B, A', B' by the shaded regions

$$\text{○} = A, \quad \text{●} = B, \quad \text{◐} = A', \quad \text{◑} = B'.$$

By uniqueness of the unitary solutions to the zig-zag equations for Y^1, Y^2 and Z^1, Z^2 respectively, u, v will automatically satisfy the relations

$$\begin{aligned} \text{○} \begin{array}{c} \overline{Y^1} \\ \downarrow \\ \overline{u^*} \\ \downarrow \\ \overline{Y^2} \end{array} &= \text{◐} \begin{array}{c} \overline{Y^1} \\ \downarrow \\ \text{○} \begin{array}{c} \overline{Y^2} \\ \downarrow \\ u \\ \downarrow \\ \overline{Y^1} \end{array} \\ \downarrow \\ \overline{Y^2} \end{array} \\ \text{◑} \begin{array}{c} \overline{Z^1} \\ \downarrow \\ \overline{v^*} \\ \downarrow \\ \overline{Z^2} \end{array} &= \text{◑} \begin{array}{c} \overline{Z^1} \\ \downarrow \\ \text{◑} \begin{array}{c} \overline{Z^2} \\ \downarrow \\ v \\ \downarrow \\ \overline{Z^1} \end{array} \\ \downarrow \\ \overline{Z^2} \end{array}. \end{aligned} \tag{2.16}$$

We leave it to the reader to define composition of 1-morphisms and the associator isomorphisms in this 2-groupoid.

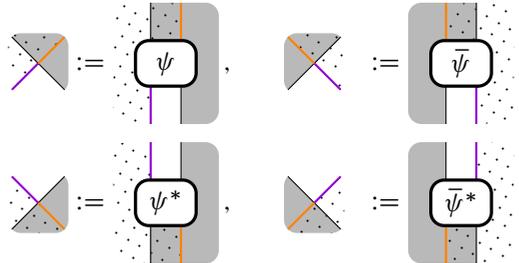
Suppose now that $(Y, Z, \psi): {}_A X^1_B \rightarrow {}_{A'} X^2_{B'}$ is a 1-isomorphism between dualizable multifactor bimodules. Denote these four von Neumann algebras A, B, A', B' by the shaded regions

$$\text{dotted circle} = A, \quad \text{solid grey circle} = B, \quad \text{dotted circle with dots} = A', \quad \text{solid grey circle with dots} = B',$$

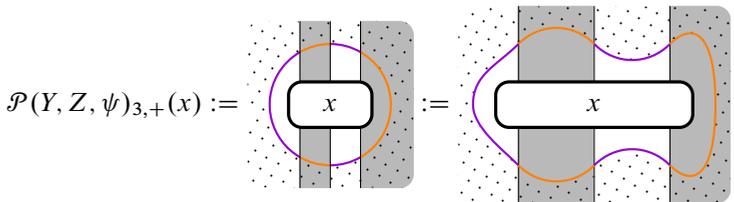
and the standard and Morita equivalence bimodules by

$$\begin{array}{|c|} \hline \text{dotted left | solid right} \\ \hline \end{array} = {}_A L^2 B_B, \quad \begin{array}{|c|} \hline \text{dotted left | dotted right with dots} \\ \hline \end{array} = {}_{A'} L^2 B'_{B'}, \quad \begin{array}{|c|} \hline \text{dotted left | dotted right} \\ \hline \end{array} = {}_{A'} Y_A, \quad \begin{array}{|c|} \hline \text{solid left | solid right with dots} \\ \hline \end{array} = {}_{B'} Z_B.$$

We denote the conjugate bimodules by the horizontal reflection, and the restriction to A, \tilde{A} respectively by changing the shading. We get a planar algebra $*$ -isomorphism by ‘encircling’ the box spaces of $\mathcal{P}({}_A X^1_B)_\bullet$ using $\psi, \psi^*, \bar{\psi}, \bar{\psi}^*$, and the standard cups and caps. For example, abbreviating isomorphisms $\psi, \bar{\psi}, \psi^*, \bar{\psi}^*$ by 4-valent vertices



given $x \in \mathcal{P}({}_A X^1_B)_{3,+}$, we define



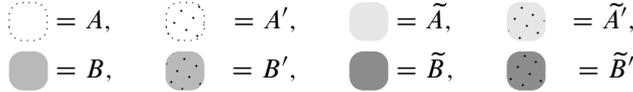
One verifies this ‘encircling’ action gives a planar algebra isomorphism using the recabling relation (2.14). It is a straightforward exercise using (2.15) and (2.16) that if there exists a 2-isomorphism $(u, v): (Y^1, Z^1, \psi_1) \Rightarrow (Y^2, Z^2, \psi_2)$, the planar algebra $*$ -isomorphisms $\mathcal{P}(Y^1, Z^1, \psi_1)_\bullet$ and $\mathcal{P}(Y^2, Z^2, \psi_2)_\bullet$ from $\mathcal{P}({}_A X^1_B)_\bullet \rightarrow \mathcal{P}({}_{A'} X^2_{B'})_\bullet$ are equal. Using this ‘encircling action’, we also get an equivalence of the non-idempotent complete full subcategories $\tilde{\mathcal{C}}({}_A X^1_B) \subset \mathcal{C}({}_A X^1_B)$ and $\tilde{\mathcal{C}}({}_{A'} X^2_{B'}) \subset \mathcal{C}({}_{A'} X^2_{B'})$ whose objects are the alternating tensor powers of X^j and \bar{X}^j for $j = 1, 2$ respectively, which descends to an equivalence of the idempotent completions $\mathcal{C}({}_A X^1_B) \simeq \mathcal{C}({}_{A'} X^2_{B'})$.

Hence the construction [12] actually gives a 1-functor from the 1-truncation of the 2-groupoid of bimodules to the 1-groupoid of unitary 2-shaded planar algebras. Moreover, the result [12, Proof of Thm. 5.4 and Rem. 5.5] can then be reinterpreted as the statement that the 1-functor from the 1-groupoid of inclusions to the 1-groupoid of planar

which only depends on w, φ . In particular, the following square commutes:

$$\begin{array}{ccc}
 \mathcal{P}(A L^2 B_B) \bullet & \xleftarrow{\cong} & \mathcal{P}(A' L^2 B'_{B'}) \bullet \\
 \mathcal{P}(L^2 \tilde{A}_\varphi, L^2 \tilde{B}_\varphi, \psi_\varphi) \bullet \downarrow & & \downarrow \mathcal{P}(L^2 \tilde{A}'_\varphi, L^2 \tilde{B}'_\varphi, \psi_\varphi) \bullet \\
 \mathcal{P}(\tilde{A} L^2 \tilde{B}_{\tilde{B}}) \bullet & \xleftarrow{\cong} & \mathcal{P}(\tilde{A}' L^2 \tilde{B}'_{\tilde{B}'}) \bullet
 \end{array} \tag{2.20}$$

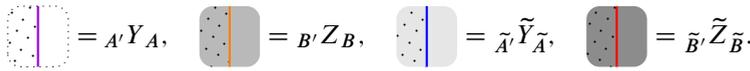
Proof. We denote the von Neumann algebras $A, A', \tilde{A}, \tilde{A}', B, B', \tilde{B}, \tilde{B}'$ by the shaded regions



the standard bimodules ${}_A L^2 B_B, {}_{A'} L^2 B'_{B'}, {}_{\tilde{A}} L^2 \tilde{B}_{\tilde{B}}, {}_{\tilde{A}'} L^2 \tilde{B}'_{\tilde{B}'}$ by



and the Morita equivalences $Y, Z, \tilde{Y}, \tilde{Z}$ by the colored strands



We denote the conjugates of these bimodules by taking horizontal reflections. We denote the restrictions of $B, B', \tilde{B}, \tilde{B}'$ -actions to $A, A', \tilde{A}, \tilde{A}'$ by changing the shading on the appropriate side. For example, when we restrict the left B, B' -action on Z, \tilde{Z} to A', \tilde{A}' , we have obvious identification isomorphisms

$$A' Z_B = A' Y \boxtimes_A L^2 B_B \quad \text{and} \quad \tilde{A}' \tilde{Z}_{\tilde{B}} = \tilde{A}' \tilde{Y} \boxtimes_{\tilde{A}} L^2 \tilde{B}_{\tilde{B}}$$

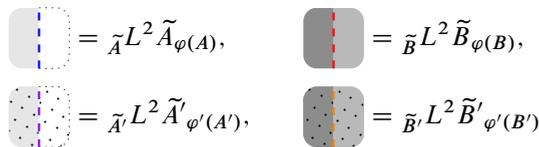
We now define a right B -linear unitary $x: Z_B \rightarrow \tilde{Z}_{\varphi(B)}$ by

$$Z_B = Y \boxtimes_A L^2 B_B \xrightarrow{w \boxtimes L^2 \varphi} \tilde{Y}_\varphi \boxtimes_A L^2 \tilde{B}_{\varphi(B)} \cong \tilde{Y} \boxtimes_{\tilde{A}} L^2 \tilde{B}_{\varphi(B)} = \tilde{Z}_{\varphi(B)}.$$

This right B -linear unitary induces an isomorphism of right B -commutants

$$\varphi' := \text{Ad}(x): B' \rightarrow \tilde{B}'.$$

By construction, $\varphi'|_{A'} = \text{Ad}(w)$. We denote the Morita equivalences $L^2 \tilde{A}_\varphi, L^2 \tilde{B}_\varphi, L^2 \tilde{A}'_{\varphi'}, L^2 \tilde{B}'_{\varphi'}$ by



their conjugates by again taking horizontal reflection, and restrictions by changing the shading. Using φ, φ' , we view w as an $A' - A$ bilinear unitary and x as a $B' - B$ bilinear unitary

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram of } w \\ \text{A box labeled } w \text{ with a vertical line through it, set against a dotted background.} \end{array} :_{A'} Y_A \rightarrow_{\varphi'(A')} L^2 \tilde{A}' \boxtimes_{\tilde{A}'} \tilde{Y} \boxtimes_{\tilde{A}} L^2 \tilde{A}_{\varphi(A)}, \\
 & \begin{array}{c} \text{Diagram of } x \\ \text{A box labeled } x \text{ with a vertical line through it, set against a shaded background.} \end{array} :_{B'} Z_B \rightarrow_{\varphi'(B')} L^2 \tilde{B}' \boxtimes_{\tilde{B}'} \tilde{Z} \boxtimes_{\tilde{B}} L^2 \tilde{B}_{\varphi(B)}.
 \end{aligned}$$

We now define our component unitaries of our 2-morphism by

$$\begin{aligned}
 u & := \begin{array}{c} \text{Diagram of } u \\ \text{A box labeled } w \text{ with a vertical line through it, set against a dotted background, with a purple dashed line above it.} \end{array} :_{\tilde{A}'} L^2 \tilde{A}'_{\varphi'} \boxtimes_{A'} Y_A \rightarrow_{\tilde{A}'} \tilde{Y} \boxtimes_{\tilde{A}} L^2 \tilde{A}_{\varphi(A)}, \\
 v & := \begin{array}{c} \text{Diagram of } v \\ \text{A box labeled } x \text{ with a vertical line through it, set against a shaded background, with a purple dashed line above it.} \end{array} :_{\tilde{B}'} L^2 \tilde{B}'_{\varphi'} \boxtimes_{B'} Z_B \rightarrow_{\tilde{B}'} \tilde{Z} \boxtimes_{\tilde{B}} L^2 \tilde{B}_{\varphi(B)}.
 \end{aligned}$$

It remains to verify (2.15) holds, which in string diagrams is as follows:

Applying isotopy and composing with obvious trivalent vertex isomorphisms, the above equation is equivalent to the following equation:

Finally, this equation holds by definition of the morphism x when restricted to an $A' - B$ bimodule map. ■

3. Distortion and extremality

In this section, we introduce the notion of the modular distortion for bimodules over finite multifactors. The notion of distortion for bimodules over a II_1 factor N is closely related with the notion of Connes–Takesaki module for an endomorphism of $B(\ell^2) \otimes N$; we refer the reader to Remark 3.10 for a detailed discussion.

Using the notion of distortion, we introduce the notion of extremality for multifactor bimodules, and we give many equivalent characterizations. We reconcile our definition based on [49] and the definition for a II_1 factor bimodule from [12, p. 51] in Corollary 3.21 below.

We connect our definition to extremality of a finite index connected inclusion of finite multifactors $A \subset (B, \text{tr}_B)$ with its Markov trace and trace-preserving conditional expectation in Section 3.3 below.

3.1. Distortion and extremality for II_1 factor bimodules

In this section, M, N, P will denote II_1 factors, unless stated otherwise.

Definition 3.1. The *modular distortion* of a dualizable $M - N$ bimodule H is defined to be

$$\delta = \delta(H) := \frac{\sqrt{\text{vNdim}_L(H)}}{\sqrt{\text{vNdim}_R(H)}}.$$

Observe that

$$\text{vNdim}_L({}_M H) = \delta(H)\Delta(H), \quad \text{vNdim}_R(H_N) = \frac{\Delta(H)}{\delta(H)}, \tag{3.1}$$

where $\Delta(H)$ is the Jones dimension. Moreover, distortion is multiplicative, i.e., if ${}_M H_N$ and ${}_N K_P$ are dualizable bimodules, then

$$\delta(H \boxtimes_N K) = \delta(H)\delta(K). \tag{3.2}$$

We say H has *constant distortion* δ if for every $M - N$ sub-bimodule $K \subset H$, we have $\delta(K) = \delta(H)$.

Remark 3.2. If H, K are both $M - N$ bimodules, the distortion

$$\delta(H \oplus K) = \sqrt{\frac{\text{vNdim}_L(H) + \text{vNdim}_L(K)}{\text{vNdim}_R(H) + \text{vNdim}_R(K)}}$$

cannot be computed from the distortions $\delta(H)$ and $\delta(K)$ alone. In particular, distortion is not additive.

Proposition 3.3. For a dualizable $M - N$ bimodule H , the following are equivalent.

- (1) H has constant distortion, and
- (2) $\Delta(H) = D(H)$, i.e., the Jones dimension is equal to the statistical dimension.

Proof. Since H is dualizable, we may write H as a finite direct sum of simple $M - N$ bimodules $H = \bigoplus_{i=1}^n K_i$. On the one hand, we have

$$\begin{aligned} D(H) &= \sum_{i=1}^n D(K_i) = \sum_{i=1}^n \Delta(K_i) = \sum_{i=1}^n \sqrt{\text{vNdim}_L(K_i) \text{vNdim}_R(K_i)} \\ &= \sum_{i=1}^n \delta(K_i) \text{vNdim}_R(K_i), \end{aligned}$$

using the fact that D is additive and the Jones dimension of each simple summand agrees with the statistical dimension. On the other hand, we have

$$\begin{aligned} \Delta(H) &= \sqrt{\left(\sum_{i=1}^n \text{vNdim}_L(K_i)\right) \left(\sum_{i=1}^n \text{vNdim}_R(K_i)\right)} \\ &= \sqrt{\left(\sum_{i=1}^n \delta(K_i)^2 \text{vNdim}_R(K_i)\right) \left(\sum_{i=1}^n \text{vNdim}_R(K_i)\right)}. \end{aligned}$$

Therefore, it is immediate that $D(H) = \Delta(H)$ when H has constant distortion.

The reverse implication follows by recognizing the expressions for $D(H)$ and $\Delta(H)$ computed above as related by the Cauchy–Schwarz inequality in \mathbb{R}^n . Indeed, if $\mathbf{e}_i \in \mathbb{R}^n$ denotes the i -th standard basis vector, setting

$$\mathbf{x} := \sum_{i=1}^n \delta(K_i) \text{vNdim}(K_i)^{1/2} \mathbf{e}_i, \quad \mathbf{y} := \sum_{i=1}^n \text{vNdim}(K_i)^{1/2} \mathbf{e}_i,$$

we have $D(H) = \langle \mathbf{x}, \mathbf{y} \rangle$ and $\Delta(H) = \|\mathbf{x}\| \|\mathbf{y}\|$. Suppose now that $D(H) = \Delta(H)$. Then we have $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\|$, and so $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \geq 0$. Hence, $\delta(K_i) = \lambda$ is constant independent of i . Since δ is constant on all simple summands of H , it follows (since vNdim_L and vNdim_R are additive) that δ is constant on all sub-bimodules of H . ■

Corollary 3.4. *If ${}_M H_N$ and ${}_N K_P$ are dualizable II_1 factor bimodules with constant distortion, then so is $H \boxtimes_N K$.*

Proof. Apply Proposition 3.3 to the equality

$$\Delta(H \boxtimes_N K) = \Delta(H) \Delta(K) = D(H) D(K) = D(H \boxtimes_N K). \quad \blacksquare$$

Remark 3.5. Given a II_1 factor N , the modular distortion δ induces a grading on the unitary tensor category $\text{Bim}_d(N)$ of dualizable $N - N$ bimodules by

$$\text{Bim}_d(N) = \bigoplus_{r>0} \delta^{-1}(r)$$

where $\delta^{-1}(r)$ is the semisimple subcategory of $\text{Bim}_d(H)$ whose simple objects have distortion r . (Observe that $\delta^{-1}(r)$ may contain only the zero object, so this $\mathbb{R}_{>0}$ -grading may

not be faithful.) By combining (3.2) and Corollary 3.4, we see that

$$\delta^{-1}(r) \otimes \delta^{-1}(s) \subset \delta^{-1}(rs).$$

By universality, we thus get a surjective homomorphism $\delta : \mathcal{U} \rightarrow \mathbb{R}_{>0}$ from the universal grading group \mathcal{U} of $\text{Bim}_d(N)$ onto $\mathbb{R}_{>0}$, which is the modular distortion up to a square root convention. By Theorem 2.5 [50], this homomorphism corresponds to a canonical unitary dual functor on $\text{Bim}_d(N)$ called the *tracial* or *von Neumann* unitary dual functor [34, §2.4]⁴ (see also [9, 12, 43]), as it is induced by the trace on N , which is used to compute the left and right von Neumann dimension of bimodules. Indeed, if $(\bar{H}, \text{ev}, \text{coev})$ is a dual for H using the tracial unitary dual functor, then

$$\text{vNdim}_L(H)\text{id}_{L^2N} = \text{ev} \circ \text{ev}^* \quad \text{and} \quad \text{vNdim}_R(H)\text{id}_{L^2N} = \text{coev}^* \circ \text{coev}.$$

We can thus view the modular distortion as a measure of how far the tracial unitary dual functor differs from the canonical spherical unitary dual functor.

Definition 3.6 ([49]). An $N - N$ bimodule H is called *extremal* if for all $N - N$ sub-bimodules $K \subset H$, we have $\text{vNdim}_L(K) = \text{vNdim}_R(K)$. When H is dualizable, H is extremal if and only if H has constant distortion equal to 1.

Corollary 3.7. *If H is an extremal dualizable $N - N$ bimodule, then $D(H) = \Delta(H)$.*

Remark 3.8. Notice that the converse of Corollary 3.7 is not true, as any simple non-distortion 1 bimodule is a counterexample.

Using the universal grading group, we get an extremely short proof of the following result, which can also be deduced from [54, §3.7.1].

Proposition 3.9. *If ${}_N H_N$ is finite depth, then ${}_N H_N$ is extremal.*

Proof. Since the unitary tensor category $\mathcal{C}(H)$ is fusion, its universal grading group \mathcal{U} is finite [13, §4.14]. Since δ induces a grading on $\mathcal{C}(H)$, δ descends to a group homomorphism $\mathcal{U} \rightarrow \mathbb{R}_{>0}$ by universality. Hence $\delta(\mathcal{U}) \subset \mathbb{R}_{>0}$ is a finite group, so it must be $\{1\}$. ■

Remark 3.10. Given a II_∞ factor M , in [30, Rem. 4.6], Izumi introduces the notion of a scalar-valued module $\text{Mod}(\rho)$ for every dualizable endomorphism $\rho \in \text{End}(M)$ determined by the formula

$$\text{Tr}_M \circ \rho = d_\rho \cdot \text{Mod}(\rho) \cdot \text{Tr}_M \tag{3.3}$$

where Tr_M is any faithful semifinite normal trace on M . When $M = B(\ell^2) \otimes N$ for a II_1 factor N , there is a well-known equivalence of rigid C^* tensor categories $\text{End}(M) \cup \{0\} \cong \text{Bim}(M)$ [26, §3.2] and an equivalence $\text{Bim}(M) \cong \text{Bim}(N)$ afforded by the Morita equivalence invertible $M - N$ bimodule ${}_M(\ell^2 \otimes L^2N)_N$. Using the composite equivalence

$$\text{End}(M) \cup \{0\} \cong \text{Bim}(N),$$

⁴We warn the reader there is a typo in [34, (10)].

we can relate $\text{Mod}(\rho)$ to the modular distortion $\delta({}_N H_N)$ where $H \in \text{Bim}(N)$ is any dualizable bimodule corresponding to the dualizable endomorphism $\rho \in \text{End}(M)$.

Fix a dualizable ${}_N H_N \in \text{Bim}(N)$, and let $\text{Tr}_M = \text{Tr} \otimes \text{tr}_N$. We get a dualizable endomorphism $\rho \in \text{End}(M)$ as follows. First choose any right N -linear unitary $u : \ell^2 \otimes H \rightarrow \ell^2 \otimes L^2 N$, and observe that for any $x \in M$, uxu^* commutes with the right N -action on $\ell^2 \otimes L^2 N$ and thus lies in M . Hence $\rho := \text{Ad}(u) \in \text{End}(M)$. Moreover, given any other right N -linear unitary $v : \ell^2 \otimes H \rightarrow \ell^2 \otimes L^2 N$, we have

$$(uxu^*)(uv^*) = uxv^* = (uv^*)v xv^*,$$

so $\text{Ad}(v)$ is equivalent to $\text{Ad}(u)$ in $\text{End}(M)$ via the unitary intertwiner uv^* .

Recall now that for any right N -linear isometry $u : H_N \rightarrow (\ell^2 \otimes L^2 N)_N$, we have $\text{Tr}_M(uu^*) = \text{vNdim}_R(H_N)$ [39, Prop. 10.1.3 and Def. 10.1.4]. This implies that for the projection $e_{11} \otimes 1_N \in M$, we may view $u(e_{11} \otimes 1_N) : e_1 \otimes H \rightarrow \ell^2 \otimes L^2 N$ as an N -linear isometry, and thus

$$\text{Tr}_M(\rho(e_{11} \otimes 1_N)) = \text{Tr}_M(u(e_{11} \otimes 1_N)u^*) = \text{vNdim}_R(H_N).$$

By (3.3), we conclude that

$$\text{Mod}(\rho) = \frac{\text{vNdim}_R(H_N)}{d_\rho} = \frac{\text{vNdim}_R(H_N)}{D(H)}.$$

When $D(H) = \Delta(H)$, by (3.1), we have

$$\text{Mod}(\rho) = \delta(H)^{-1}. \tag{3.4}$$

Now again by [30, Rem. 4.6], ρ has a Connes–Takesaki module $\text{mod}(\rho) \in \text{Aut}(Z(\tilde{M}))$ where $\tilde{M} := M \rtimes_{\sigma^{\text{Tr}_M}} \mathbb{R}$ if and only if the minimal conditional expectation $E_\rho : M \rightarrow \rho(M)$ is Tr_M -preserving. In this case, letting $\lambda^{\text{Tr}_M}(t) \in \tilde{M}$ be the unitary implementing $\sigma_t^{\text{Tr}_M}$, $\text{mod}(\rho)$ is determined by the formula

$$\text{mod}(\rho)(\lambda^{\text{Tr}_M}(t)) = \text{Mod}(\rho)^{-it} \cdot \lambda^{\text{Tr}_M}(t).$$

We claim that

- ρ has a Connes–Takesaki module if and only if H has constant distortion, and
- in this case, $\text{mod}(\rho)$ is multiplication by $\delta(H)^{it}$.

Indeed, by [30, Prop. 4.2 (2) and Rem. 4.6], it suffices to consider the case when H and ρ are simple. Since H simple implies constant distortion, we have $D(H) = \Delta(H)$, and $\text{Mod}(\rho) = \delta(H)^{-1}$ by (3.4).

3.2. Distortion and extremality for II_1 multifactor bimodules

We now extend the notions of distortion and extremality to bimodules over II_1 multifactors. Let ${}_A X_B$ be a connected dualizable bimodule, where we assume Notation 2.12.

Definition 3.11. The *modular distortion* of X , denoted $\delta = \delta(X)$, is the partially defined $a \times b$ matrix whose ij -th entry is given by

$$\delta_{ij} := \sqrt{\frac{\text{vNdim}_L(A_i(X_{ij}))}{\text{vNdim}_R((X_{ij})_{B_j})}} \quad \text{when } X_{ij} \neq 0.$$

Remark 3.12. Similar to Remark 3.5, distortion for bimodules over a II_1 multifactor A induces a $M_a(\mathbb{R}_{>0})$ -grading on $\text{Bim}_a(A)$. Here, $M_a(\mathbb{R}_{>0})$ is the groupoid with a objects such that the arrows in $i \rightarrow j$ with $i, j \in \{1, \dots, a\}$ are a $\mathbb{R}_{>0}$ -torsor. We will be interested in a notion of extremality for multifactor bimodules for which this $M_a(\mathbb{R}_{>0})$ -grading descends to a \mathcal{G}_a -grading where \mathcal{G}_a is the groupoid with a objects and a unique arrow between any two objects.

The following corollary follows immediately from Proposition 3.3.

Corollary 3.13. *The following are equivalent for a dualizable connected $A - B$ bimodule X .*

- (1) For every i, j , the bimodule X_{ij} has constant distortion (which may depend on i and j).
- (2) $D(X) = \Delta(X)$.

Definition 3.14. We call ${}_A X_B$ *extremal* if for all $1 \leq i \leq a$ and $1 \leq j \leq b$, all $A_i - A_i$ and $B_j - B_j$ bimodules generated by X and \bar{X} are extremal in the sense of Definition 3.6.

Remark 3.15. An immediate consequence of Definition 3.14 is that extremality of one of ${}_A X_B$, ${}_A X \boxtimes_B \bar{X}_A$, ${}_B \bar{X}_A$, or ${}_B \bar{X} \boxtimes_A X_B$ is equivalent to extremality of all of these bimodules.

Lemma 3.16. *Suppose ${}_A X_B$ is extremal. For any $A - B$ bimodules Y, Z generated by X and \bar{X} such that $Y_{ij} := p_i Y q_j \neq 0 \neq p_i Z q_j =: Z_{ij}$, we have*

$$\frac{\text{vNdim}_L(A_i(Y_{ij}))}{\text{vNdim}_R((Y_{ij})_{B_j})} = \frac{\text{vNdim}_L(A_i(Z_{ij}))}{\text{vNdim}_R((Z_{ij})_{B_j})}. \tag{3.5}$$

Proof. Notice that (3.5) above is equivalent to

$$\text{vNdim}_L(A_i(Y_{ij})) \text{vNdim}_R((Z_{ij})_{B_j}) = \text{vNdim}_L(A_i(Z_{ij})) \text{vNdim}_R((Y_{ij})_{B_j}),$$

which is equivalent to

$$\text{vNdim}_L(A_i(Y_{ij})) \text{vNdim}_L(B_j(\bar{Z}_{ji})) = \text{vNdim}_R((Y_{ij})_{B_j}) \text{vNdim}_R((\bar{Z}_{ji})_{A_i}),$$

which is equivalent to

$$\text{vNdim}_L(A_i(Y_{ij} \boxtimes_{B_j} \bar{Z}_{ji})) = \text{vNdim}_R((Y_{ij} \boxtimes_{B_j} \bar{Z}_{ji})_{A_i}),$$

which follows by extremality of X . ■

When ${}_A X_B$ is extremal, Lemma 3.16 allows us to uniquely extend the definition of δ_{ij} when $p_i q_j = 0$ by defining

$$\delta_{ij} := \sqrt{\frac{\text{vNdim}_L(A_i(Y_{ij}))}{\text{vNdim}_R((Y_{ij})_{B_j})}}$$

for any $A - B$ bimodule Y generated by X and \bar{X} such that $Y_{ij} := p_i Y q_j \neq 0$.

Corollary 3.17. *Suppose ${}_A X_B$ is extremal. The extension of the distortion function δ satisfies*

$$\delta_{ij} \delta_{i'j'} = \delta_{ij'} \delta_{i'j} \quad \forall 1 \leq i, i' \leq a \text{ and } 1 \leq j, j' \leq b. \tag{1.1}$$

Proof. Pick $A - B$ bimodules U, V, Y, Z generated by X and \bar{X} such that $U_{ij}, V_{i'j'}, Y_{ij'}$, and $Z_{i'j}$ are all non-zero. By Lemma 3.16, $\delta_{ij} \delta_{i'j'} = \delta_{ij'} \delta_{i'j}$ if and only if

$$\frac{\text{vNdim}_L(A_i(U_{ij}))}{\text{vNdim}_R((U_{ij})_{B_j})} \frac{\text{vNdim}_L(A_{i'}(V_{i'j'}))}{\text{vNdim}_R((V_{i'j'})_{B_{j'}})} = \frac{\text{vNdim}_L(A_i(Y_{ij'}))}{\text{vNdim}_R((Y_{ij'})_{B_{j'}})} \frac{\text{vNdim}_L(A_{i'}(Z_{i'j}))}{\text{vNdim}_R((Z_{i'j})_{B_j})}$$

which holds if and only if

$${}_A U_{ij} \boxtimes_{B_j} \bar{Z}_{j'i'} \boxtimes_{A_{i'}} V_{q_{i'j'}} \boxtimes_{B_{j'}} \bar{Y}_{j'i} A_i$$

has the same left and right von Neumann dimension, by extremality of X . ■

Lemma 3.18. *Suppose we have a connected bipartite graph Γ with a even vertices and b odd vertices and no multiple edges. Suppose we have a weighting $\delta_{ij} \in \mathbb{R}_{>0}$ for each edge $(i, j) \in \Gamma$. The following conditions are equivalent for δ .*

(1) *For any cycle $(i_1, j_1, i_2, j_2, \dots, i_n, j_n)$ in Γ we have*

$$\prod_{k=1}^n \delta_{i_k j_k} = \prod_{k=1}^n \delta_{i_{k+1} j_k} \tag{3.6}$$

where indices are taken modulo n .

- (2) *There is a weighting $\eta_i, \xi_j \in \mathbb{R}_{>0}$ for each even vertex i and odd vertex j of Γ such that $\delta_{ij} = \xi_j / \eta_i$. Moreover this weighting is unique up to simultaneous uniform scaling of all η_i, ξ_j .*
- (3) *There is an extension of δ to the complete bipartite graph $K_{a,b}$ which satisfies Condition (1.1) above. Moreover, any such extension is unique.*
- (4) *There is an extension of δ to a groupoid homomorphism $\mathcal{G}_{a+b} \rightarrow \mathbb{R}_{>0}$ where \mathcal{G}_{a+b} is the groupoid consisting of $a + b$ objects and a unique isomorphism between any two objects. Moreover, any such extension is unique.*

Proof. (1) \Rightarrow (2): Suppose (1) holds. Fix an arbitrary even vertex i_1 of Γ , and set $\eta_{i_1} := 1$. For an arbitrary odd vertex j of Γ , pick an arbitrary path $(i_1, j_1, \dots, i_n, j_n)$ from i_1 to

$j_n = j$ (one exists as Γ is connected), and define

$$\xi_j := \frac{\prod_{k=1}^n \delta_{i_k j_k}}{\prod_{\ell=1}^{n-1} \delta_{i_{\ell+1} j_\ell}}.$$

By (3.6), ξ_j is independent of the choice of path. Similarly, for an arbitrary even vertex i of Γ , we pick an arbitrary path $(i_1, j_1, \dots, i_n, j_n, i_{n+1})$ from i_1 to $i_{n+1} = i$, and we define

$$\eta_i := \frac{\prod_{k=1}^n \delta_{i_k j_k}}{\prod_{\ell=1}^n \delta_{i_{\ell+1} j_\ell}}$$

which is again independent of the choice of path by (3.6). It is clear that $\delta_{ij} = \xi_j / \eta_i$ for all $(i, j) \in \Gamma$ by construction.

Now suppose η'_i and ξ'_j is another choice of vertex weighting such that $\delta_{ij} = \xi'_j / \eta'_i$ for all $(i, j) \in \Gamma$. Setting $\lambda := \eta'_{i_1}$, we claim that $\eta'_i = \lambda \eta_i$ and $\xi'_j = \lambda \xi_j$ for all i, j . Indeed, fixing a path $(i_1, j_1, \dots, i_n, j_n)$ from i to j on Γ , we have

$$\xi_j = \xi_{j_n} = \frac{\prod_{k=1}^n \delta_{i_k j_k}}{\prod_{\ell=1}^{n-1} \delta_{i_{\ell+1} j_\ell}} = \frac{\prod_{k=1}^n \frac{\xi'_{j_k}}{\eta'_{i_k}}}{\prod_{\ell=1}^{n-1} \frac{\xi'_{j_\ell}}{\eta'_{i_{\ell+1}}}} = \frac{\xi'_{j_n}}{\eta'_{i_1}} = \frac{\xi'_j}{\lambda}$$

and thus $\xi'_j = \lambda \xi_j$ as claimed. Similarly, $\eta'_i = \lambda \eta_i$.

(2) \Rightarrow (3): Suppose (2) holds. Setting $\delta_{ij} := \xi_j / \eta_i$ for all $(i, j) \in K_{a,b}$ is an extension of δ to $K_{a,b}$ which clearly satisfies (1.1) as

$$\delta_{ij} \delta_{i'j'} = \frac{\xi_j}{\eta_i} \frac{\xi_{j'}}{\eta_{i'}} = \frac{\xi_{j'}}{\eta_i} \frac{\xi_j}{\eta_{i'}} = \delta_{i'j'} \delta_{ij}$$

for all i, i', j, j' .

Now suppose δ' is another weighting on the edges of $K_{a,b}$ satisfying (1.1) such that $\delta_{ij} = \delta'_{ij}$ for every $(i, j) \in \Gamma$. We claim that $\delta'_{ij} = \xi_j / \eta_i$ for all $(i, j) \in K_{a,b}$. Indeed, picking an arbitrary path $(i_1, j_1, \dots, i_n, j_n)$ from $i = i_1$ to $j = j_n$ in Γ , we see

$$\delta'_{ij} = \frac{\prod_{k=1}^n \delta'_{i_k j_k}}{\prod_{\ell=1}^{n-1} \delta'_{i_{\ell+1} j_\ell}} = \frac{\prod_{k=1}^n \delta_{i_k j_k}}{\prod_{\ell=1}^{n-1} \delta_{i_{\ell+1} j_\ell}} = \frac{\prod_{k=1}^n \frac{\xi_{j_k}}{\eta_{i_k}}}{\prod_{\ell=1}^{n-1} \frac{\xi_{j_\ell}}{\eta_{i_{\ell+1}}}} = \frac{\xi_{j_n}}{\eta_{i_1}} = \frac{\xi_j}{\eta_i}$$

as claimed.

(3) \Rightarrow (4): Suppose (3) holds. Setting $\delta_{ii'} := \delta_{ij} / \delta_{i'j}$ for any odd vertex j in independent of the choice of j by (1.1). We define $\delta_{jj'}$ analogously. It is straightforward to verify this is the only possible extension of δ to \mathcal{G}_{a+b} satisfying $\delta_{rs} \delta_{st} = \delta_{rt}$ for all r, s, t .

(4) \Rightarrow (1): Suppose δ extends to a groupoid homomorphism. For any cycle $(i_1, j_1, \dots, i_n, j_n)$ in Γ , we have

$$\prod_{k=1}^n \delta_{i_k j_k} \delta_{j_k i_{k+1}} = 1 \tag{3.7}$$

where indices are taken modulo n . Since $\delta_{ji} = \delta_{ij}^{-1}$, the equation (3.7) is equivalent to (3.6). ■

Similar techniques prove the following useful corollary.

Corollary 3.19. *The following are equivalent for a matrix $\delta \in M_n(\mathbb{R}_{>0})$.*

- (1) δ gives a groupoid homomorphism $\mathcal{G}_n \rightarrow \mathbb{R}_{>0}$, i.e., $\delta_{ij}\delta_{jk} = \delta_{ik}$ for all $1 \leq i, j, k \leq n$.
- (2) There are $(\lambda_i)_{i=1}^n \in \mathbb{R}_{>0}^n$ such that $\delta_{ij} = \lambda_j/\lambda_i$. Moreover this weighting is unique up to simultaneous uniform scaling of all λ_i .

Proof. Obviously (2) \Rightarrow (1). Suppose (1) holds. Setting $\lambda_i := \delta_{i1}$, we have $\delta_{ij} = \delta_{i1}\delta_{1j} = \delta_{i1}/\delta_{1i}$ as desired. If $(\eta_i) \in \mathbb{R}_{>0}^n$ is any other such vector such that $\delta_{ij} = \eta_j/\eta_i = \lambda_j/\lambda_i$, then clearly $\lambda_j/\eta_j = \lambda_i/\eta_i$ for all $1, \leq i, j \leq n$. Hence (2) holds. ■

Theorem 3.20. *The following are equivalent for a connected dualizable bimodule ${}_A X_B$.*

- (1) X is extremal.
- (2) X_{ij} has constant distortion δ_{ij} whenever $X_{ij} \neq 0$, and δ satisfies (3.6) above.
- (3) $D(X) = \Delta(X)$ and δ satisfies (3.6) above.

Proof. (1) \Rightarrow (2): Suppose X is extremal. That X_{ij} has constant distortion follows immediately from Lemma 3.16. Define a bipartite graph $\Gamma = \Gamma(X)$ with a even vertices and b odd vertices, where $(i, j) \in \Gamma$ if and only if $X_{ij} \neq 0$. By Corollary 3.17, the extension of δ to $K_{a,b}$ satisfies (1.1). Finally, we may apply Lemma 3.18 to see (3.6) holds for δ restricted back to Γ .

(2) \Rightarrow (1): Suppose (2) holds. Then as δ satisfies condition (3.6), by Lemma 3.18, δ extends uniquely to a groupoid homomorphism $\mathcal{G}_{a+b} \rightarrow \mathbb{R}_{>0}$, so $\delta_{ii} = 1 = \delta_{jj}$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$. Now since X has constant distortion, given any loop $(i_1, j_1, \dots, i_n, j_n)$ with $i_1 = i$ in Γ , the $A_i - A_i$ bimodule

$$X_{i_1 j_1} \boxtimes_{B_{j_1}} \bar{X}_{j_1 i_2} \boxtimes_{A_{i_2}} \cdots \boxtimes_{A_{i_n}} X_{i_n j_n} \boxtimes_{B_{j_n}} \bar{X}_{j_n i_1}$$

has constant distortion equal to $\delta_{ii} = 1$ by Corollary 3.4.

Moreover this constant distortion must be equal to 1 independent of the choice of loop of length $2n$ starting at i by (3.6). Hence

$$p_i (X \boxtimes_B \bar{X})^{\boxtimes A^n} p_i = \bigoplus_{\substack{\text{loops of length } 2n \\ \text{based at } i}} X_{i_1 j_1} \boxtimes_{B_{j_1}} \bar{X}_{j_1 i_2} \boxtimes_{A_{i_2}} \cdots \boxtimes_{A_{i_n}} X_{i_n j_n} \boxtimes_{B_{j_n}} \bar{X}_{j_n i_1}$$

has constant distortion equal to 1 for all $n \in \mathbb{N}$ and $1 \leq i \leq a$. Similarly, $q_j (\bar{X} \boxtimes_A X)^{\boxtimes B^n} q_j$ has constant distortion 1 for all $n \in \mathbb{N}$ and $1 \leq j \leq b$. Thus X is extremal.

(2) \Rightarrow (3) This follows immediately by Corollary 3.13. ■

As a corollary, our definition of extremality agrees with that from [12, p. 51] in the case of a dualizable bimodule over II_1 factors.

Corollary 3.21. *Suppose M, N are II_1 factors and ${}_M H_N$ is a dualizable $M - N$ bimodule. The following are equivalent.*

- ${}_M H_N$ is extremal in the sense of Definition 3.14.
- ${}_M H_N$ is extremal in the sense of [12, p. 51], i.e., the traces $\text{tr}_{N'}$ and $\text{tr}_{M'}$ agree on $N' \cap M' \cap B(H)$.

Proof. First, we note that (3.6) is always satisfied when M, N are factors, so H is extremal in the sense of Definition 3.14 if and only if it has constant distortion. Second, we note that $\text{tr}_{N'} = \text{tr}_{M'}$ on $N' \cap M'$ if and only if they agree on projections. Now observe that for every projection $p \in N' \cap M'$, we have

$$\begin{aligned} \delta(pH) &= \left(\frac{\text{vNdim}_L({}_M(pH))}{\text{vNdim}_R((pH)_N)} \right)^{1/2} = \left(\frac{\text{tr}_{M'}(p) \text{vNdim}_L({}_M H)}{\text{tr}_{N'}(p) \text{vNdim}_R(H_N)} \right)^{1/2} \\ &= \delta(H) \left(\frac{\text{tr}_{N'}(p)}{\text{tr}_{M'}(p)} \right)^{1/2}. \end{aligned}$$

Thus $\delta(pH) = \delta(H)$ if and only if $\text{tr}_{N'}(p) = \text{tr}_{M'}(p)$. We conclude that H has constant distortion if and only if $\text{tr}_{N'} = \text{tr}_{M'}$. ■

Corollary 3.22. *Every finite depth connected dualizable bimodule ${}_A X_B$ is extremal.*

Proof. We may apply Proposition 3.9 to the diagonal component unitary fusion categories $\mathcal{C}(X)_{kk}$ of the standard invariant $\mathcal{C}(X) \subset \text{Bim}(A \oplus B)$ to see that every $A_i - A_i$ and $B_j - B_j$ bimodule generated by X is extremal. ■

3.3. Distortion and extremality for multifactor inclusions

In this section, we study the special case of the bimodule $X = {}_A L^2 B_B$ for a finite index connected inclusion $A \subset B$ of finite multifactors. Our main goal for this section is to characterize extremality of ${}_A L^2 B_B$ in terms of the Markov and minimal conditional expectations.

Notation 3.23. For this section, fix a connected finite index inclusion $A \subset B$ of finite multifactors. In addition to assuming Notation 2.12, we fix the following notation. We write $\Delta = \Delta(X)$ for the Jones dimension matrix and $D = D(X)$ for the statistical dimension matrix. Since $D^T D$ has positive entries and is irreducible, by the Frobenius–Perron Theorem, there are unique row vectors $\vec{\alpha} \in \mathbb{R}_{>0}^a$ and $\vec{\beta} \in \mathbb{R}_{>0}^b$ with $\|\vec{\alpha}\|_2 = 1 = \|\vec{\beta}\|_2$ such that $\vec{\alpha} D = d_X \vec{\beta}$ and $\vec{\beta} D^T = d_X \vec{\alpha}$ where d_X^2 is the largest eigenvalue of $D^T D$ [17].

Fact 3.24. We can now express the trace matrices in terms of the modular distortion. First, observe that $p_i q_j = p_i J q_j J$, so $p_i q_j \neq 0$ if and only if $J p_i q_j J = J p_i J q_j \neq 0$. Combining (2.6), (2.8), and (2.11) respectively with (3.1), whenever $p_i q_j \neq 0$,

$$T_{ij} = \frac{\Delta_{ij}}{\delta_{ij}}, \quad \tilde{T}_{ji} = \delta_{ij} \Delta_{ij}, \quad \text{and} \quad T_{ji}^{B \subset (B,A)} = \frac{\delta_{ij} \Delta_{ij}}{\sum_{k=1}^b \delta_{ik} \Delta_{ik}}. \tag{3.8}$$

Definition 3.25. Given a conditional expectation $E : B \rightarrow A$, we define $\lambda_{ij} = \lambda_{ij}^E \in [0, \infty)$ by the formula

$$\lambda_{ij} p_i = E(p_i q_j).$$

The conditional expectation E induces conditional expectations $E_{ij} : p_i q_j B p_i q_j \rightarrow p_i q_j A$ whenever $p_i q_j \neq 0$ by

$$E_{ij}(p_i q_j b p_i q_j) := \lambda_{ij}^{-1} E(b q_j) p_i q_j.$$

We will be most interested in the unique Markov trace-preserving conditional expectation $E^{\text{Markov}} : B \rightarrow A$ and the unique minimal expectation $E^0 : B \rightarrow A$ [17, 29, 45]. By [24, Lem. 3.4], the Markov trace-preserving expectation is uniquely determined by E_{ij}^{Markov} is trace-preserving whenever $p_i q_j \neq 0$, and

$$\lambda_{ij}^{\text{Markov}} = T_{ij} \left(\frac{\text{tr}_B(q_j)}{\text{tr}_A(p_i)} \right) \stackrel{(3.8)}{=} \left(\frac{\Delta_{ij}}{\delta_{ij}} \right) \left(\frac{\text{tr}_B(q_j)}{\text{tr}_A(p_i)} \right) \quad \forall p_i q_j \neq 0 \quad (3.9)$$

where tr_B is the unique Markov trace and $\text{tr}_A = \text{tr}_B|_A$. By [17, Thm. 2.6] and [16, Thm. 2.3], the minimal expectation is uniquely determined by

$$\text{Ind}(E_{ij}^0) = D_{ij} \quad \text{and} \quad \lambda_{ij}^0 = \frac{D_{ij} \beta_j}{d \alpha_i} \quad \forall i, j. \quad (3.10)$$

Theorem 3.26. *Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors. The following are equivalent:*

- (E1) $E^{\text{Markov}} = E^0$, i.e., $A \subset B$ is extremal in the sense of [17, Def. 4.1].
- (E2) $D = \Delta$ and the Markov trace tr_B on B and its restriction $\text{tr}_A := \text{tr}_B|_A$ satisfy

$$\delta_{ij} = d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right) \quad \forall p_i q_j \neq 0. \quad (3.11)$$

- (E3) ${}_A L^2 B_B$ is extremal.

Proof. (E1) \Rightarrow (E2): Suppose that E^{Markov} is minimal. By (3.10), $\text{Ind}(E_{ij}) = \Delta_{ij} = D_{ij}$ for all i, j , so $\Delta = D$. Combining (3.10) and (3.9), whenever $p_i q_j \neq 0$,

$$\frac{D_{ij} \beta_j}{d \alpha_i} \stackrel{(3.10)}{=} \lambda_{ij}^{\text{Markov}} \stackrel{(3.9)}{=} \left(\frac{D_{ij}}{\delta_{ij}} \right) \left(\frac{\text{tr}_B(q_j)}{\text{tr}_A(p_i)} \right).$$

Solving for δ_{ij} gives (3.11).

(E2) \Rightarrow (E1): Since $D = \Delta$, $\text{Ind}(E_{ij}) = \Delta_{ij} = D_{ij}$ for all i, j . Now combining (3.9), (3.11), and $D = \Delta$, we see that

$$\lambda_{ij}^{\text{Markov}} \stackrel{(3.9)}{=} \left(\frac{D_{ij}}{\delta_{ij}} \right) \left(\frac{\text{tr}_B(q_j)}{\text{tr}_A(p_i)} \right) \stackrel{(3.11)}{=} \frac{D_{ij} \beta_j}{d \alpha_i}.$$

By (3.10), E^{Markov} is the unique minimal expectation.

- (E2) \Rightarrow (E3): By Lemma 3.18, δ satisfies (3.6). The result now follows by Theorem 3.20.

(E3) \Rightarrow (E2): Suppose ${}_A L^2 B_B$ is extremal. By Theorem 3.20, $D = \Delta$ and δ satisfies (3.6). By Lemma 3.18, there are vectors $\eta \in \mathbb{R}_{>0}^a$ and $\xi \in \mathbb{R}_{>0}^b$, unique up to uniformly scaling both η, ξ by the same positive scalar, such that $\delta_{ij} = \xi_j / \eta_i$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$.

Now choose the unique trace tr_B on B such that $\text{tr}_B(q_j) = \xi_j \beta_j / d$, where we have replaced η, ξ with the unique uniform scaling such that $\sum_{j=1}^b \text{tr}_B(q_j) = 1$. Set $\text{tr}_A := \text{tr}_B \upharpoonright_A$. We now calculate that

$$\begin{aligned} \text{tr}_A(p_i) &= \sum_{j=1}^b T_{ij} \text{tr}_B(q_j) \stackrel{(3.8)}{=} \sum_{j=1}^b \left(\frac{D_{ij}}{\delta_{ij}} \right) \frac{\xi_j \beta_j}{d} = \frac{1}{d} \sum_{j=1}^b D_{ij} \left(\frac{\eta_i}{\xi_j} \right) \xi_j \beta_j \\ &= \frac{\eta_i}{d} \sum_{j=1}^b D_{ij} \beta_j = \eta_i \alpha_i \end{aligned}$$

which implies that $\eta_i = \text{tr}_A(p_i) / \alpha_i$. Hence we have

$$\delta_{ij} = \frac{\xi_j}{\eta_i} = d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right).$$

To see that tr_B is the Markov trace, we verify (2.9):

$$\sum_{i=1}^a \tilde{T}_{ji} \text{tr}_A(p_i) \stackrel{(3.8)}{=} \sum_{i=1}^a \delta_{ij} D_{ij} \text{tr}_A(p_i) = d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) \sum_{i=1}^a D_{ij} \alpha_i = d^2 \text{tr}_B(q_j). \quad \blacksquare$$

In light of Theorem 3.26, we make the following definition.

Definition 3.27. A connected finite index multifactor inclusion is called *extremal* if either of the equivalent conditions of Theorem 3.26 holds.

Corollary 3.28. *The connected dualizable multifactor bimodule ${}_A X_B$ is extremal if and only if the finite index connected inclusion of finite multifactors $A \subset M := (B^{\text{op}})' \cap B(X)$ is extremal.*

Proof. By [59, Prop. 3.1], we have a canonical $M - M$ bimodule isomorphism $X \boxtimes_B \bar{X} \cong L^2 M$, which restricts to an $A - M$ bimodule isomorphism. By Remark 3.15, ${}_A X_B$ is extremal if and only if ${}_A X \boxtimes_B \boxtimes \bar{X}_A$ is extremal if and only if ${}_A L^2 M_A$ is extremal if and only if ${}_A L^2 M \boxtimes_M L^2 M_A$ is extremal if and only if ${}_A L^2 M_M$ is extremal. \blacksquare

3.4. Categorical explanation of distortion and extremality

We now give a completely tensor categorical explanation of the notions of distortion and extremality for a unitary multitensor category \mathcal{C} equipped with a chosen unitary dual functor \vee . The results in Section 3.2 above are the special case of $\mathcal{C} = \text{Bim}_d(A)$ for a multifactor $A = \bigoplus_{i=1}^a A_i$ equipped with the canonical *tracial* unitary dual functor with respect to $\text{tr}_A = \bigoplus_i \text{tr}_{A_i}$, i.e., $\text{tr}_A(p_i) = a^{-1}$ for all i .

We begin with the case $1_{\mathcal{C}}$ is simple, i.e., \mathcal{C} is a unitary tensor category equipped with a fixed unitary dual functor \vee . For $c \in \mathcal{C}$, we define

$$\delta(c) := \sqrt{\frac{\dim_L^\vee(c)}{\dim_R^\vee(c)}}.$$

We say c has *constant distortion* $\delta \in \mathbb{R}_{>0}$ if for every summand $a \subset c$, $\delta(a) = \delta$. As in the case of bimodules, we can define a *Jones dimension* of $c \in \mathcal{C}$ by

$$\Delta(c) := \sqrt{\dim_L^\vee(c) \dim_R^\vee(c)}.$$

Observe that the formulas in (3.1) now become

$$\dim_L^\vee(c) = \delta(c)\Delta(c), \quad \dim_R^\vee(c) = \frac{\Delta(c)}{\delta(c)}.$$

The *statistical dimension* of $c \in \mathcal{C}$ is $D(c) := \dim^{\text{sp}}(c)$, where \dim^{sp} is the dimension with respect to the unique spherical unitary dual functor.

Lemma 3.29. *The following are equivalent for $c \in \mathcal{C}$:*

- (1) c has constant distortion,
- (2) $\Delta(c) = D(c)$,
- (3) there is a $\delta > 0$ such that

$$\left(\begin{array}{c} c \\ \text{f} \\ c \end{array} \right) = \text{tr}_L^\vee(\text{f}) = \delta^2 \cdot \text{tr}_R^\vee(\text{f}) = \delta^2 \cdot \left(\begin{array}{c} c \\ \text{f} \\ c \end{array} \right) \quad \forall \text{f} \in \text{End}_{\mathcal{C}}(c). \quad (3.12)$$

Proof. (1) \Leftrightarrow (2): The proof follows *mutatis mutandis* from the proof of Proposition 3.3.

(1) \Rightarrow (3): Suppose c has constant distortion δ . Since \mathcal{C} is orthogonal projection complete, for every orthogonal projection $p : c \rightarrow c$, there is an $a \in \mathcal{C}$ and an isometry $v : a \rightarrow c$ such that $p = vv^*$. Since $\dim_L^\vee(a) = \delta^2 \dim_R^\vee(a)$, then suppressing the pivotal structure φ in the diagrams below,

$$\begin{aligned} \left(\begin{array}{c} c \\ p \\ c \end{array} \right) &= \left(\begin{array}{c} c \\ v \\ a \\ v^* \\ c \end{array} \right) = \left(\begin{array}{c} a \\ v^* \\ c \\ v \\ a \end{array} \right) = \dim_L^\vee(a) = \delta^2 \dim_R^\vee(a) = \delta^2 \cdot \left(\begin{array}{c} a \\ v^* \\ c \\ v \\ a \end{array} \right) = \delta^2 \cdot \left(\begin{array}{c} c \\ v \\ a \\ v^* \\ c \end{array} \right) \\ &= \delta^2 \cdot \left(\begin{array}{c} c \\ p \\ c \end{array} \right). \end{aligned}$$

Since every $f : c \rightarrow c$ is a linear combination of orthogonal projections, (3.12) holds.

(3) \Rightarrow (1): Assume (3.12) and suppose $v : a \rightarrow c$ is an isometry. Then, suppressing the pivotal structure φ in the diagrams below,

$$\dim_L^\vee(a) = \left(\begin{array}{c} a \\ v^* \\ c \\ v \\ a \end{array} \right) = \left(\begin{array}{c} c \\ v \\ a \\ v^* \\ c \end{array} \right) = \delta^2 \cdot \left(\begin{array}{c} c \\ v \\ a \end{array} \right) = \delta^2 \cdot \left(\begin{array}{c} a \\ v^* \\ c \\ v \\ a \end{array} \right) = \delta^2 \cdot \dim_R^\vee(a). \quad \blacksquare$$

Corollary 3.30. *If $a, b \in \mathcal{C}$ have constant distortion $\delta_a, \delta_b \in \mathbb{R}_{>0}$, then $a \otimes b$ has constant distortion $\delta_a \delta_b$.*

Proof. For all $f \in \text{End}_{\mathcal{C}}(a \otimes b)$,

$$\text{Diagram 1} = \delta_b^2 \cdot \text{Diagram 2} = \delta_b^2 \cdot \delta_a^2 \cdot \text{Diagram 3} = \delta_a^2 \delta_b^2 \cdot \text{Diagram 4}$$

The result now follows from (3.12). ■

By Corollary 3.30 above, we get a (non-faithful, in general) $\mathbb{R}_{>0}$ -grading on \mathcal{C} given by $\mathcal{C} = \bigoplus_{r \in \mathbb{R}_{>0}} \mathcal{C}_r$ where \mathcal{C}_r is the full subcategory of \mathcal{C} whose objects have constant distortion r . Observe $\mathcal{C}_r \otimes \mathcal{C}_s \subset \mathcal{C}_{rs}$, and if $c \in \mathcal{C}_r$, then $c^\vee \in \mathcal{C}_{r^{-1}}$. Denote by G the subgroup of $\mathbb{R}_{>0}$ such that $\mathcal{C}_r \neq 0$ so that \mathcal{C} is faithfully graded by G . By [50, Rem. 3.17], there is a surjective group homomorphism from the universal grading group $U_{\mathcal{C}} \twoheadrightarrow G$. Composing with the inclusion map $G \hookrightarrow \mathbb{R}_{>0}$ gives a group homomorphism $U_{\mathcal{C}} \rightarrow \mathbb{R}_{>0}$. This is one way to see that \vee restricted to \mathcal{C}_e is spherical where $e \in U_{\mathcal{C}}$ is the unit.

We say $c \in \mathcal{C}$ is *extremal* if c has constant distortion $\delta(c) = 1$, which is equivalent to saying that \vee restricted to $\langle c \rangle \subset \mathcal{C}$, the full tensor subcategory generated by c , is spherical.

Now when \mathcal{C} is a unitary multitensor category, we let $1_{\mathcal{C}} = \bigoplus_{i=1}^n 1_i$ be a decomposition into simples and $\mathcal{C}_{ij} := 1_i \otimes \mathcal{C} \otimes 1_j$ as in Notation 2.3. For $c \in \mathcal{C}$, we write $c_{ij} = 1_i \otimes c \otimes 1_j$ so that $c = \bigoplus c_{ij}$. We define distortion for objects $c \in \mathcal{C}$ which is partially defined with value in $M_r(\mathbb{R}_{>0})$ given by

$$\delta(c)_{ij} := \sqrt{\frac{\dim_L^\vee(c_{ij})}{\dim_R^\vee(c_{ij})}}$$

We say $c \in \mathcal{C}$ has *constant distortion* if for every summand $a \subset c$, $\delta(a)_{ij} = \delta(c)_{ij}$ for every i, j where $\delta(a)_{ij}$ is defined. Again we can define the *Jones matrix dimension* of $c \in \mathcal{C}$ by

$$\Delta(c)_{ij} := \sqrt{\dim_L^\vee(c_{ij}) \dim_R^\vee(c_{ij})}$$

and the *statistical matrix dimension* of $c \in \mathcal{C}$ by $D(c)_{ij} := \dim^{\text{sp}}(c_{ij})$, where \dim^{sp} is the dimension with respect to the unique spherical unitary dual functor. As in Proposition 3.3 above, $c \in \mathcal{C}$ has constant distortion if and only if $\Delta(c) = D(c)$.

Similar to Corollary 3.30 above, if $a \in \mathcal{C}_{ij}$ and $b \in \mathcal{C}_{jk}$ have constant distortion, so does $a \otimes b \in \mathcal{C}_{ik}$. Thus we get a (non-faithful, in general) $\mathcal{S}_n \times \mathbb{R}_{>0}$ -grading on \mathcal{C} given by $\mathcal{C} = \bigoplus (\mathcal{C}_{ij})_r$, where \mathcal{S}_n is the groupoid with n objects and a unique isomorphism between any two objects, and $(\mathcal{C}_{ij})_r$ is the full subcategory of \mathcal{C}_{ij} whose objects have constant distortion r . Observe that $(\mathcal{C}_{ij})_r \otimes (\mathcal{C}_{jk})_s \subset (\mathcal{C}_{ik})_{rs}$, and if $c \in (\mathcal{C}_{ij})_r$, then $c^\vee \in (\mathcal{C}_{ji})_{r^{-1}}$.

Denote by \mathcal{G} the subgroupoid of $\mathcal{E}_n \times \mathbb{R}_{>0}$ such that \mathcal{C} is faithfully graded by \mathcal{G} . Observe that the map

$$(\mathcal{C}_{ij})_r \ni c \mapsto r \in \mathbb{R}_{>0}$$

descends to a well-defined groupoid homomorphism $\mathcal{G} \rightarrow \mathbb{R}_{>0}$. By [50, Rem. 3.17], there is a surjective groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}} \twoheadrightarrow \mathcal{G}$. Composing these two homomorphisms gives a groupoid homomorphism $\mathcal{U}_{\mathcal{C}} \rightarrow \mathbb{R}_{>0}$. In summary, the map $\delta : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{R}_{>0}$ given by $c \mapsto \delta(c)$ gives a groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}} \rightarrow \mathbb{R}_{>0}$. In particular, for any idempotent $e \in \mathcal{U}_{\mathcal{C}}$, \mathcal{C}_e is spherical.

Up to square roots, δ is the classifying groupoid homomorphism π for \vee as in Theorem 2.5.⁵ However, we warn the reader that δ does not yield a homomorphism $\mathcal{E}_n \rightarrow \mathbb{R}_{>0}$ unless all objects of \mathcal{C} have constant distortion.

We now say an object $c \in \mathcal{C}$ is *extremal* if c has constant distortion and its distortion satisfies (1.1), namely

$$\delta(c)_{ij}\delta(c)_{jk} = \delta(c)_{ik} \quad \forall i, j, k.$$

Extremality ensures that the distortion matrix from our unitary dual functor \vee restricted to $\langle c \rangle \subset \mathcal{C}$, the full unitary multitensor subcategory generated by c , gives a groupoid homomorphism $\mathcal{E}_n \rightarrow \mathbb{R}_{>0}$. Thus \vee on $\langle c \rangle$ comes from a groupoid homomorphism $\mathcal{E}_n \rightarrow \mathbb{R}_{>0}$, which in turn allows us to ‘correct’ the distortion using a spherical state.

4. Modular distortion and the Jones tower/tunnel

Let $A \subset B$ be a connected finite index inclusion of finite multifactors, and assume Notations 2.12 and 3.23. In Sections 4.1 and 4.2, respectively, we characterize the behavior of the modular distortion under basic construction and downward basic construction. We then make connections to Popa’s homogeneity [57, Def. 1.2.11] in Section 4.3, and we show that the notion of *super-extremality* [17, Def. 4.1] for a finite index extremal connected Markov inclusion $A \subset (B, \text{tr}_B)$ of finite multifactors is exactly Popa’s homogeneity.

Popa’s homogeneity is a crucial ingredient in the proof of Theorem A, namely that the standard invariant is a complete invariant for a finite depth finite index connected hyperfinite II_1 multifactor inclusion which is homogeneous. Homogeneity allows us to find a generating Jones tunnel by iterating the downward Jones basic construction, analogous to the technique in [54]. An interesting new element of this section is a quantitative analysis, rather than qualitative, of when a downward basic construction exists.

4.1. The distortion dynamical system for Jones’ basic construction

We now compute the behavior of the distortion under taking Jones’ basic construction. Recall that $p_i q_j \neq 0$ if and only if $J p_i J q_j \neq 0$, and that the Jones dimension matrix for

⁵We did not use a square root in [50] as our results there also classify pivotal structures (when one exists) on a general multitensor category which is not necessarily unitary. Positivity allows us to take a unique positive square root, which is no longer possible in the non-unitary setting.

$B \subset \langle B, A \rangle$ is Δ^T , where Δ is the Jones dimension matrix of $A \subset B$ [20, Prop. 3.6.6] and $\langle B, A \rangle$ is the Jones basic construction.

Proposition 4.1. *The distortion*

$$\delta^{B \subset \langle B, A \rangle} = \delta_{(B L^2 \langle B, A \rangle_{(B, A)})}$$

is related to $\delta = \delta_{(A L^2 B_B)}$ by

$$\delta_{ji}^{B \subset \langle B, A \rangle} = \frac{1}{\delta_{ij}} \sum_{k=1}^b \delta_{ik} \Delta_{ik} \quad \forall p_i q_j \neq 0 \Leftrightarrow J p_i J q_j \neq 0. \tag{4.1}$$

Proof. Whenever $p_i q_j \neq 0 \Leftrightarrow J p_i J q_j \neq 0$, by using (3.8) twice, we have

$$\delta_{ji}^{B \subset \langle B, A \rangle} = \frac{\Delta_{ji}^T}{T_{ji}^{B \subset \langle B, A \rangle}} = \delta_{ij}^{-1} \sum_{k=1}^b \delta_{ik} \Delta_{ik}. \quad \blacksquare$$

Assumption 4.2. We assume for the remainder of this section that $A \subset B$ is extremal, i.e., $A L^2 B_B$ is extremal so that $\Delta = D$ and δ satisfies the equivalent conditions of Lemma 3.18. In particular, there are row vectors $\vec{\eta} = (\eta_i)_{i=1}^a \in \mathbb{R}_{>0}^a$ and $\vec{\xi} = (\xi_j)_{j=1}^b \in \mathbb{R}_{>0}^b$, unique up to uniform scaling both η, ξ by the same positive scalar, such that $\delta_{ij} = \xi_j / \eta_i$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$.

We now iterate the basic construction one more time to get the first 4 algebras of the Jones tower: $A = A_0 \subset B = A_1 \subset A_2 \subset A_3$.

Corollary 4.3. *The distortions $\delta^{A_1 \subset A_2} = \delta_{(A_1 L^2 A_{2A_2})}$ and $\delta^{A_2 \subset A_3} = \delta_{(A_2 L^2 A_{3A_3})}$ are related to $\delta = \delta_{(A L^2 B_B)}$ by*

$$\delta_{ji}^{A_1 \subset A_2} = \frac{(\vec{\xi} D^T)_i}{(\vec{\xi})_j}, \quad \delta_{ij}^{A_2 \subset A_3} = \frac{(\vec{\xi} D^T D)_j}{(\vec{\xi} D^T)_i}. \tag{4.2}$$

(Observe that the above formulas are independent of the choice of $\vec{\xi}$ up to uniform positive scaling.)

Proof. The first equality is proven by substituting $\delta_{ij} = \xi_j / \eta_i$ in (4.1):

$$\delta_{ji}^{A_1 \subset A_2} = \frac{\eta_i}{\xi_j} \sum_{k=1}^b \frac{\xi_k}{\eta_i} D_{ik} = \frac{1}{\xi_j} \sum_{k=1}^b \xi_k D_{ik} \quad \forall p_i q_j \neq 0 \Leftrightarrow J p_i J q_j \neq 0$$

and then extending $\delta_{ji}^{A_1 \subset A_2}$ for every j, i as in Lemma 3.18. The second equality follows by iteration with

$$D^T = D_{(A_1 L^2 A_{2A_2})}$$

in place of $D = D_{(A L^2 B_B)}$ and $\vec{\xi} D^T$ in place of $\vec{\xi}$. ■

Theorem 4.4. *Consider the topological space*

$$\{(\delta_{ij}) \in \text{Mat}_{a \times b}(\mathbb{R}_{>0}) \mid \delta_{ij}\delta_{i'j'} = \delta_{ij'}\delta_{i'j} \ \forall 1 \leq i, i' \leq a, \ \forall 1 \leq j, j' \leq b\} \cong \mathbb{R}_{>0}^{a+b-1}.$$

The distortion dynamical system

$$(\delta_{ij}) \xrightarrow{\Phi} \left(\frac{(\vec{\xi} D^T D)_j}{(\vec{\xi} D^T)_i} \right) \tag{4.3}$$

where $\vec{\xi} \in \mathbb{R}_{>0}^b$ is as in Lemma 3.18(2) for δ , has a unique fixed point σ given by $\sigma_{ij} := d_X \beta_j / \alpha_i$, where $\vec{\alpha}, \vec{\beta}, d_X$ are as in Notation 3.23.

Proof. Arguing by induction, we see that iterating this dynamical system n times starting with $\delta_{ij} = \xi_j / \eta_i$ yields

$$\Phi^n(\delta) = \left(\frac{(\vec{\xi} (D^T D)^n)_j}{(\vec{\xi} (D^T D)^{n-1} D^T)_i} \right).$$

Since $\delta_{ij} > 0$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$, it is well known that

$$d_X^{-2n} \vec{\xi} (D^T D)^n \xrightarrow{n \rightarrow \infty} c \vec{\beta}$$

for some $c > 0$ as the matrix $d_X^{-2n} (D^T D)^n$ tends to the projection onto $\mathbb{R} \vec{\beta}$. This means that given any input δ to the dynamical system, iterating yields

$$\Phi^n(\delta) \xrightarrow{n \rightarrow \infty} \frac{c(\vec{\beta} D^T D)_j}{c(\vec{\beta} D^T)_i} = \frac{d_X^2 \beta_j}{d_X \alpha_i} = \frac{d_X \beta_j}{\alpha_i}.$$

Hence there is at most one fixed point given by $\sigma_{ij} := d_X \beta_j / \alpha_i$. One immediately sees that this σ is indeed a fixed point under the dynamical system Φ . ■

Definition 4.5. Given a connected, extremal $A-B$ bimodule ${}_A X_B$ we say that X has *standard distortion* if $\delta = \sigma$, the unique fixed point under the distortion dynamical system (4.3).

4.2. Distortion and the downward basic construction

We now compare the various notions of downward basic construction for a finite index connected finite multifactor inclusion $A \subset B$ that appear in the literature.

Definition 4.6. Suppose $A \subset B$ is a finite index connected finite multifactor inclusion. Let tr_B be the unique Markov trace on B and d^2 the Markov index.

- A *trace independent downward basic construction* consists of a von Neumann sub-algebra $C \subset A$ and a unital $*$ -algebra isomorphism $\psi : B \rightarrow JC'J \cap B(L^2 A)$ such that $\psi|_A = \text{id}_A$.

- A Jones downward basic construction cf. [37, Lem. 3.1.8] consists of a projection $f \in B$ with central support 1 together with a left A -module unitary $u : L^2 A \rightarrow L^2 B f$.
- A Popa downward basic construction cf. [57, Prop. 1.2.7] consists of a projection $e \in B$ with $E_A(e) = d^{-2}$ such that setting $C := \{e\}' \cap A$, $C \subset (A, \text{tr}_A) \subset (B, \text{tr}_B, e)$ is the Jones basic construction, i.e., the map $aecb \mapsto aeb$ for $a, b \in A$ extends to a trace-preserving isomorphism $\langle A, C \rangle \cong B$. Observe that in this case, $E_C(a) := d^2 E_A(eae)$ is the unique Markov trace-preserving conditional expectation; indeed, setting $\text{tr}_C := \text{tr}_A|_C$, for all $a \in A$ and $c \in C$,

$$\begin{aligned} \text{tr}_C (E_C(a)c) &= \text{tr}_A (d^2 E_A(eae)c) = d^2 \text{tr}_B (eae c) \\ &= d^2 \text{tr}_B (eac) = d^2 \text{tr}_B (E_A(e)ac) = \text{tr}_A(ac). \end{aligned} \tag{4.4}$$

It is relatively straightforward to show that a Popa downward basic construction is a Jones downward basic construction which, in turn, is a trace independent basic construction. The main result of this section builds on [57, Prop. 1.2.7] to prove the equivalence of all these notions and further quantifies the existence of a downward basic construction in terms of the distortion δ .

Lemma 4.7. *Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors. For any projection $f \in B$,*

$$\text{vNdim}_L (A_i(p_i L^2 B f q_j)) = \pi_j \delta_{ij} \Delta_{ij} \quad \forall 1 \leq i \leq a$$

where $\pi = \text{tr}_B^Z(f)$, i.e., $\pi_j = \text{tr}_{B_j}(f q_j)$.

Proof. Fix $1 \leq i \leq a$, and recall that $\delta_{ij} \Delta_{ij} = \text{vNdim}_L(A_i(p_i L^2 B q_j))$ by (3.1). Thus by [37, (2.1.3)],

$$\text{vNdim}_L (A_i(p_i L^2 B f q_j)) = \text{tr}_{A_i'}(J f q_j J) \delta_{ij} \Delta_{ij} = \text{tr}_{JA_i' J}(f q_j) \delta_{ij} \Delta_{ij}.$$

But since $B_j \subset JA_i' J$ is a II_1 subfactor (whose unit is also q_j), we must have that

$$\text{tr}_{JA_i' J}(f q_j) = \text{tr}_{B_j}(f q_j) = \pi_j$$

by the uniqueness of the trace on B_j . The result follows. ■

Fact 4.8. Since the minimal central projections of the basic construction $\langle B, A \rangle$ are the $J p_i J$, by (2.10), the center-valued trace of $e_A \in \langle B, A \rangle$ has i -th component

$$\begin{aligned} \text{tr}_{\langle B, A \rangle_i}(e_A J p_i J) &= \frac{\text{tr}_{\langle B, A \rangle}(e_A J p_i J)}{\text{tr}_{\langle B, A \rangle}(J p_i J)} = \frac{\text{tr}_{\langle B, A \rangle}(e_A p_i)}{\text{tr}_{\langle B, A \rangle}(J p_i J)} \\ &= \frac{d^{-2} \text{tr}_A(p_i)}{\text{tr}_{\langle B, A \rangle}(J p_i J)} \stackrel{(2.10)}{=} \frac{1}{\sum_{j=1}^b \delta_{ij} \Delta_{ij}}. \end{aligned}$$

Proposition 4.9. *Let $A \subset B$ be a finite index connected inclusion of finite multifactors, and let tr_B be the unique Markov trace. For a projection $f \in B$, the following are equivalent.*

- (1) $L^2 A \cong L^2 B f$ as left A -modules.
- (2) the vector $\pi := \text{tr}_B^Z(f) \in [0, 1]^b$ satisfies $\sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij} = 1$ for all $1 \leq i \leq a$.
- (3) f is equivalent to e_A in $(B, A) \subset B(L^2(B, \text{tr}_B))$.

Proof. (1) \Leftrightarrow (2): This follows immediately from Lemma 4.7:

$$\begin{aligned} \text{vNdim}_L (A_i(p_i L^2 B f)) &= \text{vNdim}_L \left(\bigoplus_{j=1}^b A_i(p_i L^2 B f q_j) \right) \\ &= \sum_{j=1}^b \text{vNdim}_L (A_i(p_i L^2 B f q_j)) = \sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij}. \end{aligned}$$

Hence $L^2 B f \cong L^2 A$ as left A -modules if and only if $\sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij} = 1$ for all $1 \leq i \leq a$.

(2) \Rightarrow (3): Whenever $q_j J p_i J \neq 0$, by uniqueness of the trace on B_j , for all $b \in B$,

$$\text{tr}_{B_j}(b q_j) = \frac{\text{tr}_{(B,A)_i}(b q_j J p_i J)}{\text{tr}_{(B,A)_i}(q_j J p_i J)} = \frac{\text{tr}_{(B,A)_i}(b q_j J p_i J)}{T_{ij}^{B \subset (B,A)}}.$$

Hence for our projection $f \in B$,

$$\begin{aligned} \text{tr}_{(B,A)_i}(f J p_i J) &= \sum_{j=1}^b \text{tr}_{(B,A)_i}(f q_j J p_i J) = \sum_{j=1}^b \pi_j T_{ij}^{B \subset (B,A)} \\ &= \frac{\sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij}}{\sum_{k=1}^b \delta_{ik} \Delta_{ik}} \stackrel{(3.8)}{=} \frac{1}{\sum_{k=1}^b \delta_{ik} \Delta_{ik}}. \end{aligned}$$

By Fact 4.8, $\text{tr}_{(B,A)}^Z(f) = \text{tr}_{(B,A)}^Z(e_A)$.

(3) \Rightarrow (1): Recall that two projections in a finite multifactor are equivalent if and only if they are unitarily conjugate. Suppose $f \in B$ and $u \in U((B, A))$ such that $e_A = u f u^*$. Since the commutant of the left A -action on $L^2 B$ is the right- $\langle A, B \rangle$ action, and since $[J, e_A] = 0$ [20, Prop. 3.6.1 (ii)], we have isomorphisms of left A -modules

$$L^2 A \cong e_A L^2 B = L^2 B e_A = L^2 B u f u^* = (L^2 B f) u^* \cong L^2 B e. \quad \blacksquare$$

Theorem 4.10 (cf. [57, Prop. 1.2.7]). *Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors. The following are equivalent.*

- (D1) $A \subset B$ admits a trace independent downward basic construction.
- (D2) $A \subset B$ admits a Jones downward basic construction, i.e., there is a projection $f \in B$ with central support 1 such that $L^2 A \cong L^2 B f$ as left A -modules.
- (D3) There exists a projection $f \in B$ with central support 1 such that the vector $\pi := \text{tr}_B^Z(f) \in (0, 1]^b$ satisfies $\sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij} = 1$ for every $1 \leq i \leq a$.
- (D4) There is a projection $f \in B$ with central support 1 which is equivalent to $e_A \in (B, A) \subset B(L^2(B, \text{tr}_B))$ where tr_B is the unique Markov trace.

(D5) $A \subset B$ admits a Popa downward basic construction, i.e., there is a projection $e \in B$ with $E_A(e) = d^{-2}$ such that setting $C := \{e\}' \cap A$, $C \subset (A, \text{tr}_A) \subset (B, e, \text{tr}_B)$ is the Jones basic construction and $E_C^A(a) := d^2 E_A(eae)$ is the Markov trace-preserving conditional expectation.

Proof. (D1) \Rightarrow (D5): Suppose $C \subset A \subset B$ is an independent downward basic construction, and suppose there is a $*$ -isomorphism $B \cong \langle A, C \rangle$ on $L^2 A$. Identify $L^2 A = L^2(A, \text{tr}_A)$ where tr_A is the unique Markov trace for $C \subset A$, and identify $B = \langle A, C \rangle = \langle A, e_C \rangle$ where e_C is the orthogonal projection onto $L^2(C, \text{tr}_C)$ and $\text{tr}_C := \text{tr}_A|_C$. Then by [37, Props. 3.1.4 and 3.4.1], $E_A(e_C) = d^{-2}$, $C = \{e_C\}' \cap A$, and $E_C(a)e_C = e_C a e_C$ for all $a \in A$. Taking E_A , we get $E_C(a) = d^{-2} E_A(e_C a e_C)$.

(D5) \Rightarrow (D2): We set $f = e$, which we may identify with the Jones projection e_C for $C \subset (A, \text{tr}_A)$. By [37, Prop. 3.1.5 (iv)], the central support of e_C is 1. We then have the following isomorphisms of left A -modules:

$$L^2 B e_C \cong L^2 A \boxtimes_C L^2 A e_C \cong L^2 A \boxtimes_C L^2 C \cong L^2 A.$$

(D2) \Rightarrow (D1): This follows from the argument in [37, Lem. 3.1.8]. Using the isomorphism $L^2 A \cong L^2 B e$ as left A -modules, we get a normal left B -action on $L^2 A$ extending the left A -action. Now define $C := (J_A B J_A)'$ on $L^2 A$, which is manifestly a downward basic construction.

(D2) \Leftrightarrow (D3) \Leftrightarrow (D4): This follows by Proposition 4.9 together with the assumption that the central support of f is 1. ■

Remark 4.11. Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors. The Jones tower $(A_n, \text{tr}_n, e_{n+1})_{n \geq 0}$ is a *Markov tower* in the sense of [10, Def. 3.1]. Suppose $f \in B$ is a projection satisfying the equivalent conditions of Proposition 4.9 such that $E_A(f) = d^{-2}$, except the central support z of f in B is possibly not 1. We can perform a *one step downward Markov tunnel* by defining $A_{-1} := \{f\}' \cap A$, $\text{tr}_{-1} := \text{tr}_A|_{A_{-1}}$, and $e_0 := f$. Observe that $E_{-1}(a) := d^2 E_A(e_0 a e_0)$ still defines the unique trace-preserving conditional expectation by (4.4). Note, however, that $z B z$ (which is possibly not B) is the basic construction of $A_{-1} \subset (A, \text{tr}_A)$.

For (non-)examples, we refer the reader to Examples 4.18 and 4.19 below.

Corollary 4.12. *Suppose $A \subset B$ is a finite index connected II_1 multifactor inclusion such that (D3) holds. The distortion γ of the inclusion $C \subset A$ corresponding to projection $f \in B$ is given by $\gamma_{ji} = \frac{1}{\pi_j \delta_{ij}}$.*

Proof. To compute γ_{ji} , we compute

$$\left(\frac{\text{vNdim}_R((p_i L^2 A r_j)_{C_j})}{\text{vNdim}_L(A_i(p_i L^2 A r_j))} \right)^{1/2}$$

where the r_j are the minimal central projections of C . To define C , we transported the left B action on ${}_A L^2 B f$ to ${}_A L^2 A$ under the left A -module isomorphism. This means

the left A -module isomorphism extends to an $A - B'$ bimodule isomorphism, where the right action of B' is exactly the right action of C . Hence we may identify ${}_A L^2 A_C \cong {}_A L^2 B f_{Bf}$. Since $Z(fBf) = fZ(B)$, we may identify the r_j with $f q_j$.

First, by Lemma 4.7,

$$\text{vNdim}_L (A_i(p_i L^2 A r_j)) = \text{vNdim}_L (A_i(p_i L^2 B f q_j)) = \pi_j \delta_{ij} \Delta_{ij}.$$

Second, by [37, (2.1.4)] and (3.1),

$$\begin{aligned} \text{vNdim}_R ((p_i L^2 A r_j)_{C_j}) &= \text{vNdim}_R (A_i(p_i L^2 B f q_j)_{f B_j f}) \\ &= \frac{1}{\text{tr}_{B_j}(f q_j)} \text{vNdim}_R (A_i(p_i L^2 B q_j)_{B_j}) = \frac{\Delta_{ij}}{\pi_j \delta_{ij}}. \end{aligned}$$

We conclude that $\gamma_{ji} = \frac{1}{\pi_j \delta_{ij}}$ as claimed. ■

Remark 4.13. Under the hypotheses of Corollary 4.12, observe that δ and γ satisfy (4.1):

$$\begin{aligned} \frac{1}{\gamma_{ji}} \sum_{k=1}^a \gamma_{jk} \Delta_{jk}^T &= \pi_j \delta_{ij} \sum_{k=1}^a \frac{1}{\pi_j \delta_{kj}} \Delta_{kj} = \delta_{ij} \sum_{k=1}^a \frac{\Delta_{kj}}{\delta_{kj}} \\ &= \delta_{ij} \sum_{k=1}^a \text{vNdim} ((p_k L^2 B q_j)_{B_j}) \\ &= \delta_{ij} \text{vNdim} ((L^2 B q_j)_{B_j}) = \delta_{ij}. \end{aligned}$$

Moreover, we can easily see that for all $1 \leq i \leq a$,

$$\sum_{j=1}^b \pi_j \delta_{ij} \Delta_{ij} = \sum_{j=1}^b \frac{\Delta_{ji}^T}{\gamma_{ji}} = \text{vNdim} ((L^2 A p_i)_{A_i}) = 1.$$

Corollary 4.14. *Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors and suppose $e_1, e_2 \in B$ satisfy (D5), with $C_1 := \{e_1\}' \cap A$ and $C_2 := \{e_2\}' \cap A$. The following are equivalent.*

- (1) *There is a $u \in A$ such that $u e_1 u^* = e_2$.*
- (2) *There is a $u \in A$ such that $u C_1 u^* = C_2$.*
- (3) $\text{tr}_B^Z(e_1) = \text{tr}_B^Z(e_2)$.

Proof. (1) \Rightarrow (2): Suppose $u \in A$ such that $u e_1 u^* = e_2$. Then for all $c \in C_1$,

$$u c u^* e_2 = u c (u^* e_2 u) u^* = u c e_1 u^* = u e_1 c u^* = u e_1 u^* u c u^* = e_2 u c u^*,$$

so $u c u^* \in C_2$. Similarly, for all $c \in C_2$, $u^* c u \in C_1$. Hence $u C_1 u^* = C_2$.

(2) \Rightarrow (3): We prove the contrapositive. If $\text{tr}_B^Z(e_1) \neq \text{tr}_B^Z(e_2)$, then by Corollary 4.12, the distortions $\delta^{C_1 \subset A}$ and $\delta^{C_2 \subset A}$ will not agree. We conclude no such unitary can exist.

(3) \Rightarrow (1): The proof is similar to [51, Prop. 1.7 (i) and Cor. 1.8 (ii)].

First, a similar argument to [51, Prop. 1.7 (i)] proves that for any finite index connected inclusion $A \subset B$ of finite multifactors, the map $u \mapsto ue_Au^*$ descends to a bijection from $U(B)/U(A)$ to the set of projections

$$\{f \in P(\langle B, A \rangle) \mid E_B(f) = d^{-2} \text{ and } \text{tr}_{(B,A)}^Z(f) = \text{tr}_{(B,A)}^Z(e_A)\}.$$

Now suppose $e_1, e_2 \in B$ with $E_A(e_1) = E_A(e_2) = d^{-2}$ and $\text{tr}_B^Z(e_1) = \text{tr}_B^Z(e_2)$. Similar to the proof of [51, Cor. 1.8 (ii)], it follows from the above bijection applied to $C_1 \subset A$ with $\langle C_1, A \rangle = B$ that there is a unitary $u \in A$ such that $ue_1u^* = e_2$. ■

4.3. Popa’s homogeneity

We now study Popa’s notion of *homogeneity* for finite index connected inclusions $A \subset B$ of finite multifactors. We obtained the following definition by combining [57, Def. 1.2.11] and [32, Ex. 2.7 and Prop. 3.1 (1)].

Definition 4.15. We say $A \subset B$ is *homogeneous of index λ* if $\text{vNdim}_{L(A_i p_i L^2 B)} = \lambda$ for all $1 \leq i \leq a$.

Theorem 4.16. *Suppose that $A \subset B$ is a finite index extremal connected inclusion of finite multifactors. Let tr_B be the unique Markov trace, $\text{tr}_A = \text{tr}_B \upharpoonright_A$, E the unique Markov/minimal conditional expectation, and $e_1 \in B(L^2(B, \text{tr}_B))$ the Jones projection. The following are equivalent.*

- (H1) $A \subset B$ is homogeneous of index d^2 .
- (H2) $\sum_{j=1}^b \delta_{ij} D_{ij} = d^2$ for all $1 \leq i \leq a$.
- (H3) $\Phi(\delta) = \delta$, i.e., δ is a fixed point of the dynamical system (4.3) (which is unique by Theorem 4.4).
- (H4) $\delta_{ij} = d\beta_j/\alpha_i$.
- (H5) $\text{tr}_{(B,A)}^Z(e_1) \in \mathbb{C}$, where $\text{tr}_{(B,A)}^Z$ is the canonical center-valued trace;
- (H6) $\text{tr}_A(p_i) = \text{tr}_{(B,A)}(Jp_i J)$ for all i .
- (H7) $A \subset B$ is super-extremal in the sense of [17, Def. 4.1, Lem. 4.2]: $\text{tr}_A(p_i) = \alpha_i^2$ where $\vec{\alpha} \in \mathbb{R}_{>0}^a$ is the eigenvector for DD^T with eigenvalue d^2 normalized so that $\sum_{i=1}^a \alpha_i^2 = 1$.

In addition, if A and B are of type II_1 , the above are equivalent to:

- (H8) The inclusion $A \subset B$ admits an infinite Jones tunnel, i.e., we can iterate the downward basic construction infinitely many times.

Proof. (H1) \Leftrightarrow (H2): Observe that

$$\text{vNdim}_{L(A_i p_i L^2 B)} = \sum_{j=1}^b \text{vNdim}_{L(A_i p_i L^2 B q_j B_j)} = \sum_{j=1}^b \delta_{ij} D_{ij}.$$

(H2)⇒(H3): Combining (H2) and (4.1), we have $\delta_{ji}^{B \subset (B,A)} = d^2/\delta_{ij}$. Applying (4.1) again with $\delta^{B \subset (B,A)}$ in place of δ and D^T in place of D , we have

$$\Phi(\delta)_{ij} = \frac{\delta_{ij}}{d^2} \sum_{\ell=1}^a \frac{d^2}{\delta_{\ell j}} D_{j\ell}^T = \delta_{ij} \sum_{\ell=1}^a \frac{D_{\ell j}}{\delta_{\ell j}} \stackrel{(3.1)}{=} \delta_{ij} \sum_{\ell=1}^a \text{vNdim}_R(A_\ell(p_\ell L^2 B q_\ell)_{B_j}) = \delta_{ij}.$$

Hence δ is the unique fixed point under the dynamical system (4.3).

(H3)⇒(H4): This follows from Theorem 4.4.

(H4)⇒(H2): If $\delta_{ij} = d\beta_j/\alpha_i$, then for all i , $\sum_{j=1}^b \delta_{ij} D_{ij} = \sum_{j=1}^b d \frac{\beta_j}{\alpha_i} D_{ij} = d^2$.

(H2)⇔(H5): By Fact 4.8, the i -th component of the center-valued trace of e_1 is given by

$$\text{tr}_{(B,A)_i}(e_1 J p_i J) = \frac{1}{\sum_{j=1}^b \delta_{ij} \Delta_{ij}}.$$

Hence e_1 has scalar central trace if and only if $\sum_{j=1}^b \delta_{ij} \Delta_{ij}$ is the same scalar for all $1 \leq i \leq a$. Since $\text{tr}_{(B,A)}(e_1) = d^{-2}$, this scalar must be equal to d^{-2} .

(H6)⇔(H2): By (2.10) (cf. [20, (3.7.3.1)]),

$$d^2 \text{tr}_{(B,A)}(r_i) = \text{tr}_A(p_i) \sum_{j=1}^b D_{ij} \delta_{ij}.$$

Hence $\text{tr}_A(p_i) = \text{tr}_{(B,A)}(r_i)$ for all $1 \leq i \leq a$ if and only if (H2) holds.

(H4)⇔(H7): Since $A \subset B$ is extremal by Theorem 3.26, we have $D = \Delta$ and

$$\delta_{ij} = d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right) \quad \forall 1 \leq i \leq a \quad \forall 1 \leq j \leq b,$$

where tr_B is the unique Markov trace and $\text{tr}_A = \text{tr}_B|_A$. By uniqueness in Lemma 3.18 (2), $\delta_{ij} = d\beta_j/\alpha_i$ if and only if $\text{tr}_B(q_j) = \beta_j^2$ and $\text{tr}_A(p_i) = \alpha_i^2$ for all i, j .

(H5)⇔(H8): This is [57, Cor. 1.2.10]. ■

Corollary 4.17. *Let $A \subset B$ be extremal and fulfill the equivalent conditions of Theorem 4.16. Denote by $(A_n)_{n \geq 0}$ the Jones tower for $A = A_0 \subset B = A_1$. Then*

$$\delta_{ij}^{A_{2n} \subset A_{2n+1}} = \frac{d\beta_j}{\alpha_i} \quad \text{and} \quad \delta_{ji}^{A_{2n+1} \subset A_{2n+2}} = \frac{d\alpha_i}{\beta_j} \quad \forall n \geq 0, \tag{4.5}$$

i.e., the distortion matrices oscillate between two fixed matrices depending on the parity of n .

Proof. Immediate by (H4) ($\delta_{ij} = d\beta_j/\alpha_i$) and (4.2). ■

Example 4.18. Suppose $A \subset B$ is a connected inclusion of finite-dimensional von Neumann algebras. Denote by $\vec{\mu} = (\mu_i)_{i=1}^a \in \mathbb{N}^a$ and $\vec{v} = (v_j)_{j=1}^b \in \mathbb{N}^b$ the row vectors with μ_i^2 and v_j^2 equal to the algebraic dimensions of the full matrix algebras A_i and B_j . In this

case, the inclusion is always extremal [17, Cor. 3.2], and the Bratteli diagram (bipartite adjacency matrix) of the inclusion is equal to $\Delta = D$. The equality $\vec{\mu}D = \vec{v}$ corresponds to unitality of the inclusion $A \subset B$.

By [17, Prop. 4.4], $A \subset B$ is *super-extremal* if and only if $\vec{v}D^T = d^2\vec{\mu}$, so that $(d\vec{\mu} \ \vec{v})$ is a Frobenius–Perron eigenvector for right multiplication by

$$\begin{pmatrix} 0 & D \\ D^T & 0 \end{pmatrix}.$$

In particular, $d = \|\vec{v}\|_2/\|\vec{\mu}\|_2$ and the normalized Frobenius–Perron eigenvectors in this case are $\vec{\alpha} = \vec{\mu}/\|\vec{\mu}\|_2$ and $\vec{\beta} = \vec{v}/\|\vec{v}\|_2$. We then calculate from (4.5) the distortion matrices for the Jones tower:

$$\delta_{ij}^{A_{2n} \subset A_{2n+1}} = \frac{\nu_j}{\mu_i} \quad \text{and} \quad \delta_{ji}^{A_{2n+1} \subset A_{2n+2}} = \frac{\mu_i}{\nu_j} \quad \forall n \geq 0.$$

Example 4.19 ([57, Ex. 1.2.8]). Consider the finite-dimensional (and hence extremal) inclusion $P = \mathbb{C} \oplus \mathbb{C} \subset M_2(\mathbb{C}) \oplus \mathbb{C} = Q$ whose bipartite adjacency matrix and distortion matrices are given by

$$\Delta = D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.^6$$

The inclusion $A = P \otimes R \subset Q \otimes R = B$ does not admit any downward basic construction. Indeed, using Theorem 4.10, one easily verifies there is no strictly positive solution to

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi_1\delta_{11}\Delta_{11} + \pi_2\delta_{12}\Delta_{12} \\ \pi_1\delta_{21}\Delta_{21} + \pi_2\delta_{22}\Delta_{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 2\pi_1 \\ 2\pi_1 + \pi_2 \end{pmatrix}.$$

Taking the next two steps in the Jones tower $A_0 \subset A_1 \subset A_2 \subset A_3$, we get a Morita equivalent inclusion $A_2 \subset A_3$ with the same standard invariant which manifestly admits two downward basic constructions. One quickly observes these inclusions have different distortions:

$$\delta_{(A_0 L^2 A_{1A_1})} = \delta_{(P L^2 Q Q)} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad \delta_{(A_2 L^2 A_{3A_3})} = \begin{pmatrix} 5/2 & 3/2 \\ 5/3 & 1 \end{pmatrix}.$$

One calculates that for the Jones tower $(A_n)_{n \geq 0}$,

$$\delta_{(A_{2n} L^2 A_{2n+1A_{2n+1}})} = \begin{pmatrix} F_{2n+2}/F_{2n} & F_{2n+1}/F_{2n} \\ F_{2n+2}/F_{2n+1} & 1 \end{pmatrix} \xrightarrow{n \rightarrow \infty} \begin{pmatrix} \phi^2 & \phi \\ \phi & 1 \end{pmatrix},$$

where F_n is the n -th Fibonacci number ($F_0 = F_1 = 1$) and ϕ is the golden ratio.

⁶Initially, the $(1, 2)$ -entry of δ is not defined as $\Delta_{12} = 0$. This δ is the unique extension afforded by Lemma 3.18 (3).

Remark 4.20. In the previous example, although there is no downward basic construction, there is a two-step downward Markov tunnel as discussed in Remark 4.11. The projection $e_{11} \in M_2(\mathbb{C}) \oplus \mathbb{C}$ has central support $(1, 0)$, center-valued trace $(\pi_1, \pi_2) = (1/2, 0)$, and trace-preserving expectation $E_P(e_{11}) = d^{-2}$, giving a solution to

$$\sum_{j=1}^2 \pi_j \delta_{ij} \Delta_{ij} = 1 \quad \text{for } i = 1, 2.$$

Setting $N := \{e_{11}\}' \cap P \cong \mathbb{C}$ and $E_N(x) := d^{-2} E_P(e_{11} x e_{11})$, the inclusion $N \subset P$ has bipartite adjacency matrix and distortion matrix given by

$$\Delta^{N \subset P} = D^{N \subset P} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \delta^{N \subset P}.$$

There is again no strictly positive solution to

$$1 = \text{tr}_P^Z(p)_1 \cdot \delta_{11}^{N \subset P} \Delta_{11}^{N \subset P} + \text{tr}_P^Z(p)_2 \cdot \delta_{12}^{N \subset P} \Delta_{12}^{N \subset P} = \text{tr}_P^Z(p)_1 + \text{tr}_P^Z(p)_2$$

as the center-valued trace $\text{tr}_P^Z(p) \in Z(P)$ of a projection $p \in P = \mathbb{C}^2$ can only have entries in $\{0, 1\}$. Only $p := (1, 0)$ satisfies $E_N(p) = d^{-2}$ (one verifies $E_N((0, 1)) = 0$), which gives a further one-step Markov tunnel $M := \{p\}' \cap N = \mathbb{C}$ and $E_M(x) := d^{-2} E_N(p x p)$. Experts will identify $M \subset N \subset P \subset Q$ as exactly the first 4 algebras in the Temperley–Lieb–Jones A_4 subfactor planar algebra with $d = \phi$.

5. Classification of finite depth hyperfinite multifactor inclusions

In Section 5.1, we show that given an indecomposable unitary 2-shaded planar algebra \mathcal{P}_\bullet with scalar loop parameters, there is a finite index homogeneous connected hyperfinite II_1 multifactor inclusion $A \subset B$ whose standard invariant is $*$ -isomorphic to \mathcal{P}_\bullet . Then in Section 5.2, we determine how distortion varies under Morita equivalence of bimodules, and we use this to characterize all finite depth finite index connected hyperfinite II_1 multifactor inclusions.

5.1. Construction of multifactor inclusions

Suppose \mathcal{C} is an indecomposable unitary multifusion category. As in Definition 2.6, fix a 2-shading $1_{\mathcal{C}} = 1^+ \oplus 1^-$, and suppose $X = \bigoplus X_{ij} \in \mathcal{C}^{+-}$ generates \mathcal{C} . Let $D = D_X$, $d = d_X$, α , and β be as in Definition 2.7. Let $\mathcal{P}(X)_\bullet$ be the indecomposable unitary 2-shaded planar algebra constructed from (\mathcal{C}, X) from Section 2.4.

Theorem 5.1 (Existence of homogeneous inclusions). *Suppose $\mathcal{C}(X) \sim \mathcal{P}(X)_\bullet$ is a finite depth standard invariant. There exists a finite index homogeneous connected hyperfinite II_1 multifactor inclusion $A \subset B$ whose standard invariant is equivalent to $\mathcal{C}(X) \sim \mathcal{P}(X)_\bullet$. Hence by Theorem 4.16, the distortion $\delta_{(A L^2 B_B)}$ is the standard distortion σ with respect to X .*

Proof. Here, we give an outline of the proof, together with forward references for some technical lemmas whose proofs appear in Appendix A.

- (A) Suppose we are given (\mathcal{C}, X) where $X = 1^+ \otimes X \otimes 1^-$ generates \mathcal{C} and $1^+ \oplus 1^- = 1_{\mathcal{C}}$, but $1^+, 1^-$ are not necessarily simple. Choose the standard unitary dual functor \vee with respect to X from Definition 2.7. The loop parameters are both equal to d_X times the identities of 1^+ or 1^- depending on shading:

$$\begin{array}{c} \text{●} \end{array} = \text{coev}_X^* \circ \text{coev}_X = d_X \text{id}_{1^+}, \quad \begin{array}{c} \text{○} \end{array} = \text{ev}_X \circ \text{ev}_X^* = d_X \text{id}_{1^-}.$$

Moreover, for all i, j , we have (2.3):

$$\begin{array}{c} \text{○} \\ \text{○} \end{array} \begin{array}{c} \text{○} \\ \text{○} \end{array} = \frac{D_{ij} \beta_j}{\alpha_i} \begin{array}{c} \text{○} \\ \text{○} \end{array}, \quad \begin{array}{c} \text{○} \\ \text{○} \end{array} \begin{array}{c} \text{○} \\ \text{○} \end{array} = \frac{D_{ij} \alpha_i}{\beta_j} \begin{array}{c} \text{○} \\ \text{○} \end{array}.$$

There is a unique *spherical state*⁷ ψ on $\text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ given by

$$\psi(p_i) := \alpha_i^2, \quad \psi(q_j) := \beta_j^2 \tag{5.1}$$

which satisfies for all $c \in \mathcal{C}$ and $f : c \rightarrow c$, $\psi(\text{tr}_L^\vee(f)) = \psi(\text{tr}_R^\vee(f))$.

- (B) We now construct a tower of commuting squares of finite-dimensional von Neumann algebras together with a common Markov trace for canonical Jones projections. To do this, we define an alternating tensor product *to the left* by

$$\begin{aligned}
 \bar{X} \otimes X \otimes \cdots \otimes \bar{X} \otimes X &=: X^{\text{alt} \otimes 2n} \quad (2n \text{ tensorands}), \\
 X \otimes \bar{X} \otimes X \otimes \cdots \otimes \bar{X} \otimes X &=: X^{\text{alt} \otimes 2n+1} \quad (2n + 1 \text{ tensorands})
 \end{aligned}$$

We define $\bar{X}^{\text{alt} \otimes k}$ similarly. By convention, we define $X^{\text{alt} \otimes 0} = 1^-$ and $\bar{X}^{\text{alt} \otimes 0} := 1^+$.

For $n \geq 0$, we now define

$$\mathcal{Q}_{n,+} := \text{End}_{\mathcal{C}}(X^{\text{alt} \otimes n}) \quad \mathcal{Q}_{n,-} := \text{End}_{\mathcal{C}}(\bar{X}^{\text{alt} \otimes n}),$$

and observe we get a tower of commuting squares

$$\begin{array}{ccccccc}
 \cdots & \supset & \mathcal{Q}_{3,+} & \supset & \mathcal{Q}_{2,+} & \supset & \mathcal{Q}_{1,+} & \supset & \mathcal{Q}_{0,+} \\
 & & \cup & & \cup & & \cup & & \\
 \cdots & \supset & \mathcal{Q}_{2,-} & \supset & \mathcal{Q}_{1,-} & \supset & \mathcal{Q}_{0,-}. & &
 \end{array} \tag{5.2}$$

⁷Observe that the spherical state is normalized so that $\psi(\text{id}_{1_{\mathcal{C}}}) = 2$. However, this is the correct normalization so that $\psi(\text{id}_{X^{\text{alt} \otimes n}}) = 1 = \psi(\text{id}_{\bar{X}^{\text{alt} \otimes n}})$ for all n .

Above, horizontal inclusions are given by tensoring by the appropriate identity morphisms on the *left*, and vertical inclusions are given by tensoring by id_X on the *right*. Expanding the definitions, we have (5.2) is exactly

$$\begin{aligned} \cdots \supset \text{End}_{\mathcal{C}}(X \otimes \bar{X} \otimes X) &\supset \text{End}_{\mathcal{C}}(\bar{X} \otimes X) \supset \text{End}_{\mathcal{C}}(X) \supset \text{End}_{\mathcal{C}}(1^-) \\ &\cup \qquad \qquad \qquad \cup \qquad \qquad \qquad \cup \\ \cdots \supset \text{End}_{\mathcal{C}}(X \otimes \bar{X}) &\supset \text{End}_{\mathcal{C}}(\bar{X}) \supset \text{End}_{\mathcal{C}}(1^+). \end{aligned} \tag{5.3}$$

The Markov trace tr is given by the inductive limit of $d_X^{-n}(\psi \circ \text{tr}_L^{\vee})$ on $\mathcal{Q}_{n,+}$ where ψ is the spherical state from (5.1). The Jones projections $e_{n,\pm} \in \mathcal{Q}_{n+1,\pm}$ which implement the canonical trace-preserving conditional expectations $\mathcal{Q}_{n,\pm} \rightarrow \mathcal{Q}_{n-1,\pm}$ are given by

$$\begin{aligned} e_{2k+1,+} &:= \frac{1}{d} \cdot \left[\text{Diagram: a vertical line with two circles on the left side, one above and one below, and a vertical line on the right side labeled } 2k \text{ } \right] = d^{-1}(\text{ev}_X^* \circ \text{ev}_X) \otimes \text{id}_{X^{\text{alt}} \otimes 2k}, \\ e_{2k+2,+} &:= \frac{1}{d} \cdot \left[\text{Diagram: a vertical line with two circles on the left side, one above and one below, and a vertical line on the right side labeled } 2k+1 \text{ } \right] = d^{-1}(\text{coev}_X \circ \text{coev}_X^*) \otimes \text{id}_{X^{\text{alt}} \otimes 2k+1}, \\ e_{2k+1,-} &:= \frac{1}{d} \cdot \left[\text{Diagram: a vertical line with two circles on the right side, one above and one below, and a vertical line on the left side labeled } 2k \text{ } \right] = d^{-1}(\text{coev}_X \circ \text{coev}_X^*) \otimes \text{id}_{\bar{X}^{\text{alt}} \otimes 2k}, \\ e_{2k+2,-} &:= \frac{1}{d} \cdot \left[\text{Diagram: a vertical line with two circles on the right side, one above and one below, and a vertical line on the left side labeled } 2k+1 \text{ } \right] = d^{-1}(\text{ev}_X^* \circ \text{ev}_X) \otimes \text{id}_{\bar{X}^{\text{alt}} \otimes 2k+1}. \end{aligned}$$

- (C) Since \mathcal{C} is multifusion and X generates \mathcal{C} , by Lemma A.8 (see Example A.9) below, eventually these commuting squares are *nondegenerate* in the sense of Definition A.3 in Appendix A. We truncate the sequence of commuting squares (5.2) so that the first commuting square is nondegenerate.

$$\begin{aligned} M_1 = \mathcal{Q}_{2k+2,+} &\supset M_0 = \mathcal{Q}_{2k+1,+} \\ &\cup \qquad \qquad \qquad \cup \\ N_1 = \mathcal{Q}_{2k+1,-} &\supset N_0 = \mathcal{Q}_{2k,-}. \end{aligned} \tag{5.4}$$

Moreover, by Lemma A.8 (see Example A.9), the sequence of commuting squares after this point is isomorphic to the basic construction commuting square at each level. (Observe that it does not matter where we truncate as we will take inductive limits.)

- (D) The GNS construction with respect to tr_B gives an inductive limit inclusion

$$A := \varinjlim \mathcal{Q}_{n,-} \hookrightarrow \varinjlim \mathcal{Q}_{n,+} =: B,$$

of II_1 multifactors. By Lemma A.13 below, $Z(A) = N'_0 \cap N_0 = \text{End}(1^+)$ acting on the left, and $Z(B) = M'_0 \cap M_0 = \text{End}(1^-)$ acting on the right.

By the Ocneanu Compactness Theorem C.1, $A' \cap B = N'_1 \cap M_0$ inside $M_1 = \text{End}_{\mathcal{E}}(X^{\text{alt} \otimes 2k+2})$. By Corollary A.11 below, we see that

$$N'_1 \cap M_0 = \text{id}_{\bar{X}^{\text{alt} \otimes 2k+1}} \otimes \text{End}_{\mathcal{E}}(X) \cong \text{End}_{\mathcal{E}}(X).$$

Under this identification, the inclusions $Z(A), Z(B) \hookrightarrow N'_1 \cap M_0 = \text{End}_{\mathcal{E}}(X)$ are given respectively by

$$\begin{array}{|c|} \hline p \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline p \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline q \\ \hline \end{array} \mapsto \begin{array}{|c|} \hline q \\ \hline \end{array}.$$

It follows that $Z(A) \cap Z(B) = \mathbb{C}$, and the inclusion is connected.

- (E) Observe now that the sequence of commuting squares (5.3) fits into a doubly infinite lattice of commuting squares by alternately tensoring on the right by \bar{X} and X to obtain new rows above those in (5.3).

$$\begin{array}{cccc}
 \vdots & \vdots & \vdots & \vdots \\
 \cup & \cup & \cup & \cup \\
 \dots \supset \text{End}_{\mathcal{E}}(X^{\text{alt} \otimes 5}) \supset \text{End}_{\mathcal{E}}(X^{\text{alt} \otimes 4}) \supset \text{End}_{\mathcal{E}}(X^{\text{alt} \otimes 3}) \supset \text{End}_{\mathcal{E}}(\bar{X} \otimes X) & & & \\
 \cup & \cup & \cup & \cup \\
 \dots \supset \text{End}_{\mathcal{E}}(\bar{X}^{\text{alt} \otimes 4}) \supset \text{End}_{\mathcal{E}}(\bar{X}^{\text{alt} \otimes 3}) \supset \text{End}_{\mathcal{E}}(X \otimes \bar{X}) \supset \text{End}_{\mathcal{E}}(\bar{X}) & (5.5) & & \\
 \cup & \cup & \cup & \cup \\
 \dots \supset \text{End}_{\mathcal{E}}(X^{\text{alt} \otimes 3}) \supset \text{End}_{\mathcal{E}}(\bar{X} \otimes X) \supset \text{End}_{\mathcal{E}}(X) \supset \text{End}_{\mathcal{E}}(1^-) & & & \\
 \cup & \cup & \cup & \\
 \dots \supset \text{End}_{\mathcal{E}}(X \otimes \bar{X}) \supset \text{End}_{\mathcal{E}}(\bar{X}) \supset \text{End}_{\mathcal{E}}(1^+). & & &
 \end{array}$$

Moreover, truncating this lattice as in (5.4), by Lemma A.8 (see Example A.9), all the commuting squares on the left are nondegenerate, and obtained by iterated basic constructions.

$$\begin{array}{cccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \\
 \cup & \cup & \cup & \cup & \cup & \\
 A_3 \supset \dots \supset \mathcal{Q}_{2k+6,+} \supset \mathcal{Q}_{2k+5,+} \supset \mathcal{Q}_{2k+4,+} \supset \mathcal{Q}_{2k+3,+} & & & & & \\
 \cup & \cup & \cup & \cup & \cup & \\
 A_2 \supset \dots \supset \mathcal{Q}_{2k+5,-} \supset \mathcal{Q}_{2k+4,-} \supset \mathcal{Q}_{2k+3,-} \supset \mathcal{Q}_{2k+2,-} & (5.6) & & & & \\
 \cup & \cup & \cup & \cup & \cup & \\
 B = A_1 \supset \dots \supset \mathcal{Q}_{2k+4,+} \supset \mathcal{Q}_{2k+3,+} \supset \mathcal{Q}_{2k+2,+} \supset \mathcal{Q}_{2k+1,+} & & & & & \\
 \cup & \cup & \cup & \cup & \cup & \\
 A = A_0 \supset \dots \supset \mathcal{Q}_{2k+3,-} \supset \mathcal{Q}_{2k+2,-} \supset \mathcal{Q}_{2k+1,-} \supset \mathcal{Q}_{2k,-}. & & & & &
 \end{array}$$

- (F) Again by Lemma A.13 below, the inductive limit A_j of the j -th row of (5.6) is a finite direct sum of II_1 factors whose centers are isomorphic to $\text{End}_{\mathcal{C}}(1^+)$ or $\text{End}_{\mathcal{C}}(1^-)$ depending on the parity. Moreover, they come equipped with inductive limit traces tr_j coming from the normalized left traces on \mathcal{C} composed with the spherical state ψ . By the basic construction recognition lemma [41, Lem. 2.15], the tracial von Neumann algebras $(A_j, \text{tr}_j)_{j \geq 0}$ form a Jones tower where the Jones projections $f_j \in A_{j+1}$ such that $A_{j+1} = \langle A_j, f_j \rangle$ are given by

$$f_{2\ell+1} := \frac{1}{d} \cdot \left[\begin{array}{c} \text{Diagram: } 2k+2\ell \text{ strands, two circles} \end{array} \right] = d^{-1} \text{id}_{\bar{X}^{\text{alt}} \otimes (2k+2\ell)} \otimes (\text{coev}_X \circ \text{coev}_X^*) \in \mathcal{Q}_{2k+2\ell+2,-} \\ \subset A_{2\ell+2},$$

$$f_{2\ell+2} := \frac{1}{d} \cdot \left[\text{Diagram: } 2k+2\ell+1 \text{ strands, two circles} \right] = d^{-1} \text{id}_{X^{\text{alt}} \otimes (2k+2\ell+1)} \otimes (\text{ev}_X^* \circ \text{ev}_X) \in \mathcal{Q}_{2k+2\ell+3,+} \\ \subset A_{2\ell+3}.$$

- (G) For each $j \geq 2$, we can look at the composite sequence of commuting squares consisting of the 0-th and j -th rows of (5.6). Again by Lemma A.8 (see Example A.9), the first composite square is nondegenerate, and all subsequent commuting squares are obtained by iterating the basic construction. Again by the Ocneanu Compactness Theorem C.1, we have that

$$\mathcal{P}_{2n,+}^{ACB} := A'_0 \cap A_{2n} = \mathcal{Q}'_{2k+1,-} \cap \mathcal{Q}_{2k+2n,-},$$

$$\mathcal{P}_{2n+1,+}^{ACB} := A'_0 \cap A_{2n+1} = \mathcal{Q}'_{2k+1,-} \cap \mathcal{Q}_{2k+2n+1,+}$$

Similarly, looking at composite sequence of commuting squares consisting of the 1-st and j -th rows of (5.6) and applying the Ocneanu Compactness Theorem C.1, we have

$$\mathcal{P}_{2n,-}^{ACB} := A'_1 \cap A_{2n+1} = \mathcal{Q}'_{2k+2,+} \cap \mathcal{Q}_{2k+2n+1,+},$$

$$\mathcal{P}_{2n+1,-}^{ACB} := A'_1 \cap A_{2n+2} = \mathcal{Q}'_{2k+2,+} \cap \mathcal{Q}_{2k+2n+2,-}$$

- (H) By Corollary A.11, the map $\varphi_{n,+} := \text{id}_{(X \otimes \bar{X})^{\otimes k}} \otimes -$ which adds $2k$ strands to the left is a unital $*$ -algebra isomorphism

$$\mathcal{P}(X)_{2n,+} := \text{End}_{\mathcal{C}}((X \otimes \bar{X})^{\otimes n}) \rightarrow \mathcal{Q}'_{2k+1,-} \cap \mathcal{Q}_{2k+2n,-} = \mathcal{P}_{2n,+}^{ACB},$$

$$\mathcal{P}(X)_{2n+1,+} := \text{End}_{\mathcal{C}}((X \otimes \bar{X})^{\otimes n} \otimes X) \rightarrow \mathcal{Q}'_{2k+1,-} \cap \mathcal{Q}_{2k+2n+1,+} = \mathcal{P}_{2n+1,+}^{ACB},$$

which maps the Jones projections $f_n \in A'_0 \cap A_{n+1} = \mathcal{P}_{n+1,+}^{ACB}$ from (F) to $e_{n,+} \in \mathcal{P}(X)_{n+1,+}$ and is compatible with the right inclusion and the partial trace (conditional expectation).

Similarly, the map $\varphi_{n,-} := \text{id}_{(X \otimes \bar{X})^{\otimes k} \otimes X} \otimes -$ which adds $2k + 1$ strands to the left is a unital $*$ -algebra isomorphism:

$$\mathcal{P}(X)_{2n,-} := \text{End}_{\mathcal{C}}((\bar{X} \otimes X)^{\otimes n}) \rightarrow \mathcal{Q}'_{2k+2,+} \cap \mathcal{Q}_{2k+2n+1,+} = \mathcal{P}_{2n,-}^{ACB},$$

$$\mathcal{P}(X)_{2n+1,-} := \text{End}_{\mathcal{C}}((\bar{X} \otimes X)^{\otimes n} \otimes \bar{X}) \rightarrow \mathcal{Q}'_{2k+2,+} \cap \mathcal{Q}_{2k+2n+2,-} = \mathcal{P}_{2n+1,-}^{ACB}$$

which is compatible with the right inclusion and the left inclusion. Indeed, for all $y \in \mathcal{P}(X)_{n,-}$,

$$\varphi_{n+1,+}(\text{id}_X \otimes y) = \text{id}_{(X \otimes \bar{X}) \otimes k \otimes X} \otimes y = \varphi_{n,-}(y) \in \mathcal{P}_{n,-}^{ACB} \subset \mathcal{P}_{n+1,+}^{ACB}$$

as $A'_1 \cap A_{n+1} \subset A'_0 \cap A_{n+1}$.

To see that the φ_\bullet assemble into a planar $*$ -algebra isomorphism, by [41, Proof of Lem. 2.49], it suffices to check that capping on the left is compatible with $\varphi_{n,+}$. This argument (and indeed this entire part of the proof) is identical to [41, Proof of Thm. 4.1 (iii)]. For $n \geq 1$, the left capping map $\mathcal{P}_{n,+}^{ACB} \rightarrow \mathcal{P}_{n-1,-}^{ACB}$ is given by $x \mapsto d^{-1} \sum_b b x b^*$ [41, Thm. 250]. By non-degeneracy of the composite commuting squares of (5.6) and Proposition A.15, there is a Pimsner–Popa basis $\{b\}$ for $\mathcal{Q}_{2k+1,+} \subset B$ over $\mathcal{Q}_{2k,-} \subset A$ which is also a Pimsner–Popa basis for B over A . We now employ Vaughan Jones’ diagrammatic trick from [41, Proof of Thm. 4.1 (iii)]. For all $y \in \mathcal{P}(X)_{n,+}$ with $n \geq 1$,

$$\begin{aligned} \frac{1}{d} \sum_b b \cdot \varphi_{n,+}(y) \cdot b^* &= \frac{1}{d} \sum_b \begin{array}{c} \text{---} 2k \text{---} \\ | \\ \text{---} n-1 \text{---} \\ | \\ \text{---} 2k \text{---} \\ | \\ \text{---} n-1 \text{---} \end{array} \\ &= \frac{1}{d} \sum_b \begin{array}{c} \text{---} 2k \text{---} \\ | \\ \text{---} n-1 \text{---} \\ | \\ \text{---} 2k \text{---} \\ | \\ \text{---} n-1 \text{---} \end{array} \\ &= \begin{array}{c} 2k+1 \quad n-1 \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ \text{---} \quad \text{---} \\ | \quad | \\ 2k+1 \quad n-1 \end{array} = \varphi_{n-1,-} \left(\begin{array}{c} n-1 \\ | \\ \text{---} \\ | \\ n-1 \end{array} \right). \end{aligned}$$

Hence $\varphi_\bullet : \mathcal{P}(X)_\bullet \rightarrow \mathcal{P}_\bullet^{ACB}$ is a planar $*$ -algebra isomorphism.

- (I) We claim the distortion for ${}_A L^2 B_B$ is the standard distortion $\sigma_{ij} = d_X \beta_j / \alpha_i$. Indeed, since ${}_A L^2 B_B$ is finite depth and thus extremal, it suffices to prove this formula holds when $p_i q_j \neq 0$. In this case, by (2.6) above, we have $\delta_{ij} = D_{ij} / T_{ij}$. Now by (2.3) and (5.1),

$$\begin{aligned} T_{ij} = \text{tr}_{B_j}(p_i q_j) &= \frac{\text{tr}_B(p_i q_j)}{\text{tr}_B(q_j)} = \frac{d_X^{-1} \psi \left(\begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} \right)}{\beta_j^2} \\ &= \frac{d_X^{-1} \frac{D_{ij} \alpha_i}{\beta_j} \psi(q_j)}{\beta_j^2} = \frac{D_{ij} \alpha_i}{d_X \beta_j}. \end{aligned}$$

But combining (2.6), (3.1), and extremality of ${}_A L^2 B_B$, we also have $T_{ij} = \Delta_{ij} / \delta_{ij} = D_{ij} / \delta_{ij}$. The result follows. ■

Remark 5.2. Observe that the inclusion $A \subset B$ constructed above in the proof of Theorem 5.1 admits an infinite Jones tunnel. Indeed, By removing a copy of \bar{X} on the right, we can add another row below the doubly infinite lattice of commuting squares (5.5). The argument from step (F) above in the proof of Theorem 5.1 can be repeated for this new row to obtain a multifactor $A_{-1} \subset A_0$ such that A_1 is isomorphic to the basic construction algebra $\langle A_0, A_{-1} \rangle$. As one can keep removing tensor factors from the right, a simple induction argument shows that we may continue in this fashion to obtain an infinite Jones tunnel

$$\dots \subset A_{-2} \subset A_{-1} \subset A_0 \subset A_1 \subset A_2 \subset \dots.$$

5.2. Distortions realizable by multifactor inclusions

For this section, we assume both Notations 2.12 and 3.23 for a finite index inclusion $A \subset B$ of finite multifactors. Recall from Definition 2.27 in Section 2.5 that given an invertible $A' - A$ bimodule ${}_A Y_{A'}$ (so that $A' = (A^{\text{op}})' \cap B(Y)$), the Morita equivalent inclusion is $A' \subset B'$ where $B' := (B^{\text{op}})' \cap B(Z)$ and $Z_B := Y_A \boxtimes L^2 B_B$. Remember that the commutants of A and B are taken in different representations, so $A' \subset B'$ instead of the reverse inclusion.

Proposition 5.3. *Suppose $A \subset B$ is a finite index connected inclusion of finite multifactors, ${}_A Y_{A'}$ is a Morita equivalence bimodule, and $A' \subset B'$ is the induced Morita equivalent inclusion. Denote $\delta = \delta_{(A L^2 B_B)}$, and $\Delta = \Delta_{(A L^2 B_B)}$. The distortion δ' of $A' \subset B'$ is given by*

$$\delta'_{ij} = \delta_{ij} \rho_i^{-1} \sum_{h=1}^a \rho_h \Delta_{hj} \delta_{hj}^{-1}$$

where $\rho_i := \text{vNdim}_R(Y p_i A_i)$.

Proof. For $1 \leq i \leq a$, let $p'_i \in Z(A')$ be the minimal central projection corresponding to $p_i \in A$, and for $1 \leq j \leq b$, let $q'_j \in Z(B')$ be the minimal central projection in B' corresponding to $q_j \in B$. We calculate that

$$\begin{aligned} \text{vNdim}_L({}_{A'_i} p'_i L^2 B'_j q'_j) & \stackrel{[59, \text{Prop. 3.1}]}{=} \text{vNdim}_L({}_{A'_i} p'_i Y \boxtimes_A L^2 B \boxtimes_A \bar{Y} q'_j B'_j) \\ & = \text{vNdim}_L({}_{A'_i} Y p_i) \text{vNdim}_L({}_{A_i} p_i L^2 B \boxtimes_A \bar{Y} q'_j) \\ & = \rho_i^{-1} \text{vNdim}_R(q'_j Y \boxtimes_A L^2 B p_i A_i) \\ & = \rho_i^{-1} \text{vNdim}_R(Y \boxtimes_A L^2 B q_j p_i A_i) \\ & = \rho_i^{-1} \sum_{h=1}^a \text{vNdim}_R(Y p_h \boxtimes_{A_h} p_h L^2 B q_j p_i A_i) \\ & = \rho_i^{-1} \sum_{h=1}^a \text{vNdim}_R(Y p_h) \text{vNdim}_R(p_h L^2 B q_j p_i A_i) \\ & = \rho_i^{-1} \sum_{h=1}^a \rho_h \text{vNdim}_R(p_h L^2 B q_j \boxtimes_{B_j} q_j L^2 B p_i A_i) \end{aligned}$$

$$\begin{aligned}
 &= \rho_i^{-1} \sum_{h=1}^a \rho_h \text{vNdim}_R(p_h L^2 B q_j B_j) \text{vNdim}_R(q_j L^2 B p_i A_i) \\
 &= \rho_i^{-1} \sum_{h=1}^a \rho_h \frac{\Delta_{hj}}{\delta_{hj}} \text{vNdim}_L(A_i p_i L^2 B q_j) \quad (3.1) \\
 &= \rho_i^{-1} \sum_{h=1}^a \rho_h \frac{\Delta_{hj}}{\delta_{hj}} \delta_{ij} \Delta_{ij} \quad (3.1) \\
 &= \rho_i^{-1} \delta_{ij} \Delta_{ij} \sum_{h=1}^a \rho_h \frac{\Delta_{hj}}{\delta_{hj}}.
 \end{aligned}$$

By a similar calculation, we have

$$\text{vNdim}_R(p'_i L^2 B' q'_j B'_j) = \rho_i \delta_{ij}^{-1} \Delta_{ij} \left(\sum_{h=1}^a \rho_h \frac{\Delta_{hj}}{\delta_{hj}} \right)^{-1}.$$

We conclude that the distortion δ'_{ij} for $A' \subset B'$ is given by

$$\delta'_{ij} = \sqrt{\frac{\text{vNdim}_L(A'_i p'_i L^2 B' q'_j B'_j)}{\text{vNdim}_R(p'_i L^2 B' q'_j B'_j)}} = \delta_{ij} \rho_i^{-1} \sum_{h=1}^a \rho_h \Delta_{hj} \delta_{hj}^{-1}$$

as claimed. ■

Corollary 5.4. *Suppose $A \subset B$ and $A' \subset B'$ are as in the hypotheses of Proposition 5.3. If $A \subset B$ is extremal, then so is $A' \subset B'$.*

Proof. By Theorem 3.20, we must show $A' L^2 B' B'$ has constant distortion and that δ' satisfies (3.6). Observe that each of $A' Y_A$, $A L^2 B_B$, and $B' Z_B$ have constant distortion, so $A' L^2 B' B'$ has constant distortion by (2.19) and Corollary 3.4. Now to prove δ' satisfies (3.6), we use Lemma 3.18. Since $A \subset B$ is extremal, there are $\eta \in \mathbb{R}_{>0}^a$ and $\xi \in \mathbb{R}_{>0}^b$ (unique up to a simultaneous uniform scaling) such that $\delta_{ij} = \xi_j / \eta_i$. By Proposition 5.3, we have

$$\delta'_{ij} = \delta_{ij} \rho_i^{-1} \sum_{h=1}^a \rho_h \Delta_{hj} \delta_{hj}^{-1} = \frac{\xi_j \sum_{h=1}^a \rho_h \Delta_{hj} \delta_{hj}^{-1}}{\eta_i \rho_i} =: \frac{\xi'_j}{\eta'_i}.$$

Thus by Lemma 3.18 (2) again, δ' satisfies (3.6). ■

The next corollary follows immediately.

Corollary 5.5. *Suppose $A \subset B$ and $A' \subset B'$ are as in the hypotheses of Proposition 5.3. Moreover, assume $A \subset B$ is extremal and $X = A L^2 B_B$ has the standard distortion $\sigma_{ij} = d_X \beta_j / \alpha_i$. Then $A' \subset B'$ is extremal, and the distortion δ' is given by*

$$\delta'_{ij} = \alpha_i^{-1} \rho_i^{-1} \sum_{h=1}^a \rho_h D_{hj} \alpha_h.$$

Theorem 5.6. *Suppose $A \subset B$ is a finite index extremal connected II_1 multifactor inclusion. Then there is a Morita equivalence ${}_A Y_A$ such that the induced Morita equivalent inclusion $A' \subset B'$ is homogeneous. That is, $A' \subset B'$ has the standard distortion σ with respect to ${}_A L^2 B_B$.*

Proof. Let $\delta := \delta({}_A L^2 B_B)$. Since δ/σ satisfies (1.1), by Lemma 3.18 (2), there are $(\eta_i) \in \mathbb{R}_{>0}^a$ and $(\xi_j) \in \mathbb{R}_{>0}^b$ such that

$$\frac{\xi_j}{\eta_i} = \frac{\delta_{ij}}{\sigma_{ij}} \iff \eta_i \delta_{ij} = \xi_j \sigma_{ij}.$$

There exists a Morita equivalence ${}_A Y_A$ such that $\text{vNdim}_R(Y p_i A_i) = \eta_i^{-1}$. By Proposition 5.3, the distortion of the induced Morita equivalent inclusion $A' \subset B'$ is given by

$$\begin{aligned} \delta'_{ij} &= \delta_{ij} \eta_i \sum_{h=1}^a \eta_h^{-1} \delta_{hj}^{-1} D_{hj} = \xi_j \sigma_{ij} \sum_{h=1}^a \xi_j^{-1} \sigma_{hj}^{-1} D_{hj} = \sigma_{ij} \sum_{h=1}^a \sigma_{hj}^{-1} D_{hj} \\ &= \sigma_{ij} \sum_{h=1}^a \frac{\alpha_h}{d_X \beta_j} D_{hj} = \frac{\sigma_{ij}}{d_X \beta_j} \sum_{h=1}^a \alpha_h D_{hj} = \frac{\sigma_{ij}}{d_X \beta_j} d_X \beta_j = \sigma_{ij}. \end{aligned}$$

We conclude that $A' \subset B'$ has the standard distortion with respect to $X = {}_A L^2 B_B$, and thus this inclusion is homogeneous by Theorem 4.16. ■

Remark 5.7. Observe that Theorem 5.6 does not hold for $A \subset B$ finite-dimensional. Indeed, no inclusion of finite-dimensional von Neumann algebras with Bratteli diagram the A_4 Coxeter–Dynkin diagram can be homogeneous, since δ will always have rational entries, but the standard distortion function σ has irrational entries cf. Example 4.19.

Using Theorem 5.6, given a 2-shaded indecomposable unitary multifusion category \mathcal{C} and a generator $X \in \mathcal{C}^{+-}$, we can say exactly which distortions of X arise from realizations of $\mathcal{C}(X)$ as a standard invariant of a finite index connected inclusion of finite multifactors $A \subset B$.

Proposition 5.8. *Let \mathcal{C} be a 2-shaded indecomposable unitary multifusion category with generator $X \in \mathcal{C}^{+-}$ as in Definition 2.6, and let $D = D_X$, $d = d_X$, and α be as in Definition 2.7. Suppose $\delta : \{1, \dots, a\} \times \{1, \dots, b\} \rightarrow \mathbb{R}_{>0}$ is an arbitrary function satisfying (1.1), i.e.,*

$$\delta_{ij} \delta_{i'j'} = \delta_{ij'} \delta_{i'j} \quad \forall 1 \leq i, i' \leq a, \text{ and } 1 \leq j, j' \leq b.$$

The following are equivalent.

- (1) *There is a finite index connected inclusion of hyperfinite II_1 multifactors $A \subset B$ with standard invariant equivalent to \mathcal{C} such that $\delta = \delta({}_A L^2 B_B)$.*
- (2) *There exists $(\rho_i) \in \mathbb{R}_{>0}^a$ such that $\delta_{ij} = \alpha_i^{-1} \rho_i^{-1} \sum_{h=1}^a \rho_h D_{hj} \alpha_h$.*
- (3) *Writing $\delta_{ij} = \xi_j / \eta_i$ (which is unique up to uniformly scaling all ξ_j, η_i by Lemma 3.18(2)), we have $\xi_j = \sum_{h=1}^a \eta_h D_{hj}$.*

Proof. (1) \Rightarrow (2): Suppose $\delta = \delta_{(A L^2 B_B)}$ for some finite index connected inclusion $A \subset B$ of hyperfinite finite multifactors. By Corollary 3.22, $A \subset B$ is extremal, and by Theorem 5.6, it is Morita equivalent to an inclusion with standard distortion. Hence (2) follows by Corollary 5.5.

(2) \Rightarrow (1): By Theorem 5.1, there exists a finite index connected homogeneous hyperfinite II_1 multifactor inclusion $A \subset B$ with standard distortion. Let Y_A be any faithful right A -module such that $\rho_i = \text{vNdim}_R(Y p_{i A_i})$, and let $A' \subset B'$ be the induced Morita equivalent hyperfinite inclusion. By Corollary 5.5,

$$\delta'_{ij} = \alpha_i^{-1} \rho_i^{-1} \sum_{h=1}^a \rho_h D_{hj} \alpha_h$$

as desired.

(2) \Rightarrow (3): Setting $\xi_j := \sum_{h=1}^a \rho_h D_{hj} \alpha_h$ and $\eta_i := \alpha_i \rho_i$, we clearly have

$$\xi_j = \sum_{h=1}^a \eta_h D_{hj}.$$

(3) \Rightarrow (2): Setting $\rho_i := \eta_i / \alpha_i$ gives the desired formula for δ_{ij} . ■

Corollary 5.9. *Suppose $A \subset B$ is a finite depth finite index connected inclusion of finite multifactors and tr_B is the unique Markov trace on B . The distortion δ of ${}_A L^2 B_B$ is given by*

$$\delta_{ij} = \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right) \sum_{h=1}^a \left(\frac{\text{tr}_A(p_h)}{\alpha_h} \right) D_{hj}. \tag{1.2}$$

Proof. By Corollary 3.22, $A \subset B$ is extremal, so by Theorem 3.26,

$$\delta_{ij} = d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) \left(\frac{\alpha_i}{\text{tr}_A(p_i)} \right).$$

By Proposition 5.8 (3),

$$d \left(\frac{\text{tr}_B(q_j)}{\beta_j} \right) = \sum_{h=1}^a \left(\frac{\text{tr}_A(p_h)}{\alpha_h} \right) D_{hj}.$$

The result follows. ■

5.3. Classification of finite depth connected hyperfinite multifactor inclusions

We now recall Popa’s theorem for finite index finite depth homogeneous connected hyperfinite II_1 multifactor inclusions. We provide a proof here and in Appendix B for completeness and convenience of the reader.

Theorem 5.10. *Suppose $A \subset B$ and $\tilde{A} \subset \tilde{B}$ are two finite index homogeneous connected hyperfinite II_1 multifactor inclusions with isomorphic finite depth standard invariants. For every isomorphism $\varphi_\bullet : \mathcal{P}_\bullet^{A \subset B} \rightarrow \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}}$, there exists a (non-unique) $*$ -isomorphism $\varphi : B \rightarrow \tilde{B}$ taking A onto \tilde{A} which induces the original $*$ -isomorphism φ_\bullet of standard invariants.*

Proof. By Theorem B.9, there are generating tunnels $(A_{-n})_{n \in \mathbb{N}}$ and $(\tilde{A}_{-n})_{n \in \mathbb{N}}$. By tunnel/tower duality (see Fact B.1),⁸ φ_\bullet induces isomorphisms

$$\varphi_{n,+}^{\text{op}} : A'_{-n-1} \cap A \rightarrow \tilde{A}'_{-n-1} \cap \tilde{A}, \quad \varphi_{n,-}^{\text{op}} : A'_{-n} \cap B \rightarrow \tilde{A}'_{-n} \cap \tilde{B}$$

for all $n \in \mathbb{N}$ which are compatible with the inclusions, conditional expectations, and unique Markov traces. By the generating property, given this fixed generating tunnel, there is a unique unitary isomorphism

$$u_\varphi : L^2\left(\bigcup A'_{-n} \cap B, \text{tr}\right) \cong L^2 B \rightarrow L^2\left(\bigcup \tilde{A}'_{-n} \cap \tilde{B}, \text{tr}\right) \cong L^2 \tilde{B}$$

which intertwines the right $A'_{-n} \cap B$ and $\tilde{A}'_{-n} \cap \tilde{B}$ actions and the left $A'_{-n-1} \cap A$ and $\tilde{A}'_{-n-1} \cap \tilde{A}$ actions. We conclude that $\text{Ad}(u_\varphi)$ is the desired isomorphism. Since φ restricts to the isomorphisms $\varphi_{n,\pm}^{\text{op}}$ on the relative commutants of the Jones tunnel, again by tunnel/tower duality, φ induces the original $*$ -isomorphism φ_\bullet on the standard invariant. ■

Given two finite index extremal inclusions of finite multifactors $A \subset B$ and $\tilde{A} \subset \tilde{B}$ together with isomorphisms $\varphi_{0,+} : Z(A) \rightarrow Z(\tilde{A})$ and $\varphi_{0,-} : Z(B) \rightarrow Z(\tilde{B})$, we say that $\varphi_{0,\pm}$ preserves distortion if

$$\delta_{\varphi_{0,+}(p), \varphi_{0,-}(q)}^{\tilde{A} \subset \tilde{B}} = \delta_{p,q}^{A \subset B}$$

for all minimal projections $p \in Z(A)$ and $q \in Z(B)$. Here, we view $\delta^{A \subset B}$ and $\delta^{\tilde{A} \subset \tilde{B}}$ as matrices indexed by minimal projections rather than some enumerations of these projections.

Corollary 5.11. *Suppose $A \subset B$ and $\tilde{A} \subset \tilde{B}$ are two finite depth finite index connected hyperfinite II_1 multifactor inclusions. For every $*$ -isomorphism of standard invariants $\varphi_\bullet : \mathcal{P}_\bullet^{A \subset B} \rightarrow \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}}$ which preserves distortion, there exists a (non-unique) $*$ -isomorphism $\varphi : B \rightarrow \tilde{B}$ taking A onto \tilde{A} which induces the original $*$ -isomorphism φ_\bullet of standard invariants.*

Proof. Since φ_\bullet preserves distortion, using Theorem 5.6, there exist faithful right modules Y_A and $\tilde{Y}_{\tilde{A}}$ such that:

- $\text{vNdim}_R(Yp) = \text{vNdim}_R(\tilde{Y}\varphi(p))$ for every minimal projection $p \in Z(A)$, and
- the induced Morita equivalent inclusions $A' \subset B'$ and $\tilde{A}' \subset \tilde{B}'$ have the standard distortion, and are thus homogeneous by Theorem 4.16. (Recall that $A' := (A^{\text{op}})' \cap B(Y)$, $Z := Y \boxtimes_A L^2 B$, $B' := (B^{\text{op}})' \cap B(Z)$, and similarly for $\tilde{A}', \tilde{Z}, \tilde{B}'$.)

⁸We warn the reader that there are two numbering conventions for the Jones tunnel. If one sets $B = A_1$ and $A = A_0$ as we do in the main body of this article, then the box spaces of standard invariant $\mathcal{P}_{n,\pm}^{A \subset B}$ are given by $\mathcal{P}_{n,+}^{A \subset B} := A'_0 \cap A_n$ and $\mathcal{P}_{n,-}^{A \subset B} := A'_1 \cap A_{n+1}$, which are anti-isomorphic to $A'_{-n-1} \cap A_0$ and $A'_{-n} \cap A_1$ respectively. However, in Fact B.1, we set $B = B_0$ and $A = B_{-1}$, under which we get that $B'_0 \cap B_n$ is anti-isomorphic to $B'_0 \cap B_{-n}$. Thus while the first convention is more practical for discussing the standard invariant, this second convention is more practical for discussing tunnel/tower duality.

By the first bullet point above, there exists a right A -linear unitary $w : Y_A \rightarrow \tilde{Y}_{\varphi(A)}$. By Lemma 2.28, we have an isomorphism of standard invariants $\varphi'_\bullet : \mathcal{P}_\bullet^{A' \subset B'} \rightarrow \mathcal{P}_\bullet^{\tilde{A}' \subset \tilde{B}'}$ given by the following commutative diagram:

$$\begin{CD} \mathcal{P}_\bullet^{A \subset B} @<\cong><< \mathcal{P}({}_A L^2 B_B)_\bullet @>\mathcal{P}(Y, Z, \psi)_\bullet>> \mathcal{P}(A' L^2 B'_{B'})_\bullet @<\cong><< \mathcal{P}_\bullet^{A' \subset B'} \\ @VV\varphi_\bullet V @. @. @VV\varphi'_\bullet V \\ \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}} @<\cong><< \mathcal{P}(\tilde{A} L^2 \tilde{B}_{\tilde{B}})_\bullet @>\mathcal{P}(\tilde{Y}, \tilde{Z}, \tilde{\psi})_\bullet>> \mathcal{P}(\tilde{A}' L^2 \tilde{B}'_{\tilde{B}'})_\bullet @<\cong><< \mathcal{P}_\bullet^{\tilde{A}' \subset \tilde{B}'} \end{CD}$$

By the second bullet point above and Theorem 5.10, there exists an isomorphism

$$\varphi' : (A' \subset B') \rightarrow (\tilde{A}' \subset \tilde{B}')$$

such that $\mathcal{P}(\varphi'_\bullet) = \varphi'_\bullet$. By (2.18), the isomorphism $\mathcal{P}(L^2 \tilde{A}'_{\varphi'}, L^2 \tilde{B}'_{\varphi'}, \psi_{\varphi'})_\bullet$ fits in the following commutative diagram:

$$\begin{CD} \mathcal{P}_\bullet^{A \subset B} @<\cong><< \mathcal{P}({}_A L^2 B_B)_\bullet @>\mathcal{P}(Y, Z, \psi)_\bullet>> \mathcal{P}(A' L^2 B'_{B'})_\bullet @<\cong><< \mathcal{P}_\bullet^{A' \subset B'} \\ @VV\varphi_\bullet V @. @. @VV\mathcal{P}(L^2 \tilde{A}'_{\varphi'}, L^2 \tilde{B}'_{\varphi'}, \psi_{\varphi'})_\bullet V @VV\mathcal{P}(\varphi'_\bullet) = \varphi'_\bullet V \\ \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}} @<\cong><< \mathcal{P}(\tilde{A} L^2 \tilde{B}_{\tilde{B}})_\bullet @>\mathcal{P}(\tilde{Y}, \tilde{Z}, \tilde{\psi})_\bullet>> \mathcal{P}(\tilde{A}' L^2 \tilde{B}'_{\tilde{B}'})_\bullet @<\cong><< \mathcal{P}_\bullet^{\tilde{A}' \subset \tilde{B}'} \end{CD}$$

Now since A is the commutant of the right A' -action on ${}_A \bar{Y}_{A'}$ and B is the commutant of the right B' action on $\bar{Y} \boxtimes_{A'} L^2 B'_{B'} \cong \bar{Z}_{B'}$, we may invoke Proposition 2.29 with the roles of $A, B, \tilde{A}, \tilde{B}, Y, Z$ swapped with $A', B', \tilde{A}', \tilde{B}', \bar{Y}, \bar{Z}$ to get an isomorphism $\varphi : (A \subset B) \rightarrow (\tilde{A} \subset \tilde{B})$ such that the following diagram commutes:

$$\begin{CD} \mathcal{P}_\bullet^{A \subset B} @<\cong><< \mathcal{P}({}_A L^2 B_B)_\bullet @>\mathcal{P}(Y, Z, \psi)_\bullet>> \mathcal{P}(A' L^2 B'_{B'})_\bullet @<\cong><< \mathcal{P}_\bullet^{A' \subset B'} \\ @VV\varphi_\bullet V @. @. @VV\mathcal{P}(L^2 \tilde{A}_\varphi, L^2 \tilde{B}_\varphi, \psi_\varphi)_\bullet V @VV\mathcal{P}(L^2 \tilde{A}'_{\varphi'}, L^2 \tilde{B}'_{\varphi'}, \psi_{\varphi'})_\bullet V @VV\mathcal{P}(\varphi)_\bullet = \varphi'_\bullet V \\ \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}} @<\cong><< \mathcal{P}(\tilde{A} L^2 \tilde{B}_{\tilde{B}})_\bullet @>\mathcal{P}(\tilde{Y}, \tilde{Z}, \tilde{\psi})_\bullet>> \mathcal{P}(\tilde{A}' L^2 \tilde{B}'_{\tilde{B}'})_\bullet @<\cong><< \mathcal{P}_\bullet^{\tilde{A}' \subset \tilde{B}'} \end{CD}$$

Finally, by (2.18), the isomorphisms $\mathcal{P}(L^2 \tilde{A}_\varphi, L^2 \tilde{B}_\varphi, \psi_\varphi)_\bullet$ and $\mathcal{P}(\varphi)_\bullet$ satisfy a commutative square, and we conclude that $\mathcal{P}(\varphi)_\bullet = \varphi_\bullet$ as desired. ■

We now prove Theorem A in two parts.

Theorem 5.12 (Theorem A, Part I). *The map $A \subset B \mapsto (\mathcal{P}_\bullet^{A \subset B}, \text{tr}_B^{\text{Markov}} |_{Z(A)})$ descends to a well-defined bijection*

$$\frac{\left\{ \begin{array}{l} \text{Finite depth finite index connected} \\ \text{hyperfinite II}_1 \text{ multifactor inclu-} \\ \text{sions } A \subset B \end{array} \right\}}{\varphi : B_1 \xrightarrow{\sim} B_2 \text{ taking } A_1 \text{ onto } A_2} \cong \frac{\left\{ \begin{array}{l} \text{Pairs } (\mathcal{P}_\bullet, \tau) \text{ with } \mathcal{P}_\bullet \text{ a finite depth} \\ \text{indecomposable unitary 2-shaded planar} \\ \text{algebra and } \tau \text{ a faithful state on } \mathcal{P}_{0,+} \end{array} \right\}}{\varphi_\bullet : \mathcal{P}_\bullet^1 \xrightarrow{\sim} \mathcal{P}_\bullet^2 \text{ such that } \tau^2 \circ \varphi_{0,+} = \tau^1}.$$

Proof. Well-defined: Suppose $A \subset B$ and $\tilde{A} \subset \tilde{B}$ are finite index finite depth connected hyperfinite II_1 multifactor inclusions. Let tr_B and $\text{tr}_{\tilde{B}}$ denote the unique respective Markov traces, and denote by τ_A and $\tau_{\tilde{A}}$ their restrictions to $Z(A)$ and $Z(\tilde{A})$ respectively.

Suppose there is a $*$ -isomorphism $\varphi : B \rightarrow \tilde{B}$ taking A onto \tilde{A} . Then φ induces an isomorphism

$$\varphi_\bullet : \mathcal{P}_\bullet^{A \subset B} \rightarrow \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}}.$$

Moreover, by uniqueness of the Markov trace, we must have $\text{tr}_{\tilde{B}} \circ \varphi = \text{tr}_B$, so $\tau_{\tilde{A}} \circ \varphi_{0,+} = \tau_A$ on $\mathcal{P}_{0,+}^{A \subset B} = Z(A)$.

Injective: Now suppose $A \subset B$ and $\tilde{A} \subset \tilde{B}$ have isomorphic pairs, i.e., there is a planar $*$ -algebra isomorphism

$$\varphi_\bullet : \mathcal{P}_\bullet^{A \subset B} \rightarrow \mathcal{P}_\bullet^{\tilde{A} \subset \tilde{B}}$$

such that $\tau_A \circ \varphi_{0,+} = \tau_{\tilde{A}}$. Since the distortions $\delta^{A \subset B}$ and $\delta^{\tilde{A} \subset \tilde{B}}$ are determined by τ_A and $\tau_{\tilde{A}}$ respectively by (1.2), we see the isomorphism φ_\bullet preserves distortion. By Corollary 5.11, there is a $*$ -isomorphism $\varphi : B \rightarrow \tilde{B}$ which maps A onto \tilde{A} .

Surjective: Suppose \mathcal{P}_\bullet is a finite depth unitary 2-shaded planar algebra and τ is some faithful state on $\mathcal{P}_{0,+}$. By Theorem 5.1, there is a finite index homogeneous connected hyperfinite II_1 multifactor inclusion $A_0 \subset B_0$ whose standard invariant is $*$ -isomorphic to \mathcal{P}_\bullet . Taking $\eta_i := \tau(p_i)/\alpha_i$, by Proposition 5.8, there is a finite index connected hyperfinite II_1 inclusion $A \subset B$ with standard invariant $*$ -isomorphic to \mathcal{P}_\bullet with distortion $\delta_{ij} = \eta_i^{-1} \sum_{h=1}^a \eta_h D_{hj}$. By Proposition 5.8(3) and (1.2), the Markov trace tr_B on B restricts to τ on $Z(A) \cong \mathcal{P}_{0,+}$. ■

Theorem 5.13 (Theorem A, Part II). *The map $A \subset B \mapsto \mathcal{P}_\bullet^{A \subset B}$ descends to a well-defined bijection*

$$\frac{\left\{ \begin{array}{l} \text{Finite depth finite index connected} \\ \text{hyperfinite } \text{II}_1 \text{ multifactor inclusions} \\ A \subset B \end{array} \right\}}{\text{Morita equivalence}} \cong \frac{\left\{ \begin{array}{l} \text{Finite depth indecomposable} \\ \text{unitary 2-shaded planar algebras } \mathcal{P}_\bullet \end{array} \right\}}{\text{Planar } * \text{-algebra isomorphism}}.$$

Proof. Well-defined: Morita equivalent multifactor inclusions have isomorphic standard invariants by Lemma 2.28.

Injective: Suppose $A \subset B$ and $\tilde{A} \subset \tilde{B}$ are two finite depth finite index connected hyperfinite II_1 multifactor inclusions with isomorphic standard invariants. By Theorem 5.6 above, both $A \subset B$ and $\tilde{A} \subset \tilde{B}$ are Morita equivalent to homogeneous hyperfinite inclusions $A' \subset B'$ and $\tilde{A}' \subset \tilde{B}'$ respectively. By Popa’s uniqueness theorem (Theorem 5.10), there is a $*$ -algebra isomorphism $\varphi' : B' \rightarrow \tilde{B}'$ taking A' onto \tilde{A}' and inducing an isomorphism φ'_\bullet of standard invariants. Thus $L^2 \tilde{A}'_{\varphi'}$ is a Morita equivalence $\tilde{A}' - A'$ bimodule witnessing the Morita equivalence of the inclusions $A' \subset B'$ and $\tilde{A}' \subset \tilde{B}'$. We get our desired Morita equivalence by composing Morita equivalences:

$$(A \subset B) \sim (A' \subset B') \sim (\tilde{A}' \subset \tilde{B}') \sim (\tilde{A} \subset \tilde{B}).$$

Surjective: Surjectivity is immediate from the Existence Theorem 5.1. ■

6. Representations of unitary multifusion categories

For this section, \mathcal{C} denotes a unitary multitensor category, and R denotes the hyperfinite II_1 factor.

Definition 6.1 ([26, Def. 3.1]). A representation of \mathcal{C} is a tensor dagger functor $\alpha : \mathcal{C} \rightarrow \text{Bim}(M)$ for some von Neumann algebra M . Given two representations $\alpha : \mathcal{C} \rightarrow \text{Bim}(M)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(N)$, a morphism from α to β consists of an $N - M$ bimodule ${}_N\Phi_M$ together with a family of unitary natural isomorphisms

$$\{\phi_c : \Phi \boxtimes_M \alpha(c) \rightarrow \beta(c) \boxtimes_N \Phi\}_{c \in \mathcal{C}}$$

satisfying the following coherence axiom:

$$\begin{CD} \Phi \boxtimes_M \alpha(c) \boxtimes_M \alpha(d) @>\phi_c \boxtimes \text{id}>> \beta(c) \boxtimes_N \Phi \boxtimes_M \alpha(d) @>\text{id} \boxtimes \phi_d>> \beta(c) \boxtimes_N \beta(d) \boxtimes_N \Phi \\ @VV\text{id} \boxtimes \mu_{c,d}^\alpha V @. @VV\mu_{c,d}^\beta \boxtimes \text{id} V \\ \Phi \boxtimes_M \alpha(c \otimes d) @>\phi_{c \otimes d}>> \beta(c \otimes d) \boxtimes_N \Phi \end{CD} \tag{6.1}$$

We call such a morphism an *isomorphism* if ${}_N\Phi_M$ is an invertible $N - M$ bimodule.

Remark 6.2. As discussed further in [26, §5.4], there are additional structures for representations, like bi-involutivity and positivity, when \mathcal{C} has the corresponding structure. In the presence of such structures, one can ask for an additional coherence for representations. Since we are only interested in the case \mathcal{C} multifusion in this article, by [26, Thm. A], it suffices to restrict our attention to representations of unitary multitensor categories in the absence of any additional structures and coherences.

Definition 6.3. Suppose \mathcal{C} is an $n \times n$ unitary multitensor category, $\text{Irr}(\mathcal{C})$ a set of representatives of simple objects of \mathcal{C} , and $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ is a representation where R is a finite factor. The *modular distortion* of α is the function $\delta^\alpha : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{R}_{>0}$ given by $\delta^\alpha(c) := \delta_{(R\alpha(c)_R)}$.

Since the distortion is multiplicative for finite factor bimodules by (3.2), δ^α induces a grading on \mathcal{C} , and thus δ^α descends to a groupoid homomorphism $\delta^\alpha : \mathcal{U} \rightarrow \mathbb{R}_{>0}$ by universality.

When \mathcal{C} is unitary multifusion, \mathcal{U} is finite. Thus all groupoid homomorphisms $\mathcal{U} \rightarrow \mathbb{R}_{>0}$ factor uniquely through \mathcal{S}_n similar to Theorem 2.5. Hence we may identify δ^α with a groupoid homomorphism $\mathcal{S}_n \rightarrow \mathbb{R}_{>0}$.

Example 6.4. Suppose $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ is a representation with distortion δ^α . Since the fundamental group $\mathcal{F}(R)$ of the hyperfinite II_1 factor is $\mathbb{R}_{>0}$ [47], for any vector $(\lambda_i) \in \mathbb{R}_{>0}^n$, there is an invertible Morita equivalence $R^{\oplus n} - R^{\oplus n}$ bimodule Φ with $\lambda_i := \text{vNdim}_L(p_i\Phi) = \delta(p_i\Phi)$. Since distortion is multiplicative for II_1 factor bimodules by (3.2), for all $c \in \mathcal{C}_{ij}$,

$$\delta(p_i\Phi \boxtimes \alpha(c) \boxtimes \bar{\Phi} p_j) = \delta(p_i\Phi)\delta(\alpha(c))\delta(p_j\bar{\Phi}) = \lambda_i \delta_{ij}^\alpha \lambda_j^{-1}.$$

Then conjugating α by Φ yields a representation $\text{Ad}(\Phi) \circ \alpha : c \mapsto \Phi \boxtimes \alpha(c) \boxtimes \bar{\Phi}$ with distortion given by

$$\delta_{ij}^{\text{Ad}(\Phi) \circ \alpha} = \lambda_i \delta_{ij}^\alpha \lambda_j^{-1}. \tag{6.2}$$

Indeed, since Φ is invertible, by [60, Prop. 3.1] and [65, Thm. 4.7] (see also [5, §4] and [50, Cor. 3.34]), there is an $R^{\oplus n} - R^{\oplus n}$ bilinear unitary $u_\Phi : \bar{\Phi} \boxtimes_{R^{\oplus n}} \Phi \rightarrow L^2 R^{\oplus n}$, unique up to unique unitary automorphism of Φ , such that $(\bar{\Phi}, u_\Phi, \bar{u}_\Phi^*)$ exhibits $\bar{\Phi}$ as the unitary spherical dual of Φ . Denoting u_Φ by a cap, the tensorator

$$\mu_{a,b}^{\text{Ad}(\Phi) \circ \alpha} : \Phi \boxtimes \alpha(a) \boxtimes \bar{\Phi} \boxtimes \Phi \boxtimes \alpha(b) \boxtimes \bar{\Phi} \rightarrow \Phi \boxtimes \alpha(a \otimes b) \boxtimes \bar{\Phi}$$

is given by

$$\mu_{a,b}^{\text{Ad}(\Phi) \circ \alpha} := \begin{array}{c} \Phi \quad \alpha(a \otimes b) \quad \bar{\Phi} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \Phi \quad \alpha(a) \quad \bar{\Phi} \quad \Phi \quad \alpha(b) \quad \bar{\Phi} \end{array} = \text{id}_\Phi \boxtimes (\mu_{a,b}^\alpha \circ (\text{id}_{\alpha(a)} \boxtimes u_\Phi \boxtimes \text{id}_{\alpha(b)})) \boxtimes \text{id}_{\bar{\Phi}}.$$

Moreover, Φ induces a canonical isomorphism $(\Phi, \phi) : \alpha \rightarrow \text{Ad}(\Phi) \circ \alpha$ by defining

$$\phi_c := \begin{array}{c} \Phi \quad \alpha(c) \quad \bar{\Phi} \quad \Phi \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \Phi \quad \alpha(c) \end{array} = \text{id}_\Phi \boxtimes \text{id}_{\alpha(c)} \boxtimes u_\Phi^* : \Phi \boxtimes \alpha(c) \rightarrow \Phi \boxtimes \alpha(c) \boxtimes \bar{\Phi} \boxtimes \Phi.$$

6.1. Existence of representations of unitary multifusion categories

Let \mathcal{C} be an $n \times n$ unitary multifusion category, let R be a hyperfinite II_1 factor, and denote the minimal central projections of $R^{\oplus n}$ by $\{p_1, \dots, p_n\}$. It follows directly from Theorem 5.1 that representations of unitary multifusion categories exist.

Proposition 6.5. *Let \mathcal{C} be an $n \times n$ unitary multifusion category. There exists a representation $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$.*

Proof. We may reduce to the case that $n \geq 2$ by replacing \mathcal{C} with $\text{Mat}_2(\mathcal{C})$ if $n = 1$. Let $1_{\mathcal{C}} = 1^+ \oplus 1^-$ be a non-trivial 2-shading of \mathcal{C} , where $a = \dim(\text{End}_{\mathcal{C}}(1^+)) \geq 1$, $b = \dim(\text{End}_{\mathcal{C}}(1^-)) \geq 1$, and $n = a + b$. Pick an arbitrary generator $X \in \mathcal{C}^{+-}$ (e.g., we may take $X = \bigoplus_{c \in \text{Irr}(\mathcal{C}^{+-})} c$). By Theorem 5.1, there is a finite index homogeneous connected hyperfinite II_1 multifactor inclusion $A = R^{\oplus a} \subset R^{\oplus b} = B$ such that $\mathcal{C}({}_A L^2 B_B)$ is equivalent to $\mathcal{C}(X)$. The equivalence $\alpha : \mathcal{C}(X) \hookrightarrow \mathcal{C}({}_A L^2 B_B)$ is such a representation. ■

Proposition 6.6. *Suppose \mathcal{C} is an $n \times n$ unitary multifusion category and $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ is a representation. For any groupoid homomorphism $\delta : \mathcal{G}_n \rightarrow \mathbb{R}_{>0}$, there is an invertible bimodule Φ inducing an isomorphism $(\Phi, \phi) : \alpha \rightarrow \text{Ad}(\Phi) \circ \alpha$ such that $\delta^{\text{Ad}(\Phi) \circ \alpha} = \delta$.*

Proof. Since δ^α and δ are groupoid homomorphisms $\mathcal{E}_n \rightarrow \mathbb{R}_{>0}$, so is the ratio δ^α/δ . By Corollary 3.19, there exists $(\lambda_i) \in \mathbb{R}_{>0}^n$, unique up to uniform scaling, such that $\delta_{ij}^\alpha/\delta_{ij} = \lambda_j/\lambda_i$ for all $1 \leq i, j \leq n$. As discussed in Example 6.4, there is an invertible $R^{\oplus n} - R^{\oplus n}$ bimodule Φ with $\lambda_i = \delta(p_i \Phi)$ for all $1 \leq i \leq n$. By (6.2), the representation

$$\text{Ad}(\Phi) \circ \alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$$

has distortion δ as desired. ■

The next corollary now follows immediately by combining Propositions 6.5 and 6.6.

Corollary 6.7. *Suppose \mathcal{C} is an $n \times n$ unitary multifusion category. For any groupoid homomorphism $\delta : \mathcal{E}_n \rightarrow \mathbb{R}_{>0}^n$, there is a unitary tensor functor $\alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ with distortion $\delta^\alpha = \delta$.*

Remark 6.8. The realization result for unitary multifusion categories obtained in Proposition 6.5 appears also in [19, Cor. 3.10]. The proof in [19] follows different ideas and techniques, namely it is shown that every unitary multifusion category \mathcal{C} is equivalent to the category of special bimodules over a fixed standard C^* -Frobenius algebra A in a unitary fusion category, the latter can be realized in $\text{Bim}(R)$, and the realization of \mathcal{C} into $\text{Bim}(R^{\oplus n})$ is obtained by considering the non-factorial extension of R given by A , cf. Theorem 5.1. In [19] it is also shown that every unitary multitensor category, with no restriction on the cardinality of the set of isomorphism classes of simple objects by [18], and every rigid C^* 2-category with finitely decomposable horizontal units can be realized as well into $\text{Bim}(N^{\oplus n})$ where N is a II_1 factor, not necessarily hyperfinite.

6.2. Uniqueness of representations of unitary multifusion categories

In [31, Thm. 2.2], Izumi adapts Popa’s uniqueness theorem for finite depth hyperfinite III_1 subfactors [57, Cor. 6.11] to prove uniqueness of representations of unitary fusion categories as endomorphisms of the hyperfinite III_1 factor. We now adapt Izumi’s proof using Corollary 5.11 to the setting of representations of an $n \times n$ unitary multifusion category \mathcal{C} into $\text{Bim}(R^{\oplus n})$ where R is the hyperfinite II_1 factor.

Definition 6.9. Suppose \mathcal{C} be an $n \times n$ unitary multifusion category, A is a II_1 multifactor with n -dimensional center, $\alpha : \mathcal{C} \rightarrow \text{Bim}(A)$ is a representation, and $\varphi : A \rightarrow B$ is a $*$ -isomorphism. Consider the $B - A$ bimodule $L^2 B_\varphi$ with right A -action transported via the isomorphism φ . The representation induced by φ is $\text{Ad}(L^2 B_\varphi) \circ \alpha : \mathcal{C} \rightarrow \text{Bim}(B)$, and we call $L^2 B_\varphi$ the isomorphism induced by φ from α to $\text{Ad}(L^2 B_\varphi) \circ \alpha$. Since $\delta(L^2 B_\varphi)$ is the $n \times n$ identity matrix, we see that $\text{Ad}(L^2 B_\varphi) \circ \alpha$ and α have the same distortion by (6.2). Hence an isomorphism induced by a $*$ -algebra isomorphism can never change the distortion.

Theorem 6.10. *Suppose $\alpha : \mathcal{C} \rightarrow \text{Bim}(A)$ and $\beta : \mathcal{C} \rightarrow \text{Bim}(B)$ are two representations with $\delta^\alpha = \delta^\beta$, where A and B are both hyperfinite type II_1 multifactors with n -dimensional centers. Then there is an isomorphism (Φ, ϕ) from α to β which is induced by a $*$ -algebra isomorphism $\varphi : B \rightarrow A$.*

Proof. Fix an arbitrary generator $X \in \mathcal{C}$ such that every object of \mathcal{C} is isomorphic to a direct sum of summands of tensor powers of $X \otimes \bar{X}$ (e.g., we may take $X = \bigoplus_{c \in \text{Irr}(\mathcal{C})} c$). Consider the full subcategory $\tilde{\mathcal{C}} \subset \mathcal{C}$ whose objects are the tensor powers of $X \otimes \bar{X}$. To construct an equivalence from α to β , it suffices to construct an equivalence (Φ, ϕ) from $\alpha|_{\tilde{\mathcal{C}}} \rightarrow \beta|_{\tilde{\mathcal{C}}}$. The result will then follow by idempotent completion.

Define M to be the commutant of the right A -action on $\alpha(X)$, and define N to be the commutant of the right B -action on $\beta(X)$. By [60, Prop. 3.1], $\alpha(X)$ is an invertible $M - A$ bimodule, and $\beta(X)$ is an invertible $N - B$ bimodule. Since $\alpha(X)$ and $\beta(X)$ are dualizable, we immediately have that $A \subset M$ and $B \subset N$ are finite index multifactor inclusions. Since X generates \mathcal{C} , both these inclusions are connected.

Consider the amplification $\text{Mat}_2(\mathcal{C})$ with generator X_{12} corresponding to X in the component \mathcal{C}_{12} , i.e.,

$$\text{Mat}_2(\mathcal{C}) = \begin{pmatrix} \mathcal{C} & \mathcal{C} \\ \mathcal{C} & \mathcal{C} \end{pmatrix} \quad X_{12} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Denote by $\mathcal{C}(X_{12})$ the abstract standard invariant and $\mathcal{P}(X_{12})_\bullet$ the associated unitary 2-shaded planar algebra.

We now construct a distinguished isomorphism between the standard invariants of the two multifactor inclusions $\mathcal{C}({}_A L^2 M_M)$ and $\mathcal{C}({}_B L^2 N_N)$ which passes through $\mathcal{C}(X_{12})$. First, observe that amplifying α, β gives representations

$$\text{Mat}_2(\alpha) : \mathcal{C}(X_{12}) \rightarrow \text{Bim}(A \oplus A) \quad \text{and} \quad \text{Mat}_2(\beta) : \mathcal{C}(X_{12}) \rightarrow \text{Bim}(B \oplus B)$$

whose $++$ corners may be identified with α and β respectively. Moreover, $\delta^{\text{Mat}_2(\alpha)} = \delta^{\text{Mat}_2(\beta)}$ by construction. We now compose these representations with $\text{Ad}(L^2 A \oplus \alpha(X))$ and $\text{Ad}(L^2 B \oplus \beta(X))$ respectively to get representations $\tilde{\alpha} : \mathcal{C}(X_{12}) \rightarrow \text{Bim}(A \oplus M)$ and $\tilde{\beta} : \mathcal{C}(X_{12}) \rightarrow \text{Bim}(B \oplus N)$. That is,

$$\tilde{\alpha} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \alpha(a) & \alpha(b) \boxtimes_A \overline{\alpha(X)}_M \\ {}_M \alpha(X) \boxtimes_A \alpha(c) & {}_M \alpha(X) \boxtimes_A \alpha(d) \boxtimes_A \overline{\alpha(X)}_M \end{pmatrix} \in \text{Bim}(A \oplus M)$$

and similarly for $\tilde{\beta}$. Since $\delta^\alpha = \delta^\beta$ and $\delta^{\text{Mat}_2(\alpha)} = \delta^{\text{Mat}_2(\beta)}$, we also have $\delta^{\tilde{\alpha}} = \delta^{\tilde{\beta}}$. Observe now that $\tilde{\alpha}(X) = {}_A \alpha(X) \boxtimes_A \overline{\alpha(X)}_M$ which is canonically isomorphic to ${}_A L^2 M_M$ by [60, Prop. 3.1]; similarly, $\tilde{\beta}(X) \cong {}_B L^2 N_N$. We thus have the following zig-zag of isomorphisms of standard invariants:

$$\mathcal{C}({}_A L^2 M_M) \xleftarrow{\tilde{\alpha}} \mathcal{C}(X_{12}) \xrightarrow{\tilde{\beta}} \mathcal{C}({}_B L^2 N_N).$$

Inverting $\tilde{\alpha}$ on its essential image $\mathcal{C}({}_A L^2 M_M)$ and passing to planar algebras gives us a distinguished isomorphism $\varphi_\bullet : \mathcal{P}_\bullet^{A \subset M} \rightarrow \mathcal{P}_\bullet^{B \subset N}$ which preserves distortion. Since φ_\bullet came from a zig-zag of isomorphisms, we have $\varphi_{n,\pm}(\tilde{\alpha}(f)) = \tilde{\beta}(f)$ for every $f \in \mathcal{P}(X)_{n,\pm}$. Looking at the principal even part, we have $\varphi_{2n,+}(\alpha(f)) = \beta(f)$ for all $f \in \text{End}_{\tilde{\mathcal{C}}}((X \otimes \bar{X})^{\otimes n})$.

By Corollary 5.11, there is a $*$ -isomorphism $\varphi : M \rightarrow N$ taking A onto B which induces the above isomorphism of standard invariants $\mathcal{P}_{\bullet}^{A \subset M} \cong \mathcal{P}_{\bullet}^{B \subset N}$. Since the functor from the 1-groupoid of inclusions to the 1-groupoid of standard invariants factors through the 2-groupoid of standard bimodules as in (2.13), the invertible 1-morphism $(L^2 B_{\varphi}, L^2 N_{\varphi}, \psi_{\varphi}) : {}_A L^2 N_N \rightarrow {}_B L^2 M_M$ produces the isomorphism of standard invariants φ_{\bullet} by the ‘encircling’ action. Here, the isomorphism $\psi_{\varphi} : {}_B L^2 B_{\varphi} \boxtimes_A L^2 M_M \rightarrow {}_B L^2 N \boxtimes_N L^2 N_{\varphi}$ is as in (2.17).

In more detail, denote the four von Neumann algebras A, B, M, N by the shaded regions

$$\text{○} = A, \quad \text{●} = M, \quad \text{◐} = B, \quad \text{◑} = N,$$

and the standard and Morita equivalence bimodules by

$$\text{◐} \text{---} \text{◑} = {}_A L^2 M_M, \quad \text{◑} \text{---} \text{◐} = {}_B L^2 N_N, \quad \text{◐} \text{---} \text{○} = {}_B L^2 B_{\varphi(A)}, \quad \text{○} \text{---} \text{◑} = {}_N L^2 N_{\varphi(M)}.$$

We denote the conjugate bimodules by the horizontal reflection, and the restriction to A, \bar{A} respectively by changing the shading. We abbreviate the isomorphisms $\psi_{\varphi}, \bar{\psi}_{\varphi}, \psi_{\varphi}^*, \bar{\psi}_{\varphi}^*$ by 4-valent vertices:

$$\begin{aligned} \begin{array}{c} \diagup \\ \diagdown \end{array} &:= \begin{array}{c} \text{◐} \\ \psi_{\varphi} \\ \text{◑} \end{array}, & \begin{array}{c} \diagdown \\ \diagup \end{array} &:= \begin{array}{c} \text{◑} \\ \bar{\psi}_{\varphi} \\ \text{◐} \end{array}, \\ \begin{array}{c} \diagdown \\ \diagup \end{array} &:= \begin{array}{c} \text{◑} \\ \psi_{\varphi}^* \\ \text{◐} \end{array}, & \begin{array}{c} \diagup \\ \diagdown \end{array} &:= \begin{array}{c} \text{◐} \\ \bar{\psi}_{\varphi}^* \\ \text{◑} \end{array}. \end{aligned}$$

We illustrate the ‘encircling’ action for $f \in \tilde{\mathcal{C}}((X \otimes \bar{X})^{\otimes 1} \rightarrow (X \otimes \bar{X})^{\otimes 2})$:

$$\beta(f) = \begin{array}{c} \text{○} \\ \alpha(f) \\ \text{○} \end{array} = \tilde{\mathcal{C}}(\varphi)(\alpha(f)).$$

We now define ${}_B \Phi_A := {}_B L^2 B_{\varphi(A)}$ and

$$\begin{aligned} \phi &:= \phi_{X \otimes \bar{X}} := \begin{array}{c} \text{○} \\ \text{---} \\ \text{○} \end{array} \\ &:= \begin{array}{c} \text{◐} \\ \psi_{\varphi} \\ \text{---} \\ \psi_{\varphi} \\ \text{◑} \end{array} : {}_B \Phi \boxtimes_A \underbrace{\alpha(X \otimes \bar{X})}_{{}_A L^2 M_A} \xrightarrow{\sim} {}_B \underbrace{\beta(X \otimes \bar{X})}_{{}_B L^2 N_B} \boxtimes_B \Phi_A. \end{aligned}$$

For $n \geq 2$, we define $\phi_n = \phi_{(X \otimes X)^{\otimes n}}$ by concatenating n copies of ϕ along the blue string, e.g.,

$$\phi_3 = \begin{array}{c} \text{Diagram: A blue string passing through three vertical grey bars.} \end{array} : {}_B \Phi \boxtimes_A \alpha((X \otimes \bar{X})^{\otimes 3})_A \xrightarrow{\sim} {}_B \beta((X \otimes \bar{X})^{\otimes 3}) \boxtimes_B \Phi_A.$$

Now since $(X \otimes \bar{X})^{\otimes k} \otimes (X \otimes \bar{X})^{\otimes n} = (X \otimes \bar{X})^{\otimes(k+n)}$ suppressing associators, the coherence axiom (6.1) automatically holds. Naturality immediately follows by the recabling relations (2.14). ■

We now prove Theorem B in two parts.

Theorem 6.11 (Theorem B, Part I). *Let R be either the hyperfinite II_1 or II_∞ factor. The map $\alpha \mapsto \delta^\alpha$ descends to a bijection*

$$\frac{\{\text{Representations } \alpha : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})\}}{\text{Iso}(\Phi, \phi) \text{ induced by } \varphi \in \text{Aut}(R^{\oplus n})} \cong \{\text{Groupoid homomorphisms } \delta : \mathcal{E}_n \rightarrow \mathbb{R}_{>0}\}.$$

Proof. Well-defined: As discussed in Definition 6.9 above, isomorphisms induced by $*$ -algebra isomorphisms must preserve the distortion.

Injective: This follows immediately from Theorem 6.10.

Surjective: This follows immediately from Corollary 6.7. ■

Theorem 6.12 (Theorem B, Part II). *Let R be either the hyperfinite II_1 or II_∞ factor. Suppose $\alpha, \beta : \mathcal{C} \rightarrow \text{Bim}(R^{\oplus n})$ are two arbitrary representations. Then there is an isomorphism (Φ, ϕ) from α to β .*

Proof. First, since R and $B(\ell^2) \otimes R$ are Morita equivalent, $\text{Bim}(R^{\oplus n})$ and $\text{Bim}((B(\ell^2) \otimes R)^{\oplus n})$ are equivalent. Hence we may assume A and B are both II_1 multifactors. Second, by Proposition 6.6, by conjugating α and β by appropriate invertible $A - A$ and $B - B$ bimodules respectively, we may assume that $\delta^\alpha = \delta^\beta$. The result now follows by Theorem 6.10. ■

A. Commuting squares of finite index finite multifactors

Consider a quadrilateral of unital finite index inclusions of finite multifactors

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{array} \tag{A.1}$$

together with a faithful tracial state tr on M . Let $E_N^M : M \rightarrow N$, $E_P^M : M \rightarrow P$, and $E_Q^M : M \rightarrow Q$ be the canonical trace-preserving conditional expectations, where N and P are considered as subalgebras of M .

Definition A.1. The quadrilateral (A.1) is called a *commuting square* if

$$E_N^M E_P^M = E_P^M E_N^M = E_Q^M.$$

A.1. Nondegeneracy

Recall that by [20, Thm. 3.6.4 (i)], given a finite index inclusion of finite multifactors $N \subset M$ together with a faithful normal trace tr on M and the unique trace-preserving conditional expectation, there exists a (finite) Pimsner–Popa basis for M over N .

Lemma A.2 (cf. [55, Prop. in §1.1.5]). *For a commuting square of finite index inclusions of finite multifactors (A.1) the following are equivalent:*

- (N1) Every Pimsner–Popa basis for P over Q is also a Pimsner–Popa basis for M over N .
- (N2) Every Pimsner–Popa basis for N over Q is also a Pimsner–Popa basis for M over P .
- (N3) There is a Pimsner–Popa basis for P over Q which is a Pimsner–Popa basis for M over N .
- (N4) There is a Pimsner–Popa basis for N over Q which is a Pimsner–Popa basis for M over P .
- (N5) $M = \text{span } PN$.
- (N6) $M = \text{span } NP$.

Proof. Clearly (N1) \Rightarrow (N3) \Rightarrow (N5) and (N2) \Rightarrow (N4) \Rightarrow (N6), and obviously (N5) \Leftrightarrow (N6).

(N5) \Rightarrow (N1): Suppose $\{b\}$ is a Pimsner–Popa basis for P over Q , and let $x \in M$. Since $M = \text{span } PN$, we can write $x = \sum_{i=1}^n p_i n_i$. We then calculate

$$\begin{aligned} \sum_b b E_N(b^* x) &= \sum_b \sum_{i=1}^n b E_N(b^* p_i n_i) = \sum_b \sum_{i=1}^n b E_N(b^* p_i) n_i \\ &= \sum_{i=1}^n \sum_b b E_Q(b^* p_i) n_i = \sum_{i=1}^n p_i n_i = x. \end{aligned}$$

(N6) \Rightarrow (N2): This follows by an argument similar to the above swapping the roles of P and N . ■

Definition A.3 (cf. [55, §1.1.5]). A commuting square of finite index finite multifactors (A.1) is called *nondegenerate* if the equivalent conditions of Lemma A.2 hold.

Lemma A.4 (cf. [54, Proof of Lem. 6.1]). *Suppose we have a commuting square of finite index inclusions of finite multifactors as in (A.1) above such that $N \subset (M, \text{tr})$ is Markov with index d^2 . In this setting, nondegeneracy of (A.1) is equivalent to*

- (N7) The inclusion $Q \subset (P, \text{tr}|_P)$ is also Markov with index d^2 .

Proof. The canonical map $\langle P, Q \rangle = Pe_Q^P P \rightarrow \langle M, N \rangle = Me_N^M M$ by $ae_Q^P b \mapsto ae_N^M b$ is a well-defined (possibly non-unital) $*$ -homomorphism by the commuting square condition. This homomorphism preserves the canonical commutant trace on $\langle P, Q \rangle$ given by $ae_Q^P b \mapsto \text{tr}(ab)$. Hence the inclusion $Q \subset (P, \text{tr}|_P)$ is always Markov. However, it may be the case that the Markov index of $Q \subset (P, \text{tr}|_P)$ is strictly less than d^2 . The image of $1 \in \langle P, Q \rangle$ in $\langle M, N \rangle$ under the canonical map is an orthogonal projection in $\langle M, N \rangle$. We see that this projection is equal to $1 \in \langle M, N \rangle$ if and only if (N3) holds. But we also have that this projection is equal to $1 \in \langle M, N \rangle$ if and only if the Markov index of $Q \subset (P, \text{tr}|_P)$ is equal to d^2 . The result follows. ■

Facts A.5. We have the following facts concerning commuting squares.

- (Basic construction of commuting squares cf. [54] and [55, §1.1.6]) Suppose we have a nondegenerate commuting square of finite index finite multifactors as in (A.1), equipped with the unique Markov trace tr_M for the inclusion $N \subset M$. Consider the basic construction commuting square

$$\begin{array}{ccc} M & \subset & \langle M, N \rangle \\ \cup & & \cup \\ P & \subset & \langle P, Q \rangle \end{array} \tag{A.2}$$

with the canonical Markov trace $\text{tr}_{\langle M, N \rangle}$ on $\langle M, N \rangle$ where the inclusion $\langle P, Q \rangle \hookrightarrow \langle M, N \rangle$ is given by the canonical map

$$ae_Q^P b \mapsto ae_N^M b.$$

By nondegeneracy, there is a Pimsner–Popa basis for P over Q which is also a Pimsner–Popa basis for M over N , which implies the inclusion $\langle P, Q \rangle \hookrightarrow \langle M, N \rangle$ is unital. Viewing all algebras as subalgebras of $\langle M, N \rangle$ thus identifies e_Q^P with e_N^M . Thus $\langle P, Q \rangle = \text{span } Pe_N^M P$, so

$$\text{span } M \langle P, Q \rangle = \text{span } MPe_N^M P = \text{span } Me_N^M NP = \text{span } Me_N^M M = \langle M, N \rangle,$$

and the commuting square (A.2) is also nondegenerate by (N6).

If in addition all the algebras in (A.1) are finite-dimensional, then the Bratteli diagram for $\langle P, Q \rangle \hookrightarrow \langle M, N \rangle$ is the same as the Bratteli diagram for the inclusion $Q \subset N$ by [54, Lem. 6.1] or [36, Lem. 5.3.3].

- (Composite commuting squares cf. [55, Cor. in §1.1.5]) A composite of two commuting squares of finite index finite multifactors

$$\begin{array}{ccccc} T & \subset & Q & \subset & M \\ \cup & & \cup & & \cup \\ S & \subset & P & \subset & N \end{array}$$

is again a commuting square. The composite square is nondegenerate if and only if the two component commuting squares are nondegenerate.

A.2. Nondegenerate commuting squares from unitary multifusion categories

In this section, we study some nondegenerate commuting squares which arise from a unitary multifusion category \mathcal{C} . The results in this section are technical lemmas used in the proof of Theorem 5.1.

Definition A.6. For $c \in \mathcal{C}$, we denote by $1_{s(c)}$ and $1_{t(c)}$ are the *source* and *target* summands of $1_{\mathcal{C}}$ for $c \in \mathcal{C}$, i.e., the minimal subobjects of $1_{\mathcal{C}}$ such that $c = 1_{s(c)} \otimes c \otimes 1_{t(c)}$.

Assumption A.7. Fix objects $a, b, c \in \mathcal{C}$. Suppose there is a unitary dual functor \vee on \mathcal{C} such that

- $\text{ev}_a^* \circ \text{ev}_a = \lambda_a \text{id}_{1_{s(a)}}$ and $\text{coev}_c \circ \text{coev}_c^* = \rho_c \text{id}_{1_{t(c)}}$ for some positive scalars λ_a, ρ_c , and
- there is a faithful spherical state ψ on $\text{End}_{\mathcal{C}}(1_{\mathcal{C}})$ satisfying $\psi \circ \text{tr}_L^{\vee} = \psi \circ \text{tr}_R^{\vee}$ on hom categories of \mathcal{C} .

Under the above assumptions, we have a commuting square of finite-dimensional von Neumann algebras

$$\begin{array}{ccccc}
 & & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ a \quad b \end{array} & \mapsto & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ a \quad b \quad c \end{array} \\
 & & \uparrow & & \uparrow \\
 \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ a \quad b \end{array} & \text{End}_{\mathcal{C}}(a \otimes b) & \subset & \text{End}_{\mathcal{C}}(a \otimes b \otimes c) & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ a \quad b \quad c \end{array} \\
 \uparrow & \cup & & \cup & \uparrow \\
 \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ b \end{array} & \text{End}_{\mathcal{C}}(b) & \subset & \text{End}_{\mathcal{C}}(b \otimes c) & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ b \quad c \end{array} \\
 & & \uparrow & & \uparrow \\
 & & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ b \end{array} & \mapsto & \begin{array}{c} \text{---} \\ | \\ \boxed{x} \\ | \\ \text{---} \\ b \quad c \end{array}
 \end{array} \tag{A.3}$$

where we equip $\text{End}_{\mathcal{C}}(a \otimes b \otimes c)$ with the faithful trace $\text{tr} := \psi \circ \text{tr}_L^{\vee} = \psi \circ \text{tr}_R^{\vee}$. The canonical trace-preserving conditional expectations are given by the partial trace on the left/right and dividing by λ_a, ρ_c respectively, which obviously commute.

Lemma A.8. *Suppose that the set of isomorphism classes of simple summands of b and $b \otimes c \otimes c^{\vee}$ agree. Then the commuting square (A.3) is nondegenerate. Moreover, the basic construction commuting square is given by*

$$\begin{array}{ccc}
 \text{End}_{\mathcal{C}}(a \otimes b \otimes c) & \subset & \text{End}_{\mathcal{C}}(a \otimes b \otimes c \otimes c^{\vee}) \\
 \cup & & \cup \\
 \text{End}_{\mathcal{C}}(b \otimes c) & \subset & \text{End}_{\mathcal{C}}(b \otimes c \otimes c^{\vee})
 \end{array} \tag{A.4}$$

A similar statement holds if the set of isomorphism classes of simple summands of $a^{\vee} \otimes a \otimes b$ and b agree.

Proof. We prove (A.3) is nondegenerate under the condition (N3), and the second is similar. Let \mathcal{S} be a set of representatives for the common set of isomorphism classes of simple

summands of b and $b \otimes c \otimes c^\vee$. For each $s \in \mathcal{S}$, pick a basis $\{v_s\}$ of $\mathcal{C}(s \rightarrow b \otimes c \otimes c^\vee)$ consisting of isometries with orthogonal ranges so that

$$\text{id}_{b \otimes c \otimes c^\vee} = \sum_{s \in \mathcal{S}} \sum_{v_s} v_s \circ v_s^* \quad \text{and} \quad v_s^* \circ v_s = \text{id}_s \quad \forall s \in \mathcal{S}.$$

For every $s \in \mathcal{S}$, pick a single isometry $w_s \in \mathcal{C}(s \rightarrow b)$ so that

$$\text{id}_{b \otimes c \otimes c^\vee} = \sum_{s \in \mathcal{S}} \sum_{v_s} v_s \circ v_s^* = \sum_{s \in \mathcal{S}} \sum_{v_s} v_s \circ w_s^* \circ w_s \circ v_s^*.$$

Now using isotopy, for each $s \in \mathcal{S}$ and isometry basis element v_s , we set

$$u_s := \sqrt{\rho_c} \cdot \begin{array}{c} \begin{array}{c} b \quad c \\ \boxed{v_s} \\ s \end{array} \\ \begin{array}{c} \boxed{w_s^*} \\ b \end{array} \end{array} = (\text{id}_{b \otimes c} \otimes \text{coev}_c^*) \circ (v_s \otimes \text{id}_{c^\vee}).$$

Then we see that $\coprod_{s \in \mathcal{S}} \{u_s\}$ satisfies the condition

$$\text{id}_{b \otimes c \otimes c^\vee} = \frac{1}{\rho_c} \sum_{s \in \mathcal{S}} \sum_{u_s} \begin{array}{c} \boxed{u_s} \\ \boxed{u_s^*} \end{array} = \sum_{s \in \mathcal{S}} \sum_{u_s} u_s f u_s^* \tag{A.5}$$

where $f := \frac{1}{\rho_c} \cdot \text{id}_b \otimes (\text{coev}_c \circ \text{coev}_c^*)$, which immediately implies the inclusion

$$\text{End}_{\mathcal{C}}(b) \subset \text{End}_{\mathcal{C}}(b \otimes c) \stackrel{\frac{1}{\rho_c} (\text{id}_b \otimes (\text{coev}_c \circ \text{coev}_c^*))}{\subset} \text{End}_{\mathcal{C}}(b \otimes c \otimes c^\vee)$$

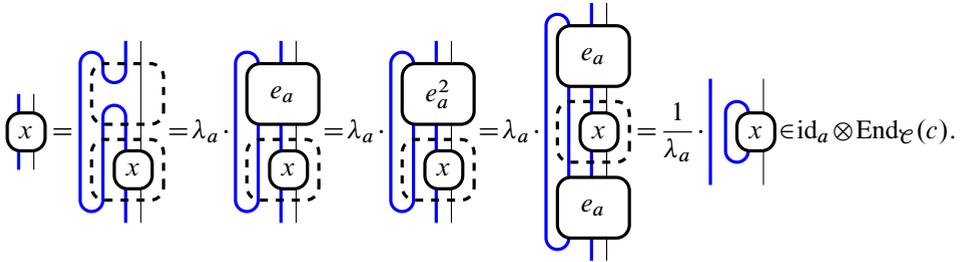
is standard, i.e., a Jones basic construction by [36, Lem. 5.3.1, Cor. 5.3.2]. The only interesting part in checking the hypotheses for this recognition lemma is checking $\text{End}_{\mathcal{C}}(b \otimes c \otimes c^\vee)$ is generated by $\text{End}_{\mathcal{C}}(b \otimes c)$ and f , which follows from (A.5) using the following graphical argument:

$$\begin{array}{c} \boxed{x} \\ \text{---} \\ \boxed{u_s} \\ \boxed{u_s^*} \\ \boxed{x} \\ \boxed{v_t} \\ \boxed{v_t^*} \end{array} = \frac{1}{\rho_c^2} \sum_{s,t \in \mathcal{S}} \sum_{u_s, v_t} \begin{array}{c} \boxed{x} \\ \boxed{u_s} \\ \boxed{u_s^*} \\ \boxed{x} \\ \boxed{v_t} \\ \boxed{v_t^*} \end{array} = \frac{1}{\rho_c^2} \sum_{s,t \in \mathcal{S}} \sum_{u_s, v_t} \begin{array}{c} \boxed{x_{s,t}} \\ \boxed{u_s} \\ \boxed{v_t^*} \end{array} = \frac{1}{\rho_c^2} \sum_{s,t \in \mathcal{S}} \sum_{u_s, v_t} \begin{array}{c} \boxed{x_{s,t}} \\ \boxed{u_s} \\ \boxed{v_t^*} \end{array}.$$

is an orthogonal projection. Suppose that $x \in \text{End}_{\mathcal{E}}(a \otimes b)$ such that

$$\text{id}_{a^\vee} \otimes x \in (\text{End}_{\mathcal{E}}(a^\vee \otimes a) \otimes \text{id}_b)' \cap (\text{id}_{a^\vee} \otimes \text{End}_{\mathcal{E}}(a \otimes b)).$$

Then since $e_a \in \text{End}_{\mathcal{E}}(a^\vee \otimes a) \otimes \text{id}_b$, $\text{id}_{a^\vee} \otimes x$ and e_a commute. Hence



The result follows. ■

Corollary A.11. *Under the hypotheses of Lemma A.10, for the commuting square*

$$\begin{array}{ccc} \text{End}_{\mathcal{E}}(a^{\text{alt}\otimes 2k} \otimes b) & \subset & \text{End}_{\mathcal{E}}(a^\vee \otimes a^{\text{alt}\otimes 2k} \otimes b) \\ \cup & & \cup \\ \text{End}_{\mathcal{E}}(a^{\text{alt}\otimes 2k}) & \subset & \text{End}_{\mathcal{E}}(a^\vee \otimes a^{\text{alt}\otimes 2k}), \end{array}$$

$$(\text{End}_{\mathcal{E}}(a^\vee \otimes a^{\text{alt}\otimes 2k}) \otimes \text{id}_b)' \cap (\text{id}_{a^\vee} \otimes \text{End}_{\mathcal{E}}(a^{\text{alt}\otimes 2k} \otimes b)) = \text{id}_{a^\vee \otimes a^{\text{alt}\otimes 2k}} \otimes \text{End}_{\mathcal{E}}(b).$$

Proof. The proof follows from Lemma A.10 by a simple induction argument. ■

A.3. Inductive limits of nondegenerate commuting squares

For this section, we fix a nondegenerate commuting square of finite-dimensional von Neumann algebras

$$\begin{array}{ccc} M_0 & \subset & M_1 \\ \cup & & \cup \\ N_0 & \subset & N_1 \end{array} \tag{A.6}$$

such that the inclusion $M_0 \subset (M_1, \text{tr}_1)$ is Markov with index d^2 . By Lemma A.4 above, $N_0 \subset (N_1, \text{tr}_1|_{N_1})$ is also Markov with index d^2 . Iterating the basic construction, we get an increasing sequence of nondegenerate commuting squares. Define $M_\infty := \varinjlim M_n$, $N_\infty := \varinjlim N_n$, and $\text{tr}_\infty := \varinjlim \text{tr}_n$ on M_∞ . Notice that tr_∞ is faithful on M_∞ since tr_n is faithful on M_n for all n by nondegeneracy.

It is straightforward to verify that M_∞ acts on $H := L^2(M_\infty, \text{tr}_\infty)$ by bounded operators, where $\|x^*x\|_{B(H)} \leq \|x^*x\|_{M_n}$ for $x \in M_n$. Let $M := M''_\infty \subset B(H)$ and $N := N''_\infty \subset B(H)$. Define tr on M by $\text{tr}(x) := \langle x\Omega, \Omega \rangle$ where $\Omega \in H$ is the image of $1 \in M_\infty$.

Lemma A.12. *The normal state tr on M is a faithful trace.*

Proof. Traciality follows by normality of tr , SOT density of M_∞ in M , and the Kaplansky density theorem. To show tr is faithful, we use the proof in [39, §6.2], which we include for completeness and convenience.

Suppose $\text{tr}(x^*x) = 0$. Then for all $m \in M_\infty$,

$$\|xm\Omega\|_2^2 = \|xR_m\Omega\|_2^2 = \|R_mx\Omega\|_2^2 \leq \|R_m\|^2\|x\Omega\|_2^2 = \|R_m\|^2\text{tr}(x^*x) = 0$$

where for $a, m \in M_\infty$, $R_m a \Omega = a m \Omega$ is the bounded right action as tr_∞ is a trace on M_∞ . We conclude $x = 0$. ■

We thus have an increasing sequence of finite-dimensional nondegenerate commuting squares with an inductive limit inclusion of finite von Neumann algebras with the canonical inductive limit faithful normal tracial state tr on M . Notice that at each iteration, identifying N_{n+1} with a subalgebra of M_{n+1} identifies the Jones projection f_n for $N_{n-1} \subset N_n$ with the Jones projection e_n for $M_n \subset M_{n-1}$.

$$\begin{array}{ccccccccccc} M_0 & \subset & M_1 & \overset{e_1}{\subset} & M_2 & \overset{e_2}{\subset} & M_3 & \overset{e_3}{\subset} & \dots & \subset & M \\ \cup & & \cup & & \cup & & \cup & & & & \cup \\ N_0 & \subset & N_1 & \overset{e_1}{\subset} & N_2 & \overset{e_2}{\subset} & N_3 & \overset{e_3}{\subset} & \dots & \subset & N \end{array} \tag{A.7}$$

Lemma A.13. *The finite von Neumann algebras M and N are finite direct sums of II_1 factors with $Z(M) = Z(M_0) \cap Z(M_1)$ and $Z(N) = Z(N_0) \cap Z(N_1)$.*

Proof. We prove the result for M and the result for N is similar. The Bratteli diagram for the inclusion $M_0 \subset M_1$ is a disjoint union of $\dim(Z(M_0) \cap Z(M_1))$ connected graphs. Denote the minimal projections of $Z(M_0) \cap Z(M_1)$ by $\{p\}$. Then the Bratteli diagram of each inclusion $pM_0 \subset pM_1$ is connected with Markov trace $x \mapsto \text{tr}_1(px) / \text{tr}_1(p)$. Iterating the basic construction, we have $M_n = \bigoplus pM_n$ for all n , and the von Neumann algebra generated by $\varinjlim pM_n$ in the GNS representation with respect to the unique Markov trace is clearly isomorphic to pM , as multiplication by p is SOT continuous. We know that each pM is a II_1 factor as the unique trace is faithful by an argument similar to Lemma A.12. Thus $M = \bigoplus_p pM$ is a finite direct sum of II_1 factors with $Z(M) = \text{span}\{p\} = Z(M_0) \cap Z(M_1)$. ■

Proposition A.14. *The Watatani index of the inductive limit hyperfinite type II_1 inclusion $N \subset M$ from (A.7) is equal to the Watatani index $\sum_b bb^*$ of the inclusion $N_0 \subset M_0$, where $\{b\}$ is any (left) Pimsner–Popa basis for M_0 over N_0 .*

Proof. The proof is identical to [36, Cor. 5.7.4]. By nondegeneracy and Facts A.5, there is a Pimsner–Popa basis $\{b\}$ for M_0 over N_0 which is also a Pimsner–Popa basis for M_n over N_n for every n . This means for every $x \in \bigcup_{n \geq 0} M_n$, we have $x = \sum_b bE_N(b^*x)$. This equation clearly varies ultraweakly continuously in x , and thus $\{b\}$ is a right Pimsner–Popa basis for M over N . We conclude that the Watatani index of $N \subset M$ is equal to $\sum_b bb^*$. ■

Remark A.15. If the commuting square (A.1) is nondegenerate and *horizontally connected*, i.e., $Z(M_0) \cap Z(M_1) = \mathbb{C}$ and $Z(N_0) \cap Z(N_1) = \mathbb{C}$, then $N \subset M$ is a II_1 subfactor whose Jones index $[M : N]$ is equal to the Watatani index $\sum_b bb^*$, which by [36, Lem. 5.3.3] or [54, Cor. 6.2], is necessarily equal to $\|\Gamma\Gamma^T\|$ where Γ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion $N_0 \subset M_0$.

B. Popa’s theorem for homogeneous connected finite depth hyperfinite II_1 multifactor inclusions

In this section, for completeness and convenience of the reader, we give a proof of Popa’s theorem that a homogeneous finite index finite depth connected hyperfinite II_1 multifactor inclusion is completely determined by its standard invariant. We adapt the proof for subfactors from [54].

Suppose $A \subset (B, \text{tr}_B)$ is a finite index connected II_1 multifactor inclusion with its unique Markov trace. As in Definition 2.18, $A \subset (B, \text{tr}_B)$ is strongly Markov.

Fact B.1 (*Tunnel-tower duality* cf. [55, §1.3.2]). Suppose $B_{-1} = A \subset B = B_0$ is a homogeneous connected II_1 multifactor inclusion of index d^2 . Consider the Jones tower of length n together with any Jones tunnel of length n :

$$B_{-n} \subset \cdots \subset B_{-1} \subset B_0 \overset{e_0}{\subset} B_1 \subset \cdots \subset B_n.$$

Identifying the tower with the multistep basic construction from Facts 2.20 on $L^2(B_0, \text{tr}_0)$, we have that $B_n = (JB_{-n}J)'$ for all $n > 0$, and conjugation by $J = J_0$ moves the Jones projections as $Je_{-n}J = e_n$ as in [54, §3.2]. In particular, we have conjugation by J is an anti-isomorphism between the centralizer algebras:

$$B'_{-n} \cap B_0 \underset{\text{anti}}{\cong} J(B'_{-n} \cap B_0)J = JB_{-n}J' \cap JB_0J = B_n \cap B'_0.$$

Lemma B.2 (cf. [7, Lem. 4.5]). *Suppose $A \subset B$ is finite index with standard invariant (\mathcal{C}, X) . Suppose $p \in A' \cap B$ is a minimal projection. Then the standard invariant of the reduced subfactor $pA \subset pBp$ is equivalent to some 2×2 unitary multifusion category generated by a single simple object of \mathcal{C} .*

Proof. Let $p_i \in Z(A)$ be the minimal projection such that $pp_i = p$, and let $q_j \in Z(B)$ be the minimal projection such that $pq_j = p$. Consider the bimodules ${}_{A_i}pL^2B_jB_j$ and ${}_{pA}L^2(pBp)_{pBp}$. Their dual Q-systems are isomorphic. Indeed, since $pBp = pB_jp$ and $pA = pA_i$, observe that

$$\begin{aligned} {}_{pBp}L^2(pBp) \boxtimes_{A_p} L^2(pBp)_{pBp} &= {}_{pBp}L^2(pB_jp) \boxtimes_{pA_i} L^2(pB_jp)_{pBp} \\ &\cong {}_{pBp}pL^2B \boxtimes_{B_j} L^2B_jp \boxtimes_{pA_i} pL^2B_j \boxtimes_{B_j} L^2Bp_{pBp} \\ &\cong {}_{pBp}pL^2B \boxtimes_{B_j} L^2B_jp \boxtimes_{A_i} pL^2B_j \boxtimes_{B_j} L^2Bp_{pBp} \end{aligned}$$

and ${}_{pBp}pL^2B_jB_j$ is an invertible bimodule. We conclude that the standard invariant of $pA \subset pBp$ is equivalent to the 2×2 unitary multifusion subcategory of \mathcal{C} generated by the simple object $A_i pL^2B_jB_j \in \mathcal{C}_{ij}$. ■

Corollary B.3 (cf. [54, Thm. 3.8]). *Suppose $A \subset B$ is finite index and finite depth with standard invariant (\mathcal{C}, X) . Define $M := \max\{\dim(c)^2 \mid c \in \text{Irr}(\mathcal{C})\} < \infty$.*

- (1) *If $(A_n)_{n \in \mathbb{N}}$ is the Jones tower of $A_0 = A \subset B = A_1$, then for every $n \in \mathbb{N}$ and any minimal projection $p \in A'_0 \cap A_n$, we have $[pA_n p : pA_0] \leq M$.*
- (2) *If moreover $A \subset B$ is homogeneous, then for any homogeneous tunnel $(A_{-n})_{n \in \mathbb{N}}$, for any $n \geq 0$ and any minimal projection $p \in A'_{-n} \cap B$, we have $[pBp : pA_{-n}] \leq M$.*

Proof. By tunnel-tower duality from Fact B.1, it suffices to prove the first statement. This is immediate by Lemma B.2. ■

Lemma B.4 (cf. [54, Thm. 4.3]). *Suppose $A \subset B$ has finite depth. There is a $\lambda > 0$ such that for any homogeneous tunnel $(A_{-n})_{n \in \mathbb{N}}$, we have $E_{A_{-n} \vee (A'_{-n} \cap B)}(x) \geq \lambda x$ for all $x \in B^+, n \in \mathbb{N}$.*

Proof. The proof of [54, Thm. 4.3] applies verbatim with Corollary B.3 in place of [54, Thm. 3.8]. (The proof of [54, Lem. 4.2] also applies verbatim when M is a II_1 multi-factor.) ■

Lemma B.5 (cf. [54, Lem. 4.4]). *Suppose $A \subset B$ is hyperfinite. Suppose we have a choice of finite homogeneous tunnel $(A_{-n})_{n=1}^j$. For any $\varepsilon > 0$ and any finite set $F \subset A_{-j} \vee (A'_{-j} \cap B)$, there is a $k \geq j$ and a homogeneous continuation of the tunnel $(A_{-n})_{n=1}^k$ such that $f \in_\varepsilon A'_{-k} \cap B$ for all $f \in F$.*

Proof. Since $A'_{-j} \cap B \subset A'_{-k} \cap B$ whenever $k \geq j$, it suffices to consider the case where $F \subset A_{-j}$.

Fix an isomorphism $A_{-j} \cong R^{\oplus c}$ for $c \in \{a, b\}$, and consider the diagonal embedding $\iota : R \hookrightarrow R^{\oplus c} \cong A_{-j}$. Let z_1, \dots, z_c be the minimal central projections in A_{-j} . Then every $x \in A_{-j}$ can be uniquely expressed in the form $x = \sum_{i=1}^c \iota(x_i)z_i$ for some $x_1, \dots, x_c \in R$. Since each $z_i \in A'_{-k} \cap B$ whenever $k \geq j$, it suffices to consider the case where $F \subset \iota(R)$.

The proof now proceeds as in [54, Lem. 4.4]. View R as generated by a sequence $(e_i)_{i=0}^\infty$ of Jones projections with λ the Markov index of $A \subset B$. Given $\varepsilon > 0$ and a finite subset $F \subset \iota(R)$, there is $N \in \mathbb{N}$ such that $F \subset_\varepsilon \text{Alg}\{1, \iota(e_1), \dots, \iota(e_{N-1})\}$.

Let $(A_{-n}^{(0)})_{n=j+1}^{j+N}$ be any homogeneous continuation of the tunnel and let $(e_{-n+1}^{(0)})_{n=j+1}^{j+N}$ be the corresponding Jones projections, which have constant center-valued trace by homogeneity. There exists a $Z(A_{-j})$ -valued trace-preserving isomorphism between the finite-dimensional algebras $\text{Alg}\{1, \iota(e_1), \dots, \iota(e_{N-1})\}$ and $\text{Alg}\{1, e_{-j-1}^{(0)}, \dots, e_{-j-N+1}^{(0)}\}$. Thus there exists a unitary $u \in A_{-j}$ such that $\iota(e_i) = ue_{-j-i}^{(0)}u^*$ for each i . Setting $A_{-n} := uA_{-n}^{(0)}u^*$ for $j+1 \leq n \leq j+N$ gives a homogeneous continuation of the tunnel with the desired property. ■

Corollary B.6 (cf. [54, Cor. 4.5]). *Suppose $A \subset B$ has finite depth and is hyperfinite. There is a choice of homogeneous tunnel $(A_{-n})_{n \in \mathbb{N}}$ such that $C := (\bigcup A'_{-n} \cap B)''$ has finite (Pimsner–Popa) index in B .*

Proof. The proof of [54, Cor. 4.5] with Lemma B.4 in place of [54, Thm. 4.3] and Lemma B.5 in place of [54, Lem. 4.4] applies verbatim, up to the final sentence. We obtain that there is a $\lambda > 0$ such that

$$\|E_C(x)\|_2^2 \geq \lambda \|x\|_2^2 \quad \forall x \in B^+. \tag{B.1}$$

Observe now that by construction, $Z(C) = Z(B)$, so $C \subset B$ is a finite direct sum of II_1 subfactors. Now each of these component II_1 subfactors satisfies (B.1), and thus each has Jones index at most λ^{-1} by [51, Thm. 2.2]. We conclude that $E_C : B \rightarrow C$ has finite Pimsner–Popa index. ■

The following lemma is well known to experts. We include a proof for the reader’s convenience.

Lemma B.7. *Suppose C is a von Neumann algebra with a faithful normal tracial state tr , and $(C_n)_{n \geq 0}$ is an increasing sequence of unital $*$ -subalgebras whose union is strongly dense in C . Let $E_n : C \rightarrow C_n$ be the unique trace-preserving conditional expectation. Then for all $x \in C$, $\|x\Omega - E_n(x)\Omega\|_2 \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Fix $x \in C$, and consider the unital $*$ -subalgebra $C^\circ := \bigcup_{n \geq 0} C_n \subset C$. Let X be the $\|\cdot\|_\infty$ -closed ball of C of radius $\|x\|_\infty$. Recall from [39, Prop. 9.1.1] that X is a complete metric space in $\|\cdot\|_2$, and the $\|\cdot\|_2$ topology on X agrees with the strong operator topology. Fix $\varepsilon > 0$, and let $B_\varepsilon(x)$ denote the open ball of radius ε in $\|\cdot\|_2$ about x . Pick an open neighborhood $U \subset C$ for the strong operator topology such that $U \cap X = B_\varepsilon(x) \cap X$.

By the Kaplansky density theorem, there is an $y \in C^\circ \cap X$ with $y \in B_\varepsilon(x)$. Let $N \in \mathbb{N}$ such that $y \in C_N$. Then since $E_n(x) \in C_n$ is the unique element in C_n closest to x in $\|\cdot\|_2$, for all $n \geq N$, $\|x\Omega - E_n(x)\Omega\|_2 \leq \|x\Omega - y\Omega\|_2 < \varepsilon$. ■

Proposition B.8. *Suppose $A \subset B$ has finite depth and $(A_{-n})_{n \in \mathbb{N}}$ is a homogeneous tunnel for $A \subset B$ such that $C := (\bigcup A'_{-n} \cap B)''$ has finite Pimsner–Popa index in B as in Corollary B.6.*

- (1) *For every $n \geq k \in \mathbb{N}$, the following are commuting squares when equipped with the Markov trace-preserving conditional expectations:*

$$\begin{array}{ccccc} A'_{-n} \cap B & \subset & C & \subset & B \\ \cup & & \cup & & \cup \\ A'_{-n} \cap A_{-k} & \subset & C \cap A_{-k} & \subset & A_{-k}. \end{array} \tag{B.2}$$

- (2) *For n sufficiently large, there is a Pimsner–Popa basis for $A'_{-n} \cap B$ over $A'_{-n} \cap A_{-k}$ which is also a Pimsner–Popa basis for both C over $C \cap A_{-k}$ and for B over A_{-k} .*
- (3) *There is a Pimsner–Popa basis for C over $C \cap A_{-k}$ which is also a Pimsner–Popa basis for B over A_{-k} .*

Proof. (1) By [53, Lem. 1.2.2], for every inclusion of von Neumann algebras $N \subset M \subset P$ with P type II_1 with a faithful normal trace, the unique trace-preserving conditional expectations $E_{N' \cap P}$ and E_M commute. Hence $E_{A'_{-n} \cap B}$ and $E_{A_{-k}}$ commute whenever $n \geq k$. This means the large composite square commutes.

Observe

$$E_{A'_{-n} \cap B} = E_{A'_{-n} \cap B}^C \circ E_C \tag{B.3}$$

by uniqueness of the trace-preserving conditional expectation. By Lemma B.7, for every $x \in B$,

$$\|E_C(x)\Omega - E_{A'_{-j} \cap B}(x)\Omega\|_2 = \|E_C(x)\Omega - E_{A'_{-j} \cap B}^C(E_C(x))\Omega\|_2 \xrightarrow{j \rightarrow \infty} 0.$$

Since $E_{A'_{-j} \cap B}$ and $E_{A_{-k}}$ commute for all $j \geq k$, E_C and $E_{A_{-k}}$ commute, so the square on the right of (B.2) commutes.

Now since $E_{A'_{-n} \cap B}^C = E_{A'_{-n} \cap B}|_C$ by (B.3) and $E_{C \cap A_{-k}}^C = E_{A_{-k}}|_C$ since the square on the right of (B.2) commutes, the square on the left of (B.2) commutes.

(2) Since $A_{-k} \subset B = A_0$ has finite depth, we can choose n large so that both

$$A_{-k} \subset A_0 \stackrel{f}{\subset} A_k \quad \text{and} \quad A'_{-n} \cap A_{-k} \subset A'_{-n} \cap A_0 \stackrel{f}{\subset} A'_{-n} \cap A_k$$

are multi-step basic constructions as in Facts 2.20 with the same Jones projection f . Hence there is a finite subset $\{b\} \subset A'_{-n} \cap A_0$ such that $\sum_b bfb^* = 1_{A'_{-n} \cap A_k} = 1_{A_k}$. Then for all $x \in B = A_0$, we have $x = \sum_b bE_{A_{-k}}(b^*x)$. Since $E_{A_{-k}}$ and E_C commute by statement (1), we see that $r = \sum_b bE_{A_{-k} \cap C}^C(b^*r)$ for all $r \in C$. Hence $\{b\}$ is the desired Pimsner–Popa basis.

(3) Observe that every inclusion in the commuting square of finite multifactors on the right of (B.2) has finite index. Indeed, $C \subset B$ was assumed to have finite index, and $A_{-k} \subset B$ has finite index as a composite of finite index submultifactors in the Jones tunnel. By statement (2), we see $C \cap A_{-k} \subset C$ has the same (finite) Watatani index as $A_{-k} \subset B$. We conclude that $C \cap A_{-k} \subset B$ has finite index.

The result now follows immediately by statement (2) and Lemma A.2. ■

Theorem B.9 (cf. [54, Thm. 4.9]). *Suppose $A \subset B$ is a homogeneous connected hyperfinite II_1 multifactor inclusion with finite depth. There is a choice of homogeneous tunnel $(A_{-n})_{n \in \mathbb{N}}$ such that $B = (\bigcup A'_{-n} \cap B)''$ and $A = (\bigcup A'_{-n} \cap A)''$.*

Proof. Suppose for contradiction that there is no such choice of generating homogeneous tunnel. By Corollary B.6, we can pick a homogeneous tunnel $(A_{-n})_{n \in \mathbb{N}}$ for $A \subset B$ such that $C := (\bigcup A'_{-n} \cap B)''$ has finite Pimsner–Popa index in B . Then by [54, Cor. 4.8], which apply verbatim to the multifactor setting, there exist a free ultrafilter ω on \mathbb{N} , an $x \in B^\omega$ with $\|x\|_2 = 1$, and an increasing sequence $(k_j) \subset \mathbb{N}$ such that

$$x \perp \left(\prod_{\omega} A_{-k_j} \right) C^\omega \left(\prod_{\omega} A_{-k_j} \right).$$

We now proceed exactly as in the proof of [54, Thm. 4.9]. For each $k \in \mathbb{N}$, we use 3 of Proposition B.8 to pick a Pimsner–Popa basis $\{b_i^k\}_{i=1}^{\ell_n}$ for A_{-k} over $C \cap A_{-k}$ which is also a Pimsner–Popa basis for B over C .

We may arrange so that $\ell_k = \ell$ is independent of $k \in \mathbb{N}$. Indeed, the Watatani indices of $C \cap A_{-k} \subset A_{-k}$ and $C \subset B$ are both given by

$$\sum_{i=1}^{\ell_n} b_i^k (b_i^n)^* \in Z(A_{-k}) \cap Z(B) = \mathbb{C},$$

i.e., they are both the same scalar d^2 . Since $Z(C) = Z(B)$ and $Z(C \cap A_{-k}) = Z(A_{-k})$, both of the inclusions $C \cap A_{-k} \subset A_{-k}$ and $C \subset B$ are finite direct sums of II_1 subfactors with the same Jones index. Hence each Pimsner–Popa basis $\{b_i^k\}$ for A_{-k} over $C \cap A_{-k}$ can be chosen to have cardinality $\lceil d^2 \rceil$ by [51].

Thus for each $n \in \mathbb{N}$, $\sum_{i=1}^{\ell} b_i^n C = B$. Setting $b_i := (b_i^{k_j}) \in \prod_{\omega} A_{-k_j}$, we have $\sum_{i=1}^{\ell} b_i C^{\omega} = B^{\omega}$. But $B^{\omega} \ni x \perp \sum_{i=1}^{\ell} b_i C^{\omega} \in (\prod_{\omega} A_{-k_j}) C^{\omega}$, a contradiction. ■

C. Ocneanu compactness

We now prove a more general version of the Ocneanu compactness theorem than that which appears in the literature. Here, we adapt the proof that appears in [36, Thm. 5.7.1] to apply to a more general class of commuting squares. This result was certainly known to Asaeda–Haagerup [2], Schou [62], and Popa [54] among other experts.

Theorem C.1 (Ocneanu compactness). *Suppose we have a nondegenerate commuting square of finite-dimensional von Neumann algebras as in (A.6) such that the inclusion $M_0 \subset (M_1, \text{tr}_1)$ is Markov. Let $N \subset M$ be the inductive limit hyperfinite type II_1 inclusion. Then the relative commutant $N' \cap M$ is equal to $N'_1 \cap M_0$ considered inside M_1 .*

Proof. The outline of the proof follows [36, Thm. 5.7.1] closely. We postpone our modified proofs of the technical lemmas, which appear below.

- (A) The same argument from [36, Thm. 5.7.1] shows that for all $1 \leq p < q$, $N'_q \cap M_p = N'_1 \cap M_0$. This immediately implies that $N'_1 \cap M_0 \subset N' \cap M$.
- (B) Let $E_n : M \rightarrow M_n$ be the canonical trace-preserving conditional expectation, and let Ω be the image of 1_M in $L^2 M$. Let $x \in M$ and set $x_n = E_n(x)$. By Lemma B.7, $\|x\Omega - x_n\Omega\|_2 \rightarrow 0$ as $n \rightarrow \infty$.
- (C) Starting with an $x \in N' \cap M$, we want to show that the sequence $(x_n)_{n \geq 0}$ from (B) is getting arbitrarily close to the finite-dimensional subspace $N'_1 \cap M_0$ in $\|\cdot\|_2$. We could then conclude by (B) that $x \in N'_1 \cap M_0$. We break this step up as follows.
 - (i) For $k \geq 0$, we have a map $\Phi_k : N'_0 \cap M_k \rightarrow N'_2 \cap M_{k+2}$ which sends $N'_j \cap M_k$ to $N'_{j+2} \cap M_{k+2}$ and $N'_j \cap N_k$ to $N'_{j+2} \cap N_{k+2}$ for all $0 \leq j \leq k$.

We define this map explicitly in Definition C.2 below, and we prove many properties about it in Proposition C.3. Of particular importance are:

- each Φ_k is a $*$ -algebra map, and
 - for all $y \in N'_0 \cap M_k$, $\Phi_k(E_{N_k}(y)) = E_{N_{k+2}}(\Phi_k(y))$.
- (ii) In general, the maps Φ_n do *not* preserve the Markov trace, and thus the Φ_n are not isometries on $(N'_0 \cap M_n)\Omega$. However, Φ_n gets closer to being an isometry as $n \rightarrow \infty$. Indeed, for $n \in \mathbb{N}$, we consider the composites $\Psi_n := \Phi_{n-1} \circ \dots \circ \Phi_1 \circ \Phi_0$ which map $N'_j \cap M_k \rightarrow N'_{2n+j} \cap M_{2n+k}$ for all $0 \leq j \leq k$. By (i), Ψ_n maps the subalgebra $N'_j \cap M_k \rightarrow N'_{2n+j} \cap M_{2n+k}$. Now setting $j = k = 0$, Ψ_n maps $Z(N_0) = N'_0 \cap N_0$ to $Z(N_{2n}) = N'_{2n} \cap N_{2n}$, which is a canonically isomorphic algebra as N_0 is Morita equivalent to N_{2n} via N_n and the multistep basic construction (see Facts 2.20). Thus starting with the trace $\tau_0 = \text{tr}_0$ on $Z(N_0)$, we obtain a sequence of traces on $Z(N_0)$ by setting $\tau_n = \text{tr}_{2n} \circ \Psi_n$. We show that each τ_n on $Z(N_0)$ is faithful, and that $(\tau_n)_{n \geq 0}$ converges to a faithful trace τ_∞ on $Z(N_0)$. We prove this result in Proposition C.7 below in the language of densities with respect to the trace tr_0 on $Z(N_0)$.
- (iii) Now since all faithful traces on a finite-dimensional algebra are comparable, for every $n \in \mathbb{N}$, there is a $C_n > 0$ such that $C_n^{-1} \text{tr}_0 \leq \tau_n \leq C_n \text{tr}_0$ on $Z(N_0) = N'_0 \cap N_0$. Since $\{\tau_n \mid n \in \mathbb{N}\} \cup \{\tau_\infty\}$ is compact by (ii), there is a $C > 0$ independent of n such that $C^{-1} \text{tr}_0 \leq \tau_n \leq C \text{tr}_0$ for all $n \in \mathbb{N}$.
- (iv) For all $y \in N'_0 \cap M_0$,

$$\begin{aligned} \|\Psi_n(y)\Omega\|_2 &= \text{tr}_{2n}(\Psi_n(y)^* \Psi_n(y)) \stackrel{(i)}{=} \text{tr}_{2n}(\Psi_n(y^*y)) \\ &= \text{tr}_{2n}(E_{N_{2n}}(\Psi_n(y^*y))) \stackrel{(i)}{=} \text{tr}_{2n}(\Psi_n(E_{N_0}(y^*y))) \\ &= \tau_n(E_{N_0}(y^*y)). \end{aligned}$$

Thus for all $n \in \mathbb{N}$ and all $y \in N'_0 \cap M_0$, by (iii) we have

$$C^{-1} \|y\Omega\|_2 \leq \|\Psi_n(y)\Omega\|_2 \leq C \|y\Omega\|_2.$$

- (v) It is a simple algebraic calculation that $x_n \in N'_n \cap M_n \subset N'_0 \cap M_n$ for all $n \geq 0$. We use the notation $H_n = (N'_n \cap M_n)\Omega$ for this finite-dimensional Hilbert space, and we see from (A) that for all $n \in \mathbb{N}$,

$$H_n \cap H_{n+1} = (N'_n \cap M_n) \cap (N'_{n+1} \cap M_{n+1}) = N'_{n+1} \cap M_n = N'_1 \cap M_0.$$

- (vi) Given a Hilbert space X with closed subspaces Y, Z , we can define two norms on $X/(Y \cap Z)$ by

$$\begin{aligned} \|\xi\|_1 &:= \|(\xi + Y, \xi + Z)\|_{X/Y \oplus_{\ell^1} X/Z} = \text{dist}(\xi, Y) + \text{dist}(\xi, Z), \\ \|\xi\|_2 &:= \|\xi + Y \cap Z\|_{X/(Y \cap Z)} = \text{dist}(\xi, Y \cap Z). \end{aligned}$$

When $X/(Y \cap Z)$ is finite-dimensional, these two norms are equivalent. Setting $X = (N'_0 \cap M_1)\Omega$, $Y = H_0 = (N'_0 \cap M_0)\Omega$ and $Z = H_1 = (N'_1 \cap M_1)\Omega$, there is a $K > 0$ such that for all $y \in N'_0 \cap M_1$,

$$\text{dist}(y\Omega, H_0 \cap H_1) \leq K(\text{dist}(y\Omega, H_0) + \text{dist}(y\Omega, H_1)).$$

In particular, for all $y \in H_0 = N'_0 \cap M_0$,

$$\text{dist}(y\Omega, H_0 \cap H_1) \leq K \text{dist}(y\Omega, H_1).$$

- (vii) Finally, we calculate for each $x_{2n} = E_{2n}(x) \in N'_{2n} \cap M_{2n}$, since $\Psi_n(H_k) = H_{2n+k}$,

$$\begin{aligned} \text{dist}(x_{2n}\Omega, (N'_1 \cap M_0)\Omega) &= \text{dist}(x_{2n}\Omega, H_{2n} \cap H_{2n+1}) && \text{(v)} \\ &\leq C \text{dist}(\Psi_n^{-1}(x_{2n})\Omega, H_0 \cap H_1) && \text{(iv)} \\ &\leq CK \text{dist}(\Psi_n^{-1}(x_{2n})\Omega, H_1) && \text{(vi)} \\ &\leq C^2K \text{dist}(x_{2n}\Omega, H_{2n+1}) && \text{(iv)} \\ &\leq C^2K \text{dist}(x_{2n}\Omega, x_{2n+1}\Omega) && \text{(v)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. && \text{(B)} \end{aligned}$$

This completes the outline of the proof. ■

The rest of the appendix consists of the technical details of the above proof.

C.1. The maps Φ_n

We now define the maps Φ_n which were the main tool for the difficult part of Theorem C.1.

Definition C.2. Let $\{b\}$ be a Pimsner–Popa basis for N_1 over N_0 . Since the commuting square (A.1) is horizontally Markov, $\{b\}$ is also a basis for M_1 over M_0 . We define Φ_n on $N'_0 \cap M_n$ by $\Phi_n(x) = d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 b^*$ (compare with the formula in [6, Thm. 2.13]). Note that Φ_n on $N'_0 \cap M_n$ is independent of the choice of basis as in [41, Rem. 2.30]. Whenever $z \in N'_0 \cap M_n$,

$$\sum_b b \otimes b^* \mapsto \sum_b b z b^*$$

is well defined, and the left-hand side is independent of the choice of $\{b\}$. Since for every $u \in U(N_0)$, $\{ub\}$ is another Pimsner–Popa basis, we see that $\Phi_n(x) \in N'_0 \cap M_{n+2}$.

Proposition C.3. *The maps Φ_n on $N'_0 \cap M_n$ enjoy the following properties:*

- (1) For all $x \in N'_0 \cap M_n$, $\Phi_n(x) \in N'_0 \cap M_{n+2}$.
- (2) $\Phi_n|_{M_k} = \Phi_k$ for all $0 \leq k \leq n$.

- (3) Φ_n is a $*$ -algebra map.
- (4) For all $0 \leq k \leq n$, if $x \in N'_k \cap M_n$, then $\Phi_n(x) \in N'_{k+2} \cap M_{n+2}$.
- (5) For all $x \in N'_0 \cap M_n$, $E_{N_{n+2}}(\Phi_n(x)) = \Phi_n(E_{N_n}(x))$.

Proof. We prove each statement in order.

- (1) Since $\{b\} \subset N_1$, if $x \in N'_0 \cap N_n$, then $\Phi_n(x) \in N'_0 \cap N_{n+2}$.
- (2) If $x \in M_k$, then $[x, e_j] = 0$ for all $k + 1 \leq j \leq n + 1$. Thus

$$\begin{aligned} \Phi_n(x) &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_k e_{k+1} \cdots e_n e_{n+1} e_n \cdots e_{k+1} x e_k \cdots e_2 e_1 b^* \\ &= d^{2k} \sum_b b e_1 e_2 \cdots e_{k+1} x e_k \cdots e_2 e_1 b^* \\ &= \Phi_k(x). \end{aligned}$$

- (3) For all $x, y \in N'_0 \cap M_n$, we have

$$\begin{aligned} &\Phi_n(x)\Phi_n(y) \\ &= d^{4n} \sum_{a,b} a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 a^* b e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{4n} \sum_{a,b} a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 E_{N_0}(a^* b) e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{4n} \sum_{a,b} a E_{N_0}(a^* b) e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x y e_n \cdots e_2 e_1 b^* \\ &= \Phi_n(xy). \end{aligned}$$

- (4) Suppose $x \in N'_k \cap M_n$ for $0 \leq k \leq n$, and suppose $y \in N_{k+2}$. We calculate

$$\begin{aligned} \Psi_n(x)y &= \Psi_n(x)y1_{N_{k+2}} \\ &= \left(d^{2n} \sum_a a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 a^* \right) y \\ &\quad \times \left(d^{2k} \sum_b b e_1 e_2 \cdots e_k e_{k+1} e_k \cdots e_2 e_1 b^* \right) \\ &= d^{2(n+k)} \sum_{a,b} (a e_1 e_2 \cdots e_n x e_{n+1} e_n \cdots e_{k+2}) \\ &\quad \times \underbrace{(e_{k+1} \cdots e_2 e_1 a^* y b e_1 e_2 \cdots e_{k+1})}_{z_{a,b} e_{k+1}} (e_k \cdots e_2 e_1 b^*). \end{aligned}$$

Since $e_{k+1}N_{k+2}e_{k+1} = N_k e_{k+1}$, for all $a \in \{a\}$ and $b \in \{b\}$, there is a $z_{a,b} \in N_k$ such that $(e_{k+1} \cdots e_2 e_1 a^* y b e_1 e_2 \cdots e_{k+1}) = z_{a,b} e_{k+1}$, as indicated in the underbrace above. Continuing the above calculation, we obtain

$$\begin{aligned} \Psi_n(x)y &= d^{2(n+k)} \sum_{a,b} (a e_1 e_2 \cdots e_n x e_{n+1} e_n \cdots e_{k+2}) (z_{a,b} e_{k+1}) (e_k \cdots e_2 e_1 b^*) \\ &= d^{2(n+k)} \sum_{a,b} a e_1 e_2 \cdots e_n x z_{a,b} e_{n+1} e_n \cdots e_2 e_1 b^*. \end{aligned} \tag{C.1}$$

Starting with $y\Phi_n(x)$, a similar calculation shows

$$\begin{aligned} y\Psi_n(x) &= 1_{N_{k+2}} y\Psi_n(x) \\ &= d^{2(n+k)} \sum_{a,b} a e_1 e_2 \cdots e_n z_{a,b} x e_{n+1} e_n \cdots e_2 e_1 b^*. \end{aligned} \tag{C.2}$$

Since each $z_{a,b} \in N_k$ and $x \in N'_k \cap M_n$, (C.1) is equal to (C.2), and we are finished.

(5) Suppose $x \in N'_0 \cap M_n$. Since

$$\begin{array}{ccc} M_n & \subset & M_{n+2} \\ \cup & & \cup \\ N_n & \subset & N_{n+2} \end{array}$$

is a commuting square, $E_{N_{n+2}}(x) = E_{N_n}(x)$. Since $E_{N_{n+2}}$ is $N_{n+2} - N_{n+2}$ bilinear, we have

$$\begin{aligned} E_{N_{n+2}}(\Phi_n(x)) &= E_{N_{n+2}}\left(d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 b^*\right) \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} E_{N_{n+2}}(x) e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} E_{N_n}(x) e_n \cdots e_2 e_1 b^* \\ &= \Phi_n(E_{N_n}(x)). \end{aligned} \quad \blacksquare$$

C.2. Behavior of the traces τ_n

For $n \in \mathbb{N}$, we define $\Psi_n = \Phi_{n-1} \circ \cdots \circ \Phi_1 \circ \Phi_0$. We now observe the behavior of the sequence of traces $\tau_n := \text{tr}_{2n} \circ \Psi_n$ on $Z(N_0) = N'_0 \cap N_0$, with $\tau_0 = \text{tr}_0$ by convention. The following lemma is a straightforward calculation.

Lemma C.4. For all $x \in Z(N_0)$ and $n \in \mathbb{N}$,

$$\tau_n(x) = d^{-2n} \sum_{b_1, \dots, b_n \in B} \text{tr}_0(x \cdot E_{N_0}(b_1^* E_{N_0}(b_2^* \cdots E_{N_0}(b_n^* b_n) \cdots b_2) b_1))$$

where B is any Pimsner–Popa basis for N_1 over N_0 .

There is a unique $k \in \mathbb{N}$ such that $Z(N_0) \cong \mathbb{C}^k$. For $x \in Z(N_0)$, we denote by \vec{x} the vector in \mathbb{C}^k corresponding to x . We define:

- Λ is the bipartite adjacency matrix of the Bratteli diagram for the inclusion $N_0 \subset N_1$, i.e., $\Lambda_{i,j}$ is the number of times the i -th simple summand of N_0 is contained in the j -th simple summand of N_1 .
- λ_i is the Markov trace column vector for N_i , whose j -th entry $\lambda_i(j)$ is the trace of a minimal projection in the j -th simple summand of N_i . This means $\Lambda \Lambda^T \lambda_0 = d^2 \lambda_0$ and $\Lambda^T \Lambda \lambda_1 = d^2 \lambda_1$. Observe that since tr_1 is a faithful Markov trace for the inclusion $N_0 \subset N_1$, the matrices $\Lambda \Lambda^T$ and $\Lambda^T \Lambda$ are both direct sums of primitive symmetric non-negative integer matrices, all of which have the same Frobenius–Perron eigenvalue d^2 .
- m_i is the dimension (row) vector for N_i , i.e., the j -th simple summand of N_i is a full matrix algebra of size $m_i(j)$. Notice that $m_i \lambda_i = 1$ for $i = 0, 1$.
- $\Delta = \text{diag}(m_0(i))_{i=1}^k$ is the diagonal $k \times k$ matrix whose (i, i) -th entry is $m_0(i)$.

Example C.5. For the A_4 inclusion $N_0 = \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) = N_1$ we have:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \lambda_0 = \frac{1}{1+2\phi} \begin{bmatrix} 1 \\ \phi \end{bmatrix}, \quad \lambda_1 = \frac{1}{2+3\phi} \begin{bmatrix} \phi \\ 1 \end{bmatrix},$$

$$m_0 = [1 \quad 2], \quad m_1 = [3 \quad 2], \quad \text{and} \quad \Delta = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Proposition C.6. *There is a Pimsner–Popa basis B for N_1 over N_0 such that for every $x \in Z(N_0)$, $\sum_{b \in B} E_{N_0}(b^* x b) \in Z(N_0)$. Moreover, under the isomorphism $Z(N_0) \cong \mathbb{C}^k$, we have*

$$\overrightarrow{\sum_{b \in B} E_{N_0}(b^* x b)} = \Delta^{-1} \Lambda \Lambda^T \Delta \vec{x}.$$

Proof. We use the loop basis for $N_0 \subset N_1$ afforded by [41, §3.1 and §3.2]. We label the edges of Λ by ε , with source $s(\varepsilon)$ an even vertex corresponding to a simple summand of N_0 , and target $t(\varepsilon)$ an odd vertex corresponding to a simple summand of N_1 . We introduce a new vertex \star with edges η , with each source $s(\eta) = \star$, and target $t(\eta)$ an even vertex corresponding to a simple summand of N_0 . The number of edges η from \star to the i -th even vertex is equal to $m_0(i)$. We denote by ε^* and η^* the edge with the reverse orientation.

We give an explicit basis for N_0 by loops of length 2 starting at \star , where adjoint is given by the conjugate linear extension of $[\eta_i \eta_j^*]^* = [\eta_j \eta_i^*]$, and multiplication is given by $[\eta_i \eta_j^*] \cdot [\eta_k \eta_\ell^*] = \delta_{j=k} [\eta_i \eta_\ell^*]$. We give an explicit basis for N_1 by loops of length 4 starting at \star , where adjoint is given by the conjugate linear extension of $[\eta_i \varepsilon_j \varepsilon_k^* \eta_\ell^*]^* = [\eta_\ell \varepsilon_k \varepsilon_j^* \eta_i^*]$, and multiplication is given by

$$[\eta_i \varepsilon_j \varepsilon_k^* \eta_\ell^*] \cdot [\eta_m \varepsilon_n \varepsilon_p^* \eta_q^*] = \delta_{\ell=m} \delta_{k=n} [\eta_i \varepsilon_j \varepsilon_p^* \eta_q^*].$$

The trace tr_1 on N_1 is given by

$$\text{tr}_1([\eta_i \varepsilon_j \varepsilon_k^* \eta_\ell^*]) = \delta_{\eta_i = \eta_\ell} \delta_{\varepsilon_j = \varepsilon_k} \lambda_1(t(\varepsilon_j)),$$

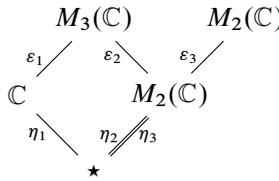
and the trace tr_0 on N_0 is given by $\text{tr}_0([\eta_i \eta_j^*]) = \delta_{\eta_i = \eta_j} \lambda_0(t(\eta_i))$. The unital inclusion $N_0 \subset N_1$ is given by

$$[\eta_i \eta_j^*] \mapsto \sum_{s(\varepsilon) = t(\eta_i)} [\eta_i \varepsilon \varepsilon^* \eta_j^*],$$

and the unique trace-preserving conditional expectation is given by

$$E_{N_0}([\eta_i \varepsilon_j \varepsilon_k^* \eta_\ell^*]) = \delta_{\varepsilon_j = \varepsilon_k} \left(\frac{\lambda_1(t(\varepsilon_j))}{\lambda_0(s(\varepsilon_j))} \right) [\eta_i \eta_\ell^*].$$

For example, the inclusion $N_0 = \mathbb{C} \oplus M_2(\mathbb{C}) \subset M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) = N_1$ from Example C.5 could be represented in the loop basis as follows:



Now by [41, Prop. 3.22] and [41, Rem. 3.23], a Pimsner–Popa basis for N_1 over N_0 is given by $B = B_1 \amalg B_2$ where

$$B_1 = \left\{ \left(\frac{\lambda_0(s(\varepsilon_2))}{d\lambda_1(t(\varepsilon_2))} \right)^{1/2} \sum_{\eta: t(\eta) = s(\varepsilon_1)} [\eta \varepsilon_1 \varepsilon_2^* \eta^*] \mid s(\varepsilon_1) = s(\varepsilon_2) \text{ and } t(\varepsilon_1) = t(\varepsilon_2) \right\},$$

$$B_2 = \left\{ \left(\frac{\lambda_0(s(\varepsilon_2))}{m_0(s(\varepsilon_2)) d\lambda_1(t(\varepsilon_2))} \right)^{1/2} [\eta_1 \varepsilon_1 \varepsilon_2^* \eta_2^*] \mid s(\varepsilon_1) \neq s(\varepsilon_2) \right\}.$$

Here, the sum in B_1 is over η such that $[\eta \varepsilon_1 \varepsilon_2^* \eta^*]$ forms a loop.

Now the minimal central projection in N_0 corresponding to the i -th simple summand is equal to $p_i = \sum_{\eta: t(\eta) = i} [\eta \eta^*]$. One calculates that

$$\sum_{b \in B_1} E_{N_0}(b^* p_i b) = \sum_j \Lambda_{i,j}^2 p_i,$$

while

$$\sum_{b \in B_2} E_{N_0}(b^* p_i b) = \sum_{i' \neq i} \sum_j \frac{m_0(i)}{m_0(i')} \Lambda_{i,j} \Lambda_{i',j} p_{i'}.$$

Hence we have that $\sum_{b \in B} E_{N_0}(b^* p_i b)$ is in $Z(N_0)$ with corresponding vector in \mathbb{C}^k equal to $\Delta^{-1} \Lambda \Lambda^T \Delta \bar{e}_i$. Now since every element of $Z(N_0)$ is a linear combination of the p_i , the result follows. ■

Equipped with this explicit Pimsner–Popa basis, we are prepared to analyze the traces τ_n . Note that an arbitrary tracial state τ on $Z(N_0)$ is always of the form $\tau(y) = \text{tr}_0(y \cdot h)$ for some positive operator $h \in Z(N_0)$ with $\text{tr}_0(h) = 1$ called the *density* of τ . Let \mathcal{T} be the topological space of traces on $Z(N_0)$, and note that we may identify the *pointed* topological space $(\mathcal{T}, \text{tr}_0)$ with $(\{h \in Z(N_0) \mid h \geq 0 \text{ and } \text{tr}(h) = 1\}, 1_{Z(N_0)})$.

We see from Lemma C.4, using the Pimsner–Popa basis B from Proposition C.6, that the densities $h_n \in Z(N_0)$ of the τ_n are given inductively by

$$h_n = d^{-2} \sum_{b \in B} E_{N_0}(b^* h_{n-1} b) \quad \text{for all } n \in \mathbb{N}.$$

Letting $\vec{h}_n \in \mathbb{C}^k$ be the vector corresponding to $h_n \in Z(N_0)$, Proposition C.6 tells us that $\vec{h}_n = d^{-2} \Delta^{-1} \Lambda \Lambda^T \Delta \vec{h}_{n-1}$ for all $n \in \mathbb{N}$. Since the density h_0 of $\tau_0 = \text{tr}_0$ is $1_{Z(N_0)}$, for all $n \in \mathbb{N}$,

$$\vec{h}_n = d^{-2n} \Delta^{-1} (\Lambda \Lambda^T)^n \Delta \vec{1}, \tag{C.3}$$

where $\vec{1} \in \mathbb{C}^k$ is the vector whose entries are all 1.

Proposition C.7. *The traces τ_n are faithful and converge to a faithful trace τ_∞ on $Z(N_0)$.*

Proof. The density vector $\vec{h}_n = d^{-2n} \Delta^{-1} (\Lambda \Lambda^T)^n \Delta \vec{1}$ from (C.3) has strictly positive entries, and thus τ_n is faithful for all n . Second, the limit of $d^{-2n} \Delta^{-1} (\Lambda \Lambda^T)^n \Delta \vec{1}$ as $n \rightarrow \infty$ is well known to be $\Delta^{-1} \vec{\lambda}$, where $\vec{\lambda}$ is a suitably normalized Frobenius–Perron eigenvector for $\Lambda \Lambda^T$. Since $\vec{\lambda}_1$ had all entries strictly positive, $\vec{\lambda}$ has all entries strictly positive. (This follows by looking at the direct sum decomposition of $\Lambda \Lambda^T$ into its primitive symmetric blocks, all which have the same Frobenius–Perron eigenvalue by the existence of a Markov trace tr_1 on $N_0 \subset N_1$.) Hence the densities \vec{h}_n converge to $\vec{h}_\infty = \Delta^{-1} \vec{\lambda}$, which gives a faithful trace τ_∞ on $Z(N_0)$. (Note that τ_∞ is not tr_0 even if $\Delta = I_k$, since its density with respect to tr_0 is $\vec{\lambda}$, which is in general not $\vec{1}$.) ■

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