Doc. Math. 30 (2025), 587–610
DOI 10.4171/DM/1008

Weighted homomorphisms between C*-algebras

Eusebio Gardella and Hannes Thiel

Abstract. We show that a bounded, linear map between C*-algebras is a weighted $*$-homomorphism (the central compression of a $*$-homomorphism) if and only if it preserves zero-products, range-orthogonality, and domain-orthogonality. It follows that a self-adjoint, bounded, linear map is a weighted $*$-homomorphism if and only if it preserves zero-products. As an application we show that a linear map between C*-algebras is completely positive, order zero in the sense of Winter–Zacharias if and only if it is positive and preserves zero-products.

1. Introduction

The study of bounded, linear maps between C*-algebras that preserve various types of orthogonality has a long history. One of the earliest results, due to Arendt [3], is the description of such maps between unital, commutative C*-algebras: every bounded, linear map $\varphi\colon C(X) \to C(Y)$ that preserves zero-products (such maps are also called *disjointness preserving*, or *separating*) is spatially implemented, that is, setting $h := \varphi(1) \in C(Y)$ and $U := \{y \in Y : h(y) \neq 0\}$, there exists a continuous map $p\colon U \to X$ such that φ is given as

$$\varphi(f)(y) = h(y) f\big(p(y)\big)$$

for every $f \in C(X)$ and every $y \in Y$. Note that p induces a $*$-homomorphism $\pi\colon C(X) \to C_b(U)$ given by $\pi(f) = f \circ p$ for all $f \in C(X)$. By embedding $C(Y)$ and $C_b(U)$ into a suitably larger (commutative) C*-algebra – the canonical choice is $C(Y)^{**}$ – we obtain that $\varphi(f) = h\pi(f) = \pi(f)h$ for all $f \in C(X)$. In particular, φ is the compression of the $*$-homomorphism π by h. For this description of φ as the compression of a $*$-homomorphism, it is crucial to allow for an enlarged target algebra. Indeed, an example of Wolff [28], shows that the image of π is not in general contained in $C(Y)$.

Generalizing this idea to the noncommutative setting, we define:

Definition A. A map $\varphi\colon A \to B$ between C*-algebras is said to be a *weighted $*$-homomorphism* if there exist a C*-algebra D containg B, a $*$-homomorphism $\pi\colon A \to D$, and an element $h \in D$ such that $\varphi(a) = \pi(a)h = h\pi(a)$ for all $a \in A$.

Mathematics Subject Classification 2020: 47B65 (primary); 47B49 (secondary).
Keywords: C*-algebras, weighted homomorphisms, zero-products, range-orthogonality, domain-orthogonality.

By the above-mentioned result of Arendt, it follows that a bounded, linear map between unital, commutative C*-algebras is a weighted $*$-homomorphism if and only if it preserves zero-products. In order to generalize Arendt's result to the noncommutative setting, we also need to consider the following types of orthogonality: two elements a and b in a C*-algebra are said to be *range-orthogonal* if $a^*b = 0$; and are said to be *domain-orthogonal* if $ab^* = 0$.

In his PhD thesis [25], Schweizer showed that for every bounded linear map $\varphi\colon A \to B$ between C*-algebras which preserves range-orthogonality, there exist $h \in B^{**}$, which can be taken to be $\varphi^{**}(1_{A^{**}})$, and a $*$-homomorphism $\pi\colon A \to B^{**}$ satisfying

$$\varphi(a) = \pi(a)h \quad \text{and} \quad h^*\varphi(a^*b) = \varphi(a)^*\varphi(b)$$

for all $a, b \in A$; see [25, Theorem 3.7] and also [22, Theorem 3.6]. See Theorem 2.2 for the complete statement. Note that this result does not show that φ is a weighted $*$-homomorphism, since it is not in general true that the element h above can be chosen to commute with the image of φ. In many cases of interest, it is important at the technical level to know that h can be chosen to commute with $\varphi(A)$, and this motivates us to pursue a characterization of this more restricted class of bounded, linear maps.

Inspired by the result mentioned above, and building on the work done in [22], we prove the following generalization of Arendt's characterization of weighted $*$-homomorphisms to the noncommutative setting; see Theorem 4.2.

Theorem B (Theorem 4.2). *A bounded, linear map between* C**-algebras is a weighted $*$-homomorphism if and only if it preserves range-orthogonality, domain-orthogonality and zero-products.*

Note that two elements in a *commutative* C*-algebra are range-orthogonal if and only if they are domain-orthogonal, and if and only if they have zero-product. In particular, our Theorem B generalizes the description of bounded, linear mpas between unital, commutative C*-algebras that preserve zero-products.

Bounded, linear maps between (noncommutative) C*-algebras that preserve zero-products have been extensively studied by Wong and coauthors, [7, 18, 29], and a general description of the structure of such maps in terms of Jordan homomorphisms is given in [7, Theorem 4.7]. More explicitly, if $\varphi\colon A \to B$ is linear, bounded, and preserves zero-products, then there exists $h \in B^{**}$, which can be taken to be $\varphi^{**}(1_{A^{**}})$, such that

$$h\varphi(a^2) = \varphi(a)^2$$

for all $a \in A$. Moreover, if h is invertible, then $\varphi \cdot h^{-1}$ is a linear Jordan homomorphism from A to B. See Theorem 4.3 for a complete statement.

Since a zero-product preserving linear map which is additionally self-adjoint (equivalently, $*$-preserving) automatically preserves domain- and range-orthogonality, we obtain the following consequence of Theorem B above.

Corollary C (Corollary 4.5). *A self-adjoint, bounded, linear map between* C**-algebras is a weighted $*$-homomorphism if and only if it preserves zero-products.*

The above corollary is the analog of a result of Wolff [28], who showed that a self-adjoint, bounded linear map between C*-algebras that preserves zero-products *among self-adjoint elements* is a weigthed Jordan $*$-homomorphisms. (Wolff showed the result under the assumption that the domain is unital. This condition can be removed using [29, Lemma 2.2].)

In [26], Winter and Zacharias introduced completely positive maps of *order zero* as those completely positive maps between C*-algebras that preserve zero-products among positive elements (equivalently, among self-adjoint elements). The main result of [26] is a characterization of completely positive maps of order zero as the *positively* weighted $*$-homomorphisms. These maps play an important role in the fine structure theory of nuclear C*-algebras; see [17, 27].

Using that a positive, linear map between C*-algebras is automatically bounded and self-adjoint, we obtain the following new characterization of completely positive, order zero maps.

Corollary D (Corollary 4.9). *A map between* C*-*algebras is completely positive, order zero if and only if it is positive and preserves zero-products.*

In particular, every positive, zero-product preserving map between C*-*algebras is completely positive (and order zero).*

As a consequence of Corollary D, we recover the result of Sato [24] that every 2-positive, order-zero map between C*-algebras is automatically completely positive; see Remark 4.10.

Our results are embedded in an enormous body of research in functional analysis on maps that preserve different notions of orthogonality between various classes of Banach algebras and Banach lattices. We refer to the survey articles [1, 16, 22] for an overview.

One major theme are automatic continuity results for certain maps preserving some concept of orthogonality. In the context of von Neumann algebras and more generally AW^*-algebras such results were obtained in [20, 21]. Another main theme are structure results for continuous, orthogonality-preserving maps, and our findings lie in this domain. An emerging area are stability results of orthogonality-preserving maps. In the context of C*-algebras, maps that only approximately preserve zero-products have been studied in [8, 19].

Zero-product preserving maps have also been studied extensively in a purely algebraic context. We refer to the book [5] for an overview, and to [13, 15] for recent results that have applications to C*-algebras.

In Section 2, we study bounded, linear maps between C*-algebras that preserve zero-products, range-orthogonality, or domain-orthogonality and we provide characterizations of these properties in terms of algebraic identities. For example, a bounded, linear map φ preserves zero-products if and only if $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all a, b, c in the domain; see Theorem 2.3. Using this, we show that preserving these types of orthogonality passes to bitransposes and tensor products.

We also consider maps preserving zero-TRO-products or 'usual' orthogonality.

In Section 3, we consider a bounded, linear map $\varphi\colon A \to B$ between C*-algebras, and a $*$-representation $B \subseteq \mathcal{B}(H)$ of B on some Hilbert space H. Assuming that π preserves zero-products, range-orthogonality, and domain-orthogonality, we construct a (canonical) $*$-homomorphism $\pi_\varphi\colon A \to \mathcal{B}(H)$ that takes image in the bicommutant $B'' \subseteq \mathcal{B}(H)$ and that satisfies $\pi_\varphi(a)\varphi(b) = \varphi(ab)$ and $\varphi(a)\pi_\varphi(b) = \varphi(ab)$ for all $a, b \in A$. This is the main technical step in the proof of our main result, which we complete in Section 4.

2. Maps preserving different notions of orthogonality

In this section, we study bounded linear maps between C*-algebras that preserve certain notions of orthogonality. We begin by defining these notions.

Definition 2.1. Let A be a C*-algebra, and let $a, b, c \in A$. We say that

- a and b have *zero-product* if $ab = 0$;
- a and b are *range-orthogonal* if $a^*b = 0$;
- a and b are *domain-orthogonal* if $ab^* = 0$;
- a and b are *orthogonal* if $a^*b = ab^* = 0$;
- a, b and c have *zero-TRO-product* if $ab^*c = 0$.

We will be interested in maps preserving certain combinations of the above notions. More explicitly, we say that a map $\varphi\colon A \to B$ between C*-algebras preserves:

(1) zero-products if $ab = 0$ implies $\varphi(a)\varphi(b) = 0$;

(2) range-orthogonality if $a^*b = 0$ implies $\varphi(a)^*\varphi(b) = 0$;

(3) domain-orthogonality if $ab^* = 0$ implies $\varphi(a)\varphi(b)^* = 0$;

(4) orthogonality if $ab^* = a^*b = 0$ implies $\varphi(a)\varphi(b)^* = \varphi(a)^*\varphi(b) = 0$;

(5) zero-TRO-products if $ab^*c = 0$ implies $\varphi(a)\varphi(b)^*\varphi(c) = 0$.

We provide characterizations of (1), (2), (3) and (5) in terms of algebraic equations (Theorem 2.3), which we then use to show that each of those properties passes to the bitranspose (Proposition 2.4) and to tensor products (Proposition 2.5). In particular, if a bounded linear map $A \to B$ preserves zero-products, then so does every amplification $M_n(A) \to M_n(B)$ – and analogously for maps preserving range-orthogonality, preserving domain-orthogonality, or preserving zero-TRO-products; see Corollary 2.6. In particular, a zero-product preserving map is automatically completely zero-product preserving (and similarly for range-orthogonality, domain-orthogonality, and zero-TRO-products).

The situation for orthogonality-preserving maps is different: As noted in [9, Section 4], the transpose map τ on $M_2(\mathbb{C})$ preserves orthogonality, but its amplification $\tau^{(2)}$ does not. Thus, orthogonality-preservation does not pass to tensor products. Nevertheless, using the structure result for orthogonality-preserving maps obtained in [6], we show that the bitranspose of an orthogonality-preserving map is again orthogonality-preserving; see Proposition 2.4.

For comparison with our results, and to faithfully illustrate the historical developments in this direction, we recall here the following result of Schweizer from [25, Theorem 3.6], whose statement was simplified in [22]:

Theorem 2.2 (Schweizer). *Let $\varphi\colon A \to B$ be a bounded, linear map between* C*-*algebras preserving range-orthogonality. Then*

$$\varphi^{**}(1)\varphi(a^*b) = \varphi(a)^*\varphi(b)$$

for all $a, b \in A$, and there is a $*$-homomorphism $\pi\colon A \to B^{**}$ satisfying $\varphi(a) = \pi(a)\varphi^{**}(1)$ for all $a \in A$. Moreover:

(1) *If $\varphi^{**}(1)$ is invertible, then $\varphi \cdot \varphi^{**}(1)^{-1}$ is a $*$-homomorphism from A to B^{**}.*

(2) *Assume that φ is bijective. If φ^{-1} also preserves range-orthogonality or if $\varphi^{**}(1)$ is normal, then $\varphi^{**}(1)$ is invertible.*

The case of domain-orthogonality is symmetric.

Part (1) of the next result follows from [2, Lemma 3.4]. (Note that by Examples 1.3 (2) and [2, Theorem 2.11], every C*-algebra has property ($\mathbb{B}$) introduced in [2].) The proof in [2, Lemma 3.4] is based on the fact that every element in a unital C*-algebra is a linear combination of unitaries and that unitaries are doubly power-bounded operators. For maps between von Neumann algebras (and more generally, C*-algebras of real rank zero), Part (1) of the next result has also been obtained in [7, Theorem 4.1], using that the linear span of projections is dense.

Part (2) of the next result also follows from [22, Theorem 3.6], which the authors of [22] trace back to the PhD thesis of Schweizer [25]. Our proof is inspired by [7, Lemma 4.4], and uses methods that go back to Wolff [28]. We include the argument for the convenience of the reader, and observe that part (1) can also be deduced from [7, Theorem 4.7] as one can show that the Jordan homomorphisms J_n and J appearing in its statement are actually $*$-homomorphisms in our setting.

Theorem 2.3. *Let $\varphi\colon A \to B$ be a bounded, linear map between* C*-*algebras. Then:*

(1) *φ preserves zero-products if and only if $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for all $a, b, c \in A$.*

(2) *φ preserves range-orthogonality if and only if $\varphi(b^*a)^*\varphi(c) = \varphi(a)^*\varphi(bc)$ for all $a, b, c \in A$.*

(3) *φ preserves domain-orthogonality if and only if $\varphi(ab)\varphi(c)^* = \varphi(a)\varphi(cb^*)^*$ for all $a, b, c \in A$.*

(4) *φ preserves zero-TRO-products if and only if the equality $\varphi(ab)\varphi(c)^*\varphi(de) = \varphi(a)\varphi(d^*cb^*)^*\varphi(e)$ holds for all $a, b, c, d, e \in A$.*

Proof. We only prove statement (2). Statements (1), (3) and (4) are shown analogously.

Backward implication. Let $a, b \in A$ satisfy $a^*b = 0$. We need to verify that

$$\varphi(a)^*\varphi(b) = 0.$$

Let $\varepsilon > 0$. Using an approximate identity in A, we can choose $c \in A$ satisfying $\|b - bc\| < \varepsilon$. Using that $b^*a = 0$ and applying the assumption at the first step, we get

$$\varphi(a)^*\varphi(bc) = \varphi(b^*a)^*\varphi(c) = 0$$

and therefore

$$\left\|\varphi(a)^*\varphi(b)\right\| = \left\|\varphi(a)^*\varphi(b) - \varphi(a)^*\varphi(bc)\right\| \leq \left\|\varphi(a)^*\right\| \|\varphi\| \|b - bc\| \leq \varepsilon \left\|\varphi(a)^*\right\| \|\varphi\|.$$

Since $\varepsilon > 0$ was arbitrary, we get $\varphi(a)^*\varphi(b) = 0$, as desired.

Forward implication. We assume that φ preserves range-orthogonality. Given $a, b, c \in A$, we need to show that

$$\varphi(b^*a)^*\varphi(c) = \varphi(a)^*\varphi(bc).$$

Using linearity of φ, we may assume that b is self-adjoint and satisfies $\|b\| < 1$.

For $k \geq 2$, let $f_k \colon \mathbb{R} \to [0, 1]$ be the continuous function that takes the value 0 on $(-\infty, \frac{1}{k}] \cup [1 + \frac{1}{k}, \infty)$, that takes the value 1 on $[\frac{2}{k}, 1]$, and that is affine on $[\frac{1}{k}, \frac{2}{k}]$ and on $[1, 1 + \frac{1}{k}]$. Let $m \geq 1$. For each $j \in \{-m, \ldots, m-1\}$ and $k \geq 2$, we define $f_{j,k} \colon \mathbb{R} \to [0, 1]$ by

$$f_{j,k}(t) := f_k(mt - j)$$

for $t \in \mathbb{R}$. Note that the support of $f_{j,k}$ is $(\frac{j}{m} + \frac{1}{mk}, \frac{j+1}{m} + \frac{1}{mk})$. In particular, $f_{j,k}$ and $f_{j',k'}$ are orthogonal if $k \neq k'$. Moreover, the sequence $(f_{j,k})_k$ converges pointwise to the characteristic function of $(\frac{j}{m}, \frac{j+1}{m}]$.

Let $\Phi := \varphi^{**} \colon A^{**} \to B^{**}$ denote the bitranspose of φ. In A^{**}, the sequence $(f_{j,k}(b))_k$ converges weak* to a projection e_j.

Claim 1. *Let $j > j'$. Then $\Phi(e_j^*a)^*\Phi(e_{j'}c) = 0$.*

To prove the claim, let $k \geq 2$. Given k' with $k \leq k'$, we have

$$\big(f_{j,k}(b)^*a\big)^*\big(f_{j',k'}(b)c\big) = 0$$

since $f_{j,k}f_{j',k'} = 0$, and thus

$$\varphi\big(f_{j,k}(b)^*a\big)^*\varphi\big(f_{j',k'}(b)c\big) = 0.$$

Using that Φ is weak*-continuous, and that multiplication in B^{**} is separately weak*-continuous, we get

$$\varphi\big(f_{j,k}(b)^*a\big)^*\Phi(e_{j'}c) = \text{wk*-}\lim_{k'} \varphi\big(f_{j,k}(b)^*a\big)^*\varphi\big(f_{j',k'}(b)c\big) = 0.$$

Since this holds for every $k \geq 2$, we get

$$\Phi(e_j^*a)^*\Phi(e_{j'}c) = \text{wk*-}\lim_{k} \Phi\big(f_{j,k}(b)^*a\big)^*\varphi(e_{j'}c) = 0.$$

This proves the claim.

Claim 2. *Let $j < j'$. Then $\Phi(e_j^* a)^* \Phi(e_{j'} c) = 0$.*

The proof is analogous to that of the previous claim. For fixed $k' \geq 2$, and $k \geq k'$, we have $(f_{j,k}(b)^* a)^*(f_{j',k'}(b)c) = 0$, and thus $\varphi(f_{j,k}(b)^* a)^* \varphi(f_{j',k'}(b)c) = 0$. We then get

$$\Phi(e_j^* a)^* \varphi\big(f_{j',k'}(b)c\big) = \text{wk*-}\lim_{k'} \varphi\big(f_{j,k}(b)^* a\big)^* \varphi\big(f_{j',k'}(b)c\big) = 0.$$

Since this holds for every $k' \geq 2$, we get

$$\Phi(e_j^* a)^* \Phi(e_{j'} c) = \text{wk*-}\lim_{k} \varphi\big(f_{j,k}(b)^* a\big)^* \Phi(e_{j'} c) = 0.$$

This proves the claim.

Using that b is selfadjoint and $\|b\| < 1$, we have

$$\left\| b - \sum_{j=-m}^{m-1} \frac{j}{m} e_j \right\| \leq \frac{1}{m}.$$

Moreover, we have $\sum_j e_j = 1$ and therefore $a = \sum_j e_j^* a$ and $c = \sum_j e_j c$. Set $K := \|\varphi\|^2 \|a\| \|c\|$. We write $x \approx_\varepsilon y$ to mean $\|x - y\| \leq \varepsilon$. Using Claims 1 and 2 at the second and third steps, we get

$$\begin{aligned}
\varphi(b^* a)^* \varphi(c) &\approx_{\frac{K}{m}} \Phi\left(\sum_{j=-m}^{m-1} \frac{j}{m} e_j^* a \right)^* \Phi\left(\sum_{j=-m}^{m-1} e_j c \right) \\
&= \sum_{j=-m}^{m-1} \frac{j}{m} \Phi(e_j^* a)^* \Phi(e_j c) \\
&= \Phi\left(\sum_{j=-m}^{m-1} e_j^* a \right)^* \Phi\left(\sum_{j=-m}^{m-1} \frac{j}{m} e_j c \right) \\
&\approx_{\frac{K}{m}} \varphi(a)^* \varphi(bc).
\end{aligned}$$

Since this holds for every $m \geq 1$, we get the desired equality. ∎

Peralta showed in [23, Proposition 3.7] that if $\varphi\colon A \to B$ preserves zero-products, then so does the restriction of the bitranspose $\varphi^{**}\colon A^{**} \to B^{**}$ to the multiplier algebra $M(A)$. We generalize this by showing that φ^{**} preserves in fact all zero-products in A^{**}. We obtain similar results for range-orthogonality, domain-orthogonality and zero-TRO-products. Using the structure result for orthogonality-preserving maps from [6], we show that orthogonality-preservation also passes to bitransposes.

Proposition 2.4. *Let $\varphi\colon A \to B$ be a bounded, linear map between* C*-*algebras. If φ preserves zero-products (range-orthogonality, domain-orthogonality, zero-TRO-products, orthogonality), then so does the bitranspose $\varphi^{**}\colon A^{**} \to B^{**}$.*

Proof. Set $\Phi = \varphi^{**}\colon A^{**} \to B^{**}$. We first show the result for the case that φ preserves zero-products. By Theorem 2.3 (1), we have $\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc)$ for every $a, b, c \in A$.

For $a, b, c \in A^{**}$, we denote by $E(a,b,c)$ the (potentially false) identity

$$\Phi(ab)\Phi(c) = \Phi(a)\Phi(bc). \qquad E(a,b,c)$$

By assumption, $E(a,b,c)$ holds whenever $a, b, c \in A$. In three steps, we will show that it holds for all $a, b, c \in A^{**}$.

Let $a \in A^{**}$ and $b, c \in A$ be given. Choose a net $(a_\lambda)_\lambda$ in A that converges weak* to a, and note that $E(a_\lambda, b, c)$ is true. Using at the first step that Φ is weak*-continuous and that the multiplication operations in A^{**} and B^{**} are separately weak*-continuous, and using $E(a_\lambda, b, c)$ at the second step, we get

$$\Phi(ab)\Phi(c) = \text{wk*-}\lim_\lambda \Phi(a_\lambda b)\Phi(c) = \text{wk*-}\lim_\lambda \Phi(a_\lambda)\Phi(bc) = \Phi(a)\Phi(bc),$$

so $E(a,b,c)$ holds whenever $a \in A^{**}$ and $b, c \in A$.

Given $a, b \in A^{**}$ and $c \in A$, we choose as above a net $(b_\lambda)_\lambda$ in A converging weak* to b. By the previous paragraph, $E(a, b_\lambda, c)$ is true for all λ. Using weak*-continuity of Φ and the fact that the multiplication operations in A^{**} and B^{**} are separately weak*-continuous, we deduce that $E(a,b,c)$ holds for the triple (a,b,c). The general case for arbitrary $a, b, c \in A^{**}$ is proved similarly. We deduce that Φ preserves zero-products.

Using the characterizations of maps preserving range-orthogonality, domain-orthogonality or zero-TRO-products from Theorem 2.3, an analogous argument shows that Φ preserves range-orthogonality, domain-orthogonality, or zero-TRO-products, whenever φ does.

Lastly, assume that φ preserves orthogonality. Recall that the triple product in a C*-algebra is defined as $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$, and that a bounded linear map θ between C*-algebras is a *triple homomorphism* if $\theta(\{a,b,c\}) = \{\theta(a), \theta(b), \theta(c)\}$ for all a, b, c in the domain of θ. Equivalently, θ satisfies

$$\theta(ab^*c + cb^*a) = \theta(a)\theta(b)^*\theta(c) + \theta(c)\theta(b)^*\theta(a)$$

for all a, b, c. Using an argument similar to the one above, it follows that the bitranspose θ^{**} is a triple homomorphism as well.

By [6, Lemma 1], two elements a, b in a C*-algebra are orthogonal if and only if $\{a,a,b\} = 0$. It follows that every triple homomorphism preserves orthogonality.

Set $h := \Phi(1) \in B^{**}$. Let $h = v|h|$ be the polar decomposition of h. The partial isometry v is also called the *range tripotent* of h, and it is denoted by $r(h)$ in [6]. By [6, Theorem 17], there exists a triple homomorphism $\pi\colon A \to B^{**}$ such that

$$\varphi(a) = hv^*\pi(a) = \pi(a)v^*h$$

for all $a \in A$. Set $\Pi := \pi^{**}\colon A^{**} \to B^{**}$, which is readily checked to be a triple homomorphism It follows that

$$\Phi(a) = hv^*\Pi(a) = \Pi(a)v^*h$$

for all $a \in A^{**}$.

To show that Φ preserves orthogonality, let $a, b \in A^{**}$ be orthogonal, that is, $ab^* = a^*b = 0$. Since Π is a triple homomorphism and therefore preserves orthogonality, we get $\Pi(a)\Pi(b)^* = \Pi(a)^*\Pi(b) = 0$. Using this at the last steps, we get

$$\Phi(a)\Phi(b)^* = \big(hv^*\Pi(a)\big)\big(hv^*\Pi(b)\big)^* = hv^*\Pi(a)\Pi(b)^*vh^* = 0$$

and

$$\Phi(a)^*\Phi(b) = \big(\Pi(a)v^*h\big)^*\big(\Pi(b)v^*h\big) = h^*v\Pi(a)^*\Pi(b)v^*h = 0,$$

as desired. ■

In the next result, we use $\otimes$ to denote the minimal tensor product of C*-algebras. We note that Proposition 2.5 also holds for maximal tensor products, with essentially the same proof.

Proposition 2.5. *Let $\varphi\colon A \to B$ and $\psi\colon C \to D$ be bounded, linear maps between C*-algebras. If φ and ψ preserve zero-products (range-orthogonality, domain-orthogonality, zero-TRO-products), then so does the tensor product map $\varphi \otimes \psi\colon A \otimes C \to B \otimes D$.*

Proof. We prove the results for zero-product preserving maps. The statements for range-orthogonality and domain-orthogonality preserving maps are shown analogously.

Assume that φ and ψ preserve zero-products and set $\alpha := \varphi \otimes \psi\colon A \odot B \to C \odot D$. Using that φ and ψ satisfy the formula from Theorem 2.3 (1), we verify that α satisfies the formula as well. Let

$$a = \sum_j u_j \otimes v_j, \quad b = \sum_k w_k \otimes x_k, \quad \text{and} \quad c = \sum_l y_l \otimes z_l$$

be finite sums of simple tensors in $A \otimes B$. Then

$$\begin{aligned}\alpha(ab)\alpha(c) &= \Big(\varphi \otimes \psi\Big(\sum_{j,k} u_j w_k \otimes v_j x_k\Big)\Big)\Big(\varphi \otimes \psi\Big(\sum_l y_l \otimes z_l\Big)\Big)\\ &= \sum_{j,k,l} \varphi(u_j w_k)\varphi(y_l) \otimes \psi(v_j x_k)\psi(z_l)\\ &= \sum_{j,k,l} \varphi(u_j)\varphi(w_k y_l) \otimes \psi(v_j)\psi(x_k z_l) = \alpha(a)\alpha(bc).\end{aligned}$$

Using that finite sums of simple tensors are dense in $A \otimes B$, and that α is continuous, it follows that $\alpha(ab)\alpha(c) = \alpha(a)\alpha(bc)$ for all $a, b, c \in A \otimes B$. Applying Theorem 2.3 (1), it follows that α preserves zero-products. ■

The next result shows that every bounded, zero-product preserving map is automatically 'completely zero-product preserving' (and similarly for range-orthogonality preserving, domain-orthogonality preserving and zero-TRO-product preserving). For orthogonality-preserving maps, this is not the case: the transpose map on $M_2(\mathbb{C})$ is orthogonality-preserving, but its amplifications are not.

Corollary 2.6. *Let $\varphi\colon A \to B$ be a bounded, linear map between C*-algebras. If φ preserves zero-products (range-orthogonality, domain-orthogonality, zero-TRO-product), then so does the amplification $\varphi^{(n)}\colon M_n(A) \to M_n(B)$ for every $n \geq 1$.*

Proof. This follows by taking $C = D = M_n$ and $\psi = \mathrm{id}_{M_n}$ in Proposition 2.5. ■

Some of the statements in this section do not involve the adjoint operation in a C*-algebra, and therefore make sense in more general settings. It would thus be interesting to find other classes of Banach algebras for which part (1) of Theorem 2.3 or Corollary 2.6 are true. For example, one could explore these questions in the context of L^p-operator algebras [10], or at least for the well-behaved class arising from groups as in [11, 12].

3. Constructing the support $*$-homomorphisms

Throughout this section, we assume that $\varphi\colon A \to B$ is a bounded, linear map between C*-algebras that preserves zero-products, range-orthogonality, and also domain-orthogonality. By Theorem 2.3, for all $a, b, c \in A$ we have

$$\varphi(ab)\varphi(c) = \varphi(a)\varphi(bc); \tag{3.1}$$

$$\varphi(b^*a)^*\varphi(c) = \varphi(a)^*\varphi(bc); \tag{3.2}$$

$$\varphi(ab)\varphi(c)^* = \varphi(a)\varphi(cb^*)^*. \tag{3.3}$$

For later use, we observe that the *adjoint* $\varphi^*\colon A \to B$ of φ, defined by $\varphi^*(a) = \varphi(a^*)^*$ for all $a \in A$, also preserves zero-products, range-orthogonality, and also domain-orthogonality. We fix a representation $B \subseteq \mathcal{B}(H)$ of B on some Hilbert space. Our goal is to construct a $*$-homomorphism $\pi_\varphi\colon A \to B'' \subseteq \mathcal{B}(H)$ such that

$$\pi_\varphi(a)\varphi(b) = \varphi(ab) = \varphi(a)\pi_\varphi(b)$$

for all $a, b \in A$. Using this, we will show in Theorem 4.2 that φ is a weighted $*$-homomorphism in the sense of Definition A.

We begin the construction by setting

$$H_0 := \operatorname{span}\big(\{\varphi(a)\xi : a \in A, \xi \in H\} \cup \{\varphi(a)^*\xi : a \in A, \xi \in H\}\big).$$

Given $a \in A$, we first show that there exists a unique operator $\pi_\varphi(a) \in \mathcal{B}(H)$ such that

$$\pi_\varphi(a)\varphi(b)\xi = \varphi(ab)\xi, \quad \text{and} \quad \pi_\varphi(a)\varphi(c)^*\eta = \varphi(ca^*)^*\eta \tag{3.4}$$

for all $b, c \in A$ and $\xi, \eta \in H$, and such that $\pi_\varphi(a)\zeta = 0$ for all $\zeta \in H_0^\perp$.

Lemma 3.1. *Let J and K be finite index sets, let $b_j \in A$ and $\xi_j \in H$ for $j \in J$, and let $c_k \in A$ and $\eta_k \in H$ for $k \in K$. Then*

$$\Big\| \sum_{j \in J} \varphi(ab_j)\xi_j + \sum_{k \in K} \varphi(c_k a^*)^*\eta_k \Big\| \leq \|a\| \Big\| \sum_{j \in J} \varphi(b_j)\xi_j + \sum_{k \in K} \varphi(c_k)^*\eta_k \Big\|$$

for all $a \in A$.

Proof. We first establish the following

Claim 1. Given $a \in A$, we have

$$\Big\| \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^* \eta_k \Big\|^2 \\ \leq \Big\| \sum_j \varphi(a^*ab_j)\xi_j + \sum_k \varphi(c_k a^* a)^* \eta_k \Big\| \Big\| \sum_j \varphi(b_j)\xi_j + \sum_k \varphi(c_k)^* \eta_k \Big\|.$$

To prove the claim, let $j \in J$ and $k' \in K$. Then

$$\begin{aligned} \langle \varphi(ab_j)\xi_j, \varphi(c_{k'}a^*)^*\eta_{k'} \rangle &= \langle \varphi(c_{k'}a^*)\varphi(ab_j)\xi_j, \eta_{k'} \rangle \\ &\overset{(3.1)}{=} \langle \varphi(c_{k'})\varphi(a^*ab_j)\xi_j, \eta_{k'} \rangle \\ &= \langle \varphi(a^*ab_j)\xi_j, \varphi(c_{k'})^*\eta_{k'} \rangle. \end{aligned}$$

Similarly, for all $j, j' \in J$ we obtain

$$\begin{aligned} \langle \varphi(ab_j)\xi_j, \varphi(ab_{j'})\xi_{j'} \rangle &= \langle \varphi(ab_{j'})^*\varphi(ab_j)\xi_j, \xi_{j'} \rangle \\ &\overset{(3.2)}{=} \langle \varphi(b_{j'})^*\varphi(a^*ab_j)\xi_j, \xi_{j'} \rangle \\ &= \langle \varphi(a^*ab_j)\xi_j, \varphi(b_{j'})\xi_{j'} \rangle. \end{aligned}$$

Also, for all $k, k' \in K$ we have

$$\begin{aligned} \langle \varphi(c_k a^*)^*\eta_k, \varphi(c_{k'}a^*)^*\eta_{k'} \rangle &= \langle \varphi(c_{k'}a^*)\varphi(c_k a^*)^*\eta_k, \eta_{k'} \rangle \\ &\overset{(3.3)}{=} \langle \varphi(c_{k'})\varphi(c_k a^* a)^*\eta_k, \eta_{k'} \rangle \\ &= \langle \varphi(c_k a^* a)^*\eta_k, \varphi(c_{k'})^*\eta_{k'} \rangle. \end{aligned}$$

Further, for each $k \in K$ and $j' \in J$ we get

$$\langle \varphi(c_k a^*)^*\eta_k, \varphi(ab_{j'})\xi_{j'} \rangle = \langle \varphi(c_k a^* a)^*\eta_k, \varphi(b_{j'})\xi_{j'} \rangle.$$

Using all of these equalities at the third step, and using the Cauchy–Schwarz inequality at the last step, we get

$$\begin{aligned} &\Big\| \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^* \eta_k \Big\|^2 \\ &= \Big\langle \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^*\eta_k, \sum_{j'} \varphi(ab_{j'})\xi_{j'} + \sum_{k'} \varphi(c_{k'}a^*)^*\eta_{k'} \Big\rangle \\ &= \sum_{j,j'} \langle \varphi(ab_j)\xi_j, \varphi(ab_{j'})\xi_{j'} \rangle + \sum_{j,k'} \langle \varphi(ab_j)\xi_j, \varphi(c_{k'}a^*)^*\eta_{k'} \rangle \\ &\quad + \sum_{k,j'} \langle \varphi(c_k a^*)^*\eta_k, \varphi(ab_{j'})\xi_{j'} \rangle + \sum_{k,k'} \langle \varphi(c_k a^*)^*\eta_k, \varphi(c_{k'}a^*)^*\eta_{k'} \rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,j'} \big\langle \varphi(a^*ab_j)\xi_j , \varphi(b_{j'})\xi_{j'} \big\rangle + \sum_{j,k'} \big\langle \varphi(a^*ab_j)\xi_j , \varphi(c_{k'})^*\eta_{k'} \big\rangle \\
&\quad + \sum_{k,j'} \big\langle \varphi(c_k a^*a)^*\eta_k , \varphi(b_{j'})\xi_{j'} \big\rangle + \sum_{k,k'} \big\langle \varphi(c_k a^*a)^*\eta_k , \varphi(c_{k'})^*\eta_{k'} \big\rangle \\
&= \Big\langle \sum_j \varphi(a^*ab_j)\xi_j + \sum_k \varphi(c_k a^*a)^*\eta_k , \sum_{j'} \varphi(b_{j'})\xi_{j'} + \sum_{k'} \varphi(c_{k'})^*\eta_{k'} \Big\rangle \\
&\leq \Big\| \sum_j \varphi(a^*ab_j)\xi_j + \sum_k \varphi(c_k a^*a)^*\eta_k \Big\| \Big\| \sum_j \varphi(b_j)\xi_j + \sum_k \varphi(c_k)^*\eta_k \Big\|.
\end{aligned}
$$

This proves the claim.

Fix $a \in A$ for the rest of the proof. We set

$$
C = \Big\| \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^*\eta_k \Big\| \quad \text{and} \quad D = \Big\| \sum_j \varphi(b_j)\xi_j + \sum_k \varphi(c_k)^*\eta_k \Big\|.
$$

For $n \in \mathbb{N}$, we denote by $I(n)$ the (potentially false) expression

$$
C^2 \leq \Big\| \sum_j \varphi\big((a^*a)^{2^n} b_j\big)\xi_j + \sum_k \varphi(c_k a^*a)^*\eta_k \Big\|^{\frac{1}{2^n}} D^{2-\frac{1}{2^n}}. \tag{$I(n)$}
$$

Claim 2. $I(n)$ is true for all $n \in \mathbb{N}$. *We will prove the claim by induction. Applying Claim 1, we get*

$$
\begin{aligned}
C^2 &= \Big\| \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^*\eta_k \Big\|^2 \\
&\leq \Big\| \sum_j \varphi(a^*ab_j)\xi_j + \sum_k \varphi(c_k a^*a)^*\eta_k \Big\| \Big\| \sum_j \varphi(b_j)\xi_j + \sum_k \varphi(c_k)^*\eta_k \Big\| \\
&= \Big\| \sum_j \varphi(a^*ab_j)\xi_j + \sum_k \varphi(c_k a^*a)^*\eta_k \Big\|^{\frac{1}{2^0}} D^{2-\frac{1}{2^0}}.
\end{aligned}
$$

In other words, $I(0)$ holds. For the induction step, assume that $I(n)$ is true for some $n \geq 0$. Applying Claim 1 with the self-adjoint element $(a^*a)^{2^n}$ in place of a, we obtain

$$
\begin{aligned}
&\Big\| \sum_j \varphi\big((a^*a)^{2^n} b_j\big)\xi_j + \sum_k \varphi\big(c_k (a^*a)^{2^n}\big)^*\eta_k \Big\| \\
&\qquad \leq \Big\| \sum_j \varphi\big((a^*a)^{2^{n+1}} b_j\big)\xi_j + \sum_k \varphi\big(c_k (a^*a)^{2^{n+1}}\big)^*\eta_k \Big\|^{\frac{1}{2}} D^{\frac{1}{2}}.
\end{aligned}
$$

Using this at the second step, we obtain

$$
\begin{aligned}
C^2 &\overset{I(n)}{\leq} \Big\| \sum_j \varphi\big((a^*a)^{2^n} b_j\big)\xi_j + \sum_k \varphi\big(c_k (a^*a)^{2^n}\big)^*\eta_k \Big\|^{\frac{1}{2^n}} D^{2-\frac{1}{2^n}} \\
&\leq \Big\| \sum_j \varphi\big((a^*a)^{2^{n+1}} b_j\big)\xi_j + \sum_k \varphi\big(c_k (a^*a)^{2^{n+1}}\big)^*\eta_k \Big\|^{\frac{1}{2^{n+1}}} D^{\frac{1}{2^{n+1}}} D^{2-\frac{1}{2^n}},
\end{aligned}
$$

which shows that $I(n+1)$ is true. This completes the induction and proves the claim.

Now, for every $n \geq 0$, we get

$$C^2 \overset{I(n)}{\leq} \Big\| \sum_j \varphi\big((a^*a)^{2^n} b_j\big)\xi_j + \sum_k \varphi(c_k a^* a)^* \eta_k \Big\|^{\frac{1}{2^n}} D^{2-\frac{1}{2^n}}$$
$$\leq \Big(\sum_j \|\varphi\| \|\xi_j\| + \sum_k \|\varphi\| \|\eta_k\| \Big)^{\frac{1}{2^n}} \|(a^*a)^{2^n}\|^{\frac{1}{2^n}} D^{2-\frac{1}{2^n}}.$$

Using that $\|(a^*a)^{2^n}\|^{\frac{1}{2^n}} = \|a\|^2$, and taking to the limit as $n \to \infty$, we deduce that $C^2 \leq \|a\|^2 D^2$, and so $C \leq \|a\| D$, as desired. ∎

Given $a \in A$, it follows from Lemma 3.1 that the map $H_0 \to H_0$, given by

$$\sum_j \varphi(b_j)\xi_j + \sum_k \varphi(c_k)^* \eta_k \mapsto \sum_j \varphi(ab_j)\xi_j + \sum_k \varphi(c_k a^*)^* \eta_k$$

is well defined, bounded and linear, and therefore extends to a bounded, linear map

$$\pi_\varphi^{(0)}(a) \colon \overline{H_0} \to \overline{H_0}$$

satisfying $\|\pi_\varphi^{(0)}(a)\| \leq \|a\|$ for all $a \in A$. We define $\pi_\varphi(a) \in \mathcal{B}(H)$ by $\pi_\varphi(a)\xi := \pi_\varphi^{(0)}(a)\xi$ for $\xi \in \overline{H_0}$ and $\pi_\varphi(a)\eta := 0$ for $\eta \in \overline{H_0}^\perp$. We note that π_φ is determined by

$$\pi_\varphi(a)\varphi(b)\xi = \varphi(ab)\xi \quad \text{and} \quad \pi_\varphi(a)\varphi(b)^*\xi = \varphi(ba^*)^*\xi \tag{3.5}$$

for all $a, b \in A$ and all $\xi \in H$. Moreover, one readily checks that $\pi_\varphi = \pi_{\varphi^*}$.

Proposition 3.2. *Let A and B be C*-algebras with $B \subseteq \mathcal{B}(H)$, and let $\varphi\colon A \to B$ be a bounded, linear map between C*-algebras that preserves zero-products, range-orthogonality, and also domain-orthogonality. Denote by $\pi_\varphi\colon A \to \mathcal{B}(H)$ the canonical bounded, linear map defined in the preceding comments. Then:*

(1) *The map π_φ is a $*$-homomorphism.*

(2) *The image of π_φ is contained in $B'' \subseteq \mathcal{B}(H)$.*

(3) *For all $a, b \in A$, we have*

$$\pi_\varphi(a)\varphi(b) = \varphi(ab) = \varphi(a)\pi_\varphi(b),$$
$$\pi_\varphi(a)\varphi(b)^* = \varphi(ba^*)^* = \varphi(a^*)^*\pi_\varphi(b^*).$$

Proof. (1) We first show that π_φ, which we will abbreviate to π throughout in the proof of this proposition, is linear and multiplicative. For a_1, a_2 and $\lambda \in \mathbb{C}$, we have

$$\pi(a_1 + \lambda a_2)\varphi(b)\xi = \varphi\big((a_1 + \lambda a_2)b\big)\xi$$
$$= \varphi(a_1 b)\xi + \lambda\varphi(a_2 b)\xi = \big(\pi(a_1) + \lambda\pi(a_2)\big)\varphi(b)\xi,$$
$$\pi(a_1)\pi(a_2)\varphi(b)\xi = \pi(a_1)\varphi(a_2 b)\xi = \varphi(a_1 a_2 b)\xi = \pi(a_1 a_2)\varphi(b)\xi$$

for all $b \in A$ and $\xi \in H$. Similarly, we have

$$\begin{aligned}\pi(a_1+\lambda a_2)\varphi(c)^*\eta &= \varphi\big(c(a_1+\lambda a_2)^*\big)^*\eta \\ &= \varphi(ca_1^*)^*\eta + \lambda\varphi(ca_2^*)^*\eta = \big(\pi(a_1)+\lambda\pi(a_2)\big)\varphi(c)^*\eta, \\ \pi(a_1)\pi(a_2)\varphi(c)^*\eta &= \pi(a_1)\varphi(ca_2^*)^*\eta = \varphi(ba_2^*a_1^*)^*\eta = \pi(a_1a_2)\varphi(c)^*\eta\end{aligned}$$

for all $c \in A$ and $\eta \in H$. Using linearity and continuity of $\pi(a_1+\lambda a_2)$ and $\pi(a_1)+\lambda\pi(a_2)$, it follows that these operators agree on $\overline{H_0}$. Since both operators vanish on $\overline{H_0}^\perp$, it follows that $\pi(a_1+\lambda a_2) = \pi(a_1)+\lambda\pi(a_2)$. Similarly, we obtain $\pi(a_1)\pi(a_2) = \pi(a_1a_2)$. Thus, π is linear and multiplicative.

To show that π preserves adjoints, let $a \in A$. Given $b, b' \in A$ and $\xi, \xi' \in H$, using (3.2) at the fourth step, we get

$$\begin{aligned}\langle \pi(a)^*\varphi(b)\xi, \varphi(b')\xi'\rangle &= \langle \varphi(b)\xi, \pi(a)\varphi(b')\xi'\rangle = \langle \varphi(b)\xi, \varphi(ab')\xi'\rangle \\ &= \langle \varphi(ab')^*\varphi(b)\xi, \xi'\rangle = \langle \varphi(b')^*\varphi(a^*b)\xi, \xi'\rangle \\ &= \langle \varphi(a^*b)\xi, \varphi(b')\xi'\rangle = \langle \pi(a^*)\varphi(b)\xi, \varphi(b')\xi'\rangle.\end{aligned}$$

Similarly, given $c, c' \in A$ and $\eta, \eta' \in H$, using (3.3) at the fourth step, we get

$$\begin{aligned}\langle \pi(a)^*\varphi(c)^*\eta, \varphi(c')^*\eta'\rangle &= \langle \varphi(c)^*\eta, \pi(a)\varphi(c')^*\eta'\rangle = \langle \varphi(c)^*\eta, \varphi(c'a^*)^*\eta'\rangle \\ &= \langle \varphi(c'a^*)\varphi(c)^*\eta, \eta'\rangle = \langle \varphi(c')\varphi(ca)^*\eta, \eta'\rangle \\ &= \langle \varphi(ca)^*\eta, \varphi(c')^*\eta'\rangle = \langle \pi(a^*)\varphi(c)^*\eta, \varphi(c')^*\eta'\rangle.\end{aligned}$$

Further, given $b, c \in A$ and $\xi, \eta \in H$, using (3.1) at the fourth step, we get

$$\begin{aligned}\langle \pi(a)^*\varphi(b)\xi, \varphi(c)^*\eta\rangle &= \langle \varphi(b)\xi, \pi(a)\varphi(c)^*\eta\rangle = \langle \varphi(b)\xi, \varphi(ca^*)^*\eta\rangle \\ &= \langle \varphi(ca^*)\varphi(b)\xi, \eta\rangle = \langle \varphi(c)^*\varphi(a^*b)\xi, \eta\rangle \\ &= \langle \varphi(a^*b)\xi, \varphi(c)^*\eta\rangle = \langle \pi(a^*)\varphi(b)\xi, \varphi(c)^*\eta\rangle.\end{aligned}$$

and analogously

$$\langle \pi(a)^*\varphi(c)^*\eta, \varphi(b)\xi\rangle = \langle \pi(a^*)\varphi(c)^*\eta, \varphi(b)\xi\rangle.$$

Using linearity and continuity of $\pi(a)^*$ and $\pi(a^*)$, we get

$$\langle \pi(a)^*\alpha, \beta\rangle = \langle \pi(a^*)\alpha, \beta\rangle$$

for all $\alpha, \beta \in \overline{H_0}$ and consequently $\pi(a)^* = \pi(a^*)$, as desired.

(2) Let $a \in A$, and let $x \in B'$. We need to show that $\pi(a)x = x\pi(a)$. Given $b \in B$ and $\xi \in H$, using at the first and third step that x commutes with $\varphi(b)$ and $\varphi(ab)$, we have

$$\pi(a)x\varphi(b)\xi = \pi(a)\varphi(b)x\xi = \varphi(ab)x\xi = x\varphi(ab)\xi = x\pi(a)\varphi(b)\xi.$$

Similarly, given $c \in B$ and $\xi \in H$, using that $\varphi(c)^*$ and $\varphi(ca^*)^*$ belong to B and that they therefore commute with x, we get

$$\pi(a)x\varphi(c)^*\xi = \pi(a)\varphi(c)^*x\xi = \varphi(ca^*)^*x\xi = x\varphi(ca^*)^*\xi = x\pi(a)\varphi(c)^*\xi.$$

Using linearity and continuity of $\pi(a)x$ and $x\pi(a)$, it follows that $\pi(a)x\xi = x\pi(a)\xi$ for all $\xi \in \overline{H_0}$.

Using that x commutes with $\varphi(b)$ and $\varphi(c)^*$ for all $b, c \in A$, one readily checks that $x(\overline{H_0}) \subseteq \overline{H_0}$. Similarly, since x^* commutes with $\varphi(b)$ and $\varphi(c)^*$ for all $b, c \in A$, we get $x^*(\overline{H_0}) \subseteq \overline{H_0}$. It follows that x leaves $\overline{H_0}^\perp$ invariant, and thus

$$\pi(a)x\eta = 0 = x\pi(a)\eta$$

for all $\eta \in \overline{H_0}^\perp$. In conclusion, we obtain $\pi(a)x = x\pi(a)$, as desired.

(3) Let $a, b \in A$. We have

$$\pi(a)\varphi(b)\xi \overset{(3.5)}{=} \varphi(ab)\xi$$

for all $\xi \in H$, and therefore $\pi(a)\varphi(b) = \varphi(ab)$.

To check that $\varphi(a)\pi(b) = \varphi(ab)$, we prove that these operators agree both on $\overline{H_0}$ and on its orthogonal complement. Given $c \in A$, we have

$$\varphi(a)\pi(b)\varphi(c)\xi \overset{(3.5)}{=} \varphi(a)\varphi(bc)\xi \overset{(3.1)}{=} \varphi(ab)\varphi(c)\xi. \tag{3.6}$$

Similarly,

$$\varphi(a)\pi(b)\varphi(c)^*\xi \overset{(3.5)}{=} \varphi(a)\varphi(cb^*)^*\xi \overset{(3.3)}{=} \varphi(ab)\varphi(c)^*\xi. \tag{3.7}$$

It follows from (3.6) and (3.7) that the linear maps $\varphi(a)\pi(b)$ and $\varphi(ab)$ agree on $\overline{H_0}$. Let $\eta \in \overline{H_0}^\perp$. Then $\pi(b)\eta = 0$. On the other hand, we have

$$\langle\varphi(ab)\eta, \varphi(ab)\eta\rangle = \langle\eta, \varphi(ab)^*\varphi(ab)\eta\rangle = 0,$$

where at the last step we use that $\varphi(ab)^*\varphi(ab)\eta$ belongs to $\overline{H_0}$. This shows that $\varphi(ab)$ and $\varphi(a)\pi(b)$ agree on $\overline{H_0}^\perp$, and hence $\varphi(a)\pi(b) = \varphi(ab)$.

The last two equalities are proved analogously. (They also follow by replacing φ with φ^*, which satisfies the same assumptions as φ and has $\pi_\varphi = \pi_{\varphi^*}$.) ∎

Notation 3.3. Let A and B be C*-algebras and let $\varphi\colon A \to B \subseteq \mathcal{B}(H)$ be a bounded, linear map that preserves zero-products, range-orthogonality, and also domain-orthogonality. Denote by $\pi_\varphi\colon A \to B''$ the canonical $*$-homomorphism provided by Proposition 3.2. We denote by $\Phi, \Pi_\varphi\colon A^{**} \to \mathcal{B}(H)$ the (unique) extensions of $\varphi, \pi_\varphi\colon A \to \mathcal{B}(H)$ to weak*-continuous, bounded linear maps, which are explicitly constructed as follows. Let $S_1(H)$ denote the space of trace-class of operators on H, which we naturally identify with the (unique) isometric predual of $\mathcal{B}(H)$. Let $\kappa\colon S_1(H) \to S_1(H)^{**}$ be the natural inclusion of $S_1(H)$ into its bidual. Then the transpose map

$$\kappa^*\colon \mathcal{B}(H)^{**} \cong S_1(H)^{***} \to S_1(H)^* \cong \mathcal{B}(H)$$

is a weak*-continuous $*$-homomorphism, and

$$\Phi = \kappa^* \circ \varphi^{**}\colon A^{**} \to \mathcal{B}(H), \quad \text{and} \quad \Pi_\varphi = \kappa^* \circ \pi_\varphi^{**}\colon A^{**} \to \mathcal{B}(H).$$

By Proposition 2.4, φ^{**} preserves range-orthogonality, domain-orthogonality and zero-products, and hence so does Φ. Similarly, Π_φ is a $*$-homomorphism. Note that the images of Φ and Π_φ are contained in $B'' \subseteq \mathcal{B}(H)$.

Proposition 3.4. *Let A and B be* C*-*algebras and let $\varphi\colon A \to B \subseteq \mathcal{B}(H)$ be a bounded, linear map that preserves zero-products, range-orthogonality, and also domain-orthogonality. Denote by $\pi_\varphi\colon A \to B''$ the canonical $*$-homomorphism provided by Proposition 3.2, and let $\Phi, \Pi_\varphi\colon A^{**} \to B''$ the maps from Notation 3.3.*

(1) *For all $a, b \in A$, we have*

$$\Pi_\varphi(a)\Phi(b) = \Phi(ab) = \Phi(a)\Pi_\varphi(b),$$
$$\Pi_\varphi(a)\Phi(b)^* = \Phi(ba^*)^* = \Phi(a^*)^*\Pi_\varphi(b^*).$$

Let C be the (not necessarily self-adjoint) closed subalgebra of B generated by the image of φ, and let $D :=$ C$(\varphi(A))$ be the sub-*C*-*algebra of B generated by $\varphi(A)$.*

(2) *Let $a \in M(A) \subseteq A^{**}$. Then $\Phi(a)$ and $\Pi_\varphi(a)$ normalize C and D, that is, $\Phi(a)x$, $x\Phi(a)$, $\Pi_\varphi(a)x$, and $x\Pi_\varphi(a)$ belong to C (respectively, to D) for every $x \in C$ (respectively, $x \in D$).*

(3) *If a belongs to the center of $M(A)$, then $\Phi(a)$ belongs to C'.*

Proof. (1) This follows from part (3) of Proposition 3.2 by applying the same argument as in the proof of Proposition 2.4, using twice that multiplication on $\mathcal{B}(H)$ is separately weak*-continuous.

(2) We only show the normalization for D. The proof for C is similar, but easier. Set

$$G := \{\varphi(b)\colon b \in A\} \cup \{\varphi(c)^*\colon c \in A\}.$$

Then the set of finite linear sums of finite products of elements in G is dense in D.

We first show that $\Pi_\varphi(a)$ is a left normalizer of D, that is, $\Pi_\varphi(a)D \subseteq D$. Using that $\Pi_\varphi(a)$ is a bounded, linear operator, it suffices to show that $\Pi_\varphi(a)x \in D$ whenever $x \in G^n$ is a finite product of elements in G. Let $x = gy$ with $g \in G$ and $y \in G^n$, for some $n \in \mathbb{N}$. If $g = \varphi(b)$ for some $b \in A$, then applying part (1) of Proposition 3.4, and using that $ab \in A$, we obtain

$$\Pi_\varphi(a)x = \Pi_\varphi(a)gy = \Pi_\varphi(a)\Phi(b)y = \Phi(ab)y = \varphi(ab)y \in G^{n+1} \subseteq D.$$

Similarly, if $g = \varphi(c)^*$ for some $c \in A$, using that $ca^* \in A$, we obtain

$$\Pi_\varphi(a)x = \Pi_\varphi(a)gy = \Pi_\varphi(a)\Phi(c)^*y = \Phi(ca^*)^*y = \varphi(ca^*)^*y \in G^{n+1} \subseteq D.$$

Analogously, one verifies that $\Pi_\varphi(a)$ is a right normalizer of D, that is, we have $D\Pi_\varphi(a) \subseteq D$.

Next, we show that $\Phi(a)$ is a left normalizer of D. Again, it suffices to verify that $\Phi(a)x \in D$ for $x = gy$ with $g \in G$ and $y \in G^n$ a product of n elements in G. If $g = \varphi(b)$ for some $b \in A$, then choose $b_1, b_2 \in A$ with $b = b_1 b_2$. Using that Φ preserves zero-products and therefore satisfies the formula in Theorem 2.3 (1), and using that $ab_1 \in A$, we obtain

$$\begin{aligned}\Phi(a)x &= \Phi(a)gy = \Phi(a)\Phi(b_1b_2)y = \Phi(ab_1)\Phi(b_2)y \\ &= \varphi(ab_1)\varphi(b_2)y \in G^{n+2} \subseteq D.\end{aligned}$$

Similarly, if $g = \varphi(c)^*$ for some $c \in A$, then choose $c_1, c_2 \in A$ with $c = c_1 c_2$. Using that Φ preserves domain-orthogonality and therefore satisfies the formula in part (3) of Theorem 2.3, and using that $ac_2^* \in A$, we obtain

$$\begin{aligned}\Phi(a)x &= \Phi(a)gy = \Phi(a)\Phi(c_1c_2)^*y = \Phi(ac_2^*)\Phi(c_1)^*y \\ &= \varphi(ac_2^*)\varphi(c_1)^*y \in G^{n+2} \subseteq D.\end{aligned}$$

Analogously, one verifies that $\Phi(a)$ is a left normalizer for D.

(3) Let $a \in M(A)$ be central. Let $b \in A$ and denote by 1_A the unit of $M(A)$. Using that Φ preserves zero-products and therefore satisfies the formula in Theorem 2.3 (1), we get

$$\begin{aligned}\Phi(a)\varphi(b) &= \Phi(1_Aa)\Phi(b) = \Phi(1_A)\Phi(ab) = \Phi(1_A)\Phi(ba) \\ &= \Phi(1_Ab)\Phi(a) = \varphi(b)\Phi(a).\end{aligned}$$

We deduce that $\Phi(a)$ commutes with every finite linear combination of products of finitely many elements in $\varphi(A)$. Since such elements are dense in C, we have $\Phi(a) \in C'$. ∎

4. Weighted $*$-homomorphisms

In this section, we prove the main result of the paper (Theorem 4.2), where we obtain an intrinsic and algebraic characterization of weighted $*$-homomorphisms between C*-algebras. Our characterization simplifies considerably for self-adjoint and positive maps; see Corollaries 4.5 and 4.9.

Lemma 4.1. *Let $\varphi\colon A \to B$ be a bounded, linear map between C*-algebras that preserves range-orthogonality, domain-orthogonality and zero-products. Assume that $B = C^*(\varphi(A))$, that is, $\varphi(A)$ is not contained in a proper sub-C*-algebra of B. Let C be the closed subalgebra of B generated by the image of φ. Set $h := \varphi^{**}(1) \in B^{**}$.*

Then there exists a canonical $*$-homomorphism $\pi_\varphi\colon A \to B^{**}$ such that

$$\varphi(a) = h\pi_\varphi(a) = \pi_\varphi(a)h \quad \text{for all } a \in A,$$

and such that $\pi_\varphi(a)$ normalizes both B and C for every $a \in A$. Further, h normalizes B and C, and commutes with every element of C. In particular, we may view π_φ as a $*$-homomorphism $\pi_\varphi\colon A \to M(B) \cap \{h\}'$, and h belongs to $M(B) \cap C'$.

Proof. Use [4, Section III.5.2] to choose a (universal) representation $B \subseteq \mathcal{B}(H)$ such that, with $\kappa^*\colon \mathcal{B}(H)^{**} \to \mathcal{B}(H)$ denoting the $*$-homomorphism described in Notation 3.3, the restriction of κ^* to B^{**} is an isomorphism onto B''. The situation is shown in the following commutative diagram:

$$\begin{tikzcd}
& B^{**} \arrow[rr, hook] \arrow[dr, "\kappa^*|_{B^{**}}"] \arrow[d, phantom, shift left=4ex, "\cong"] & & \mathcal{B}(H)^{**} \arrow[d, "\kappa^*"] \\
A \arrow[ur, "\varphi^{**}"] \arrow[r, "\varphi"] & B \arrow[u, hook] \arrow[r, hook] & B'' \arrow[r, hook] & \mathcal{B}(H).
\end{tikzcd}$$

By Proposition 3.2, there is a canonical *-homomorphism $\pi_\varphi\colon A \to B''$ such that

$$\pi_\varphi(a)\varphi(b) = \varphi(ab) = \varphi(a)\pi_\varphi(b)$$

for all $a, b \in A$. Let $\Phi = \varphi^{**}\colon A^{**} \to B^{**} \subseteq \mathcal{B}(H)$ be the unique extension of φ to a weak-* continuous map. After identifying B^{**} with B'', the map Φ is simply the bitranspose of φ. Hence, applying part (1) of Proposition 3.4 at the second steps, we obtain

$$\varphi(a) = \Phi(1a) = \Phi(1)\pi_\varphi(a) = h\pi_\varphi(a),$$
$$\varphi(a) = \Phi(a1) = \pi_\varphi(a)\Phi(1) = \pi_\varphi(a)h$$

for every $a \in A$.

It follows from part (2) of Proposition 3.4 that $\pi_\varphi(a)$ normalizes both B and C, for each $a \in A$. Moreover, $h = \Phi(1)$ normalizes both B and C by part (3) of Proposition 3.4, and clearly belongs to C'. ∎

The following theorem characterizes weighted $*$-homomorphisms (Definition A) in terms of orthogonality-preservation properties. Note that part (2) gives canonical choices for the algebra D and $h \in D$.

Theorem 4.2. *Let $\varphi\colon A \to B$ be a bounded, linear map between* C*-*algebras. Then the following are equivalent:*

(1) *φ is a weighted $*$-homomorphism, namely: there exist a* C*-*algebra D with $B \subseteq D$, a $*$-homomorphism $\pi\colon A \to D$ and $h \in D$ such that $\varphi(a) = h\pi(a) = \pi(a)h$ for all $a \in A$;*

(2) *there exists a (canonical) $*$-homomorphism $\pi_\varphi\colon A \to B^{**}$ such that we have $\varphi(a) = \varphi^{**}(1)\pi_\varphi(a) = \pi_\varphi(a)\varphi^{**}(1)$ for all $a \in A$;*

(3) *φ preserves range-orthogonality, domain-orthogonality and zero-products.*

Proof. It is clear that (2) implies (1). To show that (1) implies (3), assume that there exist a C*-algebra D with $B \subseteq D$, a $*$-homomorphism $\pi\colon A \to D$ and $h \in D$ such that $\varphi(a) = h\pi(a) = \pi(a)h$ for all $a \in A$. To verify that φ preserves range-orthogonality, let $a, b \in A$ satisfy $a^*b = 0$. Then

$$\varphi(a)^*\varphi(b) = \big(\pi(a)h\big)^*\big(\pi(b)h\big) = h^*\pi(a)^*\pi(b)h = h^*\pi(a^*b)h = 0.$$

Similarly, one verifies that φ preserves domain-orthogonality and zero-products.

To show that (3) implies (2), assume that φ preserves range-orthogonality, domain-orthogonality and zero-products. Let $B_0 := C^*(\varphi(A))$ be the sub-C*-algebra of B generated by the image of φ. Let $\varphi_0\colon A \to B_0$ be the corestriction, and set $h_0 := \varphi_0^{**}(1) \in B_0^{**}$. Applying Lemma 4.1, we obtain a $*$-homomorphism $\pi_0\colon A \to B_0^{**}$ such that $\varphi_0(a) = h_0\pi_0(a) = \pi_0(a)h_0$ for all $a \in A$. The inclusion $\iota\colon B_0 \hookrightarrow B$ induces a natural inclusion $\iota^{**}\colon B_0^{**} \hookrightarrow B^{**}$ that identifies h_0 with $\varphi^{**}(1)$. Then the $*$-homomorphism $\pi_\varphi := \iota^{**} \circ \pi_0\colon A \to B^{**}$ has the desired properties. ∎

For the sake of comparison, we state here a result due to Chebotar, Ke, Lee, and Wong from [7] which was mentioned in the introduction and largely motivated our work.

Theorem 4.3 ([7, Theorem 4.7], see also [22, Theorem 3.5]). *Let $\varphi\colon A \to B$ be a bounded, linear map between C*-algebras preserving zero-produces in A_{sa}. Then $\varphi^{**}(1)$ commutes with $\varphi(A)$, and*

$$\varphi^{**}(1)\varphi(a^2) = \varphi(a)^2$$

for all $a \in A$. Moreover:

(1) *If $\varphi^{**}(1)$ is invertible, then $\varphi \cdot \varphi^{**}(1)^{-1}$ is a linear Jordan homomorphism from A to B^{**}.*

(2) *If $\varphi^{**}(1)$ is normal with support projection $p \in B^{**}$, then there is a sequence $(J_n)_{n\in\mathbb{N}}$ of bounded, linear Jordan homomorphisms $J_n\colon A \to B^{**}$ such that*

$$\mathrm{SOT} - \lim_{n\to\infty} J_n(a)\varphi^{**}(1) = \varphi(a)p$$

for all $a \in A$.

(3) *If φ is surjective, then $\varphi^{**}(1)$ is invertible.*

Remark 4.4. Let us clarify the relationship between our notion of 'weighted $*$-homomorphism' from Definition A and the concept of 'weighted homomorphism' used in the theory of (Banach) algebras.

A map $\varphi\colon A \to B$ between Banach algebras is said to be a *weighted homomorphism* if there exists a homomorphism $\pi\colon A \to B$ and an invertible centralizer W on B such that $\varphi = W\pi$. Here, a *centralizer* on B is a linear map $W\colon B \to B$ satisfying $W(ab) = aW(b) = W(a)b$ for all $a, b \in B$. We use $\Gamma(B)$ to denote the algebra of centralizers on B.

If B is *faithful* (that is, every element $b \in B$ satisfying $bB = \{0\}$ or $Bb = \{0\}$ is zero), then centralizers correspond to central multipliers: A *multiplier* (also called a *double centralizer*) on B is a pair (L, R) of linear maps $L, R\colon B \to B$ such that $aL(b) = R(a)b$ for all $a, b \in B$. We obtain a natural map from $\Gamma(B)$ to the multiplier algebra $M(B)$ given by $W \mapsto (W, W)$. If B is faithful, then this map defines an isomorphism between $\Gamma(B)$ and $Z(M(B))$, the center of the multiplier algebra.

Now let $\varphi\colon A \to B$ be a weighted $*$-homomorphism between C*-algebras. Let $B_0 := C^*(\varphi(A))$ be the sub-C*-algebra of B generated by the image of φ, and let C denote the closed subalgebra of B generated by $\varphi(A)$. By Lemma 4.1, there exists a $*$-homomorphism $\pi\colon A \to M(B_0)$ and a multiplier $h \in M(B_0) \cap C'$ such that $\varphi(a) = h\pi(a) = \pi(a)h$ for all $a \in A$. However, while h commutes with elements in the image of φ (and as a consequence belongs to C'), it may not commute with the *adjoints* of the elements in $\varphi(A)$, and therefore does not necessarily belong to $Z(M(B_0))$.

If h is normal (which is automatically the case if φ is self-adjoint), then Fuglede's theorem implies that h also commutes with the adjoints of elements in $\varphi(A)$, and consequently belongs to $M(B_0) \cap B_0'$, and therefore to $Z(M(B_0))$. Similarly, if $\varphi(A)$ is a self-adjoint subset of B (which is automatically the case if φ is surjective), then h belongs to $Z(M(B_0))$. In both cases, we see that the map $\varphi\colon A \to B_0 \subseteq M(B_0)$ is a weighted homomorphism in the algebraic sense.

On the other hand, we may view φ as a map from A to the Banach algebra C. Note that multiplication by h defines a centralizer on C, and that $\pi(a)$ defines a multiplier on C for each $a \in A$. However, C can be very pathological. For example, if $h^2 = 0$, then multiplication by h defines the zero centralizer on C, the product of any two elements in C is zero, C is not faithful, and the canonical map $C \to M(C)$ is the zero map.

Corollary 4.5. *Let $\varphi\colon A \to B$ be a self-adjoint, bounded, linear map between* C*-*algebras. Then φ is a weighted $*$-homomorphism if and only if φ preserves zero-products.*

Proof. Assume that φ preserves zero-products. For $a, b \in A$ with $a^*b = 0$, we have

$$\varphi(a)^*\varphi(b) = \varphi(a^*)\varphi(b) = 0.$$

That is, φ preserves range-orthogonality. One similarly shows that φ preserves domain-orthogonality, and the statement thus follows from Theorem 4.2. ∎

The above corollary motivates the following problem.

Problem 4.6. Characterize weighted $*$-homomorphisms with normal weights.

The next result is a generalization to the setting of self-adjoint, zero-product preserving maps of the structure theorem of Winter and Zacharias [26, Theorem 3.3] for completely positive, order zero maps.

Corollary 4.7. *Let $\varphi\colon A \to B$ be a self-adjoint, bounded, linear map between* C*-*algebras that preserves zero-products. Set $C := \mathrm{C}^*(\varphi(A)) \subseteq B$. Then $h := \varphi^{**}(1)$ belongs to $M(C) \cap C' \subseteq C^{**} \subseteq B^{**}$, and there exists a canonical $*$-homomorphism $\pi_\varphi\colon A \to M(C) \cap \{h\}'$ such that $\varphi(a) = h\pi_\varphi(a)$ for all $a \in A$.*

Proof. By Corollary 4.5, φ is a weighted $*$-homomorphism and thus preserves range-orthogonality, domain-orthogonality and zero-products by Theorem 4.2. Since φ is self-adjoint, it follows that C agrees with the closed subalgebra of B generated by $\varphi(A)$. Now it follows from Lemma 4.1 that there exists a exists a $*$-homomorphism $\pi_\varphi\colon A \to M(C) \cap \{h\}'$ such that $\varphi(a) = h\pi_\varphi(a)$ for all $a \in A$, and such that h belongs to $M(C) \cap C'$. ∎

We stress the fact that the map π_φ that we obtain in the previous corollary is natural. This has the following consequence:

Remark 4.8. Adopt the notation and assumptions of Corollary 4.7. Let G be a topological group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be continuous actions, and suppose that the map $\varphi\colon A \to B$ is equivariant. Then h is G-invariant, and there is a canonical continuous action of G on C, and hence there is a (not necessarily continuous) action $\gamma\colon G \to \mathrm{Aut}(M(C) \cap \{h\}')$. Using naturality of π_φ at the first and third steps, and equivariance of φ at the second step, we get

$$\pi_\varphi \circ \alpha_g = \pi_{\varphi\circ\alpha_g} = \pi_{\beta_g\circ\varphi} = \gamma_g \circ \pi_\varphi.$$

In other words, the $*$-homomorphism $\pi_\varphi\colon A \to M(C) \cap \{h\}'$ is equivariant. Since the action on A is assumed to be continuous, the range of this map is contained in the γ-*continuous part* of $M(C) \cap \{h\}'$, namely

$$M(C)_\gamma \cap \{h\}' = \big\{x \in M(C)\colon xh = hx \text{ and } g \mapsto \gamma_g(x) \text{ is norm-continuous}\big\}.$$

Recall a linear map $\varphi\colon A \to B$ between C*-algebras is *positive* if $\varphi(A_+) \subseteq B_+$; it is n-*positive* if the amplification $\varphi^{(n)} = \varphi \otimes \mathrm{id}_{M_n}\colon M_n(A) \to M_n(B)$ is positive; and it is *completely positive* if it is n-positive for every $n \in \mathbb{N}$.

Note that a weighted $*$-homomorphism with weight h is self-adjoint (positive) if and only if h is self-adjoint (positive). It is easy to see that a positively weighted $*$-homomorphism is even completely positive. Hence, we obtain the following characterization of completely positive, order zero maps.

Corollary 4.9. *A map between* C*-*algebras is completely positive, order zero if and only if it is positive and preserves zero-products. In particular, every positive, zero-product preserving map between* C*-*algebras is automatically completely positive.*

Remark 4.10. Following Sato, [24], we say that a positive (but not necessarily completely positive) map between C*-algebras is *order zero* if it preserves zero-products of positive elements. By [24, Corollary 3.7], every 2-positive, order zero map is completely positive. On the other hand, not every positive, order zero map is automatically completely positive, and the transpose map on $M_2(\mathbb{C})$ is a counterexample.

An alternative proof of [24, Corollary 3.7] can be obtained from Corollary 4.9 by noting that every 2-positive, order zero map preserves zero-products. Indeed, we first note that elements a, b in a C*-algebra satisfy $ab = 0$ if and only if $(a^*a)(bb^*) = 0$. Now, let $\varphi\colon A \to B$ be a 2-positive, order zero map. Then φ satisfies the Kadison inequality $\varphi(a)^*\varphi(a) \leq \|\varphi\|\varphi(a^*a)$ for all $a \in A$. Hence, if $a, b \in A$ satisfy $ab = 0$, then $(a^*a)(bb^*) = 0$ and therefore

$$\varphi(a^*a)\varphi(bb^*) = 0, \quad \varphi(a)^*\varphi(a) \leq \|\varphi\|\varphi(a^*a), \quad \text{and} \quad \varphi(b)\varphi(b)^* \leq \|\varphi\|\varphi(bb^*).$$

This implies $\varphi(a)^*\varphi(a)\varphi(b)\varphi(b)^* = 0$, and thus $\varphi(a)\varphi(b) = 0$, as desired.

Proposition 4.11. *Let $\varphi\colon A \to B$ be a bounded, linear map between* C*-*algebras. If φ preserves range-orthogonality, domain-orthogonality and zero-products, then φ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\| = \|\varphi^{**}(1)\|$.*

Proof. Set $h := \varphi^{**}(1) \in B^{**}$. By Theorem 4.2, there exists a $*$-homomorphism $\pi\colon A \to B^{**}$ such that $\varphi(a) = h\pi(a)$ for all $a \in A$. We have

$$\|\varphi^{**}(1)\| \leq \|\varphi^{**}\| = \|\varphi\| \leq \|\varphi\|_{cb}.$$

Given $n \geq 1$, set $h^{(n)} = 1_{M_n} \otimes h \in M_n(B^{**})$, which is the diagonal matrix with diagonal entries all equal to h. It follows that the amplification $\varphi^{(n)}$ satisfies $\varphi^{(n)}(x) = h^{(n)}\pi^{(n)}(x)$ for all $x \in M_n(A)$. Using that $\pi^{(n)}$ is a $*$-homomorphism, we deduce that

$$\|\varphi^{(n)}(x)\| = \|h^{(n)}\pi^{(n)}(x)\| \leq \|h^{(n)}\|\|\pi^{(n)}(x)\| \leq \|h\|\|x\|$$

and thus $\|\varphi^{(n)}\| \leq \|h\|$. Since this holds for every n, we obtain $\|\varphi\|_{cb} \leq \|h\|$, as desired. ■

Remark 4.12. In [14], we show that Proposition 4.11 also holds for bounded linear maps that only preserve range-orthogonality or domain-orthogonality. In particular, a bounded, range-orthogonality preserving map is automatically completely bounded. We also show that a range-orthogonality preserving map from a unital C*-algebra that has no one-dimensional irreducible representations is automatically bounded, and hence completely bounded.

Funding. The first named author was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through an Eigene Stelle, and by the Swedish Research Council Grant 2021-04561. The second named author was partially supported by the ERC Consolidator Grant No. 681207, and also by the Knut and Alice Wallenberg Foundation (KAW 2021.0140).

References

[1] Y. A. Abramovich and A. K. Kitover, Inverses of disjointness preserving operators. *Mem. Amer. Math. Soc.* **143** (2000), no. 679, viii+162 Zbl 0974.47032 MR 1639940

[2] J. Alaminos, M. Brešar, J. Extremera, and A. R. Villena, Maps preserving zero products. *Studia Math.* **193** (2009), no. 2, 131–159 Zbl 1168.47029 MR 2515516

[3] W. Arendt, Spectral properties of Lamperti operators. *Indiana Univ. Math. J.* **32** (1983), no. 2, 199–215 Zbl 0488.47016 MR 0690185

[4] B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras*. Encyclopaedia Math. Sci. 122, Springer, Berlin, 2006 Zbl 1092.46003 MR 2188261

[5] M. Brešar, *Zero product determined algebras*. Front. Math., Birkhäuser/Springer, Cham, 2021 Zbl 1505.16001 MR 4311188

[6] M. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno, and A. M. Peralta, Orthogonality preservers in C^*-algebras, JB*-algebras and JB*-triples. *J. Math. Anal. Appl.* **348** (2008), no. 1, 220–233 Zbl 1156.46045 MR 2449340

[7] M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong, Mappings preserving zero products. *Studia Math.* **155** (2003), no. 1, 77–94 Zbl 1032.46063 MR 1961162

[8] G. Dolinar, Stability of disjointness preserving mappings. *Proc. Amer. Math. Soc.* **130** (2002), no. 1, 129–138 Zbl 1001.46033 MR 1855629

[9] J. J. Garcés, Complete orthogonality preservers between C*-algebras. *J. Math. Anal. Appl.* **483** (2020), no. 1, article no. 123596 Zbl 1498.47083 MR 4019103

[10] E. Gardella, A modern look at algebras of operators on L^p-spaces. *Expo. Math.* **39** (2021), no. 3, 420–453 Zbl 1487.22006 MR 4314026

[11] E. Gardella and H. Thiel, Group algebras acting on L^p-spaces. *J. Fourier Anal. Appl.* **21** (2015), no. 6, 1310–1343 Zbl 1334.22007 MR 3421918

[12] E. Gardella and H. Thiel, Representations of p-convolution algebras on L^q-spaces. *Trans. Amer. Math. Soc.* **371** (2019), no. 3, 2207–2236 Zbl 1461.43002 MR 3894050

[13] E. Gardella and H. Thiel, Zero-product balanced algebras. *Linear Algebra Appl.* **670** (2023), 121–153 Zbl 1523.16045 MR 4579922

[14] E. Gardella and H. Thiel, Automatic continuity for orthogonality-preserving maps between C*-algebras. 2025, in preparation

[15] E. Gardella and H. Thiel, The zero-product structure of rings and C*-algebras. 2025, in preparation

[16] P. Grover and S. Singla, Birkhoff–James orthogonality and applications: a survey. In *Operator theory, functional analysis and applications*, pp. 293–315, Oper. Theory Adv. Appl. 282, Birkhäuser/Springer, Cham, 2021 Zbl 1482.46017 MR 4248023

[17] I. Hirshberg, E. Kirchberg, and S. White, Decomposable approximations of nuclear C^*-algebras. *Adv. Math.* **230** (2012), no. 3, 1029–1039 Zbl 1256.46019 MR 2921170

[18] W.-F. Ke, B.-R. Li, and N.-C. Wong, Zero product preserving maps of operator-valued functions. *Proc. Amer. Math. Soc.* **132** (2004), no. 7, 1979–1985 Zbl 1044.46035 MR 2053969

[19] T. Kochanek, Approximately order zero maps between C*-algebras. *J. Funct. Anal.* **281** (2021), no. 2, article no. 109025 Zbl 1478.46051 MR 4242962

[20] C.-W. Leung, C.-W. Tsai, and N.-C. Wong, Linear disjointness preservers of W*-algebras. *Math. Z.* **270** (2012), no. 3-4, 699–708 Zbl 1250.47041 MR 2892919

[21] J.-H. Liu, C.-Y. Chou, C.-J. Liao, and N.-C. Wong, Disjointness preservers of AW^*-algebras. *Linear Algebra Appl.* **552** (2018), 71–84 Zbl 1398.46046 MR 3804477

[22] J.-H. Liu, C.-Y. Chou, C.-J. Liao, and N.-C. Wong, Linear disjointness preservers of operator algebras and related structures. *Acta Sci. Math. (Szeged)* **84** (2018), no. 1-2, 277–307 Zbl 1413.47063 MR 3792777

[23] A. M. Peralta, A note on 2-local representations of C*-algebras. *Oper. Matrices* **9** (2015), no. 2, 343–358 Zbl 1321.46060 MR 3338568

[24] Y. Sato, 2-positive almost order zero maps and decomposition rank. *J. Operator Theory* **85** (2021), no. 2, 505–526 Zbl 1513.46112 MR 4249561

[25] J. Schweizer, *Interplay between noncommutative topology and operators on* C**-algebras*. Ph.D. thesis, University of Tübingen, 1996

[26] W. Winter and J. Zacharias, Completely positive maps of order zero. *Münster J. Math.* **2** (2009), 311–324 Zbl 1190.46042 MR 2545617

[27] W. Winter and J. Zacharias, The nuclear dimension of C^*-algebras. *Adv. Math.* **224** (2010), no. 2, 461–498 Zbl 1201.46056 MR 2609012

[28] M. Wolff, Disjointness preserving operators on C^*-algebras. *Arch. Math. (Basel)* **62** (1994), no. 3, 248–253 Zbl 0803.46069 MR 1259840

[29] N.-C. Wong, Zero product preservers of C^*-algebras. In *Function spaces*, pp. 377–380, Contemp. Math. 435, American Mathematical Society, Providence, RI, 2007 Zbl 1146.46046 MR 2359445

Communicated by Wilhelm Winter

Received 17 April 2022; revised 13 August 2024.

Eusebio Gardella
Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, 41296 Gothenburg, Sweden; gardella@chalmers.se

Hannes Thiel
Department of Mathematical Sciences, University of Gothenburg and Chalmers University of Technology, 41296 Gothenburg, Sweden; hannes.thiel@chalmers.se