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t-motivic interpretations for special values of Thakur hypergeometric functions and Kochubei multiple polylogarithms

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Abstract. In 1995, Thakur invented and studied positive characteristic analogues of hypergeometric functions. In this paper, we interpret the special values of those functions by rigid analytic trivializations for some pre-t-motives. As a consequence, we show their transcendence and linear independence results by using Chang's refined version of the Anderson–Brownawell–Papanikolas criterion. Furthermore, we show some linear independence results among the special values of Kochubei multiple polylogarithms according to our t-motivic interpretation and the corresponding refined criterion.

1. Introduction

1.1. Thakur hypergeometric functions

Let N be a set of positive integers. Let $\mathbb{F}_q$ be a fixed finite field with q elements, where q is a power of a prime number p. Let $\mathbb{P}^1$ be a projective line defined over $\mathbb{F}_q$ with a fixed point at infinity $\infty \in \mathbb{P}^1(\mathbb{F}_q)$. Let A be the ring of regular functions on $\mathbb{P}^1$ away from ∞, with k as its fraction field. Let k_∞ be the completion of k at ∞, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of k_∞. With the variable θ, we can identify A with the polynomial ring $\mathbb{F}_q[\theta]$ and k with the rational function field $\mathbb{F}_q(\theta)$. Thakur defined and studied the positive characteristic analogues of the classical hypergeometric functions (HGFs) in [26]. His definition is motivated by Barns integral representation [26, Section 3.4] of the HGFs. For $D_i := \prod_{j=1}^{i}(\theta^{q^j} - \theta)^{q^{i-j}}$ ($D_0 := 1$) and $L_i := \prod_{j=1}^{i}(\theta - \theta^{q^j})$ ($L_0 := 1$) with $i \in \mathbb{Z}_{\geq 0}$, he found the following analogue of the Pochhammer symbols:

$$(a)_n := \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{if } a \geq 1, \\ 1/L_{-a-n}^{q^n} & \text{if } 0 \geq a \text{ and } -a \geq n, \\ 0 & \text{if } n > -a \geq 0. \end{cases}$$

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Then, based on the characteristic 0 case, he defined the following analogue of HGFs by using the above symbols: for $a_1, \ldots, a_r \in \mathbb{Z}$ and $b_1, \ldots, b_s \in \mathbb{N}$,

$$ {}_rF_s(z) := {}_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s)(z) := \sum_{n \geq 0} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n D_n} z^{q^n} \in \mathbb{C}_\infty[\![z]\!]. \tag{1.1} $$

Throughout this paper, we call these analogues of HGFs the Thakur hypergeometric functions (THGFs) and without loss of generality, we assume that $a_i \leq a_j$ and $b_i \leq b_j$ for $i \leq j$. The THGFs ${}_rF_s(z)$ have three cases of the convergence domain as follows:

$$ {}_rF_s(z) \text{ are defined for } \begin{cases} z = 0 & \text{if } r > s+1, \\ z \in \mathbb{C}_\infty & \text{if } r < s+1, \\ z \in \mathbb{C}_\infty \text{ with } |z|_\infty < q^{\sum_{j=1}^{s}(b_j-1) - \sum_{j=1}^{r}(a_j-1)} & \text{if } r = s+1. \end{cases} \tag{1.2} $$

Here, $|\cdot|_\infty$ is the absolute value on $\mathbb{C}_\infty$ such that $|\theta|_\infty = q$. Thakur also showed that THGFs satisfy an analogue of the hypergeometric differential equation by using the Carlitz differential operator Δ_a and the Carlitz derivative d_F. The case of Δ_0 is originated in [5]. They are $\mathbb{F}_q$-linear operators defined on $\mathbb{F}_q$-linear functions $f(z)$ by

$$ \Delta_a\big(f(z)\big) := f(\theta z) - \theta^{q^{-a}} f(z) \text{ for } a \in \mathbb{Z}, \quad d_F\big(f(z)\big) := \Delta_0\big(f(z)\big)^{1/q}. $$

We can consider operator Δ_a and d_F to be the positive characteristic analogue of $z(d/dz) + a$ and d/dz respectively. For more details and studies, see [26, Section 3.1] and [27].

Then, Thakur demonstrated the following differential equation [26, (10)] which is seen to be an analogue of the hypergeometric differential equation:

$$ d_F \circ \Delta_{a_1} \circ \cdots \circ \Delta_{a_r}\big({}_rF_s(z)\big) = \Delta_{b_1-1} \circ \cdots \circ \Delta_{b_s-1}\big({}_rF_s(z)\big). $$

Furthermore, he discovered several properties of ${}_rF_s(z)$, including the analogue of contiguous relations, the summation formula, specializations to exponential functions as well as the Bessel functions, the Jacobi/Legendre polynomials in positive characteristic, and the connection to the tensor powers of the Carlitz modules in [26].

The second analogue of the HGF is also defined in [26] by using the positive characteristic analogue of the binomial coefficients. In this paper, we discuss only the first analogue recalled in (1.1).

Later, Thakur, Wen, Yao, and Zhao [28] obtained a sufficient condition for the special values of ${}_rF_s(z)$ $(r < s+1)$ and an equivalent condition for the special values of ${}_rF_s(z)$ $(r = s+1)$ to be transcendental over k. These transcendence results were the consequence of their Diophantine criterion for transcendence in positive characteristic, which generalized [31, Theorem 1]. Moreover, there are some approaches for solving transcendence/linear independence problems via certain pre-t-motives (see Definition 2.4), as developed by Anderson, Brownawell, and Papanikolas [1], namely, the so-called ABP criterion. For example, Carlitz multiple polylogarithms are firstly defined in [7] as follows: for $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{z} = (z_1, \ldots, z_r) \in \{(z_1, \ldots, z_r) \in \mathbb{C}_\infty^r \mid |z_1/\theta^{\frac{qs_1}{q-1}}|_\infty^{q^{i_1}} \cdots |z_r/\theta^{\frac{qs_r}{q-1}}|_\infty^{q^{i_r}} \to 0$ as $0 \leq i_r < \cdots < i_1 \to \infty\}$,

$$Li_{C,\mathfrak{s}}(\mathbf{z}) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}} \in \mathbb{C}_\infty,$$

and we can have a t-motivic interpretation of their special values at algebraic points by [7, 23], that is, the values appear in entries of a matrix so-called rigid analytic trivialization (see Definition 2.5) associated to certain pre-t-motives after specialization $t = \theta$. For each index $\mathfrak{s} \in \mathbb{N}^r$ ($r > 0$), we set the weight as $\mathrm{wt}(\mathfrak{s}) := s_1 + \cdots + s_r$ and the depth as $\mathrm{dep}(\mathfrak{s}) := r$. Analogous to the classical case, the Carlitz multiple polylogarithm $Li_{C,\mathfrak{s}}(\mathbf{z})$ includes the Carlitz polylogarithm introduced in [2] and the Carlitz logarithm introduced in [5] as $\mathrm{dep}(\mathfrak{s}) = 1$ case and $\mathrm{dep}(\mathfrak{s}) = \mathrm{wt}(\mathfrak{s}) = 1$ case respectively. The ABP criterion is applied to the linear independence of the Carlitz multiple polylogarithms at algebraic points with different weights in [7], and the criterion for Eulerian Carlitz multiple polylogarithms at algebraic points in [11].

Our motivation in this paper is to develop a t-motivic interpretation for the special values of ${}_{s+1}F_s(z)$ with some q-th powers and to provide linear independence results among these values by using Chang's refined version of ABP criterion (see Theorem 4.2). Moreover, we present a t-motivic interpretation and the linear independence results for the special values of Kochubei multiple polylogarithms (KMPLs).

1.2. Main results

In the characteristic 0 case, it is known that the HGFs can be given by periods of algebraic varieties in some cases, particularly with rational parameters. For example, the special values of $\pi {}_2F_1(1/2, 1/2; 1)(z)$ and $\pi\sqrt{-1}{}_2F_1(1/2, 1/2; 1)(1-z)$ are known to be periods related to the elliptic curve $y^2 = x(x-1)(x-z)$ (cf. [20, Section 2.2]). These observations give rise to the connection between HGFs of ${}_{s+1}F_s$ case and certain pure motives over $\mathbb{Q}$ which are called hypergeometric motives. For the details, see the survey [24]. There is also the study about the transcendence of special values of HGFs. For example, Schwarz determined the list of HGFs of ${}_2F_1$ cases, which are algebraic functions in [25] later generalized to the ${}_{s+1}F_s$ case by Beukers and Heckman [4]. Furthermore, there are studies about the linear independence of the special values of HGFs. For the recent works, Fischler, Rivoal [14] and David, Hirata-Kohno, Kawashima [13] proved the linear independence among the values of HGFs with some different algebraic points or some different rational parameters.

Our main results include a t-motivic interpretation of the special values of the THGFs (Theorem 3.4), the transcendence and linear independence results of some special values of the THGFs (Theorems 4.6, 4.7 and 4.9) and the linear independence results of the KMPLs at algebraic points (Theorems 4.11–4.12). These independence results are addressed by using Chang's remarkable works, in particular, the refined ABP criterion

(Theorem 4.2) invented in [6] and the techniques of computing Frobenius difference equations described in [7, Sections 4.1 and 4.2]. We note that our Theorem 4.6 gives equivalent conditions for the special values of ${}_{s+1}F_s(z)$ to be transcendental over k, and it is also proved by [28, Theorem 4] but proofs are different. Indeed we use a t-motivic interpretation for the special values of the THGFs. About our linear independence results among the special values of ${}_{s+1}F_s(z)$, which are not discussed in [28].

We can show that the q-th power of the THGFs at algebraic points are related to a rigid analytic trivialization coming from a specific pre-t-motive $M_{\mathbf{a},\mathbf{b},d}$ defined by (3.8). We set

$$\Omega := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty}\left(1 - \frac{t}{\theta^{q^i}}\right) \in k_\infty(\theta^{\frac{1}{q-1}})[\![t]\!]$$

where we fix a $(q-1)$-th root of $-\theta$. The Carlitz period $\tilde{\pi}$ is defined by $(\Omega|_{t=\theta})^{-1}$.

Let $d \geq \max\{b_1, b_2, \ldots, b_s\}$ for given $b_1, b_2, \ldots, b_s \in \mathbb{N}$. We further set the following power series:

$$\begin{aligned} P_{\mathbf{b},d} &:= (-\theta)^{\frac{-\sum_{j=1}^{s}(b_j-1)q^{d-1}}{q-1}} \prod_{l=1}^{\infty} \prod_{\substack{j=1\\ b_j\geq 2}}^{s} \left\{\left(1-\frac{t}{\theta^{q^l}}\right)^{q^{b_j-2}}\left(1-\frac{t}{\theta^{q^{l+1}}}\right)^{q^{b_j-3}}\cdots\left(1-\frac{t}{\theta^{q^{l+b_j-2}}}\right)\right\}^{q^{d-b_j}} \\ &= \prod_{j=1}^{s} \frac{\prod_{i=1}^{b_j-2}\prod_{h=i}^{b_j-2}\theta^{q^{d-2-h+i}}}{(\mathbb{D}_{b_j-2}\cdots\mathbb{D}_2\mathbb{D}_1)^{q^{d-b_j}}} \prod_{i=2}^{b_j}\Omega^{q^{d-i}} \in k_\infty(\theta^{\frac{1}{q-1}})[\![t]\!]. \end{aligned} \tag{1.3}$$

Here, we set $\mathbb{D}_n := \prod_{i=1}^{n}(\theta^{q^i} - t)^{q^{n-i}}$ for $n > 0$ and $\mathbb{D}_n := 1$ for $n \leq 0$.

Remark 1.1. When all b_i $(i = 1, 2, \ldots, s)$ and d are equal to 2, we obtain $P_{\mathbf{b},d} = \Omega^s$.

Then, our t-motivic interpretation for the special values of the THGFs is stated as follows (stated again as Theorem 3.4).

Theorem 1.2. *Let $a_i, b_j \in \mathbb{N}$ $(1 \leq i \leq s+1,\ 1 \leq j \leq s)$. Then, for $\alpha \in \bar{k}$ with $|\alpha|_\infty < q^{\sum_{j=1}^{s}(b_j-1)-\sum_{i=1}^{s+1}(a_i-1)}$ and $d \in \mathbb{Z}$ with $d \geq \max_{i,j}\{a_i, b_j\}$, the special value of THGF ${}_{s+1}F_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s)(\alpha)^{q^{d-1}}$ multiplied by*

$$P_{\mathbf{b},d}|_{t=\theta} = \prod_{j=1}^{s} \frac{\prod_{i=1}^{b_j-2}\prod_{h=i}^{b_j-2}\theta^{q^{d-2-h+i}}}{(D_{b_j-2}\cdots D_2 D_1)^{q^{d-b_j}}} \prod_{i=2}^{b_j}\tilde{\pi}^{-q^{d-i}}$$

is an entry of a rigid analytic trivialization at $t = \theta$, which is related to the pre-t-motive $M_{\mathbf{a},\mathbf{b},d}$.

Remark 1.3. We also give a t-motivic interpretation of the special values of the THGFs without $P_{\mathbf{b},d}|_{t=\theta}$ in Remark 3.7.

According to Theorem 1.2 and the refined ABP criterion, we obtain the following linear independence results, each of which is restated later in Theorems 4.6, 4.7 and 4.9.

For the definition of $c(\cdot)$ and the necessity of its conditions in Theorem 1.4, we explain later by Proposition 4.3 and Remark 4.5 respectively.

Theorem 1.4. *We denote all m satisfying $d \geq m \geq 0$ by m_i $(i = 1, \dots, n)$ where $d = \max_{\substack{1 \leq i \leq s+1 \\ 1 \leq j \leq s}} \{a_i, b_j\}$.*

(i) *We set $\mathbf{a} = (a_1, \dots, a_{s+1}) \in \mathbb{N}^{s+1}$, $\mathbf{b} = (b_1, \dots, b_s) \in \mathbb{N}^s$ and $\alpha \in \bar{k}^{\times}$ with $|\alpha|_\infty < q^{\sum_{j=1}^{s}(b_j - 1) - \sum_{i=1}^{s+1}(a_i - 1)}$. Then, ${}_{s+1}F_s(a_1, \dots, a_{s+1}; b_1, \dots, b_s)(\alpha)$ is transcendental over k if and only if $b_j > a_{j+1}$ for some j.*

(ii) *Fix $\mathbf{a}_s = (a_1, \dots, a_{s+1}) \in \mathbb{N}^{s+1}$, $\mathbf{b}_s = (b_1, \dots, b_s) \in \mathbb{N}^s$ such that $b_1 > a_{s+1}$. We take m_r satisfying $b_1 - 1 \geq m_r$. Let $\mathbf{a}_h = (a_1, \dots, a_{h+1})$, $\mathbf{b}_h = (b_1, \dots, b_h)$ $(h = 1, \dots, s)$ and $\alpha_h \in \bar{k}^{\times}$ with $|\alpha_h|_\infty < q^{\sum_{j=1}^{h}(b_j - 1) - \sum_{i=1}^{h+1}(a_i - 1)}$. Then, if $\min_{1 \leq i \leq n,\, i \neq r}\{c(m_i) q^{d - m_i}\} > c(m_r) q^{d - m_r}$, ${}_{h+1}F_h(\mathbf{a}_h; \mathbf{b}_h)(\alpha_h)$ $(1 \leq h \leq s)$ are $\bar{k}$-linearly independent.*

(iii) *For any $\mathbf{a} = (a_1, \dots, a_{s+1}) \in \mathbb{N}^{s+1}$ and $\mathbf{b} = (b_1, \dots, b_s) \in \mathbb{N}^s$ such that $b_j > a_{j+1}$ for some j and that $\min_{1 \leq i \leq n,\, i \neq u}\{c(m_i) q^{d - m_i}\} > c(m_u) q^{d - m_u}$ for some u, let $\alpha_i \in \bar{k}^{\times}$ $(i = 1, \dots, r)$ with $|\alpha_i|_\infty < q^{\sum_{j=1}^{s}(b_j - 1) - \sum_{i=1}^{s+1}(a_i - 1)}$. If $\alpha_1, \dots, \alpha_r$ are k-linearly independent, then ${}_{s+1}F_s(\mathbf{a}; \mathbf{b})(\alpha_1), \dots, {}_{s+1}F_s(\mathbf{a}; \mathbf{b})(\alpha_r)$ are $\bar{k}$-linearly independent.*

The first result (Theorem 4.6) is already proved in [28, Theorem 4]. However, in this paper, we prove it by different tools, pre-t-motive and refined ABP criterion, while [28] use Diophantus approximation.

One may think that the conditions of Theorem 1.4 are restrictive due to $c(\cdot)$. These are required to complete our computations in the proofs, as explained in Remark 4.5. We will try to weaken them in the future. Moreover, several analogues of HGFs are developed after THGFs [26]. For example, the definition (1.1) of THGF is extended by using fractional parameters in $\mathbb{Q}$ [28]. Later, Yao [32] proved that transcendence results of THGFs in [28] are generalized to the case of these fractional THGFs. On the other hand, generalizing Theorems 1.2 and 1.4 to the fractional THGFs is an open problem. Also, HGF with characteristic p parameters [19, 26] and Hasegawa's exponential (logarithmic) type HGFs [17] are invented as positive characteristic analogues of HGFs. One can consider the transcendence/linear independence problems about special values of these analogues via t-motivic interpretations. We hope to address these issues in the future.

By Propositions 2.2 and 2.3, the above results can be applied to the case of Kochubei polylogarithms (KPLs), which were defined and studied by Kochubei [18] as follows:

$$Li_{K,(s)}(z) := \sum_{i \geq 1} \frac{z^{q^i}}{(\theta^{q^i} - \theta)^s} \in \mathbb{C}_\infty$$

for $z \in \mathbb{C}_\infty$ with $|z|_\infty < q^s$. The case of $s = z = 1$ was discussed by Wade [29], who found that $\sum_{i \geq 1} 1/(\theta^{q^i} - \theta)$ is transcendental over k. The KPLs are considered to be another positive characteristic analogue of the polylogarithms, with a different motivation from that of the Carlitz polylogarithms. On the one hand, the Carlitz polylogarithms are generalizations of the Carlitz logarithm which is defined by the formal inverse of the Carlitz exponential; on the other hand, Kochubei's idea was to obtain the analogue of the classical polylogarithm $Li_s(z) := \sum_{n>0} z^n/n^s$ by finding a function that satisfies the analogue of the differential equation $zd/dzLi_s(z) = Li_{s-1}(z)$. Notably, our definition of the KPLs is given in the ∞-adic case, while Kochubei [18] defined them in the v-adic case (v is a monic irreducible polynomial in $\mathbb{F}_q[\theta]$). In [18], he also defined the analogues of the Riemann zeta values by $\zeta_K(\theta^{-n}) := Li_{K,n}(1)$ which we call the Kochubei zeta values in this paper.

Based on the following setting, we can define the KMPLs as $\mathfrak{s} := (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{z} := (z_1, \dots, z_r) \in \mathbb{C}_\infty^r$ such that $|z_i|_\infty < q^{s_i}$:

$$Li_{K,\mathfrak{s}}(\mathbf{z}) := \sum_{i_1 > \cdots > i_r > 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{(\theta q^{i_1} - \theta)^{s_1} \cdots (\theta q^{i_r} - \theta)^{s_r}} \in \mathbb{C}_\infty.$$

In this paper, we denote the 1-variable case by

$$Li_{K,\mathfrak{s}}(z) = \sum_{i_1 > \cdots > i_r > 0} \frac{z^{q^{i_1}}}{(\theta q^{i_1} - \theta)^{s_1} \cdots (\theta q^{i_r} - \theta)^{s_r}}.$$

Similar to the cases of classical multiple polylogarithm [30] and the Carlitz multiple polylogarithm [7, Section 5.2], the KMPLs also satisfy the sum-shuffle relation by their series expressions. We can describe the relation in the same way of the Carlitz case [7, Section 5.2]. For a given $\mathfrak{s}_1 \in \mathbb{N}^{r_1}$ and $\mathfrak{s}_2 \in \mathbb{N}^{r_2}$ it is described by

$$Li_{K,\mathfrak{s}_1}(\mathbf{z}_1) Li_{K,\mathfrak{s}_2}(\mathbf{z}_2) = \sum_{(\mathbf{v}_1, \mathbf{v}_2)} Li_{K,\mathbf{v}_1+\mathbf{v}_2}(\mathbf{z}_3). \tag{1.4}$$

Here, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{Z}_{\geq 0}^{r_3}$ satisfying $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{N}^{r_3}$ with $\max\{r_1, r_2\} \leq r_3 \leq r_1 + r_2$ and $\mathbf{v}_i$ $(i = 1, 2)$ is obtained by inserting $(r_3 - r_i)$ zeros into $\mathfrak{s}_i$ in all possible ways, including in front and the end of $\mathfrak{s}_i$. The pair $(\mathbf{v}_1, \mathbf{v}_2)$ runs over all such expressions for all r_3 with $\max\{r_1, r_2\} \leq r_3 \leq r_1 + r_2$. For every such $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{N}^{r_3}$, the m-th component z_{3m} of $\mathbf{z}_3$ is z_{in} if the m-th component of $\mathbf{v}_i$ is s_{in}, while the m-th component of $\mathbf{v}_j$ $(i \neq j)$ is 0 or z_{3m} is $z_{1n} z_{2l}$ if the m-th component of both $\mathbf{v}_1$ and $\mathbf{v}_2$ are z_{1n} and z_{2l}. For example, we have

$$Li_{K,(s_1)}(z_1) Li_{K,(s_2)}(z_2) = Li_{K,(s_1+s_2)}(z_1 z_2) + Li_{K,(s_1,s_2)}(z_1, z_2) + Li_{K,(s_2,s_1)}(z_2, z_1).$$

While there is an algebraic dependence result of the special values of KMPLs such as the sum-shuffle relation, in this paper we will also obtain linear independence results among them. Precisely, those results are stated as follows (each of them is described again later in Theorems 4.11–4.14).

Theorem 1.5. *The following statements* (i)–(iv) *hold:*

(i) *For indices $\mathfrak{s} = (s_1, s_2) \in \mathbb{N}^2$ with $\mathrm{wt}(\mathfrak{s}) = w$ and $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < q^{s_1}$, $Li_{K,\mathfrak{s}}(\alpha)$ are $\bar{k}$-linearly independent.*

(ii) *For given $n \in \mathbb{N}$, let $\alpha, \beta \in \bar{k}^\times$ such that $|\alpha|_\infty < q^n$, $|\beta|_\infty \in q^{nq/(q-1)}$. Then, $Li_{K,(n)}(\alpha)$, $Li_{C,(n)}(\beta)$ are $\bar{k}$-linearly independent.*

(iii) *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $\mathrm{wt}(\mathfrak{s}) = w$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in (\bar{k}^\times)^r$ with $|\alpha_i|_\infty < q^{s_i}$ $(i = 1, \dots, r)$, such that $Li_{K,\mathfrak{s}}(\boldsymbol{\alpha}) \neq 0$. Then $Li_{K,\mathfrak{s}}(\boldsymbol{\alpha})$ and $\widetilde{\pi}^w$ are $\bar{k}$-linearly independent.*

(iv) *For $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ such that $\mathrm{wt}(\mathfrak{s}) = w$, let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in (\bar{k}^\times)^r$ with $|\alpha_i|_\infty < q^{s_i}$ $(i = 1, \dots, r)$ and $\beta \in \bar{k}^\times$ with $|\beta|_\infty < q^w$. Then, $Li_{K,\mathfrak{s}}(\boldsymbol{\alpha})$ and $Li_{K,w}(\beta)$ are $\bar{k}$-linearly independent.*

As noted before, there are Carlitz' analogue of polylogarithms. Theorem 1.5 (i) implies that KMPLs may not satisfy $\bar{k}$-linear relations as much as Carlitz multiple polylogarithms case, in particular, comparing to the results of [8] and [22]. Theorem 1.5 (ii) says that we may not easily apply the linear/algebraic independence results of Carlitz polylogarithms to KPLs. We can define Kochubei multizeta values by $\zeta_K(\theta^{-s_1}, \dots, \theta^{-s_r}) := Li_{K,s_1,\dots,s_r}(1, \dots, 1)$, as generalizations of Kochubei zeta values. According to the Thakur multizeta values case [21], we can also define Eulerian/zeta-like indices for Kochubei multizeta values, but Theorem 1.5 (iii) and (iv) imply the non-existence of such indices.

As a future project, we can study linear/algebraic independence among KMPLs in more generality, based on previous studies of Carlitz multiple polylogarithms. For instance, we can try to prove that non-zero values as specializations of KMPLs at algebraic points with distinct weights are $\bar{k}$-linearly independent. This is already proven in the Carlitz multiple polylogarithm case by [7, Theorem 5.4.3]. Also, we can try to prove that if KPLs at algebraic points are k-linearly independent, then they are algebraically independent over $\bar{k}$. This is also already shown in the Carlitz polylogarithm case by [12, Corollary 3.2].

Finally, this paper is organized as follows. In Section 2, we recall fundamental notations and the definition of rigid analytic trivializations together with pre-t-motives. We also present the relation which shows that s are q-th power of HGFs with certain parameters. In Section 3, we consider the deformation of THGFs and KMPLs to obtain Theorem 1.2 and (3.15), the t-motivic interpretation of the values of THGFs and KMPLs. In Section 4, we recall the refined ABP criterion and present $(\theta^{q^i} - t)$-expansion of the deformation of THGFs. These enable us to prove Theorem 1.4 which is explained in Section 4.1. Further, in Section 4.2, we deal with linear independence problems among the special values of KMPLs and conclude the section with our proof of Theorem 1.5.

2. Preliminaries

2.1. Notations

We fix the following symbols.

$\mathbb{N}$:= the set of positive integers.

q := a power of a prime number p.

$\mathbb{F}_q$:= a finite field with q elements.

θ, t := independent variables.
A := the polynomial ring $\mathbb{F}_q[\theta]$.
A_+ := the set of monic polynomials in A.
k := the rational function field $\mathbb{F}_q(\theta)$.
k_∞ := $\mathbb{F}_q((\frac{1}{\theta}))$, the completion of k at infinite place ∞.
$\overline{k_\infty}$:= a fixed algebraic closure of k_∞.
$\mathbb{C}_\infty$:= the completion of $\overline{k_\infty}$ at infinity ∞.
$\bar{k}$:= a fixed algebraic closure of k in $\mathbb{C}_\infty$.
$|\cdot|_\infty$:= a fixed absolute value for the completed field $\mathbb{C}_\infty$ such that $|\theta|_\infty = q$.
$\mathbb{T}$:= the Tate algebra over $\mathbb{C}_\infty$, which is the subring of $\mathbb{C}_\infty[\![t]\!]$ that consists of power series convergent on the closed unit disc $|t|_\infty \leq 1$.
$\mathbb{L}$:= the fraction field of $\mathbb{T}$.
$\|\cdot\|$:= the norm on $\mathbb{T}$ defined as $\|f\| := \max_i |a_i|_\infty$ for $f = \sum_i a_i t^i \in \mathbb{T}$.
$\mathbb{E}$:= $\{\sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[\![t]\!] \mid \lim_{i\to\infty} |a_i|_\infty^{1/i} = 0,\ [k_\infty(a_0, a_1, \ldots) : k_\infty] < \infty\}$.
D_i := $\prod_{j=1}^{i-1}([j])^{q^{i-j}} \in A_+$ where $[j] := \theta^{q^j} - \theta$ and $D_0 := 1$.
L_i := $\prod_{j=1}^{i}(-[j]) \in A_+$ and $L_0 := 1$.

For $n \in \mathbb{Z}$, we define the following automorphism, which is known as the n-fold Frobenius twist:

$$\mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$$
$$f := \sum_i a_i t^i \mapsto \sum_i a_i^{q^n} t^i =: f^{(n)}.$$

Definition 2.1. For $s \geq 0$, we set $z \in \mathbb{C}_\infty$ with $|z|_\infty < q^s$ and define the following power series:

$$\mathcal{L}_{K,(s)}(z) := \sum_{i \geq 1} \frac{z^{q^i}}{(\theta^{q^i} - t)^s} \in \mathbb{C}_\infty[\![t]\!].$$

This series is specialized to $Li_{K,s}(\alpha)$ with $t = \theta$ and satisfies the following Frobenius difference equation:

$$\mathcal{L}_{K,(s)}(z)^{(-1)} = \frac{z}{(\theta - t)^s} + \mathcal{L}_{K,(s)}(z). \tag{2.1}$$

We propose the following relation, inspired by the well-known relation for Lerch transcendents and HGFs in the classical case.

Proposition 2.2. *For $m \in \mathbb{N}$ and $z \in \mathbb{C}_\infty$ with $|z|_\infty < q^{s+m-1}$, we have*

$$\left({}_{s+1}F_s(1, m, \ldots, m; 1+m, \ldots, 1+m)(z^{q^{-m+1}})\right)^{q^m} = \sum_{i \geq 0} \frac{z^{q^{i+1}}}{[i+m]^s}.$$

Proof. By using the relation $(1+m)_i = [i+m]^{q^{-m}}(m)_i$ introduced in [26, (12)], we obtain

$$\begin{aligned}
&\big({}_{s+1}F_s(1,m,\dots,m;1+m,\dots,1+m)(z^{q^{-m+1}})\big)^{q^m}\\
&= \Big(\sum_{i\geq 0} \frac{D_i(m)_i\cdots(m)_i}{(1+m)_i\cdots(1+m)_i D_i} z^{q^{i-m+1}}\Big)^{q^m}\\
&= \Big(\sum_{i\geq 0} \frac{1}{[i+m]^{nq^{-m}}} z^{q^{i-m+1}}\Big)^{q^m}\\
&= \sum_{i\geq 0} \frac{z^{q^{i+1}}}{[i+m]^s}. \qquad \blacksquare
\end{aligned}$$

When $m=1$, the above proof gives the relation for the KPLs and THGFs, which is considered to be an analogue of the formula for the HGF ${}_{s+1}F_s(1,\dots,1;2,\dots,2)(z)$ and the classical polylogarithm $Li_s(z)$ as follows:

$$z\big({}_{s+1}F_s(1,\dots,1;2,\dots,2)(z)\big) = Li_s(z).$$

Restricting $m=1$, Thakur et al. proved the $s=1,q-1$ cases of (2.2) in [28], and later Nagoya University student Daichi Matsuzuki generalized them to the $s>0$ case.

Proposition 2.3 (Matsuzuki and [28, p. 154]). *For $s>0$, we have*

$${}_{s+1}F_s(1,\dots,1;2,\dots,2)(z)^q = Li_{K,(s)}(z).$$

2.2. Pre-t-motives and rigid analytic trivializations

We denote $\bar{k}(t)[\sigma,\sigma^{-1}]$ by the non-commutative $\bar{k}(t)$-algebra generated by σ and σ^{-1}, which is subject to the following relation:

$$\sigma f = f^{(-1)}\sigma, \quad f\in\bar{k}(t).$$

Definition 2.4 ([23, Section 3.2.1]). A pre-t-motive is a left $\bar{k}(t)[\sigma,\sigma^{-1}]$-module that is finite-dimensional over $\bar{k}(t)$.

Once we fix a $\bar{k}(t)$-vector space M of rank d with a fixed $\bar{k}(t)$-basis $\mathbf{m}=(m_1,\dots,m_d)^{\mathrm{tr}}$ and $\Phi\in\mathrm{GL}_d(\bar{k}(t))$, we can uniquely determine the pre-t-motive structure on M by setting $\sigma\mathbf{m}=\Phi\mathbf{m}$ (cf. [23, Section 3.2.3]). In this case, we call M the pre-t-motive defined by Φ.

The notion of rigid analytically trivial pre-t-motives is defined as follows:

Definition 2.5 ([23, Proposition 3.3.9 (a)]). Let M be a pre-t-motive defined by $\Phi\in\mathrm{GL}_d(\bar{k}(t))$. If there exists $\Psi\in\mathrm{GL}_d(\mathbb{L})$ such that

$$\Psi^{(-1)} = \Phi\Psi,$$

Ψ is called a rigid analytic trivialization of Φ.

3. t-motivic interpretations of the THGFs and the KMPLs

We set

$$\begin{aligned}\mathbb{D}_i &:= \prod_{j=1}^{i}(\theta^{q^j}-t)^{q^{i-j}} \quad \text{for } i>0 \text{ and } \mathbb{D}_i := 1 \text{ for } i \le 0,\\ \mathbb{L}_i &:= \prod_{j=1}^{i}(t-\theta^{q^j}) \quad \text{for } i>0 \text{ and } \mathbb{L}_i := 1 \text{ for } i \le 0.\end{aligned} \tag{3.1}$$

Further, we set the following symbols for $n \in \mathbb{Z}_{\geq 0}$:

$$\langle a\rangle_n := \begin{cases} \mathbb{D}_{n+a-1}^{q^{-(a-1)}} & \text{if } a \ge 1,\\ 1/\mathbb{L}_{-a-n}^{q^n} & \text{if } 0 \ge a \text{ and } -a \ge n,\\ 0 & \text{if } n > -a \ge 0.\end{cases}$$

These symbols are specialized to the Pochhammer–Thakur symbols, at $t=\theta$. By using the above symbol, we can define the deformation series of ${}_rF_s(z)$ as follows.

Definition 3.1. For $a_1,\dots,a_r \in \mathbb{Z}$ and $b_1,\dots,b_s \in \mathbb{N}$, we set

$$\begin{aligned}{}_r\mathcal{F}_s(z) &:= {}_r\mathcal{F}_s(a_1,\dots,a_r;b_1,\dots,b_s)(z)\\ &:= \sum_{n\ge 0}\frac{\langle a_1\rangle_n\cdots\langle a_r\rangle_n}{\langle b_1\rangle_n\cdots\langle b_s\rangle_n\mathbb{D}_n}z^{q^n} \in \mathbb{C}_\infty[\![t^{q^{-d+1}},z]\!]\end{aligned} \tag{3.2}$$

where $d=\max_{\substack{1\le i\le r\\ 1\le j\le s}}\{a_i,b_j\}$.

We also assume throughout this paper that for a given ${}_r\mathcal{F}_s(a_1,\dots,a_r;b_1,\dots,b_s)(z)$, its parameters satisfy $a_i \le a_j$ and $b_i \le b_j$ for $i \le j$ without loss of generality.

The formal power series ${}_r\mathcal{F}_s(a_1,\dots,a_r;b_1,\dots,b_s)(z)$ at $t=\theta$ is equal to THGFs ${}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s)(z)$. Furthermore, we have

$${}_{s+1}\mathcal{F}_s(1,\dots,1;2,\dots,2)(\alpha)^q = \mathcal{L}_{K,(s)}(\alpha) \quad \big(\alpha\in\bar{k} \text{ and } |\alpha|_\infty < q^s\big). \tag{3.3}$$

This can be solved in the same manner as the proof for Proposition 2.3 by using

$$\langle 2\rangle_n = \langle 1\rangle_n(\theta^{q^{n+1}}-t)^{1/q}.$$

We can show that the q-th power of this power series is a non-zero element of $\mathbb{T}$.

Proposition 3.2. *Let $a_i, b_j \in \mathbb{N}$ $(1\le i\le s+1, 1\le j\le s)$. Then, for $\alpha\in\mathbb{C}_\infty^\times$ with $|\alpha|_\infty < q^{\sum_{j=1}^{s}(b_j-1)-\sum_{j=1}^{s+1}(a_j-1)}$ and $d\in\mathbb{Z}$ with $d \ge \max_{i,j}\{a_i,b_j\}$,*

$${}_{s+1}\mathcal{F}_s(\alpha)^{q^{d-1}} \in \mathbb{T}\setminus\{0\}. \tag{3.4}$$

Proof. Since $\|\langle a\rangle_m^{q^{d-1}}\| = \|\mathbb{D}_{m+a-1}^{q^{d-a}}\| = q^{(m+a-1)q^{m+d-1}}$, we can compute the value of each term of ${}_{s+1}\mathscr{F}_s(\alpha)^{q^{d-1}}$ as follows:

$$\begin{aligned}&\frac{\|\langle a_1\rangle_m^{q^{d-1}}\cdots\langle a_{s+1}\rangle_m^{q^{d-1}}\|}{\|\langle b_1\rangle_m^{q^{d-1}}\cdots\langle b_s\rangle_m^{q^{d-1}}\mathbb{D}_m^{q^{d-1}}\|}\|\alpha^{q^{m+d-1}}\|\\ &\quad= \left(q^{(a_1+m-1+\cdots+a_{s+1}+m-1)-(b_1+m-1+\cdots+b_s+m-1+m)}\right)^{q^{m+d-1}}|\alpha|_\infty^{q^{m+d-1}}\\ &\quad= \left(q^{\sum_{i=1}^{s+1}(a_i-1)-\sum_{j=1}^{s}(b_j-1)}|\alpha|_\infty\right)^{q^{m+d-1}}. \end{aligned}\tag{3.5}$$

Therefore, ${}_{s+1}\mathscr{F}_s(\alpha)^{q^{d-1}}$ converges on $|t|_\infty \le 1$ since $q^{\sum_{j=1}^{s+1}(a_j-1)-\sum_{j=1}^{s}(b_j-1)}|\alpha|_\infty < 1$. Furthermore, the above computation shows that the largest term of ${}_{s+1}\mathscr{F}_s(\alpha)^{q^{d-1}}$ with respect to $\|-\|$ is

$$\frac{\langle a_1\rangle_1^{q^{d-1}}\cdots\langle a_{s+1}\rangle_1^{q^{d-1}}}{\langle b_1\rangle_1^{q^{d-1}}\cdots\langle b_s\rangle_1^{q^{d-1}}\mathbb{D}_1^{q^{d-1}}}\alpha^{q^{d+1-1}}.$$

Thus, ${}_{s+1}\mathscr{F}_s(\alpha)^{q^{d-1}}$ is not zero. ∎

Since $|(a)_m^{q^{d-1}}|_\infty = |D_{m+a-1}^{q^{d-a}}|_\infty = q^{(m+a-1)q^{m+d-1}}$, we can show that the largest term of ${}_{s+1}F_s(\alpha)^{q^{d-1}}$ is

$$\frac{(a_1)_1^{q^{d-1}}\cdots(a_{s+1})_1^{q^{d-1}}}{(b_1)_1^{q^{d-1}}\cdots(b_s)_1^{q^{d-1}}D_1^{q^{d-1}}}\alpha^{q^d}$$

for $\alpha \in \mathbb{C}_\infty$ with $|\alpha|_\infty < q^{\sum_{j=1}^{s}(b_j-1)-\sum_{j=1}^{s+1}(a_j-1)}$ by the same calculation as in the above proof. Then, we obtain

$${}_{s+1}F_s(a_1,\ldots,a_{s+1};b_1,\ldots,b_s)(\alpha) \neq 0. \tag{3.6}$$

Next we show that $P_{\mathbf{b},d}$ defined by (1.3) is an entire function. Later, this helps us to check that the matrix Ψ belongs to $\mathrm{GL}_2(\mathbb{L})$ and thus is the rigid analytic trivialization of $\Phi_{\mathbf{a},\mathbf{b},d}$. We will see the details in the paragraph above Theorem 3.4, and in Theorem 3.4.

Proposition 3.3. *For $\mathbf{b} = (b_1, b_2, \ldots, b_s) \in \mathbb{N}^s$, we set $d \in \mathbb{Z}$ such that*

$$d \ge \max\{b_1, b_2, \ldots, b_s\}.$$

Then $P_{\mathbf{b},d} \in \mathbb{E}$ holds.

Proof. Because $[k_\infty(\theta^{\frac{1}{q-1}}) : k_\infty] < \infty$, it is enough to prove that $P_{\mathbf{b},d} \in \bar{k}[\![t]\!]$ and that $P_{\mathbf{b},d}$ is entire. Based on the definition (1.3), it follows that

$$P_{\mathbf{b},d}^{(-1)} = (-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}}\prod_{\substack{j=1\\ b_j\ge 2}}^{s}\left\{(\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}\right\}P_{\mathbf{b},d}. \tag{3.7}$$

We can expand

$$
\begin{aligned}
&(-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}} \prod_{\substack{j=1\\ b_j\geq 2}}^{s} \left\{(\theta-t)^{q^{d-2}} \mathbb{D}_{b_j-2}^{q^{d-b_j}}\right\} \\
&= (-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}} \prod_{\substack{j=1\\ b_j\geq 2}}^{s} \left\{(\theta-t)^{q^{d-1}} \left((\theta^{q}-t)^{q^{b_j-3}} \cdots (\theta^{q^{b_j-2}}-t)\right)^{q^{d-b_j}}\right\} \\
&= \sum_{m=0}^{N} f_m t^m \in A[t]
\end{aligned}
$$

by some $f_m \in A$ and some $N \geq 0$. We can also expand $P_{\mathbf{b},d} = \sum_{l\geq 0} g_l t^l \in k_\infty(\theta^{\frac{1}{q-1}})[\![t]\!]$ by some $g_l \in k_\infty(\theta^{\frac{1}{q-1}})$. Then, (3.7) can be written as

$$
\sum_{l\geq 0} g_l^{(-1)} t^l = \sum_{m=0}^{N} f_m t^m \sum_{l\geq 0} g_l t^l .
$$

By comparing the coefficients, we find that

$$
g_l^{(-1)} = \sum_{\substack{m_1+m_2=l\\ N\geq m_1\geq 0,\ m_2\geq 0}} f_{m_1} g_{m_2} .
$$

Thus, $g_l \in \bar{k}$ holds by the induction on l and $P_{\mathbf{b},d} \in \bar{k}[\![t]\!]$. The entireness of $P_{\mathbf{b},d}$ follows from the Weierstrass factorization theorem introduced in [15, Theorem 2.14]. ∎

Let $a_i, b_j \in \mathbb{N}$ $(1 \leq i \leq n+1, 1 \leq j \leq n)$ and $\alpha \in \bar{k}$ with $|\alpha|_\infty < q^{\sum_{j=1}^{n}(b_j-1)-\sum_{i=1}^{n+1}(a_i-1)}$. We set $M_{\mathbf{a},\mathbf{b}}$ to be the pre-t-motive defined by

$$
\begin{aligned}
\Phi_{\mathbf{a},\mathbf{b},d} &:= (-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}} \\
&\quad\times \begin{pmatrix} \prod_{\substack{j=1\\ b_j\geq 2}}^{s} \left((\theta-t)^{q^{d-2}} \mathbb{D}_{b_j-2}^{q^{d-b_j}}\right) & 0 \\ \prod_{\substack{j=1\\ a_j\geq 2}}^{s+1} \left((\theta-t)^{q^{d-2}} \mathbb{D}_{a_j-2}^{q^{d-a_j}}\right)\alpha^{q^{d-2}} & \prod_{\substack{j=1\\ b_j\geq 2}}^{s} \left((\theta-t)^{q^{d-2}} \mathbb{D}_{b_j-2}^{q^{d-b_j}}\right) \end{pmatrix} \\
&\quad\in \operatorname{Mat}_2\left(\bar{k}[t]\right) \cap \operatorname{GL}_2\left(\bar{k}(t)\right).
\end{aligned} \tag{3.8}
$$

We also define the following matrix:

$$
\Psi := \begin{pmatrix} P_{\mathbf{b},d} & 0 \\ P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}} & P_{\mathbf{b},d} \end{pmatrix}.
$$

Clearly, $\Psi \in \operatorname{Mat}_2(\mathbb{T})$ by using (3.4) and Proposition 3.3.

Furthermore, since

$$\frac{1}{\mathbb{D}_n} = \prod_{i=1}^{n}\Big(\sum_{l\geq 1}\frac{t^{l-1}}{\theta^{lq^i}}\Big)^{q^{n-i}},$$

one can show $1/(\mathbb{D}_{b_j-2}\cdots\mathbb{D}_2\mathbb{D}_1)\in\mathbb{T}\backslash\{0\}$ similarly to the proof of Proposition 3.2. Then $\Psi\in\mathrm{GL}_2(\mathbb{L})$ since $\det\Psi = P^2_{\mathbf{b},d}\in\mathbb{T}\backslash\{0\}$ by combining $1/(\mathbb{D}_{b_j-2}\cdots\mathbb{D}_2\mathbb{D}_1)\in\mathbb{T}\backslash\{0\}$, $\Omega\in\mathbb{T}^\times$ (cf. [23, Section 3.3.4]), and (1.3).

Theorem 3.4. *Let $a_i, b_j\in\mathbb{N}$ $(1\leq i\leq n+1,\ 1\leq j\leq n)$. Then, for $\alpha\in\bar{k}$ with $|\alpha|_\infty < q^{\sum_{j=1}^n(b_j-1)-\sum_{i=1}^{n+1}(a_i-1)}$ and $d\in\mathbb{Z}$ with $d\geq\max_{i,j}\{a_i,b_j\}$, Ψ is a rigid analytic trivialization of $\Phi_{\mathbf{a},\mathbf{b},d}$.*

Proof. According to the definition (3.1), each element of $\mathbb{D}_i$ satisfies

$$\mathbb{D}_i^{(-1)} = \begin{cases}(\theta-t)^{q^{i-1}}\mathbb{D}_{i-1} & \text{if } i>0,\\ 1 & \text{if } i\leq 0.\end{cases} \tag{3.9}$$

Then, we can obtain the Frobenius difference equation based on (3.9):

$$\begin{aligned}
&\big({}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}\big)^{(-1)}\\
&= \Bigg(\frac{\langle a_1\rangle_0^{q^{d-1}}\cdots\langle a_{s+1}\rangle_0^{q^{d-1}}}{\langle b_1\rangle_0^{q^{d-1}}\cdots\langle b_s\rangle_0^{q^{d-1}}\mathbb{D}_0^{q^{d-1}}}\alpha^{q^{d-1}} + \sum_{m\geq 1}\frac{\langle a_1\rangle_m^{q^{d-1}}\cdots\langle a_{s+1}\rangle_m^{q^{d-1}}}{\langle b_1\rangle_m^{q^{d-1}}\cdots\langle b_s\rangle_m^{q^{d-1}}\mathbb{D}_m^{q^{d-1}}}\alpha^{q^{m+d-1}}\Bigg)^{(-1)}\\
&= \frac{\prod_{\substack{i=1\\ a_i\geq 2}}^{s+1}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{a_i-2}^{q^{d-a_i}}\big)}{\prod_{\substack{j=1\\ b_j\geq 2}}^{s}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}\big)}\alpha^{q^{d-2}}\\
&\quad + \sum_{m\geq 1}\frac{\prod_{i=1}^{s+1}\big((\theta-t)^{q^{m+a_i-2}}\mathbb{D}_{m+a_i-2}\big)^{q^{d-a_i}}}{\big\{\prod_{j=1}^{s}\big((\theta-t)^{q^{m+b_j-2}}\mathbb{D}_{m+b_j-2}\big)^{q^{d-b_j}}\big\}\big((\theta-t)^{q^{m-1}}\mathbb{D}_{m-1}\big)^{q^{d-1}}}\alpha^{q^{m+d-2}}\\
&= \frac{\prod_{\substack{i=1\\ ba_i\geq 2}}^{s+1}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{a_i-2}^{q^{d-a_i}}\big)}{\prod_{\substack{j=1\\ b_j\geq 2}}^{s}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}\big)}\alpha^{q^{d-2}} + \sum_{m\geq 1}\frac{\mathbb{D}_{m+a_1-2}^{q^{d-a_1}}\cdots\mathbb{D}_{m+a_{s+1}-2}^{q^{d-a_{s+1}}}}{\mathbb{D}_{m+b_1-2}^{q^{d-b_1}}\cdots\mathbb{D}_{m+b_s-2}^{q^{d-b_s}}\mathbb{D}_{m-1}^{q^{d-1}}}\alpha^{q^{m+d-2}}\\
&= \frac{\prod_{\substack{i=1\\ a_i\geq 2}}^{s+1}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{a_i-2}^{q^{d-a_i}}\big)}{\prod_{\substack{j=1\\ b_j\geq 2}}^{s}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}\big)}\alpha^{q^{d-2}} + {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\big({}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}\big)^{(-1)}\\
&\quad = \frac{\prod_{\substack{i=1\\ a_i\geq 2}}^{s+1}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{a_i-2}^{q^{d-a_i}}\big)}{\prod_{\substack{j=1\\ b_j\geq 2}}^{s}\big((\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}\big)}\alpha^{q^{d-2}} + {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}.
\end{aligned} \tag{3.10}$$

Finally, we have

$$\begin{aligned}&\big(P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}\big)^{(-1)}\\&\quad=(-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}}\prod_{\substack{i=1\\a_i\geq 2}}^{s+1}(\theta-t)^{q^{d-2}}\mathbb{D}_{a_i-2}^{q^{d-a_i}}\alpha^{q^{d-2}}P_{\mathbf{b},d}\\&\qquad+(-1)^{\sum_{j=1}^{s}(b_j-1)q^{d-2}}\prod_{\substack{j=1\\b_j\geq 2}}^{s}(\theta-t)^{q^{d-2}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}.\end{aligned}\tag{3.11}$$

Therefore, $\Psi^{(-1)}=\Phi_{\mathbf{a},\mathbf{b},d}\Psi$ follows from (3.10). Accordingly, we can obtain the rigid analytic trivialization Ψ of $\Phi_{\mathbf{a},\mathbf{b},d}$ so that

$$\Psi|_{t=\theta}=\begin{pmatrix}P_{\mathbf{b},d}|_{t=\theta} & 0\\ P_{\mathbf{b},d}|_{t=\theta}\ {}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}} & P_{\mathbf{b},d}|_{t=\theta}\end{pmatrix}.\qquad\blacksquare$$

Remark 3.5. In the above theorem, $\Phi_{\mathbf{a},\mathbf{b},d}\in\mathrm{Mat}_2(\bar{k}[t])$ and $\Psi\in\mathrm{Mat}_2(\mathbb{T})\cap\mathrm{GL}_2(\mathbb{L})$ such that $\det(\Phi|_{t=0})\neq 0$ and $\Psi^{(-1)}=\Phi_{\mathbf{a},\mathbf{b},d}\Psi$. Then, we get $\Psi\in\mathrm{Mat}_2(\mathbb{E})$ by [1, Proposition 3.1.3].

Theorem 3.4 also presents a t-motivic interpretation of the KPLs with $\tilde{\pi}$.

Example 3.6. For $\mathbf{a}=(1,\dots,1)$, $\mathbf{b}=(2,\dots,2)$ and $d=2$, the pre-t-motive $M_{\mathbf{a},\mathbf{b},d}$ is more precisely the dual t-motive introduced in [1] defined by the matrix

$$\Phi_{\mathbf{a},\mathbf{b},d}=\begin{pmatrix}(t-\theta)^s & 0\\(-1)^s\alpha & (t-\theta)^s\end{pmatrix}.$$

Then, $\Phi_{\mathbf{a},\mathbf{b},d}$ satisfies the relation $\Psi^{(-1)}=\Phi_{\mathbf{a},\mathbf{b},d}\Psi$, where $\Psi:=\begin{pmatrix}\Omega^s & 0\\ \Omega^s\mathcal{L}_{K,(s)}(\alpha) & \Omega^s\end{pmatrix}$ by using (2.1) and Theorem 3.4. Thus, Ψ is a rigid analytic trivialization of $M_{\mathbf{a},\mathbf{b},d}$ whose entries specialized to $\tilde{\pi}^{-s}$ and $L_{K,(s)}(\alpha)$. We can also obtain the period matrix of $M_{\mathbf{a},\mathbf{b},d}$ as $\Psi^{-1}|_{t=\theta}$ (see [9, p. 267]) described by

$$\Psi^{-1}|_{t=\theta}=\begin{pmatrix}\tilde{\pi}^s & 0\\-\tilde{\pi}^s Li_{K,(s)}(\alpha) & \tilde{\pi}^s\end{pmatrix}.$$

Remark 3.7. As long as we focus on only the t-motivic interpretation of the THGFs, we do not need to consider the power series $P_{\mathbf{b},d}$. Indeed,

$$\Phi'_{\mathbf{a},\mathbf{b},d}:=\begin{pmatrix}1 & 0\\ \dfrac{\prod_{\substack{j=1\\a_j\geq 2}}^{s+1}(\theta-t)^{q^{d-1}}\mathbb{D}_{a_j-2}^{q^{d-a_j}}}{\prod_{\substack{j=1\\b_i\geq 2}}^{s}(\theta-t)^{q^{d-1}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}}\alpha^{q^{d-2}} & 1\end{pmatrix}\in\mathrm{GL}_2\big(\bar{k}(t)\big)$$

defines a pre-t-motive. According to the equation (3.10), it satisfies $\Psi^{(-1)} = \Phi'_{\mathbf{a},\mathbf{b},d}\Psi$ with

$$\Psi := \begin{pmatrix} 1 & 0 \\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{T}).$$

Thus, we can obtain a simpler t-motivic interpretation of the THGFs:

$$\Psi|_{t=\theta} = \begin{pmatrix} 1 & 0 \\ {}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}} & 1 \end{pmatrix}.$$

However, for our proof of the transcendence/linear independence results, we should assign the representation matrix to be in $\mathrm{Mat}_n(\bar{k}[t])$ to apply Chang's refined ABP criterion (Theorem 4.2, [6, Theorem 1.2]). Thus, we modify the interpretation with $P_{\mathbf{b},d}$ as Theorem 3.4.

Remark 3.8. The pre-t-motive $M_{\mathbf{a},\mathbf{b},d}$ in Example 3.6 was also considered by Taelman and the group of Anglès, Ngo Dac, Tavares Ribeiro to develop a counterexample to Taelman's conjecture. See [3] for more details.

We can extend Example 3.6 to the KMPL case as follows, which is similar to the Carlitz multiple polylogarithm case.

Definition 3.9. Set $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$. Then, for $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}_\infty^r$ with $|z_i|_\infty < q^{s_i}$, we define the following power series:

$$\mathcal{L}_{K,\mathfrak{s}}(\mathbf{z}) := \sum_{i_1 > i_2 > \cdots > i_r > 0} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}} \cdots z_r^{q^{i_r}}}{(\theta^{q^{i_1}} - t)^{s_1}(\theta^{q^{i_2}} - t)^{s_2} \cdots (\theta^{q^{i_r}} - t)^{s_r}}$$

which belong to $\mathbb{T}$ since $\|z_1^{q^{i_1}} z_2^{q^{i_2}} \cdots z_r^{q^{i_r}} / (\theta^{q^{i_1}} - t)^{s_1}(\theta^{q^{i_2}} - t)^{s_2} \cdots (\theta^{q^{i_r}} - t)^{s_r}\| \to 0$ as $1 \leq i_r < \cdots < i_1 \to \infty$. The following holds according to the definition of the Frobenius (-1)-fold twist and the above series expression:

$$\mathcal{L}_{K,\mathfrak{s}}(\mathbf{z})^{(-1)} = \frac{z_r}{(\theta - t)^{s_r}} \mathcal{L}_{K,(s_1,\ldots,s_{r-1})}(z_1, \ldots, z_{r-1}) + \mathcal{L}_{K,\mathfrak{s}}(\mathbf{z}). \tag{3.12}$$

We also define the series

$$\mathcal{L}^*_{K,\mathfrak{s}}(\mathbf{z}) := \sum_{i_1 \geq i_2 \geq \cdots \geq i_r > 0} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}} \cdots z_r^{q^{i_r}}}{(\theta^{q^{i_1}} - t)^{s_1}(\theta^{q^{i_2}} - t)^{s_2} \cdots (\theta^{q^{i_r}} - t)^{s_r}} \in \mathbb{T}$$

which is specialized to the star-version of the KMPLs at $t = \theta$. The star-version of the KMPL is defined by

$$Li^*_{K,\mathfrak{s}}(\mathbf{z}) := \sum_{i_1 \geq i_2 \geq \cdots \geq i_r > 0} \frac{z_1^{q^{i_1}} z_2^{q^{i_2}} \cdots z_r^{q^{i_r}}}{[i_1]^{s_1}[i_2]^{s_2} \cdots [i_r]^{s_r}}.$$

Then, in the same way as the proof for the star-versions of the Carlitz multiple polylogarithms by [10, 16], we obtain the following equations for $1 \le l \le j \le r$ by the inclusion–exclusion principle:

$$\begin{aligned}(-1)^l &\mathcal{L}^*_{K,(s_j,\dots,s_l)}(\alpha_j,\dots,\alpha_l) \\ &= \sum_{i=l+1}^{j} (-1)^{i-1} \mathcal{L}_{K,(s_l,\dots,s_{i-1})}(\alpha_l,\dots,\alpha_{i-1}) \mathcal{L}^*_{K,(s_j,\dots,s_i)}(\alpha_j,\dots,\alpha_i) \\ &\quad + (-1)^j \mathcal{L}_{K,(s_l,\dots,s_j)}(\alpha_l,\dots,\alpha_j), \end{aligned} \tag{3.13}$$

$$\begin{aligned}(-1)^j &\mathcal{L}^*_{K,(s_j,\dots,s_l)}(\alpha_j,\dots,\alpha_l) \\ &= \sum_{i=l+1}^{j} (-1)^{i} \mathcal{L}_{K,(s_i,\dots,s_j)}(\alpha_i,\dots,\alpha_j) \mathcal{L}^*_{K,(s_{i-1},\dots,s_l)}(\alpha_{i-1},\dots,\alpha_l) \\ &\quad + (-1)^l \mathcal{L}_{K,(s_l,\dots,s_j)}(\alpha_l,\dots,\alpha_j). \end{aligned} \tag{3.14}$$

Based on (3.12), it follows that

$$\Psi^{(-1)}_{\mathfrak{s},\boldsymbol{\alpha}} = \Phi_{\mathfrak{s},\boldsymbol{\alpha}} \Psi_{\mathfrak{s},\boldsymbol{\alpha}} \tag{3.15}$$

where

$$\Phi_{\mathfrak{s},\boldsymbol{\alpha}} = \begin{pmatrix} (t-\theta)^w & 0 & \cdots & 0 & 0 \\ (-1)^{s_r}\alpha_r(t-\theta)^{w-s_r} & (t-\theta)^w & \ddots & \vdots & \vdots \\ 0 & (-1)^{s_{r-1}}\alpha_{r-1}(t-\theta)^{w-s_{r-1}} & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & & (t-\theta)^w & 0 \\ 0 & 0 & & (-1)^{s_1}\alpha_1(t-\theta)^{w-s_1} & (t-\theta)^w \end{pmatrix}$$
$$\in \mathrm{Mat}_{r+1}(\bar{k}[t]) \cap \mathrm{GL}_{r+1}\left(\bar{k}(t)\right)$$

and

$$\Psi_{\mathfrak{s},\boldsymbol{\alpha}} = \begin{pmatrix} \Omega^w & 0 & \cdots & 0 & 0 \\ \Omega^w \mathcal{L}_{K,(s_r)}(\alpha_r) & \Omega^w & \ddots & \vdots & \vdots \\ \Omega^w \mathcal{L}_{K,(s_{r-1},s_r)}(\alpha_{r-1},\alpha_r) & \Omega^w \mathcal{L}_{K,(s_{r-1})}(\alpha_{r-1}) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \Omega^w & 0 \\ \Omega^w \mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) & \Omega^w \mathcal{L}_{K,(s_1,\dots,s_{r-1})}(\alpha_1,\dots,\alpha_{r-1}) & \cdots & \Omega^w \mathcal{L}_{K,(s_1)}(\alpha_1) & \Omega^w \end{pmatrix}$$
$$\in \mathrm{GL}_{r+1}(\mathbb{L}).$$

We remark that $\Psi_{\mathfrak{s},\boldsymbol{\alpha}} \in \mathrm{Mat}_{r+1}(\mathbb{E})$ by [1, Proposition 3.1.3], that is, $\Psi_{\mathfrak{s},\boldsymbol{\alpha}} \in \mathrm{GL}_{r+1}(\mathbb{L}) \cap \mathrm{Mat}_{r+1}(\mathbb{E})$.

Furthermore, by using (3.13) and (3.14), $\Psi_{\mathfrak{s},\boldsymbol{\alpha}}^{-1}$ can be written as follows:

$$\Psi_{\mathfrak{s},\boldsymbol{\alpha}}^{-1} = \Omega^{-w} I_{r+1} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\mathcal{L}^*_{K,(s_r)}(\alpha_r) & 1 & \ddots & \vdots & \vdots \\ (-1)^2\mathcal{L}^*_{K,(s_r,s_{r-1})}(\alpha_r,\alpha_{r-1}) & -\mathcal{L}^*_{K,(s_{r-1})}(\alpha_{r-1}) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ \vdots & \vdots & & 1 & 0 \\ (-1)^r\mathcal{L}^*_{K,\overleftarrow{\mathfrak{s}}}(\overleftarrow{\boldsymbol{\alpha}}) & (-1)^{r-1}\mathcal{L}^*_{K,(s_{r-1},\dots,s_1)}(\alpha_{r-1},\dots,\alpha_1) & \cdots & -\mathcal{L}^*_{K,(s_1)}(\alpha_1) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{L}).$$

Here, we set I_{r+1} the identity matrix of size $r+1$, and set $\overleftarrow{\mathfrak{s}} = (s_r, s_{r-1}, \dots, s_1)$, $\overleftarrow{\boldsymbol{\alpha}} = (\alpha_r, \alpha_{r-1}, \dots, \alpha_1)$. The matrix $\Phi_{\mathfrak{s},\boldsymbol{\alpha}}$ defines a pre-t-motive M and thus $\Psi_{\mathfrak{s},\boldsymbol{\alpha}}$ is its rigid analytic trivialization. Then $\Psi_{\mathfrak{s},\boldsymbol{\alpha}}^{-1}|_{t=\theta}$ is a period matrix of M whose entries are expressed by $\tilde{\pi}^w$ and

$$(-1)^{\mathrm{dep}(s_j,\dots,s_l)}\tilde{\pi}^w Li^*_{K,(s_j,\dots,s_l)}(\alpha_j,\dots,\alpha_l) \quad (1 \le l \le j \le r).$$

4. Linear independence results of the THGFs and the KMPLs

In this section, we discuss the transcendence and linear independence results derived by using a refined version of the Anderson–Brownawell–Papanikolas' linear independence criterion. The original version is given in the following statement.

Theorem 4.1 ([1, Theorem 3.1.1]). *Fix $\Phi \in \mathrm{Mat}_d(\bar{k}[t])$ such that $\det\Phi = c(t-\theta)^s$ for some $c \in \bar{k}^\times$ and some $s \in \mathbb{Z}_{\ge 0}$. Suppose that there exists a vector $\psi \in \mathrm{Mat}_{d\times 1}(\mathbb{E})$ that satisfies*

$$\psi^{(-1)} = \Phi\psi.$$

For every $\rho \in \mathrm{Mat}_{1\times d}(\bar{k})$ such that $\rho\psi(\theta) = 0$, there exists a $P \in \mathrm{Mat}_{1\times d}(\bar{k}[t])$ such that $P(\theta) = \rho$ and $P\psi = 0$.

By the definition, $\det(\Phi_{\mathbf{a},\mathbf{b},d})$ is a polynomial in $\bar{k}[t]$ but generally it cannot be written by some powers of $(t-\theta)$ multiplied with a non-zero constant in $\bar{k}$. Therefore, we need to employ the following refined version of Theorem 4.1.

Theorem 4.2 ([6, Theorem 1.2]). *We fix a matrix $\Phi = \Phi(t) \in \mathrm{Mat}_l(\bar{k}[t])$ such that $\det\Phi$ is a polynomial in t that satisfies $\det\Phi(0) \neq 0$. Fix a vector $\psi = [\psi_1(t), \dots, \psi_l(t)]^{\mathrm{tr}} \in \mathrm{Mat}_{l\times 1}(\mathbb{E})$ that satisfies the functional equation $\psi^{(-1)} = \Phi\psi$. Let $\xi \in \bar{k}^\times \backslash \overline{\mathbb{F}_q}^\times$ satisfy*

$$\det\Phi(\xi^{(-i)}) \neq 0 \quad \textit{for all } i = 1, 2, \dots.$$

Then the following properties hold.

(1) *For every vector* $\rho \in \mathrm{Mat}_{1\times l}(\bar{k})$ *such that* $\rho\psi(\xi) = 0$*, there exists a vector* $P = P(t) \in \mathrm{Mat}_{1\times l}(\bar{k}[t])$ *such that* $P(\xi) = \rho$ *and* $P\psi = 0$*,*

(2) $\mathrm{tr.deg}_{\bar{k}(t)}\bar{k}(t)(\psi_1(t), \dots, \psi_l(t)) = \mathrm{tr.deg}_{\bar{k}}\bar{k}(\psi_1(\xi), \dots, \psi_l(\xi))$.

Furthermore, for our proof, we use the following $(\theta^{q^i} - t)$-expansion of ${}_r\mathcal{F}_s(\alpha)$, which follows from the method described in [28, p. 143]. We again remark that for a given ${}_r\mathcal{F}_s(a_1, \dots, a_r; b_1, \dots, b_s)(\alpha)$, we assume throughout this paper that its parameters satisfy $a_i \le a_j$ and $b_i \le b_j$ for $i \le j$ without loss of generality.

Proposition 4.3. *For a given* ${}_r\mathcal{F}_s(\alpha) = {}_r\mathcal{F}_s(a_1, \dots, a_r; b_1, \dots, b_s)(\alpha)$ *with* $\alpha \in \bar{k}$ *satisfying* (1.2) *and for* $j \in \mathbb{Z}$*, we define*

$$\begin{aligned} a(j) &= r - u + 1 \quad \text{if } a_{u-1} \le j \le a_u - 1, \\ b(j) &= s - v + 1 \quad \text{if } b_{v-1} \le j \le b_v - 1, \\ c(j) &= a(j) - b(j) \end{aligned}$$

by setting $b_0 = 1$*,* $a_0 = b_{-1} = -\infty$ *and* $a_{r+1} = b_{s+1} = +\infty$*. Then, we have*

$$\left({}_r\mathcal{F}_s(\alpha)\right)^{q^d} = \sum_{n=0}^{\infty}\left(\prod_{m=1}^{n+d-1}(\theta^{q^m} - t)^{c(m-n)q^{n+d-m}}\right)\alpha^{q^{n+d}} \tag{4.1}$$

where $d = \max\{a_1, \dots, a_r, b_1, \dots, b_s\}$.

Remark 4.4. For $l \ge \max_{\substack{1\le i\le r \\ 1\le j\le s}}\{a_i, b_j\}$,

$$c(l) = a(l) - b(l) = 0 - 0 = 0$$

holds according to the definition. Especially when $r = s + 1$, for $l \le 0$, we again obtain $c(l) = a(l) - b(l) = s + 1 - (s + 1) = 0$.

Later, in our proofs of Theorems 4.7 and 4.9, we need to assume some conditions for $c(j)$ due to the following observation.

Remark 4.5. Let $N > 0$. For each $\alpha^{q^{n+d}}$ $(N > n \ge N - d + 1)$ in (4.1), the coefficient

$$\prod_{m=1}^{n+d-1}(\theta^{q^m} - t)^{c(m-n)q^{n+d-m}}$$

of $\alpha^{q^{n+d}}$ has a pole or zero at $t = \theta^{q^N}$ with order $|c(N-n)|q^{n+d-N}$. By the definition, the range of the quantity $c(j)$ depends on the parameters $a_1, \dots, a_r$ and $b_1, \dots, b_s$ and we do not have $c(j_1) > c(j_2)$ for $j_1 > j_2$ or $j_2 > j_1$ in general. Thus for some large enough r and s, there may exist distinct $n_1, n_2, \dots, n_l$ with $N > n_1, n_2, \dots, n_l \ge N - d + 1$ so that

$$c(N - n_1)q^{n_1+d-N} = \cdots = c(N - n_l)q^{n_l+d-N}.$$

By the expression (4.1),

$$\begin{aligned}\left({}_r\mathscr{F}_s(\alpha)\right)^{q^d} &= \sum_{n=N}^{\infty}\left(\prod_{m=1}^{n+d-1}(\theta^{q^m}-t)^{c(m-n)q^{n+d-m}}\right)\alpha^{q^{n+d}} \\ &\quad + \sum_{n=N-d+1}^{N-1}\left(\prod_{m=1}^{n+d-1}(\theta^{q^m}-t)^{c(m-n)q^{n+d-m}}\right)\alpha^{q^{n+d}} \\ &\quad + \sum_{n=0}^{N-d}\left(\prod_{m=1}^{n+d-1}(\theta^{q^m}-t)^{c(m-n)q^{n+d-m}}\right)\alpha^{q^{n+d}}. \end{aligned} \tag{4.2}$$

By multiplying $(\theta^{q^N}-t)^{c(N-n_1)q^{n_1+d-N}}$ on both side of (4.2) and substituting $t=\theta^{q^N}$, we get

$$\begin{aligned}&\left((\theta^{q^N}-t)^{c(N-n_1)q^{n_1+d-N}}\left({}_r\mathscr{F}_s(\alpha)\right)^{q^d}\right)\Big|_{t=\theta^{q^N}} \\ &\quad = \sum_{i=1}^{l}\prod_{\substack{m=1\\ m\neq N}}^{n_i+d-1}(\theta^{q^m}-\theta^{q^N})^{c(m-n_i)q^{n_i+d-m}}\alpha^{q^{n_i+d}}.\end{aligned}$$

Thus we obtain the k-linear combination of α with some distinct powers. If we do not assign the conditions to $c(j)$, l may not be equal to 1 in general. Then the right-hand side of the above equation cause problems in our proofs of Theorems 4.7 and 4.9, in showing contradictions to k-linear independence of $\alpha_1,\dots,\alpha_r\in\bar{k}$ and non-vanishing of $\alpha\in\bar{k}$.

4.1. Applications to the special values of the THGFs

In this section, we describe the equivalent conditions for the transcendence of the THGFs, which is already given by Thakur et al. in [28]. We reprove it via the t-motivic interpretation of the values of the THGFs and Chang's refined ABP criterion. Furthermore, we show the linear independence of some THGFs at algebraic points, which are specialized to the results of the KPLs.

Here we again recall our assumptions in Sections 1.1 and 2.1. We assume throughout this paper that for a given THGF ${}_rF_s(a_1,\dots,a_r;b_1,\dots,b_s)(z)$ (resp. ${}_r\mathscr{F}_s$ case), its parameters satisfy $a_i\le a_j$ and $b_i\le b_j$ for $i\le j$ without loss of generality.

Theorem 4.6. *Let $a_i,b_j\in\mathbb{N}$ $(1\le i\le s+1,\ 1\le j\le s)$ and let $\alpha\in\bar{k}^\times$ satisfying that*

$$|\alpha|_\infty < q^{\sum_{j=1}^{s}(b_j-1)-\sum_{i=1}^{s+1}(a_i-1)}.$$

Then, ${}_{s+1}F_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)$ is transcendental over k if and only if $b_j>a_{j+1}$ for some j.

Proof. First, we prove the transcendence of ${}_{s+1}F_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)$ with $b_j>a_{j+1}$ for some j. Suppose on the contrary that

$$f + {}_{s+1}F_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)=0$$

for some $f \in \bar{k}^{\times}$. This is equivalent to saying

$$P_{\mathbf{b},d}|_{t=\theta} f^{q^d} + P_{\mathbf{b},d}|_{t=\theta}\ {}_{s+1}F_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)^{q^d} = 0.$$

Then, by Theorem 3.4 and 4.2, we can lift this to the relation

$$(g_1, g_2)\begin{pmatrix} P_{\mathbf{b},d} \\ P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)^{q^d} \end{pmatrix} = 0 \tag{4.3}$$

where $g_i(t) \in \bar{k}[t]$ $(i = 1, 2)$ such that $g_1(\theta) = f^{q^d}$ and $g_2(\theta) = 1$. This is written without $P_{d,\mathbf{b}}$ as

$$g_1(t) + g_2(t)_{s+1}\mathcal{F}_s(a_1,\dots,a_{s+1};b_1,\dots,b_s)(\alpha)^{q^d} = 0. \tag{4.4}$$

Let $N \in \mathbb{N}$ such that $g_2(\theta^{q^N}) \neq 0$. With the expansion (4.1), Remark 4.4 and changing a variable from m to $l + n$, we have

$$\begin{aligned}\left({}_{s+1}\mathcal{F}_s(\alpha)\right)^{q^d} &= \sum_{n=0}^{\infty}\left(\prod_{m=1}^{n+d-1}(\theta^{q^m} - t)^{c(m-n)q^{n+d-m}}\alpha^{q^{n+d}}\right) \\ &= \sum_{n=0}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right).\end{aligned} \tag{4.5}$$

Because $b_j > a_{j+1}$ for some $s+1 > j > 0$, there exists $d - 1 \geq l \geq 1$ such that $b_j > l \geq a_{j+1}$. Then $a_u - 1 \geq m \geq a_{u-1}$ for some $s + 3 > u \geq j + 2$ and $b_v - 1 \geq l \geq b_{v-1}$ for some $j \geq v \geq 1$. Here, we assume that $b_0 = 1, a_0 = b_{-1} = -\infty$ and $a_{s+2} = b_{s+1} = +\infty$ as in Proposition 4.3. Thus, we obtain $c(l) < 0$ for these j. Indeed, $c(l) = a(l) - b(l) = (s - u + 2) - (s - v + 1) = v + 1 - u \leq j + 1 - u \leq -1$. We decompose ${}_{s+1}\mathcal{F}_s(\alpha)^{q^d}$ as follows:

$$\begin{aligned}\left({}_{s+1}\mathcal{F}_s(\alpha)\right)^{q^d} &= \sum_{n=N}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right) \\ &\quad + \sum_{n=0}^{N-1}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right) \\ &= \sum_{n=N}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right) \\ &\quad + \sum_{n=0}^{N-d}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right) \\ &\quad + \sum_{n=N+1-d}^{N-1}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right).\end{aligned}$$

Then, we denote all $l \in \mathbb{Z}$ such that $d-1 \geq l \geq 1$ and $c(l) < 0$ by l_i $(i = 1, \ldots, r)$ and decompose the above as

$$
\begin{aligned}
&= \sum_{n=N}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right) + \sum_{n=0}^{N-d}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)\\
&\quad + \sum_{\substack{n=N+1-d\\ n\neq N-l_1,\ldots,N-l_r}}^{N-1}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)\\
&\quad + \sum_{n=N-l_1,\ldots,N-l_r}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right).
\end{aligned}
$$

Thus, (4.4) can be rewritten as

$$
\begin{aligned}
g_1(t) &+ g_2(t)\sum_{n=N}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)\\
&+ g_2(t)\sum_{n=0}^{N-d}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)\\
&+ g_2(t)\sum_{\substack{n=N+1-d\\ n\neq N-l_1,\ldots,N-l_r}}^{N-1}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)\\
&= g_2(t)\sum_{n=N-l_1,\ldots,N-l_r}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right).
\end{aligned}
$$

At $t = \theta^{q^N}$, the left-hand side of the above equation is regular, while

$$
\sum_{n=N-l_1,\ldots,N-l_r}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)
$$

(this sum is non-zero because the largest term with respect to $\|\cdot\|$ is $\prod_{l=1}^{d-1}(\theta^{q^{l+N-l_h}} - t)^{c(l)q^{d-l}}\alpha^{q^{N-l_h+d}}$ where $l_h = \min\{l_1, \ldots, l_r\}$) on the right-hand side has a pole. Indeed, on the left-hand side, the 1st term $g_1(t)$ is a polynomial, the 2nd sum

$$
\sum_{n=N}^{\infty}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)
$$

is $\left({}_{s+1}F_s(\alpha)^{q^d}\right)^{q^N}$ at $t = \theta^{q^N}$, and $c(l)$ for $l = N - n$ are not negative in both the 3rd sum

$$
\sum_{n=0}^{N-d}\left(\prod_{l=1}^{d-1}(\theta^{q^{l+n}} - t)^{c(l)q^{d-l}}\alpha^{q^{n+d}}\right)
$$

and 4th sum

$$\sum_{\substack{n=N+1-d \\ n\neq N-l_1,\dots,N-l_r}}^{N-1} \left(\prod_{l=1}^{d-1} (\theta^{q^{l+n}} - t)^{c(l)q^{d-l}} \alpha^{q^{n+d}} \right).$$

Thus, $g_2(t)$ must have a zero at $t = \theta^{q^N}$, so we obtain a contradiction.

Next, we prove that ${}_{s+1}F_s(\alpha)$ is algebraic when $b_j \le a_{j+1}$ for all j. Due to the former part of the proof, in this case, $c(l) \ge 0$ for any l. Indeed, if there exists $i > 0$ such that $c(i) < 0$, we obtain $a(i) - b(i) = (s+1-u+1) - (s-v+1) = 1-u+v < 0$ for some $1 \le u \le s+1$, $1 \le v \le s$. Then, we have $b_{v-1} \le i \le b_v - 1$ and $a_{u-1} \le i \le a_u - 1$ thus, in particular we have $a_{u-1} < b_v$. However, this contradicts $1-u+v < 0$, that is, $b_v \le a_{u-1}$.

By using the expression (4.5), we have

$$\begin{aligned}
\left({}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^d}\right)^{(-1)} &= \left(\sum_{n\ge 0} \prod_{l=1}^{d-1} (\theta^{q^{n+d}} - t)^{c(l)q^{d-l}} \alpha^{q^{n+d}} \right)^{(-1)} \\
&= \prod_{l=1}^{d-1} (\theta^{q^{d-1}} - t)^{c(l)q^{d-l}} \alpha^{q^{d-1}} + {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^{d-1}}.
\end{aligned}$$

Thus, by setting

$$\Phi_{\mathbf{a},\mathbf{b},d} = \begin{pmatrix} 1 & 0 \\ \prod_{l=1}^{d-1} (\theta^{q^{d-1}} - t)^{c(l)q^{d-l}} \alpha^{q^{d-1}} & 1 \end{pmatrix} \in \mathrm{Mat}_2\left(\bar{k}[t]\right),$$

$$\psi = \begin{pmatrix} 1 \\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^d} \end{pmatrix} \in \mathrm{Mat}_{2\times 1}(\mathbb{T}),$$

we have $\psi^{(-1)} = \Phi_{\mathbf{a},\mathbf{b},d}\,\psi$ and then, $\psi \in \mathrm{Mat}_{2\times 1}(\mathbb{E})$ by [1, Proposition 3.1.3]. This allows us to apply Theorem 4.2 without $P_{\mathbf{b},d}$.

By expanding $\prod_{j=1}^{d-1} (\theta^{q^{n+d}} - t^{q^{d-j}})^{c(j)} \alpha^{q^{n+d}}$, we obtain the finite $\mathbb{F}_q[t]$-linear combination $\sum_{H\ge h\ge 1} f_h(t) \theta^{hq^{n+d}} \alpha^{q^{n+d}}$ with $f_h(t) \in \mathbb{F}_q[t]$. Thus we can write ${}_{s+1}\mathcal{F}_s(\alpha)^{q^d} \in \mathbb{T}\setminus\{0\}$ (see (3.4)) by

$$\begin{aligned}
{}_{s+1}\mathcal{F}_s(\alpha)^{q^d} &= \sum_{n\ge 0} \sum_{H\ge h\ge 1} f_h(t) \theta^{hq^{n+d}} \alpha^{q^{n+d}} \\
&= \sum_{H\ge h\ge 1} f_h(t) \sum_{n\ge 0} \theta^{hq^{n+d}} \alpha^{q^{n+d}}.
\end{aligned} \tag{4.6}$$

Then we have the algebraic relation

$$\left(\sum_{n\ge 0} \theta^{hq^{n+d}} \alpha^{q^{n+d}} \right)^q = \sum_{n\ge 0} \theta^{hq^{n+d}} \alpha^{q^{n+d}} - \theta^{hq^d} \alpha^{q^d}$$

which implies that the sum $\sum_{n\geq 0}\theta^{hq^{n+d}}\alpha^{q^{n+d}}$ is in $\bar{k}$. Thus the expression (4.6) shows that ${}_{s+1}\mathscr{F}_s(\alpha)^{q^d}\in\bar{k}[t]$ and Theorem 4.2 (2) yields

$$0=\operatorname{tr.deg}_{\bar{k}(t)}\bar{k}(t)\{1,\ {}_{s+1}\mathscr{F}_s(\mathbf{a};\mathbf{b})(\alpha)^{q^d}\}=\operatorname{tr.deg}_{\bar{k}}\bar{k}\{1,\ {}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha)^{q^d}\}.$$

Therefore, ${}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha)$ is algebraic over k. ∎

In the following, we set

$$d_h=\max_{\substack{1\leq i\leq h+1\\ 1\leq j\leq h}}\{a_i,b_j\}\quad(h=1,\ldots,s)$$

and we denote all m satisfying $d_s\geq m\geq 0$ by m_i $(i=1,\ldots,n)$ where $n:=d_s+1$.

Theorem 4.7. *Fix $\mathbf{a}_s=(a_1,\ldots,a_{s+1})\in\mathbb{N}^{s+1}$, $\mathbf{b}_s=(b_1,\ldots,b_s)\in\mathbb{N}^s$ such that $b_1>a_{s+1}$. We take $n\geq r\geq 1$ such that $b_1-1\geq m_r$. Let $\mathbf{a}_h=(a_1,\ldots,a_{h+1})$, $\mathbf{b}_h=(b_1,\ldots,b_h)$ $(h=1,\ldots,s)$ and let $\alpha_h\in\bar{k}^\times$ with*

$$|\alpha_h|_\infty<q^{\sum_{j=1}^{h}(b_j-1)-\sum_{i=1}^{h+1}(a_i-1)}.$$

If $\min_{1\leq i\leq n,i\neq r}\{c(m_i)q^{d-m_i}\}>c(m_r)q^{d-m_r}$, then ${}_{h+1}F_h(\mathbf{a}_h;\mathbf{b}_h)(\alpha_h)$ $(1\leq h\leq s)$ are $\bar{k}$-linearly independent.

Proof. Suppose on the contrary that

$$f_0+f_1\,{}_2F_1(\mathbf{a}_1;\mathbf{b}_1)(\alpha_1)^{q^{d_s}}+\cdots+f_s\,{}_{s+1}F_s(\mathbf{a}_s;\mathbf{b}_s)(\alpha_s)^{q^{d_s}}=0$$

for some $f_i\in\bar{k}$ $(i=0,1,\ldots,s)$ which are not all zero. Without loss of generality, we assume that $f_s\neq 0$. We consider

$$\Phi=\begin{pmatrix}\Phi_1&&&\\&\Phi_2&&\\&&\ddots&\\&&&\Phi_s\end{pmatrix}\quad\text{and}\quad\psi=\begin{pmatrix}\psi_1\\\psi_2\\\vdots\\\psi_s\end{pmatrix}$$

where

$$\begin{aligned}\Phi_h&=(-1)^{\sum_{j=h+1}^{s}(b_j-1)q^{d-2}}\prod_{\substack{j=h+1\\ b_j\geq 2}}^{s}(\theta-t)^{q^{d_s-1}}\mathbb{D}_{b_j-2}^{q^{d_s-b_j}}\Phi_{\mathbf{a}_h,\mathbf{b}_h,d_s}\\&\in\operatorname{Mat}_2\left(\bar{k}[t]\right)\cap\operatorname{GL}_2\left(\bar{k}(t)\right)\end{aligned}$$

and

$$\psi_h=\begin{pmatrix}P_{\mathbf{b}_s,d_s}\\P_{\mathbf{b}_s,d_s}\,{}_{h+1}\mathscr{F}_h(\mathbf{a}_h;\mathbf{b}_h)(\alpha_h)^{q^{d_s}}\end{pmatrix}\in\operatorname{Mat}_{2\times 1}(\mathbb{E}).$$

According to Theorem 3.4, $\psi_h^{(-1)} = \Phi_h \psi_h$ is true for each h; thus we have $\psi^{(-1)} = \Phi\psi$. Then, by using Theorem 4.2, we have the following $\bar{k}[t]$-linear relation:

$$\begin{aligned} P_{\mathbf{b}_s,d_s} g_0(t) + P_{\mathbf{b}_s,d_s} g_1(t)\, {}_2\mathcal{F}_1(\mathbf{a}_1;\mathbf{b}_1)(\alpha_1)^{q^d} \\ + \cdots + P_{d_s,\mathbf{b}_s} g_s(t)\, {}_{s+1}\mathcal{F}_s(\mathbf{a}_s;\mathbf{b}_s)(\alpha_s)^{q^d} = 0 \end{aligned} \tag{4.7}$$

for $g_i(t) \in \bar{k}[t]$ such that $g_i(\theta) = f_i$ $(i = 1, \ldots, s)$. In particular, $g_s(t) \neq 0$ since $g_s(\theta) = f_s \neq 0$. We can rewrite the above as follows by using (4.1):

$$\begin{aligned} P_{\mathbf{b}_s,d_s} \Bigg\{ g_0(t) + g_1(t) \left(\sum_{n=0}^{\infty} \left(\prod_{l=1}^{d_1-1} (\theta^{q^{l+n}} - t)^{c_1(l) q^{d-l}} \alpha_1^{q^{n+d}} \right) \right) \\ + \cdots + g_s(t) \left(\sum_{n=0}^{\infty} \left(\prod_{l=1}^{d_s-1} (\theta^{q^{l+n}} - t)^{c_s(l) q^{d-l}} \alpha_s^{q^{n+d}} \right) \right) \Bigg\} = 0. \end{aligned}$$

Here each $c_h(\cdot)$ is associated to ${}_{h+1}\mathcal{F}_h(\mathbf{a}_h;\mathbf{b}_h)(\alpha_h)$ and thus $c_s(\cdot)$ is nothing but $c(\cdot)$.

We note that $c_h(m_r) = -h$ for each $h = 1, \ldots, s$. Indeed, $b_1 - 1 \geq m_r \geq a_{s+1}$ implies $b_1 - 1 \geq m_r \geq a_{h+1}$ and then,

$$c_h(m_r) = a_h(m_r) - b_h(m_r) = h + 1 - (h+2) + 1 - (h - 1 + 1) = -h.$$

We set N to be a positive integer such that $g_s(\theta^{q^N}) \neq 0$. Then, by multiplying

$$\left(P_{\mathbf{b}_s,d_s} (\theta^{q^N} - t)^{c_s(m_r) q^{d-m_r}} \right)^{-1}$$

on both sides of (4.7) and we have

$$\begin{aligned} &(\theta^{q^N} - t)^{-c_s(m_r) q^{d-m_r}} g_0(t) \\ &\quad + (\theta^{q^N} - t)^{-c_s(m_r) q^{d-m_r}} g_1(t) \left(\sum_{n=0}^{\infty} \left(\prod_{l=1}^{d_1-1} (\theta^{q^{l+n}} - t)^{c_1(l) q^{d-l}} \alpha_1^{q^{n+d}} \right) \right) \\ &\quad \vdots \\ &\quad + (\theta^{q^N} - t)^{-c_s(m_r) q^{d-m_r}} g_{s-1}(t) \left(\sum_{n=0}^{\infty} \left(\prod_{l=1}^{d_{s-1}-1} (\theta^{q^{l+n}} - t)^{c_{s-1}(l) q^{d-l}} \alpha_{s-1}^{q^{n+d}} \right) \right) \\ &\quad + (\theta^{q^N} - t)^{-c_s(m_r) q^{d-m_r}} g_s(t) \left(\sum_{\substack{n=0 \\ n \neq N - m_r}}^{\infty} \left(\prod_{l=1}^{d_s-1} (\theta^{q^{l+n}} - t)^{c_s(l) q^{d-l}} \alpha_s^{q^{n+d}} \right) \right) \\ &\quad + (\theta^{q^N} - t)^{-c_s(m_r) q^{d-m_r}} g_s(t) \prod_{l=1}^{d_s-1} (\theta^{q^{l+N-m_r}} - t)^{c_s(l) q^{d-l}} \alpha_s^{q^{N-m_r+d}} = 0. \end{aligned}$$

Since we assume the condition $\min_{1\le i\le n, i\ne r}\{c(m_i)q^{d-m_i}\} > c(m_r)q^{d-m_r}$, by substituting $t = \theta^{q^N}$ into the above equation, we obtain

$$\begin{aligned}
&(\theta^{q^N} - t)^{-c_s(m_r)q^{d-m_r}} g_s(t) \prod_{l=1}^{d_s-1} (\theta^{q^{l+N-m_r}} - t)^{c_s(l)q^{d-l}} \alpha_s^{q^{N-m_r+d}} |_{t=\theta^{q^N}} \\
&= g_s(t) \prod_{\substack{l=1 \\ l\ne m_r}}^{d_s-1} (\theta^{q^{l+N-m_r}} - t)^{c_s(l)q^{d-l}} \alpha_s^{q^{N-m_r+d}} |_{t=\theta^{q^N}} \\
&= g_s(\theta^{q^N})\alpha_s^{q^N} = 0.
\end{aligned}$$

Therefore, we obtain a contradiction $\alpha_s \ne 0$ and then, the desired result holds. ∎

Remark 4.8. For a given $s > 0$, if we specialize $\mathbf{a}_s = (1,\dots,1) \in \mathbb{N}^{s+1}$ and $\mathbf{b}_s = (2,\dots,2) \in \mathbb{N}^s$ in the above theorem, it shows that 1, $Li_{K,s}(\alpha_s)$, $Li_{K,s-1}(\alpha_{s-1}),\dots,$ $Li_{K,1}(\alpha_1)$ are $\bar{k}$-linearly independent. Accordingly, there are no $\bar{k}$-linear relations among 1 and the KPLs at algebraic points with different weights.

In the following, we denote all m satisfying $d \ge m \ge 0$ by m_i $(i = 1,\dots,n)$ where $n := d + 1$.

Theorem 4.9. *For any* $\mathbf{a} = (a_1,\dots,a_{s+1}) \in \mathbb{N}^{s+1}$ *and* $\mathbf{b} = (b_1,\dots,b_s) \in \mathbb{N}^s$ *satisfying that* $b_j > a_{j+1}$ *for some* j*, and that* $\min_{1\le i\le n, i\ne u}\{c(m_i)q^{d-m_i}\} > c(m_u)q^{d-m_u}$ *for some* u*, let* $\alpha_i \in \bar{k}^\times$ $(i = 1,\dots,r)$ *with* $|\alpha_i|_\infty < q^{\sum_{j=1}^{s}(b_j-1)-\sum_{i=1}^{s+1}(a_i-1)}$. *If* $\alpha_1,\dots,\alpha_r$ *are* k*-linearly independent, then* ${}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha_1),\dots,$ ${}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha_r)$ *are* $\bar{k}$*-linearly independent.*

Proof. We assume on the contrary that there exists a non-trivial $\bar{k}$-linear relation:

$$f_1\, {}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + f_r\, {}_{s+1}F_s(\mathbf{a};\mathbf{b})(\alpha_r)^{q^d} = 0.$$

We define the matrices Φ and ψ as

$$\Phi = \begin{pmatrix}
\prod_{j=1}^{s}(\theta-t)^{q^{d-1}}\mathbb{D}_{b_j-2}^{q^{d-b_j}} & & & \\
\prod_{j=1}^{s+1}(\theta-t)^{q^{d-1}}\mathbb{D}_{a_i-2}^{q^{d-a_j}}\alpha_1^{q^{d-2}} & \prod_{j=1}^{s}(\theta-t)^{q^{d-1}}\mathbb{D}_{b_j-2}^{q^{d-b_j}} & & \\
\vdots & & \ddots & \\
\prod_{j=1}^{s+1}(\theta-t)^{q^{d-1}}\mathbb{D}_{a_j-2}^{q^{d-a_j}}\alpha_r^{q^{d-2}} & & & \prod_{j=1}^{s}(\theta-t)^{q^{d-1}}\mathbb{D}_{b_j-2}^{q^{d-b_j}}
\end{pmatrix}$$

$$\in \mathrm{Mat}_{r+1}\left(\bar{k}[t]\right) \cap \mathrm{GL}_{r+1}\left(\bar{k}(t)\right)$$

and

$$\psi = \begin{pmatrix} P_{\mathbf{b},d} \\ P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} \\ \vdots \\ P_{\mathbf{b},d}\ {}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_r)^{q^d} \end{pmatrix} \in \mathrm{Mat}_{(r+1)\times 1}(\mathbb{E}).$$

According to Theorem 3.4, $\psi^{(-1)} = \Phi\psi$; then, we can apply Theorem 4.2 and obtain the following:

$$g_1(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + g_r(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_r)^{q^d} = 0 \tag{4.8}$$

where $g_i(t) \in \bar{k}[t]$ with $g_i(\theta) = f_i$. We assume $g_r(t) \neq 0$ without loss of generality and set $g_i'(t) = g_i(t)/g_r(t)$. Next, we transform (4.8). We divide both sides of (4.8) by $g_r(t)$ and obtain

$$g_1'(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + g_r'(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_r)^{q^d} = 0. \tag{4.9}$$

Then, based on Theorem 3.10 and (4.5), the (-1)-fold Frobenius twist of (4.9) is

$$\begin{aligned} &g_1'(t)^{(-1)}\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + g_r'(t)^{(-1)}\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_r)^{q^d} \\ &+ g_1'(t)^{(-1)}\prod_{j=1}^{d-1}(\theta^{q^j}-t)^{c(j)q^{d-j}}\alpha_1^{q^d} + \cdots + g_r'(t)^{(-1)}\prod_{j=1}^{d-1}(\theta^{q^j}-t)^{c(j)q^{d-j}}\alpha_r^{q^d} = 0. \end{aligned} \tag{4.10}$$

According to (4.9) and (4.10), we have the following:

$$h_1(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + h_{r-1}(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_{r-1})^{q^d} + R = 0$$

where

$$\begin{aligned} h_i(t) &:= g_i'(t) - g_i'(t)^{(-1)}, \\ R &:= -\prod_{m=1}^{d-1}(\theta^{q^m}-t)^{c(m)q^{d-m}}\left(g_1'(t)^{(-1)}\alpha_1^{q^d} + \cdots + g_{r-1}'(t)^{(-1)}\alpha_{r-1}^{q^d} + \alpha_r^{q^d}\right). \end{aligned}$$

The j-th repetition of this transformation gives the following equation:

$$h_{1,j}(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + h_{r-j,j}(t)\,{}_{s+1}\mathcal{F}_s(\mathbf{a};\mathbf{b})(\alpha_{r-j})^{q^d} + R_j = 0$$

where

$$\begin{aligned} h_{i.j+1}(t) &= \frac{h_{i,j}(t)}{h_{r-j,j}(t)} - \left(\frac{h_{i,j}(t)}{h_{r-j,j}(t)}\right)^{(-1)} \in \bar{k}(t), \\ R_{j+1} &= \frac{R_j}{h_{r-j,j}(t)} - \left(\frac{R_j}{h_{r-j,j}(t)}\right)^{(-1)} \in \bar{k}(t) \end{aligned}$$

with $h_{i,1}(t) = h_i(t)$, $R_1 = R$. We repeat the above transformation until (i) $j = r-1$ and $h_{1,r-1}(t) \neq 0$ or (ii) j is equal to some j' such that

$$h_{1,j'} = \cdots = h_{r-j',j'} = 0.$$

For the case (i), we have

$$h_{1,r-1}(t)\, {}_{s+1}\mathscr{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + R_{r-1} = 0. \tag{4.11}$$

We set $N > 0$ such that $h_{1,r-1}(\theta^{q^N}) \neq 0$ and R_{r-1} is regular at $t = \theta^{q^N}$. Multiplying by $(\theta - t)^{-c(m_u)q^{q^{d-m_u}}}$ and then substituting $t = \theta^{q^N}$ on both sides of (4.11), we obtain the equation

$$h_{1,r-1}(\theta^{q^N}) \prod_{\substack{d-1 \geq j \geq 1 \\ j \neq m_u}} (\theta^{q^{N-m_u+j}} - \theta^{q^N})^{c(j)q^{d-j}} \alpha_1^{q^{N-m_u+d}} = 0.$$

This contradicts to $\alpha_1 \neq 0$. For the case (ii), first we set $N > n + d$ such that

$$h_{r-j'+1,j'-1}(\theta^{q^N}) \neq 0$$

is non-zero and $R_{j'-1}$ is regular at $t = \theta^{q^N}$. Due to our assumption of the minimality of $c(m_u)q^{d-m_u}$, multiplying by $(\theta^{q^N} - t)^{c(m_u)q^{d-m_u}}$ on both sides of

$$h_{1,j'-1}(t)\, {}_{s+1}\mathscr{F}_s(\mathbf{a};\mathbf{b})(\alpha_1)^{q^d} + \cdots + h_{r-j'+1,j'-1}(t)\, {}_{s+1}\mathscr{F}_s(\mathbf{a};\mathbf{b})(\alpha_{r-j'+1})^{q^d} + R_{j'-1} = 0$$

gives the following at $t = \theta^{q^N}$:

$$\begin{aligned} &h_{1,j'-1}(\theta^{q^N}) \left(\prod_{\substack{l=1 \\ l \neq m_u}}^{d-1} (\theta^{q^{l+N-m_u}} - \theta^{q^N})^{c(l)q^{d-l}} \alpha_1^{q^{n+d}} \right) \\ &\quad + \cdots + h_{r-j'+1,j'-1}(\theta^{q^N}) \left(\prod_{\substack{l=1 \\ l \neq m_u}}^{d-1} (\theta^{q^{l+N-m_u}} - \theta^{q^N})^{c(l)q^{d-l}} \alpha_{r-j'+1}^{q^{n+d}} \right) = 0. \end{aligned} \tag{4.12}$$

Since $h_{1,j'} = \cdots = h_{r-j',j'} = 0$, we get

$$h_{i,j'-1}(t)/h_{r-j'+1,j'-1}(t) \in \mathbb{F}_q(t) \quad (1 \leq i \leq r - j').$$

Thus they belong to $\mathbb{F}_q(\theta^{q^N})$ at $t = \theta^{q^N}$. Therefore by dividing with $h_{r-j'+1,j'-1}(\theta^{q^N})$ and taking q^{n+d}-th root of both sides of the relation (4.12), it gives a k-linear relation between $\alpha_1, \ldots, \alpha_r$, and we obtain a contradiction. ■

Remark 4.10. When $\mathbf{a} = (1, \ldots, 1)$ and $\mathbf{b} = (2, \ldots, 2)$, the above theorem shows that if $\alpha_1, \ldots, \alpha_r \in \bar{k}^\times$ with $|\alpha_i|_\infty < q^s$ are k-linearly independent, $Li_{K,s}(\alpha_1), \ldots, Li_{K,s}(\alpha_r)$ are $\bar{k}$-linearly independent.

4.2. Linear independence results of the KMPLs

As applications of Theorem 4.2 and our t-motivic interpretation of the KMPLs in (3.15), we discuss a linear independence result among the depth 2 KMPLs. Furthermore, we compare the KMPLs with other quantities and show that the Kochubei multizeta values do not have Eulerian/zeta-like indices with Kochubei zeta values and Carlitz periods.

For the depth 2 KMPLs, we have the following linear independence result.

Theorem 4.11. *For $\alpha \in \bar{k}^{\times}$ with $|\alpha|_\infty < q$ and $w \in \mathbb{N}$, the following set*

$$\left\{Li_{K,\mathfrak{s}}(\alpha) \mid \mathfrak{s} \in \mathbb{N}^2 \text{ with } \operatorname{wt}(\mathfrak{s}) = w\right\}$$

is linearly independent over $\bar{k}$.

Proof. We assume on the contrary that among $Li_{K,\mathfrak{s}_i}(\alpha)$ $(i = 1, \ldots, r)$ with $\mathfrak{s}_i = (s_{i1}, s_{i2})$ $(\mathfrak{s}_i \neq \mathfrak{s}_j$ for $i \neq j)$, there exists a $\bar{k}$-linear relation:

$$f_1 Li_{K,\mathfrak{s}_1}(\alpha) + \cdots + f_r Li_{K,\mathfrak{s}_r}(\alpha) = 0 \tag{4.13}$$

for some $f_i \in \bar{k}^{\times}$. We define

$$\Phi_i = \begin{pmatrix} (t-\theta)^w & 0 & 0 \\ (-1)^{s_{i2}}(t-\theta)^{s_{i1}} & (t-\theta)^w & 0 \\ 0 & (-1)^{s_{i1}}\alpha(t-\theta)^{s_{i2}} & (t-\theta)^w \end{pmatrix} \in \operatorname{Mat}_3\left(\bar{k}[t]\right) \cap \operatorname{GL}_3\left(\bar{k}(t)\right)$$

and

$$\psi_i = \begin{pmatrix} \Omega^w \\ \Omega^w \mathcal{L}_{K,(s_{i1})}(\alpha) \\ \Omega^w \mathcal{L}_{K,\mathfrak{s}_i}(\alpha) \end{pmatrix} \in \operatorname{Mat}_{3\times 1}(\mathbb{E}),$$

put

$$\Phi = \begin{pmatrix} \Phi_1 & & & \\ & \Phi_2 & & \\ & & \ddots & \\ & & & \Phi_r \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \end{pmatrix}.$$

Then, we can apply Theorem 4.2 (1) to (4.13) and obtain the following $\bar{k}[t]$-linear relation

$$g_1(t)\mathcal{L}_{K,\mathfrak{s}_1}(\alpha) + \cdots + g_r(t)\mathcal{L}_{K,\mathfrak{s}_r}(\alpha) = 0$$

with $g_i(t) \in \bar{k}[t]$ such that $g_i(\theta) = f_i$. We set

$$s = \max\{s_{ij} \mid i = 1, \ldots, r \text{ and } j = 1, 2\}.$$

For some i, there are indices $\mathfrak{s}_i = (w-s, s)$ or $(s, w-s)$ with $s \geq w-s$. When $s = w-s$, the equation (4.13) becomes $f_i Li_{K,\mathfrak{s}_i}(\alpha) = 0$; however, this contradicts Theorem 4.6. When $s \neq w-s$, we have three cases and again get contradictions as follows.

Case 1. If $\mathfrak{s}_i = (w-s, s)$ for some i and $\mathfrak{s}_j \neq (s, w-s)$ with $j \neq i$, we set $N > 0$ such that $g_i(\theta^{q^N}) \neq 0$. Then, we can set $i = 1$ without loss of generality and obtain

$$\begin{aligned}
&(\theta^{q^N} - t)^s \big(g_1(t)\mathcal{L}_{K,\mathfrak{s}_1}(\alpha) + \cdots + g_r(t)\mathcal{L}_{K,\mathfrak{s}_r}(\alpha)\big) \\
&= (\theta^{q^N} - t)^s \Bigg(g_1(t) \sum_{i_1 = N > i_2 > 0} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^s (\theta^{q^{i_2}} - t)^{w-s}} \\
&\qquad + g_1(t) \sum_{\substack{i_1 > i_2 > 0 \\ i_1 \neq N}} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^s (\theta^{q^{i_2}} - t)^{w-s}} \\
&\qquad + g_2(t)\mathcal{L}_{K,\mathfrak{s}_2}(\alpha) + \cdots + g_r(t)\mathcal{L}_{K,\mathfrak{s}_r}(\alpha) \Bigg) = 0.
\end{aligned}$$

By substituting $t = \theta^{q^N}$, we obtain $g_1(\theta^{q^N}) \sum_{N > i_2 > 0} \alpha^{q^N} / (\theta^{q^{i_2}} - \theta^{q^N})^{w-s} = 0$. This contradicts the assumption that $g_1(\theta^{q^N}) \neq 0$ and that α and $\sum_{N > i_2 > 0} \alpha^{q^N} / (\theta^{q^{i_2}} - \theta^{q^N})^{w-s}$ are non-zero.

Case 2. If $\mathfrak{s}_i = (s, w-s)$ for some i and $\mathfrak{s}_j \neq (w-s, s)$ with $j \neq i$, in the same way of Case 1, we obtain a contradiction from $g_1(\theta^{q^N}) \sum_{N > i_2 > 0} \alpha^{q^N} / (\theta^{q^{i_2}} - \theta^{q^N})^{w-s} = 0$.

Case 3. If $\mathfrak{s}_i = (s, w-s)$ and $\mathfrak{s}_j = (w-s, s)$ for some i and j, we can set $i = 1$, $j = 2$ without loss of generality and set $N > 0$ such that both $g_1(\theta^{q^N})$, $g_2(\theta^{q^N})$ are non-zero. We have

$$\begin{aligned}
&(\theta^{q^N} - t)^s \big(g_1(t)\mathcal{L}_{K,\mathfrak{s}_1}(\alpha) + \cdots + g_r(t)\mathcal{L}_{K,\mathfrak{s}_r}(\alpha)\big) \\
&= (\theta^{q^N} - t)^s \Bigg(g_1(t) \sum_{i_1 = N > i_2 > 0} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^s (\theta^{q^{i_2}} - t)^{w-s}} \\
&\qquad + g_1(t) \sum_{\substack{i_1 > i_2 > 0 \\ i_1 \neq N}} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^s (\theta^{q^{i_2}} - t)^{w-s}} \\
&\qquad + g_2(t) \sum_{i_1 > N = i_2 > 0} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^{w-s} (\theta^{q^{i_2}} - t)^s} \\
&\qquad + g_2(t) \sum_{\substack{i_1 > i_2 > 0 \\ i_2 \neq N}} \frac{\alpha^{q^{i_1}}}{(\theta^{q^{i_1}} - t)^{w-s} (\theta^{q^{i_2}} - t)^s} \\
&\qquad + g_3(t)\mathcal{L}_{K,\mathfrak{s}_3}(\alpha) + \cdots + g_r(t)\mathcal{L}_{K,\mathfrak{s}_r}(\alpha) \Bigg) = 0.
\end{aligned}$$

By substituting $t = \theta^{q^N}$ into the above equation, we obtain

$$g_1(\theta^{q^N}) \sum_{N > i_2 > 0} \frac{\alpha^{q^N}}{(\theta^{q^{i_2}} - \theta^{q^N})^{w-s}} + g_2(\theta^{q^N}) Li_{K,(w-s)}(\alpha)^{q^N} = 0.$$

This contradicts to the transcendence of $Li_{K,(w-s)}(\alpha)$ shown by Theorem 4.6.

Thus, we obtain the desired result. ∎

By this theorem, the dimension of the $\bar{k}$-linear space

$$\mathrm{Span}_{\bar{k}}\{Li_{\mathfrak{s}}(\alpha) \mid \mathrm{dep}(\mathfrak{s}) = 2 \text{ and } \mathrm{wt}(\mathfrak{s}) = w\}$$

for fixed $w \in \mathbb{N}$ and fixed $\alpha \in \bar{k}$ with $|\alpha|_\infty < q^w$, is 2^{w-1}.

Furthermore, we can compare Carlitz polylogarithms and KPLs at algebraic points as follows.

Theorem 4.12. *Given $n \in \mathbb{N}$, let $\alpha, \beta \in \bar{k}^\times$ such that $|\alpha|_\infty < q^n$, $|\beta|_\infty < q^{nq/(q-1)}$ and $Li_{C,(n)}(\beta) \neq 0$. Then $Li_{K,(n)}(\alpha), Li_{C,(n)}(\beta)$ are $\bar{k}$-linearly independent.*

Proof. For the pre-t-motive defined by

$$\Phi = \begin{pmatrix} (t-\theta)^n & 0 & 0 \\ (-1)^n\alpha & (t-\theta)^n & 0 \\ \beta^{(-1)}(t-\theta)^n & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_3\left(\bar{k}[t]\right) \cap \mathrm{GL}_3\left(\bar{k}(t)\right),$$

we have $\psi^{(-1)} = \Phi\psi$ for

$$\psi = \begin{pmatrix} \Omega^n \\ \Omega^n \mathcal{L}_{K,(n)}(\alpha) \\ \Omega^n \mathcal{L}_{C,(n)}(\beta) \end{pmatrix} \in \mathrm{Mat}_{3\times 1}(\mathbb{E})$$

where $\mathcal{L}_{C,(n)}(\beta) = \beta + \sum_{i>0} \beta^{q^i}/\mathbb{L}_i^n$ that satisfies

$$\mathcal{L}_{C,(n)}(\beta)^{(-1)} = \beta^{(-1)} + \frac{\mathcal{L}_{C,(n)}(\beta)}{(t-\theta)^n}.$$

We assume on the contrary that $f_1 Li_{K,(n)}(\alpha) + f_2 Li_{C,(n)}(\beta) = 0$ for some $f_i \in \bar{k}^\times$ $(i = 1, 2)$. Then, we have $\tilde{\pi}^{-n}(f_1 Li_{K,(n)}(\alpha) + f_2 Li_{C,(n)}(\beta)) = 0$. Based on Theorem 4.1, this relation can be extended as follows:

$$g_1(t)\Omega^n \mathcal{L}_{K,(n)}(\alpha) + g_2(t)\Omega^n \mathcal{L}_{C,(n)}(\beta) = 0 \tag{4.14}$$

for some $g_i(t) \in \bar{k}[t]$ with $g_i(\theta) = f_i$ $(i = 1, 2)$. Let $N \in \mathbb{N}$ such that $g_1(\theta^{q^N}) \neq 0$. By multiplying by $\mathbb{L}_N^n \Omega^{-n}$ on both sides, we can rewrite (4.14) as follows:

$$\begin{aligned} &g_1(t)\left\{\mathbb{L}_N^n \sum_{i=1}^{N-1} \frac{\alpha^{q^i}}{(\theta^{q^i}-t)^n} + (-1)^n \mathbb{L}_{N-1}^n \alpha^{q^N} + \mathbb{L}_N^n \sum_{i>N} \frac{\alpha^{q^i}}{(\theta^{q^i}-t)^n}\right\} \\ &+ g_2(t)\left\{(t-\theta^q)^n \cdots (t-\theta^{q^N})^n \beta + (t-\theta^{q^2}) \cdots (t-\theta^{q^N})^n \beta^q + \cdots + (t-\theta^{q^N})^n \beta^{q^{N-1}}\right. \\ &\qquad \left. + \sum_{i>N} \frac{\beta^{q^i}}{(t-\theta^{q^{N+1}})^n \cdots (t-\theta^{q^i})^n}\right\} = 0. \end{aligned} \tag{4.15}$$

We have

$$\sum_{i>N} \frac{\beta^{q^i}}{(t-\theta^{q^{N+1}})^n \cdots (t-\theta^{q^i})^n}\Big|_{t=\theta^{q^N}} = Li_{C,(n)}(\beta)^{q^N} - \beta^{q^N}.$$

Therefore, by substituting $t = \theta^{q^N}$, the equation (4.15) becomes

$$g_1(\theta^{q^N})\big((\theta^{q^N}-\theta^q)^n \cdots (\theta^{q^N}-\theta^{q^{N-1}})^n\big)\alpha^{q^N} + g_2(\theta^{q^N})Li_{C,n}(\alpha)^{q^N} - g_2(\theta^{q^N})\beta^{q^N} = 0.$$

This forces $Li_{C,n}(\beta) = g_2(\theta^{q^N})^{-1/q^N}\{g_1(\theta^{q^N})\beta^{q^N} - g_2(\theta^{q^N})\}^{1/q^N} \in \bar{k}$ while $Li_{C,n}(\beta)$ is transcendental over k by [7, Theorem 5.4.3]. Therefore, we obtain a contradiction and the desired $\bar{k}$-linear independence result is proven. ■

We conclude this section with the following two theorems and their proofs. Then we get the linear independence results among the KMPLs, Carlitz period, and KPLs, as introduced in Theorem 1.5 (iii)–(iv).

Theorem 4.13. *Given $w \in \mathbb{N}$, let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $wt(\mathfrak{s}) = w$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in (\bar{k}^\times)^r$ with $|\alpha_i|_\infty < q^{s_i}$ $(i = 1, \dots, r)$. Then, $Li_{K,\mathfrak{s}}(\boldsymbol{\alpha})$ and $\tilde{\pi}^w$ are $\bar{k}$-linearly independent.*

Proof. We assume on the contrary that there exists a non-trivial $\bar{k}$-linear equation $f_1\tilde{\pi}^w + f_2 Li_{K,\mathfrak{s}}(\boldsymbol{\alpha}) = 0$. With (3.15), we have $\psi^{(-1)} = \Phi\psi$, where

$$\Phi = \begin{pmatrix} \Phi_{\mathfrak{s},\boldsymbol{\alpha}} & \\ & 1 \end{pmatrix} \in \mathrm{Mat}_{r+2}\big(\bar{k}[t]\big) \cap \mathrm{GL}_{r+2}\big(\bar{k}(t)\big),$$

and

$$\psi = \begin{pmatrix} \Omega^w \\ \Omega^w \mathcal{L}_{K,(s_r)}(\alpha_r) \\ \Omega^w \mathcal{L}_{K,(s_{r-1},s_r)}(\alpha_{r-1},\alpha_r) \\ \vdots \\ \vdots \\ \Omega^w \mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) \\ 1 \end{pmatrix} \in \mathrm{Mat}_{r+2\times 1}(\mathbb{E}).$$

Then, by using Theorem 4.1, we can obtain the following $\bar{k}[t]$-linear relation

$$g_1(t)\Omega^w \mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) + g_2(t)\cdot 1 = 0 \tag{4.16}$$

for some $g_i(t) \in \bar{k}[t]$ $(i = 1, 2)$ such that $g_i(\theta) = f_i$. By taking the (-1)-fold Frobenius twist of (4.16), we obtain

$$\begin{aligned} g_1(t)^{(-1)}(-1)^{s_r}(t-\theta)^{w-s_r}\Omega^w \alpha_r \mathcal{L}_{K,(s_1,\dots,s_{r-1})}(\alpha_1,\dots,\alpha_{r-1}) \\ + g_1(t)^{(-1)}(t-\theta)^w \Omega^w \mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) + g_2(t)^{(-1)} = 0. \end{aligned} \tag{4.17}$$

We set $N \in \mathbb{N}$ such that $g_2(t)^{(-1)}$ is non-zero at $t = \theta^{q^N}$. By definition, Ω^w has a zero at $t = \theta^{q^N}$ with an order w, while the terms of both $\mathcal{L}_{K,(s_1,\dots,s_{r-1})}(z_1,\dots,z_{r-1})$ and $\mathcal{L}_{K,\mathfrak{s}}(\mathbf{z})$ have poles at $t = \theta^{q^N}$ with orders strictly less than w. Thus, $\Omega^w\mathcal{L}_{K,(s_1,\dots,s_{r-1})}(z_1,\dots,z_{r-1})$ and $\Omega^w\mathcal{L}_{K,\mathfrak{s}}(\mathbf{z})$ vanish at $t = \theta^{q^N}$. Then, by substituting $t = \theta^{q^N}$ on both sides of (4.17), we obtain the contradiction

$$g_2(t)^{(-1)}|_{t=\theta^{q^N}} = 0.$$

Therefore, we obtain the desired result. ∎

Theorem 4.14. *Given $w \in \mathbb{N}$, let $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ such that $wt(\mathfrak{s}) = w$. For $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_r) \in (\bar{k}^\times)^r$ and $\beta \in \bar{k}^\times$ with $|\alpha_i|_\infty < q^{s_i}$ and $|\beta|_\infty < q^w$, $Li_{K,\mathfrak{s}}(\boldsymbol{\alpha})$ and $Li_{K,w}(\beta)$ are $\bar{k}$-linearly independent.*

Proof. We assume on the contrary that there exists a non-trivial $\bar{k}$-linear relation for some $f_i \in \bar{k}$ $(i = 1, 2)$:

$$f_1 Li_{K,\mathfrak{s}}(\boldsymbol{\alpha}) + f_2 Li_{K,(w)}(\beta) = 0.$$

We define the matrices

$$\Phi = \begin{pmatrix} \Phi_{\mathfrak{s},\boldsymbol{\alpha}} & \\ \boldsymbol{\beta} & (t-\theta)^w \end{pmatrix} \in \operatorname{Mat}_{r+2}\left(\bar{k}[t]\right) \cap \operatorname{GL}_{r+2}\left(\bar{k}(t)\right)$$

where $\boldsymbol{\beta} = (\beta, 0, \dots, 0) \in \operatorname{Mat}_{1\times r}(\bar{k})$ and

$$\psi = \begin{pmatrix} \Omega^w \\ \Omega^w\mathcal{L}_{K,(s_r)}(\alpha_r) \\ \Omega^w\mathcal{L}_{K,(s_{r-1},s_r)}(\alpha_{r-1},\alpha_r) \\ \vdots \\ \vdots \\ \Omega^w\mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) \\ \Omega^w\mathcal{L}_{K,(w)}(\beta) \end{pmatrix} \in \operatorname{Mat}_{r+2\times 1}(\mathbb{E}).$$

By using Theorem 4.1, we obtain a $\bar{k}$-linear equation

$$g_1(t)\Omega^w\mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) + g_2(t)\Omega^w\mathcal{L}_{K,(w)}(\beta) = 0,$$

and then we have

$$g_1(t)\mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) + g_2(t)\mathcal{L}_{K,(w)}(\beta) = 0.$$

By taking the (-1)-fold Frobenius twist, we obtain

$$\begin{aligned} g_1(t)^{(-1)}(\theta - t)^{-s_r}\alpha_r\mathcal{L}_{K,(s_1,\dots,s_{r-1})}(\alpha_1,\dots,\alpha_{r-1}) + g_1(t)^{(-1)}\mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha}) \\ + g_2(t)^{(-1)}(\theta - t)^{-w}\beta + g_2(t)^{(-1)}\mathcal{L}_{K,(w)}(\beta) = 0. \quad (4.18) \end{aligned}$$

We set $N > 0$ such that $g_2(t)^{(-1)}|_{t=\theta^{q^N}} \neq 0$. After multiplying by $(\theta^{q^N} - t)^w$ on both sides of (4.18), we obtain

$$(\theta^{q^N} - t)^w \big(g_1(t)^{(-1)}(\theta - t)^{-s_r}\alpha_n \mathcal{L}_{K,(s_1,\ldots,s_{r-1})}(\alpha_1,\ldots,\alpha_{r-1}) + g_1(t)^{(-1)}\mathcal{L}_{K,\mathfrak{s}}(\boldsymbol{\alpha})\big)$$
$$+ (\theta^{q^N} - t)^w g_2(t)^{(-1)}(\theta - t)^{-w}\beta + (\theta^{q^N} - t)^w g_2(t)^{(-1)} \sum_{\substack{i>0 \\ i\neq N}} \frac{\beta^{q^i}}{(\theta^{q^i} - t)^w}$$
$$+ g_2(t)^{(-1)}\beta^{q^N} = 0.$$

By substituting $t = \theta^{q^N}$, we obtain a relation $g_2(t)^{(-1)}|_{t=\theta^{q^N}}\beta^{q^N} = 0$. This contradicts $g_2(t)^{(-1)}|_{t=\theta^{q^N}} \neq 0$ and $\beta \neq 0$. Therefore we obtain the desired result. ∎

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References

[1] G. W. Anderson, W. D. Brownawell, and M. A. Papanikolas, Determination of the algebraic relations among special Γ-values in positive characteristic. *Ann. of Math. (2)* **160** (2004), no. 1, 237–313 Zbl 1064.11055 MR 2119721

[2] G. W. Anderson and D. S. Thakur, Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)* **132** (1990), no. 1, 159–191 Zbl 0713.11082 MR 1059938

[3] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro, On special L-values of t-modules. *Adv. Math.* **372** (2020), article no. 107313 Zbl 1458.11095 MR 4128574

[4] F. Beukers and G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Invent. Math.* **95** (1989), no. 2, 325–354 Zbl 0663.30044 MR 0974906

[5] L. Carlitz, On certain functions connected with polynomials in a Galois field. *Duke Math. J.* **1** (1935), no. 2, 137–168 Zbl 61.0127.01 MR 1545872

[6] C.-Y. Chang, A note on a refined version of Anderson–Brownawell–Papanikolas criterion. *J. Number Theory* **129** (2009), no. 3, 729–738 Zbl 1187.11022 MR 2488599

[7] C.-Y. Chang, Linear independence of monomials of multizeta values in positive characteristic. *Compos. Math.* **150** (2014), no. 11, 1789–1808 Zbl 1306.11058 MR 3279257

[8] C.-Y. Chang, Linear relations among double zeta values in positive characteristic. *Camb. J. Math.* **4** (2016), no. 3, 289–331 Zbl 1421.11065 MR 3550284

[9] C.-Y. Chang, Frobenius difference equations and difference Galois groups. In *t-motives: Hodge structures, transcendence and other motivic aspects*, pp. 261–295, EMS Ser. Congr. Rep., EMS Publ. House, Berlin, 2020 Zbl 1440.11098 MR 4321967

[10] C.-Y. Chang and Y. Mishiba, On multiple polylogarithms in characteristic p: v-adic vanishing versus ∞-adic Eulerianness. *Int. Math. Res. Not. IMRN* **2019** (2019), no. 3, 923–947 Zbl 1479.11158 MR 3910478

[11] C.-Y. Chang, M. A. Papanikolas, and J. Yu, An effective criterion for Eulerian multizeta values in positive characteristic. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 2, 405–440 Zbl 1417.11139 MR 3896206

[12] C.-Y. Chang and J. Yu, Determination of algebraic relations among special zeta values in positive characteristic. *Adv. Math.* **216** (2007), no. 1, 321–345 Zbl 1123.11025 MR 2353259

[13] S. David, N. Hirata-Kohno, and M. Kawashima, Generalized hypergeometric G-functions take linear independent values. 2022, arXiv:2203.00207v1

[14] S. Fischler and T. Rivoal, Linear independence of values of G-functions. *J. Eur. Math. Soc. (JEMS)* **22** (2020), no. 5, 1531–1576 Zbl 1446.11136 MR 4081728

[15] D. Goss, *Basic structures of function field arithmetic*. Ergeb. Math. Grenzgeb. (3) 35, Springer, Berlin, 1996 Zbl 0874.11004 MR 1423131

[16] N. Green and T. Ngo Dac, On log-algebraic identities for Anderson t-modules and characteristic p multiple zeta values. *Int. Math. Res. Not. IMRN* **2023** (2023), no. 16, 13687–13756 Zbl 07794865 MR 4631419

[17] T. Hasegawa, Logarithmic-type and exponential-type hypergeometric functions for function fields. *J. Number Theory* **233** (2022), 87–111 Zbl 1498.11227 MR 4356846

[18] A. N. Kochubei, Polylogarithms and a zeta function for finite places of a function field. In *Ultrametric functional analysis*, pp. 157–167, Contemp. Math. 384, American Mathematical Society, Providence, RI, 2005 Zbl 1153.11325 MR 2174783

[19] A. N. Kochubei, Evolution equations and functions of hypergeometric type over fields of positive characteristic. *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), no. 5, 947–959 Zbl 1197.12004 MR 2378999

[20] M. Kontsevich and D. Zagier, Periods. In *Mathematics unlimited—2001 and beyond*, pp. 771–808, Springer, Berlin, 2001 Zbl 1039.11002 MR 1852188

[21] J. A. Lara Rodríguez and D. S. Thakur, Zeta-like multizeta values for $\mathbb{F}_q[t]$. *Indian J. Pure Appl. Math.* **45** (2014), no. 5, 787–801 Zbl 1365.11104 MR 3286087

[22] Y. Mishiba, Algebraic independence of the Carlitz period and the positive characteristic multizeta values at n and (n, n). *Proc. Amer. Math. Soc.* **143** (2015), no. 9, 3753–3763 Zbl 1322.11082 MR 3359567

[23] M. A. Papanikolas, Tannakian duality for Anderson–Drinfeld motives and algebraic independence of Carlitz logarithms. *Invent. Math.* **171** (2008), no. 1, 123–174 Zbl 1235.11074 MR 2358057

[24] D. P. Roberts and F. Rodriguez Villegas, Hypergeometric motives. *Notices Amer. Math. Soc.* **69** (2022), no. 6, 914–929 Zbl 1506.11097 MR 4442789

[25] H. A. Schwarz, Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. *J. Reine Angew. Math.* **75** (1873), 292–335 Zbl 05.0146.03 MR 1579568

[26] D. S. Thakur, Hypergeometric functions for function fields. *Finite Fields Appl.* **1** (1995), no. 2, 219–231 Zbl 0838.11043 MR 1337745

[27] D. S. Thakur, Hypergeometric functions for function fields. II. *J. Ramanujan Math. Soc.* **15** (2000), no. 1, 43–52 Zbl 1024.11039 MR 1751925

[28] D. S. Thakur, Z.-Y. Wen, J.-Y. Yao, and L. Zhao, Transcendence in positive characteristic and special values of hypergeometric functions. *J. Reine Angew. Math.* **657** (2011), 135–171 Zbl 1227.11088 MR 2824786

[29] L. I. Wade, Certain quantities transcendental over $GF(p^n, x)$. *Duke Math. J.* **8** (1941), 701–720 Zbl 0063.08101 MR 0006157

[30] M. Waldschmidt, Multiple polylogarithms: an introduction. In *Number theory and discrete mathematics (Chandigarh, 2000)*, pp. 1–12, Trends Math., Birkhäuser, Basel, 2002 Zbl 1035.11033 MR 1952273

[31] J.-Y. Yao, A transcendence criterion in positive characteristic and applications. *C. R. Math. Acad. Sci. Paris* **343** (2006), no. 11–12, 699–704 Zbl 1156.11028 MR 2284696

[32] J.-Y. Yao, Special values of fractional hypergeometric functions for function fields. *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 17–18, 801–806 Zbl 1264.11066 MR 2989380

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