Enseign. Math. (2) 71 (2025), 411–431
DOI 10.4171/LEM/1080

Stabilisation, scanning, and handle cancellation

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Abstract. In this note, we describe a family of arguments that link the homotopy type of (a) the diffeomorphism group of the disc D^n, (b) the space of co-dimension one embedded spheres in S^n, and (c) the homotopy type of the space of co-dimension two trivial knots in S^n. We also describe some natural extensions to these arguments. We begin with Cerf's "upgraded" proof of Smale's theorem, showing that the diffeomorphism group of S^2 has the homotopy type of the isometry group. This entails a cancelling-handle construction, related to recently studied "scanning" maps of spaces of embeddings $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega^j \mathrm{Emb}(D^{n-1-j}, S^1 \times D^{n-1})$. We further give a Bott-style variation on Cerf's construction and a related embedding calculus framework for these constructions. We use these arguments to prove that the monoid of Schönflies spheres $\pi_0 \mathrm{Emb}(S^{n-1}, S^n)$ is a group with respect to the connected-sum operation for all $n \geq 2$. This last result is perhaps only interesting when $n = 4$, as when $n \neq 4$, it follows from the resolution of the various generalised Schönflies problems.

Mathematics Subject Classification 2020: 57M99 (primary); 57R52, 57R50, 57N50 (secondary).

Keywords: embeddings, diffeomorphisms.

1. Introduction

In Cerf's landmark paper [9], somewhat overlooked is a novel proof of Smale's theorem, that the group of diffeomorphisms of the 2-sphere, $\mathrm{Diff}(S^2)$, has the homotopy type of its linear subgroup O_3. The core of Cerf's argument is the proof that the Smale–Hirsch map (pointwise derivative) $\mathrm{Diff}(D^2) \to \Omega^2 \mathrm{GL}_2(\mathbb{R})$ has a left homotopy inverse. Cerf states his theorem in the language of homotopy groups; i.e., the homotopy groups of $\mathrm{Diff}(D^2)$ inject into the homotopy groups of $\Omega^2 \mathrm{GL}_2(\mathbb{R})$. Since the latter homotopy groups are trivial and diffeomorphism groups of compact manifolds have the homotopy type of countable CW complexes [27], this allows Cerf to conclude that $\mathrm{Diff}(D^2)$ is contractible via the Whitehead theorem. In this paper, we use the notation that if N is a manifold with boundary, $\mathrm{Diff}(N)$ denotes the group of diffeomorphisms of N that restrict to the identity on ∂N. We will use the same conventions for embedding spaces;

i.e., $\mathrm{Emb}(N, M)$ denotes the space of smooth embeddings of N in M, and if N and M have boundary, these maps will all restrict to one given map $\partial N \to \partial M$.

Smale's proof that $\mathrm{Diff}(D^2)$ is contractible uses the Poincaré–Bendixson theorem to guarantee the flow of the vector fields he uses terminate in finite time. As the Poincaré–Bendixson theorem is only available in dimension two, it limits the applicability of Smale's argument. We should note that there have been attempts to broaden the applicability of a Smale-type argument by studying spaces of closed 1-forms. See, for example, the two papers of Laudenbach and Blank [23, 24] for a sampling. Since Cerf's argument does not depend on Poincaré–Bendixson theorem, it allows for greater applicability. The headline consequences of Cerf's arguments are that the diffeomorphism group $\mathrm{Diff}(D^n)$ has the same homotopy type as $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$ and also that the embedding space $\mathrm{Emb}(D^{n-1}, D^n)$ is a homotopy retract of $\Omega\,\mathrm{Emb}(D^{n-2}, D^n)$. Putting these two results together, the homotopy groups of $\mathrm{Diff}(D^n)$ inject into the homotopy groups of $\Omega^2\,\mathrm{Emb}(D^{n-2}, D^n)$ for all n. While Cerf states these as his theorems, his techniques prove much more. It is the purpose of this paper to outline the consequences of his techniques.

1.1. Cerf's techniques. To give Cerf's results some context, we first mention how the spaces he studies are related to some more commonly discussed objects. A linearisation argument [2] shows that the diffeomorphism group $\mathrm{Diff}(S^n)$ has the homotopy type of $O_{n+1} \times \mathrm{Diff}(D^n)$; indeed, the homotopy equivalence comes from considering $\mathrm{Diff}(D^n)$ as the subgroup of $\mathrm{Diff}(S^n)$ that is the identity on a fixed hemi-sphere, and the homotopy equivalence $O_{n+1} \times \mathrm{Diff}(D^n) \to \mathrm{Diff}(S^n)$ is given by the group multiplication in $\mathrm{Diff}(S^n)$. There is an analogous homotopy equivalence

$$\mathrm{Emb}(S^j, S^n) \simeq \mathrm{SO}_{n+1} \times_{\mathrm{SO}_{n-j}} \mathrm{Emb}(D^j, D^n)$$

when $n > j$. If we let $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$ denote the space of smooth embeddings of D^{n-1} in $S^1 \times D^{n-1}$ which agree with the standard inclusion $D^{n-1} \to \{1\} \times D^{n-1}$ on the boundary sphere, then there is a "handle-filling" homotopy equivalence $\mathrm{Diff}(S^1 \times D^{n-1}) \simeq \mathrm{Diff}(D^n) \times \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$.

In Cerf's paper [9], the main results we highlight concern three maps.

(1) $\mathrm{Diff}(D^n) \to \Omega\,\mathrm{Emb}(D^{n-1}, D^n)$.

(2) $\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$: this is the map given by attaching a 1-handle to D^n so that the attaching sphere links the standard S^{n-2} in $\partial D^n \equiv S^{n-1}$; i.e., we think of $S^1 \times D^{n-1}$ as D^n with a 1-handle attached; thus, the map comes from simply changing the codomain of the embedding.

(3) $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega\,\mathrm{Emb}^\nu(D^{n-2}, D^n)$, where ν indicates the embeddings that are required to have an everywhere non-zero normal vector field, and the vector fields are some standard (constant) on the boundary.

Cerf's result is that the maps (1) and (3) are homotopy-equivalences, while (2) is a homotopy retract, i.e., has a left homotopy inverse. The definitions of the maps (1) and (3) are analogous and will be described precisely in Section 2. The rough idea of these maps is to fibre the domain of the embedding by a 1-parameter family of co-dimension one discs and restrict the map to these fibres, appropriately changing the codomain of the family of embeddings, via a filling. In the case of (3), the fibring construction would give a 1-parameter family of embeddings of D^{n-2} in $S^1 \times D^{n-1}$, but we carefully fill with a cancelling 2-handle to construct an element of $\Omega\,\mathrm{Emb}(D^{n-2}, D^n)$.

1.2. Extrapolating from Cerf. In Section 2, we observe that Cerf's argument, unchanged, gives a homotopy equivalence

$$\mathrm{Emb}(D^j, S^{n-j} \times D^j) \to \Omega\,\mathrm{Emb}^\nu(D^{j-1}, D^n).$$

Cerf's results (1) and (3) above correspond to the $j = n$ and $j = n-1$ cases of this homotopy equivalence. If we think of $S^{n-j} \times D^j$ as D^n union an $(n-j)$-handle, then the domain of our map, $\mathrm{Emb}(D^j, S^{n-j} \times D^j)$, is the space of all cocores, i.e., smooth embeddings of D^j in $S^{n-j} \times D^j$ that agree with a standard linear inclusion $D^j \to \{*\} \times D^j$ on the boundary. The codomain is the loop space of the space of smooth embeddings $D^{j-1} \to D^n$ that carry a nowhere-zero normal vector field; moreover, the embedding and the vector field are standard linear embeddings on the boundary. The base point of the embedding space $\mathrm{Emb}^\nu(D^{j-1}, D^n)$ is the linear (i.e., boundary parallel) embedding.

It is here where authors noticed a connection to recent "lightbulb theorems" in low-dimensional topology [4,12,21]. The above equivalence can be recast slightly, using the same argument but applying it to a strictly larger class of spaces. Let N be a compact n-manifold with non-empty boundary, and let $\natural$ denote the boundary connected-sum operation. Think of the boundary connected-sum $N \natural (S^{n-j} \times D^j)$ as N with a trivial $(n-j)$-handle attached; then the space of cocores of this attached handle, which we could denote by $\mathrm{Emb}(D^j, N \natural (S^{n-j} \times D^j))$, has the same homotopy type as the loop space $\Omega\,\mathrm{Emb}^\nu(D^{j-1}, N)$, which is the loop space of the space of embedded D^{j-1} discs with normal vector field in the manifold N – the space of embeddings we give the base point of a boundary-parallel embedding. This version is emphasised in [21].

1.3. Related expositions. Another way to look at this paper is that it is both an addendum to [2] and a paper that highlights methods from [4, 9] that deserve to be singled out. Both [4, 9] are long papers with many results, so it is easy to overlook this technique. We hope a shorter-format paper devoted to one tool does the ideas the justice they deserve. In [2], an attempt was made to describe the most basic relations between the homotopy type of diffeomorphism groups and embedding spaces for the smallest manifolds, such as spheres and discs. These Cerf techniques were known to the author, but perhaps indicative of the techniques, the only consequences the author knew at the time were already known, by other methods. So, they were removed from the paper before publication.

For example, the connection between the homotopy type of the component of the unknot $\mathrm{Emb}_u(S^1, S^3)$ and the homotopy type of $\mathrm{Diff}(S^3)$, which is immediate from Cerf's perspective, is historically derived using Hatcher's work on spaces of incompressible surfaces [17] (see the final pages). In Section 3, we describe the relation between Cerf's half-disc fibrations and the more commonly used restriction fibration $\mathrm{Diff}(S^n) \to \mathrm{Emb}(S^j, S^n)$.

1.4. Schönflies. An interesting observation in [4] is that the "stacking" operation, while appearing to be just a monoid structure on the space $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$, using Cerf's argument one can show that the space is group-like; i.e., the induced monoid structure on $\pi_0 \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$ is that of a group for all $n \geq 2$. One consequence of this is an argument that the monoid of Schönflies spheres $\pi_0 \mathrm{Emb}(S^{n-1}, S^n)$ is a group using the relative connected-sum operation. There is a classical argument due to Kervaire–Milnor that this monoid has inverses. Our argument is characteristically different, in that we construct an onto homomorphism from a group; i.e., in a weak sense, we give a presentation of the monoid of Schönflies spheres. This appears in Section 4.

1.5. High co-dimension scanning. A scanning technique was proposed for studying the homotopy type of $\mathrm{Diff}(S^1 \times D^n)$, by considering the chain of maps

$$\begin{aligned}\mathrm{Diff}(S^1 \times D^{n-1}) \to \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) &\to \Omega\, \mathrm{Emb}(D^{n-2}, S^1 \times D^{n-1}) \to \cdots \\ &\to \Omega^{n-2} \mathrm{Emb}(D^1, S^1 \times D^{n-1})\end{aligned}$$

in the sequence [3,4]. Interestingly, an infinitely generated subgroup of $\pi_{n-4} \mathrm{Diff}(S^1 \times D^{n-1})$ survives to the end of the sequence

$$\pi_{n-4}\Omega^{n-2} \mathrm{Emb}(D^1, S^1 \times D^{n-1}) \equiv \pi_{2n-6} \mathrm{Emb}(D^1, S^1 \times D^{n-1})$$

for all $n \geq 4$. At present, little is known about Cerf's scanning maps $\mathrm{Diff}(D^n) \to \Omega^j \mathrm{Emb}(D^{n-j}, D^n)$ when $j \geq 3$, but these results suggest that such maps have the potential to be homotopically non-trivial and could be used to deduce results even about $\pi_0 \mathrm{Diff}(D^n)$ for $n \geq 4$. However, we now know the map

$$\mathrm{Diff}(D^n) \to \Omega^{n-1} \mathrm{Emb}(D^1, D^n)$$

is null-homotopic [3]. The transitional map

$$\mathrm{Emb}(D^{n-2}, D^n) \to \Omega\,\mathrm{Emb}(D^{n-3}, D^n)$$

is perhaps of greatest interest, as the target space can be studied with techniques such as the embedding calculus, while we have little in the way of general theory to study the homotopy type of $\mathrm{Emb}(D^{n-2}, D^n)$. It would be more precise to say that we have general theory when $n < 4$, but when $n \geq 4$, separating the path-components of $\mathrm{Emb}(D^{n-2}, D^n)$ is a difficult mathematical problem. Similarly, little is known about $\pi_1 \mathrm{Emb}(D^2, D^4)$ at present. If one allows the embeddings to have trivialised normal bundles (normal framings), one has scanning maps of the form

$$\mathrm{Diff}(D^n) \to \Omega^j\,\mathrm{Emb}^{\mathrm{fr}}(D^{n-j}, D^n) \to \Omega^n\,\mathrm{GL}_n(\mathbb{R}),$$

where the space on the right is the terminal $j = n$ case. The map $\mathrm{Diff}(D^n) \to \Omega^n\,\mathrm{GL}_n(\mathbb{R})$ is known as the Smale–Hirsch map, i.e., the pointwise derivative map. Whether or not this Smale–Hirsch map is homotopically non-trivial has been an open problem for some time. Interestingly, it has recently been shown to be homotopically non-trivial in the $n = 11$ case [10].

One other impetus for studying such scanning maps is that these embedding spaces are highly structured objects. For example, $\mathrm{Diff}(D^n)$ is homotopy equivalent to the space $\mathrm{EC}(n, *)$, called the “cubically supported embedding space”. If M is a compact manifold, $\mathrm{EC}(j, M)$ denotes the space of smooth embeddings $f : \mathbb{R}^j \times M \to \mathbb{R}^j \times M$, where the support $\mathrm{supp}(f)$ is constrained to be a subset of $I^j \times M$; i.e.,

$$\mathrm{supp}(f) = \{p \in \mathbb{R}^j \times M : f(p) \neq p\} \subset I^j \times M.$$

The space $\mathrm{EC}(j, M)$ admits an action of the operad of $(j+1)$-cubes; thus, it is not far away from being an $(j+1)$-fold loop space. The way to think about this operad action is that there is an action of the j-cubes operad on $\mathrm{EC}(j, M)$ due to the affine structure on the $\mathbb{R}^j$ factors of $\mathbb{R}^j \times M$. The space $\mathrm{EC}(j, M)$ is also a monoid under composition of functions, and these two operations can be promoted naturally to a $(j+1)$-cubes action, described in [2].

The space $\mathrm{EC}(j, D^{n-j})$ fibres over $\mathrm{Emb}(D^j, D^n)$ with fibre $\Omega^j\,\mathrm{SO}_{n-j}$ – indeed, the spaces $\mathrm{EC}(j, D^{n-j})$ are homotopy equivalent to $\mathrm{Emb}^{\mathrm{fr}}(D^j, D^n)$. There are scanning maps

$$\begin{aligned}\mathrm{EC}(n, *) \to \Omega\,\mathrm{EC}(n-1, D^1) \to \cdots \to \Omega^j\,\mathrm{EC}(n-j, D^j) \to \cdots \\ \to \Omega^{n-1}\,\mathrm{EC}(1, D^{n-1}) \to \Omega^n\,\mathrm{GL}_n(\mathbb{R})\end{aligned}$$

which commute with the action of the $(n+1)$-cubes operad. While [2] allows one to see these cubes’ actions explicitly, there are also ways of describing the iterated loop space structure using smoothing theory. Thus, the ability of scanning maps to detect homotopy in diffeomorphism groups and embedding spaces is closely connected to the question of to what extent the Smale–Hirsch map for $\mathrm{Diff}(D^n)$ is non-trivial. To add some additional context, iterated loop spaces are highly structured objects, and finding maps between them is somewhat analogous to finding a homomorphism between other highly structured objects like rings or modules: if the map is not zero, it is often highly non-trivial.

In this paper, we outline what is known about such scanning maps and where some potentially interesting future computations sit.

2. Canceling handles

The space $\mathrm{Emb}(D^j, N)$ denotes the space of embeddings of D^j in N, where the boundary of D^j is mapped to ∂N in some fixed, prescribed manner. In the case of $\mathrm{Emb}(D^j, D^n)$, the embedding is required to restrict to the standard inclusion $x \mapsto (x, 0)$ on the boundary.

Cerf constructs an isomorphism [9, Proposition 5, p. 128] for all $i \geq 0$ and $n \geq 1$ (see also [8, Theorem 4]):

$$\pi_i \,\mathrm{Diff}(D^n) \simeq \pi_{i+1} \,\mathrm{Emb}(D^{n-1}, D^n),$$

which we promote to a homotopy equivalence

$$\mathrm{Diff}(D^n) \simeq \Omega \,\mathrm{Emb}(D^{n-1}, D^n).$$

Equivalently, this homotopy equivalence can be stated as a description of the classifying space of $\mathrm{Diff}(D^n)$,

$$B \,\mathrm{Diff}(D^n) \simeq \mathrm{Emb}_u(D^{n-1}, D^n).$$

The subscript u indicates the component of the unknot in $\mathrm{Emb}(D^{n-1}, D^n)$, i.e., the component of the linear embedding. The above results were stated at least as far back as [2], but it would not be surprising if this observation had been written down earlier.

The map $\mathrm{Diff}(D^n) \to \Omega \,\mathrm{Emb}(D^{n-1}, D^n)$ has a simple description thinking of $\mathrm{Diff}(D^n)$ as the diffeomorphisms of $\mathbb{R}^n$ with support contained in D^n. One then considers D^n as a subset of $I \times D^{n-1}$. Restriction to the fibres $\{t\} \times D^{n-1}$ gives the 1-parameter family of embeddings of D^{n-1} into D^n. After suitably translating and scaling the embedding family to have fixed boundary conditions, this is an element of $\Omega \,\mathrm{Emb}(D^{n-1}, D^n)$.

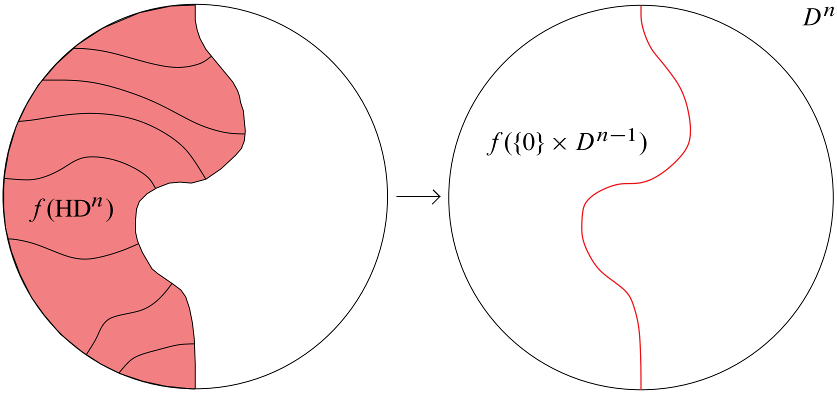

FIGURE 1
The half-disc fibration.

The map back $\Omega\,\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Diff}(D^n)$ is defined by an elementary isotopy-extension construction. Following Cerf, let HD^j denote the j-dimensional *half-disc*; i.e.,

$$\mathrm{HD}^j = \left\{(x_1, \dots, x_j) \in \mathbb{R}^j : \sum_{i=1}^{j} x_i^2 \le 1,\ x_1 \le 0\right\}.$$

The boundary $\partial\,\mathrm{HD}^j$ consists of the two subspaces: the subspace (1) $\partial D^j \cap \mathrm{HD}^j$, called the *round face*, and the subspace (2) satisfying $x_1 = 0$ called the *flat face*.

Let $\mathrm{Emb}(\mathrm{HD}^n, D^n)$ be the space of embeddings of HD^n into D^n that restrict to the identity map on $\mathrm{HD}^n \cap \partial D^n$, i.e., acting as the identity on the round face. The map given by restriction to the flat face is a Serre fibration (see Figure 1) [7]:

$$\mathrm{Diff}(\mathrm{HD}^n) \to \mathrm{Emb}(\mathrm{HD}^n, D^n) \to \mathrm{Emb}(D^{n-1}, D^n).$$

Moreover, via an argument directly analogous to the homotopy classification of collar neighbourhoods or tubular neighbourhoods, one can show that $\mathrm{Emb}(\mathrm{HD}^n, D^n)$ is contractible [8]. The rough idea is that every such embedding is isotopic to its restriction to a small neighbourhood of the round face, where you can approximate the embedding by the standard linear inclusion – indeed, the straight-line homotopy between the embedding and the standard linear inclusion is an isotopy, at least in a sufficiently small neighbourhood of the round face.

The proof that the above map is a Serre fibration is a version of the isotopy-extension theorem “with parameters”; i.e., the proof of isotopy extension given in Hirsch’s text [18] suffices to also prove that such maps are Serre fibrations. We should also mention that Palais also has shown [27] that a broad class of spaces of embeddings and diffeomorphism groups, including all the spaces discussed in this paper, have the homotopy type of countable CW complexes. The rough idea of the proof is that such embedding spaces are homeomorphic to open subsets of a Hilbert cube (consider, for example, representing smooth functions via something like a Fourier expansion), and open subsets of Hilbert cubes admit CW structures, in a manner analogous to open subsets of $\mathbb{R}^n$.

The total space $\mathrm{Emb}(\mathrm{HD}^n, D^n)$ is contractible, as sketched above and proven by Cerf [9]. This tells us that the connecting map

$$\Omega\,\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Diff}(\mathrm{HD}^n)$$

is a homotopy equivalence. The inclusion $\mathrm{Diff}(\mathrm{HD}^n) \to \mathrm{Diff}(D^n)$ is a homotopy equivalence via a rounding-the-corners argument. The definition of the connecting map $\Omega\,\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Diff}(\mathrm{HD}^n)$ comes from observing that an element of $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$ via currying can be thought of as a map $[0,1] \times D^{n-1} \to D^n$ which is continuous globally but smooth on the $\{t\} \times D^{n-1}$ fibres. A smoothing construction [18] allows us to perturb this map to be globally smooth, not affecting the restriction of the map to the boundary of $[0,1] \times D^{n-1}$. It is with this smoothing that we apply the isotopy-extension construction. Specifically, this smoothing argument tells us the subspace of $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$ such that the associated map $[0,1] \times D^{n-1} \to D^n$ is smooth; this subspace has the same homotopy type as $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$. There is an alternative approach that is formally analogous to the result that the loop space of a manifold has the same homotopy type as its subspace of smooth loops. We view $\mathrm{Emb}(D^{n-1}, D^n)$ as a smooth Banach or Fréchet manifold (depending on the order of differentiability of the embeddings, C^k with k finite or infinite, respectively). From this perspective, a smooth map $[0,1] \to \mathrm{Emb}(D^{n-1}, D^n)$ via currying produces a smooth map $[0,1] \times D^{n-1} \to D^n$. This has been made precise in several places in the literature; see [19] or [25].

We can justify why scanning $\mathrm{Diff}(D^n) \to \Omega\,\mathrm{Emb}(D^{n-1}, D^n)$ is the homotopy inverse to the connecting map $\Omega\,\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Diff}(D^n)$ via Figure 2. We have a central square whose horizontal axis is labeled t and whose vertical axis is labeled α. Given $t \in [0,1]$, let $f_t : \mathrm{HD}^n \to D^n$ denote the embedding whose restriction to $\{0\} \times D^{n-1}$ is a given element of $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$. Let $\simeq$ be the equivalence relation on $[0,1] \times D^{n-1}$ generated by the equivalence classes $[0,1] \times \{p\}$ for all $p \in \partial D^{n-1}$; thus, $[0,1] \times D^{n-1}$ can be identified with HD^n; i.e., we collapse all the edges $[0,1] \times \{p\}$ for all $p \in \partial D^{n-1}$. Under the identification of $[0,1] \times D^{n-1}/\sim$ with HD^n, the corner strata correspond to the collapsed edges, $\{0\} \times D^{n-1}$ to the round face, and $\{1\} \times D^{n-1}$ to the flat face. In Figure 2, $f_t(\{\alpha\} \times D^{n-1})$ is denoted via a thick red curve. The upper line of our square therefore denotes $f_t(\{0\} \times D^{n-1})$, our element of $\Omega\,\mathrm{Emb}(D^{n-1}, D^n)$. This is homotopic to the concatenation of the other

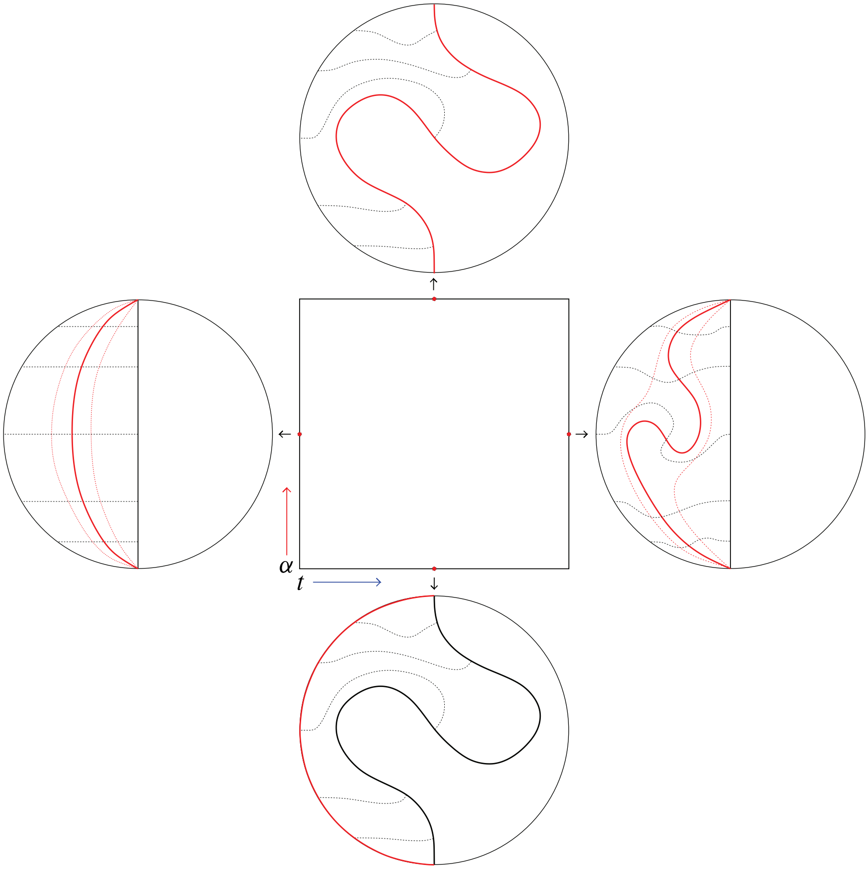

FIGURE 2
Homotopy inverse of isotopy extension.

three boundary segments of the square. The rightmost segment of the square is the "swept-out" portion of scanning, and the leftmost segment is the swept-out portion of the standard inclusion. The lower edge is constant.

To extrapolate, let $\mathrm{Emb}^{\nu}(D^{j-1}, N)$ denote the space of smooth embeddings of D^{j-1} in N such that the boundary is sent to the boundary in a prescribed manner, and the embedding comes equipped with a normal vector field (standard on the boundary); then. we have a restriction (Serre) fibration

$$\mathrm{Emb}(D^j, N \setminus \nu D^{j-1}) \to \mathrm{Emb}(\mathrm{HD}^j, N) \to \mathrm{Emb}^{\nu}(D^{j-1}, N).$$

The total space is the space of smooth embeddings of HD^j in N such that the round face is sent to ∂N in a prescribed manner. The space νD^{j-1} indicates an open tubular

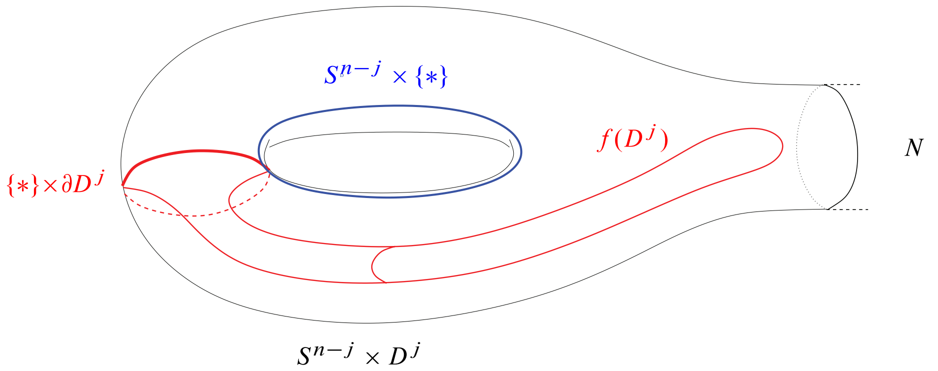

Figure 3
Cocore embedding $f \in \mathrm{Emb}(D^j, (S^{n-j} \times D^j)\natural N)$ in red. If one drills a tubular neighbourhood of a linearly embedded $D^{j-1} \to D^n$, one has a manifold diffeomorphic to $S^{n-j} \times D^j$, which gives the equivalence $(S^{n-j} \times D^j)\natural N \simeq N \setminus \nu D^{j-1}$.

neighbourhood in N corresponding to the base point element of $\mathrm{Emb}^\nu(D^{j-1}, N)$. We keep track of the normal vector field in the base space, as otherwise the fibre would be an embedding space where the discs are not neatly embedded. One can of course argue that the above is not literally the fibre – it should be the subspace of $\mathrm{Emb}(\mathrm{HD}^j, N)$ that agrees with a fixed embedding on the flat boundary. That said, blowing up the flat boundary or a tubular neighbourhood argument together with drilling the open tubular neighbourhood completes the identification of the fibre.

This gives us an analogous homotopy equivalence

$$\Omega\, \mathrm{Emb}^\nu(D^{j-1}, N) \simeq \mathrm{Emb}(D^j, N \setminus \nu D^{j-1}).$$

The space $N \setminus \nu D^{j-1}$ is N with a $(j-1)$-handle drilled out, and the embedding of D^j is a cancelling handle for the $(j-1)$-handle; thus, the $(j-1)$-handle is parallel to the boundary. As another model for $N \setminus \nu D^{j-1}$, we turn the handle upside down and think of this manifold as $N \cup H^{n-j}$, i.e., N union a $(n-j)$-handle. Since the handle attachment is trivial, this manifold is diffeomorphic to $(S^{n-j} \times D^j)\natural N$ (see Figure 3). From this perspective, the embeddings of $\mathrm{Emb}(D^j, (S^{n-j} \times D^j)\natural N)$ can be thought of as a space of embeddings of cocores for the $(n-j)$-handle attachment of the boundary connected-sum $(S^{n-j} \times D^j)\natural N$; i.e., these cocores are allowed to reach into the N summand.

This last interpretation is perhaps the most convenient for stating the homotopy equivalence

$$\Omega\, \mathrm{Emb}^\nu(D^{j-1}, N) \simeq \mathrm{Emb}(D^j, (S^{n-j} \times D^j)\natural N),$$

as the boundary condition on the latter embedding space sends ∂D^j to $\{p\} \times \partial D^j \subset S^{n-j} \times D^j$. By design, the embeddings in $\mathrm{Emb}^\nu(D^{j-1}, N)$ are isotopically trivial on the boundary $S^{j-2} \to \partial N$.

Theorem 2.1. *There is a homotopy equivalence*

$$\Omega\,\mathrm{Emb}^\nu(D^{j-1}, N) \simeq \mathrm{Emb}(D^j, (S^{n-j} \times D^j)\natural N),$$

where $\mathrm{Emb}^\nu(D^{j-1}, N)$ *is the space of smooth embeddings of D^{j-1} in N such that the pre-image of the boundary of N is the boundary of D^{j-1}. The embedding of ∂D^{j-1} is required to be a fixed embedding and isotopically trivial, i.e., bounding an embedded $D^{j-1} \to \partial N$. The base point of* $\mathrm{Emb}^\nu(D^{j-1}, N)$ *can be chosen to be any embedding that is parallel to an embedding in ∂N (rel ∂), where ν indicates that the embedding comes equipped with a normal vector field, standard on the boundary. The space* $\mathrm{Emb}(D^j, (S^{n-j} \times D^j)\natural N)$ *is a space of cocores for the handle attachment $(S^{n-j} \times D^j)\natural N = N \cup H^{n-j}$; i.e., it is the space of smooth embeddings of D^j in $(S^{n-j} \times D^j)\natural N$ such that the boundary of D^j is sent to $\{*\} \times \partial D^j$, where $* \in S^{n-j}$ is some point disjoint from the mid-ball of the boundary connected sum.*

Alternatively, one could express the theorem in the "reductionist" form

$$\mathrm{Emb}(D^j, M) \simeq \Omega\,\mathrm{Emb}^\nu(D^{j-1}, M \cup H^{n-j+1}),$$

i.e., by writing $M = (S^{n-j} \times D^j)\natural N$, then $N = M \cup H^{n-j+1}$; i.e., we derive N from M by adding a cancelling handle. Thus, for the homotopy equivalence to hold, we need M to admit a cancelling handle; i.e., for an element $f \in \mathrm{Emb}(D^j, M)$, the restriction to the boundary is an embedding $f_{|\partial D^j} : S^{j-1} \to \partial M$ and there must admit an embedding $S^{n-j} \to \partial M$ with a trivial normal bundle that transversely intersects $f_{|\partial D^j}$ in a single point. In the recent "light-bulb theorem", literature the embedded S^{n-j} is simply called a *transverse sphere* [12]. This version of Theorem 2.1 appears in [21].

A homotopy equivalence can be expressed as a map in either direction. The map $\Omega\,\mathrm{Emb}^\nu(D^{j-1}, N) \to \mathrm{Emb}(D^j, N \setminus \nu D^{j-1})$ is induced by isotopy extension; i.e., one lifts the element of $\Omega\,\mathrm{Emb}^\nu(D^{j-1}, N)$ to a path in $\mathrm{Emb}(\mathrm{HD}^j, N)$, starting at the base point of $\mathrm{Emb}(\mathrm{HD}^j, N)$. Drilling the flat face from the endpoint of this path gives the element of $\mathrm{Emb}(D^j, N \setminus \nu D^{j-1})$.

The map back $\mathrm{Emb}(D^j, N \setminus \nu D^{j-1}) \to \Omega\,\mathrm{Emb}^\nu(D^{j-1}, N)$ involves thinking of D^j as fibered by parallel copies of D^{j-1} and taking those restrictions and composing with the inclusion $N \setminus \nu D^{j-1} \to N$. The paper [4] gives a detailed account in the $\mathrm{Emb}(\mathrm{HD}^j, D^n)$ case, and [21] gives a detailed account using the "reductionist" perspective for $\mathrm{Emb}(\mathrm{HD}^j, N)$.

Proposition 2.2. *The co-dimension 2 scanning map*

$$\mathrm{Diff}(D^n) \to \Omega^2\,\mathrm{Emb}^\nu(D^{n-2}, D^n)$$

induces a split injection on all homotopy and homology groups for $n \geq 2$. The map admits a left homotopy inverse.

Proposition 2.2 is a space-level statement of [9, Proposition 6]. When $n = 2$, the two-fold scanning map $\mathrm{Diff}(D^2) \to \Omega^2\,\mathrm{Emb}^\nu(D^0, D^2) \equiv \Omega^2\,\mathrm{GL}_2(\mathbb{R})$ is the Smale–Hirsch map. Since $\Omega^2\,\mathrm{GL}_2(\mathbb{R})$ is contractible, this is Smale's theorem that $\mathrm{Diff}(D^2)$ is contractible. Since $\mathrm{Diff}(S^2) \simeq O_3 \times \mathrm{Diff}(D^2)$ (this is a standard linearisation argument; see [2]), this proves Smale's theorem $\mathrm{Diff}(S^2) \simeq O_3$.

When $n > 3$, the forgetful map $\mathrm{Emb}^\nu(D^{n-2}, D^n) \to \mathrm{Emb}(D^{n-2}, D^n)$ is a homotopy equivalence, since the fibre has the homotopy type of $\Omega^{n-2}S^1$. When $n = 2$ or $n = 3$, the double looping of the map

$$\Omega^2\,\mathrm{Emb}^\nu(D^{n-2}, D^n) \to \Omega^2\,\mathrm{Emb}(D^{n-2}, D^n)$$

is a homotopy equivalence, as the fibre has the homotopy type of $\Omega^2\Omega^{n-2}S^1$.

Corollary 2.3 (Smale). $\mathrm{Diff}(D^2)$ *is contractible; i.e.,* $\mathrm{Diff}(S^2)$ *has the homotopy type of its linear subgroup O_3.*

Proof of Proposition 2.2. The proof follows from forming a composite of functions involving the homotopy equivalence $\mathrm{Diff}(D^n) \to \Omega\,\mathrm{Emb}(D^{n-1}, D^n)$ (i.e., Theorem 2.1, $N = D^n$, $j = n$) with the induced map on loop spaces from Theorem 2.1, where $N = D^n$ and $j = n - 1$,

$$\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega\,\mathrm{Emb}^\nu(D^{n-2}, D^n).$$

Given that the unit normal fibres are copies of S^1, we can discard the normal vector fields; i.e., the forgetful map $\Omega^2\,\mathrm{Emb}^\nu(D^{n-2}, D^n) \to \Omega^2\,\mathrm{Emb}(D^{n-2}, D^n)$ is a homotopy equivalence. Thinking of $S^1 \times D^{n-1}$ as D^n union a 1-handle, this gives an inclusion $\mathrm{Emb}(D^{n-1}, D^n) \to \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$. Thus, we have a composable triple

$$\mathrm{Diff}(D^n) \to \Omega\,\mathrm{Emb}(D^{n-1}, D^n) \to \Omega\,\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega^2\,\mathrm{Emb}(D^{n-2}, D^n).$$

The left homotopy inverse of the map in the middle comes from thinking of the universal cover of $S^1 \times D^{n-1}$ as a copy of $\mathbb{R} \times D^{n-1}$ which could also be thought of as D^n removing two points from its boundary; i.e., we have a map back $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \mathrm{Emb}(D^{n-1}, D^n)$. Since the two maps on the ends are homotopy-equivalences, this gives us the result. ∎

Cerf's proof of Smale's theorem (Corollary 2.3) is also highlighted in [20, Section 6.2.4]. When the co-dimension of the embeddings is three or larger, sharp connectivity estimates for the scanning map exist. See, for example, [5, pp. 23–25], and the initial pages of Goodwillie's Ph.D thesis [14]. The paper [13] also includes a detailed analysis of scanning maps for spaces of concordance embeddings, when the co-dimension is at least three.

The deloopings of the spaces $\mathrm{Diff}(D^n)$ and $\mathrm{Emb}(D^j, D^n)$ are studied in [28, 29]. It would be interesting to see if there are analogous retraction results for the deloopings of the scanning maps $\mathrm{Diff}(D^n) \to \Omega^{n-j}\,\mathrm{Emb}^{\mathrm{fr}}(D^j, D^n)$. It is perhaps unlikely, but it is a basic question that deserves investigation.

Theorem 2.4. *The scanning map*

$$\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega\,\mathrm{Emb}^{\nu}(D^{n-2}, S^1 \times D^{n-1})$$

is the inclusion portion of a homotopy-retraction; i.e., it induces split injections on all homotopy-groups for all $n \geq 2$. When $n > 2$, ν can be dropped from the target space; i.e., the theorem remains true for embeddings without a normal vector field.

Proof. By Theorem 2.1, scanning gives us a homotopy equivalence

$$\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega\,\mathrm{Emb}^{\nu}(D^{n-2}, D^n).$$

We construct an inclusion map $\mathrm{Emb}^{\nu}(D^{n-2}, D^n) \to \mathrm{Emb}^{\nu}(D^{n-2}, S^1 \times D^{n-1})$ by attaching a trivial 1-handle to D^n, i.e., thinking of $S^1 \times D^{n-1}$ as D^n union a 1-handle. This inclusion is the inclusion portion of a homotopy-retract; i.e., it has a left homotopy inverse. The left homotopy inverse comes from lifting an embedding $D^{n-2} \to S^1 \times D^{n-1}$ to the universal cover, which we identify with a copy of D^n with two points removed from the boundary. ■

The proof of the above theorem is largely a duplicate of the proof of Proposition 2.2. Similarly, this argument allows us to identify the map $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \to \Omega\,\mathrm{Emb}(D^{n-2}, S^1 \times D^{n-1})$ with the scanning map.

Notice when $n = 2$, the above scanning map is a homotopy equivalence by Gramain [16]. When $n = 3$, it follows from Hatcher's work [17] that the scanning map is a homotopy equivalence; indeed, both spaces are contractible.

When $n \geq 4$, far less is known about such scanning maps. In [3, 4], the mapping-class group $\pi_0\,\mathrm{Diff}(S^1 \times D^3)$ was shown to be not finitely generated via the map $\pi_0\,\mathrm{Diff}(S^1 \times D^3) \to \pi_2\,\mathrm{Emb}(D^1, S^1 \times D^3)$. Above, we see that the intermediate map

$$\pi_0\,\mathrm{Diff}(S^1 \times D^3) \to \pi_1\,\mathrm{Emb}(D^2, S^1 \times D^3)$$

has kernel isomorphic to $\pi_0 \operatorname{Diff}(D^4)$; this follows from "handle-attachment homotopy equivalence" $\operatorname{Diff}(S^1 \times D^{n-1}) \simeq \operatorname{Diff}(D^n) \times \operatorname{Emb}(D^{n-1}, S^1 \times D^{n-1})$ described in [4]. The study of our scanning map $\pi_0 \operatorname{Diff}(S^1 \times D^3) \to \pi_2 \operatorname{Emb}(D^1, S^1 \times D^3)$ is thus reduced to the final step $\pi_1 \operatorname{Emb}(D^2, S^1 \times D^3) \to \pi_2 \operatorname{Emb}(D^1, S^1 \times D^3)$, i.e., the loop space functor applied to the scanning map

$$\operatorname{Emb}(D^2, S^1 \times D^3) \to \Omega \operatorname{Emb}(D^1, S^1 \times D^3).$$

One might attempt to apply the reductionist version of Theorem 2.1 to construct a homotopy equivalence $\operatorname{Emb}(D^2, S^1 \times D^3) \simeq \Omega \operatorname{Emb}^\nu(D^1, S^1 \times D^3 \cup H^3)$, but since the boundary circle for the embeddings of $\operatorname{Emb}(D^2, S^1 \times D^3)$ is homologically trivial, it does not have the required 2-sphere intersecting the embedding transversely at a single point. Alternatively, the embeddings in $\operatorname{Emb}(D^2, S^1 \times D^3)$ are not the cocore of a 2-handle attachment, so we cannot appeal to the primary version of Theorem 2.1, either. That said, we do know that the map $\operatorname{Emb}(D^2, S^1 \times D^3) \to \Omega \operatorname{Emb}(D^1, S^1 \times D^3)$ is homotopically non-trivial [3, 4] as the induced map on π_1 maps onto an infinitely generated subgroup, so further study of these scanning maps is warranted.

The work of Fresse–Turchin–Willwacher [11] describes the delooping of the homotopy fibre of the map from embeddings to immersions:

$$\operatorname{Emb}(D^j, D^n) \to \operatorname{Imm}(D^j, D^n) \simeq \Omega^j V_{n,j},$$

description of its rational homotopy type in the language of graph complexes when $n - j > 2$. In principle, this should give us some useful information on the co-dimension one scanning map $\operatorname{Emb}(D^j, D^n) \to \Omega \operatorname{Emb}(D^{j-1}, D^n)$ in rational homotopy, although our lack of understanding of the induced map $\operatorname{Emb}(D^j, D^n) \to \operatorname{Imm}(D^j, D^n)$ in rational homotopy (when $j > 1$) may be a limiting factor at present. A related topic is the "Freudenthal suspension map" $\operatorname{Emb}(D^1, D^n) \to \Omega \operatorname{Emb}(D^1, D^{n+1})$ [2] which is defined via two canonical unknotting operations. This map is known to be zero on rational homotopy (unpublished at this time), yet the map itself could potentially be homotopically non-trivial.

3. Bott handles and miscellany

The homotopy equivalence $\operatorname{Diff}(D^n) \simeq \Omega \operatorname{Emb}(D^{n-1}, D^n)$ can be extrapolated to a homotopy equivalence $\operatorname{Diff}(I \times N) \simeq \Omega \operatorname{Emb}(\{\frac{1}{2}\} \times N, I \times N)$ and scanning maps

$$\operatorname{Diff}(D^k \times N) \to \Omega \operatorname{Emb}(D^{k-1} \times N, D^k \times N) \to \cdots \to \Omega^j \operatorname{Emb}(D^{k-j} \times N, D^k \times N).$$

Whereas the scanning of Section 2 could be viewed as an argument where the intermediate space is that of the space of cancelling handles, i.e., vanilla Morse theory, the scanning above has intermediate space the space Bott-style cancelling handles, i.e., the kinds of handles that occur with Bott-style Morse functions (functions on manifolds where the critical point sets are manifolds and the Hessian is non-degenerate on these critical submanifolds [1]). For Bott-style Morse functions, "handle" attachments are disc bundles over manifolds, whereas in standard Morse theory one attaches disc bundles over points, i.e., plain discs. Specifically, an adjunction where one attaches a disc-bundle over M, $M \ltimes D^k$ to another manifold N along an embedding

$$M \ltimes \partial D^k \to \partial N$$

is what is called a Bott-style handle attachment [1], as these sorts of attachments occur for Bott-type Morse functions, i.e., functions $W \to \mathbb{R}$ whose critical points are manifolds and the Hessian is non-degenerate on the normal bundle fibres. Bott-style Morse functions typically occur when functions have symmetry; for example, the trace of a matrix is a Bott-style Morse function on the orthogonal group O_n. The critical points of this function are the square roots of the identity matrix I, thus copies of Grassman manifolds. As a concrete example, the trace functional expresses SO_3 as the tautological line bundle over $\mathbb{R}P^2$ union a 3-handle.

The analogue to Theorem 2.1 in the Bott case has the form of a homotopy equivalence

$$\mathrm{Emb}(M \ltimes D^k, N \setminus \nu(M \ltimes D^{k-1})) \simeq \Omega\, \mathrm{Emb}(M \ltimes D^{k-1}, N).$$

Given that our scanning maps are highly structured, they would appear to be a potentially useful device for exploring the homotopy-types of diffeomorphism groups like $\mathrm{Diff}(D^n)$, $\mathrm{Diff}(S^1 \times D^n)$ and generally product manifolds $\mathrm{Diff}(N \times D^k)$, in particular for studying spaces of pseudo-isotopies. From this perspective, there is perhaps a similarly overlooked element of embedding calculus [26, 31] that is relevant.

For example, given a manifold M, let $\mathcal{O}_k(M)$ be the category of open subsets of M diffeomorphic to a disjoint union of at most k open balls, arrows given by inclusion maps. Given $U \in \mathcal{O}_k(M)$, let $F(U)$ be $\mathrm{Emb}(U \times D^j, M \times D^j)$, i.e., smooth embeddings of $U \times D^j$ in $M \times D^j$ that restrict to the standard inclusion on $U \times \partial D^j$. The k-th stage of the Taylor tower could be taken to be $T_k F(U) = \mathrm{holim}_{V \in \mathcal{O}_k(U)} F(V)$. From this perspective, the scanning map is the evaluation map to the first stage of the Taylor tower. Higher stages of the Taylor tower are built from spaces of generalised string links (in the sense that the Goodwillie–Weiss–Klein embedding calculus is built from configuration spaces), and similarly, the layers will be a relative section space. This Taylor tower maps to the GWK–Taylor tower, so it should converge when the co-dimensions of the embeddings are sufficiently large. Minimally from the above it will have embeddings as a homotopy retract. The rate of convergence of this Taylor tower we suspect will often be greater – for example, by Cerf's theorem,

$$\mathrm{Diff}(D^n) \simeq \Omega\, \mathrm{Emb}(D^{n-1}, D^n),$$

the first stage when $M = I$ is homotopy equivalent to $\mathrm{Diff}(I \times D^{j-1}) \simeq \mathrm{Diff}(D^j)$. The potential for this framework is that it may provide more manageable inductive steps for practical computations of homotopy and homology groups of embedding spaces, as one is no longer comparing an embedding space directly with configuration spaces. Spaces of string links have been the subject of some recent investigations by Koytcheff [22], Turchin, and Tsopméné [22, 30], including a description of some of their low-dimensional homotopy groups [22] as well as an operad action [6], so we may not be far removed from being able to analyze these *string link Taylor towers.*

String links appear in two essential ways in both [3, 4]. Specifically, barbell diffeomorphism families are the induced diffeomorphisms coming from the low-dimensional homotopy groups of spaces of 2-component string links. Moreover, the map we use to detect our diffeomorphisms of $S^1 \times D^{n-1}$ has the form

$$\mathrm{Diff}(S^1 \times D^{n-1}) \to \Omega^{n-2}\,\mathrm{Emb}(D^1, S^1 \times D^{n-1}).$$

If we take the lifts of an element of $\mathrm{Emb}(D^1, S^1 \times D^{n-1})$ to the universal cover, we get an equivariant, infinite-component string link in $\mathbb{R} \times D^{n-1}$. Thus, string links would appear to be a relatively efficient machine for investigating embedding spaces and diffeomorphism groups. It would be very interesting to see the relative rate of convergence of the above Taylor towers, compared to the standard embedding calculus [15].

There is a small comment on the relationship between the restriction maps

$$\mathrm{Diff}(S^n) \to \mathrm{Emb}(S^j, S^n)$$

and the Cerf half-disc fibrations. When $j < n$, these fibrations are null-homotopic via a “shrinking support” argument [2]. This is closely related to the half-disc fibration. Specifically, if we replace the above diffeomorphism group and embedding space with their “long” version and require the embeddings to have trivialized normal bundles, we have the fibration $\mathrm{Diff}(D^n) \to \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n)$. This fibration has fibre homotopy equivalent to $\mathrm{Diff}(S^{n-j-1} \times D^{j+1})$. There is a cancelling-handle homotopy equivalence

$$\mathrm{Diff}(S^{n-j-1} \times D^{j+1}) \simeq \mathrm{Diff}(D^n) \times \mathrm{Emb}^{\mathrm{fr}}(D^{j+1}, S^{n-j-1} \times D^{j+1}).$$

Lastly, let $\mathrm{Emb}^{\mathrm{fr}}(\mathrm{HD}^{j+1}, D^n)$ be the half-disc embedding space where the half-discs are equipped with trivialized normal bundles. Then, we have a fibre sequence

$$\mathrm{Emb}^{\mathrm{fr}}(D^{j+1}, S^{n-j-1} \times D^{j+1}) \to \mathrm{Emb}^{\mathrm{fr}}(\mathrm{HD}^{j+1}, D^n) \to \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n).$$

Like in the unframed case, the space $\mathrm{Emb}^{\mathrm{fr}}(\mathrm{HD}^{j+1}, D^n)$ is contractible.

This gives us a little commutative diagram of homotopy fibre sequences (three top vertical maps are fibrations; three rightmost horizontal maps are also fibrations):

$$
\begin{array}{ccccc}
\mathrm{Emb}^{\mathrm{fr}}(D^{j+1}, S^{n-j-1} \times D^{j+1}) & \longrightarrow & \mathrm{Emb}^{\mathrm{fr}}(\mathrm{HD}^{j+1}, D^n) & \longrightarrow & \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Diff}(S^{n-j-1} \times D^{j+1}) & \longrightarrow & \mathrm{Diff}(D^n) & \longrightarrow & \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Diff}(D^n) & \longrightarrow & \mathrm{Diff}(D^n) & \longrightarrow & \{*\}
\end{array}
$$

i.e., we are asserting that the fibration $\mathrm{Diff}(D^n) \to \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n)$ is simply the half-disc fibration $\mathrm{Emb}^{\mathrm{fr}}(\mathrm{HD}^{j+1}, D^n) \to \mathrm{Emb}^{\mathrm{fr}}(D^j, D^n)$ but where we have inserted a trivial $\mathrm{Diff}(D^n)$ factor in the total space and fibre.

4. The Schönflies monoid

We end with the observation, implicit in [4], that the monoid $\pi_0 \,\mathrm{Emb}(S^{n-1}, S^n)$, using the connected-sum operation, is a group for all $n \geq 2$, as it is unclear if a proof of this statement exists in the literature. For $n \neq 4$, this group is known to be isomorphic to $\pi_0 \,\mathrm{Diff}(D^{n-1})$. In dimension $n = 4$, the Schönflies problem is equivalent to stating that this group is trivial.

The connected-sum operation on $\pi_0 \,\mathrm{Emb}(S^{n-1}, S^n)$ has a description as a relative surgery (i.e., performing surgery on both the ambient manifold and submanifold at the same time). One embeds pairs (D^n, D^{n-1}) in the pairs $(S^n, f(S^{n-1}))$ and $(S^n, g(S^{n-1}))$, respectively. Given that our embeddings are parametrised this requires a linearisation operation relative to the functions f and g about the embeddings $D^{n-1} \to f(S^{n-1})$ and $D^{n-1} \to g(S^{n-1})$, respectively, as well as an identification of $S^n \# S^n$ with S^n.

To minimise the overhead of formalism, we will assume the homotopy equivalence [2]

$$\mathrm{Emb}(S^{n-1}, S^n) \simeq \mathrm{SO}_{n+1} \times \mathrm{Emb}(D^{n-1}, D^n),$$

which follows from a linearisation argument.

This homotopy equivalence tells us $\pi_0 \,\mathrm{Emb}(S^{n-1}, S^n) \simeq \pi_0 \,\mathrm{Emb}(D^{n-1}, D^n)$, allowing us to define the monoid structure on $\pi_0 \,\mathrm{Emb}(D^{n-1}, D^n)$.

The space $\mathrm{Emb}(D^{n-1}, D^n)$ can be thought of as the smooth embeddings $\mathbb{R}^{n-1} \to \mathbb{R}^n$ that agrees with the standard inclusion $\{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ outside of D^{n-1} and maps D^{n-1} into D^n. We endow $\mathrm{Emb}(D^{n-1}, D^n)$ with a binary operation (indeed many such) by stacking embeddings. To stack two elements of $\operatorname{Emb}(D^{n-1}, D^n)$, one needs rescaling and translation to make the operation precise [2].

If one combines all such operations, one has an action of the operad of $(n-1)$-discs on $\operatorname{Emb}(D^{n-1}, D^n)$. The connected-sum operation is the induced monoidal structure on $\pi_0 \operatorname{Emb}(D^{n-1}, D^n)$. The neutral element is the linear embedding. This connected-sum operation generalises directly to all embedding spaces $\pi_0 \operatorname{Emb}(S^j, S^n)$. When $n = j + 2$, it is the classical connected-sum operation on co-dimension two knots, and when $j = n$, it is the composition operation on $\pi_0 \operatorname{Diff}(S^n)$.

The proof is a small variation on the proofs of Proposition 2.2 and Theorem 2.4. The inclusion from Proposition 2.2

$$\operatorname{Emb}(D^{n-1}, D^n) \to \operatorname{Emb}(D^{n-1}, S^1 \times D^{n-1})$$

is compatible with stacking; i.e., it induces a map of monoids on path components. The space $\operatorname{Emb}(D^{n-1}, S^1 \times D^{n-1})$ has the homotopy type of $\Omega \operatorname{Emb}^\nu(D^{n-2}, D^n)$ by Theorem 2.1. The space $\Omega \operatorname{Emb}^\nu(D^{n-2}, D^n)$ has *two* stacking operations; i.e., one can "stack" using the loop-space parameter or stack using the analogous stacking operation on the space $\operatorname{Emb}^\nu(D^{n-2}, D^n)$. These two operations are homotopic. In introductory algebraic topology courses, one uses this type of argument to show that the fundamental group of a topological group must be abelian. It is often called an Eckmann–Hilton argument. Another way to say this is that the space $\Omega \operatorname{Emb}^\nu(D^{n-2}, D^n)$ has an action of the operad of 2-cubes, where the action restricts to either concatenation construction, depending on the position of the cubes.

Theorem 4.1. *The monoid structure on* $\pi_0 \operatorname{Emb}(S^{n-1}, S^n)$ *comes from the connected-sum operation; this is a group for all* $n \geq 2$. *Moreover, there is an onto-homomorphism*

$$\pi_1 \operatorname{Emb}^\nu(D^{n-2}, D^n) \to \pi_0 \operatorname{Emb}(D^{n-1}, D^n) \simeq \pi_0 \operatorname{Emb}(S^{n-1}, S^n).$$

When $n = 1$, the set $\pi_0 \operatorname{Emb}(S^0, S^1)$ is also known to be a group, as it has only a single element. The group $\pi_1 \operatorname{Emb}^\nu(D^{n-2}, D^n)$ is known to be non-trivial when $n = 4$ [4] although all presently known elements map to zero in $\pi_0 \operatorname{Emb}(D^{n-1}, D^n)$.

The homomorphism $\pi_1 \operatorname{Emb}^\nu(D^{n-2}, D^n) \to \pi_0 \operatorname{Emb}(D^{n-1}, D^n)$ has this description. Take a linearly embedded copy of HD^{n-1} in D^n, i.e., the half-disc in $D^{n-1} \times \{0\} \subset D^n$. Given a loop of embeddings of D^{n-2} (with normal vector field) in D^n, lift that path of embeddings to a path in $\operatorname{Emb}(\mathrm{HD}^{n-1}, D^n)$ that begins at the linear embedding. At the end of this path, we have a smooth embedding $\mathrm{HD}^{n-1} \to D^n$ which agrees with our standard inclusion on the boundary, including its normal derivative. Via a small isotopy, we can ensure that this embedding $\mathrm{HD}^{n-1} \to D^n$ agrees with the standard inclusion in a neighbourhood of the boundary. Drill the flat face of the embedded HD^{n-1} from D^n; this results in a copy of $S^1 \times D^{n-1}$ together with a smoothly embedded $D^{n-1} \to S^1 \times D^{n-1}$ which agrees with the standard inclusion $\{1\} \times D^{n-1} \subset S^1 \times D^{n-1}$ on the boundary. Lift this embedding to the universal cover of $S^1 \times D^{n-1}$ and identify the universal cover with a subspace of D^n (D^n with two boundary points removed). This embedding $D^{n-1} \to D^n$ is the value of our map $\pi_1 \operatorname{Emb}^\nu(D^{n-2}, D^n) \to \pi_0 \operatorname{Emb}(D^{n-1}, D^n)$.

There is a Kervaire–Milnor style argument that the monoid $\pi_0 \operatorname{Emb}(S^{n-1}, S^n)$ has inverses. Given an embedding $f : S^{n-1} \to S^n$ drill a small open ball from S^{n-1} and consider a tubular neighbourhood of this manifold. It is diffeomorphic to $D^{n-1} \times I$, and so, the boundary of this manifold is diffeomorphic to the connected-sum of $f(S^{n-1})$ with its mirror reverse. Since the embedding bounds a copy of $D^{n-1} \times I \simeq D^n$ (after rounding corners), we have that $f(S^{n-1}) \# f(-S^{n-1})$ is standard; thus, f and its mirror reverse are inverses of each other. The relative advantage of Theorem 4.1 is that it provides a group $\pi_1 \operatorname{Emb}^\nu(D^{n-2}, D^n)$ that maps onto the Schönflies monoid $\pi_0 \operatorname{Emb}(S^{n-1}, S^n)$; i.e., it gives us a prescription for how one can construct all Schönflies spheres.

The resolution of the Schönflies problem in dimension different from four gives another argument that the monoid of Schönflies spheres $\pi_0 \operatorname{Emb}(S^{n-1}, S^n)$ is a group, when $n \neq 4$, as this tells us the reparametrisations of the linear embedding gives an onto homomorphism $\pi_0 \operatorname{Diff}(S^{n-1}) \to \pi_0 \operatorname{Emb}(S^{n-1}, S^n)$. The triviality of $\pi_0 \operatorname{Emb}(S^{n-1}, S^n)$ when $n = 4$ is equivalent to the Schönflies problem, as $\operatorname{Diff}(D^3)$ is contractible [17].

Acknowledgements. The author would like to thank David Gabai, Robin Koytcheff, Victor Turchin, Hyam Rubinstein, and Alexander Kupers for helpful comments. In particular, this paper is largely an exposition of a subset of results from a joint paper with David Gabai [4]. The anonymous referee was particularly helpful.

Funding. This work was partially supported by an NSERC (Canada) Discovery Grant.

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(Reçu le 28 avril 2023)

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