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Enriched Koszul duality for dg categories

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Abstract. It is well known that the category of small dg categories dgCat, though it is monoidal, does not form a monoidal model category. In this paper we construct a monoidal model structure on the category of pointed curved coalgebras ptdCoa^* over a field $\mathbf{k}$ and show that the Quillen equivalence relating it to dgCat is monoidal. We also show that dgCat is a ptdCoa^*-enriched model category. As a consequence, the homotopy category of dgCat is closed monoidal and is equivalent as a closed monoidal category to the homotopy category of ptdCoa^*. In particular, this gives a conceptual construction of a derived internal hom in dgCat which we establish over a general PID. This proves Kontsevich's characterization of the internal hom in terms of A_∞-functors. As an application we obtain a new description of simplicial mapping spaces in dgCat (over a field) and a calculation of their homotopy groups in terms of Hochschild cohomology groups, reproducing a well-known result of Toën. Comparing our approach to Toën's, we also obtain a description of the core of Lurie's dg nerve in terms of the ordinary nerve of a discrete category.

1. Introduction

The category dgCat of small differential graded (dg) categories has an internal homotopy theory, underpinned by a Quillen model structure constructed by Tabuada [22] whose weak equivalences are quasi-equivalences. There is also a closed monoidal structure on dgCat, however since the tensor product of two dg categories is not a Quillen bifunctor, e.g. the tensor product of two cofibrant dg categories is not itself cofibrant, the two structures are not compatible. Thus, the internal hom in dgCat does not determine an internal hom in the homotopy category Ho(dgCat), even though the tensor product does lift to Ho(dgCat). Nevertheless, Toën showed in [24] that Ho(dgCat) does have a closed monoidal structure, cf. also [6] for an alternative approach. The resulting derived internal hom is constructed using bimodules of special kind ('quasi-representable functors'). Borrowing the terminology of algebraic geometry, this result shows that all functors between dg-categories are of Fourier–Mukai type, cf. [5] regarding this point of view. For the Morita model structure on dgCat Tabuada gave a construction of internal homs by considering a category of localizing pairs [23].

Looking at this issue from a different angle, Kontsevich suggested that the category of unital A_∞-functors between two dg categories can be taken as the derived internal hom

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between them. A complete proof of this statement was obtained only recently and only when working over a field, in [4].

The purpose of this paper is to provide another, more structured version of the derived hom for dg categories. Our starting point is categorical Koszul duality developed in [13] and which states, roughly, that the category of (small) dg categories is Quillen equivalent to the category of coalgebras of a special kind. At the same time it is known that the category of coalgebras (unlike that of algebras) does possess an internal hom making it a closed monoidal category (this point of view and its various ramifications are explained in [1]). One can hope, therefore, that the internal hom in coalgebras is homotopically better behaved than that in dg categories, and admits a straightforward lift to homotopy categories. Provided that Koszul duality is compatible with monoidal structures on coalgebras, this would give a derived internal hom for dg categories.

The programme thus outlined is carried out in the present paper. More precisely, it establishes a closed monoidal structure on the category ptdCoa^* of pointed curved coalgebras and shows that this structure is compatible with the model structure on ptdCoa^* and the Koszul adjunction to dg categories. Kontsevich's characterization of the derived internal hom in dgCat is an immediate consequence of these results.

Note that there is a close analogy between dg-categories and ∞-categories: just like in dg-categories, simplicial categories do not possess a well-behaved internal hom, and this problem is resolved by replacing simplicial categories by simplicial sets with the Joyal model structure which does possess a monoidal model structure and thus, a well-behaved internal hom. In fact, this is more than an analogy; there exists a direct relationship between the differential graded and simplicial pictures, it is explained in Remark 5.8.

Recall that a coalgebra C is pointed if its coradical is a direct sum of copies of the ground field $\mathbf{k}$ that we fix throughout the paper. A pointed curved coalgebra is a curved coalgebra that is pointed and has a splitting of the coradical satisfying some compatibilities which we recall below, see Definition 1.3. We denote by ptdCoa^* the category of pointed curved coalgebras equipped with a final object.

According to [13] there is a natural model structure on ptdCoa^* and a Quillen equivalence $\Omega\colon \mathsf{ptdCoa}^* \rightleftarrows \mathsf{dgCat}'\colon \mathrm{B}$ between pointed curved coalgebras and dg categories (with the Dwyer–Kan model structure). Here dgCat' is an equivalent model for dgCat that will be defined below. Ω and B are suitable versions of the cobar and bar construction.

In this setting we prove the following results:

Theorem 5.4. *There is a closed monoidal model structure on* ptdCoa^*.

Moreover, the cobar construction is quasi-strong monoidal in the sense that it induces a strong monoidal functor on homotopy categories (Lemma 5.1), and it induces an equivalence of monoidal category $\mathsf{Ho}(\mathsf{ptdCoa}^*) \cong \mathsf{Ho}(\mathsf{dgCat})$ (Corollary 6.1).

The compatibility between ptdCoa^* and dgCat goes beyond the homotopy categories. While dgCat is not a monoidal model category, it is enriched, tensored and cotensored (powered) over ptdCoa^* and this structure is compatible with the model structures of dgCat' and ptdCoa^*. We say for short dgCat is a ptdCoa^*-enriched model category.

Theorem 6.2. *The category* dgCat' *is a* ptdCoa^**-enriched model category.*

Our results have immediate consequences for describing internal homs in $\mathsf{Ho}(\mathsf{dgCat})$.

Corollary 6.5. *The internal hom in* $\mathsf{Ho}(\mathsf{dgCat})$ *may be computed as (a small modification of) the Maurer–Cartan category of a convolution category* $R\,\underline{\mathrm{Hom}}(D, D') \simeq \overline{\mathsf{MC}}\{BD, D'\}$.

In fact, this result still holds when we no longer work over a field but only assume that **k** is a principal ideal domain. Unravelling definitions, this internal hom is identified with the category of unital A_∞-functors, see Corollary 6.7. This was proposed by Kontsevich and previously only shown rigorously over a field [4].

We also obtain (over a field) a new description of mapping spaces in dgCat, that differs from the classical description in [24].

Theorem 6.9. *Given two dg categories* D, D' *the mapping space* $\mathrm{Map}(D, D')$ *is weakly equivalent to the core of* $\mathrm{N}_{\mathrm{dg}}\, R\,\underline{\mathrm{Hom}}(D, D')$ *where* N_{dg} *denotes Lurie's dg nerve.*

Combining this with Toën's characterization of the mapping space, we obtain a new description of the core of the dg nerve N_{dg} in terms of the *ordinary* nerve of the category of weak equivalences (Corollary 6.11 below).

By defining a Morita model structure on ptdCoa^* we may apply our techniques also to the Morita model structure on dgCat and compute internal homs similarly.

As an application, we compute the homotopy groups of simplicial mapping spaces between dg categories in terms of Hochschild cohomology, reproducing over a field the well-known result of Toën [24] (which holds over an arbitrary commutative ring).

Theorem 7.4. *Let* $G: D \to D'$ *be a functor of dg categories. We then have* $\mathsf{HH}^0(D, D')^\times \cong \pi_1(\mathrm{Map}(D, D'), G)$ *and* $\mathsf{HH}^i(D, D') \cong \pi_{1-i}(\mathrm{Map}(D, D'), G)$ *for* $i < 0$.

Finally, we mention that analogous results hold for ordinary dg Koszul duality (i.e. for the Quillen equivalence between augmented dg algebras and conilpotent dg coalgebras). In this context, the monoidal structure on coalgebras is given by a smash-product (as opposed to a tensor product) and it lacks a monoidal unit, which brings about specific subtleties. This theory is being developed in [12].

1.1. Outline

We first recall some concepts and definitions as well as the main result of [13] in the remainder of this section. Then Section 2 provides some auxiliary results on resolving dg categories by free dg categories and pointed curved coalgebras by cofree ones. In Section 3 we construct the convolution category structure on homs from a pointed curved coalgebra to a dg category, see Definition 3.2. This is the main ingredient in constructing the MC category $\overline{\mathsf{MC}}\{C, D\}$ (Definition 4.2) and the closed monoidal structure on pointed curved coalgebras in Section 4. The compatibility with the model structure on ptdCoa^* is provided in Section 5. Section 6 then compares the monoidal structures on ptdCoa^* and dgCat' and shows that dg categories form a ptdCoa^*-enriched model category. We show the same is true for the Morita model structure on dgCat' if we adjust the model structure on ptdCoa^*. We conclude with applications to Hochschild cohomology in Section 7.

1.2. Definitions, notation and conventions

The symbol $\mathbf{k}$ stands for a fixed ground ring that is always assumed to be a principal ideal domain. We will specify when $\mathbf{k}$ is assumed to be a field. In either case, $\mathbf{k}$-linear Hom sets and Hom complexes will be denoted by Hom while internal hom objects will be denoted by $\underline{\mathrm{Hom}}$. The modifier '$\mathbf{k}$-linear' will usually be omitted later on as no other ground rings or fields will be considered (e.g. an 'algebra' will stand for a '$\mathbf{k}$-algebra' etc.) Mapping spaces in model categories will be denoted as $\mathrm{Map}(-,-)$.

We refer to [13] for a more detailed overview of the background material described below.

Let dgCat be the category of *differential graded (dg) categories* over $\mathbf{k}$, considered as a model category with its Dwyer–Kan(–Bergner–Tabuada) model structure where weak equivalences are given by quasi-equivalences [22]. (When considering the Morita model structure on dgCat at the end of Section 6 we will make this explicit.)

We will mainly consider dgCat', the full subcategory of dgCat consisting of dg categories without zero objects together with the dg category $\mathbf{0}$ with one object and a single zero morphism. The reason for this minor modification is that dgCat' fits better with categorical Koszul duality of [13] than dgCat. The category dgCat' inherits the Dwyer–Kan model structure and the associated ∞-categories of dgCat and dgCat' are equivalent.

There is a natural right Quillen functor from dgCat to ∞-categories. We write qCat for simplicial sets with the Joyal model structure, then the *dg nerve* $\mathrm{N}_{\mathrm{dg}}\colon \mathsf{dgCat} \to \mathsf{qCat}$ is constructed [18, Section 1.3.1]; it was further analyzed in [20, Section 4] and [13, Section 4]. It may be restricted to dgCat'.

The monoidal structure on dgCat may easily be extended to dgCat' with one adjustment: We need to define $\mathbf{0} \otimes D$ to be $\mathbf{0}$ for all dg categories D (the standard definition would be a dg category with the same objects as D but only zero morphisms and this is not an object of dgCat'). The inclusion $\mathsf{dgCat}' \to \mathsf{dgCat}$ is lax monoidal and induces a strong monoidal functor on homotopy categories. We call such functors *quasi-strong monoidal.*

Over a ground ring $\mathbf{k}$ that is not a field we may also consider the category $\mathsf{dgCat}_{\mathrm{fr}}$ of $\mathbf{k}$-free dg categories, which are those whose hom spaces have $\mathbf{k}$-free underlying graded modules. This is a relative category whose weak equivalences are given by quasi-equivalences.

A *curved category* is a graded category equipped with a degree 1 map d on all morphism spaces satisfying the Leibniz rule, and a degree 2 curvature endomorphism h_X for each object X such that for $f\colon X \to Y$ we have $d^2 f = h_Y f - f h_X$.

Let D be a dg category with the set of objects $\mathrm{Ob}\, D$ and consider the coalgebra $D_0 := \mathbf{k}[\mathrm{Ob}\, D]$ spanned by grouplike elements, one for every object in D. Then D can be viewed it as a monoid in bicomodules over the cosemisimple coalgebra D_0. This is an example of a *semialgebra*.

In particular, the space of morphisms between two objects d, d' is given by the cotensor product $\mathbf{k}_d \square_{D_0} D \square_{D_0} \mathbf{k}_{d'}$ where $\mathbf{k}_d$ is the 1-dimensional comodule whose coaction map is induced by the inclusion of d into $\mathrm{Ob}\, D$.

Let C be a coalgebra over $\mathbf{k}$. If $\mathbf{k}$ is not a field we shall always assume that C is $\mathbf{k}$-free, i.e. the underlying $\mathbf{k}$-module is free.

We denote by C_0 is its coradical, i.e. its maximal cosemisimple subcoalgebra and set $\bar{C} := C/C_0$. We say C is a *pointed coalgebra* if C_0 is a direct sum of copies of the ground field. Such coalgebras are also called cocomplete augmented cocategories. We then denote the set of grouplike elements by $\mathrm{Ob}\, C$ so that $C_0 \cong \mathbf{k}[\mathrm{Ob}\, C]$.

A pointed coalgebra C is *split* if it is equipped with a section $\varepsilon\colon C \to C_0$ of the inclusion $C_0 \to C$.

We will now consider curved coalgebras.

As in [13] many statements become easier when moving from the category of coalgebras to the opposite category of *pseudocompact algebras* by taking continuous duals. Recall that a pseudocompact algebra is a topological algebra that is the projective limit of (discrete) finite-dimensional algebras.

Recall the following definition, analogous to [19, Section 3.1].

Definition 1.1. A *curved pseudocompact algebra* $A = (A, d, h)$ is a graded pseudocompact algebra supplied with a derivation $d\colon A \to A$ (a differential) of degree 1 and an element $h \in A^2$ called the *curvature* of A, such that $d^2(x) = [h, x]$ and $d(h) = 0$ for any $x \in A$.

A *curved morphism* between two curved pseudocompact algebras $A \to B$ is a pair (f, b) where $f\colon A \to B$ is a map of graded algebras of degree zero and $b \in B^1$ so that:

(1) $f(d_A x) = d_B f(x) + [b, f(x)]$;

(2) $f(h_A) = h_B + d_B(b) + b^2$.

Two such morphisms (f, b) and (g, c) are composed as $(g, c) \circ (f, b) = (g \circ f, c + g(b))$. In particular, every map (f, b) can be decomposed $(f, b) = (\mathrm{id}, b) \circ (f, 0)$.

Definition 1.2. A *curved coalgebra* is a coalgebra C equipped with an odd coderivation d and a homogeneous linear function $h\colon C \to \mathbf{k}$ of degree 2, called the *curvature*, such that its dual (C^*, d^*, h^*) is a curved pseudocompact algebra.

A morphism of curved dg coalgebras from (C, d_C, h_C) to (D, d_D, h_D) is given by the data (f, a) where $f\colon C \to D$ is a morphism of graded coalgebras and $a\colon C \to \mathbf{k}$ is a linear map of degree 1, such that (f^*, a^*) is a curved morphisms $D^* \to C^*$. The composition rule is $(g, b) \circ (f, a) = (g \circ f, b \circ f + a)$.

Definition 1.3. A *pointed curved coalgebra* is a tuple $(C, \Delta_{\mathbf{k}}, \varepsilon_{\mathbf{k}}, d, h_{\mathbf{k}}, \varepsilon_C)$ such that

- $(C, \Delta_{\mathbf{k}}, \varepsilon_{\mathbf{k}}, d, h_{\mathbf{k}})$ is a curved coalgebra (over $\mathbf{k}$)
- the restriction of d to the coradical $C_0 \hookrightarrow C$ is zero,
- $\varepsilon_C\colon C \to C_0$ is a coalgebra map compatible with the differential d, which is left inverse to $i\colon C_0 \hookrightarrow C$.

We will often write simply C or (C, ε_C) for $(C, \Delta_{\mathbf{k}}, \varepsilon_{\mathbf{k}}, d, h_{\mathbf{k}}, \varepsilon_C)$ when it does not cause confusion.

The map ε together with the comultiplication induce the structure of a C_0-bicomodule on C. It follows from coassociativity that $\Delta_{\mathbf{k}}$ factors as $C \xrightarrow{\Delta} C \square_{C_0} C \to C \otimes C$. Thus, $\Delta: C \to C \square_{C_0} C$ and ε exhibit C as a comonoid in C_0-bicomodules. The inclusion $C_0 \hookrightarrow C$ provides a coaugmentation of this comonoid. The differential d is compatible with the C_0-bicomodule structure and the comonoid structure given by Δ and ε. Note also that there is automatically a curvature h with values in C_0, obtained by factorizing the curvature $h_{\mathbf{k}}: C \to \mathbf{k}$ as $\varepsilon_{\mathbf{k}} \circ (h_{\mathbf{k}} \otimes \mathrm{id}_{C_0}) \circ \rho_C : C \to C \otimes C_0 \to \mathbf{k} \otimes C_0 \to \mathbf{k}$, where ρ_C is the right coaction. We define h as $(h_{\mathbf{k}} \otimes \mathrm{id}_{C_0}) \circ \rho_C$.

We note that in particular the zero vector space 0 with coradical 0 and all defining maps equal to the zero map is a pointed curved coalgebra which we will also denote by 0.

Definition 1.4. A morphism $(f, a): (C, \varepsilon) \to (D, \delta)$ of pointed curved coalgebras consists of

- a morphism $(f, a_{\mathbf{k}})$ of curved coalgebras
- a factorization of $a_{\mathbf{k}}$ as the composition $C \xrightarrow{a} D_0 \to \mathbf{k}$

such that

- $\delta \circ f = f \circ \varepsilon$,
- f and a are D_0-bicomodule maps,

where the D_0-bicomodule structure on C is induced by the map $f: C_0 \to D_0$ on coradicals induced by f (this is a slight abuse of notation, but it will be clear from context what the domain of f is).

The composition is then defined as

$$(g, b) \circ (f, a) = (g \circ f, b \circ f + g \circ a).$$

The category of pointed curved coalgebras will be denoted by ptdCoa. The same category together with a final object $*$ will be denoted by ptdCoa^*.

Recall that when $\mathbf{k}$ is not a field the category ptdCoa^* consists of $\mathbf{k}$-free curved pointed coalgebras.

Given a pair of grouplike elements x, y in a $C \in \mathsf{ptdCoa}^*$ we write $C(x, y)$ for the complex $\mathbf{k}_x \square_{C_0} C \square_{C_0} \mathbf{k}_y$, analogous to the hom space in a category.

1.3. Recalling categorical Koszul duality

In [13], we proved there is a bar-cobar an adjunction $\Omega \dashv \mathrm{B}$ between pointed curved coalgebras and $\mathbf{k}$-free dg categories. Over a field $\mathbf{k}$ this induces a Quillen equivalence between the categories ptdCoa^* and dgCat' or suitable model structures.

Given $C \in \mathsf{ptdCoa}^*$ the cobar construction ΩC is defined to have underlying graded category given by the tensor monoid $T_{C_0} \overline{C}[-1]$ in C_0-bicomodules and differential induced by differential, comultiplication and curvature. A *weak equivalence* of pointed curved coalgebras is a map $f: C \to C'$ such that $\Omega(f)$ is a quasi-equivalence. This is part of a model structure when $\mathbf{k}$ is a field. When $\mathbf{k}$ is a PID we will consider ptdCoa^* with these weak equivalences as a relative category.

Similarly, given a dg category D viewed as a monoid in D_0-bicomodules the bar construction is defined as the tensor coalgebra $T_{D_0}\overline{D}[1]$ where $D \cong D_0 \oplus \overline{D}$ is a choice of splitting of D as a bicomodule. The differential and curvature are induced by differential and composition in D. Note that the splitting is not canonical and this is a source of substantial technical difficulties (also present in the one-object situation), for the details we refer to [13, Section 3].

The adjunction is naturally written as

$$\mathrm{Hom}(C, \mathrm{B}D) \cong \mathsf{MC}\big(\{\overline{C}, D\}\big) \cong \mathrm{Hom}(\Omega C, D)$$

where the middle term is the set of Maurer–Cartan (MC) elements in the reduced convolution category $\{\overline{C}, D\}$ which will be discussed in detail below, see Definitions 3.2 and 4.2.

We note the following special cases. Let $\mathbf{0}$ denote the dg category with one object and only the zero morphism and $\emptyset$ the empty dg category. Recall furthermore the initial object 0 and the final object $*$ in ptdCoa^*. Then $\mathrm{B}\emptyset = 0$ and $\Omega 0 = \emptyset$ by unravelling the definitions. Furthermore, we define $\mathrm{B}\mathbf{0} = *$ and $\Omega * = \mathbf{0}$.

2. Quivers, bicomodules and resolutions

We recall the category of *graded* $\mathbf{k}$-*quivers* following Keller [15]. A graded $\mathbf{k}$-quiver V consists of a set of vertices or *objects* $\mathrm{Ob}\, V$ and a graded $\mathbf{k}$-module $V(x, y)$ of *arrows* for every pair of objects x, y. (We shall always assume $V(x, y)$ is $\mathbf{k}$-free.)

A morphism $V \to W$ consists of a map on objects $f: \mathrm{Ob}\, V \to \mathrm{Ob}\, W$ and for each pair $x, y \in \mathrm{Ob}\, V$ a morphism of arrows $V(x, y) \to W(fx, fy)$.

The category of graded quivers will be denoted by grQuiv.

One sees that the data of a graded quiver is equivalent to a pair (N, C) where C is a coalgebra of the form $\oplus\mathbf{k}$ and N is a graded bicomodule N over it. One defines the correspondence by $N = \bigoplus_{x,y} V(x, y)$ and $C = \mathbf{k}[\mathrm{Ob}\, V]$.

Lemma 2.1. *There is a closed monoidal structure on* grQuiv*, with*

$$\mathrm{Ob}(V \otimes W) = \mathrm{Ob}(V) \times \mathrm{Ob}(W) \quad \textit{and} \quad (V \otimes W)(x, y) = V(x) \otimes V(y).$$

The internal hom is given by

$$\mathrm{Ob}\, \underline{\mathrm{Hom}}(V, W) = \mathrm{Hom}(\mathrm{Ob}\, V, \mathrm{Ob}\, W)$$

and

$$\mathrm{Hom}(f, g) = \bigoplus_{x,y \in \mathrm{Ob}\, V} \mathrm{Hom}\big(V(x, y), W(fx, gy)\big).$$

Proof. Immediate from unravelling definitions. ∎

We also consider the category of *augmented graded* **k**-*quivers* $\mathsf{grQuiv}^{\mathrm{aug}}$ whose objects are graded quivers V together with a factorization $\mathbf{k}[\mathrm{Ob}\,V] \xrightarrow{\eta} V \xrightarrow{\varepsilon} \mathbf{k}[\mathrm{Ob}\,V]$ of the identity on $\mathbf{k}[\mathrm{Ob}\,V]$, which is defined as the quiver with $V(x,y) = \mathbf{k}\delta_{x,y}$. Morphisms of augmented quivers are morphisms of quivers compatible with the augmentation maps η and ε.

Remark 2.2. The tensor product defined as above of two augmented quivers is naturally augmented and $\mathsf{grQuiv}^{\mathrm{aug}}$ is a closed monoidal category. The internal hom is somewhat delicate, it is described in detail in [15, Section 5.1].

We now show that arbitrary objects in ptdCoa^* can be resolved by cofree coalgebras, and in fact by bar constructions of dg categories. This will be important later.

Let $\mathsf{ptdgrCo}$ denote the category of *pointed graded coalgebras*, i.e. graded coalgebras C with coradical C_0 of the form $\oplus\mathbf{k}$ that are coaugmented comonoids over C_0.

Lemma 2.3. *There is a forgetful-cofree adjunction* $U\colon \mathsf{ptdgrCo} \rightleftarrows \mathsf{grQuiv}^{\mathrm{aug}}\colon G$*, where the cofree functor sends a quiver* Q *to the tensor coalgebra over* $Q_0 = \mathbf{k}[\mathrm{Ob}\,Q]$.

Proof. For any fixed pair of objects C, Q we have the cofree-forgetful adjunction of coalgebras in Q_0-bicomodules, thus we find

$$\mathrm{Hom}(UC, Q) = \bigoplus_{f:C_0\to Q_0} \mathrm{Hom}_{Q_0}(f_* UC, Q)$$

but as f commutes with U this is

$$\bigoplus_{f:C_0\to Q_0} \mathrm{Hom}_{Q_0}(Uf_*C, Q) \cong \bigoplus_{f:C_0\to Q_0} \mathrm{Hom}_{Q_0}(f_*C, T_{Q_0}Q) \cong \mathrm{Hom}(C, GQ)$$

and this bijection is natural. ∎

Let $\mathsf{ptdCoa}^{\mathrm{str}}$ denote the category of strict pointed curved coalgebras, with the same objects as ptdCoa^* but morphisms only the strict, i.e. uncurved, morphisms $f\colon (C,d,h) \to (C',d',h')$ given by $f\colon C \to C'$ compatible with differential and curvature.

Lemma 2.4. *There is a comonadic adjunction* $V\colon \mathsf{ptdCoa}^{\mathrm{str}} \rightleftarrows \mathsf{ptdgrCo}\colon H$ *with left adjoint* V *given by the functor forgetting differential and curvature.*

Proof. For simplicity we formulate the proof for **k**-free pseudocompact algebras, which is contravariantly equivalent by dualizing. Thus we claim that there is a monadic adjunction $H^*\colon \mathsf{ptdgrpcAlg} \rightleftarrows \mathsf{ptdcupcAlg}^{\mathrm{str}}\colon V^*$ with the right adjoint V^* forgetting differential and curvature.

Given a pointed graded pseudocompact algebra A we define H^*A to be freely generated by an element h in degree 2 as well as differentials da in degree $|a|+1$ for every $a \in A$ satisfying the rules for derivations and compatibility with the bimodule structure. Then $d^2(a) := [h,a]$ and this is easily seen to be a left adjoint of V^*.

It is clear that V^* reflects isomorphisms, thus the adjunction is monadic by the Barr–Beck theorem if we can show that $\mathsf{ptdcupcAlg}^{\mathrm{str}}$ has and V^* preserves V^*-split coequalizers.

In fact, V^* preserves all coequalizers that exist as coequalizers in the strict curved category are given by coequalizers in the graded category equipped with the induced differential and curvature. The existence of coequalizers amounts to the existence of equalizers in the category of pointed curved coalgebras. The only concern is **k**-freeness, but submodules of free **k**-modules are free as **k** is a PID.

Dualizing we see that $V \dashv H$ is comonadic. ∎

Lemma 2.5. *Any pointed curved coalgebra C is the equalizer of a diagram of cofree pointed curved coalgebras, i.e. pointed curved coalgebras of the form HGV for a graded augmented quiver V.*

Proof. We consider C as an object in $\mathsf{ptdCoa}^{\mathrm{str}}$. From the composition of the comonadic adjunctions $\mathsf{ptdCoa}^{\mathrm{str}} \rightleftarrows \mathsf{ptdgrCo}$ from Lemma 2.4 and $\mathsf{ptdgrCo} \rightleftarrows \mathsf{grQuiv}$ from Lemma 2.3 we obtain a comonad $K = HGUV$ on $\mathsf{ptdCoa}^{\mathrm{str}}$ and $C = \mathrm{eq}(KC \rightrightarrows KKC)$.

Note that a composition of comonadic functors is not necessarily comonadic, but it is if the first functor (V in our case) satisfies the crude monadicity theorem (which is true as V preserves all equalizers), see [2, Theorem 3.5.1]. We conclude by noting that this diagram is also an equalizer diagram in ptdCoa^*, which follows from the construction of equalizers in curved coalgebras (or equivalently coequalizers in curved pseudocompact algebras) in the proof of [13, Lemma 3.30]. ∎

Lemma 2.6. *Any cofree coalgebra arises as the bar construction of a dg category.*

Proof. Let V be a graded augmented quiver and write $V_0 = \mathbf{k}[\mathrm{Ob}\, V]$ for the quiver given by a copy of the ground field at every object and $\overline{V} = V/V_0$ where $\mathrm{Ob}\, \overline{V} = \mathrm{Ob}\, V$ and the quotient is taken for each pair of objects (it is free as V is augmented).

Then we define a differential graded quiver $V' = V_0 \oplus \overline{V}[-1] \oplus \overline{V} \oplus V_0[1]$ with differential given by the identity on $V_0[1]$ and $\overline{V}$ respectively. We note that there is a map of quivers $V_0 \to V'$, but the natural projection is not compatible with the differential, so this is not an augmentation.

Then define the dg category D by setting all composition zero except that each $V_0(x,x) \cong \mathbf{k}$ is the unit at $x \in \mathrm{Ob}\, V'$.

It now follows from the definitions that $\mathrm{B}D \cong HGV$. ∎

Corollary 2.7. *Every object of* ptdCoa^* *is of the form* $\mathrm{eq}(\mathrm{B}D_1 \rightrightarrows \mathrm{B}D_2)$ *for suitable dg categories D_1 and D_2.*

Proof. For a pointed curved coalgebra this is immediate from combining Lemma 2.5 and Lemma 2.6. The final object $*$ is by definition the bar construction of the dg category with one object and only the zero morphism. ∎

A similar result holds for dg categories:

Lemma 2.8. *Any small dg category D is of the form* $\mathrm{coeq}(\Omega C_1 \rightrightarrows \Omega C_2)$ *for pointed curved coalgebras C_1, C_2.*

Proof. We again construct a composition of monadic adjunctions. Forgetting the differential provides an adjunction $U\colon \mathsf{dgCat} \rightleftarrows \mathsf{grCat}\colon D$, which satisfies the crude monadicity theorem, as U preserves all coequalizers. (Coequalizers of differential graded categories can be computed on the underlying graded category and then equipped with a suitable differential.)

There is also a monadic adjunction from graded categories to graded reflexive quivers (quivers V equipped with a unit $\mathbf{k}[\mathrm{Ob}\, V] \to V$), see [26].

Together these give a monad T on dgCat and thus any dg category is the coequalizer of free dg categories.

It remains to observe that any free dg category on a reflexive quiver V is the cobar construction of a coalgebra C obtained by equipping V with the zero comultiplication on V/V_0. ∎

3. Convolution

We now introduce a convolution structure on the maps between a pointed curved coalgebra and a dg category. We will need the construction for not necessarily counital coalgebras, so let us define a *non-counital pointed graded coalgebra* to be a pair (C_0, C) where C_0 is a coalgebra of the form $\oplus\mathbf{k}$ and C is a graded C_0 bicomodule which has a coassociative comultiplication $C \to C \square_{C_0} C$, which is not necessarily counital.

Definition 3.1. A *non-counital pointed curved coalgebra* is a non-counital pointed graded coalgebra that moreover has a differential d and a curvature $h\colon C \to C_0$ such that the square of the differential is given by the coaction of the curvature.

In particular, any pointed curved coalgebra C is non-counital pointed curved coalgebra by forgetting the counit, and the quotient C/C_0 is also a non-counital pointed curved coalgebra (which is $\mathbf{k}$-free as $C_0 \to C$ has a section).

Given a dg category D and a possibly non-counital pointed curved coalgebra C we will now construct a convolution category $\{C, D\}$ as follows. Note that unless C is counital this is a non-unital category, i.e. a category without units.

The set of objects of $\{C, D\}$ is given by $\mathrm{Hom}(C_0, D_0) \cong \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Ob}\, C, \mathrm{Ob}\, D)$. Given two objects f, g in $\mathrm{Hom}(D_0, C_0)$ we make C into a D_0-bicomodule $(f, g)_*C$ via the composition

$$C \to C_0 \otimes C \otimes C_0 \xrightarrow{f \otimes \mathrm{id}_C \otimes g} D_0 \otimes C \otimes D_0.$$

Note that for $(f, f)_*C$ we will write f_*C.

Then we define

$$\mathrm{Hom}_{\{C,D\}}(f, g) = \mathrm{Hom}_{D_0}\big((f, g)_*C, D\big).$$

We note this agrees with the internal hom of quivers from Lemma 2.1.

To define the composition note that for fixed f, h the comultiplication $\Delta\colon C \to C \square_{C_0} C$ induces a map $(f,h)_* C \to (f,g)_* C \square_{D_0} (g,h)_* C$ of D_0-bicomodules for each g, put differently $(f,h)_*(C \square_{C_0} C) \subset (f,g)_* C \square_{D_0} (g,h)_* C$ (by definition of the cotensor product).

The composition $\mathrm{Hom}_{\{C,D\}}(f,g) \otimes \mathrm{Hom}_{\{C,D\}}(g,h) \to \mathrm{Hom}_{\{C,D\}}(f,h)$, spelled out $\mathrm{Hom}_{D_0}((f,g)_* C, D) \otimes \mathrm{Hom}_{D_0}((g,h)_* C, D) \to \mathrm{Hom}_{D_0}((f,h)_* C, D)$ is then defined by convolution:

$$\phi \circ \psi\colon C \xrightarrow{\Delta} C \square_{C_0} C \hookrightarrow C \square_{D_0} C \xrightarrow{\phi \otimes \psi} D \square_{D_0} D \xrightarrow{\mu} D$$

or, spelling out the D_0-bicomodule structures:

$$(f,h)_* C \xrightarrow{\Delta} (f,h)_*(C \square_{C_0} C) \hookrightarrow (f,g)_* C \square_{D_0} (g,h)_* C \xrightarrow{\phi \otimes \psi} D \square_{D_0} D \xrightarrow{\mu} D$$

The composition is associative as Δ and μ are (co)associative. Thus $\{C, D\}$ is a (non-unital) monoid in $\mathbf{k}[\mathrm{Hom}(C_0, D_0)]$-bicomodules.

The differential on C and D induces a natural differential on $\{C, D\}$. Let C have curvature $h_C\colon C \to C_0$ and let $\eta\colon D_0 \to D$ be the unit map of D. Then at an object $f \mapsto \eta \circ f \circ h_C\colon \mathrm{Hom}(C_0, D_0) \to \mathrm{Hom}(C, D)$ defines a curvature on the convolution category, compatible with the differential.

Definition 3.2. Given a dg category D and a possibly non-counital pointed curved coalgebra C the *convolution category* $\{C, D\}$ is the possibly non-unital curved category of maps from C to D, with objects $\mathrm{Hom}(C_0, D_0)$, morphisms

$$\mathrm{Hom}_{\{C,D\}}(f,g) = \mathrm{Hom}_{D_0}\big((f,g)_* C, D\big)$$

and composition given by the convolution product as defined above.

If C has a counit $\varepsilon\colon C \to C_0$ and D has a unit $D_0 \to D$ then every object f has a unit $\eta \circ f \circ \varepsilon$ and $\{C, D\}$ is a (unital) curved category.

We note that we may also define a curved convolution monoid if D is a curved category as long as C is counital, then the curvature will acquire an additional summand $f \mapsto h_D \circ f \circ \varepsilon_C$.

Lemma 3.3. *Let C, C' be (possibly non-counital) pointed graded coalgebras and D a graded category. Then there is an isomorphism of (possibly non-unital) graded convolution categories $\{C, \{C', D\}\} \cong \{C \otimes C', D\}$.*

Proof. The statement is true on the underlying bicomodules by Lemma 2.1. Next we compare the convolution products, which essentially follows by naturality of the coproduct: If $f^\#, g^\#\colon C \to \mathrm{Hom}(C', D)$ are adjoint to $f, g\colon C \otimes C' \to D$ we apply the two compositions to $c \otimes c'$. We obtain $f^\#(c^{(1)})(c'^{(1)}).g^\#(c^{(2)})(c'^{(2)})$ and $f(c^{(1)} \otimes c'^{(1)}).g(c^{(2)} \otimes c'^{(2)})$ in Sweedler notation, which of course agree. ∎

The curved version of the lemma needs an extra assumption as $\{C', D\}$ will be curved in general.

Corollary 3.4. *Let C, C' be (possibly non-counital) pointed curved coalgebras and D a curved category. Then there is an isomorphism of (possibly non-unital) curved convolution categories $\{C, \{C', D\}\} \cong \{C \otimes C', D\}$ in either of the following three cases:*

- *C is counital, D has no curvature,*
- *C' and D have no curvature,*
- *C and C' are both counital.*

Proof. The conditions of the corollary are exactly those needed to define the curvature on both sides. It is then straightforward to check that the induced differential and curvature are compatible with the adjunction and agree on both sides. ∎

4. Closed monoidal structure on coalgebras

In order to define a closed monoidal structure on coalgebras we need to introduce the Maurer–Cartan (MC) category of a curved category.

Definition 4.1. Given a curved category D we construct its *dg MC category* $\mathsf{MC}_{dg}(D)$ whose objects are pairs (X, ξ) with X an object of D and ξ an MC element in $\mathrm{End}(X)$, i.e. ξ satisfies $d\xi + \xi^2 + h_X = 0$ where h_X is the curvature of D at X.

The hom spaces are given by the complex of morphisms between twisted elements:

$$\mathrm{Hom}_{MC}((X, \xi), (X', \xi')) = \mathrm{Hom}_D(X, X')^{[\xi, \xi']} := (\mathrm{Hom}_D(X, X'), d^{[\xi, \xi']})$$

where $d^{[\xi,\xi']}(f) = df + \xi' f - (-1)^{|f|} f \xi$.

We also write $\mathrm{End}_D(X)^{\xi}$ for $\mathrm{Hom}_D(X)^{[\xi,\xi]}$.

Note that even if D is curved $\mathsf{MC}_{dg}(D)$ is naturally a dg category. We denote by $\mathsf{MC}(D)$ the set of objects of $\mathsf{MC}_{dg}(D)$.

Definition 4.2. Given $C \in \mathsf{ptdCoa}^*$ and $D \in \mathsf{dgCat}$ we define the Maurer–Cartan category $\overline{\mathsf{MC}}\{C, D\}$ to be the full subcategory of $\mathsf{MC}_{dg}(\{C, D\})$ whose objects are the objects of the non-unital dg category $\mathsf{MC}_{dg}(\{\overline{C}, D\})$.

Thus, an object of $\overline{\mathsf{MC}}\{C, D\}$ consists of a map $f : \mathrm{Ob}\, C \to \mathrm{Ob}\, D$ together with a Maurer–Cartan element in $\xi \in \mathrm{End}_{\mathsf{MC}_{dg}(\{\bar{C}, D\})}(f) = \mathrm{Hom}_{D_0}(f_* \overline{C}, D)$.

Remark 4.3. Given a pointed curved coalgebra C and a dg category D Definition 4.2 categorifies the set of MC elements in maps from $\overline{C}$ to D which mediates the Koszul adjunction $\Omega \dashv \mathrm{B}$. This leads naturally to consider the reduced convolution category $\{\overline{C}, D\}$, however as this is a non-unital category we need to consider morphisms coming from the larger (unital) category $\{C, D\}$.

The category of pointed curved coalgebras admits a symmetric monoidal structure given by the ordinary tensor product. Indeed, if C, C' are objects in ptdCoa^* with curvature functions $h\colon C \to C_0$ and $h'\colon C' \to C_0'$ and counits $\varepsilon\colon C \to C_0$ and $\varepsilon'\colon C' \to C_0'$ then $C \otimes C'$ is likewise pointed curved with the coradical $\mathbf{k}[\mathrm{Ob}\, C] \otimes \mathbf{k}[\mathrm{Ob}\, C']$ and curvature $h \otimes \varepsilon' + \varepsilon \otimes h'\colon C \otimes C' \to C_0 \otimes C_0'$. Furthermore, we define $C \otimes * := * \otimes C := *$.

We are now ready for our first main result.

Theorem 4.4. *The tensor product defines a closed monoidal structure on* ptdCoa^* *where the internal hom* $\underline{\mathrm{Hom}}(C, C')$ *is defined as* $\underline{\mathrm{Hom}}(C, \mathrm{B}D) = \mathrm{B}\overline{\mathsf{MC}}\{C, D\}$ *whenever* $C' = \mathrm{B}D$. *We also define* $\underline{\mathrm{Hom}}(C, *) = *$ *and* $\underline{\mathrm{Hom}}(*, C) = 0$ *if* $C \neq *$.

Note that from this definition $\underline{\mathrm{Hom}}(C, 0) = \mathrm{B}\overline{\mathsf{MC}}\{C, \emptyset\} = \mathrm{B}\emptyset = 0$ and $\underline{\mathrm{Hom}}(0, \mathrm{B}D) = \mathrm{B}\overline{\mathsf{MC}}\{0, D\} = \mathrm{B}\mathbf{0} = *$. From the latter it follows (see the proof of Theorem 5.4 below) that $\underline{\mathrm{Hom}}(0, C) = *$ for all C.

Remark 4.5. It is easy to see that the tensor product makes ptdCoa^* into a monoidal category and one may check that $\otimes$ commutes with all colimits. As ptdCoa^* is locally presentable, the existence of an internal hom as a right adjoint to the bifunctor $\otimes$ then follows. The explicit description that we give is more complicated but turns out to be very fruitful.

We prove some lemmas before turning to the proof of Theorem 4.4.

Lemma 4.6. *Given* $C, C' \in \mathsf{ptdCoa}^*$ *and* $D \in \mathsf{dgCat}$ *there is a decomposition*

$$\mathsf{MC}\big(\{\overline{C \otimes C'}, D\}\big) \cong \coprod_{f \in \mathrm{Hom}(C_0 \otimes C_0', D)} \; \coprod_{\phi \in \mathsf{MC}(\mathrm{End}_{\{C_0 \otimes \bar{C}', D\}}(f))} \mathsf{MC}(\mathrm{End}_{\{\bar{C} \otimes C', D\}}(f)^{\phi})$$

where (f, ϕ) *is an object in* $\mathsf{MC}(\{C_0 \otimes \bar{C}', D\})$ *and the twist by* ϕ *is induced by the natural action of* $\{C_0 \otimes \bar{C}', D\}$ *on* $\{\bar{C} \otimes C', D\}$.

Proof. The decomposition $\overline{C \otimes C'} \cong C_0 \otimes \bar{C}' \oplus \bar{C} \otimes C'$ induces a bicomodule decomposition of $\{\overline{C \otimes C'}, D\}$ and the product decomposes as

$$(\phi, \psi).(\phi', \psi') = (\phi.\phi', \phi.\psi' + \psi.\phi' + \psi.\psi'),$$

i.e. $\{C_0 \otimes \bar{C}', D\}$ is a subsemialgebra and $\{\bar{C} \otimes C', D\}$ is an ideal of the convolution category. In particular, the second summand has an action by the first.

Thus an MC element in $\{\overline{C \otimes C'}, D\}$ is given by an object $f \in \mathrm{Hom}(C_0 \otimes C_0', D_0)$ together with MC elements $\phi \in \mathrm{End}_{\{C_0 \otimes \bar{C}', D\}}(f)$ and $\psi \in \mathrm{End}_{\{\bar{C} \otimes C', D\}}(f)^{\phi}$. ∎

The following lemma is the heart of constructing the closed monoidal structure. The proof looks quite complicated, but it consists mostly in unravelling notations and bookkeeping in order to reduce the result to the usual tensor hom adjunction.

Lemma 4.7. *Given* $C, C' \in \mathsf{ptdCoa}^*$ *and* $D \in \mathsf{dgCat}$ *there is a natural isomorphism in* ptdCoa^**:*

$$\mathrm{Hom}(C \otimes C', \mathrm{B}D) \cong \mathrm{Hom}\big(C, \underline{\mathrm{Hom}}(C', \mathrm{B}D)\big) \tag{4.1}$$

Proof. To make the argument clearer we first consider the case where C, C' and D have one object. In this case we may discard the object $f\colon \mathrm{Ob}\, C \times \mathrm{Ob}\, C' \to \mathrm{Ob}\, D$ in our considerations.

By the Koszul adjunction the left-hand side of (4.1) is the set of MC elements in the convolution algebra $\{\overline{C \otimes C'}, D\}$. There is a decomposition of vector spaces

$$\{\overline{C \otimes C'}, D\} \cong \{\bar{C} \otimes C', D\} \times \{\overline{\mathbf{k} \otimes C'}, D\}$$

with $\{\overline{\mathbf{k} \otimes C'}, D\}$ a subalgebra. From Lemma 4.6 we obtain

$$\begin{aligned} \mathsf{MC}\big(\{\overline{C \otimes C'}, D\}\big) &\cong \coprod_{\phi \in \mathsf{MC}(\{\mathbf{k} \otimes \bar{C}', D\})} \mathsf{MC}\big(\{\bar{C} \otimes C', D\}^{\phi}\big) \\ &\cong \coprod_{\phi \in \mathsf{MC}(\{\mathbf{k} \otimes \bar{C}', D\})} \mathsf{MC}\big(\{\bar{C}, \{C', D\}^{\phi'}\}\big) \end{aligned}$$

where the second line follows from Lemma 3.3. To be precise, Lemma 3.3 only identifies the underlying graded algebras. It remains to compare the differential and curvature on both sides. Here ϕ' is defined as the image of ϕ in $\{C', D\}$ under the map induced by $C' \to \bar{C}'$. Then ϕ' is MC and thus the twisted convolution algebra $\{C', D\}^{\phi'}$ is a dg algebra and $\{\bar{C}, \{C', D\}^{\phi'}\}$ is a well-defined curved algebra. Unravelling definitions matches up differentials and curvatures in the two different convolution algebras and thus the MC elements agree.

Thus we write an element of the left-hand side of (4.1) as a pair (ϕ, ψ) with $\phi \in \mathsf{MC}(\{\mathbf{k} \otimes \bar{C}', D\})$ and $\psi \in \mathsf{MC}(\{\bar{C}, \{C', D\}^{\phi'}\})$.

On the right-hand side of (4.1) we have MC elements in $\{\bar{C}, \overline{\mathsf{MC}}\{C', D\}\}$, i.e. pairs (X, η) where $X \in \mathrm{Ob}\{\bar{C}, \overline{\mathsf{MC}}\{C', D\}\}$ and $\eta \in \mathrm{End}(X)$ is MC. Thus X is an MC element of $\{\bar{C}', D\}$, equivalent to ϕ on the LHS. The MC element η lives in

$$\mathrm{End}(X) \cong \mathrm{Hom}\big(\bar{C}, \underline{\mathrm{Hom}}(C', D)^{X'}\big),$$

writing X' for the image of X. Thus after identifying X and ϕ we may identify η with ψ.

We consider arbitrary objects next. Again we decompose the data of an element of the left-hand side of (4.1) first. An object of $\underline{\mathrm{Hom}}(\overline{C \otimes C'}, BD)$ is an MC element of $\{\overline{C \otimes C'}, D\}$, so it consists of an object f, given by a map $f\colon \mathrm{Ob}\, C \times \mathrm{Ob}\, C' \to \mathrm{Ob}\, D$, and an MC element in the endomorphism algebra of f, i.e. $\Phi \in \mathrm{Hom}_{D_0}(f_* \overline{C \otimes C'}, D)$ satisfying $d\Phi + \Phi^2 + h = 0$ where h is the curvature induced by the curvatures of C and C'.

As above, we use Lemma 4.6 to identify Φ with a pair

$$\phi \in \mathsf{MC}\big(\mathrm{Hom}_{D_0}\big(f_*(C_0 \otimes \bar{C}'), D\big)\big), \quad \psi \in \mathsf{MC}\big(\mathrm{Hom}_{D_0}\big(f_*(\bar{C} \otimes C'), D\big)^{\phi}\big)$$

and consider ϕ and ψ separately. Here we spelled out the endomorphisms of f as bicomodule maps,

$$\mathrm{End}_{\{C_0 \otimes \bar{C}', D\}}(f) := \mathrm{Hom}_{D_0}\big(f_*(C_0 \otimes \bar{C}'), D\big).$$

Using the definition of comodule maps we find:

$$\begin{aligned}
\phi &\in \operatorname{Hom}_{D_0}\left(f_*(C_0 \otimes \bar{C}'), D\right) \\
&\cong \bigoplus_{d_1,d_2\in \operatorname{Ob} D} \bigoplus_{\substack{c_1,c_1' \\ f(c_1,c_1')=d_1}} \bigoplus_{\substack{c_2,c_2' \\ f(c_2,c_2')=d_2}} \operatorname{Hom}\left(C_0(c_1,c_2)\otimes \bar{C}'(c_1',c_2'), D(d_1,d_2)\right) \\
&\cong \bigoplus_{c\in \operatorname{Ob} C} \bigoplus_{d_1,d_2\in \operatorname{Ob} D} \bigoplus_{\substack{c_1',c_2'\in \operatorname{Ob} C' \\ f(c,c_1')=d_1,\, f(c,c_2')=d_2}} \operatorname{Hom}\left(\bar{C}'(c_1',c_2'), D(d_1,d_2)\right)
\end{aligned}$$

where we used $C_0(c_1,c_2) = \mathbf{k}.\delta_{c_1,c_2}$. The MC condition says that each $\phi(c)$ is an MC element in $\operatorname{Hom}_{D_0}(f^\sharp(c)_*\bar{C}', D)$. We also have

$$\begin{aligned}
\psi &\in \operatorname{Hom}_{D_0}\left(f_*(\bar{C}\otimes C'), D\right)^\phi \\
&= \bigoplus_{d_1,d_2\in \operatorname{Ob} D} \bigoplus_{\substack{c_1,c_2\in \operatorname{Ob} C,\, c_1',c_2'\in \operatorname{Ob} C' \\ f(c_1,c_1')=d_1,\, f(c_2,c_2')=d_2}} \operatorname{Hom}\left(\bar{C}(c_1,c_2)\otimes C'(c_1',c_2'), D(d_1,d_2)\right)^\phi \\
&= \bigoplus_{d_1,d_2\in \operatorname{Ob} D} \bigoplus_{\substack{c_1,c_2\in \operatorname{Ob} C,\, c_1',c_2'\in \operatorname{Ob} C' \\ f(c_1,c_1')=d_1,\, f(c_2,c_2')=d_2}} \operatorname{Hom}\Big(\bar{C}(c_1,c_2), \\
&\qquad\qquad \operatorname{Hom}\left(C'(c_1',c_2'), D(d_1,d_2)\right)^{\phi(c_1),\phi(c_2)}\Big)
\end{aligned}$$

where we used Lemmas 3.3 and 4.6 again to identify the last two lines as graded algebras and then observe that differential and curvature also match up.

On the right-hand side of (4.1) we unravel similarly. By definition an MC element of the internal hom $\underline{\operatorname{Hom}}(C, \mathrm{B}\overline{\mathsf{MC}}\{C', D\})$ is given by an MC element of the convolution category $\{C, \overline{\mathsf{MC}}\{C', D\}\}$, i.e. an object $\Xi\colon \operatorname{Ob} C \to \operatorname{Ob}(\overline{\mathsf{MC}}\{C', D\})$ with an MC element η in $\operatorname{End}(\Xi) = \operatorname{Hom}_{M_0}(\Xi_* C, \overline{\mathsf{MC}}\{C', D\})$ where M_0 is the coalgebra $\mathbf{k}[\overline{\mathsf{MC}}\{C', D\}]$.

Thus Ξ sends any $c \in \operatorname{Ob} C$ to a pair $(f^\sharp(c), \xi(c))$ with $f^\sharp(c)\colon \operatorname{Ob} C' \to \operatorname{Ob} D$ an object and $\xi(c)$ an MC element

$$\xi(c) \in \operatorname{End}_{\overline{\mathsf{MC}}\{C',D\}}(f^\sharp) \cong \operatorname{Hom}_{D_0}(f^\sharp{}_*\bar{C}', D)$$

Here we write $f^\sharp\colon \operatorname{Ob} C \to \operatorname{Hom}(\operatorname{Ob} C', \operatorname{Ob} D)$ as it may be seen as the adjoint of $f\colon \operatorname{Ob} C \times \operatorname{Ob} C' \to \operatorname{Ob} D$ from the LHS.

Summing over c we find that ξ is equivalent to an MC element in

$$\begin{aligned}
\bigoplus_{c\in \operatorname{Ob} C} \operatorname{End}_{\{\bar{C}',D\}}\left(f^\sharp(c)\right) &\cong \bigoplus_{c\in \operatorname{Ob} C} \operatorname{Hom}_{D_0}\left((f^\sharp)_*\bar{C}', D\right) \\
&\cong \bigoplus_{c\in \operatorname{Ob} C} \bigoplus_{d_1,d_2\in \operatorname{Ob} D} \bigoplus_{\substack{c_1',c_2'\in \operatorname{Ob} C \\ f^\sharp(c)(c_i')=d_i'}} \operatorname{Hom}\left(\bar{C}'(c_1',c_2'), D(d_1,d_2)\right)
\end{aligned}$$

which is exactly the same curved algebra that ϕ lives in.

For the final step we need to match up η with ψ. By definition η is an MC element in

$$
\begin{aligned}
&\mathrm{Hom}_{D_0}\left(\Xi_*\bar{C}, \overline{\mathsf{MC}}\{C', D\}\right) \\
&\cong \bigoplus_{\zeta_1,\zeta_2\in \mathrm{Ob}\,\overline{\mathsf{MC}}\{C',D\}} \bigoplus_{\substack{c_1,c_2\in \mathrm{Ob}\,C \\ \Xi(c_i)=\zeta_i}} \mathrm{Hom}\left(\bar{C}(c_1,c_2), \overline{\mathsf{MC}}\{C', D\}(\zeta_1,\zeta_2)\right) \\
&\cong \bigoplus_{c_1,c_2\in \mathrm{Ob}\,C} \mathrm{Hom}\left(\bar{C}(c_1,c_2), \mathrm{Hom}_{\{C',D\}}\left(f^\sharp(c_1), f^\sharp(c_2)\right)^{\xi(c_1),\xi(c_2)}\right) \\
&\cong \bigoplus_{c_1,c_2\in \mathrm{Ob}\,C} \mathrm{Hom}\left(\bar{C}(c_1,c_2), \mathrm{Hom}_{D_0}\left(\left(f^\sharp(c_1), f^\sharp(c_2)\right)_* C', D\right)^{\xi(c_1),\xi(c_2)}\right) \\
&\cong \bigoplus_{c_1,c_2\in \mathrm{Ob}\,C} \bigoplus_{\substack{d_1,d_2\in \mathrm{Ob}\,D \\ c_1',c_2'\in \mathrm{Ob}\,C' \\ f^\sharp(c_i)(c_i')=d_i}} \mathrm{Hom}\left(\bar{C}(c_1,c_2), \mathrm{Hom}\left(C'(c_1',c_2'), D(d_1,d_2)\right)^{\xi(c_1),\xi(c_2)}\right)
\end{aligned}
$$

where we used that the hom spaces in $\overline{\mathsf{MC}}\{C, D\}$ are defined in terms of the non-reduced coalgebra C'.

Matching ϕ with ξ this shows that ψ and η are MC elements in isomorphic curved algebras and the construction of the bijection is complete.

It remains to consider the special case that C is $*$. Then, the left-hand side is $\underline{\mathrm{Hom}}(*, \mathrm{B}D)$ which is 0 unless $D = \emptyset$ in which case it equals $*$. The right-hand side $\underline{\mathrm{Hom}}(*, \underline{\mathrm{Hom}}(C', \mathrm{B}D))$ is 0 unless $\underline{\mathrm{Hom}}(C', \mathrm{B}D)$ is $*$, which by definition is only possible if $D = \emptyset$. A similar argument applies if $C' = *$. As all objects here are initial or final naturality is immediate. ■

Proof of Theorem 4.4. We observe first that the construction in Lemma 4.7 is functorial on the full subcategory whose objects are bar constructions of dg categories. This is immediate for the first variable, for the second variable it follows from the Yoneda embedding: it suffices to construct a map

$$\mathrm{Hom}\left(C, \underline{\mathrm{Hom}}(C', BD)\right) \to \mathrm{Hom}\left(C, \underline{\mathrm{Hom}}(C', BD')\right)$$

for any map $\mathrm{B}D \to BD'$. But such a map is equivalent to

$$\mathrm{Hom}(C \otimes C', BD) \to \mathrm{Hom}(C \otimes C', BD'),$$

induced by functoriality of the ordinary hom.

Thus the theorem follows from Lemma 4.7 as we are able to rewrite an arbitrary pointed curved coalgebra in terms of bar constructions. Indeed, using Corollary 2.7 we write $C' = \lim_i \mathrm{B}D_i$ and define

$$\underline{\mathrm{Hom}}(C, C') := \lim_i \underline{\mathrm{Hom}}(C, \mathrm{B}D_i) \cong \lim_i \left(\mathrm{B}\overline{\mathsf{MC}}\{C, D_i\}\right).$$

This satisfies the adjointness isomorphism for fixed C' and thus is functorial in maps $\lim_i \mathrm{B}D_i \to \lim_i \mathrm{B}D_i'$ by the Yoneda lemma again. Naturality of adjointness follows. ■

Remark 4.8. Note that the compatibility with limits also holds for $\underline{\mathrm{Hom}}(*, -)$ since we have

$$\underline{\mathrm{Hom}}(*, \lim C_i) = \lim \underline{\mathrm{Hom}}(*, C_i) = 0$$

unless all C_i equal $*$, in which case we get $\underline{\mathrm{Hom}}(*, *) = *$.

Corollary 4.9. *For $C, C', C'' \in \mathsf{ptdCoa}^*$ we have an isomorphism of pointed curved coalgebras* $\underline{\mathrm{Hom}}(C, \underline{\mathrm{Hom}}(C', C'')) \cong \underline{\mathrm{Hom}}(C \otimes C', C'')$.

Proof. This follows from Theorem 4.4 using the Yoneda lemma. ∎

Remark 4.10. Our construction may be compared to the internal hom of (cocomplete augmented) cocategories in terms of the internal hom of augmented quivers that is considered by Keller, see [15, Theorem 5.3] and the following discussion. Note that [15] considers uncurved cocategories and uses them to express functors of augmented A_∞ categories. As usual in curved Koszul duality the introduction of curved coalgebras allows us to remove the augmentation.

5. A monoidal model category

We next show that ptdCoa^* is in fact a monoidal model category with its model structure defined in [13]. For this we will have to consider the case that $\mathbf{k}$ is a field, but first we prove some preliminary results when $\mathbf{k}$ is a PID.

We will first compare the monoidal structure on ptdCoa^* and dgCat'.

Let C and C' be pointed curved coalgebras. Consider the canonical MC elements $\xi_C \in \overline{\mathsf{MC}}\{C, \Omega C\}$ and $\xi_{C'} \in \overline{\mathsf{MC}}\{C', \Omega C'\}$ corresponding to the identity maps $\Omega C \to \Omega C$ and $\Omega C' \to \Omega C'$. We take their images in the curved categories $\{C, \Omega C\}$ and $\{C', \Omega C'\}$ induced by $C \to \bar{C}$ and $C' \to \bar{C}'$. Then the element $\xi_C \otimes 1 + 1 \otimes \xi_{C'}$ is an MC element in the curved category $\{C, \Omega C\} \otimes \{C', \Omega C'\}$. Consider the natural map of curved semialgebras

$$m\colon \{C, \Omega C\} \otimes \{C', \Omega C'\} \to \{C \otimes C', \Omega C \otimes \Omega C'\} \to \{\overline{C \otimes C'}, \Omega C \otimes \Omega C'\}$$

where the second map is induced by the inclusion of the kernel of the counit. Then, $m(\xi_C \otimes 1 + 1 \otimes \xi_{C'})$ is an MC element in $\mathrm{Hom}(\overline{C \otimes C'}, \Omega C \otimes \Omega C')$ and thus by Koszul duality, it determines a map of dg categories

$$M\colon \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C').$$

Lemma 5.1. *The map M defined above is a quasi-equivalence of dg categories. Thus the cobar construction is a quasi-strong monoidal.*

Proof. Viewing all categories as semialgebras it suffices to prove that M induces a quasi-isomorphism.

Assume first that the pointed curved coalgebras C, C' have no curvature. Consider the cosimplicial complexes $T_{C_0}C = \{C^{\square_{C_0} i}\}_{i=0}^{\infty}$ and $T_{C_0'}C' = \{C'^{\square_{C_0'} i}\}_{i=0}^{\infty}$. The cosimplicial structures comes from considering the standard cosimplicial resolution $\{C^{\square_{C_0} i}\}_{i=2}^{\infty}$ and cotensoring on the left and on the right with C_0. Thus the coface map come from the comultiplication on C (respectively C') and the comodule coaction $C_0 \to C$. The codegeneracy maps are induced by the counit.

Then ΩC and $\Omega C'$ are the totalizations of the normalized cochain complexes of these cosimplicial complexes, we have $\Omega(C) \cong N(T_{C_0}C)$, $\Omega(C') \cong N(T_{C_0'}C')$ and $\Omega(C \otimes C') \cong N(T_{C_0 \otimes C_0'}C \otimes C')$. Next we observe that $T_{C_0 \otimes C_0'}C \otimes C' \cong T_{C_0}C \otimes T_{C_0'}C'$. We then claim that we can identify the map $M\colon \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C')$ with the dual Eilenberg–Zilber map. As such it induces a quasi-isomorphism on total complexes, proving the lemma in this special case.

To prove this claim we recall the dual Eilenberg–Zilber map

$$EZ^*\colon N(A \otimes B) \to N(A) \otimes N(B)$$

that sends $a \otimes b \in A_n \otimes B_n$ to

$$\sum_{p+q=n} \sum_{(\lambda,\mu) \in Sh(p,q)} \varepsilon(\lambda,\mu)\lambda^*(a) \otimes \mu^*(b)$$

where we sum over all shuffles and $\varepsilon(\lambda,\mu)$ denotes the sign. In the case at hand $A = T_{C_0}C$ and $B = T_{C_0'}C'$ and we can check that EZ^* is an algebra map.

Indeed, unravelling definitions we just have to match up shuffles on n objects with pairs of shuffles on k and on $n-k$ objects (with the correct signs), but this is a classical computation, see [11, Section 17].

Since M is an algebra map by construction and on generators $C \otimes C'$ the maps EZ^* and M are easily seen to agree (both send $c \otimes e$ to $c \otimes 1 + 1 \otimes e$) we have shown the claim.

We turn to the general case where C, C' may have curvature. We note that the coradical filtration on any pointed curved coalgebra has the property that its associated graded has no curvature. Thus, consider the coradical filtrations on C, C' and $C \otimes C'$ and the induced filtrations on $\Omega C \otimes \Omega C'$ and $\Omega(C \otimes C')$. The map $M\colon \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C')$ is compatible with these filtrations, and thus it induces a map on associated graded. These are nothing but the cobar constructions of the associated graded coalgebras. By the special case proved above, this induced map is a quasi-isomorphism and so, F is a quasi-isomorphism to begin with.

As usual, we need to consider the special case $* \in \mathsf{ptdCoa}^*$, but both $*$ and $\Omega(*) = \mathbf{0}$ are absorbing for the respective tensor product. ■

Remark 5.2. The argument above using the Eilenberg–Zilber map (albeit in the dual setting of bar-constructions of algebras over a field) goes back to Cartan and Eilenberg, [7, Chapter XI, Section 6].

Corollary 5.3. *The tensor product with a pointed curved coalgebra preserves all weak equivalences of pointed curved coalgebras.*

Proof. Let E, C, C' be pointed curved coalgebras and $C \to C'$ a weak equivalence. Then we claim $E \otimes C \to E \otimes C'$ is also a weak equivalence. By definition, we need to check $\Omega(E \otimes C) \simeq \Omega(E \otimes C')$, or, by Lemma 5.1, $\Omega(E) \otimes \Omega(C) \simeq \Omega(E) \otimes \Omega(C')$.

Thus the result follows if tensoring with a cofibrant dg category preserves quasi-equivalences. But this is readily verified, both quasi-full faithfulness and quasi-essential surjectivity are easy to check. ∎

Theorem 5.4. *Let* **k** *be a field. The category* ptdCoa^* *is a monoidal model category.*

Proof. Recall the model structure on pointed curved coalgebras from [13]. The cofibrations are given by injections and a map $f: C \to C'$ is a weak equivalence exactly if $\Omega(f)$ is a quasi-equivalence.

It remains to show that this is compatible with the closed monoidal structure from Theorem 4.4. The unit axiom follows from Corollary 5.3 as tensor product preserves all weak equivalences.

It remains to check the pushout-product axiom, i.e. let $E \to E'$ and $C \to C'$ be cofibrations in ptdCoa^*. We first check that $(E \otimes C') \amalg_{E \otimes C} (E' \otimes C) \to E' \otimes C'$ is a cofibration. Indeed, rewriting the pushout as a coequalizer it follows from the description of coequalizers in [13, Lemma 3.30] that the coequalizer of any two parallel arrows with a cone has underlying space the coequalizer in graded augmented quivers. This suffices to show that the canonical map to $E' \otimes C'$ is injective, and thus a cofibration.

Next we assume that $C \to C'$ is, moreover, a weak equivalence. By the first part (or by inspection) $E \otimes C \to E \otimes C'$ and $E \otimes C \to E' \otimes C$ are cofibrations. As Ω preserves cofibrations we see that $\Omega(E \otimes C') \amalg_{\Omega(E \otimes C)} \Omega(E' \otimes C)$ is in fact a homotopy colimit, and weakly equivalent to the homotopy colimit $(\Omega C \otimes \Omega E') \amalg^h_{\Omega C \otimes \Omega E} (\Omega C' \otimes \Omega E)$. But the latter is weakly equivalent to $\Omega C \otimes \Omega E'$ as $\Omega C \otimes \Omega E \to \Omega C' \otimes \Omega E$ is a weak equivalence. Thus the canonical map to $\Omega(C' \otimes E') \simeq \Omega C' \otimes \Omega E'$ is also a weak equivalence. ∎

Remark 5.5. It is well known that dgCat is not a monoidal model category. As Ω is only quasi-monoidal there is no contradiction to ptdCoa^* being a monoidal model category.

In order to consider mapping spaces, we recall an adjunction between simplicial sets with the Joyal model structure (which we denote qCat for quasi-categories) and pointed curved coalgebras. Define $F(C) := \mathrm{Hom}_{\mathsf{ptdCoa}^*}(\widetilde{C}_*(\Delta^\bullet), C)$ where C is a pointed curved coalgebra, K is a simplicial set and $\widetilde{C}(K)$ is the twisted chain coalgebra of K (the detailed definition of is found in [13, Section 4]). Then there is a Quillen adjunction $\widetilde{C}_*: \mathsf{qCat} \leftrightarrows \mathsf{ptdCoa}: F$.

To consider mapping spaces we will be interested in the 'maximal subgroupoid' of an ∞-category.

Definition 5.6. We call the maximal Kan subset of a quasi-category K the *core* of K.

It is clear that the core is right adjoint to the inclusion of Kan complexes into quasi-categories (weak Kan complexes).

Corollary 5.7. *Let C, C' be pointed curved coalgebras over a field* $\mathbf{k}$*. Then their mapping space* $\mathrm{Map}(C, C')$ *in the model category* ptdCoa^* *is weakly equivalent to the core of the* ∞*-category* $F\,\underline{\mathrm{Hom}}(C, C')$.

Proof. We define a cosimplicial simplicial set $E^\bullet$ by letting E^n be the nerve of the groupoid with object set $[n]$ and one arrow connecting each pair of objects. By [9, Section 4.1], this is a Reedy cosimplicial resolution of the point in the Joyal model structure i.e. a cosimplicial resolution of the trivial ∞-category.

We claim that tensoring with $\widetilde{C}_*(E^\bullet)$ gives a cosimplicial resolution in ptdCoa^*. Indeed, as $\widetilde{C}_*$ and the tensor product are left Quillen, they preserve colimits (and thus latching objects) and cofibrations, and thus Reedy cofibrant objects.

We compute

$$\begin{aligned}\mathrm{Map}(C, C') &\cong \mathrm{Hom}^*_{\mathsf{ptdCoa}}\left(C \otimes \widetilde{C}_*(E^\bullet), C'\right)\\ &\cong \mathrm{Hom}^*_{\mathsf{ptdCoa}}\left(\widetilde{C}_*(E^\bullet), \underline{\mathrm{Hom}}(C, C')\right)\\ &\cong \mathrm{Hom}_{\mathsf{sSet}}\left(E^\bullet, F\,\underline{\mathrm{Hom}}(C, C')\right)\end{aligned}$$

by the closed monoidal structure of ptdCoa^*. But the expression in the final line above is, by construction, the core of $F\,\underline{\mathrm{Hom}}(C, C')$. (We may also view it as $\mathrm{Map}_{\mathsf{qCat}}(*, F\,\underline{\mathrm{Hom}}(C, C'))$, which is known to be the core.) ■

Remark 5.8. We recall from [13, Section 4.2] the relation of Koszul duality and the coherent nerve $\mathrm{N}_{\mathrm{coh}}$ construction sending simplicial categories to quasi-categories. There is the following diagram, where the inner and outer square commute up to homotopy.

$$\begin{array}{ccc}\mathsf{qCat} & \underset{\mathfrak{C}}{\overset{\mathrm{N}_{\mathrm{coh}}}{\leftrightarrows}} & \mathsf{sCat}\\ \widetilde{C}_* \downarrow \uparrow F & & G_* \downarrow \uparrow H\\ \mathsf{ptdCoa}^* & \underset{\Omega}{\overset{\mathrm{B}}{\leftrightarrows}} & \mathsf{dgCat}'\end{array}$$

Here the horizontal arrows are ∞-equivalences and downward arrows are induced by normalized chain functors, the upwards maps are right adjoints (on the level of homotopy categories). We also showed in [13] that the dg nerve N_{dg}, the natural functor from dgCat' to qCat which is equivalent to $\mathrm{N}_{\mathrm{coh}} \circ H$, factors through B.

On the left-hand side of the above diagram we have simplicial sets with the Joyal model structure and pointed curved coalgebras, which both possess monoidal model structures. In contrast, simplicial categories or dg categories on the right-hand side do not have monoidal model structures.

Note that, moreover, $\widetilde{C}_*$ has a lax monoidal structure given by the Eilenberg–Zilber map together with the isomorphism $\widetilde{C}_*(K) \cong C_*(K)$ of curved coalgebras. Thus $\widetilde{C}_*$ also has a lax closed structure. This closed structure is far from being quasi-strong. Indeed, restricting to the subcategory of spaces in qCat it is clear that the map

$$\widetilde{C}_* \operatorname{Map}(X, Y) \to \underline{\operatorname{Hom}}(\widetilde{C}_* X, \widetilde{C}_* Y)$$

does not become an isomorphism in the homotopy category: $\underline{\operatorname{Hom}}(\widetilde{C}_* X, \widetilde{C}_* Y)$ ignores the homotopy cocommutative nature of chain coalgebras and so cannot faithfully reflect maps between the corresponding spaces.

6. Coalgebras and dg categories

Corollary 6.1. *There is an equivalence of closed monoidal categories* $\mathsf{Ho}(\mathsf{ptdCoa}^*) \cong \mathsf{Ho}(\mathsf{dgCat})$ *(for* $\mathbf{k}$ *an arbitrary PID).*

Proof. The equivalences are by [13, Theorem 3.40 and Corollary 3.41]. Note that we replaced the subcategory of $\mathbf{k}$-free dg categories in dgCat' by the usual dgCat as the two have equivalent homotopy categories (and the natural inclusions $\mathsf{dgCat}'_{\mathrm{fr}} \to \mathsf{dgCat}' \to \mathsf{dgCat}$ are quasi-strong monoidal). The equivalences are strong monoidal by Lemma 5.1 and thus also closed by general principles. ■

In fact, before passing to homotopy categories there is a very strong relation between the monoidal model category ptdCoa^* and the model category dgCat.

Recall that given a model category $\mathcal{M}$ and a monoidal model category $\mathcal{C}$ we say $\mathcal{M}$ is a $\mathcal{C}$*-enriched model category* [17, Definition A.3.1.5] if $\mathcal{M}$ is enriched in $\mathcal{C}$, tensored and cotensored over $\mathcal{C}$ and moreover the tensor action $\widetilde{\otimes}\colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is a left Quillen bifunctor (which ensures that enrichment and cotensor are also compatible with the model structures). This is a natural generalization of the notion of simplicial model category.

Theorem 6.2. *Let* $\mathbf{k}$ *be a field. The category* dgCat' *is a* ptdCoa^**-enriched model category.*

Proof. We define the external hom of a coalgebra C and dg category D as

$$\overline{\operatorname{Hom}}(\Omega C, D) := \mathrm{B}\overline{\mathsf{MC}}\{C, D\}$$

and extend to all dg categories by writing a dg category as a colimit of cobar constructions using Lemma 2.8.

We then define the cotensoring $D^C := \overline{\mathsf{MC}}\{C, D\}$ and the tensoring by $C \mathbin{\widetilde{\otimes}} \Omega C' := \Omega(C \otimes C')$ for dg categories in the image of Ω, extended by colimits to all dg categories.

In both cases functoriality follows from the Yoneda lemma as in the proof of Theorem 4.4. E.g. for fixed C, D any map $\Omega C' \to \Omega C''$ induces a map

$$\operatorname{Hom}\big(\Omega C'', \overline{\mathsf{MC}}\{C, D\}\big) \to \operatorname{Hom}\big(\Omega C', \overline{\mathsf{MC}}\{C, D\}\big)$$

and thus

$$\operatorname{Hom}\big(\Omega(C'' \otimes C), D\big) \to \operatorname{Hom}\big(\Omega(C' \otimes C), D\big).$$

To show compatibility for the cotensoring we need to check that $\overline{\mathrm{Hom}}(D', D^C) \cong \underline{\mathrm{Hom}}(C, \overline{\mathrm{Hom}}(D', D))$. Assume first $D' = \Omega C'$. We have

$$\begin{aligned}\underline{\mathrm{Hom}}\left(C, \overline{\mathrm{Hom}}(\Omega C', D)\right) &\cong \underline{\mathrm{Hom}}\left(C, \mathrm{B}\overline{\mathsf{MC}}\{C', D\}\right)\\ &\cong \underline{\mathrm{Hom}}\left(C, \underline{\mathrm{Hom}}(C', \mathrm{B}D)\right)\\ &\cong \underline{\mathrm{Hom}}(C \otimes C', \mathrm{B}D)\\ &\cong \underline{\mathrm{Hom}}\left(C', \underline{\mathrm{Hom}}(C, \mathrm{B}D)\right)\\ &\cong \underline{\mathrm{Hom}}\left(\Omega C', \overline{\mathsf{MC}}\{C, D\}\right)\end{aligned}$$

and the last term is $\overline{\mathrm{Hom}}(D, D^C)$ by definition. The general case follows by applying Lemma 2.8 and observing that the construction on both sides is compatible with colimits as in the proof of Theorem 4.4.

The compatibility condition of the tensor product is

$$\overline{\mathrm{Hom}}(C \mathbin{\tilde{\otimes}} D', D) \cong \underline{\mathrm{Hom}}\left(C, \overline{\mathrm{Hom}}(D', D)\right).$$

Again it suffices to show this if $D' = \Omega C'$. Then we have

$$\begin{aligned}\overline{\mathrm{Hom}}(C \mathbin{\tilde{\otimes}} \Omega C, D) &\cong \overline{\mathrm{Hom}}\left(\Omega(C \otimes C', D)\right)\\ &\cong \underline{\mathrm{Hom}}(C \otimes C', \mathrm{B}D)\\ &\cong \underline{\mathrm{Hom}}\left(C, \underline{\mathrm{Hom}}(C', \mathrm{B}D)\right)\\ &\cong \underline{\mathrm{Hom}}\left(C, \overline{\mathrm{Hom}}(\Omega C', D)\right)\end{aligned}$$

completing the proof of compatibility. Here the second line follows as $\underline{\mathrm{Hom}}(C, \mathrm{B}D) \cong B\overline{\mathsf{MC}}\{C, D\} \cong \overline{\mathrm{Hom}}(\Omega C, D)$ as pointed curved coalgebras by definition.

It remains to show that the enrichment is compatible with the model structures, i.e. that $\tilde{\otimes}$ is a left Quillen bifunctor, associating to a pair of cofibrations $f : C \to C'$ and $g : D \to D'$ a cofibration

$$f \square g : (C \mathbin{\tilde{\otimes}} D') \amalg_{C \tilde{\otimes} D} (C' \mathbin{\tilde{\otimes}} D) \to C' \mathbin{\tilde{\otimes}} D'$$

that is acyclic if f or g is.

We show first that $f \square g$ is a cofibration. By [14, Corollary 4.25] it suffices to check this only in the case that f and g are generating cofibrations. But the generating cofibrations in dgCat' are images of cofibrations under Ω, as was shown in the proof of [13, Proposition 3.33].

Thus we may assume $D \to D'$ is of the form $\Omega(\tilde{g})$ for a cofibration $\tilde{g} : E \to E'$ and we consider

$$\Omega(C \otimes E') \amalg_{\Omega(C \otimes E)} \Omega(C' \otimes E) \to \Omega(C' \otimes E')$$

which is a cofibration as it is an image under Ω of $f \square \tilde{g}$, which in turn is a cofibration by Theorem 5.4.

Similarly, to check the case one of f, g is acyclic it suffices to check on generating trivial cofibrations in dgCat. But these also lie in the image of Ω, so the same argument applies. ∎

The following observation was used in the proof of Theorem 6.2, it is worth re-stating.

Corollary 6.3. *The Koszul adjunction* $\Omega\colon \mathsf{ptdCoa}^* \rightleftarrows \mathsf{dgCat}\colon \mathrm{B}$ *is enriched in* ptdCoa^**, i.e. there is a natural isomorphism* $\underline{\mathrm{Hom}}(C, \mathrm{B}D) \cong \overline{\mathrm{Hom}}(\Omega C, D)$ *of pointed curved coalgebras enhancing the adjunction isomorphism.*

Remark 6.4. In particular, note that the MC elements constructed in an ad-hoc way to prove categorical Koszul duality in [13] are just the objects of the MC category $\overline{\mathsf{MC}}\{C, D\}$.

We obtain the following corollary over an arbitrary PID.

Corollary 6.5. *Let* D *and* D' *be* $\mathbf{k}$*-free dg categories over a PID* $\mathbf{k}$*. Then the internal hom in* $\mathsf{Ho}(\mathsf{dgCat})$ *may be computed as*

$$R\,\underline{\mathrm{Hom}}(D, D') \simeq \overline{\mathsf{MC}}\{BD, D'\}.$$

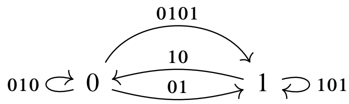
Proof. Let first $\mathbf{k}$ be a field. On the level of homotopy categories, $L\Omega$ is a strong monoidal equivalence of categories by Corollary 6.1, thus it identifies internal homs. It follows that $L\Omega(R\,\underline{\mathrm{Hom}}(\mathrm{B}D, \mathrm{B}D'))$ is an internal hom in the homotopy category. Since $\mathrm{B}D'$ is always fibrant and all pointed curved coalgebras are cofibrant we may rewrite the internal hom underived as $\Omega(\underline{\mathrm{Hom}}(\mathrm{B}D, \mathrm{B}D')) \simeq \overline{\mathsf{MC}}\{\mathrm{B}D, D'\}$.

If $\mathbf{k}$ is only a PID we do not have the model structures available. We still have, for any C in ptdCoa^*, an adjunction between $-\otimes C$ and $\underline{\mathrm{Hom}}(C, -)$ as endofunctors of the relative category $(\mathsf{ptdCoa}^*, \simeq)$. Note that the weak equivalences of pointed curved coalgebars satisfy 2-out-of-6 as they are the preimage of weak equivalences in a model category. Thus we may view $(\mathsf{ptdCoa}^*, \simeq)$ as a homotopical category in the sense of [10]. Moreover, the adjunction is deformable in the sense of [10, Section 43.1]: By Corollary 5.3 the left adjoint is homotopical (i.e. preserves weak equivalences), and the right adjoint is right deformable, see [10, Section 40.1], i.e. the unit $\mathrm{id} \to B\Omega$ induces a natural transformation to the homotopical functor $\underline{\mathrm{Hom}}(C, \mathrm{B}\Omega-)$. We have to check that $\underline{\mathrm{Hom}}(C, \mathrm{B}\Omega-)$ is indeed homotopical. For a weak equivalence $f\colon C' \to C''$ in ptdCoa^* the induced map $\mathrm{B}\overline{\mathsf{MC}}\{C, \Omega C'\} \simeq \mathrm{B}\overline{\mathsf{MC}}\{C, \Omega C''\}$ is a weak equivalence as $\overline{\mathsf{MC}}\{C, -\}$ preserves homotopy equivalences and thus quasi-equivalences of cofibrant dg categories. We prove this in Lemma 6.6 below. ∎

Lemma 6.6. *For any* $C \in \mathsf{ptdCoa}^*$ *the functor* $\overline{\mathsf{MC}}\{C, -\}\colon \mathsf{dgCat} \to \mathsf{dgCat}$ *sends homotopy equivalences of* $\mathbf{k}$*-free dg categories to quasi-equivalences.*

Proof. We fix two homotopy equivalent $\mathbf{k}$-free dg categories D, D'. We will first show that the homotopy equivalence identifies the isomorphism classes of objects in the homotopy categories of $\overline{\mathsf{MC}}\{C, D\}$ and $\overline{\mathsf{MC}}\{C, D'\}$. We have the Koszul adjunction $\Omega\colon \mathsf{ptdCoa}^* \rightleftarrows \mathsf{dgCat}_{\mathrm{fr}}\colon \mathrm{B}$ mediated by the MC elements, see the proof of [13, Theorem 3.23] which remains valid if $\mathbf{k}$ is not a field. This gives $\mathrm{Hom}(\Omega C, D) \cong \overline{\mathsf{MC}}\{C, D\}$ and we also know that $\mathrm{Hom}_{\mathsf{Ho}(\mathsf{dgCat})}(\Omega C, D) \cong \mathrm{Hom}_{\mathsf{Ho}(\mathsf{dgCat})}(\Omega C, D')$ as ΩC is cofibrant and D, D' are fibrant. Thus our claim about objects follows if we identify homotopy equivalences of objects in $\overline{\mathsf{MC}}\{C, D\}$ with homotopy equivalences of functors $\Omega C \to D$.

We define the simplicial set $K \subset N(0 \leftrightarrows 1)$ consisting of 2 0-simplices $\{0, 1\}$, two 1-simplices $\{01, 10\}$, two 2-simplices $\{010, 101\}$ and one 3-simplex $\{0101\}$. Then $\Omega(K)$ is Drinfeld's interval object in dg categories,



which is easily seen to be quasi-equivalent to the one-object dg category $\mathbf{k}$, thus $K \simeq \mathbf{k}$. It follows from Corollary 5.3 that $\Omega(C \otimes K) \to \Omega C$ is a weak equivalence. Moreover, Ω takes injections to cofibrations, see the proof of Proposition 3.33, condition (4), in [13] (the argument is unaffected by working over $\mathbf{k}$ not a field as C and K are $\mathbf{k}$-free). Thus $\Omega(C \otimes K)$ is indeed a cylinder object for ΩC.

Assume now we have a homotopy $H\colon \Omega(C \otimes K) \to D$ between functors H_0, H_1. We may represent H by an MC element $\mathbf{h} = (h, \eta)$ where $h\colon \mathrm{Ob}\, C \times \{0, 1\} \to D$ is a function and η is an MC element in $\bigoplus_{(x,i),(y,j)} \mathrm{Hom}(C(x, y) \otimes K(i, j), \mathrm{Hom}_D(h_i(x), h_j(y)))$ where $i \in \{0, 1\}$, such that the restrictions $\mathbf{h}_0$ and $\mathbf{h}_1$ represent the functors H_0 and H_1. (Here $\mathbf{h}_i = (h_i, \eta_i)$ are defined as the images under the map induced by the inclusions $i \to K$.) Then the further components of η which we may denote $\eta_{01}, \eta_{10}, \eta_{010}, \eta_{101}$ and η_{0101} provide exactly an isomorphism in the homotopy category of $\overline{\mathsf{MC}}\{C, D\}$ between $\mathbf{h}_0$ and $\mathbf{h}_1$: By [21, Lemma 3.6] such a homotopy equivalence is given by 5-tuple of maps

$$
\begin{aligned}
f &\in \mathrm{Hom}(\mathbf{h}_0, \mathbf{h}_1) & g &\in \mathrm{Hom}(\mathbf{h}_1, \mathbf{h}_0)\\
r_0 &\in \mathrm{Hom}(\mathbf{h}_0, \mathbf{h}_0)[1] & r_1 &\in \mathrm{Hom}(\mathbf{h}_1, \mathbf{h}_1)[1]\\
r_{01} &\in \mathrm{Hom}(\mathbf{h}_0, \mathbf{h}_1)[2]
\end{aligned}
$$

with $fg = \mathrm{id}_0 + dr_0$, $fg = \mathrm{id}_1 + dr_1$ and $fr_0 - r_1 f = dr_{01}$. We identify

$$
\begin{aligned}
f &\leftrightarrow \eta_{01} & g &\leftrightarrow \eta_{01}\\
r_0 &\leftrightarrow \eta_{010} & r_1 &\leftrightarrow \eta_{101}\\
r_{01} &\leftrightarrow \eta_{0101}
\end{aligned}
$$

and the conditions on the differential agree exactly with the MC condition. For example $fr_0 - r_1 f = dr_{01}$ becomes

$$\eta_{01} \cdot \eta_{010} - \eta_{101} \cdot \eta_{01} = d^{[\eta_1, \eta_0]} \eta_{0101} = \eta_1 \eta_{0101} - \eta_{0101} \eta_0 + d\eta_{0101}$$

which is the 0101-component of the MC equation for η. Note that the curvature of C appears only in the MC condition for η_0 and η_1.

It remains to show that any homotopy equivalence $T\colon D \to D'$ induces quasi-isomorphisms on hom spaces. Let $\mathbf{K}$ be an arbitrary field over $\mathbf{k}$. Fix objects

$$(f, \phi), (g, \psi) \in \overline{\mathsf{MC}}\{C, D\}.$$

Then T induces a natural map of hom spaces

$$\begin{array}{ccc} \operatorname{Hom}_{\overline{\mathsf{MC}}\{C,D\}}\big((f,\phi),(g,\psi)\big) & \longrightarrow & \operatorname{Hom}_{\overline{\mathsf{MC}}\{C,D'\}}\big((T\circ f, T_*\phi),(T\circ g, T_*\psi)\big) \\ \| & & \| \\ \bigoplus_{x,y} \operatorname{Hom}\big(C(x,y), D(fx,gy)\big)^{[\psi,\phi]} & & \bigoplus_{x,y} \operatorname{Hom}\big(C(x,y), D'\big(T(fx),T(gy)\big)\big)^{[T_*\psi,T_*\phi]} \end{array}$$

and $T \otimes_{\mathbf{k}} \mathbf{K}$ induces a map between the hom spaces tensored with $\mathbf{K}$. We claim now that

$$\bigoplus_{x,y} \operatorname{Hom}\big(C(x,y), D(fx,gy)\big)^{[\psi,\phi]} \otimes_{\mathbf{k}} \mathbf{K} \cong \bigoplus_{x,y} \operatorname{Hom}\big(C(x,y)\otimes_{\mathbf{k}} \mathbf{K}, D(fx,gy)\otimes_{\mathbf{k}} \mathbf{K}\big)^{[\psi\otimes_{\mathbf{k}}\mathbf{K},\phi\otimes_{\mathbf{k}}\mathbf{K}]}$$

which is immediate as C, D are $\mathbf{k}$-free. But the right-hand side is a hom space in $\overline{\mathsf{MC}}\{C\otimes_{\mathbf{k}}\mathbf{K}, D\otimes_{\mathbf{k}}\mathbf{K}\}$. We already established Corollary 6.5 over a ground field, thus we have a quasi-equivalence $\overline{\mathsf{MC}}\{C\otimes_{\mathbf{k}}\mathbf{K}, D\otimes_{\mathbf{k}}\mathbf{K}\} \simeq \overline{\mathsf{MC}}\{C\otimes_{\mathbf{k}}\mathbf{K},' D\otimes_{\mathbf{k}}\mathbf{K}\}$. By [8, Lemma 3.6] we can test quasi-isomorphisms for free dg modules over a PID $\mathbf{k}$ by tensoring with all fields over $\mathbf{k}$. The result follows. ∎

Unravelling the definition of $R\,\underline{\operatorname{Hom}}(D,D') \simeq \overline{\mathsf{MC}}\{BD,D'\}$ allows us to recover the description of $R\,\underline{\operatorname{Hom}}(D,D')$ as the dg category of *unital A_∞-functors*. This description was proposed by Kontsevich and has so far only been proved rigorously in [4] in the case that $\mathbf{k}$ is a field.

Corollary 6.7. *Let D, D' be $\mathbf{k}$-free dg categories over a PID $\mathbf{k}$. The objects in the category $\overline{\mathsf{MC}}\{BD,D'\}$ are unital A_∞ functors $D\to D'$. The morphisms in $\overline{\mathsf{MC}}\{\mathrm{B}D,D'\}$ are unital A_∞ transformations between the corresponding A_∞ functors.*

Proof. Considering the definition of unital A_∞-functors from D to D', as for example in [4, Definitions 1.6, 1.8], we see that they are exactly given by maps from the bar construction of D to D' satisfying the MC condition: The degree n part of the map sends a string of composable non-identity morphisms in D of length n, i.e. an object in $T^n\bar{D}$ to a morphism in D' shifted in degree by $1-n$. This is a degree 1 map from the underlying graded of $\mathrm{B}D'$ to D. The MC condition on differentials unravels exactly to the relation satisfied by A_∞ functors between dg categories.

Similarly, the morphisms in $\overline{\mathsf{MC}}\{\mathrm{B}D,D'\}$ are unital A_∞ transformations between the functors. ∎

Remark 6.8. Note that [4] considers not necessarily unital A_∞ transformations between functors, but the two notions agree. This was shown in [16, Lemma 8.2.1.3], it also follows from Lemma 7.2 below, the difference between unital and non-unital transformations corresponds exactly to the difference between reduced and non-reduced Hochschild cochains.

While the embedding of dg categories into A_∞ categories thus computes the correct internal hom object, it is worth recalling that the category of A_∞ categories does not have a sensible monoidal structure.

Let now $\mathbf{k}$ be a field for the remainder of this section.

Theorem 6.9. *Given two dg categories D, D' the mapping space $\mathrm{Map}(D, D')$ is weakly equivalent to the core of $\mathrm{N_{dg}}\, R\,\underline{\mathrm{Hom}}(D, D')$.*

Proof. By the main results of [13] we may compute $\mathrm{Map}(D, D')$ as $\mathrm{Map}(\mathrm{B}D, \mathrm{B}D')$ in ptdCoa^*. By Theorem 4.4 and Corollary 5.7 this is given by the core of

$$F\mathrm{B}\overline{\mathsf{MC}}\{\mathrm{B}D, D'\} \simeq \mathrm{N_{dg}}\,\overline{\mathsf{MC}}\{\mathrm{B}D, D'\} \simeq \mathrm{N_{dg}}\, R\,\underline{\mathrm{Hom}}(D, D'),$$

where we used $\mathrm{N_{dg}} \simeq F\mathrm{B}$ from [13, Theorem 4.16] as well as Corollary 6.5. ∎

Remark 6.10. The theorem may also be deduced directly from the existence of an internal hom in $\mathsf{Ho}(\mathsf{dgCat})$ and some other standard results. We denote the left adjoint of the dg nerve by L and note that $L(*) = \mathbf{k}$ (considered as a dg category with one object). Then the core of $\mathrm{N_{dg}}(R\,\underline{\mathrm{Hom}}(D, D'))$ may be computed as

$$\mathrm{Map}_{\mathsf{qCat}}\big(*, N_{dg}\big(R\,\underline{\mathrm{Hom}}(D, D')\big)\big) \simeq \mathrm{Map}_{\mathsf{dgCat}}\big(L*, R\,\underline{\mathrm{Hom}}(D, D')\big) \simeq \mathrm{Map}_{\mathsf{dgCat}}(\mathbf{k} \otimes D, D').$$

Here we use that the closed structure dgCat induces a weak equivalence of mapping spaces even though it is not Quillen. This follows from the Yoneda lemma by taking hom out an arbitrary simplicial set and using the simplicial enrichment on dgCat [24, Section 5].

We may compare Theorem 6.9 with Toën's characterization of $\mathrm{Map}(D, D')$ as the (classical) nerve of the category of weak equivalences in the category of right quasi-representable cofibrant $D \otimes D'^{\mathrm{op}}$-modules [24, Theorem 1.1]. In the special case of the mapping space $\mathrm{Map}(\mathbf{k}, D)$ this gives the following corollary:

Corollary 6.11. *For a dg category D the core of $\mathrm{N_{dg}}(D)$ is equivalent to the nerve of the 1-category of quasi-isomorphisms between cofibrant quasi-representable D^{op}-modules.*

This corollary is already interesting in the case that the dg category has one object. Then it says that the core of the simplicial Maurer–Cartan set of a dg algebra A (as considered in [13, Section 4.1]) is given by the A-component of the nerve of the category of quasi-isomorphisms between cofibrant A-modules.

We have considered the Dwyer–Kan model structure on dgCat so far, but similar results apply for the Morita model structure. We denote by $\mathsf{dgCat}'_{\mathrm{Mor}}$ the left Bousfield localization of dgCat' at all *Morita equivalences*, i.e. functors inducing equivalences of derived categories.

Lemma 6.12. *There is a model structure on ptdCoa^* whose weak equivalences are maps $C \to C'$ inducing equivalences of coderived categories $\mathsf{D}^{\mathrm{co}}(C) \cong \mathsf{D}^{\mathrm{co}}(C')$ and whose cofibrations are injections.*

Proof. By [13, Proposition 3.33], ptdCoa^* is a left proper combinatorial model category, thus the left Bousfield localization at Morita equivalences exists. ∎

We call this the *Morita model structure* and the weak equivalences *Morita equivalences* of pointed curved coalgebras. We denote pointed curved coalgebras with the Morita model structure by $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$.

Proposition 6.13. *The model category* $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$ *is Quillen equivalent to* $\mathsf{dgCat}'_{\mathrm{Mor}}$.

Proof. By [13, Theorem 3.43], a map $f: C \to C'$ of coalgebras is a Morita equivalence if and only if Ωf is a Morita equivalence, and similarly a functor between dg categories is a Morita equivalence if and only if its bar construction is a Morita equivalence.

As the cobar construction on $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$ preserves cofibrations and all weak equivalences it defines a left Quillen functor to $\mathsf{dgCat}'_{\mathrm{Mor}}$. As it induces an equivalence on homotopy categories this is a Quillen equivalence ∎

Lemma 6.14. *The tensor product makes* $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$ *into a monoidal model category whose homotopy category is monoidally equivalent to* $\mathsf{Ho}(\mathsf{dgCat}'_{\mathrm{Mor}})$.

Proof. The closed monoidal structure is the one considered in Theorem 5.4.

We first note that Lemma 5.1 implies that Ω is quasi-strong monoidal also for the Morita model structures. We then observe that tensor product of dg categories preserves Morita equivalences. This is well known, see [25, Exercise 32]. ∎

Proposition 6.15. *The category* $\mathsf{dgCat}'_{\mathrm{Mor}}$ *is a* $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$*-enriched model category.*

Proof. Given $f: C \to C'$ in $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$ and $g: D \to D'$ in $\mathsf{dgCat}'_{\mathrm{Mor}}$ we consider

$$f \square g: (C \mathbin{\tilde{\otimes}} D') \amalg_{C \tilde{\otimes} D} (C' \mathbin{\tilde{\otimes}} D) \to C' \mathbin{\tilde{\otimes}} D'$$

As the cofibrations in dgCat and $\mathsf{dgCat}'_{\mathrm{Mor}}$ agree we see that $f \square g$ is a cofibration if f and g are, by the proof of Theorem 6.2.

We now have to check that $f \square g$ is a Morita equivalence if f or a g is a (Morita) acyclic cofibration. It would suffice to show that all generating acyclic cofibrations in $\mathsf{dgCat}'_{\mathrm{Mor}}$ lie in the image of Ω. We expect this to be true, but to avoid excessive computations we use another argument.

By imitating the proof of Theorem 5.4, it suffices to show that the action $\tilde{\otimes}$ preserves Morita equivalences in $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$ and $\mathsf{dgCat}'_{\mathrm{Mor}}$. Morita equivalences will be denoted by $\simeq$ in the following.

The proof uses the simple observation that for any dg category D we have a natural quasi-equivalence $\Omega\mathrm{B}D \xrightarrow{\sim} D$, thus by Theorem 6.2 we have $C \mathbin{\tilde{\otimes}} \Omega\mathrm{B}D \simeq C \mathbin{\tilde{\otimes}} D$ for any $C \in \mathsf{ptdCoa}^*$.

Let $C \xrightarrow{\sim} C'$ be an acyclic cofibration in $\mathsf{ptdCoa}^*_{\mathrm{Mor}}$. As Ω is quasi-strong monoidal and $\otimes$ preserves Morita equivalences, see Lemma 6.14, we have the following zig-zag of

Morita equivalences:

$$C \tilde{\otimes} D \xleftarrow{\sim} C \tilde{\otimes} \Omega \mathrm{B} D \xleftarrow{\sim} \Omega C \otimes \Omega \mathrm{B} D \xrightarrow{\sim} \Omega C' \otimes \Omega \mathrm{B} D \xrightarrow{\sim} C' \otimes \Omega \mathrm{B} D \xrightarrow{\sim} C' \tilde{\otimes} D'$$

and thus $C \tilde{\otimes} D \simeq C' \tilde{\otimes} D$.

Similarly, let $D \to D'$ be a Morita equivalence. As $\Omega \mathrm{B} D \simeq D$ it follows from 2-out-of-3 that $\Omega \mathrm{B} D \simeq \Omega \mathrm{B} D'$. Putting this together with the previous observations we get the following Morita equivalences

$$C \tilde{\otimes} D \xleftarrow{\sim} C \tilde{\otimes} \Omega \mathrm{B} D \xleftarrow{\sim} \Omega C \otimes \Omega \mathrm{B} D \xrightarrow{\sim} \Omega C \otimes \Omega \mathrm{B} D' \xrightarrow{\sim} C \tilde{\otimes} \Omega \mathrm{B} D' \xrightarrow{\sim} C \tilde{\otimes} D'.$$

Thus $C \tilde{\otimes} D \simeq C \tilde{\otimes} D'$ and the proposition follows. ■

Corollary 6.16. *The internal hom in* $\mathsf{Ho}(\mathsf{dgCat}'_{\mathrm{Mor}})$ *is computed by*

$$R\,\underline{\mathrm{Hom}}(D, D') \simeq \overline{\mathsf{MC}}\{BD, D'\}$$

whenever D' is Morita fibrant.

Proof. As for Corollary 6.5, except that in order to ensure $\mathrm{B}D'$ is fibrant we need to assume D' is fibrant in $\mathsf{dgCat}'_{\mathrm{Mor}}$. ■

We may use this to obtain an alternative description of the internal hom between derived categories of perfect complexes on schemes with very mild assumptions.

Example 6.17. Assume we are given two schemes and their dg categories $\mathsf{Perf}(X)$ and $\mathsf{Perf}(Y)$ enhancing the derived category of perfect complexes. Assume that X is quasi-separated and quasi-compact, so that it has a compact generator Q with endomorphism dg algebra E such that $\mathsf{Perf}(X)$ is Morita equivalent to E, see [3].

Then $R\,\underline{\mathrm{Hom}}(\mathsf{Perf}(X), \mathsf{Perf}(Y))$ (computed with respect to the DK or Morita model structure) has as objects pairs $(\mathcal{F}, \phi)$ where $\mathcal{F} \in \mathsf{Perf}(Y)$ and ϕ is an MC element in $\underline{\mathrm{Hom}}(\mathrm{B}E, R\,\mathrm{End}(\mathcal{F}))$. Morphisms from $(\mathcal{F}, \phi)$ to $(\mathcal{G}, \psi)$ are given by two-sided twistings: $\mathrm{Hom}(\mathrm{B}E, R\,\mathrm{Hom}(\mathcal{F}, \mathcal{G}))^{[\psi,\phi]}$.

Equivalently ϕ can be viewed as an A_∞-morphism between the dg algebras E and $R\,\mathrm{End}(\mathcal{F})$, and morphisms are A_∞-natural transformations.

This follows from Corollary 6.16 using that $\mathsf{Perf}(X)$ is Morita equivalent to E and that $\mathsf{Perf}(Y)$ is Morita fibrant.

7. Hochschild cohomology

One key application of the construction of the derived internal hom of dg categories is the computation of homotopy groups in terms of Hochschild cohomology by Toën [24]. We recreate this computation in our setting in somewhat greater generality.

Definition 7.1. The Hochschild cochain complex of a dg category D with coefficients in a bimodule $M\colon D \otimes D^{\mathrm{op}} \to \mathsf{dgVect}$ is defined as

$$C^*_{\mathsf{HH}}(D, M) := R\,\mathrm{Hom}_{D \otimes D^{\mathrm{op}}}(D, M).$$

It is well known that $C^*_{\mathsf{HH}}(D, M)$ may be computed by the Hochschild complex of the dg category D with coefficients in M. Writing $D(d_0, d_1)$ for $\mathrm{Hom}_D(d_0, d_1)$ for better legibility this complex is

$$\prod M(d_0, d_1) \to \prod \mathrm{Hom}\big(D(d_0, d_1), M(d_0, d_1)\big) \\ \to \prod \mathrm{Hom}\big(D(d_1, d_2) \otimes D(d_0, d_1), M(d_0, d_2)\big) \to \cdots$$

with a differential induced by the internal differentials, composition in D and the action of D on M. One may equivalently compute with the reduced Hochschild complex replacing D by $\bar{D}$ everywhere.

We specialize now to the case where M is given by another dg category D' with a pair of functors $F, G\colon D \to D'$, i.e. we consider the bimodule ${}_F D'_G$, or by abuse of notation just D', sending $d_1 \otimes d_0$ to $\mathrm{Hom}_{D'}(F(d_0), G(d_1))$.

Any dg functor $F\colon D \to D'$ gives rise to an object in $R\,\underline{\mathrm{Hom}}(D, D')$ in a natural way. We use this to state the following lemma.

Lemma 7.2. *Let $F, G\colon D \to D'$ be functors of dg categories, then Hochschild cohomology of D with coefficients in D' is given by*

$$C^*_{\mathsf{HH}}(D, {}_F D'_G) \cong \mathrm{Hom}_{R\,\underline{\mathrm{Hom}}(D,D')}(F, G).$$

Proof. By definition $\underline{\mathrm{Hom}}_{R\,\mathrm{Hom}(D,D')}(F, G) \cong \mathrm{Hom}_{\overline{\mathsf{MC}}\{\mathrm{B}D,D'\}}(F, G)$ is the twisted hom space $\mathrm{Hom}_{\underline{\mathrm{Hom}}(\mathrm{B}D,D')}(F, G)^{[\xi_F,\xi_G]}$ where ξ_F, ξ_G are the MC elements in the convolution category corresponding to $F, G\colon D \to D'$. Unravelling definitions, we recognize this as the reduced Hochschild complex which we may write as

$$D' \to \mathrm{Hom}_{D'_0}(\bar{D}, D') \to \mathrm{Hom}_{D'_0}(\bar{D} \square_{D_0} \bar{D}, D') \to \cdots$$

with differential induced by internal differentials, the composition in D and action of D on D' (the latter corresponding to the twist by ξ_F and ξ_G). ■

If $F = G = \mathrm{id}_D$ this specializes to the well-known equivalence

$$C^*_{\mathsf{HH}}(D) \cong \mathrm{End}_{R\,\underline{\mathrm{Hom}}(D,D)}(\mathrm{id}).$$

Remark 7.3. One can also compare $\mathrm{Hom}_{R\,\underline{\mathrm{Hom}}(D,D')}(F, G)$ to $R\,\mathrm{Hom}_{D\otimes D^{\mathrm{op}}}(D, {}_F D'_G)$ without reference to any resolutions, as follows.

We use that the functor $F\colon D \to D'$ gives rise to a $D \otimes D'^{\mathrm{op}}$-bimodule D'_F sending $d, d' \mapsto \mathrm{Hom}_{D'}(d', F(d))$. The functor $\widehat{F} = \mathrm{id} \otimes F^{\mathrm{op}}\colon D \otimes D^{\mathrm{op}} \to D \otimes D'^{\mathrm{op}}$ induces an adjunction on module categories. Considering the $D \otimes D^{\mathrm{op}}$-module D and the $D \otimes D'^{\mathrm{op}}$-module $D'_G\colon d \otimes d' \mapsto \mathrm{Hom}(d', d)$ we have

$$R\,\mathrm{Hom}_{D\otimes D^{\mathrm{op}}}\big(D, \widehat{F}^*(D'_G)\big) \cong R\,\mathrm{Hom}_{D\otimes D'^{\mathrm{op}}}(\widehat{F}_! D, D'_G)$$

By definition $\widehat{F}^*({}_G D')$ is ${}_F D'_G$ and moreover one may check $\widehat{F}_! D = D'_F$, for example by rewriting $\widehat{F}_!$, which by definition is a left Kan extension, as a coend and computing it.

The proof is completed by recalling from [24] that the functor category $R\,\underline{\mathrm{Hom}}(D, D')$ is weakly equivalent to a full subcategory of (fibrant cofibrant) $D \otimes D'^{\mathrm{op}}$-modules, and in particular the map $F \mapsto D'_F$ induces weak equivalences of hom spaces. Thus we have

$$\mathrm{Hom}_{R\,\underline{\mathrm{Hom}}(D,D')}(F, G) \simeq R\,\mathrm{Hom}_{D\otimes D'^{\mathrm{op}}}(D'_F, D'_G) \simeq R\,\mathrm{Hom}_{D,D}(D, {}_F D'_G).$$

This is just a categorical version of the classical formula $\mathsf{HH}^*(A, B) \cong \mathrm{Ext}^*_{A\otimes B^{\mathrm{op}}}(B, B)$ for algebras A and B.

We now specialize to $F = G$ and consider the bimodule $D' = {}_G D'_G$.

Theorem 7.4. *Let $G: D \to D'$ be a functor of dg categories. We then have* $\mathsf{HH}^0(D, D')^\times \cong \pi_1(\mathrm{Map}(D, D'), G)$ *and* $\mathsf{HH}^i(D, D') \cong \pi_{1-i}(\mathrm{Map}(D, D'), G)$ *for* $i < 0$.

Proof. By Theorem 6.9, the space $\mathrm{Map}(D, D')$ is weakly equivalent to the core of $\mathrm{N_{dg}}\, R\,\underline{\mathrm{Hom}}(D, D')$.

It now follows, cf. [18, Remark 1.3.1.12], that the image of $\mathrm{Hom}_{R\,\underline{\mathrm{Hom}}(D,D')}(G, G)$ under Dold–Kan is the infinity categorical mapping space from G to G in $F\mathrm{B}\overline{\mathsf{MC}}\{\mathrm{B}D, D'\}$. Taking homotopy groups and using Lemma 7.2 we have

$$\mathsf{HH}^i(D, D') = \pi_{-i}(\Omega\,\mathrm{Map}(D, D'), G)$$

for $i > 1$ as the higher homotopy groups of mapping spaces are unaffected by taking the core. Finally, $\mathsf{HH}^0(D, G)$ is given by $\pi_0\,\mathrm{Map}_{F\,\Omega\overline{\mathsf{MC}}\{BD,D\}'}(G, G)$ and taking units on both sides we obtain $\mathsf{HH}^0(D, D')^* \cong \pi_0\,\mathrm{Map}_{\mathrm{Map}(D,D')}(G, G) = \pi_1(\mathrm{Map}(D, D'), G)$. ∎

Corollary 7.5 ([24, Corollary 8.2]). *We have* $\mathsf{HH}^i(D) \cong \pi_{1-i}(\mathrm{Map}(D, D), \mathrm{id})$ *for* $i < 0$ *and* $\mathsf{HH}^0(D)^\times \cong \pi_1(\mathrm{Map}(D, D), \mathrm{id}_D)$.

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