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On n-ADC integral quadratic lattices over algebraic number fields

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Abstract. In the paper, we extend the ADC property to the representation of quadratic lattices by quadratic lattices, which we define as n-ADC-ness. We explore the relationship between n-ADC-ness, n-regularity, and n-universality for integral quadratic lattices. Also, for $n \geq 2$, we give necessary and sufficient conditions for an integral quadratic lattice over arbitrary non-archimedean local fields to be n-ADC. Moreover, we show that over any algebraic number field F, an integral $\mathcal{O}_F$-lattice with rank $n+1$ is n-ADC if and only if it is $\mathcal{O}_F$-maximal of class number one.

1. Introduction

The problem of representing quadratic forms by quadratic forms was first studied by Mordell. In [27], he proved that the sum of five squares represents all binary quadratic forms. Building on this work, B. M. Kim, M.-H. Kim, and S. Raghavan [22] defined a positive definite classic integral quadratic form (i.e., a form with even cross terms) to be n-universal if it represents all n-ary classic integral quadratic forms. When $n = 1$, this notion agrees with the concept of universal quadratic forms, which dates back to Lagrange's four-square theorem and has been extensively studied by mathematicians such as Ramanujan, Dickson and so on. Among the most famous are the 15-theorem by Conway and Schneeberger for classic integral quadratic forms and the 290-theorem by Bhargava and Hanke for quadratic forms with integer coefficients. Similar results have also been shown for n-universal quadratic forms in [8, 21]. Another important topic is the study of regular quadratic forms, which represent all integers represented by their genus. This concept was first introduced and systematically studied by Dickson in [11]. Since then, a significant amount of research has been devoted to classifying them in the ternary case (cf. [18, 20, 25, 29]). Similarly, Earnest [12] introduced n-regular quadratic forms and showed that there are only finitely many primitive quaternary 2-regular quadratic forms up to equivalence, which were partly classified by Oh [28]. For $n \geq 2$, Chan and Oh [7] extended Earnest's result to $(n+2)$-ary (resp. $(n+3)$-ary) n-regular quadratic forms.

A classical theorem, due to Aubry, Davenport and Cassels, states: if $Q(x)$ is a positive definite classic n-ary quadratic form such that for all $x \in \mathbb{Q}^n$ there exists $y \in \mathbb{Z}^n$ such that $Q(x-y) < 1$, then $Q(x)$ satisfies the property: for all $c \in \mathbb{Z}$, if the equation

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$Q(x) = c$ has a solution in $\mathbb{Q}$, then it has a solution in $\mathbb{Z}$. Based on this result, Clark [9] introduced the concept of ADC quadratic forms, which satisfy the property "solvable over rationals implies solvable over integers". In general, he defined ADC and Euclidean quadratic forms over normed ring, and investigated their relationship. In [10], Clark and Jagy further determined all ADC forms in non-dyadic local fields and obtained some partial results in 2-adic local fields. Additionally, they completely enumerated all n-ary ADC integral forms for $1 \le n \le 4$ and all Euclidean integral forms.

As for universality and regularity, studying higher-dimensional analogues of the ADC property is a natural generalization, which motivates the introduction of n-ADC lattices (defined in Definitions 1.1 and 1.2) in this paper. Also, we find that such notion plays an important role between n-universality and n-regularity (Theorems 1.3 and 1.4 (iii)). We will investigate n-ADC-ness from local fields to global fields. Precisely, we characterize n-ADC lattices of rank $\ge n$ over arbitrary non-archimedean local fields for $n \ge 2$ (Theorems 1.4 (i)–(ii), 1.5 and 1.9), and give a counting formula (Theorem 1.10). The case $n = 1$ requires a different approach than that for $n \ge 3$ odd, as discussed in Section 7, and it will be treated in a future paper. Due to the complexity of Jordan splittings, we will use a non-classical but effective theory, developed by Beli [2–4], to treat higher dimensional representations of quadratic lattices over general dyadic local fields (see [14,15] for recent progress). By virtue of these local classifications, we establish the equivalent condition on n-ADC lattices of rank $n + 1$ over algebraic number fields (Theorem 1.7). Based on the work of previous researchers [13,23,28], we determine all positive definite n-ADC lattices of rank $n + 1$ over totally real number fields (Corollary 1.8), and partially classify 2-ADC lattices over $\mathbb{Q}$ (Theorem 1.11).

First of all, we briefly introduce the arithmetic theory of quadratic forms. Any unexplained notations or definitions can be found in [31]. For short, by local fields, we always mean non-archimedean local fields (cf. [31, §32:1 Definition]).

General settings. Let F be an algebraic number field or a local field with char $F \neq 2$, $\mathcal{O}_F$ the ring of integers of F and $\mathcal{O}_F^{\times}$ the group of units. Let V be a non-degenerate quadratic space over F together with the symmetric bilinear form $B\colon V \times V \to F$ and set $Q(x) := B(x, x)$ for all $x \in V$. We call L an *$\mathcal{O}_F$-lattice* in V if it is a finitely generated $\mathcal{O}_F$-submodule of V, and say that L is on V if $V = FL$, i.e., V is spanned by L over F. For an $\mathcal{O}_F$-lattice L, we denote by $\mathfrak{s}(L)$ (resp. $\mathfrak{n}(L)$, $\mathfrak{v}(L)$) the scale (resp. the norm, the volume) of L as usual. We call L *integral* if $\mathfrak{n}(L) \subseteq \mathcal{O}_F$. For a non-zero fractional ideal $\mathfrak{a}$ in F, we also call L *$\mathfrak{a}$-maximal* if $\mathfrak{n}(L) \subseteq \mathfrak{a}$ and there is no $\mathcal{O}_F$-lattice L' on FL with $\mathfrak{n}(L') \subseteq \mathfrak{a}$ such that $L \subsetneq L'$. We denote by $\mathcal{L}_{F,n}$ the set of all integral $\mathcal{O}_F$-lattices of rank n and by $\mathcal{M}_n$ the set of all $\mathcal{O}_F$-maximal lattices of rank n. When F is an algebraic number field, we also denote by Ω_F (resp. ∞_F) the set of all primes (resp. all archimedean primes) of F.

Local settings. When F is a local field, write $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}_F$, $\pi \in \mathfrak{p}$ for a uniformizer and $N\mathfrak{p}$ for the number of elements in the residue class field of F. Set $\mathfrak{p}^0 = \mathcal{O}_F$ for convention. For $c \in F^{\times}$, let $c = \varepsilon\pi^k$ with $\varepsilon \in \mathcal{O}_F^{\times}$ and $k \in \mathbb{Z}$. We denote by

$\operatorname{ord}(c) = k$ the *order* of c and formally put $\operatorname{ord}(0) = \infty$. Put $e := \operatorname{ord}(2)$. For a fractional or zero ideal $\mathfrak{c}$ of F, we put $\operatorname{ord}(\mathfrak{c}) = \min\{\operatorname{ord}(c) \mid c \in \mathfrak{c}\}$. We fix $\Delta \in \mathcal{O}_F^\times$ such that $F(\sqrt{\Delta})/F$ is quadratic unramified. If F is non-dyadic, then Δ is an arbitrary non-square unit; if F is dyadic, then Δ is a non-square unit of the form $\Delta = 1 - 4\rho$, with $\rho \in \mathcal{O}_F^\times$.

If F is dyadic, we define the *quadratic defect* of c by $\mathfrak{d}(c) := \bigcap_{x \in F}(c - x^2)\mathcal{O}_F$ and the *order of relative quadratic defect* by the map d from $F^\times/F^{\times 2}$ to $\mathbb{N} \cup \{\infty\}$: $d(c) := \operatorname{ord}(c^{-1}\mathfrak{d}(c))$. Recall some properties of the map d:

(i) The image of d is $\{0, 1, 3, \dots, 2e-1, 2e, \infty\}$.

(ii) For $c \in F^\times$, $d(c) = 0$ if and only if $\operatorname{ord}(c)$ is odd, $d(c) = 2e$ if and only if $c \in \Delta F^{\times 2}$, and $d(c) = \infty$ if and only if $c \in F^{\times 2}$.

(iii) The domination principle: $d(ab) \geq \min\{d(a), d(b)\}$ for all $a, b \in F^\times$.

Also, we denote by $\mathcal{U}$ a complete system of representatives of $\mathcal{O}_F^\times/\mathcal{O}_F^{\times 2}$ such that $d(\delta) = \operatorname{ord}(\delta - 1)$ for all $\delta \in \mathcal{U}$, and by $\mathcal{V} := \mathcal{U} \cup \pi\mathcal{U} = \{\delta, \pi\delta \mid \delta \in \mathcal{U}\}$ a set of representatives of $F^\times/F^{\times 2}$. If F is non-dyadic, we put $\mathcal{U} = \{1, \Delta\}$ and $\mathcal{V} = \{1, \Delta, \pi, \Delta\pi\}$.

We write $V \cong [a_1, \dots, a_n]$ (resp. $L \cong \langle a_1, \dots, a_n\rangle$) if $V = Fx_1 \perp \cdots \perp Fx_n$ (resp. $L = \mathcal{O}_F x_1 \perp \cdots \perp \mathcal{O}_F x_n$) with $Q(x_i) = a_i$. For $\gamma \in F^\times$ and $\xi, \mu \in F$, we denote by $\gamma A(\xi, \mu)$ the binary $\mathcal{O}_F$-lattice associated with the Gram matrix $\gamma\left(\begin{smallmatrix} \xi & 1 \\ 1 & \mu \end{smallmatrix}\right)$. Write $\mathbf{H} = 2^{-1}A(0, 0)$ and $\mathbf{A} = 2^{-1}A(2, 2\rho)$. When F is non-dyadic, we have $\mathbf{H} = \langle 1, -1\rangle$ and $\mathbf{A} = \langle 1, -\Delta\rangle$. Let $\mathbb{H}$ denote the usual hyperbolic plane. Clearly, $\mathbb{H} = F\mathbf{H}$. We further denote by $\mathbf{H}^k$ (resp. $\mathbb{H}^k$) the orthogonal sum of k copies of $\mathbf{H}$ (resp. $\mathbb{H}$) for any positive integer k.

If instead of a given local field F we talk about the localization $F_\mathfrak{p}$ of an algebraic number field F at the finite prime $\mathfrak{p}$, then we will add the subscript $\mathfrak{p}$ to the notations π, ord, e, $\mathfrak{d}$, d, $\mathcal{U}$ and $\mathcal{V}$.

Definition 1.1. Let n be a positive integer. Let M be an integral $\mathcal{O}_F$-lattice over a local field F. Then

(i) M is called n-universal if it represents all lattices N in $\mathcal{L}_{F,n}$.

(ii) M is called n-ADC if it represents every lattice N in $\mathcal{L}_{F,n}$ for which FM represents FN.

In the rest of this section, we assume that F is an algebraic number field, V is a quadratic space over F, and M is an integral $\mathcal{O}_F$-lattice on V. For $\mathfrak{p} \in \Omega_F$, let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$. Then write $M_\mathfrak{p} := \mathcal{O}_{F_\mathfrak{p}} \otimes M$ when $\mathfrak{p} \in \Omega_F \backslash \infty_F$, and set $M_\mathfrak{p} := F_\mathfrak{p} \otimes M = V_\mathfrak{p}$ for convention when $\mathfrak{p} \in \infty_F$. Thus $M_\mathfrak{p}$ is always n-ADC for $\mathfrak{p} \in \infty_F$. Then we say that M is *locally n-ADC* (resp. *locally n-universal*) if $M_\mathfrak{p}$ is n-ADC (resp. n-universal) for all $\mathfrak{p} \in \Omega_F \backslash \infty_F$.

Definition 1.2. Let n be a positive integer. Then

(i) M is called globally n-universal, or simply n-universal, if it represents all lattices N in $\mathcal{L}_{F,n}$ with compatible signatures, i.e., with $N_\mathfrak{p} \rightarrowtail M_\mathfrak{p}$ at all real primes $\mathfrak{p} \in \infty_F$.

(ii) M is called globally n-ADC, or simply n-ADC, if it represents every lattice N in $\mathcal{L}_{F,n}$ for which FM represents FN.

Recall that an $\mathcal{O}_F$-lattice M (that may not be integral) is called *n-regular* if it represents every lattice N in $\mathcal{L}_{F,n}$ for which $M_\mathfrak{p}$ represents $N_\mathfrak{p}$ for each $\mathfrak{p} \in \Omega_F$. The n-ADC property can be viewed as a transition between n-universality and n-regularity. More specifically, an $\mathcal{O}_F$-lattice that is n-universal must be n-ADC from definition, and an n-ADC $\mathcal{O}_F$-lattice is n-regular. In fact, we have the following equivalent condition for n-ADC $\mathcal{O}_F$-lattices, which is a generalization of [9, Theorem 25] with $R = \mathcal{O}_F$.

Theorem 1.3. *Let n be a positive integer. Then M is globally n-ADC if and only if it is locally n-ADC and n-regular.*

Theorem 1.4. *Suppose* $\operatorname{rank} M \geq n + 3 \geq 4$. *Let $\mathfrak{p} \in \Omega_F \backslash \infty_F$. Then*

(i) *$M_\mathfrak{p}$ is n-ADC if and only if it is n-universal.*

(ii) *M is locally n-ADC if and only if it is locally n-universal.*

(iii) *M is globally n-ADC if and only if it is globally n-universal.*

For $n \geq 1$, all n-universal lattices over non-dyadic/dyadic local fields have been completely determined in [6,15,16,33]. Hence, from Theorems 1.3 and 1.4 (i), determining the n-regularity for a given $\mathcal{O}_F$-lattice M is crucial for its n-ADC-ness when $\operatorname{rank} M \geq n + 3$. Although it was shown in [16, Theorem 1.1 (1)] that local-global principle holds for indefinite n-universality[1] with $n \geq 3$, it is difficult to verify n-regularity of a quadratic lattice in general for definite cases.

Theorem 1.5. *Suppose* $\operatorname{rank} M = n \geq 2$ *or* $\operatorname{rank} M = n + 1 \geq 3$. *Let $\mathfrak{p} \in \Omega_F \backslash \infty_F$. Then*

(i) *$M_\mathfrak{p}$ is n-ADC if and only if it is $\mathcal{O}_{F_\mathfrak{p}}$-maximal.*

(ii) *M is locally n-ADC if and only if it is $\mathcal{O}_F$-maximal.*

Remark 1.6. For $\mathcal{O}_{F_\mathfrak{p}}$-maximal lattices, we note that

(i) All $\mathcal{O}_{F_\mathfrak{p}}$-maximal lattices have been explicitly listed in [15, 16]. See Lemmas 4.7 (i) and 4.9 (i) (or [15, Proposition 3.7] described in terms of minimal norm splittings).

(ii) Theorem 19 of [9] states that $M_\mathfrak{p}$ is $\mathcal{O}_{F_\mathfrak{p}}$-maximal if and only if it is Euclidean with respect to the canonical norm (see [9, §4.2] for Euclidean property over local fields). Therefore, in Theorems 1.5 and 1.9, the term "$\mathcal{O}_{F_\mathfrak{p}}$-maximal" can be smoothly replaced with "Euclidean".

For $n \geq 2$, the class number of an n-regular $\mathcal{O}_F$-lattice M may not be equal to one in general, but it is exactly one when the rank is $n + 1$, as proved by Kitaoka in [24, Corollary 6.4.1] for $F = \mathbb{Q}$ and M is positive definite, which was extended by Meyer

[1] In the indefinite case, the notion of n-universal defined in this paper does not coincide with that of indefinite n-universal introduced in [16, Definition 1.4 (3)] for $n \geq 2$.

[26, Corollary 5.3] to the case when F is totally real and M is definite. This is also true for indefinite cases (Corollary 8.3). Based on these and Theorem 1.5, we provide more explicit equivalent conditions on n-ADC $\mathcal{O}_F$-lattices with rank $n+1$.

Theorem 1.7. *If* rank $M = n+1 \geq 3$, *then M is n-ADC if and only if it is $\mathcal{O}_F$-maximal of class number one.*

When F is totally real, all positive definite $\mathcal{O}_F$-maximal lattices with rank ≥ 3 of class number one were enumerated by Hanke [13] for $F = \mathbb{Q}$ (115 in total) and by Kirschmer [23] for $F \neq \mathbb{Q}$ (471 in total), respectively. Thus, from Theorem 1.7, we have the following finiteness result.

Corollary 1.8. *Up to isometry, there are* 586 *positive definite n-ADC integral $\mathcal{O}_F$-lattices of rank $n+1 \geq 3$ in total, when F varies through all totally real number fields.*

Theorem 1.9. *Suppose* rank $M = n+2 \geq 4$. *Let $\mathfrak{p} \in \Omega_F \backslash \infty_F$.*

(i) *If $\mathfrak{p}$ is non-dyadic, then $M_{\mathfrak{p}}$ is n-ADC if and only if it is $\mathcal{O}_{F_{\mathfrak{p}}}$-maximal.*

(ii) *If $\mathfrak{p}$ is dyadic and n is even, then $M_{\mathfrak{p}}$ is n-ADC if and only if it is either $\mathcal{O}_{F_{\mathfrak{p}}}$-maximal or isometric to the non $\mathcal{O}_{F_{\mathfrak{p}}}$-maximal lattice*

$$\mathbf{H} \perp 2^{-1}\pi_{\mathfrak{p}} A(2\pi_{\mathfrak{p}}^{-1}, 2\rho_{\mathfrak{p}}\pi_{\mathfrak{p}}).$$

(iii) *If $\mathfrak{p}$ is dyadic and n is odd, then $M_{\mathfrak{p}}$ is n-ADC if and only if it is either $\mathcal{O}_{F_{\mathfrak{p}}}$-maximal or isometric to*

$$\mathbf{H}^{\frac{n-1}{2}} \perp \pi_{\mathfrak{p}}^{-l_{\mathfrak{p}}} A\big(\pi_{\mathfrak{p}}^{l_{\mathfrak{p}}}, -(\delta_{\mathfrak{p}}-1)\pi_{\mathfrak{p}}^{-l_{\mathfrak{p}}}\big) \perp \langle \varepsilon_{\mathfrak{p}}\pi_{\mathfrak{p}}^{k_{\mathfrak{p}}}\rangle$$

or

$$\mathbf{H}^{\frac{n-1}{2}} \perp \delta_{\mathfrak{p}}^{\#}\pi_{\mathfrak{p}}^{-l_{\mathfrak{p}}} A\big(\pi_{\mathfrak{p}}^{l_{\mathfrak{p}}}, -(\delta_{\mathfrak{p}}-1)\pi_{\mathfrak{p}}^{-l_{\mathfrak{p}}}\big) \perp \langle \varepsilon_{\mathfrak{p}}\pi_{\mathfrak{p}}^{k_{\mathfrak{p}}}\rangle,$$

with

$$\delta_{\mathfrak{p}} \in \mathcal{U}_{\mathfrak{p}}\backslash\{1, \Delta_{\mathfrak{p}}\},\ 2l_{\mathfrak{p}} = d_{\mathfrak{p}}(\delta_{\mathfrak{p}}) - 1 \leq 2e_{\mathfrak{p}} - 2,\ \varepsilon_{\mathfrak{p}} \in \mathcal{U}_{\mathfrak{p}} \text{ and } k_{\mathfrak{p}} \in \{0, 1\},$$

where $\delta_{\mathfrak{p}}^{\#} = 1 + 4\rho_{\mathfrak{p}}(\delta_{\mathfrak{p}}-1)^{-1}$.
Moreover, if $M_{\mathfrak{p}}$ is simultaneously $\mathcal{O}_{F_{\mathfrak{p}}}$-maximal and has the described orthogonal splitting, then it is isometric to

$$\mathbf{H}^{\frac{n-1}{2}} \perp 2^{-1}\pi_{\mathfrak{p}} A(2, 2\rho_{\mathfrak{p}}) \perp \langle \Delta_{\mathfrak{p}}\varepsilon_{\mathfrak{p}}\rangle,$$

with $\varepsilon_{\mathfrak{p}} \in \mathcal{U}_{\mathfrak{p}}$.

If $m \geq n+3$, then from Theorem 1.4, the notions of n-ADC-ness and n-universality coincide. Because the n-universality was treated in [15], in this paper we deal with the remaining cases, with $n \leq m \leq n+2$. In these cases the number of n-ADC lattices is finite and it can be calculated as follows.

Theorem 1.10. *Let $n \geq 2$. Denote by $B(m,n)$ the number of n-ADC $\mathcal{O}_F$-lattices with rank $m \in \{n, n+1, n+2\}$ over a local field F. Then $B(m,n)$ is given by*

$$\begin{cases} 8(N\mathfrak{p})^e - 1 + 0 & \text{if } m = n = 2, \\ 8(N\mathfrak{p})^e + 1 & \text{if } m = n+2 = 4 \text{ and } e \geq 1, \\ 8(N\mathfrak{p})^e + (8e-2)(N\mathfrak{p})^e & \text{if } m = n+2 \geq 5 \text{ with odd } n \text{ and } e \geq 1, \\ 8(N\mathfrak{p})^e + 0 & \text{otherwise}, \end{cases}$$

where the second addend counts the number of those lattices that are n-ADC, but not $\mathcal{O}_F$-maximal.

From Theorem 1.3, one can determine whether an n-regular $\mathcal{O}_F$-lattice with rank $n+2$ is n-ADC by virtue of Theorem 1.9. In particular, we classify the case $\mathcal{O}_F = \mathbb{Z}$ and $n = 2$ based on Oh's classification for stable 2-regular quaternary $\mathbb{Z}$-lattices [28].

Theorem 1.11. *There are exactly 21 quaternary positive definite 2-ADC $\mathbb{Z}$-lattices up to isometry, which are enumerated in Table 2. Each 2-ADC $\mathbb{Z}$-lattice L_i in the table is obtained by scaling some lattice $\mathcal{L}_j$ in Table 1 by $1/2$.*

Moreover, all of the lattices have class number one, and all except for L_{10} are $\mathbb{Z}$-maximal.

Remark 1.12. All of the ternary 2-ADC lattices have been determined by Theorem 1.7. For the quinary case, Theorem 1.4 (iii) indicates that the 2-ADC property is equivalent to 2-universality. However, currently it is only known from [19, Theorem 2.4] that there are at most 55 quinary 2-universal $\mathbb{Z}$-lattices M with $2\mathfrak{s}(M) = \mathbb{Z}$, of which the 2-universality has not been completely confirmed yet.

The rest of the paper is organized as follows. We first prove Theorems 1.3 and 1.4 in Section 2. Then, we review Beli's BONGs theory of quadratic forms in Section 3. In Section 4, we study some basic notions including quadratic spaces and maximal lattices, and the related results in local fields. In Sections 5, 6 and 7, we establish equivalent conditions on n-ADC lattices in non-dyadic local fields, and in dyadic local fields for even and odd n, respectively. In the last section, we will prove our main results, including Theorems 1.5, 1.7, 1.9, 1.10 and 1.11.

Here and subsequently, all lattices under consideration are assumed to be integral.

2. Proof of Theorems 1.3 and 1.4

To show Theorems 1.3 and 1.4, we need some lemmas.

Lemma 2.1. *Suppose that F is an algebraic number field or a local field. Let M be an $\mathcal{O}_F$-lattice. Then M is n-ADC if and only if M represents every lattice N in $\mathcal{M}_n$ for which FM represents FN.*

Proof. Necessity is trivial. Suppose that FM represents FN. By [31, 82:18], there exists some lattice N' inside $\mathcal{M}_n$ on FN such that $N \subseteq N'$. Since FM represents $FN \cong FN'$, by the n-ADC-ness, M represents N', and therefore represents N. ∎

Lemma 2.2. *Suppose that F is an algebraic number field. Let V be a quadratic space over F and $\mathfrak{p} \in \Omega_F \backslash \infty_F$. Given a subspace $U(\mathfrak{p}) \subseteq V_\mathfrak{p}$, there exists a subspace $U \subseteq V$ such that $U_\mathfrak{p} \cong U(\mathfrak{p})$.*

Proof. We prove the statement by induction on $\dim U(\mathfrak{p})$. When $\dim U(\mathfrak{p}) = 1$, then $U(\mathfrak{p}) = F_\mathfrak{p} u(\mathfrak{p})$ for some $u(\mathfrak{p}) \in V_\mathfrak{p}$. Recall from [31, 63:1b Corollary, 21:1] that $F_\mathfrak{p}^{\times 2}$ is open in $F_\mathfrak{p}$ and V is dense in $V_\mathfrak{p}$. Then there exists $u \in V$ such that $Q(u) \in Q(u(\mathfrak{p}))F_\mathfrak{p}^{\times 2}$. Thus, $Q(u) = c^2 Q(u(\mathfrak{p})) = Q(cu(\mathfrak{p}))$ for some $c \in F_\mathfrak{p}^\times$. Take $U := Fu$. Then $U \subseteq V$ and $F_\mathfrak{p} U = F_\mathfrak{p} u = F_\mathfrak{p}(cu(\mathfrak{p})) \cong F_\mathfrak{p} u(\mathfrak{p}) = U(\mathfrak{p})$.

For $\dim U(\mathfrak{p}) > 1$, we may let $U(\mathfrak{p}) = W(\mathfrak{p}) \perp F_\mathfrak{p} u(\mathfrak{p})$. Then, by inductive assumption, there exists $W \subseteq V$ such that $W_\mathfrak{p} \cong W(\mathfrak{p})$. Since $U(\mathfrak{p})$ is non-degenerate, and so is $W(\mathfrak{p})$. Thus W is also non-degenerate. It follows that $V = W \perp W^\perp$, where $W^\perp := \{v \in V \mid B(v, W) = 0\}$. This yields $V_\mathfrak{p} = W_\mathfrak{p} \perp F_\mathfrak{p} W^\perp$. We also have $V_\mathfrak{p} = W(\mathfrak{p}) \perp W(\mathfrak{p})^\perp$. By Witt's cancellation theorem, $F_\mathfrak{p} W^\perp \cong W(\mathfrak{p})^\perp$. Thus one can find $u' \in W_\mathfrak{p}^\perp \cong W(\mathfrak{p})^\perp \cong F_\mathfrak{p} W^\perp$ such that $Q(u') = Q(u(\mathfrak{p}))$. By the one-dimensional case of the lemma, there exists $u \in W^\perp$ such that $F_\mathfrak{p} u \cong F_\mathfrak{p} u' \cong F_\mathfrak{p} u(\mathfrak{p})$. Now take $U = W \perp Fu$, as desired. ∎

Proof of Theorem 1.3. For sufficiency, suppose that FM represents FN for some $N \in \mathcal{L}_{F,n}$. By [31, 66:3 Theorem], $FM_\mathfrak{p}$ represents $FN_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_F$, so $M_\mathfrak{p}$ represents $N_\mathfrak{p}$ by the n-ADC-ness of $M_\mathfrak{p}$. Hence M represents N by the n-regularity of M.

For necessity, we will first prove that M is locally n-ADC, i.e., $M_\mathfrak{p}$ is n-ADC for each $\mathfrak{p} \in \Omega_F \backslash \infty_F$. By Lemma 2.1, it is sufficient to show that $M_\mathfrak{p}$ represents every $\mathcal{O}_{F_\mathfrak{p}}$-maximal lattice $N(\mathfrak{p})$ for which $FM_\mathfrak{p}$ represents $FN(\mathfrak{p})$.

We may assume $FN(\mathfrak{p}) \subseteq FM_\mathfrak{p}$. By Lemma 2.2, there exists a subspace $U \subseteq FM$ such that $U_\mathfrak{p} \cong FN(\mathfrak{p})$. Hence, by [31, 82:18], FM represents FL for some $\mathcal{O}_F$-maximal lattice L on U. So M represents L from the n-ADC-ness of M. Thus $M_\mathfrak{p}$ represents $L_\mathfrak{p}$. Note from [31, §82K] that $L_\mathfrak{p}$ is $\mathcal{O}_{F_\mathfrak{p}}$-maximal, so $L_\mathfrak{p} \cong N(\mathfrak{p})$ by [31, 91:2 Theorem]. Hence $M_\mathfrak{p}$ represents $N(\mathfrak{p})$, as desired.

To show the n-regularity of M, let $N \in \mathcal{L}_{F,n}$. Suppose that $M_\mathfrak{p}$ represents $N_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_F$. Then $FM_\mathfrak{p}$ represents $FN_\mathfrak{p}$ for all $\mathfrak{p} \in \Omega_F$. Hence, by [31, 66:3 Theorem], FM represents FN. So M represents N from the n-ADC-ness of M. ∎

Proof of Theorem 1.4. (i) Let $\mathfrak{p} \in \Omega_F \backslash \infty_F$. Since $\dim FM_\mathfrak{p} = \operatorname{rank} M_\mathfrak{p} \geq n + 3$, by [16, Theorem 2.3 (1)], $FM_\mathfrak{p}$ represents all n-dimensional quadratic spaces. Hence for every lattice $N_\mathfrak{p}$ in $\mathcal{L}_{F_\mathfrak{p},n}$, $FM_\mathfrak{p}$ represents $FN_\mathfrak{p}$, so $M_\mathfrak{p}$ represents $N_\mathfrak{p}$ by the n-ADC-ness of $M_\mathfrak{p}$, i.e., $M_\mathfrak{p}$ is n-universal, as desired.

(ii) This is clear from the definition and (i).

(iii) Note that M is locally n-ADC (resp. locally n-universal) if and only if it is globally n-ADC (resp. globally n-universal) when M is n-regular. Then we are done by Theorem 1.3 and (ii). ∎

3. Lattices in terms of BONGs

In this section, following Beli's work [1–6], we use bases of norm generators (abbr. BONGs) to describe the lattices in arbitrary dyadic local fields instead of Jordan splittings. Let us first review his BONGs theory and recent development [14, 15].

Unless otherwise stated, we always assume F to be dyadic, i.e., $e \geq 1$. We write $[h,k]^E$ (resp. $[h,k]^O$) for the set of all even (resp. odd) integers i such that $h \leq i \leq k$. For $c_i \in F^\times$, we also write $c_{i,j} = c_i \cdots c_j$ for short and put $c_{i,i-1} = 1$.

The vectors $x_1, \ldots, x_m$ of FM is called a *BONG* for M if $\mathfrak{n}(M) = Q(x_1)\mathcal{O}_F$ and $x_2, \ldots, x_m$ is a BONG for $\mathrm{pr}_{x_1^\perp} M$, and it is said to be *good* if $\mathrm{ord}(Q(x_i)) \leq \mathrm{ord}(Q(x_{i+2}))$ for $1 \leq i \leq m-2$. We denote by $M \cong \prec a_1, \ldots, a_m \succ$ if $x_1, \ldots, x_m$ forms a BONG for M with $Q(x_i) = a_i$.

Lemma 3.1 ([15, Lemma 2.2]). *Let $x_1, \ldots, x_m$ be pairwise orthogonal vectors in V with $Q(x_i) = a_i$ and $R_i = \mathrm{ord}(a_i)$. Then $x_1, \ldots, x_m$ forms a good BONG for some lattice is equivalent to the conditions*

$$R_i \leq R_{i+2} \quad \text{for all } 1 \leq i \leq m-2 \tag{3.1}$$

and

$$R_{i+1} - R_i + d(-a_i a_{i+1}) \geq 0 \quad \text{and} \quad R_{i+1} - R_i \geq -2e \quad \text{for all } 1 \leq i \leq m-1. \tag{3.2}$$

Corollary 3.2 ([15, Corollary 2.3]). *Suppose* $1 \leq i \leq m-1$.

(i) *If $R_{i+1} - R_i$ is odd, then $R_{i+1} - R_i$ must be positive.*

(ii) *If $R_{i+1} - R_i = -2e$, then $d(-a_i a_{i+1}) \geq 2e$ and $\prec a_i, a_{i+1} \succ \cong 2^{-1}\pi^{R_i} A(0,0)$ or $2^{-1}\pi^{R_i} A(2, 2\rho)$. Consequently, $[a_i, a_{i+1}] \cong \mathbb{H}$ or $\pi^{R_i}[1, -\Delta]$.*

Let $M \cong \prec a_1, \ldots, a_m \succ$ be an $\mathcal{O}_F$-lattice relative to some good BONG. Define the *R_i-invariants* $R_i(M) := \mathrm{ord}(a_i)$ for $1 \leq i \leq m$ and the *α_i-invariants*

$$\alpha_i(M) := \min\Big(\big\{(R_{i+1} - R_i)/2 + e\big\} \cup \big\{R_{i+1} - R_j + d(-a_j a_{j+1}) \mid 1 \leq j \leq i\big\} \\ \cup \big\{R_{j+1} - R_i + d(-a_j a_{j+1}) \mid i \leq j \leq m-1\big\}\Big)$$

for $1 \leq i \leq m-1$. Both are independent of the choice of the good BONG (cf. [2, Lemma 4.7], [4, §2]).

We give some useful properties for R_i and α_i without proof (cf. [15] or [6]).

Proposition 3.3. *Suppose* $1 \leq i \leq m-1$.

(i) *$R_{i+1} - R_i > 2e$ (resp. $= 2e$, $< 2e$) if and only if $\alpha_i > 2e$ (resp. $= 2e$, $< 2e$).*

(ii) *If $R_{i+1} - R_i \geq 2e$ or $R_{i+1} - R_i \in \{-2e, 2-2e, 2e-2\}$, then $\alpha_i = (R_{i+1} - R_i)/2 + e$.*

(iii) *If $R_{i+1} - R_i \leq 2e$, then $\alpha_i \geq R_{i+1} - R_i$. Also, the equality holds if and only if $R_{i+1} - R_i = 2e$ or $R_{i+1} - R_i$ is odd.*

Proposition 3.4. *Suppose* $1 \le i \le m-1$.

(i) *Either* $0 \le \alpha_i \le 2e$ *and* $\alpha_i \in \mathbb{Z}$, *or* $2e < \alpha_i < \infty$ *and* $2\alpha_i \in \mathbb{Z}$; *thus* $\alpha_i \ge 0$.

(ii) $\alpha_i = 0$ *if and only if* $R_{i+1} - R_i = -2e$.

(iii) $\alpha_i = 1$ *if and only if either* $R_{i+1} - R_i \in \{2-2e, 1\}$, *or* $R_{i+1} - R_i \in [4-2e, 0]^E$ *and* $d[-a_{i,i+1}] = R_i - R_{i+1} + 1$.

(iv) *If* $\alpha_i = 0$, *i.e.,* $R_{i+1} - R_i = -2e$, *then* $d[-a_{i,i+1}] \ge 2e$.

(v) *If* $\alpha_i = 1$, *then* $d[-a_{i,i+1}] \ge R_i - R_{i+1} + 1$. *Also, the equality holds if* $R_{i+1} - R_i \ne 2-2e$.

Proposition 3.5. *Suppose that* M *is integral.*

(i) *We have* $R_j \ge R_i \ge 0$ *for all odd integers* i, j *with* $j \ge i$ *and* $R_j \ge R_i \ge -2e$ *for all even integers* i, j *with* $j \ge i$.

(ii) *If* $R_j = 0$ *for some* $j \in [1, m]^O$, *then* $R_i = 0$ *for all* $i \in [1, j]^O$ *and* R_i *is even for all* $1 \le i \le j$.

(iii) *If* $R_j = -2e$ *for some* $j \in [1, m]^E$, *then for each* $i \in [1, j]^E$, *we have* $R_{i-1} = 0$, $R_i = -2e$ *and* $d(-a_{i-1}a_i) \ge d[-a_{i-1,i}] \ge 2e$. *Consequently,* $d[(-1)^{j/2}a_{1,j}] \ge 2e$.

(iv) *If* $R_j = -2e$ *for some* $j \in [1, m]^E$, *then* $[a_1, \dots, a_j] \cong \mathbb{H}^{j/2}$ *or* $\mathbb{H}^{(j-2)/2} \perp [1, -\Delta]$, *according as* $d((-1)^{j/2}a_{1,j}) = \infty$ *or* $2e$.

(v) *If* $R_j = -2e$ *and* R_{j+1} *is even for some* $j \in [1, m]^E$, *then* $[a_1, \dots, a_{j+1}] \cong \mathbb{H}^{j/2} \perp [\varepsilon]$ *for some* $\varepsilon \in \mathcal{O}_F^{\times}$ *with* $\varepsilon \in a_{j+1}F^{\times 2} \cup \Delta a_{j+1}F^{\times 2}$.

Let $N \cong \prec b_1, \dots, b_n \succ$ be another $\mathcal{O}_F$-lattice relative to some good BONG, $S_i = R_i(N)$ and $\beta_i = \alpha_i(N)$. For $0 \le i \le m$ and $0 \le j \le n$, we define

$$d[ca_{1,i}b_{1,j}] := \min\{d(ca_{1,i}b_{1,j}), \alpha_i, \beta_j\}, \quad c \in F^{\times},$$

where α_i is ignored if $i \in \{0, m\}$ and β_j is ignored if $j \in \{0, n\}$. In particular, if $M = N$ and $0 \le i-1 \le j \le m$, then we define

$$d[ca_{i,j}] := d[ca_{1,i-1}a_{1,j}] = \min\{d(ca_{i,j}), \alpha_{i-1}, \alpha_j\}.$$

Here we ignore α_{i-1} if $i \in \{1, m+1\}$ and we ignore α_j if $j \in \{0, m\}$. Recall that the invariants $d[ca_{1,i}b_{1,j}]$ satisfy the same domination principles as their $d(ca_{1,i}b_{1,j})$ correspondents. (See [6, §1.4].) With this notation, the invariant α_i can be neatly expressed as

$$\alpha_i = \min\{(R_{i+1} - R_i)/2 + e, R_{i+1} - R_i + d[-a_{i,i+1}]\} \tag{3.3}$$

(cf. [4, Corollary 2.5 (i)]). For any $1 \le i \le \min\{m-1, n\}$, we define

$$A_i = A_i(M, N) := \min\{(R_{i+1} - S_i)/2 + e, R_{i+1} - S_i + d[-a_{1,i+1}b_{1,i-1}], \\ R_{i+1} + R_{i+2} - S_{i-1} - S_i + d[a_{1,i+2}b_{1,i-2}]\},$$

where the term $R_{i+1} + R_{i+2} - S_{i-1} - S_i + d[a_{1,i+2}b_{1,i-2}]$ is ignored if $i = 1$ or $m-1$.

Our main tool is the representation theorem due to Beli [6, Theorem 1.2 and Remarks 1] (see [4, Theorem 4.5] and [5] for more details).

Theorem 3.6. *Suppose $n \le m$. Then $N \rightarrowtail M$ if and only if $FN \rightarrowtail FM$ and the following conditions hold:*

(i) *For any $1 \le i \le n$, we have either $R_i \le S_i$, or $1 < i < m$ and $R_i + R_{i+1} \le S_{i-1} + S_i$.*

(ii) *For any $1 \le i \le \min\{m-1, n\}$, we have $d[a_{1,i}b_{1,i}] \ge A_i$.*

(iii) *For any $1 < i \le \min\{m-1, n+1\}$, if*

$$R_{i+1} > S_{i-1} \quad \text{and} \quad d[-a_{1,i}b_{1,i-2}] + d[-a_{1,i+1}b_{1,i-1}] > 2e + S_{i-1} - R_{i+1},$$

then $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_i]$.

(iv) *For any $1 < i \le \min\{m-2, n+1\}$ such that $S_i \ge R_{i+2} > S_{i-1} + 2e \ge R_{i+1} + 2e$, we have $[b_1, \dots, b_{i-1}] \rightarrowtail [a_1, \dots, a_{i+1}]$. (If $i = n+1$, the condition $S_i \ge R_{i+2}$ is ignored.)*

4. Preliminaries over local fields

Unless otherwise stated, we always assume that F is a local field and n is a positive integer in this section. Clearly, $e = 0$ if F is non-dyadic and $e \ge 1$ if F is dyadic.

First, we extend [15, Definitions 3.4 and 3.6, and Proposition 3.5] to the non-dyadic case, including $n = 1$.

Definition 4.1. Let $n \ge 1$. For $c \in \mathcal{V}$, we define the n-dimensional quadratic space over F:

$$W_1^n(c) := \begin{cases} \mathbb{H}^{\frac{n-2}{2}} \perp [1, -c] & \text{if } n \text{ is even,} \\ \mathbb{H}^{\frac{n-1}{2}} \perp [c] & \text{if } n \text{ is odd,} \end{cases}$$

and define the n-dimensional quadratic space $W_2^n(c)$ with $\det W_2^n(c) = \det W_1^n(c)$ and $W_2^n(c) \not\cong W_1^n(c)$ if $n \ne 1$ and $(n, c) \ne (2, 1)$. We further define the $\mathcal{O}_F$-maximal lattice on $W_\nu^n(c)$ by $N_\nu^n(c)$ provided that $W_\nu^n(c)$ is defined.

From Definition 4.1, the notations $W_\nu^n(c)$ and $N_\nu^n(c)$ are defined only if

$$(n, \nu) \ne (1, 2) \quad \text{and} \quad (n, \nu, c) \ne (2, 2, 1), \tag{4.1}$$

which is in essential due to [31, 63:22 Theorem]. Hereafter, we always assume that the conditions (4.1) hold when a quadratic space $W_\nu^n(c)$ or an $\mathcal{O}_F$-maximal lattice $N_\nu^n(c)$ is discussed.

If F is dyadic, for $c \in \mathcal{V} \backslash \{1, \Delta\}$, we let $c\pi^{-\operatorname{ord}(c)} = \xi^2(1 + \mu\pi^{d(c)})$ with $\xi, \mu \in \mathcal{O}_F^\times$ when $\operatorname{ord}(c)$ is even. To describe $W_2^n(c)$ and $N_2^n(c)$ explicitly, we put

$$c^\# := \begin{cases} \Delta & \text{if } \operatorname{ord}(c) \text{ is odd,} \\ 1 + 4\rho\mu^{-1}\pi^{-d(c)} & \text{if } \operatorname{ord}(c) \text{ is even,} \end{cases} \tag{4.2}$$

as in [15, Definition 3.1]. From [15, Proposition 3.2], we also have the properties for $c^{\#}$:

$$d(c^{\#}) = 2e - d(c) \quad \text{and} \quad (c^{\#}, c)_{\mathfrak{p}} = -1. \tag{4.3}$$

Proposition 4.2. *Let $n \geq 1$, $\nu \in \{1, 2\}$ and $c \in \mathcal{V}$.*

(i) *The quadratic space $W_\nu^n(c)$ is given by the following table,*

n	c	$W_1^n(c)$	$W_2^n(c)$
Even	1	$\mathbb{H}^{\frac{n}{2}}$	$\mathbb{H}^{\frac{n-4}{2}} \perp [1, -\Delta, \pi, -\Delta\pi]$
	Δ	$\mathbb{H}^{\frac{n-2}{2}} \perp [1, -\Delta]$	$\mathbb{H}^{\frac{n-2}{2}} \perp [\pi, -\Delta\pi]$
	$\delta, \delta \in \mathcal{U} \backslash \{1, \Delta\}$	$\mathbb{H}^{\frac{n-2}{2}} \perp [1, -\delta]$	$\mathbb{H}^{\frac{n-2}{2}} \perp [\delta^{\#}, -\delta^{\#}\delta]$
	$\delta\pi, \delta \in \mathcal{U}$	$\mathbb{H}^{\frac{n-2}{2}} \perp [1, -\delta\pi]$	$\mathbb{H}^{\frac{n-2}{2}} \perp [\Delta, -\Delta\delta\pi]$
Odd	$\delta, \delta \in \mathcal{U}$	$\mathbb{H}^{\frac{n-1}{2}} \perp [\delta]$	$\mathbb{H}^{\frac{n-3}{2}} \perp [\pi, -\Delta\pi, \Delta\delta]$
	$\delta\pi, \delta \in \mathcal{U}$	$\mathbb{H}^{\frac{n-1}{2}} \perp [\delta\pi]$	$\mathbb{H}^{\frac{n-3}{2}} \perp [1, -\Delta, \Delta\delta\pi]$

where the third case is ignored when $e = 0$. (If $e = 0$, then $\mathcal{U} \backslash \{1, \Delta\} = \emptyset$.)

(ii) *Every n-dimensional quadratic space over F is isometric to one of the spaces in the table above.*

(iii) *For every n-dimensional quadratic space W, up to isometry, there is exactly one $(n+2)$-dimensional quadratic space V representing all n-dimensional quadratic spaces except for W. Precisely, if $W = W_\nu^n(c)$ with $(n, \nu) \neq (1, 2)$ and $(n, \nu, c) \neq (2, 2, 1)$, then $V = W_{3-\nu}^{n+2}(c)$.*

Remark 4.3. All n-dimensional quadratic spaces have been exhausted by the above table from Proposition 4.2 (ii). Also, on each space $W_\nu^n(c)$, by [31, 91:2 Theorem], there is exactly one $\mathcal{O}_F$-maximal lattice, up to isometry. Hence $\mathcal{M}_n$ consists of all the $\mathcal{O}_F$-maximal lattices $N_\nu^n(c)$ for $\nu \in \{1, 2\}$ and $c \in \mathcal{V}$ such that $(n, \nu) \neq (1, 2)$ and $(n, \nu, c) \neq (2, 2, 1)$. So, by Proposition 4.2 (i) and [31, 63:9], one can count the number of $\mathcal{O}_F$-maximal lattices with rank n:

$$|\mathcal{M}_n| = \begin{cases} 8(N\mathfrak{p})^e & \text{if } n \geq 3, \\ 8(N\mathfrak{p})^e - 1 & \text{if } n = 2, \\ 4(N\mathfrak{p})^e & \text{if } n = 1. \end{cases} \tag{4.4}$$

Next we show Lemma 4.4, which refines [31, 63:21 Theorem] slightly and provides an alternative proof for Proposition 4.2 (ii) and (iii), covering the case $n = 1$.

Lemma 4.4. *Let $n \geq 1$, $\nu, \nu' \in \{1, 2\}$ and $c, c' \in \mathcal{V}$.*

(i) *$W_{\nu'}^n(c')$ represents $W_\nu^n(c)$, i.e., $W_{\nu'}^n(c') \cong W_\nu^n(c)$ if and only if $(\nu', c') = (\nu, c)$.*

(ii) *$W_{\nu'}^{n+1}(c')$ represents $W_\nu^n(c)$ if and only if $(c', c)_{\mathfrak{p}} = (-1)^{\nu'+\nu}$.*

(iii) *$W_{\nu'}^{n+2}(c')$ represents $W_\nu^n(c)$ if and only if $c' \neq c$ or $(\nu', c') = (\nu, c)$.*

Proof. (i) This is clear from Definition 4.1.

(ii) Let $D = \det W_1^{n+1}(c') \det W_1^n(c)$. Then $D = \det W_{\nu'}^{n+1}(c') \det W_\nu^n(c)$ for any $\nu, \nu' \in \{1, 2\}$. By [31, 63:21 Theorem], $W_\nu^n(c) \rightarrowtail W_{\nu'}^{n+1}(c')$ if and only if $W_{\nu'}^{n+1}(c') \cong W_\nu^n(c) \perp [D]$. Since $\det W_{\nu'}^{n+1}(c') = \det(W_\nu^n(c) \perp [D])$, this is equivalent to

$$S_\mathfrak{p}\big(W_{\nu'}^{n+1}(c')\big) = S_\mathfrak{p}\big(W_\nu^n(c) \perp [D]\big).$$

Consider the case $\nu = \nu' = 1$. If n is even, then $W_1^n(c) = \mathbb{H}^{n/2-1} \perp [1, -c]$ and $W_1^n(c') = \mathbb{H}^{n/2} \perp [c']$, so $W_1^n(c) \rightarrowtail W_1^{n+1}(c')$ if and only if $[1, -c] \rightarrowtail \mathbb{H} \perp [c'] \cong [c, -c, c']$, which is equivalent to $1 \rightarrowtail [c, c']$, i.e., $(c, c')_\mathfrak{p} = 1$. If n is odd, then $W_1^n(c) = \mathbb{H}^{(n-1)/2} \perp [c]$ and $W_1^{n+1}(c') = \mathbb{H}^{(n-1)/2} \perp [1, -c']$, so $W_1^n(c) \rightarrowtail W_1^{n+1}(c')$ if and only if $c \rightarrowtail [1, -c']$, which is equivalent to $(c, c')_\mathfrak{p} = 1$. Hence, regardless of the parity of n, we have

$$S_\mathfrak{p}\big(W_1^{n+1}(c')\big) = S_\mathfrak{p}\big(W_1^n(c) \perp [D]\big) \iff W_1^n(c) \rightarrowtail W_1^{n+1}(c') \iff (c, c')_\mathfrak{p} = 1.$$

It follows that

$$S_\mathfrak{p}\big(W_1^{n+1}(c')\big) = (c, c')_\mathfrak{p} S_\mathfrak{p}\big(W_1^n(c) \perp [D]\big). \tag{4.5}$$

Also, we have

$$S_\mathfrak{p}\big(W_1^n(c) \perp [D]\big) = (-1)^{\nu-1} S_\mathfrak{p}\big(W_\nu^n(c) \perp [D]\big). \tag{4.6}$$

Indeed, if $\nu = 1$, this is trivial. And if $\nu = 2$, then $\det(W_1^n(c) \perp [D]) = \det(W_2^n(c) \perp [D])$, but $W_1^n(c) \perp [D] \ncong W_2^n(c) \perp [D]$, so $S_\mathfrak{p}(W_1^n(c) \perp [D]) = -S_\mathfrak{p}(W_2^n(c) \perp [D])$.

Similarly, $S_\mathfrak{p}(W_1^{n+1}(c')) = -S_\mathfrak{p}(W_2^{n+1}(c'))$, so for $\nu' \in \{1, 2\}$, we have

$$S_\mathfrak{p}\big(W_1^{n+1}(c')\big) = (-1)^{\nu'-1} S_\mathfrak{p}\big(W_{\nu'}^{n+1}(c')\big). \tag{4.7}$$

Plugging (4.6) and (4.7) into (4.5), we deduce that

$$S_\mathfrak{p}\big(W_{\nu'}^{n+1}(c')\big) = (-1)^{\nu+\nu'} (c, c')_\mathfrak{p} S_\mathfrak{p}\big(W_\nu^n(c) \perp [D]\big).$$

Hence

$$W_\nu^n(c) \rightarrowtail W_{\nu'}^{n+1}(c') \iff S_\mathfrak{p}\big(W_{\nu'}^{n+1}(c')\big) = S_\mathfrak{p}\big(W_\nu^n(c) \perp [D]\big) \iff (-1)^{\nu+\nu'}(c, c')_\mathfrak{p} = 1.$$

(iii) If V and W are two quadratic spaces such that $\dim V - \dim W = 2$, then $W \rightarrowtail V$ if and only if either $\det V \neq -\det W = \det(W \perp \mathbb{H})$ or $V \cong W \perp \mathbb{H}$. In our case, since $W_\nu^n(c) \perp \mathbb{H} = W_\nu^{n+2}(c)$, we have that $W_{\nu'}^{n+2}(c')$ represents $W_\nu^n(c)$ if and only if either $\det W_{\nu'}^{n+2}(c') \neq \det W_\nu^{n+2}(c)$, i.e., $c' \neq c$, or $W_{\nu'}^{n+2}(c') \cong W_\nu^{n+2}(c)$, i.e., $(\nu', c') = (\nu, c)$. ∎

Lemma 4.5. *Let V be a quadratic space over F. Let W_1 and W_2 be n-dimensional quadratic spaces over F such that* $\det W_1 = \det W_2 = D$ *and* $W_1 \ncong W_2$.

(i) *For $n \geq 2$, suppose either* $\dim V = n + 1$, *or* $\dim V = n + 2$ *with* $\det V = -D$. *Then V represents precisely one of W_1 and W_2.*
In particular, for every c, V represents exactly one of $W_1^n(c)$ and $W_2^n(c)$.

(ii) *For* $n \geq 3$*, suppose either* $\dim V = n-1$*, or* $\dim V = n-2$ *with* $\det V = -D$*. Then* V *is represented by precisely one of* W_1 *and* W_2*.*
In particular, for every c*,* V *is represented by exactly one of* $W_1^n(c)$ *and* $W_2^n(c)$*.*

Proof. (i) See [15, Lemma 3.13].

(ii) Let $\dim V = n-1$. Recall from [31, 63:21 Theorem] that $V \rightarrowtail W$ if and only if $W \cong V \perp [\det W \det V]$. Since W_1 and W_2 are the exactly non-isometric n-dimensional spaces with the same determinant D, we see that $V \perp [D \det V] \cong W_1$ or W_2, but not both. Hence V is represented by precisely one of W_1 and W_2.

If $\dim V = n-2$ and $\det V = -D$, then, by [31, 63:21 Theorem], $V \rightarrowtail W$ if and only if $W \cong V \perp \mathbb{H}$. Similar to the previous case, we see that $V \perp \mathbb{H} \cong W_1$ or W_2, but not both, as desired. ∎

Under the n-ADC assumption, we also have the lattice versions of Lemma 4.5 (i) and Proposition 4.2 (iii).

Lemma 4.6. *Let* $n \geq 2$*,* $\nu \in \{1,2\}$ *and* $c \in \mathcal{V}$*. Let* M *be an* n*-ADC* $\mathcal{O}_F$*-lattice.*

(i) *If either* $\operatorname{rank} M = n+1$*, or* $\operatorname{rank} M = n+2$ *and* $\det FM = -\det W_\nu^n(c)$*, then* M *represents exactly one of* $N_1^n(c)$ *and* $N_2^n(c)$*.*

(ii) *If* $FM \cong W_\nu^{n+2}(c)$*, then* M *represents every lattice* N *in* $\mathcal{L}_{F,n}$ *with* $FN \not\cong W_{3-\nu}^n(c)$*. In particular,* M *represents every* N *in* $\mathcal{M}_n$ *with* $N \not\cong N_{3-\nu}^n(c)$*.*

Proof. This follows from Lemma 4.5 (i), Proposition 4.2 (iii) and the n-ADC-ness of M. ∎

We treat the non-dyadic and dyadic case separately.

Case I. F is non-dyadic. Recall from [31, 92:2 Theorem] that a lattice L over F has a unique Jordan splitting. Hence we may denote by $J_k(L)$ the Jordan component of L, with possible zero rank and scale $\mathfrak{p}^k$, and write $J_{i,j}(L) := J_i(L) \perp J_{i+1}(L) \perp \cdots \perp J_j(L)$ for integers $i \leq j$.

We reformulate [16, Proposition 3.2] as below.

Lemma 4.7. *Let* $n \geq 1$*,* $\nu \in \{1,2\}$ *and* $c \in \mathcal{V}$*.*

(i) *The* $\mathcal{O}_F$*-maximal lattice* $N_\nu^n(c)$ *is given by the following table.*

n	c	$N_1^n(c)$	$N_2^n(c)$
Even	1	$\mathbf{H}^{\frac{n}{2}}$	$\mathbf{H}^{\frac{n-4}{2}} \perp \langle 1, -\Delta, \pi, -\Delta\pi \rangle$
	Δ	$\mathbf{H}^{\frac{n-2}{2}} \perp \langle 1, -\Delta \rangle$	$\mathbf{H}^{\frac{n-2}{2}} \perp \langle \pi, -\Delta\pi \rangle$
	$\delta\pi, \delta \in \mathcal{U} = \{1, \Delta\}$	$\mathbf{H}^{\frac{n-2}{2}} \perp \langle 1, -\delta\pi \rangle$	$\mathbf{H}^{\frac{n-2}{2}} \perp \langle \Delta, -\Delta\delta\pi \rangle$
Odd	$\delta, \delta \in \mathcal{U} = \{1, \Delta\}$	$\mathbf{H}^{\frac{n-1}{2}} \perp \langle \delta \rangle$	$\mathbf{H}^{\frac{n-3}{2}} \perp \langle \pi, -\Delta\pi, \Delta\delta \rangle$
	$\delta\pi, \delta \in \mathcal{U} = \{1, \Delta\}$	$\mathbf{H}^{\frac{n-1}{2}} \perp \langle \delta\pi \rangle$	$\mathbf{H}^{\frac{n-3}{2}} \perp \langle 1, -\Delta, \Delta\delta\pi \rangle$

(ii) *The set* $\mathcal{M}_n$ *is a minimal testing set for* n*-universality.*

Lemma 4.8. *Let $N = N_\nu^n(c)$ be an $\mathcal{O}_F$-maximal lattice of rank n. Then $J_{0,1}(N) = N$.*

Thus, an $\mathcal{O}_F$-lattice M represents N if and only if $FJ_0(M)$ represents $FJ_0(N)$ and $FJ_{0,1}(M)$ represents FN.

Proof. Recall that $\mathbf{H} = \langle 1, -1\rangle$ is unimodular, and so is $\mathbf{H}^k$. For each $N = N_\nu^n(c)$ in Lemma 4.7 (i), one may obtain the Jordan splittings by collecting or reordering the components according to their scales. From these splittings, it follows that $J_{0,1}(N) = N$ for each $N = N_\nu^n(c)$. Furthermore, the second assertion holds by [30, Theorem 1]. ∎

Case II. F is dyadic. We rephrase [15, Theorem 1.2] in terms of BONGs by virtue of [15, Remark 3.9, Lemmas 3.10 and 3.11].

Lemma 4.9. *Let $n \geq 1$, $\nu \in \{1, 2\}$ and $c \in \mathcal{V}$.*

(i) *The $\mathcal{O}_F$-maximal lattice $N_\nu^n(c)$ is given by the following table,*

n	c	$N_1^n(c)$	$N_2^n(c)$
Even	1	$\mathbf{H}^{\frac{n}{2}}$	$\mathbf{H}^{\frac{n-4}{2}} \perp \prec 1, -\Delta\pi^{-2e}, \pi, -\Delta\pi^{1-2e} \succ$
	Δ	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec 1, -\Delta\pi^{-2e} \succ$	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec \pi, -\Delta\pi^{1-2e} \succ$
	$\delta, \delta \in \mathcal{U}\backslash\{1, \Delta\}$	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec 1, -\delta\pi^{1-d(\delta)} \succ$	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec \delta^\#, -\delta^\#\delta\pi^{1-d(\delta)} \succ$
	$\delta\pi, \delta \in \mathcal{U}$	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec 1, -\delta\pi \succ$	$\mathbf{H}^{\frac{n-2}{2}} \perp \prec \Delta, -\Delta\delta\pi \succ$
Odd	$\delta, \delta \in \mathcal{U}$	$\mathbf{H}^{\frac{n-1}{2}} \perp \prec \delta \succ$	$\mathbf{H}^{\frac{n-3}{2}} \perp \prec \delta\kappa^\#, -\delta\kappa^\#\kappa\pi^{2-2e}, \delta\kappa \succ$
	$\delta\pi, \delta \in \mathcal{U}$	$\mathbf{H}^{\frac{n-1}{2}} \perp \prec \delta\pi \succ$	$\mathbf{H}^{\frac{n-3}{2}} \perp \prec 1, -\Delta\pi^{-2e}, \Delta\delta\pi \succ$

where κ is a fixed unit with $d(\kappa) = 2e-1$ and $\mathbf{H} \cong \prec 1, -\pi^{-2e} \succ$ (cf. [15, Lemma 3.9 (i)]).

(ii) *The set $\mathcal{M}_n$ is a minimal testing set for n-universality.*

Remark 4.10. Each $\mathcal{O}_F$-maximal lattice $N_\nu^n(c)$ can be written as the form $\mathbf{H}^k \perp L \cong \prec 1, -\pi^{-2e}, \ldots, 1, -\pi^{-2e}, c_1, \ldots, c_\ell \succ$ relative to a good BONG, where k, ℓ are nonnegative integers and $L \cong \prec c_1, \ldots, c_\ell \succ$ relative to a good BONG. Hence the above table gives the values of the invariants $R_i(N_\nu^n(c))$ (cf. [15, Lemma 3.11]).

The next two lemmas indicate that the invariants $R_i(N)$ $(1 \leq i \leq n)$ and the space FN determine whether an $\mathcal{O}_F$-lattice N of rank n is $\mathcal{O}_F$-maximal or not. The proofs are the same as that of [15, Proposition 3.7]. (See also [15, Lemma 3.11].)

Lemma 4.11. *Let N be an $\mathcal{O}_F$-lattice of even rank $n \geq 2$ and put $S_i = R_i(N)$.*

(i) *If $FN \cong W_1^n(c)$ with $c \in \{1, \Delta\}$, then $N \cong N_1^n(c)$ if and only if $S_i = 0$ for $i \in [1, n]^O$ and $S_i = -2e$ for $i \in [1, n]^E$.*

(ii) *If $FN \cong W_2^n(c)$ with $c \in \{1, \Delta\}$, then $N \cong N_2^n(c)$ if and only if $S_i = 0$ for $i \in [1, n-2]^O$, $S_i = -2e$ for $i \in [1, n-2]^E$, $S_{n-1} = 1$ and $S_n = 1 - 2e$.*

(iii) *If $FN \cong W_\nu^n(c)$, with $\nu \in \{1, 2\}$ and $c \in \mathcal{V}\backslash\{1, \Delta\}$, then $N \cong N_\nu^n(c)$ if and only if $S_i = 0$ for $i \in [1, n-1]^O$, $S_i = -2e$ for $i \in [1, n-1]^E$ and $S_n = 1 - d(c)$.*

Lemma 4.12. *Let N be an $\mathcal{O}_F$-lattice of odd rank $n \geq 1$ and put $S_i = R_i(N)$.*

(i) *If $FN \cong W_1^n(\delta)$ with $\delta \in \mathcal{U}$, then $N \cong N_1^n(\delta)$ if and only if $S_i = 0$ for $i \in [1,n]^O$ and $S_i = -2e$ for $i \in [1,n]^E$.*

(ii) *If $FN \cong W_2^n(\delta)$ with $\delta \in \mathcal{U}$, then $N \cong N_2^n(\delta)$ if and only if $S_i = 0$ for $i \in [1,n]^O$, $S_i = -2e$ for $i \in [1,n-2]^E$ and $S_{n-1} = 2-2e$.*

(iii) *If $FN \cong W_\nu^n(\delta\pi)$, with $\nu \in \{1,2\}$ and $\delta \in \mathcal{U}$, then $N \cong N_\nu^n(\delta\pi)$ if and only if $S_i = 0$ for $i \in [1,n-1]^O$, $S_i = -2e$ for $i \in [1,n-1]^E$ and $S_n = 1$.*

Proposition 4.13. *Let $N \cong \prec b_1,\dots,b_n \succ$ be an $\mathcal{O}_F$-maximal lattice of odd rank $n \geq 3$. Put $S_i = R_i(N)$ and $\beta_i = \alpha_i(N)$. Then*

(i) *$S_i = 0$ for $i \in [1,n-2]^O$, $S_i = -2e$ for $i \in [1,n-2]^E$ and $S_{n-1} \in \{-2e, 2-2e\}$.*

(ii) *If $S_{n-1} = -2e$, then $S_n \in \{0,1\}$, $\beta_{n-2} = 0$ and $\beta_{n-1} \geq d[-b_{n-2,n-1}] \geq 2e$.*

(iii) *If $S_{n-1} = 2-2e$, then $S_n = 0$, $\beta_{n-2} = 1$ and $d[-b_{n-2,n-1}] = \beta_{n-1} = 2e-1$.*

Proof. (i) It is clear from Lemma 4.12.

(ii) If $S_{n-1} = -2e$, then $S_n \in \{0,1\}$ by Lemma 4.12. Since $S_{n-1} - S_{n-2} = -2e$, by Proposition 3.4 (ii) and (iv), we have $\beta_{n-2} = 0$ and $\beta_{n-1} \geq d[-b_{n-2,n-1}] \geq 2e$.

(iii) If $S_{n-1} = 2-2e$, then $S_n = 0$ by Lemma 4.12. Since $S_{n-1} - S_{n-2} = 2-2e$ and $S_n - S_{n-1} = 2e-2$, by Proposition 3.3 (ii), we have $\beta_{n-2} = 1$ and $\beta_{n-1} = 2e-1$. Hence

$$2e-1 = (2e-2)+1 = S_{n-2} - S_{n-1} + \beta_{n-2} \leq d[-b_{n-2,n-1}] \leq \beta_{n-1} = 2e-1$$

by (3.3), as desired. ■

We return to the case where F is a local field. The following lemma shows that, over local fields, maximality implies n-ADC-ness.

Lemma 4.14. *Let M be an $\mathcal{O}_F$-maximal lattice. If FM represents FN, then M represents N; thus M is n-ADC for $1 \leq n \leq \operatorname{rank} M$.*

Proof. If FM represents FN, by [30, Proposition 1], $FM \cong FN \perp V$ for some quadratic space. Take an integral lattice L on V. Then $\mathfrak{n}(N \perp L) \subseteq \mathcal{O}_F$. By [31, 82:18] and [31, 91:2 Theorem], there must be an $\mathcal{O}_F$-maximal lattice M' of rank n on FM such that $N \subseteq N \perp L \subseteq M' \cong M$. Thus M represents N. ■

The following proposition characterizes n-ADC $\mathcal{O}_F$-lattices of rank n, thereby proving the simple case of Theorem 1.5 (i).

Proposition 4.15. *Let M be an $\mathcal{O}_F$-lattice of rank $n \geq 2$. Then M is n-ADC if and only if M is $\mathcal{O}_F$-maximal.*

Proof. Sufficiency is clear from Lemma 4.14. From Remark 4.3, we may choose an $\mathcal{O}_F$-maximal lattice N of rank n such that $FN \cong FM$. Then $FN \rightarrowtail FM$ by [31, 63:21 Theorem]. Since M is n-ADC, we have $N \rightarrowtail M$, so $M \cong N$ by the maximality of N. ■

Using Lemmas 4.7 (i) and 4.9 (i) with $n = 4$, one can easily prove the following proposition for quaternary maximal lattices, which will be used in the proof of Theorem 1.11.

Proposition 4.16. *Let N be a quaternary $\mathcal{O}_F$-maximal lattice. Then N represents $\mathbf{H}$ except when $N = N_2^4(1)$. In the exceptional case, $N \cong \mathbf{A} \perp \mathbf{A}^{(\pi)}$, where $\mathbf{A}^{(\pi)}$ denotes the lattice $\mathbf{A}$ scaled by π.*

5. *n*-ADC lattices over non-dyadic local fields

Throughout this section, let n be an integer with $n \geq 2$. We assume that F is a non-dyadic local field and M is an $\mathcal{O}_F$-lattice. In this case, we have $\mathcal{U} = \{1, \Delta\}$ and $\mathcal{V} = \{1, \Delta, \pi, \Delta\pi\}$.

Theorem 5.1. *If* rank $M = n + 1$ *or* $n + 2$, *then M is n-ADC if and only if M is $\mathcal{O}_F$-maximal.*

Lemma 5.2. *Suppose that $M = J_{0,1}(M)$ and* rank $J_1(M) \leq 1$. *Then M is $\mathcal{O}_F$-maximal; thus it is n-ADC.*

In particular, if M is unimodular, then it is $\mathcal{O}_F$-maximal and n-ADC.

Proof. By the hypothesis, we have $\mathfrak{n}(M) = \mathfrak{s}(M) = \mathcal{O}_F$ and $\mathfrak{v}(M) \supseteq \mathfrak{p}$. It follows from [31, 82:19] that M is $\mathcal{O}_F$-maximal. Hence it is n-ADC from Lemma 4.14.

If M is unimodular, then $M = J_0(M)$ and rank $J_1(M) = 0$, so the first assertion applies. ∎

Lemma 5.3. *We have the following statements:*

(i) *If for every $\delta \in \mathcal{U}$ there is some $\nu_\delta \in \{1, 2\}$ such that M represents $N_{\nu_\delta}^n(\delta\pi)$, then* rank $J_0(M) \geq n - 1$ *and* rank $J_{0,1}(M) \geq n + 1$.

(ii) *If M represents $N_1^n(\delta)$ for some $\delta \in \mathcal{U}$, then* rank $J_0(M) \geq n$.

(iii) *If M represents $N_1^n(\delta)$ for all $\delta \in \mathcal{U}$, then* rank $J_0(M) \geq n + 1$.

(iv) *If M represents both $N_1^n(c)$ and $N_2^n(c)$ for some $c \in \mathcal{V}$, then* rank $J_{0,1}(M) \geq n + 2$.

Proof. (i) By Lemma 4.8, $FJ_0(M)$ represents $FJ_0(N_{\nu_\delta}^n(\delta\pi))$, which implies that

$$\text{rank } J_0(M) \geq \text{rank } J_0\big(N_{\nu_\delta}^n(\delta\pi)\big) = n - 1.$$

Also, $FJ_{0,1}(M)$ represents $FN_{\nu_\delta}^n(\delta\pi) = W_{\nu_\delta}^n(\delta\pi)$ for $\delta = 1, \Delta$. Since $W_{\nu_1}^n(\pi) \not\cong W_{\nu_\Delta}^n(\Delta\pi)$ by Lemma 4.4 (i), this implies that

$$\text{rank } J_{0,1}(M) = \dim FJ_{0,1}(M) \geq n + 1.$$

(ii) Observe from Lemma 4.7 (i) that $N_1^n(\varepsilon)$ is unimodular for any $\varepsilon \in \mathcal{U}$, so $J_0(N_1^n(\varepsilon)) = N_1^n(\varepsilon)$. By Lemma 4.8, $FJ_0(M)$ represents $FJ_0(N_1^n(\delta)) = FN_1^n(\delta) = W_1^n(\delta)$, which implies that

$$\text{rank } J_0(M) \geq \text{rank } N_1^n(\delta) = n.$$

(iii) By Lemma 4.8, $FJ_0(M)$ represents $FJ_0(N_1^n(\delta)) = W_1^n(\delta)$ for $\delta = 1, \Delta$. Since $W_1^n(1) \not\cong W_1^n(\Delta)$ by Lemma 4.4 (i), this implies that

$$\operatorname{rank} J_0(M) = \dim FJ_0(M) \geq n + 1.$$

(iv) By Lemma 4.8, $FJ_{0,1}(M)$ represents $FN_\nu^n(c) = W_\nu^n(c)$ for $\nu = 1, 2$. This contradicts Lemma 4.5 (i) if $\dim FJ_{0,1}(M) = n + 1$. Hence

$$\operatorname{rank} J_{0,1}(M) = \dim FJ_{0,1}(M) \geq n + 2. \qquad \blacksquare$$

Lemma 5.4. *Suppose that M is n-ADC of rank $n + 1$ or $n + 2$. Then $M = J_{0,1}(M)$ and* $\operatorname{rank} J_1(M) \leq 2$*.*

Proof. Let $\operatorname{rank} M = n + 1$. For each $\delta \in \{1, \Delta\}$, by Lemma 4.6 (i), there is some $\nu_\delta \in \{1, 2\}$ such that M represents $N_{\nu_\delta}^n(\delta\pi)$. Hence, by Lemma 5.3 (i), we have

$$\operatorname{rank} J_0(M) \geq n - 1 \quad \text{and} \quad \operatorname{rank} J_{0,1}(M) \geq n + 1.$$

So $\operatorname{rank} J_{0,1}(M) = n + 1$. Thus $J_{0,1}(M) = M$ and $\operatorname{rank} J_1(M) \leq 2$.

Let $\operatorname{rank} M = n + 2$ and $FM \cong W_\nu^{n+2}(c)$. Since $|\mathcal{U}| = 2$ and $|\mathcal{V}| = 4$, we have $\mathcal{U}\backslash\{c\} \neq \emptyset$ and $\mathcal{V}\backslash\{1, c\} \neq \emptyset$. Let $\delta \in \mathcal{U}\backslash\{c\}$ and $c' \in \mathcal{V}\backslash\{1, c\}$. Since $\delta \neq c$, by Lemma 4.4 (iii), FM represents $W_1^n(\delta)$. Since $c' \neq 1$, both $W_1^n(c')$ and $W_2^n(c')$ are defined (including the case $n = 2$) and, since $c' \neq c$, by Lemma 4.4 (iii), FM represents both of them. Then, since M is n-ADC, it represents $N_1^n(\delta)$, $N_1^n(c')$ and $N_2^n(c')$. By Lemma 5.3 (ii) and (iv), we get

$$\operatorname{rank} J_0(M) \geq n \quad \text{and} \quad \operatorname{rank} J_{0,1}(M) \geq n + 2.$$

So $\operatorname{rank} J_{0,1}(M) = n + 2$. Thus $J_{0,1}(M) = M$ and $\operatorname{rank} J_1(M) \leq 2$. $\blacksquare$

Proof of Theorem 5.1. Sufficiency is clear from Lemma 4.14. Let $m = \operatorname{rank} M \in \{n+1, n + 2\}$. By Lemma 5.4, $M = J_{0,1}(M)$ and $\operatorname{rank} J_1(M) \leq 2$. If $\operatorname{rank} J_1(M) \leq 1$, then we are done by Lemma 5.2.

Assume $\operatorname{rank} J_1(M) = 2$ and let $M = J_0(M) \perp M'^{(\pi)}$, where M' is unimodular of rank 2. Since $J_0(M)$ and M' are unimodular, both are $\mathcal{O}_F$-maximal from Lemma 5.2. Hence, by Lemma 4.7 (i),

$$J_0(M) \cong N_1^{m-2}(1) \text{ or } N_1^{m-2}(\Delta), \quad \text{and} \quad M' \cong \mathbf{H} \text{ or } \langle 1, -\Delta \rangle.$$

If $M' \cong \langle 1, -\Delta \rangle$, then $M \cong N_2^m(1)$ or $N_2^m(\Delta)$, as desired. Assume $M' \cong \mathbf{H}$. Then

$$FM \cong W_1^m(\eta') \quad \text{for some } \eta' \in \{1, \Delta\}.$$

Hence FM represents $W_1^n(\eta)$ with $\eta \in \{1, \Delta\}$, by Lemma 4.4 (ii), for $m = n + 1$ (we have $(\eta, \eta')_\mathfrak{p} = 1$) and, by Lemma 4.4 (iii), for $m = n + 2$, so M represents both of $N_1^n(1)$ and $N_1^n(\Delta)$ by n-ADC-ness of M. By Lemma 5.3 (iii), we have $\operatorname{rank} J_0(M) \geq n + 1$, a contradiction. $\blacksquare$

6. n-ADC lattices over dyadic local fields I

In this section, let n be an even integer with $n \geq 2$. We assume that F is a dyadic local field and $M \cong \prec a_1, \ldots, a_m \succ$ is an $\mathcal{O}_F$-lattice of rank $m \geq n$, relative to some good BONG. Let $R_i = R_i(M)$ for $1 \leq i \leq m$ and $\alpha_i = \alpha_i(M)$ for $1 \leq i \leq m-1$. We also suppose that $N \cong \prec b_1, \ldots, b_n \succ$ is an $\mathcal{O}_F$-lattice of rank n, relative to some good BONG, and denote its associated invariants by $S_i = R_i(N)$ and $\beta_i = \alpha_i(N)$ when an $\mathcal{O}_F$-lattice N with rank n is discussed.

Theorem 6.1. *If* rank $M = n+1$, *then M is n-ADC if and only if M is $\mathcal{O}_F$-maximal.*

Theorem 6.2. *If* rank $M = n+2$, *then M is n-ADC if and only if either M is $\mathcal{O}_F$-maximal, or $n = 2$ and*

$$M \cong \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ \cong \mathbf{H} \perp 2^{-1}\pi A(2\pi^{-1}, 2\rho\pi),$$

which is not $\mathcal{O}_F$-maximal.

Remark 6.3. When $e = 1$, by [2, Corollary 3.4 (ii)] and [15, Lemma 3.10], we also have $\mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ \cong \mathbf{H} \perp \langle 1, -\Delta \rangle$.

Lemma 6.4. *We have the following statements:*

(i) *If M represents $N_1^n(\Delta)$ (resp. $N_1^n(1)$), then $R_{i-1} = R_i + 2e = 0$ for $i \in [1,n]^E$. If moreover $R_{n+1} > 0$, then $d((-1)^{n/2}a_{1,n}) = 2e$ (resp. $d((-1)^{n/2}a_{1,n}) = \infty$).*

(ii) *If M represents $N_1^n(1)$ and $N_1^n(\Delta)$, then $R_{i-1} = R_i + 2e = 0$ for $i \in [1,n]^E$ and $R_{n+1} = 0$.*

(iii) *If M represents $N_2^n(\Delta)$ (resp. $N_2^n(1)$, with $n \geq 4$), then $R_{i-1} = R_i + 2e = 0$ for $i \in [1, n-2]^E$ and either $R_{n-1} = 0$ and $R_n \in \{-2e, 2-2e\}$ or $R_{n-1} = R_n + 2e = 1$.*

(iv) *If M represents one of $N_1^n(1)$, $N_1^n(\Delta)$ and $N_2^n(\Delta)$, and one of $N_1^n(\kappa)$ and $N_2^n(\kappa)$, then $R_{n+1} \in \{0, 1, 2\}$. (Here κ is the unit with $d(\kappa) = 2e-1$ from Lemma 4.9 (i).)*

Proof. (i) Let $N = N_1^n(\Delta)$ or $N_1^n(1)$. Then $S_{n-1} = S_n + 2e = 0$ by Lemma 4.11 (i). By Proposition 3.5 (i), we have $R_{n-1} \geq 0$ and $R_n \geq -2e$. If M represents N, then

$$-2e \leq R_n \leq R_{n-1} + R_n \leq S_{n-1} + S_n = -2e$$

by [3, Lemma 4.6 (i)]. So $R_n = -2e$ and hence $R_{i-1} = R_i + 2e = 0$ for $i \in [1,n]^E$, by Proposition 3.5 (iii).

If $R_{n+1} > 0$, then $R_{n+1} - S_n > 2e$. By [5, Corollary 2.10], we have $a_{1,n}b_{1,n} \in F^{\times 2}$ and thus $a_{1,n} = b_{1,n}$ in $F^\times/F^{\times 2}$. So $(-1)^{n/2}a_{1,n} = (-1)^{n/2}b_{1,n} = \Delta$ or 1, i.e., $d((-1)^{n/2}a_{1,n}) = 2e$ or ∞, according as $N = N_1^n(\Delta)$ or $N = N_1^n(1)$.

(ii) The first statement is clear from (i). By Proposition 3.5 (i), we have $R_{n+1} \geq 0$. Assume $R_{n+1} > 0$. If M represents $N_1^n(1)$ and $N_1^n(\Delta)$, then $d((-1)^{n/2}a_{1,n}) = \infty$ and $d((-1)^{n/2}a_{1,n}) = 2e$ by (i). This is impossible.

(iii) Let $N = N_2^n(\Delta)$, or $N_2^n(1)$, with $n \geq 4$. Then $S_{n-3} = S_{n-2} + 2e = 0$ and $S_{n-1} = S_n + 2e = 1$ by Lemma 4.11 (ii). Similar to (i), we have $R_{i-1} = R_i + 2e = 0$ for $i \in [1, n-2]^E$. Applying [3, Lemma 4.6 (i)], we see that

$$-2e + R_{n-1} = R_{n-2} + R_{n-1} \leq S_{n-2} + S_{n-1} = 1 - 2e, \tag{6.1}$$

$$R_{n-1} + R_n \leq S_{n-1} + S_n = 2 - 2e. \tag{6.2}$$

Hence $R_{n-1} \in \{0, 1\}$ by (6.1). If $R_{n-1} = 0$, then $-2e \leq R_n \leq 2 - 2e$ by (3.2) and (6.2), so $R_n \in \{-2e, 2-2e\}$ by Corollary 3.2 (i). If $R_{n-1} = 1$, then $R_n = 1 - 2e$ similarly.

(iv) Assume $R_{n+1} > 2$. If $N = N_\nu^n(\varepsilon)$ is any of the five lattices under consideration, then, by Lemma 4.11 (i) and (iii), we have $S_n \leq 2 - 2e$, so $R_{n+1} - S_n > 2e$. Then, same as in the proof of (ii), we get $d((-1)^{n/2}a_{1,n}) = d((-1)^{n/2}b_{1,n}) = d(\varepsilon)$.

Since M represents $N_1^n(1)$, $N_1^n(\Delta)$ or $N_2^n(\Delta)$, we have $d((-1)^{n/2}a_{1,n}) = \infty$ or $2e$. Since M also represents $N_1^n(\kappa)$ or $N_2^n(\kappa)$, we have

$$d\big((-1)^{n/2}a_{1,n}\big) = d(\kappa) = 2e - 1,$$

a contradiction. ∎

Lemma 6.5. *Suppose $m = n + 1$ and $R_{i-1} = R_i + 2e = 0$ for all $i \in [1, n-2]^E$. Let $N = N_\nu^n(\varepsilon\pi)$, with $\nu \in \{1, 2\}$ and $\varepsilon \in \mathcal{U}$.*

(i) *If $R_{n-1} = 0$, $R_n \in \{-2e, 2-2e\}$ and $R_{n+1} \geq 2$, then Theorem 3.6 (ii) fails at $i = n$.*

(ii) *If $R_{n-1} = R_n + 2e = 1$, then Theorem 3.6 (ii) fails at $i = n - 1$.*

Proof. By Lemma 4.11 (iii), we have $S_i = S_{i+1} + 2e = 0$ for $i \in [1, n-2]^O$, $S_{n-1} = 0$ and $S_n = 1$.

(i) If $R_{n-1} = 0$ and $R_n \in \{-2e, 2-2e\}$, then $\mathrm{ord}(a_{1,n}b_{1,n})$ is odd and thus $d[a_{1,n}b_{1,n}] = 0$. Also, $R_{n+1} - S_n \geq 2 - 1 = 1$ and $d[-a_{1,n+1}b_{1,n-1}] \geq 0$. Hence

$$\begin{aligned} A_n &= \min\big\{(R_{n+1} - S_n)/2 + e, R_{n+1} - S_n + d[-a_{1,n+1}b_{1,n-1}]\big\} \geq \min\{1/2 + e, 1\} \\ &= 1 > 0 = d[a_{1,n}b_{1,n}]. \end{aligned}$$

(ii) If $R_{n-1} = 1$, then $\mathrm{ord}(a_{1,n-1}b_{1,n-1})$ is odd and so $d[a_{1,n-1}b_{1,n-1}] = 0$. Since $R_n - R_{n-1} = -2e$, by Proposition 3.4 (iv), we have $d[-a_{n-1,n}] \geq 2e$. Since $R_{n-2} = S_{n-2} = -2e$, by Proposition 3.5 (iii), we have

$$d\big[(-1)^{(n-2)/2}a_{1,n-2}\big] \geq 2e \quad \text{and} \quad d\big[(-1)^{(n-2)/2}b_{1,n-2}\big] \geq 2e.$$

So, by the domination principle, we see that

$$d[-a_{1,n}b_{1,n-2}] \geq 2e.$$

Also, $R_n - S_{n-1} = (1 - 2e) - 0 = 1 - 2e$, and

$$R_{n+1} - S_{n-2} + d[a_{1,n+1}b_{1,n-3}] \geq R_{n+1} - S_{n-2} \geq R_{n-1} - S_{n-2} = 1 - (-2e) = 2e + 1$$

from (3.1). Hence

$$\begin{aligned}A_{n-1} &= \min\big\{(R_n - S_{n-1})/2 + e, R_n - S_{n-1} + d[-a_{1,n}b_{1,n-2}],\\ &\qquad R_n - S_{n-1} + R_{n+1} - S_{n-2} + d[a_{1,n+1}b_{1,n-3}]\big\}\\ &\geq \min\big\{(1-2e)/2 + e, (1-2e) + 2e, (1-2e) + (2e+1)\big\}\\ &= 1/2 > 0 = d[a_{1,n-1}b_{1,n-1}].\end{aligned}$$

∎

Proof of Theorem 6.1. Sufficiency follows from Lemma 4.14. We claim that $R_{i-1} = R_i + 2e = 0$ for $i \in [1, n-2]^E$, $R_{n-1} = 0$, $R_n \in \{-2e, 2-2e\}$ and $R_{n+1} \in \{0, 1\}$. By Lemma 4.6 (i), M represents either $N_1^n(\Delta)$ or $N_2^n(\Delta)$. In both cases, by Lemma 6.4 (i) and (iii), we have $R_{i-1} = R_i + 2e = 0$ for $i \in [1, n-2]^E$ and either $R_{n-1} = 0$ and $R_n \in \{2-2e, -2e\}$ or $R_{n-1} = R_n + 2e = 1$.

Next, take $\varepsilon \in \mathcal{U}$. By Lemma 4.6 (i), M represents $N_\nu^n(\varepsilon\pi)$ for some $\nu \in \{1, 2\}$. Hence M and $N_\nu^n(\varepsilon\pi)$ satisfy the conditions (i)–(iv) of Theorem 3.6. Then, by Lemma 6.5 (i) and (ii), we cannot have $R_{n-1} = 0$, $R_n \in \{2-2e, -2e\}$ and $R_{n+1} \geq 2$ or $R_{n-1} = R_n + 2e = 1$, because Theorem 3.6 (ii) would fail at $i = n$ or $n-1$. Hence we are left with the case when $R_{n-1} = 0$, $R_n \in \{2-2e, -2e\}$ and $R_{n+1} \leq 1$. The claim is proved.

If $R_n = -2e$, then $[a_1, \ldots, a_n] \cong W_1^n(\eta)$ with $\eta \in \{1, \Delta\}$ by Proposition 3.5 (iv). For any $\varepsilon \in \mathcal{U}$, since $(\eta, \varepsilon)_\mathfrak{p} = 1 \neq -1$, by Lemma 4.4 (ii),

$$[a_1, \ldots, a_n] \cong W_1^n(\eta) \not\rightarrowtail W_2^{n+1}(\varepsilon).$$

Hence $FM \not\cong W_2^{n+1}(\varepsilon)$, so FM is isometric to one of the remaining three types: $W_1^{n+1}(\delta)$, $W_1^{n+1}(\delta\pi)$ and $W_2^{n+1}(\delta\pi)$, with $\delta \in \mathcal{U}$ (cf. Proposition 4.2 (i)).

Recall that $R_{n+1} \in \{0, 1\}$. If $FM \cong W_1^{n+1}(\delta)$, then $\operatorname{ord}(a_{1,n+1})$ is even, so $R_{n+1} = 0$ and hence M is $\mathcal{O}_F$-maximal by Lemma 4.12 (i); if $FM \cong W_1^{n+1}(\delta\pi)$ or $W_2^{n+1}(\delta\pi)$, then $\operatorname{ord}(a_{1,n+1})$ is odd, so $R_{n+1} = 1$ and hence M is $\mathcal{O}_F$-maximal by Lemma 4.12 (iii).

If $R_n = 2-2e$, let $FM \cong W_\nu^{n+1}(c)$. Assume that $W_1^n(\eta) \rightarrowtail FM$ for some $\eta \in \{1, \Delta\}$. Then, by n-ADC-ness, $N_1^n(\eta) \rightarrowtail M$, which, by Lemma 6.4 (i), implies $R_n = -2e$. Contradiction. Hence for $\eta \in \{1, \Delta\}$, we have $W_1^n(\eta) \not\rightarrowtail FM \cong W_\nu^{n+1}(c)$, which, by Lemma 6.4 (ii), is equivalent to $(1, c)_\mathfrak{p} = (\Delta, c)_\mathfrak{p} = -(-1)^{1+\nu}$, i.e., $1 = (\Delta, c)_\mathfrak{p} = (-1)^\nu$. But this happens precisely when $\nu = 2$ and $c = \delta$ for some $\delta \in \mathcal{U}$. Thus $FM \cong W_2^{n+1}(\delta)$. Recall that $R_{n+1} \in \{0, 1\}$ from the claim. Hence $R_{n+1} = 0$ by the parity of $\operatorname{ord}(a_{1,n+1})$ and so M is $\mathcal{O}_F$-maximal by Lemma 4.12 (ii). ∎

Lemma 6.6. *Suppose $m = n+2$, $R_{i-1} = R_i + 2e = 0$ for all $i \in [1, n]^E$, $R_{n+2} \geq 2-2e$ and $d[-a_{n+1,n+2}] > 1 - R_{n+2}$.*

(i) *For $n \geq 2$, if R_{n+1} is even or $d((-1)^{n/2}a_{1,n}) = 2e$, then Theorem* 3.6 (iii) *fails at $i = n+1$ for M and $N_2^n(\Delta)$.*

(ii) *For $n \geq 4$, if R_{n+1} is even or $d((-1)^{n/2}a_{1,n}) = \infty$, then Theorem* 3.6 (iii) *fails at $i = n+1$ for M and $N_2^n(1)$.*

Proof. Let $N = N_2^n(\eta)$, with $\eta \in \{1, \Delta\}$. Then $S_{n-1} = S_n + 2e = 1$, by Lemma 4.11 (ii). Thus $R_{n+2} \geq 2 - 2e > S_n$.

Since $S_n - S_{n-1} = -2e$, by Proposition 3.4 (iv), we have $d[-b_{n-1,n}] \geq 2e$. Since $R_n = S_{n-2} = -2e$, by Proposition 3.5 (iii), we have

$$d\big[(-1)^{n/2}a_{1,n}\big] \geq 2e \quad \text{and} \quad d\big[(-1)^{(n-2)/2}b_{1,n-2}\big] \geq 2e.$$

Hence $d[a_{1,n}b_{1,n}] \geq 2e > 1 - R_{n+2}$ by the domination principle. Combining with the assumption $d[-a_{n+1,n+2}] > 1 - R_{n+2}$, we deduce that

$$d[-a_{1,n+2}b_{1,n}] > 1 - R_{n+2}$$

by the domination principle again. This, combined with $d[-a_{1,n+1}b_{1,n-1}] \geq 0$, shows that

$$\begin{aligned} d[-a_{1,n+1}b_{1,n-1}] + d[-a_{1,n+2}b_{1,n}] > 0 + (1 - R_{n+2}) &= 2e + (1-2e) - R_{n+2} \\ &= 2e + S_n - R_{n+2}. \end{aligned}$$

Now it remains to show that $[a_1, \ldots, a_{n+1}]$ fails to represent $[b_1, \ldots, b_n] \cong FN$, which, under hypothesis (i) (resp. (ii)), is isometric to $W_2^n(\Delta)$ (resp. $W_2^n(1)$). Equivalently, by Lemma 4.5 (i), we must show that $[a_1, \ldots, a_{n+1}]$ represents $W_1^n(\Delta)$ (resp. $W_1^n(1)$).

If $R_{n+1} = \mathrm{ord}(a_{n+1})$ is even, then, by Proposition 3.5 (v), $[a_1, \ldots, a_{n+1}] \cong \mathbb{H}^{n/2} \perp [\varepsilon] = W_1^{n+1}(\varepsilon)$ for some $\varepsilon \in \mathcal{O}_F^\times$. For $\eta \in \{1, \Delta\}$, since $(\eta, \varepsilon)_\mathfrak{p} = 1$, by Lemma 4.4 (ii), $W_1^n(\eta) \twoheadrightarrow W_1^{n+1}(\varepsilon)$, as required.

If $d((-1)^{n/2}a_{1,n}) = 2e$, then, by Proposition 3.5 (iv),

$$W_1^n(\Delta) \cong [a_1, \ldots, a_n] \twoheadrightarrow [a_1, \ldots, a_{n+1}],$$

so (i) holds. And if $d((-1)^{n/2}a_{1,n}) = \infty$, then, by Proposition 3.5 (iv) again, $W_1^n(1) \cong [a_1, \ldots, a_n] \twoheadrightarrow [a_1, \ldots, a_{n+1}]$, so (ii) holds. ∎

Lemma 6.7. *Suppose $m = n+2$, $R_{i-1} = R_i + 2e = 0$ for all $i \in [1,n]^E$ and $R_{n+2} \geq 2 - 2e$.*

(i) *Suppose that either R_{n+1} is even or $d((-1)^{n/2}a_{1,n}) = 2e$. If M represents $N_2^n(\Delta)$, then either $\alpha_{n+1} = 0$, or $\alpha_{n+1} = 1$ and $d(-a_{n+1}a_{n+2}) = d[-a_{n+1,n+2}] = 1 - R_{n+2}$.*

(ii) *Suppose that $n \geq 4$ and either R_{n+1} is even, or $d((-1)^{n/2}a_{1,n}) = \infty$. If M represents $N_2^n(1)$, then either $\alpha_{n+1} = 0$, or $\alpha_{n+1} = 1$ and $d(-a_{n+1}a_{n+2}) = d[-a_{n+1,n+2}] = 1 - R_{n+2}$.*

Proof. (i) Assume $d[-a_{n+1,n+2}] > 1 - R_{n+2}$. Then Theorem 3.6 (iii) fails at $i = n+1$ for $N = N_2^n(\Delta)$ by Lemma 6.6 (i). But this contradicts the fact that M represents $N_2^n(\Delta)$. Thus $d[-a_{n+1,n+2}] \leq 1 - R_{n+2}$. By Proposition 3.5 (i), we have $R_{n+1} \geq 0$. Hence, by (3.3), we deduce that

$$\alpha_{n+1} \leq R_{n+2} - R_{n+1} + d[-a_{n+1,n+2}] \leq R_{n+2} + d[-a_{n+1,n+2}] \leq 1,$$

which implies that $\alpha_{n+1} \in \{0, 1\}$ by Proposition 3.4 (i), and $d[-a_{n+1,n+2}] = 1 - R_{n+2}$ if $\alpha_{n+1} = 1$.

Since $R_{n+1} - R_n \geq 2e$, by Proposition 3.3 (i) and the hypothesis that $R_{n+2} \geq 2-2e$, we have $\alpha_n \geq 2e > 1 - R_{n+2} = d[-a_{n+1,n+2}] = \min\{d(-a_{n+1,n+2}), \alpha_n\}$. (We have $n + 2 = m$, so α_{n+2} is ignored.) It follows that $d(-a_{n+1,n+2}) = d[-a_{n+1,n+2}] = 1 - R_{n+2}$.

(ii) Similar to (i). ∎

Lemma 6.8. *Suppose that $m = n + 2$ and M is n-ADC.*

(i) *If $FM \cong W_1^{n+2}(1)$, then $M \cong N_1^{n+2}(1)$.*

(ii) *If $n \geq 4$ and $FM \cong W_1^{n+2}(\Delta)$, then $M \cong N_1^{n+2}(\Delta)$.*

(iii) *If $FM \cong W_2^{n+2}(1)$, then $M \cong N_2^{n+2}(1)$.*

(iv) *If $FM \cong W_2^{n+2}(\Delta)$, then $M \cong N_2^{n+2}(\Delta)$.*

(v) *If $c \in \mathcal{V}\backslash\{1, \Delta\}$ and $FM \cong W_1^{n+2}(c)$, then $M \cong N_1^{n+2}(c)$.*

(vi) *If $c \in \mathcal{V}\backslash\{1, \Delta\}$ and $FM \cong W_2^{n+2}(c)$, then $M \cong N_2^{n+2}(c)$.*

Proof. (i) If $n = 2$, then FM is 2-universal by [16, Theorem 2.3] and so M is 2-universal by 2-ADC-ness. So $M \cong \mathbf{H}^2 = N_1^4(1)$ by [15, Remark 6.4]. Suppose $n \geq 4$. Then, by Lemma 4.6 (ii), M represents every N in $\mathcal{M}_n$ with $N \ncong N_2^n(1)$. Since M represents $N_1^n(1)$ and $N_1^n(\Delta)$, by Lemma 6.4 (ii), we have

$$R_i = 0 \text{ for } i \in [1, n+1]^O \quad \text{and} \quad R_i = -2e \text{ for } i \in [1, n]^E. \tag{6.3}$$

If $R_{n+2} = R_{n+2} - R_{n+1} \geq 2-2e$, then $\alpha_{n+1} \neq 0$, by Proposition 3.4 (ii). Since $R_{n+1} = 0$ is even and M represents $N_2^n(\Delta)$, we have $\alpha_{n+1} = 1$ and $d[-a_{n+1,n+2}] = 1 - R_{n+2}$ by Lemma 6.7 (i). Since $R_n = -2e$, we also have $d[(-1)^{n/2}a_{1,n}] \geq 2e$ by Proposition 3.5 (iii). So

$$d\big((-1)^{(n+2)/2}a_{1,n+2}\big) = d\big[(-1)^{(n+2)/2}a_{1,n+2}\big] = 1 - R_{n+2} < 2e$$

by the domination principle. However, $FM \cong W_1^{n+2}(1)$ and so $d((-1)^{(n+2)/2}a_{1,n+2}) = \infty$, a contradiction. Hence $R_{n+2} - R_{n+1} < 2-2e$.

Note that $R_{n+2} - R_{n+1} \neq 1-2e$ by Corollary 3.2 (i). Hence $R_{n+2} = R_{n+2} - R_{n+1} = -2e$ by (3.2). Combining with (6.3), we conclude that $N \cong N_1^{n+2}(1)$ by Lemma 4.11 (i).

(ii) If $n \geq 4$, then $N_2^n(1)$ is defined. By Lemma 4.6 (ii), M represents every N in $\mathcal{M}_n$ with $N \ncong N_2^n(\Delta)$. In particular, it represents $N_1^n(1)$, $N_1^n(\Delta)$ and $N_2^n(1)$. We repeat the reasoning from (i), but in this case we use Lemma 6.7 (ii) instead of Lemma 6.7 (i). Again we see that M satisfies (6.3) and $R_{n+2} = -2e$. Since $FM \cong W_1^{n+2}(\Delta)$, we deduce that $N \cong N_1^{n+2}(\Delta)$ by Lemma 4.11 (i).

(iii)–(iv) First, M represents every N in $\mathcal{M}_n$ with $N \ncong N_1^n(1)$ (resp. $N \ncong N_1^n(\Delta)$) by Lemma 4.6 (ii). Since M represents $N_1^n(\Delta)$ (resp. $N_1^n(1)$) and $N_1^n(\kappa)$, we see that

$$R_i = 0 \text{ for } i \in [1, n]^O \quad \text{and} \quad R_i = -2e \text{ for } i \in [1, n]^E \tag{6.4}$$

by Lemma 6.4 (i) and $R_{n+1} \in \{0, 1, 2\}$ by Lemma 6.4 (iv).

By Proposition 3.5 (i), we have $R_{n+2} \geq -2e$. We assert $R_{n+2} = 1 - 2e$.

If $R_{n+2} = -2e$, then $FM \cong W_1^{n+2}(1)$ or $W_1^{n+2}(\Delta)$ by Proposition 3.5 (iv). This contradicts $FM \cong W_2^{n+2}(1)$ (resp. $FM \cong W_2^{n+2}(\Delta)$).

If $R_{n+2} \geq 2-2e$, by Lemma 6.4 (i), we have either $R_{n+1} \in \{0,2\}$, or

$$R_{n+1} = 1 \quad \text{and} \quad d\big((-1)^{n/2}a_{1,n}\big) = 2e$$

(resp. $R_{n+1} = 1$ and $d((-1)^{n/2}a_{1,n}) = \infty$). Hence the hypothesis of Lemma 6.7 (i) (resp. Lemma 6.7 (ii)) is satisfied. Since M represents $N_2^n(\Delta)$ (resp. $N_2^n(1)$ with $n \geq 4$), we see that either $\alpha_{n+1} = 0$, or $\alpha_{n+1} = 1$ and $d(-a_{n+1}a_{n+2}) = 1 - R_{n+2}$ by Lemma 6.7 (i) (resp. Lemma 6.7 (ii)).

Case I: $\alpha_{n+1} = 0$. By Proposition 3.4 (ii), $R_{n+2} - R_{n+1} = -2e$. Since $R_{n+1} \leq 2$ and, by our assumption, $R_{n+2} \geq 2-2e$, we must have $R_{n+1} = 2$ and $R_{n+2} = 2-2e$. This combined with (6.4) shows that for every $i \in [1, n+1]^O$, we have $R_{i+1} - R_i = -2e$ and R_i is even. So, by Corollary 3.2 (ii), $[a_i, a_{i+1}] \cong \mathbb{H}$ or $[1,-\Delta]$. It follows that $FM \cong \mathbb{H}^{n/2} \perp [1,-\eta] = W_1^{n+2}(\eta)$ for $\eta = 1$ or Δ. This contradicts $FM \cong W_2^{n+2}(1)$ (resp. $FM \cong W_2^{n+2}(\Delta)$).

Case II: $\alpha_{n+1} = 1$. Since $R_n = -2e$, we have $d((-1)^{n/2}a_{1,n}) \geq 2e$ by Proposition 3.5 (iii). Hence

$$d\big((-1)^{(n+2)/2}a_{1,n+2}\big) = d(-a_{n+1}a_{n+2}) = 1 - R_{n+2} < 2e$$

by the domination principle. This contradicts $FM \cong W_2^{n+2}(1)$ (resp. $FM \cong W_2^{n+2}(\Delta)$) again.

With above discussion, the assertion is proved and thus $R_{n+2} = 1-2e$.

Recall that $R_{n+1} \in \{0,1,2\}$ and so $R_{n+1} = 1$ by Corollary 3.2 (i). Combining with (6.4), we deduce that $M \cong N_2^{n+2}(1)$ (resp. $N_2^{n+2}(\Delta)$) by Lemma 4.11 (ii).

(v) Let $c \in \mathcal{V}\backslash\{1,\Delta\}$. By Lemma 4.6 (ii), M represents every N in $\mathcal{M}_n$ with $N \not\cong N_2^n(c)$. In particular, M represents $N_1^n(1)$ and $N_1^n(\Delta)$, so it satisfies (6.3) by Lemma 6.4 (ii).

By Proposition 3.5 (i), we have $R_{n+2} \geq -2e$. If $R_{n+2} = -2e$, then $FM \cong W_1^{n+2}(1)$ or $W_1^{n+2}(\Delta)$ by Proposition 3.5 (iv), which contradicts $FM \cong W_1^{n+2}(c)$. Thus $R_{n+2} > -2e$. Since $R_{n+1} = 0$, Corollary 3.2 (i) implies $R_{n+2} \neq 1-2e$. Hence $R_{n+2} \geq 2-2e$ and so $\alpha_{n+1} \neq 0$ by Proposition 3.4 (ii).

Now, we see that $R_{n+1} = 0$ is even, M represents $N_2^n(\Delta)$ and $\alpha_{n+1} \neq 0$, so $1 - R_{n+2} = d(-a_{n+1}a_{n+2})$ by Lemma 6.7 (i). Since $R_n = -2e$, we also have

$$d\big((-1)^{n/2}a_{1,n}\big) \geq d\big[(-1)^{n/2}a_{1,n}\big] \geq 2e$$

by Proposition 3.5 (iii). On the other hand, $FM \cong W_1^{n+1}(c)$, so in $F^\times/F^{\times 2}$ we have $a_{1,n+2} = \det FM = (-1)^{(n+2)/2}c$. It follows that

$$d\big((-1)^{(n+2)/2}a_{1,n+2}\big) = d(c) < 2e = d\big((-1)^{n/2}a_{1,n}\big).$$

By the domination principle, this implies $1 - R_{n+2} = d(-a_{n+1}a_{n+2}) = d(c)$. Combining with (6.3), we conclude $M \cong N_1^{n+2}(c)$ by Lemma 4.11 (iii).

(vi) Similar to (v). ∎

Lemma 6.9. *Suppose $m = 4$ and $R_1 = R_3 = R_2 + 2e = 0$. If $FM \cong W_1^4(\Delta)$ and M represents both $N_1^2(\kappa)$ and $N_2^2(\kappa)$, then $R_4 \in \{-2e, 2-2e\}$.*

Proof. Let $N = N_\nu^2(\kappa)$, $\nu \in \{1,2\}$. Then $S_1 = 0$ and $S_2 = 2-2e$ by Lemma 4.11 (iii). Suppose $R_4 > 2-2e$. Then $R_4 > S_2$. Since $R_4 - R_3 = R_4 > 2-2e > -2e$ and $S_2 - S_1 = 2-2e$, we have $\alpha_3 \geq 1 = \beta_1$ by Proposition 3.4 (ii) and (iii). Since $\mathrm{ord}(a_{1,3}b_1)$ is even, we also have $d(-a_{1,3}b_1) \geq 1$. Combining these, we see that

$$d[-a_{1,3}b_1] = \min\left\{d(-a_{1,3}b_1), \alpha_3, \beta_1\right\} = 1.$$

Also, $d[-a_{1,4}b_{1,2}] = d(-a_{1,4}b_{1,2}) = d(\Delta\kappa) = d(\kappa) = 2e-1$ by the domination principle. So

$$d[-a_{1,3}b_1] + d[-a_{1,4}b_{1,2}] = 1 + (2e-1) > 2e + (2-2e) - R_4 = 2e + S_2 - R_4.$$

By definition, $[b_1, b_2] = FN \cong W_1^2(\kappa)$ or $W_2^2(\kappa)$. But, by Lemma 4.5 (i), $[a_1, a_2, a_3]$ represents exactly one of $W_1^2(\kappa)$ and $W_2^2(\kappa)$. Hence Theorem 3.6 (iii) fails at $i = 3$ for either $N = N_1^2(\kappa)$ or $N_2^2(\kappa)$. This contradicts the hypothesis that M represents both $N_1^2(\kappa)$ and $N_2^2(\kappa)$. Hence $R_4 \leq 2-2e$.

By Proposition 3.5 (i), we have $R_4 \geq -2e$. Recall that $R_3 = 0$ and so $R_4 \neq 1-2e$ by Corollary 3.2 (i). Hence $R_4 \in \{-2e, 2-2e\}$. ∎

Lemma 6.10. *If $FM \cong W_1^4(\Delta)$ and $R_1 = R_3 = R_2 + 2e = R_4 + 2e - 2 = 0$, then $M \cong \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$.*

Proof. Let $N = \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$. By [15, Lemma 3.10], $N \cong \prec 1, -\pi^{-2e}, 1, -\Delta\pi^{2-2e} \succ$, $S_1 = S_3 = 0$, $S_2 = -2e$ and $S_4 = 2-2e$.

To show $M \cong N$, we only need to verify that conditions (i)–(iv) in [3, Theorem 3.2] are satisfied. We have $R_2 - R_1 = -2e$, $R_3 - R_2 = 2e$ and $R_4 - R_3 = 2-2e$. Hence $(\alpha_1, \alpha_2, \alpha_3) = (0, 2e, 1)$ by Proposition 3.3 (ii). Since $R_i = S_i$ for $1 \leq i \leq 4$, we have $(\beta_1, \beta_2, \beta_3) = (0, 2e, 1)$ similarly. Hence conditions (i) and (ii) hold. For $i = 1, 3$, since $\mathrm{ord}(a_{1,i}b_{1,i})$ is even, we have $d(a_{1,i}b_{1,i}) \geq 1 \geq \alpha_i$. Since $R_2 - R_1 = -2e$, we have $d(-a_1a_2) \geq 2e$ by Corollary 3.2 (ii). Similarly, $d(-b_1b_2) \geq 2e$. Hence $d(a_{1,2}b_{1,2}) \geq 2e \geq \alpha_2$ by the domination principle. Thus condition (iii) is checked. Since $\alpha_1 + \alpha_2 = 2e$ and $\alpha_2 + \alpha_3 = 2e+1$, we only need to show that $[b_1, b_2] \rightarrowtail [a_1, a_2, a_3]$ for condition (iv). By definition, $[b_1, b_2] \cong W_1^2(1)$. By Proposition 3.5 (v), $[a_1, a_2, a_3] \cong W_1^3(\varepsilon)$ for some $\varepsilon \in \mathcal{U}$. Hence $[b_1, b_2] \rightarrowtail [a_1, a_2, a_3]$ by Lemma 4.4 (ii). ∎

Lemma 6.11. *Suppose that M is 2-ADC of rank 4. If $FM \cong W_1^4(\Delta)$, then $M \cong N_1^4(\Delta)$ or $\mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$.*

Proof. By Lemma 4.6 (ii), M represents every N in $\mathcal{M}_2$ with $N \ncong N_2^2(\Delta)$. Since M represents $N_1^2(1)$ and $N_1^2(\Delta)$, we have $R_1 = R_3 = R_2 + 2e = 0$ by Lemma 6.4 (ii). Since M represents $N_1^2(\kappa)$ and $N_2^2(\kappa)$, we also have $R_4 \in \{-2e, 2-2e\}$ by Lemma 6.9. If $R_4 = -2e$, then $M \cong N_1^4(\Delta)$ by Lemma 4.11 (i). If $R_4 = 2-2e$, then $M \cong \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$ by Lemma 6.10. ∎

Lemma 6.12. *Let* $M \cong \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$. *Then*

(i) *M is 2-ADC, but not $\mathcal{O}_F$-maximal.*

(ii) *M is not 3-ADC.*

Proof. (i) We have $FM \cong W_1^4(\Delta)$ and $R_4(M) = 2 - 2e$. Hence M is not $\mathcal{O}_F$-maximal from Lemma 4.11 (i).

By Proposition 4.2 (iii), FM represents FN for every N in $\mathcal{M}_2$ with $N \not\cong N_2^2(\Delta)$. So, by Lemma 2.1, it suffices to show that M represents all N in $\mathcal{M}_2$ except for $N \cong N_2^2(\Delta)$. To do so, we will verify conditions (i)–(iv) in Theorem 3.6 for those N. Note that their invariants S_i are clear from Lemma 4.11.

Let $\nu \in \{1,2\}$, $\eta \in \{1,\Delta\}$ and $c \in \mathcal{V}\backslash\{1,\Delta\}$. Then $d(c) < 2e$. For condition (i), we have $R_1 = 0 \le S_1$ and $R_2 = -2e \le S_2$ for every N in $\mathcal{M}_2$. Since $S_1 + 2e \ge 2e > 2 - 2e = R_4$, condition (iv) is verified.

To verify condition (ii), for every N in $\mathcal{M}_2$, we have

$$A_1 \le \frac{R_2 - S_1}{2} + e = \frac{-2e - S_1}{2} + e = \frac{-S_1}{2} \le 0 \le d[a_1 b_1].$$

Thus condition (ii) holds at $i = 1$ for these N.

For $N = N_1^2(\eta)$, since $R_2 = S_2 = -2e$, by Proposition 3.5 (iii), we have $d[-a_{1,2}] \ge 2e$ and $d[-b_{1,2}] \ge 2e$. So $d[a_{1,2}b_{1,2}] \ge 2e$ by the domination principle. Hence

$$A_2 \le \frac{R_3 - S_2}{2} + e = \frac{0 - (-2e)}{2} + e = 2e \le d[a_{1,2}b_{1,2}].$$

For $N = N_\nu^2(c)$, by the domination principle, we have $d[a_{1,2}b_{1,2}] = d[-b_{1,2}] = d(-b_1 b_2) = d(c) < 2e$. Since $S_1 = 0$ and $S_2 = 1 - d(c)$, (3.3) gives $\beta_1 = 1$. Hence

$$A_2 \le R_3 - S_2 + d[-a_{1,3}b_1] \le R_3 - S_2 + \beta_1 = 0 - \big(1 - d(c)\big) + 1 = d(c) = d[a_{1,2}b_{1,2}].$$

Hence condition (ii) also holds at $i = 2$ for every N in $\mathcal{M}_2$. Thus condition (ii) is verified.

To verify condition (iii), we have $R_3 = 0 \le S_1$ for every N in $\mathcal{M}_2$. Thus condition (iii) holds at $i = 2$ for these N.

For $N = N_1^2(\eta)$, we have $[b_1, b_2] \cong W_1^2(1)$ or $W_1^2(\Delta)$. Also, $[a_1, a_2, a_3] \cong W_1^3(\varepsilon)$ for some $\varepsilon \in \mathcal{U}$ by Proposition 3.5 (v). Hence $[b_1, b_2] \rightarrow [a_1, a_2, a_3]$ by Lemma 4.4 (ii). For $N = N_\nu^2(c)$, in $F^\times/F^{\times 2}$ we have $a_{1,4} = \det FM = \Delta$ and $b_{1,2} = \det FN = -c$. Since $d(\Delta) = 2e > d(c)$, by the domination principle, we have $d(-a_{1,4}b_{1,2}) = d(\Delta c) = d(c) = 1 - S_2$. Hence

$$d[-a_{1,3}b_1] + d[-a_{1,4}b_{1,2}] \le \beta_1 + d(-a_{1,4}b_{1,2}) = 1 + (1 - S_2) \le 2e + S_2 - R_4,$$

where the last inequality holds from $2 - 2e \le S_2$ and $R_4 = 2 - 2e$. Hence condition (iii) also holds at $i = 3$ for every N in $\mathcal{M}_2$. Thus condition (iii) is verified.

(ii) Suppose that M is 3-ADC. Let $\varepsilon \in \mathcal{U}$. By Lemma 4.6 (i), M represents $N_\nu^3(\varepsilon\pi)$ for some $\nu \in \{1,2\}$. Then $S_1 = S_2 + 2e = 0$ and $S_3 = 1$ by Lemma 4.12 (iii). Since

$\mathrm{ord}(a_{1,3}b_{1,3})$ is odd, $d[a_{1,3}b_{1,3}] = 0$. By definition, we have $d(a_{1,4}) = d(\Delta) = 2e$. Since $S_2 - S_1 = -2e$, we also have $d(-b_1b_2) \geq 2e$ by Corollary 3.2. Hence $d(-a_{1,4}b_{1,2}) \geq 2e$ by the domination principle. Since $S_3 - S_2 = 2e+1$, Proposition 3.3 (ii) implies $\beta_2 = 2e + 1/2$. So $d[-a_{1,4}b_{1,2}] = \min\{d(-a_{1,4}b_{1,2}), \beta_2\} \geq 2e$. Also, $R_4 - S_3 = (2-2e) - 1 = 1 - 2e$. Hence

$$\begin{aligned} A_3 &= \min\left\{(R_4 - S_3)/2 + e, R_4 - S_3 + d[-a_{1,4}b_{1,2}]\right\} \\ &\geq \min\left\{(1-2e)/2 + e, (1-2e) + 2e\right\} = 1/2 > 0 = d[a_{1,3}b_{1,3}]. \end{aligned}$$

Thus Theorem 3.6 (ii) fails at $i = 3$, which contradicts the fact that M represents N. ■

Proof of Theorem 6.2. Sufficiency follows by Lemmas 4.14 and 6.12 (i). Suppose that M is n-ADC. Then, by Proposition 4.2 (ii), $FM \cong W_\nu^n(c)$ for some $\nu \in \{1,2\}$ and $c \in \mathcal{V}$. So, by Lemmas 6.8 and 6.11, $M \cong N_\nu^n(c)$ or $\mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$. Also, $\prec 1, -\Delta\pi^{2-2e} \succ \cong 2^{-1}\pi A(2\pi^{-1}, 2\rho\pi)$ by [2, Corollary 3.4 (iii)] and [31, 93:17 Example]. ■

7. *n*-ADC lattices over dyadic local fields II

In this section, we keep the setting as the previous section, but let n be an odd integer with $n \geq 3$.

Theorem 7.1. *If* $\mathrm{rank}\, M = n+1$, *then* M *is* n*-ADC if and only if* M *is* $\mathcal{O}_F$*-maximal.*

Proof. Sufficiency is clear from Lemma 4.14. Suppose that M is n-ADC. Then it is $(n-1)$-ADC. Since $n-1$ is even, M is $\mathcal{O}_F$-maximal except for $n-1 = 2$ and $M \cong \mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$ by Theorem 6.2. However, $\mathbf{H} \perp \prec 1, -\Delta\pi^{2-2e} \succ$ is not 3-ADC by Lemma 6.12 (ii). So the exceptional case cannot happen. ■

Theorem 7.2. *If* $\mathrm{rank}\, M = n+2$, *then* M *is* n*-ADC if and only if either* M *is* $\mathcal{O}_F$*-maximal, or*

$$M \cong N_\nu^{n+1}(\delta) \perp \langle \varepsilon\pi^k \rangle,$$

with $\nu \in \{1,2\}$, $\delta \in \mathcal{U}\backslash\{1,\Delta\}$, $\varepsilon \in \mathcal{U}$ *and* $k \in \{0,1\}$.

Also, if M *is simultaneously* $\mathcal{O}_F$*-maximal and isometric to the described orthogonal splitting, then* $M \cong N_2^{n+2}(\varepsilon)$ *with* $\varepsilon \in \mathcal{U}$.

Remark 7.3. For the lattice $N_\nu^{n+1}(\delta)$ given in Theorem 7.2, we see from Lemma 4.9 and [15, Remark 3.8, Lemma 3.9] that

$$\begin{aligned} N_1^{n+1}(\delta) &= \mathbf{H}^{(n-1)/2} \perp N_1^2(\delta) \cong \mathbf{H}^{(n-1)/2} \perp \pi^{-l} A\big(\pi^l, -(\delta-1)\pi^{-l}\big) \quad \text{and} \\ N_2^{n+1}(\delta) &= \mathbf{H}^{(n-1)/2} \perp N_2^2(\delta) \cong \mathbf{H}^{(n-1)/2} \perp \delta^{\#}\pi^{-l} A\big(\pi^l, -(\delta-1)\pi^{-l}\big), \end{aligned}$$

with $\delta \in \mathcal{U}\backslash\{1,\Delta\}$ and $2l = d(\delta) - 1 \leq 2e - 2$, where $\delta^{\#} = 1 + 4\rho(\delta-1)^{-1}$. Similarly, we also see that

$$N_2^{n+2}(\varepsilon) = \mathbf{H}^{(n-1)/2} \perp N_2^3(\varepsilon) \cong \mathbf{H}^{(n-1)/2} \perp 2^{-1}\pi A(2, 2\rho) \perp \langle \Delta\varepsilon \rangle,$$

with $\varepsilon \in \mathcal{U}$.

Before showing Theorem 7.2, we first prove the following theorem, which characterizes the n-ADC lattices with odd n. In the remainder of this section, we assume rank $M = n+2$.

Theorem 7.4. *M is n-ADC if and only if $R_i = 0$ for $i \in [1,n]^O$, $R_i = -2e$ for $i \in [1,n]^E$, $R_{n+1} \in [-2e, 0]^E$ and $R_{n+2}, \alpha_n \in \{0,1\}$.*

Proof. We will show that the theorem is equivalent to Lemma 7.5 below.

For necessity, by Proposition 3.4 (ii), $\alpha_n = 0$ if and only if $R_{n+1} = -2e < 0$. Hence the conditions follows from Lemma 7.5 (i), (ii) and (iv).

For sufficiency, from the hypothesis, we have $R_{n+1} \geq -2e$ and $R_{n+2} \leq 1$. It follows that $R_{n+2} - R_{n+1} \leq 2e+1$, and the equality holds if and only if $R_{n+1} = -2e$ and $R_{n+2} = 1$. This shows Lemma 7.5 (iii). If $\alpha_n = 1$, then $R_{n+1} = R_{n+1} - R_n \in [2-2e, 0]^E \cup \{1\}$ by Proposition 3.4 (iii), but $R_{n+1} \leq 0$ and so $R_{n+1} \in [2-2e, 0]^E$. Hence Lemma 7.5 (i), (ii) and (iv) follow from the hypothesis except for the condition $R_{n+1} + d[-a_{n,n+1}] = 1$.

Since $\alpha_n = 1$, by Proposition 3.4 (v), we see that $d[-a_{n,n+1}] \geq 1 - R_{n+1}$ and the equality holds when $R_{n+1} \neq 2-2e$. Assume $R_{n+1} = 2-2e$. Since $R_{n+2} - R_{n+1} \leq 1-(2-2e) = 2e-1$, Proposition 3.3 (i) implies that

$$d[-a_{n,n+1}] \leq \alpha_{n+1} \leq 2e-1 = 1 - R_{n+1}.$$

Hence $d[-a_{n,n+1}] = 1 - R_{n+1}$, as desired. ∎

Lemma 7.5. *M is n-ADC if and only if the following conditions hold:*

(i) *$R_i = 0$ for $i \in [1,n]^O$ and $R_i = -2e$ for $i \in [1,n]^E$.*

(ii) *Either $\alpha_n = 0$ or $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$.*

(iii) *If $R_{n+2} - R_{n+1} > 2e$, then $R_{n+1} = -2e$ and $R_{n+2} = 1$.*

(iv) *If $\alpha_n = 1$, then $R_{n+1} \in [2-2e, 0]^E$ and $R_{n+2} \in \{0,1\}$.*

To establish Lemma 7.5, we need a series of lemmas. First, we review the invariants $S_i = R_i(N)$ from Proposition 4.13 for N in $\mathcal{M}_n$. Precisely, we have

$$\begin{aligned} S_i = 0 \quad \text{for } i \in [1, n-2]^O, \quad S_i = -2e \quad \text{for } i \in [1, n-2]^O, \\ S_{n-1} \in \{-2e, 2-2e\} \quad \text{and} \quad S_n \in \{0,1\}, \end{aligned} \tag{7.1}$$

which will be repeatedly used for the argument in Lemmas 7.6, 7.7 and 7.8.

Lemma 7.6. *Suppose that $R_i = 0$ for $i \in [1,n]^O$ and $R_i = -2e$ for $i \in [1,n]^E$. For any N in $\mathcal{M}_n$, the following statements hold:*

(i) *$d[a_{1,i}b_{1,i}] \geq 2e$ for $i \in [1, n-2]^E$.*

(ii) *If $S_{n-1} = -2e$, then $d[a_{1,n-1}b_{1,n-1}] \geq 2e$; if $S_{n-1} = 2-2e$, then*

$$d[a_{1,n-1}b_{1,n-1}] = 2e-1.$$

(iii) *If* $\alpha_n = 1$, *then* $d[a_{1,n}b_{1,n}] = 1 - S_n$.

(iv) *If* $S_{n-1} = -2e$, *then* $d[-a_{1,n}b_{1,n-2}] = 0$; *if* $S_{n-1} = 2-2e$, *then*

$$d[-a_{1,n}b_{1,n-2}] \le 1.$$

(v) *If* $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, *then* $d[-a_{1,n+1}b_{1,n-1}] = 1 - R_{n+1}$.

Proof. (i) For $i \in [1, n-2]^E$, since $R_i = S_i = -2e$, Proposition 3.5 (iii) implies that $d[(-1)^{i/2}a_{1,i}] \ge 2e$ and $d[(-1)^{i/2}b_{1,i}] \ge 2e$. Hence $d[a_{1,i}b_{1,i}] \ge 2e$ by the domination principle.

(ii) Since $R_{n-1} = S_{n-3} = -2e$, by Proposition 3.5 (iii), we have

$$d\big[(-1)^{(n-1)/2}a_{1,n-1}\big] \ge 2e \quad \text{and} \quad d\big[(-1)^{(n-3)/2}b_{1,n-3}\big] \ge 2e.$$

If $S_{n-1} = -2e$, then $d[-b_{n-2,n-1}] \ge 2e$ by Proposition 4.13 (ii). If $S_{n-1} = 2-2e$, then $d[-b_{n-2,n-1}] = 2e-1$ by Proposition 4.13 (iii). Hence

$$d[a_{1,n-1}b_{1,n-1}] \begin{cases} \ge 2e & \text{if } S_{n-1} = -2e, \\ = 2e-1 & \text{if } S_{n-1} = 2-2e, \end{cases}$$

by the domination principle.

(iii) First, $\mathrm{ord}(a_{1,n})$ is even from hypothesis and $\mathrm{ord}(b_{1,n-1})$ is also even from (7.1). If $S_n = 1$, then $d[a_{1,n}b_{1,n}] = d(a_{1,n}b_{1,n}) = 0$; if $S_n = 0$, then $d(a_{1,n}b_{1,n}) \ge 1 = \alpha_n$, so $d[a_{1,n}b_{1,n}] = \min\{d(a_{1,n}b_{1,n}), \alpha_n\} = 1$. In both cases, $d[a_{1,n}b_{1,n}] = 1 - S_n$.

(iv) If $S_{n-1} = -2e$, then $\beta_{n-2} = 0$, by Proposition 4.13 (ii); if $S_{n-1} = 2-2e$, then $\beta_{n-2} = 1$, by Proposition 4.13 (iii). Note that $0 \le d[-a_{1,n}b_{1,n-2}] \le \beta_{n-2}$ and we are done.

(v) If $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, then $R_{n+1} = R_{n+1} - R_n \ge 2-2e$, by Proposition 3.4 (iii).

By (ii), we have $d[a_{1,n-1}b_{1,n-1}] \ge 2e-1 \ge 1 - R_{n+1}$. Moreover, the first inequality is strict unless $S_{n-1} = 2-2e$ and the second is strict unless $R_{n+1} = 2-2e$. Therefore,

$$d[a_{1,n-1}b_{1,n-1}] > 1 - R_{n+1} = d[-a_{n,n+1}],$$

unless $S_{n-1} = R_{n+1} = 2-2e$. Hence, by the domination principle, $d[-a_{1,n+1}b_{1,n-1}] = 1 - R_{n+1}$ holds except for $S_{n-1} = R_{n+1} = 2-2e$.

In the exceptional case $S_{n-1} = R_{n+1} = 2-2e$, we have

$$d[a_{1,n-1}b_{1,n-1}] \ge 2e-1 = 1 - R_{n+1} = d[-a_{n,n+1}],$$

so $d[-a_{1,n+1}b_{1,n-1}] \ge 2e-1$, by the domination principle. But, by Proposition 4.13 (iii), we have

$$d[-a_{1,n+1}b_{1,n-1}] \le \beta_{n-1} = 2e-1.$$

Hence $d[-a_{1,n+1}b_{1,n-1}] = 2e-1 = 1-R_{n+1}$. ∎

Lemma 7.7. *Suppose that $R_i = 0$ for $i \in [1, n]^O$ and $R_i = -2e$ for $i \in [1, n]^E$. Then*

(i) *Theorem 3.6 (i) holds for every N in $\mathcal{M}_n$.*

(ii) *If $\alpha_n = 0$ or $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, then Theorem 3.6 (ii) holds for every N in $\mathcal{M}_n$.*

(iii) *If $\alpha_n \in \{0, 1\}$ and $R_{n+2} - R_{n+1} \leq 2e$, then Theorem 3.6 (iv) holds for every N in $\mathcal{M}_n$.*

Proof. (i) By Proposition 3.5 (i), if i is odd, then $R_i = 0 \leq S_i$, and if i is even, then $R_i = -2e \leq S_i$. Hence Theorem 3.6 (i) holds for $1 \leq i \leq n$.

(ii) For $i \in [1, n-2]^O$, note that $S_i = R_{i+1} + 2e = 0$ and so

$$A_i \leq \frac{R_{i+1} - S_i}{2} + e = \frac{-2e - 0}{2} + e = 0 \leq d[a_{1,i}b_{1,i}].$$

For $i \in [1, n-2]^E$, since $R_{i+1} = S_i + 2e = 0$, we have

$$A_i \leq \frac{R_{i+1} - S_i}{2} + e = \frac{0 - (-2e)}{2} + e = 2e \leq d[a_{1,i}b_{1,i}]$$

by Lemma 7.6 (i). For $i = n-1$, by Lemma 7.6 (ii), we have

$$d[a_{1,n-1}b_{1,n-1}] \begin{cases} \geq 2e & \text{if } S_{n-1} = -2e, \\ = 2e - 1 & \text{if } S_{n-1} = 2 - 2e. \end{cases}$$

By Lemma 7.6 (iv), we also have

$$d[-a_{1,n}b_{1,n-2}] \begin{cases} = 0 & \text{if } S_{n-1} = -2e, \\ \leq 1 & \text{if } S_{n-1} = 2 - 2e. \end{cases}$$

So

$$\begin{aligned} A_{n-1} &\leq R_n - S_{n-1} + d[-a_{1,n}b_{1,n-2}] \\ &\begin{cases} = 0 - (-2e) + 0 = 2e \leq d[a_{1,n-1}b_{1,n-1}] & \text{if } S_{n-1} = -2e, \\ \leq 0 - (2 - 2e) + 1 = 2e - 1 = d[a_{1,n-1}b_{1,n-1}] & \text{if } S_{n-1} = 2 - 2e. \end{cases} \end{aligned}$$

For $i = n$, if $\alpha_n = 0$, then $R_{n+1} = -2e$ by Proposition 3.4 (ii) and so

$$A_n \leq \frac{R_{n+1} - S_n}{2} + e = \frac{-S_n}{2} \leq 0 \leq d[a_{1,n}b_{1,n}].$$

If $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, then $d[-a_{1,n+1}b_{1,n-1}] = 1 - R_{n+1}$ by Lemma 7.6 (v). Also, $d[a_{1,n}b_{1,n}] = 1 - S_n$ by Lemma 7.6 (iii). So

$$A_n \leq R_{n+1} - S_n + d[-a_{1,n+1}b_{1,n-1}] = R_{n+1} - S_n + (1 - R_{n+1}) = 1 - S_n = d[a_{1,n}b_{1,n}].$$

Hence Theorem 3.6 (ii) holds for $1 \leq i \leq n$.

(iii) Since $\alpha_n \leq 1$, Proposition 3.3 (i) implies $R_{n+1} - R_n < 2e$. Combining with the hypothesis, for every $2 \leq i \leq n$, we have $R_{i+2} - R_{i+1} \leq 2e$, so Theorem 3.6 (iv) holds. ∎

Lemma 7.8. *Suppose that $R_i = 0$ for $i \in [1,n]^O$, $R_i = -2e$ for $i \in [1,n]^E$ and $R_{n+2} - R_{n+1} \le 2e$. If either $\alpha_n = 0$, or $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, $R_{n+1} \in [2-2e, 0]^E$ and $R_{n+2} \in \{0,1\}$, then Theorem 3.6 (iii) holds for every N in $\mathcal{M}_n$.*

Proof. By (7.1), we have $R_{i+1} = 0 = S_{i-1}$ for $i \in [2, n-1]^E$ and $R_{i+1} = -2e = S_{i-1}$ for $i \in [2, n-1]^O$, Theorem 3.6 (iii) holds trivially for $2 \le i \le n-1$.

For $i = n$, if $\alpha_n = 0$, then $R_{n+1} = -2e \le S_{n-1}$. If $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$, when $S_{n-1} = 2-2e$, we have

$$d[-a_{1,n}b_{1,n-2}] + d[-a_{1,n+1}b_{1,n-1}] \le 1 + (1 - R_{n+1}) = 2e + S_{n-1} - R_{n+1}$$

by Lemma 7.6 (iv) and (v); when $S_{n-1} = -2e$, note that $[b_1, \dots, b_{n-1}] \cong W_1^{n-1}(1)$ or $W_1^{n-1}(\Delta)$ from Proposition 3.5 (iv), and $[a_1, \dots, a_n] \cong W_1^n(\varepsilon)$ with $\varepsilon \in \mathcal{O}_F^\times$ from Proposition 3.5 (v). Hence $[b_1, \dots, b_{n-1}] \rightarrowtail [a_1, \dots, a_n]$ by Lemma 4.4 (ii).

For $i = n+1$, we may assume $R_{n+2} > S_n \ge 0$. If $R_{n+1} = -2e$, then, by hypothesis, $R_{n+2} \le R_{n+1} + 2e = 0$, a contradiction. Hence $R_{n+1} \ne -2e$, i.e., $\alpha_n \ne 0$. So

$$\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1.$$

Hence $R_{n+1} \in [2-2e, 0]^E$ and $R_{n+2} = 1$. Now, we have $1 = R_{n+2} > S_n \ge 0$, so $S_n = 0$. It follows that $\operatorname{ord}(a_{1,n+2}b_{1,n})$ is odd and so $d[-a_{1,n+2}b_{1,n}] = 0$. Since $R_{n+2} \le 2-2e$, we have

$$d[-a_{1,n+1}b_{1,n-1}] + d[-a_{1,n+2}b_{1,n}] = (1 - R_{n+1}) + 0 \le 2e - 1 = 2e + S_n - R_{n+2},$$

by Lemma 7.6 (v). Hence Theorem 3.6 (iii) holds for $2 \le i \le n+1$. ∎

Now, we are ready to show the sufficiency of Lemma 7.5.

Proof of sufficiency of Lemma 7.5. If $R_{n+2} - R_{n+1} > 2e$, then $R_{n+1} = -2e$ and $R_{n+2} = 1$. Then $\det FM$ has an odd order and so $FM \cong W_\nu^{n+2}(\delta\pi)$ for some $\delta \in \mathcal{U}$ and $\nu \in \{1,2\}$. Then, by Lemma 4.12 (iii), we have $M \cong N_\nu^{n+2}(\delta\pi)$. So M is $\mathcal{O}_F$-maximal and thus is n-ADC by Lemma 4.14.

Assume $R_{n+2} - R_{n+1} \le 2e$. By Lemma 2.1, it is sufficient to show that for every N in $\mathcal{M}_n$, if FM represents FN, then M represents N. To do so, we need to verify that Theorem 3.6 (i)–(iv) hold for M and N. But this follows from Lemmas 7.7 and 7.8. ∎

Lemma 7.9. *If M is n-ADC, then it is $(n-1)$-universal.*

Proof. Let N be an $\mathcal{O}_F$-lattice of rank $n-1$. We take a non-zero element $c \in \mathcal{O}_F$ such that $c \ne -\det FM \det FN$, i.e., $c \det FN \ne -\det FM$. Define $N' := N \perp \langle c \rangle$. Then N' is integral and $\det FN' = c \det FN \ne -\det FM$. Since $\dim FM - \dim FN' = 2$, it follows from [31, 63:21 Theorem] that $FN' \rightarrowtail FM$. Since M is n-ADC, we have $N' \rightarrowtail M$. Since also $N \rightarrowtail N'$, we have $N \rightarrowtail M$. Thus M is $(n-1)$-universal by the arbitrariness of N. ∎

In view of Lemma 7.9 and the classification for $(n-1)$-universality in [15, Theorem 4.1], we further have the lemma.

Lemma 7.10. *Suppose that M is n-ADC. Then*

(i) $R_i = 0$ *for* $i \in [1,n]^O$ *and* $R_i = -2e$ *for* $i \in [1,n]^E$.

(ii) *Either* $\alpha_n = 0$ *or* $\alpha_n = R_{n+1} + d[-a_{n,n+1}] = 1$.

(iii) *If* $R_{n+2} - R_{n+1} > 2e$, *then* $R_{n+1} = -2e$; *and if moreover* $n \geq 5$, *or* $n = 3$ *and* $d(a_{1,4}) = 2e$, *then* $R_{n+2} = 1$.

Lemma 7.11. *Suppose* $n = 3$, $d(a_{1,4}) = \infty$, $R_1 = R_3 = R_2 + 2e = R_4 + 2e = 0$ *and* $R_5 > 1$. *Then Theorem* 3.6 (iii) *fails at* $i = 4$ *for all* $N = N_2^3(c)$ *with* $c \in \mathcal{V}$.

Proof. Since $R_4 - R_3 = -2e$, we have $d[-a_{3,4}] \geq 2e$ by Proposition 3.4 (iv).

We have $c = \varepsilon$ or $\varepsilon\pi$ for some $\varepsilon \in \mathcal{U}$. For $N = N_2^3(\varepsilon)$, we have $S_1 = 0$, $S_2 = 2-2e$ and $S_3 = 0$ by Lemma 4.12 (ii), so $d[a_{1,2}b_{1,2}] = 2e - 1$ by Lemma 7.6 (ii). For $N = N_2^3(\varepsilon\pi)$, we have $S_1 = 0$, $S_2 = -2e$ and $S_3 = 1$ by Lemma 4.12 (iii), so $d[a_{1,2}b_{1,2}] \geq 2e$ by Lemma 7.6 (ii). Since also $d[-a_{3,4}] \geq 2e$, by the domination principle, we have $d[-a_{1,4}b_{1,2}] = 2e - 1$ or $\geq 2e$, according as $S_3 = 0$ or 1. So, in both cases,

$$d[-a_{1,4}b_{1,2}] \geq 2e - 1 + S_3.$$

So we conclude that $R_5 > 1 \geq S_3$ and

$$d[-a_{1,4}b_{1,2}] + d[-a_{1,5}b_{1,3}] \geq (2e - 1 + S_3) + 0 > 2e + S_3 - R_5. \tag{7.2}$$

It remains to show that $[a_1, a_2, a_3, a_4]$ fails to represent $[b_1, b_2, b_3]$. Since $a_{1,4} \in F^{\times 2}$, $[a_1, a_2, a_3, a_4] \cong W_1^4(1) = \mathbb{H}^2$ by Proposition 3.5 (iv). Also, $[b_1, b_2, b_3] \cong W_2^3(c)$. Hence $[b_1, b_2, b_3] \not\rightarrow [a_1, a_2, a_3, a_4]$ by Lemma 4.4 (ii). ∎

If M is n-ADC, then it is $(n-1)$-universal by Lemma 7.9 (iii) and thus M satisfies the hypothesis of [15, Lemma 5.8] from Lemma 7.10. Hence we have the following lemma.

Lemma 7.12. *Suppose that M is n-ADC. If* $\alpha_n = 1$ *and either* $R_{n+1} = 1$ *or* $R_{n+2} > 1$, *then*

$$d\left((-1)^{(n+1)/2}a_{1,n+1}\right) = 1 - R_{n+1} < 2e,$$

$((-1)^{(n+1)/2}a_{1,n+1})^\#$ *is a unit and* $d(((-1)^{(n+1)/2}a_{1,n+1})^\#) = 2e + R_{n+1} - 1$.

Lemma 7.13. *Suppose that M is n-ADC and* $FM \cong W_\nu^{n+2}(c)$. *Thus*

$$c = (-1)^{(n+1)/2}a_{1,n+2}.$$

Let $\tilde{c} = (-1)^{(n+1)/2}a_{1,n+1}$ *and let* $N = N_\nu^n(c)$ *or* $N_\nu^n(c\tilde{c}^\#)$.

If $\alpha_n = 1$ *and either* $R_{n+1} = 1$ *or* $R_{n+2} > 1$, *then*

(i) $R_{n+2} > S_n$ *and* $d[-a_{1,n+1}b_{1,n-1}] + d[-a_{1,n+2}b_{1,n}] > 2e + S_n - R_{n+2}$.

(ii) $[a_1, \ldots, a_{n+1}]$ *fails to represent* $FN = [b_1, \ldots, b_n]$.

Thus, Theorem 3.6 (iii) *fails at* $i = n + 1$.

Proof. (i) First, $\mathrm{ord}(a_{1,n})$ is even from Lemma 7.10 (i) and $\tilde{c}^{\#}$ is a unit from Lemma 7.12. Hence $\mathrm{ord}(c) = \mathrm{ord}(c\tilde{c}^{\#}) \equiv R_{n+2} - R_{n+1} \pmod 2$. Therefore, by Lemma 4.12, both when $N = N_\nu^n(c)$ or $N_\nu^n(c\tilde{c}^{\#})$ we have

$$S_n = \begin{cases} 1 & \text{if } R_{n+2} - R_{n+1} \text{ is odd,} \\ 0 & \text{if } R_{n+2} - R_{n+1} \text{ is even.} \end{cases}$$

Note that $R_{n+2} \geq 0$ by Proposition 3.5 (i). If $R_{n+2} = 0$, then $R_{n+1} = 1$ by the hypothesis. This contradicts Corollary 3.2 (i). Thus $R_{n+2} \geq 1$. If $R_{n+2} = 1$, then $R_{n+1} = 1$ by the hypothesis. Then $R_{n+2} - R_{n+1}$ is even, so $S_n = 0$ and thus $R_{n+2} = 1 > 0 = S_n$. If $R_{n+2} > 1$, then $R_{n+2} > 1 \geq S_n$. So, in all cases, $R_{n+2} > S_n$.

Secondly, from the hypothesis, we have

$$a_{1,n+2} = (-1)^{(n+1)/2} c \quad \text{and} \quad b_{1,n} = (-1)^{(n-1)/2} c \quad \text{or} \quad (-1)^{(n-1)/2} c\tilde{c}^{\#}.$$

By Lemma 7.12, we also have $d(\tilde{c}^{\#}) = 2e + R_{n+1} - 1$. Hence

$$d[-a_{1,n+2}b_{1,n}] = d(-a_{1,n+2}b_{1,n}) = \begin{cases} d(c^2) = \infty & \text{if } N = N_\nu^n(c), \\ d(c^2\tilde{c}^{\#}) = 2e + R_{n+1} - 1 & \text{if } N = N_\nu^n(c\tilde{c}^{\#}). \end{cases}$$

Also, by Lemma 7.6 (v), $d[-a_{1,n+1}b_{1,n-1}] = 1 - R_{n+1}$. Thus

$$\begin{aligned} d[-a_{1,n+1}b_{1,n-1}] + d[-a_{1,n+2}b_{1,n}] &\geq (1 - R_{n+1}) + (2e + R_{n+1} - 1) = 2e \\ &> 2e + S_n - R_{n+2}. \end{aligned}$$

(Recall that we have shown $R_{n+2} > S_n$.)

(ii) Let $V = [a_1, \ldots, a_{n+1}]$. Then $\det V = a_{1,n+1} = (-1)^{(n+1)/2}\tilde{c}$, so $V \cong W_{\nu'}^{n+1}(\tilde{c})$, with $\nu' \in \{1, 2\}$. Assume that V represents $[b_1, \ldots, b_n] \cong FN$ for $N = N_\nu^n(c)$ and $N = N_\nu^n(c\tilde{c}^{\#})$, i.e., $W_{\nu'}^{n+1}(\tilde{c})$ represents both $W_\nu^n(c)$ and $W_\nu^n(c\tilde{c}^{\#})$. Then, by Lemma 4.4 (ii), we have

$$(c, \tilde{c})_{\mathfrak{p}} = (-1)^{\nu+\nu'} = (c\tilde{c}^{\#}, \tilde{c})_{\mathfrak{p}} = (c, \tilde{c})_{\mathfrak{p}} (\tilde{c}^{\#}, \tilde{c})_{\mathfrak{p}},$$

which implies $(\tilde{c}^{\#}, \tilde{c})_{\mathfrak{p}} = 1$. This contradicts (4.3). ■

Proof of necessity of Lemma 7.5. Let $FM \cong W_\nu^{n+2}(c)$, where $\nu \in \{1, 2\}$ and $c \in \mathcal{V}$. Suppose that M is n-ADC. Then (i) and (ii) coincide with Lemma 7.10 (i)–(ii).

For (iii), assume that $R_{n+2} - R_{n+1} > 2e$. Then, by Lemma 7.10 (iii), $R_{n+1} = -2e$. If, moreover, either $n \geq 5$ or $n = 3$ and $d(a_{1,4}) = 2e$, then also $R_{n+2} = 1$. So (iii) holds.

Suppose now that in the remaining case, $n = 3$ and $d(a_{1,4}) \neq 2e$, and (iii) fails, i.e., $R_5 \neq 1$. Again, by Lemma 7.10 (i) and (iii), $R_1 = R_3 = R_2 + 2e = R_4 + 2e = 0$. Since $d(a_{1,4}) \neq 2e$, Proposition 3.5 (iv) implies $d(a_{1,4}) = \infty$. Also from $R_5 - R_4 > 2e$ we see that $R_5 > R_4 + 2e = 0$, so $R_5 \neq 1$ implies $R_5 > 1$.

Let $c' \in \mathcal{V} \backslash \{c\}$ and let $N = N_2^3(c')$. Since $c' = c$, we have $N \not\cong N_{3-\nu}^3(c)$. So, by Lemma 4.6 (ii), M represents N. But, by Lemma 7.11, Theorem 3.6 (iii) fails for M and N, so M cannot represent N. Contradiction. Hence $R_5 = 1$ and (iii) is proved.

For (iv), suppose $\alpha_n = 1$ and either $R_{n+1} = 1$ or $R_{n+2} > 1$. By Lemma 4.4 (i), $N_\nu^n(c) \not\cong N_{3-\nu}^n(c)$ and $N_\nu^n(c\tilde{c}^\#) \not\cong N_{3-\nu}^n(c)$, so, by Lemma 4.6 (ii), M represents $N_\nu^n(c)$ and $N_\nu^n(c\tilde{c}^\#)$. But, by Lemma 7.13, Theorem 3.6 (iii) fails for either $N = N_\nu^n(c)$ or $N = N_\nu^n(c\tilde{c}^\#)$. Hence $R_{n+1} = R_{n+1} - R_n \neq 1$. Since $\alpha_n = 1$, Proposition 3.4 (iii) implies $R_{n+1} \in [2-2e, 0]^E$. Also, $R_{n+2} \leq 1$, i.e., $R_{n+2} \in \{0, 1\}$. Thus (iv) is proved. ■

Unlike in the even case, there are many n-ADC lattices that are not $\mathcal{O}_F$-maximal when n is odd. Thus, to complete the proof of Theorem 7.2, we need to determine the structures of n-ADC lattices explicitly, as presented in Lemma 7.20.

First, recall from Theorem 7.4 that M is n-ADC of rank $n+2$ if and only if

$$\begin{aligned} &\text{(a) } R_i = 0 \text{ for } i \in [1,n]^O \quad \text{and} \quad R_i = -2e \text{ for } i \in [1,n]^E; \\ &\text{(b) } R_{n+1} \in [-2e, 0]^E; \quad \text{(c) } \alpha_n \in \{0,1\}; \quad \text{(d) } R_{n+2} \in \{0,1\}. \end{aligned} \tag{7.3}$$

Lemma 7.14. *Let $\nu \in \{1,2\}$ and $\varepsilon \in \mathcal{U}$. Suppose that M is n-ADC.*

(i) *If $FM \cong W_\nu^{n+2}(\varepsilon)$, then $R_{n+2} = 0$.*

(ii) *If $FM \cong W_\nu^{n+2}(\varepsilon\pi)$, then $R_{n+2} = 1$.*

Proof. Clearly, $\mathrm{ord}(a_{1,n+2})$ is even or odd, according as $FM \cong W_\nu^{n+2}(\varepsilon)$ or $W_\nu^{n+2}(\varepsilon\pi)$. By (7.3) (a)–(b), we have

$$\mathrm{ord}(a_{1,n+2}) \equiv \sum_{i=1}^{n+2} R_i \equiv R_{n+2} \pmod 2.$$

By (d), we further have

$$R_{n+2} = \begin{cases} 0 & \text{if } \mathrm{ord}(a_{1,n+2}) \text{ is even,} \\ 1 & \text{if } \mathrm{ord}(a_{1,n+2}) \text{ is odd,} \end{cases}$$

as desired. ■

Lemma 7.15. *Let M, M' be two n-ADC $\mathcal{O}_F$-lattices of rank $n+2$. Then $M \cong M'$ if and only if $FM \cong FM'$ and $R_{n+1}(M) = R_{n+1}(M')$.*

Proof. We only need to show the sufficiency. Let $M \cong \prec a_1, \ldots, a_{n+2} \succ$, $R_i = R_i(M)$ and $\alpha_i = \alpha_i(M)$. Since M is n-ADC, the conditions (a)–(d) in (7.3) hold. Let $M' \cong \prec b_1, \ldots, b_{n+2} \succ$, $S_i = R_i(M')$ and $\beta_i = \alpha_i(M')$. The same conditions (a')–(d') hold for the corresponding invariants S_i and β_i of M'.

By (7.3) (a) and (a'), we have $R_i = S_i$ for $1 \leq i \leq n$. By hypothesis, $R_{n+1} = S_{n+1}$. And, by Lemma 7.14, $R_{n+2} = S_{n+2} = 0$ or 1, according as $FM \cong FM' \cong W_\nu^{n+2}(\varepsilon)$ or $W_\nu^{n+2}(\varepsilon\pi)$ for some $\varepsilon \in \mathcal{U}$. Thus

$$R_i = S_i \tag{7.4}$$

for $1 \leq i \leq n+2$, i.e., the condition (i) of [4, Theorem 3.1] is fulfilled.

Suppose $R_{n+1}=-2e$. If $FM\cong FM'\cong W_1^{n+2}(\varepsilon)$ with $\varepsilon\in\mathcal{U}$, by Lemma 7.14 (i), we have $R_{n+2}=0$. Hence, by Lemma 4.12 (i), $M\cong M'\cong N_1^{n+2}(\varepsilon)$. If $FM\cong FM'\cong W_\nu^{n+2}(\varepsilon\pi)$, with $\nu\in\{1,2\}$ and $\varepsilon\in\mathcal{U}$, by Lemma 7.14 (ii), we have $R_{n+2}=1$. Hence, by Lemma 4.12 (iii), $M\cong M'\cong N_\nu^{n+2}(\varepsilon\pi)$.

Now, assume that $R_{n+1}\neq -2e$, i.e., $\alpha_n\neq 0$. By (7.4), M and M' satisfy the condition (i) of [4, Theorem 3.1], so we are left to verify that the conditions (ii)–(iv) are fulfilled.

By (c), we have $\alpha_n=1$. By (a) and Proposition 3.3 (i)–(ii), we have

$$\alpha_i=\begin{cases}0 & \text{if } i\in[1,n-1]^O,\\ 2e & \text{if } i\in[1,n-1]^E.\end{cases}\tag{7.5}$$

If $R_{n+2}=1$, then $R_{n+2}-R_{n+1}$ is odd, so Proposition 3.3 (iii) implies $\alpha_{n+1}=R_{n+2}-R_{n+1}=1-R_{n+1}$; if $R_{n+2}=0$, then $R_n=R_{n+2}$, so $R_{n+1}+\alpha_{n+1}=R_n+\alpha_n=1$ by [4, Corollary 2.3(i)], i.e., $\alpha_{n+1}=1-R_{n+1}$. Hence, in both cases, we have

$$\alpha_{n+1}=1-R_{n+1}.$$

The same argument combined with (7.4) gives the values of β_i's. Thus $\alpha_i=\beta_i$ for $1\le i\le n+1$ and so [4, Theorem 3.1 (ii)] holds for M and M'.

For $i\in[1,n]^O$, we have $\alpha_i\le 1$ ($\alpha_i=0$ for $i\in[1,n-1]^O$ and $\alpha_n=1$). Since $R_i=S_i$, $\mathrm{ord}(a_{1,i}b_{1,i})=\sum_{k=1}^{i}(R_k+S_k)$ is even, so $d(a_{1,i}b_{1,i})\ge 1\ge\alpha_i$. For $i\in[1,n]^E$, since $R_i=-2e$, by Proposition 3.5 (iii), we have $d((-1)^{i/2}a_{1,i})\ge d[(-1)^{i/2}a_{1,i}]\ge 2e$. Similarly, $d((-1)^{i/2}b_{1,i})\ge 2e$. Hence, by the domination principle, $d(a_{1,i}b_{1,i})\ge 2e=\alpha_i$. For $i=n+1$, by Proposition 3.4 (v), we have $d(-a_na_{n+1})\ge d[-a_{n,n+1}]=1-R_{n+1}$. Since $d((-1)^{(n-1)/2}a_{1,n-1})\ge 2e$, by the domination principle, we see that

$$\begin{aligned}d\big((-1)^{(n+1)/2}a_{1,n+1}\big)&\ge\min\big\{d\big((-1)^{(n-1)/2}a_{1,n-1}\big),d(-a_na_{n+1})\big\}\\&\ge\min\{2e,1-R_{n+1}\}=1-R_{n+1}.\end{aligned}$$

Similarly, $d((-1)^{(n+1)/2}b_{1,n+1})\ge 1-R_{n+1}$. So, by the domination principle again, we conclude that $d(a_{1,n+1}b_{1,n+1})\ge 1-R_{n+1}=\alpha_{n+1}$. Thus [4, Theorem 3.1 (iii)] holds for M and M'.

By (7.5), we have $\alpha_i+\alpha_{i+1}=2e$ for $1\le i\le n-2$. Recall that $\alpha_n=1$, $\alpha_{n+1}=1-R_{n+1}$ and $R_{n+1}\in[2-2e,0]^E$. We also have $\alpha_n+\alpha_{n+1}=1+(1-R_{n+1})=2-R_{n+1}\le 2e$. For $i=n-1$, since $\alpha_{n-1}+\alpha_n=2e+1>2e$, we need to prove that $[b_1,\dots,b_{n-1}]\twoheadrightarrow[a_1,\dots,a_n]$. By Proposition 3.5 (iv) and (v), $[b_1,\dots,b_{n-1}]\cong W_1^{n-1}(\eta)$, with $\eta\in\{1,\Delta\}$, and $[a_1,\dots,a_n]\cong W_1^n(\delta)$ for some $\delta\in\mathcal{O}_F^\times$. Then $W_1^{n-1}(\eta)\twoheadrightarrow W_1^n(\delta)$ follows from Lemma 4.4 (ii). (Both when $\eta=1$ or Δ, we have $(\eta,\delta)_\mathfrak{p}=1$.) Thus [4, Theorem 3.1 (iv)] holds for M and M'. ∎

Definition 7.16. Let $\nu\in\{1,2\}$, $r\in\{0,\dots,e\}$ and $c\in\mathcal{V}$. We denote by $M_{\nu,r}^{n+2}(c)$ the only n-ADC lattice M with $FM\cong W_\nu^{n+2}(c)$ and $R_{n+1}(M)=-2r$, provided that such lattice exists.

Remark 7.17. By Lemma 7.15, such lattice is unique up to isometry, if it exists.

If M is n-ADC of rank $n+2$, from (7.3) (b) we have $R_{n+1}(M) \in [-2e, 0]^E$, i.e., $R_{n+1}(M) = -2r$ for some $0 \le r \le e$. Hence $M \cong M_{\nu,r}^{n+2}(c)$, where $FM \cong W_\nu^{n+2}(c)$. Thus every n-ADC lattice of rank $n+2$ is isometric to $M_{\nu,r}^{n+2}(c)$ for some, $\nu \in \{1,2\}$, $r \in \{0,\dots,e\}$ and $c \in \mathcal{V}$.

Lemma 7.18. *Suppose that M is n-ADC. If $FM \cong W_2^{n+2}(\varepsilon)$ for some $\varepsilon \in \mathcal{U}$, then $R_{n+1} \neq -2e$. Equivalently, $M_{2,e}^{n+2}(\varepsilon)$ is not defined.*

Proof. Assume that $R_{n+1} = -2e$. Since $FM \cong W_2^{n+2}(\varepsilon)$, by Lemma 7.14 (i), $R_{n+2} = 0$. Proposition 3.5 (v), with $j = n+1$, implies that

$$FM \cong \mathbb{H}^{(n+1)/2} \perp [\varepsilon'] = W_1^{n+2}(\varepsilon')$$

for some $\varepsilon' \in \mathcal{U}$, a contradiction. ∎

Lemma 7.19. *Let $\nu \in \{1,2\}$, $c \in \mathcal{V}$ and $\delta \in \mathcal{U}$, with $d(\delta) < 2e$. Then $M = N_\nu^{n+1}(\delta) \perp \langle c \rangle$ is n-ADC and $R_{n+1}(M) = 1 - d(\delta) \in [2-2e, 0]^E$.*

Proof. By Lemma 4.9 and Remark 4.10, we have $N_\nu^{n+1}(\delta) \cong \prec a_1,\dots,a_{n+1} \succ$ relative to a good BONG, with $(a_1,\dots,a_{n-1}) = (1, -\pi^{-2e},\dots,1,-\pi^{-2e})$ and $(a_n, a_{n+1}) = (1, -\delta\pi^{1-d(\delta)})$ or $(\delta^\#, -\delta^\#\delta\pi^{1-d(\delta)})$, according as $\nu = 1$ or 2. Put $R_i = R_i(N_\nu^{n+1}(\delta))$. Then, by Lemma 4.11 (iii), $R_i = 0$ for $i \in [1,n]^O$, $R_i = -2e$ for $i \in [1,n]^E$ and $R_{n+1} = 1 - d(\delta)$. Since $c \in \mathcal{V}$, we have $\mathrm{ord}(c) \in \{0,1\}$. Hence if $a_{n+2} := c$ and $R_{n+2} := \mathrm{ord}(a_{n+2})$, then $R_{n+2} \in \{0,1\}$.

Since $R_{n+2} \ge 0 = R_n$ and $R_{n+2} \ge 0 \ge 1 - d(\delta) = R_{n+1}$, by [2, Corollary 4.4 (v)], we have

$$M \cong \prec a_1,\dots,a_{n+1} \succ \perp \prec a_{n+2} \succ \cong \prec a_1,\dots,a_{n+1},a_{n+2} \succ$$

relative to a good BONG and $R_i(M) = R_i$. In particular, since $\delta \in \mathcal{U}\backslash\{1,\Delta\}$, we have $d(\delta) \in [1, 2e-1]^O$, so $R_{n+1}(M) = 1 - d(\delta) \in [2-2e, 0]^E$.

Write $\alpha_n = \alpha_n(M)$. Since $R_{n+1} - R_n = R_{n+1} > -2e$, Proposition 3.4 (ii) implies that $\alpha_n \ge 1$. On the other hand, for $\nu \in \{1,2\}$, in $F^\times/F^{\times 2}$ we have $-a_n a_{n+1} = \delta$, so

$$\alpha_n \le R_{n+1} - R_n + d(-a_n a_{n+1}) = \big(1 - d(\delta)\big) - 0 + d(\delta) = 1.$$

Hence $\alpha_n = 1$.

With above discussion, we have shown the conditions (a)–(d) in (7.3). By Theorem 7.4, M is n-ADC. ∎

Let $c \in F^\times$. For convenience, we also write $c = \mathcal{U}$ (resp. $c \neq \mathcal{U}$) for $c \in \mathcal{U}$ (resp. $c \notin \mathcal{U}$) temporarily.

Lemma 7.20. *Let $\nu \in \{1,2\}$, $r \in \{0,\dots,e\}$ and $c \in \mathcal{V}$. Then $M_{\nu,r}^{n+2}(c)$ is defined except for $(\nu,r,c) = (2,e,\mathcal{U})$.*

(i) *If $r = e$ and $(\nu, c) \neq (2,\mathcal{U})$, then $M_{\nu,e}^{n+2}(c) \cong N_\nu^{n+2}(c)$.*

(ii) *If* $r = e - 1$ *and* $(\nu, c) = (2, \mathcal{U})$, *then* $M^{n+2}_{2,e-1}(c) \cong N^{n+2}_{2}(c)$.

(iii) *If* $0 \le r \le e - 1$, *then* $M^{n+2}_{\nu,r}(c) \cong N^{n+1}_{\nu'}(\omega_r) \perp \langle \omega_r c\rangle$, *where* $\omega_r \in \mathcal{U}$ *is arbitrary such that* $d(\omega_r) = 2r + 1$ *and* $\nu' \in \{1, 2\}$ *satisfies* $(-1)^{\nu'} = (-1)^{\nu}(\omega_r, c)_{\mathfrak{p}}$.[2]

Proof. First, by Lemma 7.18, $M^{n+2}_{2,e}(c)$ is undefined for every $c \in \mathcal{U}$. Next, we will show the assertions (i)–(iii), thereby confirming that the lattice $M^{n+2}_{\nu,r}(c)$ is defined for $(\nu, r, c) \neq (2, e, \mathcal{U})$.

For (i) and (ii), by Lemma 4.14, $N^{n+2}_{\nu}(c)$ is $\mathcal{O}_F$-maximal and thus is n-ADC. We also have $FN^{n+2}_{\nu}(c) \cong W^{n+2}_{\nu}(c)$. If $(\nu, c) \neq (2, \mathcal{U})$, then, by Lemma 4.12 (i) and (iii), $R_{n+1}(N^{n+2}_{\nu}(c)) = -2e$. So, by Definition 7.16, $N^{n+2}_{\nu}(c) \cong M^{n+2}_{\nu,e}(c)$. If $(\nu, c) = (2, \mathcal{U})$, then, by Lemma 4.12 (ii), $R_{n+1}(N^{n+2}_{2}(c)) = 2 - 2e$. So, by Definition 7.16, $N^{n+2}_{2}(c) \cong M^{n+2}_{2,e-1}(c)$.

For (iii), let $M = N^{n+1}_{\nu'}(\omega_r) \perp \langle \omega_r c\rangle$ and $0 \le r \le e - 1$. Since $(\omega_r, c)_{\mathfrak{p}} = (-1)^{\nu+\nu'}$, by Lemma 4.4 (ii), we have $W^{n+1}_{\nu'}(\omega_r) \rightarrowtail W^{n+2}_{\nu}(c)$. Since $\det W^{n+1}_{\nu'}(\omega_r) \det W^{n+2}_{\nu}(c) = \omega_r c$, we get $FM \cong W^{n+1}_{\nu'}(\omega_r) \perp [\omega_r c] \cong W^{n+2}_{\nu}(c)$. Also, by Lemma 7.19, M is n-ADC and $R_{n+1}(M) = 1 - d(\omega_r) = -2r$. Then, by Definition 7.16, $M \cong M^{n+2}_{\nu,r}(c)$. ∎

Corollary 7.21. *Up to isometry, there are* $(8e + 6)(N\mathfrak{p})^e$ *n-ADC lattices of rank* $n + 2$ *with odd* $n \ge 3$, *of which* $(8e - 2)(N\mathfrak{p})^e$ *are not* $\mathcal{O}_F$*-maximal.*

Proof. If M is n-ADC of the form (i) in Lemma 7.20, then $M \cong N^{n+2}_{\nu}(c)$ with $(\nu, c) \neq (2, \mathcal{U})$, and the number of these $\mathcal{O}_F$-maximal lattices is given by

$$3|\mathcal{U}| = 3[\mathcal{O}_F^{\times} : \mathcal{O}_F^{\times 2}] = 6(N\mathfrak{p})^e$$

from (4.4) and [31, 63:9]. If M is n-ADC of the form (iii) in Lemma 7.20, then the number of such lattices is given by

$$4e|\mathcal{U}| = 4e[\mathcal{O}_F^{\times} : \mathcal{O}_F^{\times 2}] = 8e(N\mathfrak{p})^e.$$

Then excluding out the $\mathcal{O}_F$-maximal lattices of the form (ii) in Lemma 7.20, i.e., $N^{n+2}_{2}(\varepsilon)$ with $\varepsilon \in \mathcal{U}$, gives the number for the non $\mathcal{O}_F$-maximal lattices: $4e|\mathcal{U}| - |\mathcal{U}| = 8e(N\mathfrak{p})^e - 2(N\mathfrak{p})^e$, as desired. ∎

Proof of Theorem 7.2. This follows from Definition 7.16, Remark 7.17 and Lemma 7.20. ∎

8. Proof of Theorems 1.5, 1.7, 1.9, 1.10 and 1.11

We first prove Theorems 1.5, 1.9 and 1.10.

Proof of Theorem 1.5. (i) Combine Proposition 4.15 and Theorems 5.1, 6.1 and 7.1.

(ii) This follows from (i) and [31, §82K]. ∎

[2] I am thankful to the referee for suggesting this improved version for the case $\alpha_n = 1$, which refines the original version of Theorem 7.2.

Proof of Theorem 1.9. Combine Theorems 5.1, 6.2 and 7.2 and Remark 7.3. ∎

Proof of Theorem 1.10. Let M be an integral $\mathcal{O}_F$-lattice of rank m over a local field F.

If $m \in \{n, n+1\}$, or $m = n+2$ and F is non-dyadic, then, by Theorems 1.5 (i) and 1.9 (i), M is n-ADC if and only if it is $\mathcal{O}_F$-maximal. Hence, by (4.4), $B(m,n) = |\mathcal{M}_m| = 8(N\mathfrak{p})^e$ or $8(N\mathfrak{p})^e - 1$, according as $m \geq 3$ or $m = 2$, as required.

Assume that $m = n+2$ and F is dyadic. If n is odd, then we are done by Corollary 7.21. If n is even, then, by Theorem 1.9 (ii), M is n-ADC if and only if M is $\mathcal{O}_F$-maximal or $n = 2$ and it is not $\mathcal{O}_F$-maximal. Consequently, $B(m,n) = 8(N\mathfrak{p})^e$ or $8(N\mathfrak{p})^e + 1$. ∎

In the rest of the paper, we always assume that F is an algebraic number field and M is an $\mathcal{O}_F$-lattice. To show Theorem 1.7, we need some results on the class number of M.

Lemma 8.1. *Suppose that M has class number one.*

(i) *If M is locally n-ADC, then it is globally n-ADC.*

(ii) *If M is $\mathcal{O}_F$-maximal, then it is globally n-ADC.*

Proof. Since the class number of M is one, M is n-regular.

(i) If M is locally n-ADC, then it is globally n-ADC by Theorem 1.3.

(ii) If M is $\mathcal{O}_F$-maximal, then for each $\mathfrak{p} \in \Omega_F \backslash \infty_F$, $M_\mathfrak{p}$ is $\mathcal{O}_{F_\mathfrak{p}}$-maximal by [31, §82K] and so it is n-ADC by Lemma 4.14. Hence M is locally n-ADC, so it is globally n-ADC by (i). ∎

Based on Xu's work [32, §1], we extend [26, Theorem 5.2 and Corollary 5.3] to the indefinite case. (Also see [17, §4].)

Theorem 8.2. *Suppose* rank $M = n + 1 \geq 3$. *Then there exists an $\mathcal{O}_F$-lattice N of rank n such that*

(i) $N \rightarrowtail M$;

(ii) *if $N \rightarrowtail M'$ for some lattice M' in* gen(M), *then $M' \cong M$.*

Proof. This is clear from [26, Theorem 5.2] when M is definite. Assume that M is indefinite. Let $V = FM$ and take $H = O_{\mathbb{A}}(M)O(V)O'_{\mathbb{A}}(V)$ in [32, Theorem 1.5']. Then there exists an $\mathcal{O}_F$-lattice $N \subseteq M$ with rank n such that

$$X_{M/N}O(V)O'_{\mathbb{A}}(V) = O_{\mathbb{A}}(M)O(V)O'_{\mathbb{A}}(V).$$

From the one-to-one correspondence in [32, p. 181], there is only one spinor genus in gen(M) representing N. Since M is indefinite, by [31, 104:5 Theorem], there is exactly one class in gen(M) representing N. ∎

Corollary 8.3. *Suppose* rank $M = n + 1 \geq 3$. *If M is n-regular, then M has class number one.*

Proof. Let M' be a lattice in gen(M). Then there exists some lattice N of rank n such that $N \rightarrowtail M'$ and if $N \rightarrowtail M$ for some lattice M in gen(M'), then $M \cong M'$.

Since $N \twoheadrightarrow M'$, we see that $N_{\mathfrak{p}} \twoheadrightarrow M'_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \Omega_F$. Since M is n-regular, it follows that $N \twoheadrightarrow M$. So $M \cong M'$ and thus the class number of M is one. ■

Proof of Theorem 1.7. Sufficiency is clear from Lemma 8.1 (ii). To show necessity, suppose that M is n-ADC of rank $n+1$. Then, by Theorem 1.3, it is locally n-ADC. So, by Theorem 1.5 (ii), M is $\mathcal{O}_F$-maximal. Again by Theorem 1.3, M is n-regular. Hence, by Corollary 8.3, the class number of M is one. ■

Now, we consider the case $F = \mathbb{Q}$ and $n = 2$. Let p be a prime number. For $\gamma \in \mathbb{Q}$, denote by $N^{(\gamma)}$ the $\mathbb{Z}$-lattice (resp. $\mathbb{Z}_p$-lattice) N scaled by γ (cf. [31, §82J]). Assume that M is a positive definite quaternary $\mathbb{Z}$-lattice. Following [28], a lattice is called *primitive* if $\mathfrak{s}(M) = \mathbb{Z}$. Also note that if $p = 2$, then $\mathbb{H}$ and $\mathbb{A}$ from [28] are $A(0,0)$ and $A(2,2)$, so they coincide with our $\mathbf{H}^{(2)}$ and $\mathbf{A}^{(2)}$. (We have $\mathbf{H} = 2^{-1}A(0,0)$ and $\mathbf{A} = 2^{-1}A(2,2)$.) If $p > 2$, they are the same as our $\mathbf{H}$ and $\mathbf{A}$. But 2 is a unit in $\mathbb{Q}_p$, so $\langle 1,-1\rangle \cong \langle 2,-2\rangle$ and $\langle 1,-\Delta\rangle \cong \langle 2,-2\Delta\rangle$, i.e., $\mathbf{H} \cong \mathbf{H}^{(2)}$ and $\mathbf{A} \cong \mathbf{A}^{(2)}$. So again $\mathbb{H}$ and $\mathbb{A}$ from [28] coincide with $\mathbf{H}^{(2)}$ and $\mathbf{A}^{(2)}$. Then as defined in [28], we call $\mathcal{L}$ *stable at* p if $\mathfrak{n}(\mathcal{L}_p) = 2\mathbb{Z}_p$ and $\mathbf{H}^{(2)} \twoheadrightarrow \mathcal{L}_p$ or $\mathcal{L}_p \cong \mathbf{A}^{(2)} \perp \mathbf{A}^{(2p)}$. Moreover, we call $\mathcal{L}$ *stable* if it is stable at every prime p.

Lemma 8.4. *If M is 2-ADC, then $M^{(2)}$ is 2-regular and stable.*

Proof. If M is 2-ADC, then $\mathfrak{n}(M_p) = \mathbb{Z}_p$, so $\mathfrak{n}(M_p^{(2)}) = 2\mathbb{Z}_p$. By Theorem 1.3, we see that M is 2-regular and locally 2-ADC. Clearly, $M^{(2)}$ is 2-regular because 2-regularity is invariant under scaling. For any prime p, since M_p is 2-ADC, by Theorem 6.2 and Proposition 4.16, $\mathbf{H} \twoheadrightarrow M_p$ or $M_p \cong \mathbf{A} \perp \mathbf{A}^{(p)}$. This implies that $\mathbf{H}^{(2)} \twoheadrightarrow M_p^{(2)}$ or $M_p^{(2)} \cong \mathbf{A}^{(2)} \perp \mathbf{A}^{(2p)}$, so $M_p^{(2)}$ is p-stable. Thus $M^{(2)}$ is stable. ■

By Theorem 1.3 and Lemma 8.4, we have the following corollary.

Corollary 8.5. *M is 2-ADC if and only if it is locally 2-ADC and isometric to $\mathcal{L}^{(1/2)}$ for some stable 2-regular lattice $\mathcal{L}$.*

As in [28], we put $\mathcal{L} \cong [a,b,c,d,f_1,f_2,f_3,f_4,f_5,f_6]$ if

$$\mathcal{L} \cong \begin{pmatrix} a & f_1 & f_2 & f_4 \\ f_1 & b & f_3 & f_5 \\ f_2 & f_3 & c & f_6 \\ f_4 & f_5 & f_6 & d \end{pmatrix}.$$

Table 1 adopted from [28, §4] enumerates all primitive stable 2-regular quaternary $\mathbb{Z}$-lattices, where we list all the primes for which $(\mathcal{L}_i^{(1/2)})_p$ is not 2-ADC in the last column. Then, we relabel these lattices $\mathcal{L}_j^{(1/2)}$, as shown in the first two columns of Table 2. The third and fourth columns provide the local structures of each L_i for the primes p, where $(L_i)_p$ is not unimodular.

$\mathscr{L}$	$[a,b,c,d,f_1,f_2,f_3,f_4,f_5,f_6]$	$d\mathscr{L}$	The primes p where $\mathscr{L}_p^{(1/2)}$ is not 2-ADC
$\mathscr{L}_1$	$[2,2,2,2,0,0,0,1,1,1]$	2^2	None
$\mathscr{L}_2$	$[2,2,2,2,1,0,0,1,0,1]$	5	None
$\mathscr{L}_3$	$[2,2,2,2,0,0,0,1,1,0]$	2^3	None
$\mathscr{L}_4$	$[2,2,2,2,1,0,0,0,0,1]$	3^2	None
$\mathscr{L}_5$	$[2,2,2,4,1,1,0,1,0,0]$	$2^2\cdot 3$	None
$\mathscr{L}_6$	$[2,2,2,2,1,0,0,0,0,0]$	$2^2\cdot 3$	None
$\mathscr{L}_7$	$[2,2,2,4,1,1,0,0,1,0]$	13	None
$\mathscr{L}_8$	$[2,2,2,4,1,0,0,1,0,1]$	17	None
$\mathscr{L}_9$	$[2,2,2,4,0,0,0,1,1,1]$	$2^2\cdot 5$	None
$\mathscr{L}_{10}$	$[2,2,2,4,1,0,0,1,0,0]$	$2^2\cdot 5$	None
$\mathscr{L}_{11}$	$[2,2,2,4,1,0,0,0,0,1]$	$3\cdot 7$	None
$\mathscr{L}_{12}$	$[2,2,2,6,1,1,0,0,1,0]$	$3\cdot 7$	None
$\mathscr{L}_{13}$	$[2,2,2,4,0,0,0,1,1,0]$	$2^3\cdot 3$	None
$\mathscr{L}_{14}$	$[2,2,4,4,1,1,0,1,1,2]$	5^2	None
$\mathscr{L}_{15}$	$[2,2,4,4,1,1,0,0,1,1]$	$2^2\cdot 7$	None
$\mathscr{L}_{16}$	$[2,2,2,6,1,0,0,1,0,0]$	2^5	2
$\mathscr{L}_{17}$	$[2,2,4,4,0,0,0,1,1,2]$	2^5	2
$\mathscr{L}_{18}$	$[2,2,4,4,1,1,0,1,0,0]$	2^5	2
$\mathscr{L}_{19}$	$[2,2,4,4,0,1,1,1,0,2]$	$3\cdot 11$	None
$\mathscr{L}_{20}$	$[2,2,2,10,1,1,0,1,0,0]$	$2^2\cdot 3^2$	3
$\mathscr{L}_{21}$	$[2,2,2,6,0,0,0,1,1,1]$	$2^2\cdot 3^2$	3
$\mathscr{L}_{22}$	$[2,2,2,6,1,0,0,0,0,0]$	$2^2\cdot 3^2$	2
$\mathscr{L}_{23}$	$[2,2,4,4,1,0,0,0,0,2]$	$2^2\cdot 3^2$	3
$\mathscr{L}_{24}$	$[2,2,4,4,0,1,1,1,1,1]$	$2^2\cdot 3^2$	2
$\mathscr{L}_{25}$	$[2,2,4,4,1,0,0,0,0,1]$	$3^2\cdot 5$	None
$\mathscr{L}_{26}$	$[2,2,4,4,0,1,0,0,1,1]$	$3^2\cdot 5$	3
$\mathscr{L}_{27}$	$[2,2,4,6,1,1,0,0,1,1]$	$2^4\cdot 3$	2
$\mathscr{L}_{28}$	$[2,2,4,4,0,1,1,0,0,0]$	$2^4\cdot 3$	2
$\mathscr{L}_{29}$	$[2,4,4,4,0,0,0,1,2,2]$	$2^4\cdot 3$	2
$\mathscr{L}_{30}$	$[2,2,4,4,0,1,0,0,1,0]$	7^2	None
$\mathscr{L}_{31}$	$[2,4,4,4,1,0,2,0,1,2]$	$2^2\cdot 3\cdot 5$	None
$\mathscr{L}_{32}$	$[2,2,4,6,0,1,0,1,1,0]$	$3\cdot 23$	None
$\mathscr{L}_{33}$	$[2,4,4,4,1,1,0,1,0,0]$	$2^4\cdot 5$	2
$\mathscr{L}_{34}$	$[2,2,4,8,0,1,0,0,0,2]$	$2^5\cdot 3$	2
$\mathscr{L}_{35}$	$[2,4,4,4,0,0,0,1,1,1]$	$2^5\cdot 3$	2
$\mathscr{L}_{36}$	$[2,2,6,6,0,1,1,1,1,1]$	$2^2\cdot 5^2$	5
$\mathscr{L}_{37}$	$[2,4,4,6,1,0,2,0,1,2]$	$2^2\cdot 5^2$	2
$\mathscr{L}_{38}$	$[2,4,4,6,0,0,2,-1,1,-1]$	$2^2\cdot 3^3$	3
$\mathscr{L}_{39}$	$[2,4,4,6,1,0,2,0,1,0]$	$2^4\cdot 7$	2
$\mathscr{L}_{40}$	$[2,2,6,8,0,1,1,1,0,3]$	5^3	5
$\mathscr{L}_{41}$	$[2,4,4,8,1,0,2,0,2,0]$	2^7	2
$\mathscr{L}_{42}$	$[2,4,4,6,1,1,0,0,1,1]$	2^7	2
$\mathscr{L}_{43}$	$[2,4,4,8,1,1,0,1,2,2]$	$2^4\cdot 3^2$	2
$\mathscr{L}_{44}$	$[2,4,4,8,1,0,1,1,1,2]$	13^2	None
$\mathscr{L}_{45}$	$[2,4,6,6,0,1,1,1,2,1]$	$3^3\cdot 7$	3
$\mathscr{L}_{46}$	$[2,4,6,6,1,0,1,0,-1,2]$	$2^6\cdot 3$	2
$\mathscr{L}_{47}$	$[2,4,6,10,0,1,2,0,2,1]$	$2^2\cdot 3^4$	3
$\mathscr{L}_{48}$	$[2,4,6,12,0,1,0,0,2,0]$	$2^2\cdot 11^2$	11

Table 1. Quaternary positive definite stable 2-regular integral $\mathbb{Z}$-lattices $\mathscr{L}_i$.

L	$\mathscr{L}^{(1/2)}$	L_p is not unimodular		$\mathbb{Z}$-maximal
		$p=2$	$p>2$	
L_1	$\mathscr{L}_1^{(1/2)}$	$N_2^4(1)$		True
L_2	$\mathscr{L}_2^{(1/2)}$	$N_1^4(5)$	$N_2^4(5)$, $p=5$	True
L_3	$\mathscr{L}_3^{(1/2)}$	$N_2^4(2)$		True
L_4	$\mathscr{L}_4^{(1/2)}$	$N_1^4(1)$	$N_2^4(1)$, $p=3$	True
L_5	$\mathscr{L}_5^{(1/2)}$	$N_2^4(3)$	$N_1^4(3)$, $p=3$	True
L_6	$\mathscr{L}_6^{(1/2)}$	$N_1^4(3)$	$N_2^4(3)$, $p=3$	True
L_7	$\mathscr{L}_7^{(1/2)}$	$N_1^4(5)$	$N_2^4(13)$, $p=13$	True
L_8	$\mathscr{L}_8^{(1/2)}$	$N_1^4(1)$	$N_2^4(17)$, $p=17$	True
L_9	$\mathscr{L}_9^{(1/2)}$	$N_2^4(5)$	$N_1^4(5)$, $p=5$	True
L_{10}	$\mathscr{L}_{10}^{(1/2)}$	$\mathbf{H}\perp\langle 1,-5\rangle$	$N_2^4(5)$, $p=5$	False
L_{11}	$\mathscr{L}_{11}^{(1/2)}$	$N_1^4(5)$	$N_2^4(3)$, $p=3$ $N_1^4(7\Delta_7)$, $p=7$	True
L_{12}	$\mathscr{L}_{12}^{(1/2)}$	$N_1^4(5)$	$N_1^4(3)$, $p=3$ $N_2^4(7\Delta_7)$, $p=7$	True
L_{13}	$\mathscr{L}_{13}^{(1/2)}$	$N_1^4(6)$	$N_2^4(3\Delta_3)$, $p=3$	True
L_{14}	$\mathscr{L}_{14}^{(1/2)}$	$N_1^4(1)$	$N_2^4(1)$, $p=5$	True
L_{15}	$\mathscr{L}_{15}^{(1/2)}$	$N_2^4(7)$	$N_1^4(7)$, $p=7$	True
L_{16}	$\mathscr{L}_{19}^{(1/2)}$	$N_1^4(1)$	$N_2^4(3\Delta_3)$, $p=3$ $N_1^4(11)$, $p=11$	True
L_{17}	$\mathscr{L}_{25}^{(1/2)}$	$N_1^4(5)$	$N_2^4(5)$, $p=3$ $N_1^4(5)$, $p=5$	True
L_{18}	$\mathscr{L}_{30}^{(1/2)}$	$N_1^4(1)$	$N_2^4(1)$, $p=7$	True
L_{19}	$\mathscr{L}_{31}^{(1/2)}$	$N_2^4(7)$	$N_2^4(3\Delta_3)$, $p=3$ $N_2^4(5\Delta_5)$, $p=5$	True
L_{20}	$\mathscr{L}_{32}^{(1/2)}$	$N_1^4(5)$	$N_2^4(3\Delta_3)$, $p=3$ $N_1^4(23)$, $p=23$	True
L_{21}	$\mathscr{L}_{44}^{(1/2)}$	$N_1^4(1)$	$N_2^4(1)$, $p=13$	True

Table 2. Quaternary positive definite 2-ADC integral $\mathbb{Z}$-lattices L_i.

Proof of Theorem 1.11. As mentioned before, Table 1 lists all stable 2-regular quaternary $\mathbb{Z}$-lattices. Hence, by Corollary 8.5, the lattices $\mathscr{L}_i^{(1/2)}$ ($i=1,\dots,48$) in Table 1 are all possible candidates, and it suffices to determine which of them are locally 2-ADC. For each prime $p>2$ with $p\nmid d\mathscr{L}$, L_p is unimodular and so $\mathbb{Z}_p$-maximal. Thus it is 2-ADC. Therefore, one only needs to check if L_p is 2-ADC for $p=2$ and the primes $p>2$ with $p\mid d\mathscr{L}$, and we complete the verification by hand. ∎

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