

Locally analytic vectors and rings of periods

Léo Poyeton

Abstract. In this paper, we try to extend Berger’s and Colmez’s point of view, using locally analytic vectors in order to generalize classical cyclotomic theory, in higher rings of periods. We also explain how the formalism of locally analytic vectors recovers the ring \mathbf{B}_{Sen} of Colmez, and extends to Sen theory in the de Rham case, and to classical (φ, Γ) -modules theory. We explain what happens when we try to generalize constructions of (φ, Γ) -modules to arbitrary infinitely ramified p -adic Lie extensions, and provide a conjecture on the structure of the locally analytic vectors in the corresponding rings. We also highlight the fact that the situation should be very different, depending on whether the p -adic Lie extension “contains a cyclotomic extension” or not. Finally, we explain how some of these constructions may be related to the construction of a ring of trianguline periods.

1. Introduction

Let p be a prime, and let K be a finite extension of \mathbf{Q}_p . We fix $\bar{\mathbf{Q}}_p = \bar{K}$ an algebraic closure of K , and we let $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$ be its absolute Galois group.

A classical idea in p -adic Hodge theory in order to study p -adic representations of \mathcal{G}_K is to use an intermediate extension K_∞/K such that K_∞/K is nice enough but such that it contains “most of the ramification” of $\bar{\mathbf{Q}}_p/K$, so that $\bar{\mathbf{Q}}_p/K_\infty$ is almost étale in the sense of Faltings (which is the same as saying that the p -adic completion of K_∞ is perfectoid). The main example of such an extension is the cyclotomic extension $K(\mu_{p^\infty})$ of K , which has been thoroughly used in p -adic Hodge theory, notably in Sen theory and (φ, Γ) -modules theory.

In some sense, Kummer extensions are simpler than the cyclotomic extension, and work from Breuil [9] and Kisin [30] show that Kummer extensions are very useful in order to study semistable representations. However, Kummer extensions are never Galois and this implies that we usually have to replace them by their Galois closure which increases the difficulty of the situation. Lubin–Tate extensions attached to uniformizers of K , of which the cyclotomic extension when $K = \mathbf{Q}_p$ is a particular case, trivialize local class field theory and thus seem particularly useful in order to extend the p -adic Langlands correspondence to $\text{GL}_2(K)$ (see for example [5, 26, 31] for work in this direction). More generally, the interesting framework should be the one of infinitely ramified Galois extensions whose Galois group is a p -adic Lie group, with potential applications in Iwasawa theory [44].

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Let V be a p -adic representation of \mathcal{G}_K , and let $K_\infty = K(\mu_{p^\infty})$, $H_K = \text{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$ and $\Gamma_K = \text{Gal}(K_\infty/K)$. Recall that the cyclotomic character $\chi_{\text{cycl}}: \Gamma_K \rightarrow \mathbf{Z}_p^\times$ identifies Γ_K with an open subgroup of \mathbf{Z}_p^\times . Since $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale, $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K} \otimes_{\widehat{K_\infty}} \mathbf{C}_p \simeq V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$, so that the study of the \mathbf{C}_p -representation $V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ is reduced to the one of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$. The idea of Sen to study such a representation [43] is to consider the subspace $D_{\text{Sen}}(V)$ of K -finite vectors, which are elements of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$ which belong to finite-dimensional sub- K -vector spaces stable by Γ_K . This is a sub- K_∞ -vector space of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$, and Sen proved that $D_{\text{Sen}}(V) \otimes_{K_\infty} \widehat{K_\infty} \simeq (V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$.

If K_∞ is any infinitely ramified p -adic Lie extension K_∞/K , and if V is a \mathbf{Q}_p -representation of \mathcal{G}_K , then since $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale, we still have an isomorphism $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K} \otimes_{\widehat{K_\infty}} \mathbf{C}_p \simeq V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$, but if the dimension of $\Gamma_K = \text{Gal}(K_\infty/K)$ as a p -adic Lie group is greater or equal to 2, then the space of K -finite vectors of this semi-linear $\widehat{K_\infty}$ -representation of Γ_K is no longer suitable, as shown by [7, Prop. 1.5].

In order to generalize Sen theory to any infinitely ramified p -adic Lie extension K_∞/K , Berger and Colmez suggested replacing the space of K -finite vectors and the use of normalized Tate's traces maps (which no longer exist in general [25]) by the space of locally analytic vectors, which are elements x such that the orbit map $g \mapsto g(x)$ is a locally analytic function on Γ_K . This gives a decompletion of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{\text{Gal}(\overline{\mathbf{Q}}_p/K_\infty)}$ into a $\widehat{K_\infty}^{\text{la}}$ -vector space of dimension $\dim_{\mathbf{Q}_p} V$, but in general $\widehat{K_\infty}^{\text{la}}$ strictly contains K_∞ .

Recall that the strategy developed by Fontaine (see [23]) to study p -adic representations of \mathcal{G}_K is to construct some p -adic rings of periods B , which are topological \mathbf{Q}_p -algebras endowed with an action of \mathcal{G}_K and additional structures such that if V is a p -adic representation of \mathcal{G}_K , then the $B^{\mathcal{G}_K}$ -module $D_B(V) := (B \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is endowed with the structures coming from those on B , and such that the functor $V \mapsto D_B(V)$ gives some interesting invariants attached to V . For Fontaine's strategy to work, one requires that these rings of periods B are \mathcal{G}_K -regular in the sense of [23, §1.4.1] (this implies in particular that $B^{\mathcal{G}_K}$ is a field). We then say that a p -adic representation V of \mathcal{G}_K of dimension d is B -admissible if $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as B -representations. The strategy of Fontaine then consists of classifying p -adic representations according to the rings of periods for which they are admissible. In the case where V is admissible, $D_B(V)$ can usually be used to recover V , or at least $V|_{\mathcal{G}_L}$ for some finite extension L of K .

Colmez has constructed in [12] a ring of periods \mathbf{B}_{Sen} which recovers Sen's theory in the cyclotomic setting. Precisely, he defines $\mathbf{B}_{\text{Sen}}^n$ as the set of power series in the variable u over \mathbf{C}_p , with radius of convergence $\geq p^{-n}$, and endows it with an action of $\text{Gal}(\overline{\mathbf{Q}}_p/K(\mu_{p^n}))$ by $g(u) = u + \log \chi_{\text{cycl}}(g)$ (this makes sense since $\log \chi_{\text{cycl}}(g) \in p^n \mathcal{O}_K$ if $g \in \mathcal{G}_{K_n}$). He then shows that $(\mathbf{B}_{\text{Sen}}^n)^{\mathcal{G}_{K(\mu_{p^n})}} = K(\mu_{p^n})$ and that $K_\infty \otimes_{K_n} (\mathbf{B}_{\text{Sen}}^n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K(\mu_{p^n})}}$ is isomorphic to $\mathbf{D}_{\text{Sen}}(V)$ for n big enough.

One other key ingredient in the study of p -adic representations of \mathcal{G}_K is the theory of (φ, Γ_K) -modules, which provides an equivalence of categories $V \mapsto D(V)$ between the category of all p -adic representations of \mathcal{G}_K and the category of étale (φ, Γ_K) -modules. In Fontaine's theory, (φ, Γ_K) -modules are finite-dimensional vector spaces, defined over

a 2-dimensional local ring \mathbf{B}_K and endowed with semilinear actions of a Frobenius φ and of Γ_K which commutes one to another.

One variant of the theory, which has been used with many useful applications, is the theory of (φ, Γ_K) -modules over the Robba ring $\mathbf{B}_{\text{rig}, K}^\dagger$. The theorem of Cherbonnier–Colmez [11] shows that the category of étale (φ, Γ_K) -modules over \mathbf{B}_K actually embeds into the category of (φ, Γ_K) -modules over $\mathbf{B}_{\text{rig}, K}^\dagger$ of slope 0, and the slope filtration theorem of Kedlaya [28] shows that this is an equivalence of categories.

One interesting feature of the Robba ring is that it can be used as a bridge between the classical theory of (φ, Γ_K) -modules and p -adic Hodge theory, as its elements can be embedded inside \mathbf{B}_{dR}^+ . In particular, Berger has shown [2] how to recover the invariants attached to a p -adic representation V in p -adic Hodge theory from its (φ, Γ_K) -module on the Robba ring.

Kisin and Ren have defined Lubin–Tate (φ_q, Γ_K) -modules [31] and proved that the category of Lubin–Tate étale (φ_q, Γ_K) -modules is equivalent to the one of \mathbf{Q}_p -representations, but unfortunately a result from Fourquaux and Xie [26] shows that those (φ_q, Γ_K) -modules are usually not overconvergent. Results from Berger [3, 5] suggest that the right objects to consider are once again the locally analytic vectors inside some higher rings of periods.

In this paper, we try to understand what happens if we use the point of view of Berger–Colmez of locally analytic vectors in “higher rings of periods”.

Our first remark, which follows from the formalism of locally analytic vectors, is that Colmez’s construction of \mathbf{B}_{Sen} can be generalized to construct rings of periods which “compute the cyclotomic theory”. More precisely, if \mathbf{B} is a \mathbf{Q}_p -Banach (or Fréchet) ring endowed with an action of \mathcal{G}_K , such that the functor

$$V \mapsto \mathbf{D}_{\mathbf{B}}(V)^{\text{la}} := (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{\text{Gal}(\overline{\mathbf{Q}}_p/K(\mu_{p^\infty})), \Gamma_K^{-\text{la}}}$$

gives interesting invariants of V , where V is a p -adic representation of \mathcal{G}_K , then the ring $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B})_1$, the stalk at the identity of the sheaf of locally analytic functions on Γ_K with coefficients in \mathbf{B} , “computes” the functor $V \mapsto \mathbf{D}_{\mathbf{B}}(V)^{\text{la}}$, in the sense that

$$(\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B})_1 \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K} \simeq ((\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K^{-\text{la}}}.$$

In particular, this allows us to provide constructions recovering cyclotomic (φ, Γ) -modules and cyclotomic Sen theory for \mathbf{B}_{dR}^+ -representations in this spirit, which extend to the F -analytic Lubin–Tate case as remarked by Berger [5, §8, §9, §10] and Porat [37, §3].

In order to generalize (φ, Γ_K) -modules theory to any infinitely ramified p -adic Lie extension, one would like to understand the structure of the rings $(\tilde{\mathbf{B}}^I)^{H_K, \Gamma_K^{-\text{la}}}$, where the rings $\tilde{\mathbf{B}}^I$ are some higher rings of periods which are properly defined in Section 2. For the theory to behave well and indeed generalize, we should expect that $(\tilde{\mathbf{B}}^I)^{H_K, \Gamma_K^{-\text{la}}}$ can be interpreted as a ring of power series in d variables, where d is the dimension of Γ_K as a p -adic Lie group. We expect that if K_∞ contains a twist by an unramified character of the cyclotomic extension, in which case we say that the extension K_∞/K contains a cyclotomic extension, then the theory does generalize and the rings $(\tilde{\mathbf{B}}^I)^{H_K, \text{la}}$ can be interpreted as rings of power series in d variables.

Conjecture. *If K_∞/K contains a cyclotomic extension, then the rings $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ can be interpreted as rings of power series in d variables, with some convergence condition.*

More precisely, we expect that, for $n \gg 0$, there exist d elements $x_{1,n}, \dots, x_{d,n}$ in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ such that $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ is the set of power series $\sum_{\mathbf{i}=(i_1, \dots, i_d) \in \mathbb{N}^d} a_{\mathbf{i}} x_{j,n}^{i_j}$ in the variables $(x_{i,n})_{i \in \{1, \dots, d\}}$ with coefficients in K such that the series $\sum_{\mathbf{i}=(i_1, \dots, i_d) \in \mathbb{N}^d} a_{\mathbf{i}} x_{j,n}^{i_j}$ converge in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$.

It was proven by Berger in [5, Thm. 4.4] that this conjecture holds when K_∞/K is a Lubin–Tate extension. In the particular case of the cyclotomic extension, Berger’s result shows that $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ is a ring of power series in one variable with coefficients in K , such that the series converge on some annulus depending only on n and I . Moreover, this variable is, up to some power of the Frobenius, exactly the one used in the construction of cyclotomic (φ, Γ) -modules. In particular, just as in [7], we notice that locally analytic vectors applied to the cyclotomic setting recover the classical theory.

In this paper, we are able to generalize Berger’s result and to prove our conjecture in a particular case.

Theorem. *Let K_∞/K be an infinitely ramified p -adic Lie extension which is a successive extension of \mathbf{Z}_p -extensions and contains a cyclotomic extension. Then the conjecture above is true for K_∞/K .*

The fact that we expect the need to contain a cyclotomic extension follows from the following, which shows that for p -adic Lie extensions which do not contain a cyclotomic extension, the situation looks different.

Theorem. *Let $K_\infty/\mathbf{Q}_{p^2}$ be the anticyclotomic extension, where \mathbf{Q}_{p^2} is the unramified extension of \mathbf{Q}_p of degree 2. Then the rings $(\tilde{\mathbf{B}}^I)^{H_K, \Gamma_K-\text{la}}$ are equal to \mathbf{Q}_{p^2} if $0 \in I$.*

If W is a Fréchet representation of a p -adic Lie group, the space of locally analytic vectors W^{la} can be defined but is too small in general to be able to recover W from W^{la} . We provide in this paper computations of locally analytic vectors for Robba rings in the F -analytic Lubin–Tate case, which highlights this fact. We also show that taking F -locally analytic vectors in the (φ_q, Γ_K) -modules on Robba rings recovers modules defined by Colmez in [16] through different methods.

Theorem. *Let V be an F -analytic representation of \mathcal{G}_K , and let $\mathbf{D}_{\text{rig}}^\dagger(V)$ be its attached (φ_q, Γ_K) -module over the Robba ring $\mathbf{B}_{\text{rig}, K}^\dagger$. We have the following:*

- $(\tilde{\mathbf{B}}_{\text{rig}, K}^\dagger)^{\Gamma_K-\text{la}} = (\mathbf{B}_{\text{rig}, K}^\dagger)^{\Gamma_K-\text{la}} = K\langle\langle t_\pi \rangle\rangle$;
- $\mathbf{D}_{\text{rig}}^\dagger(V)^{\Gamma_K-\text{la}} = \bigcap_{n \geq 0} \varphi^n(\mathbf{D}_{\text{rig}}^\dagger(V))$ and is a free $K\langle\langle t_\pi \rangle\rangle$ -module of rank $\leq \dim_{\mathbf{Q}_p} V$; where t_π is the “Lubin–Tate analog of t ” and $K\langle\langle T \rangle\rangle$ denote the set of power series in T with coefficients in K and infinite radius of convergence.

This theorem alongside [16, Thm. 3.23] show that in general the rank of $\mathbf{D}_{\text{rig}}^\dagger(V)^{\text{la}}$ as a $K\langle\langle t_\pi \rangle\rangle$ -module is strictly smaller than $\dim_{\mathbf{Q}_p} V$ and is thus too small to recover $\mathbf{D}_{\text{rig}}^\dagger(V)$.

Finally, we highlight the fact that thinking of the rings $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B})_1$ as rings of periods could have applications in order to define rings of periods for trianguline representations: a trianguline representation is a representation such that its attached (φ, Γ_K) -module on the Robba ring is a successive extension of rank 1 (φ, Γ_K) -modules, but that does not mean that the corresponding representation itself is a successive extension of rank 1 representations, because the (φ, Γ_K) -modules of rank 1 that appear in the decomposition do not need to be étale. Trianguline representations are assumed to be related to representations coming from global geometric objects (see for example [19, 29]) and for example the representations attached to overconvergent modular forms of finite slope are trianguline.

In order to better understand and parametrize trianguline representations, it would make sense to construct a ring which would be to trianguline representations what \mathbf{B}_{crys} is to crystalline representations, and we try to offer candidate rings for that purpose.

Note that, since unramified representations of \mathcal{G}_K are crystalline and thus trianguline, we expect such rings to contain $\mathbf{Q}_p^{\text{unr}}$. Moreover, when we talk about rings of periods \mathbf{B} , we expect that the corresponding modules attached to p -adic representations V and defined by $\mathbf{D}_{\mathbf{B}}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ satisfy at least the three following properties:

- (1) \mathbf{B} is reduced;
- (2) for any p -adic representation V of \mathcal{G}_K , $\mathbf{D}_{\mathbf{B}}(V)$ is a free $\mathbf{B}^{\mathcal{G}_K}$ -module;
- (3) the \mathbf{B} -linear map $\alpha_V: \mathbf{B} \otimes_{\mathbf{B}^{\mathcal{G}_K}} \mathbf{D}_{\mathbf{B}}(V) \rightarrow \mathbf{B} \otimes_{\mathbf{Q}_p} V$ deduced from the inclusion $\mathbf{D}_{\mathbf{B}}(V) \subset \mathbf{B} \otimes_{\mathbf{Q}_p} V$ by extending the scalars to \mathbf{B} , is injective.

The reason why one would have to define several rings is the following.

Proposition. *There is no ring of periods \mathbf{B} satisfying those three properties and containing $\mathbf{Q}_p^{\text{unr}}$ such that for any finite extension K of \mathbf{Q}_p , \mathbf{B} is a trianguline period ring for \mathcal{G}_K .*

Therefore, our ring of trianguline periods of \mathcal{G}_K has to be dependent on K . In the case $K = \mathbf{Q}_p$, since every rank 1 representation is trianguline, our ring has to contain every $\exp(\alpha \log t)$ with $\alpha \in E$, a field of coefficients. In particular, the ring $\mathbf{B}_{\text{tri}, \mathbf{Q}_p}^{\text{an}}$ we define is to $\tilde{\mathbf{B}}_{\text{rig}}^+$ what the ring \mathbf{B}_{Sen} introduced in [12] is to \mathbf{C}_p : $\mathbf{B}_{\text{tri}, \mathbf{Q}_p}^{\text{an}}$ is the ring $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$, which is also the inductive limit of the rings $\mathcal{C}^{\text{an}}(\Gamma_{K_n}, \tilde{\mathbf{B}}_{\text{rig}}^+)$. Proposition 8.1 shows that

$$(\mathbf{B}_{\text{tri}, \mathbf{Q}_p}^{\text{an}})^{\mathcal{G}_{K_n}} = \mathbf{Q}_p \langle\langle t \rangle\rangle,$$

the set of power series in t with infinite radius of convergence. The module $\mathbf{D}_{\text{tri}, \mathbf{Q}_p}^{\text{an}}(V)$ is therefore a module over $\mathbf{Q}_p \langle\langle t \rangle\rangle$ and is also endowed with a Frobenius φ coming from the one on $\tilde{\mathbf{B}}_{\text{rig}}^+$ and an operator ∇ coming from the action of the Lie algebra of Γ_K and commutes with the action of φ . We then extend these constructions to the F -analytic case, constructing a ring $\mathbf{B}_{\text{tri}, K}^{\text{an}}$ in the same fashion, and we extend Fontaine’s classical formalism of admissibility to take this setting into account.

Generalizing the notion of refinements of p -adic representations [1, 34] to our setting, we prove the following.

Theorem. *Let V be an F -analytic representation of \mathcal{G}_K which is $\mathbf{B}_{\text{tri}, K}^{\text{an}}$ -admissible. Then V is trianguline.*

While the ring $\mathbf{B}_{\text{tri},K}^{\text{an}}$ is too small to contain the periods of all F -analytic trianguline representations of \mathcal{G}_K , we could adapt our constructions to “add a log to our ring”, which would cover the semistable periods, but we would still be missing the “nongeometric” periods of trianguline representations, which appear in item (ii) of [16, Thm. 3.23]. It is not yet clear how many periods one would have to add to $\mathbf{B}_{\text{tri},K}^{\text{an}}$ to get a ring of trianguline periods.

Structure of the paper

The first section of the paper recalls the theory of classical rings of periods and the theory of (φ, Γ) -modules and the rings it involves. The second section recalls the theory of locally and pro-analytic vectors. In Section 4, we recall the main results from [7]. We explain in Section 5 how this framework recovers classical Sen theory for \mathbf{B}_{dR}^+ -representations, and we compute what $(\mathbf{B}_{\text{dR}}^+)^{H_K, \text{la}}$ looks like in some particular cases with emphasis on the Lubin–Tate one. In Section 6 we explain how (φ, Γ) -modules theory is recovered through our framework. In Section 7, we explain what we expect to happen in general when trying to generalize (φ, Γ) -modules theory by using locally analytic vectors, prove the particular case of the conjecture and highlight some problems which may arise in the anticyclotomic case. The computations of locally analytic vectors in Robba rings is done in Section 8. Finally, Section 9 is devoted to the applications to trianguline representations and towards a construction of rings of trianguline periods.

2. Classical p -adic rings of periods and (φ, Γ) -modules

2.1. Fontaine’s strategy and some rings of periods

Let p be a prime, let K be a finite extension of \mathbf{Q}_p and let $\mathcal{G}_K = \text{Gal}(\bar{K}/K)$ be its absolute Galois group. Let k be the residual field $\text{ok } K$ and let $F = W(k)[1/p]$ be the maximal unramified extension of \mathbf{Q}_p inside K . Let \mathbf{C}_p be the p -adic completion of \bar{K} . Let $F_\infty = \mathbf{Q}_p(\mu_{p^\infty})$ be the cyclotomic extension of \mathbf{Q}_p . For $n \geq 1$ let $K_n = K(\mu_{p^n})$ be the extension of K generated by the p^n -th roots of unity, and let $K_\infty = \bigcup_{n \geq 1} K(\mu_{p^n}) = K \cdot F_\infty$ be the cyclotomic extension of K . Let $H_{\mathbf{Q}_p} = \text{Gal}(\bar{\mathbf{Q}}_p/F_\infty)$ and $\Gamma_{\mathbf{Q}_p} = \text{Gal}(F_\infty/\mathbf{Q}_p)$. Let $H_K = \text{Gal}(\bar{K}/K_\infty)$ and $\Gamma_K = \text{Gal}(K_\infty/K)$. Recall that the cyclotomic character $\chi_{\text{cycl}}: \mathcal{G}_K \rightarrow \mathbf{Z}_p^\times$ factors through Γ_K and identifies it with an open subset of \mathbf{Z}_p^\times . We also let K_0 denote the maximal unramified extension of \mathbf{Q}_p inside K_∞ .

Recall that the strategy developed by Fontaine (see [23]) to study p -adic representations of \mathcal{G}_K is to construct some p -adic rings of periods \mathbf{B} , which are topological \mathbf{Q}_p -algebras endowed with an action of \mathcal{G}_K and additional structures such that if V is a p -adic representation of \mathcal{G}_K , then the $\mathbf{B}^{\mathcal{G}_K}$ -module $\mathbf{D}_{\mathbf{B}}(V) := (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is endowed with the structures coming from those on \mathbf{B} , and such that the functor $\mathbf{B} \mapsto \mathbf{D}_{\mathbf{B}}(V)$ gives some interesting invariants attached to V . In Fontaine’s original setting, one requires that these

rings of periods \mathbf{B} are \mathcal{G}_K -regular in the sense of [23, §1.4.1] (this implies in particular that $\mathbf{B}^{\mathcal{G}_K}$ is a field). We then say that a p -adic representation V of \mathcal{G}_K of dimension d is \mathbf{B} -admissible if $\mathbf{B} \otimes_{\mathbf{Q}_p} V \simeq \mathbf{B}^d$ as \mathbf{B} -representations. The strategy of Fontaine then consists of classifying p -adic representations according to the rings of periods for which they are admissible. In the case where V is admissible, $\mathbf{D}_{\mathbf{B}}(V)$ can usually be used to recover V , or at least $V|_{\mathcal{G}_L}$ for some finite extension L of K .

We now recall the construction of some rings of periods.

Let $\tilde{\mathbf{E}}^+ = \varprojlim_{x^{\dagger} \mapsto x^p} \mathcal{O}_{\mathbf{C}_p} = \{(x^{(0)}, \dots) \in \mathcal{O}_{\mathbf{C}_p}^{\mathbf{N}} : (x^{(n+1)})^p = x^{(n)}\}$ and recall [45, Thm. 4.1.2] that this ring is naturally endowed with a ring structure which makes it a perfect ring of characteristic p which is complete for the valuation $v_{\tilde{\mathbf{E}}}$ defined by $v_{\tilde{\mathbf{E}}}(x) = v_p(x^{(0)})$. Let $\tilde{\mathbf{E}}$ be its field of fractions and note that it is algebraically closed. We denote by φ the absolute Frobenius $x \mapsto x^p$ on $\tilde{\mathbf{E}}^+$ and $\tilde{\mathbf{E}}$. The action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\mathcal{O}_{\mathbf{C}_p}$ induces a continuous action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\tilde{\mathbf{E}}$.

Choose a sequence $\varepsilon = (\varepsilon^n)_{n \in \mathbf{N}} \in \tilde{\mathbf{E}}^+$ of compatible p^n -th roots of unity (with $\varepsilon^{(1)} \neq 1$). Let $\bar{v} = \varepsilon - 1 \in \tilde{\mathbf{E}}^+$ and let $\mathbf{E}_{\mathbf{Q}_p} := \mathbf{F}_p((\bar{v})) \subset \tilde{\mathbf{E}}$. Let $\mathbf{E} = \mathbf{E}_{\mathbf{Q}_p}^{\text{sep}}$ be the separable closure of $\mathbf{E}_{\mathbf{Q}_p}$ inside $\tilde{\mathbf{E}}$. The field $\mathbf{E}_{\mathbf{Q}_p}$ is left invariant by the action of $H_{\mathbf{Q}_p}$ so that we have a morphism $H_{\mathbf{Q}_p} \rightarrow \text{Gal}(\mathbf{E}/\mathbf{E}_{\mathbf{Q}_p})$. By [45, Thm. 3.2.2], it is actually an isomorphism. We also let $\mathbf{E}_K = \mathbf{E}^{H_K}$. Note that Γ_K acts on \mathbf{E}_K , and that the action of $\mathcal{G}_{\mathbf{Q}_p}$ on \bar{v} is given by $g(\bar{v}) = (1 + \bar{v})^{\chi_{\text{cycl}}(g)} - 1$.

Let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ and let $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$. We also let $\tilde{\mathbf{B}} = \text{Frac}(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}[1/p]$ and $\tilde{\mathbf{B}}^+ = \tilde{\mathbf{A}}^+[1/p]$. By functoriality of Witt vectors, the action of $\mathcal{G}_{\mathbf{Q}_p}$ extends to an action on $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ that commutes with the Frobenius φ . If L is a finite extension of \mathbf{Q}_p , we let $\tilde{\mathbf{B}}_L = \tilde{\mathbf{B}}^{H_L}$ and $\tilde{\mathbf{A}}_L = \tilde{\mathbf{A}}^{H_L}$, where $H_L = \text{Gal}(\bar{\mathbf{Q}}_p/L(\mu_{p^\infty}))$.

Note that any element x of $\tilde{\mathbf{A}}^+$ can be written as $x = \sum_{k \geq 0} p^k [x_k]$ where the x_k belong to $\tilde{\mathbf{E}}^+$ and $[\cdot]$ denotes the Teichmüller lift. Recall [22, §1.5.1] that we have a surjective morphism of rings $\theta: \tilde{\mathbf{A}}^+ \rightarrow \mathcal{O}_{\mathbf{C}_p}$ given by $\theta(x) = \sum_{k \geq 0} p^k x_k^{(0)}$ and whose kernel is a principal maximal ideal of $\tilde{\mathbf{A}}^+$. This morphism θ naturally extends to $\tilde{\mathbf{B}}^+$ to a surjective morphism that we still denote by $\theta: \tilde{\mathbf{B}}^+ \rightarrow \mathbf{C}_p$. For $m \in \mathbf{N}$, we let \mathbf{B}_m be the ring $\tilde{\mathbf{B}}^+ / \ker(\theta)^m \tilde{\mathbf{B}}^+$ and we endow it with the structure of a p -adic Banach ring by taking the image of $\tilde{\mathbf{A}}^+$ as its ring of integers. We let $\mathbf{B}_{\text{dR}}^+ = \varprojlim_{m \in \mathbf{N}} \mathbf{B}_m$ be the completion of $\tilde{\mathbf{B}}^+$ for the $\ker(\theta)$ -adic topology and we endow it with the Fréchet topology of the projective limit. By construction, θ extends to a continuous morphism $\theta: \mathbf{B}_{\text{dR}}^+ \rightarrow \mathbf{C}_p$ and the action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\tilde{\mathbf{B}}^+$ extends by continuity to a continuous action on \mathbf{B}_{dR}^+ . We let \mathbf{B}_{dR} be the fraction field of \mathbf{B}_{dR}^+ . The power series defining $\log[\varepsilon]$ converges in \mathbf{B}_{dR}^+ to an element t that generates the maximal ideal $\ker(\theta: \mathbf{B}_{\text{dR}}^+ \rightarrow \mathbf{C}_p)$ of \mathbf{B}_{dR}^+ , so that $\mathbf{B}_{\text{dR}} = \mathbf{B}_{\text{dR}}^+[1/t]$. Note that the action of $\mathcal{G}_{\mathbf{Q}_p}$ on t is given by $g(t) = \chi_{\text{cycl}}(g) \cdot t$. We endow \mathbf{B}_{dR} with a filtration by setting $\text{Fil}^i \mathbf{B}_{\text{dR}} = t^i \mathbf{B}_{\text{dR}}^+$. We call representations that are \mathbf{B}_{dR} -admissible “de Rham representations”.

Fontaine has also defined several other rings of periods, among which \mathbf{B}_{crys} and \mathbf{B}_{st} , in order to study p -adic representations. Recall that \mathbf{B}_{crys} is endowed with a Frobenius φ , \mathbf{B}_{st} contains \mathbf{B}_{crys} , is endowed with a Frobenius φ and a monodromy operator N such that

$\mathbf{B}_{\text{crys}} = \mathbf{B}_{\text{st}}^{N=0}$, and \mathbf{B}_{dR} is a field endowed with a filtration $\{\text{Fil}^i \mathbf{B}_{\text{dR}}\}_{i \in \mathbb{Z}}$ and such that there is an injective map $\mathbf{B}_{\text{st}} \rightarrow \mathbf{B}_{\text{dR}}$. Moreover, these rings all contain the element t , and there exist rings $\mathbf{B}_{\text{crys}}^+$ and \mathbf{B}_{st}^+ such that $\mathbf{B}_{\text{crys}} = \mathbf{B}_{\text{crys}}^+[1/t]$ and $\mathbf{B}_{\text{st}} = \mathbf{B}_{\text{st}}^+[1/t]$. Representations that are \mathbf{B}_{crys} -admissible and \mathbf{B}_{st} -admissible are respectively called crystalline and semistable representations. The relations between those rings imply that crystalline representations are semistable and that semistable representations are de Rham. We do not recall the proper definitions of \mathbf{B}_{crys} and \mathbf{B}_{st} as they are not needed in this note.

2.2. Cyclotomic (φ, Γ) -modules

Let us now recall briefly the theory of (φ, Γ) -modules and some of the rings involved in the theory. Let $v = [\varepsilon] - 1$. Let $\mathbf{A}_{\mathbf{Q}_p}$ be the p -adic completion of $\mathbf{Z}_p((v))$ inside $\tilde{\mathbf{A}}$. This is a discrete valuation ring with residue field $\mathbf{E}_{\mathbf{Q}_p}$. Since

$$\varphi(v) = (1 + v)^p - 1 \quad \text{and} \quad g(v) = (1 + v)^{\chi_{\text{cycl}}(g)} - 1 \quad \text{if } g \in \mathcal{G}_{\mathbf{Q}_p},$$

the ring $\mathbf{A}_{\mathbf{Q}_p}$ and its field of fractions $\mathbf{B}_{\mathbf{Q}_p} = \mathbf{A}_{\mathbf{Q}_p}[1/p]$ are both stable by φ and $\mathcal{G}_{\mathbf{Q}_p}$.

For $r > 0$, we define $\tilde{\mathbf{B}}^{\dagger,r}$ the subset of overconvergent elements of “radius” r of $\tilde{\mathbf{B}}$, by

$$\tilde{\mathbf{B}}^{\dagger,r} = \left\{ x = \sum_{n \ll -\infty} p^n [x_n] \text{ such that } \lim_{k \rightarrow +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{p-1}k = +\infty \right\}$$

and we let $\tilde{\mathbf{B}}^{\dagger} = \bigcup_{r>0} \tilde{\mathbf{B}}^{\dagger,r}$ be the subset of all overconvergent elements of $\tilde{\mathbf{B}}$.

Let $\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r}$ be the subset of $\mathbf{B}_{\mathbf{Q}_p}$ given by

$$\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r} = \left\{ \sum_{i \in \mathbb{Z}} a_i v^i, a_i \in \mathbf{Q}_p \text{ such that the } a_i \text{ are bounded} \right. \\ \left. \text{and } \lim_{i \rightarrow -\infty} v_p(a_i) + i \frac{pr}{p-1} = +\infty \right\},$$

and note that $\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r} = \mathbf{B}_{\mathbf{Q}_p} \cap \tilde{\mathbf{B}}^{\dagger,r}$.

Let $\mathbf{B}_{\mathbf{Q}_p}^{\dagger} = \bigcup_{r>0} \mathbf{B}_{\mathbf{Q}_p}^{\dagger,r}$. By [33, §2], this is a Henselian field, and its residue ring is still $\mathbf{E}_{\mathbf{Q}_p}$. Since $\mathbf{B}_{\mathbf{Q}_p}^{\dagger}$ is Henselian, there exists a finite unramified extension $\mathbf{B}_K^{\dagger}/\mathbf{B}_{\mathbf{Q}_p}^{\dagger}$ inside $\tilde{\mathbf{B}}$, of degree f and whose residue field is \mathbf{E}_K . Therefore, there exists $r(K) > 0$ and elements x_1, \dots, x_f in $\mathbf{B}_K^{\dagger,r(K)}$ such that $\mathbf{B}_K^{\dagger,s} = \bigoplus_{i=1}^f \mathbf{B}_{\mathbf{Q}_p}^{\dagger,s} \cdot x_i$ for all $s \geq r(K)$. We let \mathbf{B}_K be the p -adic completion of \mathbf{B}_K^{\dagger} and we let \mathbf{A}_K be its ring of integers for the p -adic valuation. One can show that \mathbf{B}_K is a subfield of $\tilde{\mathbf{B}}$ stable under the action of φ and Γ_K (see for example [14, Prop. 6.1]). Let \mathbf{A} be the p -adic completion of $\bigcup_{K/\mathbf{Q}_p} \mathbf{A}_K$, taken over all the finite extensions K/\mathbf{Q}_p . Let $\mathbf{B} = \mathbf{A}[1/p]$. Note that \mathbf{A} is a complete discrete valuation ring whose field of fractions is \mathbf{B} and with residue field \mathbf{E} . Once again, both \mathbf{A} and \mathbf{B} are stable by φ and $\mathcal{G}_{\mathbf{Q}_p}$. Moreover, we have $\mathbf{A}^{H_K} = \mathbf{A}_K$ and $\mathbf{B}_K = \mathbf{B}^{H_K}$, so that \mathbf{A}_K is a complete discrete valuation ring with residue field \mathbf{E}_K and fraction field $\mathbf{B}_K = \mathbf{A}_K[1/p]$. If L is a finite extension of K , then $\mathbf{B}_L/\mathbf{B}_K$ is an unramified extension of degree $[L_{\infty} : K_{\infty}]$ and if L/K is Galois then so is $\mathbf{B}_L/\mathbf{B}_K$, and we have the following isomorphisms: $\text{Gal}(\tilde{\mathbf{B}}_L/\tilde{\mathbf{B}}_K) = \text{Gal}(\mathbf{B}_L/\mathbf{B}_K) = \text{Gal}(\mathbf{E}_L/\mathbf{E}_K) = \text{Gal}(L_{\infty}/K_{\infty}) = H_K/H_L$.

Definition 2.1. If K is a finite extension of \mathbf{Q}_p , a (φ, Γ_K) -module D on \mathbf{A}_K (resp. \mathbf{B}_K) is an \mathbf{A}_K -module of finite rank (resp. a finite-dimensional \mathbf{B}_K -vector space) endowed with semilinear actions of Γ_K and φ that commute one to another.

It is said to be étale if $1 \otimes \varphi: \varphi^* D \rightarrow D$ is an isomorphism (resp. if there exists a basis of D such that $\text{Mat}(\varphi) \in \text{GL}_d(\mathbf{A}_K)$).

If K is a finite extension of \mathbf{Q}_p and if V is a p -adic representation of \mathcal{G}_K , we set

$$D(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}.$$

Note that $D(V)$ is a (φ, Γ_K) -module. Moreover, if V is a p -adic representation of \mathcal{G}_K , then $D(V)$ is étale and $(\mathbf{B} \otimes_{\mathbf{B}_K} D(V))^{\varphi=1}$ is canonically isomorphic to V (see [21, Prop. 1.2.6]). The functors $V \mapsto D(V)$ and $D \mapsto (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$ then induce an equivalence of Tannakian categories between p -adic representations of \mathcal{G}_K and étale (φ, Γ_K) -modules.

For $r > 0$, we define a valuation $V(\cdot, r)$ on $\tilde{\mathbf{B}}^+[1/[\bar{v}]]$ by setting

$$V(x, r) = \inf_{k \in \mathbf{Z}} \left(k + \frac{p-1}{pr} v_{\mathbf{E}}(x_k) \right)$$

for $x = \sum_{k \gg -\infty} p^k [x_k]$.

For $r = 0$, we let $V(\cdot, 0)$ be the p -adic valuation on $\tilde{\mathbf{B}}^+[1/[\bar{v}]]$.

If I is a closed subinterval of $[0; +\infty[$, $I \neq [0, 0]$, we let $V(x, I) = \inf_{r \in I, r \neq 0} V(x, r)$ (one can take a look at [27, Rem. 2.1.9] to understand why we avoid defining $V(\cdot, 0)$). We then define the ring $\tilde{\mathbf{B}}^I$ as the completion of $\tilde{\mathbf{B}}^+[1/[\bar{v}]]$ for the valuation $V(\cdot, I)$ if $0 \notin I$, and as the completion of $\tilde{\mathbf{B}}^+$ for $V(\cdot, I)$ if $I = [0; r]$. We will write $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger, r}$ for $\tilde{\mathbf{B}}^{[r, +\infty[}$ and $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger}$ for $\tilde{\mathbf{B}}^{[0, +\infty[}$. We also define $\tilde{\mathbf{B}}_{\text{rig}}^{\dagger} = \bigcup_{r \geq 0} \tilde{\mathbf{B}}_{\text{rig}}^{\dagger, r}$.

Let I be a subinterval of $[0, +\infty[$ which is either a subinterval of $]1, +\infty[$ or of the form $[0, r]$, for some $r > 0$. Let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $v_p(a_k) + k/\rho \rightarrow +\infty$ when $|k| \rightarrow +\infty$ for all $\rho \in I$. The series $f(v)$ converges in $\tilde{\mathbf{B}}^I$ and we let $\mathbf{B}_{\mathbf{Q}_p}^I$ denote the set of all $f(\pi)$ with f as above. It is a subring of $\tilde{\mathbf{B}}_{\mathbf{Q}_p}^I$.

We also write $\mathbf{B}_{\text{rig}, \mathbf{Q}_p}^{\dagger, r}$ for $\mathbf{B}_{\mathbf{Q}_p}^{[r; +\infty[}$. It is a subring of $\mathbf{B}_{\mathbf{Q}_p}^{[r; s]}$ for all $s \geq r$ and note that the set of all $f(v) \in \mathbf{B}_{\text{rig}, \mathbf{Q}_p}^{\dagger, r}$ such that the sequence $(a_k)_{k \in \mathbf{Z}}$ is bounded is exactly the ring $\mathbf{B}_{\mathbf{Q}_p}^{\dagger, r}$. Let $\mathbf{B}_{\mathbf{Q}_p}^{\dagger} = \bigcup_{r \gg 0} \mathbf{B}_{\mathbf{Q}_p}^{\dagger, r}$.

Recall that, for K a finite extension of \mathbf{Q}_p , there exists a separable extension $\mathbf{E}_K/\mathbf{E}_{\mathbf{Q}_p}$ of degree $f = [K_{\infty} : F_{\infty}]$ and an attached unramified extension $\mathbf{B}_K^{\dagger}/\mathbf{B}_{\mathbf{Q}_p}^{\dagger}$ of degree f with residue field \mathbf{E}_K , so that there exists $r(K) > 0$ and elements $x_1, \dots, x_f \in \mathbf{B}_K^{\dagger, r(K)}$ such that

$$\mathbf{B}_K^{\dagger, s} = \bigoplus_{i=1}^f \mathbf{B}_{\mathbf{Q}_p}^{\dagger, s} \cdot x_i \quad \text{for all } s \geq r(K).$$

If $r(K) \leq \min(I)$, we let \mathbf{B}_K^I be the completion of $\mathbf{B}_K^{\dagger, r(K)}$ for $V(\cdot, I)$, so that $\mathbf{B}_K^I = \bigoplus_{i=1}^f \mathbf{B}_{\mathbf{Q}_p}^I \cdot x_i$.

We actually have a better description of the rings $\mathbf{B}_{\text{rig},K}^{\dagger,r}$ in general.

Proposition 2.2. *Let K be a finite extension of \mathbf{Q}_p .*

- (1) *There exists $v_K \in \mathbf{A}_K^{\dagger,r(K)}$ whose image modulo p is a uniformizer of \mathbf{E}_K and such that, for $r \geq r(K)$, every element $x \in \mathbf{B}_K^{\dagger,r}$ can be written as $x = \sum_{k \in \mathbf{Z}} a_k v_K^k$, where $a_k \in F' = \mathbf{Q}_p^{\text{unr}} \cap K_\infty$, and the power series $\sum_{k \in \mathbf{Z}} a_k T^k$ is holomorphic and bounded on $\{p^{-1/e_K r} \leq |T| < 1\}$.*
- (2) *Let $\mathcal{H}_{F'}^\alpha(T)$ be the set of power series $\sum_{k \in \mathbf{Z}} a_k T^k$ where $a_k \in F'$ and such that, for all $\rho \in [\alpha; 1[$, $\lim_{k \rightarrow \pm\infty} |a_k| \rho^k = 0$ and let $\alpha_K^r = p^{-1/e_K r}$. Then the map $\mathcal{H}_{F'}^\alpha(T) \rightarrow \mathbf{B}_{\text{rig},K}^{\dagger,r}$ sending f to $f(v_K)$ is an isomorphism.*

Proof. The first item is proved in [14, Prop. 7.5] and the second one in [14, Prop. 7.6]. Be careful that the notations for the rings and the normalizations of the valuations used in Colmez’s paper are a bit different from ours. ■

The following theorem is the main result of [11] and shows that every étale (φ, Γ_K) -module is the base change to \mathbf{B}_K of an overconvergent module.

Theorem 2.3. *If D is an étale (φ, Γ_K) -module, then the set of free sub- \mathbf{B}_K^\dagger -modules of finite type stable by φ and Γ_K admits a bigger element D^\dagger and one has $D = \mathbf{B}_K \otimes_{\mathbf{B}_K^\dagger} D^\dagger$.*

In particular, if V is a p -adic representation of \mathcal{G}_K , then there exists an étale (φ, Γ_K) -module over \mathbf{B}_K^\dagger which we will denote by $\mathbf{D}^\dagger(V)$ and such that $D(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V)$. We let $\mathbf{D}_{\text{rig}}^\dagger(V) = \mathbf{B}_{\text{rig},K}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V)$.

If E is a finite extension of \mathbf{Q}_p , we can make the following definition.

Definition 2.4. A (φ, Γ_K) -module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K}^\dagger$ is a finite module D over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K}^\dagger$, equipped with a semilinear Frobenius φ_D and a continuous semilinear action of Γ_K such that D is free as a $\mathbf{B}_{\text{rig},K}^\dagger$ -module, $\text{id} \otimes \varphi_D: \mathbf{B}_{\text{rig},K}^\dagger \otimes_{\varphi, \mathbf{B}_{\text{rig},K}^\dagger} D \rightarrow D$ is an isomorphism and that the actions of φ_D and Γ_K commute.

By [35, Lem. 1.30], a (φ, Γ_K) -module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K}^\dagger$ is free as an $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K}^\dagger$ -module. We say that a (φ, Γ_K) -module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K}^\dagger$ is étale if its underlying φ -module over $\mathbf{B}_{\text{rig},K}^\dagger$ is étale.

2.3. Lubin–Tate (φ, Γ) -modules

We now recall the theory of (φ, Γ) -modules in the Lubin–Tate setting. We let F be a finite extension of \mathbf{Q}_p , π a uniformizer of \mathcal{O}_F and LT be a Lubin–Tate formal \mathcal{O}_F -module attached to the uniformizer π of \mathcal{O}_F . Let q be the cardinal of the residue field of F and let h be such that $q = p^h$. Let $F_0 = F \cap \mathbf{Q}_p^{\text{unr}}$. We let F_n denote the extension of F generated by the points of π^n -torsion of LT for $n \geq 1$, and $F_\infty = \bigcup_{n \geq 1} F_n$. We let $\Gamma_F = \text{Gal}(F_\infty/F)$ and $H_F = \text{Gal}(F_\infty/F)$. By Lubin–Tate’s theory [32, Thm. 2], the Lubin–Tate character $\chi_\pi: \mathcal{G}_F \rightarrow \mathcal{O}_F^\times$ induces an isomorphism $\Gamma_F \simeq \mathcal{O}_F^\times$. For $a \in \mathcal{O}_F$, we let $[a](T)$ denote the power series that corresponds to the multiplication by a map on LT. Let $v_0 = 0$ and for each $n \geq 1$, let $v_n \in \overline{\mathbf{Q}_p}$ be such that $[\pi](v_n) = v_{n-1}$, with $v_1 \neq 0$.

Recall that we defined rings $\tilde{\mathbf{A}}, \tilde{\mathbf{A}}^+, \tilde{\mathbf{A}}^I$ and $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}^+, \tilde{\mathbf{B}}^I$ previously, and in what follows we will keep the same notations for those rings tensored over F_0 (resp. \mathcal{O}_{F_0} in the case of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}^I$), by F (resp. \mathcal{O}_F). We let $\varphi_q = \varphi^{\circ h}$ and we let $r_k = p^{kh-1}(p-1)$ for $k \geq 1$.

Recall that by [13, §9.2], there exists $v \in \tilde{\mathbf{A}}^+$ whose image in $\tilde{\mathbf{E}}^+$ is (v_0, v_1, \dots) , where $\tilde{\mathbf{E}}^+ = \lim_{\leftarrow x^i \rightarrow x^q} \mathcal{O}_{\mathbf{C}_p}/\pi$ (by [10, Prop. 4.3.1], this is the same ring $\tilde{\mathbf{E}}^+$ as before) and such that

$$g(v) = [\chi_\pi(g)](v) \quad \text{and} \quad \varphi_q(v) = [\pi](v).$$

We also let $t_\pi = \log_{\text{LT}}(v) \in \tilde{\mathbf{B}}_{\text{rig}}^+$ so that

$$g(t_\pi) = \chi_\pi(g) \cdot t_\pi \quad \text{and} \quad \varphi_q(t_\pi) = \pi t_\pi.$$

Note that when $F = \mathbf{Q}_p$ and $\pi = p$, this is exactly the classical t of p -adic Hodge theory.

For $\rho > 0$, let $\rho' = \rho \cdot e \cdot p/(p-1) \cdot (q-1)/q$, where e is the ramification index of F/\mathbf{Q}_p . Let I be a subinterval of $[0, +\infty[$ which is either a subinterval of $]1, +\infty[$ or of the form $[0, r]$, for some $r > 0$. Let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $v_p(a_k) + k/\rho' \rightarrow +\infty$ when $|k| \rightarrow +\infty$ for all $\rho \in I$. The series $f(v)$ converges in $\tilde{\mathbf{B}}^I$ and we let \mathbf{B}_F^I denote the set of all $f(v)$ with f as above. It is a subring of $\tilde{\mathbf{B}}_F^I$. We also write $\mathbf{B}_{\text{rig},F}^{\dagger,r}$ for $\mathbf{B}_F^{[r;+\infty[}$.

We let \mathbf{A}_F denote the p -adic completion of $\mathcal{O}_F((v))$ inside $\tilde{\mathbf{A}}$, and we let $\mathbf{B}_F = \mathbf{A}_F[1/p]$. As in the cyclotomic case, to any extension L/F finite, there corresponds extensions $\mathbf{A}_L/\mathbf{A}_F$ and $\mathbf{B}_L/\mathbf{B}_F$, of degree $[L_\infty : F_\infty]$ where $L_\infty = L \cdot F_\infty$, equipped with actions of φ_q and $\Gamma_L := \text{Gal}(L_\infty/L)$. As in the cyclotomic case, we fix once and for all a finite extension K of F and will apply the theory to p -adic representations of $\mathcal{G}_K = \text{Gal}(\bar{\mathbf{Q}}_p/K)$. There is also a theory of (φ_q, Γ_K) -modules over \mathbf{B}_K , which are finite-dimensional \mathbf{B}_K vector spaces endowed with commuting semilinear actions of Γ_K and φ_q . Once again, such a (φ_q, Γ_K) -module is said to be étale if there exists a basis in which $\text{Mat}(\varphi_q)$ belongs to $\text{GL}_d(\mathbf{A}_K)$. By specializing Fontaine’s constructions [21, Prop. A.1.2.6 and Thm. A.3.4.3], Kisin and Ren prove the following, which is [31, Thm. 1.6].

Theorem 2.5. *There is a Tannakian equivalence of categories between F -linear representations of \mathcal{G}_K and étale (φ_q, Γ_K) -modules over \mathbf{B}_K .*

However, unlike in the cyclotomic case, these (φ_q, Γ_K) modules are rarely overconvergent. Berger showed in [5] that the right subcategory of representations corresponding to overconvergent (φ_q, Γ_K) -modules was the one of F -analytic representations (note however that there are representations which are not F -analytic but whose attached (φ_q, Γ_K) -module is overconvergent). An E -representation V of \mathcal{G}_K , where $E \supset F^{\text{Gal}}$, is said to be F -analytic if for any $\tau \in \text{Emb}(E, \bar{\mathbf{Q}}_p)$, $\tau \neq \text{id}$, the semilinear \mathbf{C}_p -representation $\mathbf{C}_p \otimes^\tau V$ is trivial. In that case, [5, Thm. 10.4] shows that one can attach to V an étale F -analytic (φ_q, Γ_K) -module $\mathbf{D}_{\text{rig}}^\dagger(V)$ on $\mathbf{B}_{\text{rig},K}^\dagger$, which means that the operator $\frac{\log g}{\log \chi_\pi(g)}$ is F -linear on $\mathbf{D}_{\text{rig}}^\dagger(V)$. Note that, when $F = \mathbf{Q}_p$, every representation of \mathcal{G}_K is \mathbf{Q}_p -analytic.

For an F -analytic character $\delta: K^\times \rightarrow E^\times$, we let $w(\delta)$ denote its weight, which is defined by $w(\delta) = \delta'(1)$.

Lemma 2.6. *Let \mathbf{D} be a rank 1 F -analytic (φ_q, Γ_K) -module over $E \otimes_K \mathbf{B}_{\text{rig}, K}^\dagger$. Then there exists δ an F -analytic character $K^\times \rightarrow E^\times$ and a basis e of \mathbf{D} in which $g(e) = \delta(\chi_\pi(g)) \cdot e$ and $\varphi_q(e) = \delta(\pi) \cdot e$.*

Proof. This is the same as in [15, Prop. 3.1], using [5, Thm. 10.4]. ■

3. Locally and pro-analytic vectors

Here, we recall some of the theory of locally- and pro-analytic vectors, following the presentation of Emerton in [20] and of Berger in [5].

Let G be a p -adic Lie group, and let W be a \mathbf{Q}_p -Banach representation of G . Let H be an open subgroup of G such that there exists coordinates $c_1, \dots, c_d: H \rightarrow \mathbf{Z}_p$ giving rise to an analytic bijection $\mathbf{c}: H \rightarrow \mathbf{Z}_p^d$. We say that $w \in W$ is an H -analytic vector if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}^d}$ such that $w_{\mathbf{k}} \rightarrow 0$ in W and such that $g(w) = \sum_{\mathbf{k} \in \mathbf{N}^d} \mathbf{c}(g)^{\mathbf{k}} w_{\mathbf{k}}$ for all $g \in H$. We let $W^{H\text{-an}}$ be the space of H -analytic vectors. This space injects into $\mathcal{C}^{\text{an}}(H, W)$, the space of all analytic functions $f: H \rightarrow W$. Note that $\mathcal{C}^{\text{an}}(H, W)$ is a Banach space equipped with its usual Banach norm, so that we can endow $W^{H\text{-an}}$ with the induced norm, that we will denote by $\|\cdot\|_H$. With this definition, we have $\|w\|_H = \sup_{\mathbf{k} \in \mathbf{N}^d} \|w_{\mathbf{k}}\|$ and $(W^{H\text{-an}}, \|\cdot\|_H)$ is a Banach space.

The space $\mathcal{C}^{\text{an}}(H, W)$ is endowed with an action of $H \times H \times H$, given by

$$((g_1, g_2, g_3) \cdot f)(g) = g_1 \cdot f(g_2^{-1} g g_3)$$

and one can recover $W^{H\text{-an}}$ as the closed subspace of $\mathcal{C}^{\text{an}}(H, W)$ of its $\Delta_{1,2}(H)$ -invariants, where $\Delta_{1,2}: H \rightarrow H \times H \times H$ denotes the map $g \mapsto (g, g, 1)$ (we refer the reader to [20, §3.3] for more details).

We say that a vector w of W is locally analytic if there exists an open subgroup H as above such that $w \in W^{H\text{-an}}$. Let W^{la} be the space of such vectors, so that $W^{\text{la}} = \bigcup_H W^{H\text{-an}}$, where H runs through a sequence of open subgroups of G . The space W^{la} is naturally endowed with the inductive limit topology, so that it is an LB space.

Lemma 3.1. *If W is a ring such that $\|xy\| \leq \|x\| \cdot \|y\|$ for $x, y \in W$, then*

- (1) $W^{H\text{-an}}$ is a ring, and $\|xy\|_H \leq \|x\|_H \cdot \|y\|_H$ if $x, y \in W^{H\text{-an}}$;
- (2) if $w \in W^\times \cap W^{\text{la}}$, then $1/w \in W^{\text{la}}$. In particular, if W is a field, then W^{la} is also a field.

Proof. See [7, Lem. 2.5]. ■

It is often useful to choose a specific fundamental system of open neighborhoods of G : let G_0 be a compact open subgroup of G which is p -valued and saturated (see [41, §26 and §27] for the definition and proof of existence), with coordinates \mathbf{c} , and set

$$G_n = G^{p^n} = \{g^{p^n} : g \in G_0\} \quad \text{for } n \geq 0.$$

These are subgroups [41, Rem. 26.9] which have induced coordinates $\mathbf{c}: G_n \xrightarrow{\sim} (p^n \mathbf{Z}_p)^d$. The normalization is such that for $w \in W^{G_n\text{-an}}$ we can write

$$g(w) = \sum_{\mathbf{k} \in \mathbb{N}^d} c(g)^{\mathbf{k}} w_{\mathbf{k}}$$

for $g \in G_n$ and $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}^d}$ with $p^{n|\mathbf{k}|} w_{\mathbf{k}} \rightarrow 0$, and the Banach norm is given by

$$\|w\|_{G_n\text{-an}} = \sup_{\mathbf{k}} \|p^{n\mathbf{k}} w_{\mathbf{k}}\|.$$

It is easy to check if $w \in W^{G_n\text{-an}}$ then

$$\|w\|_{G_m\text{-an}} \leq \|w\|_{G_{m+1}\text{-an}} \text{ for } m \geq n \quad \text{and} \quad \|w\|_{G_m\text{-an}} = \|w\| \text{ for } m \gg n$$

(see [7, Lem. 2.4]).

Let W be a Fréchet space whose topology is defined by a sequence $\{p_i\}_{i \geq 1}$ of seminorms. Let W_i be the Hausdorff completion of W at p_i , so that

$$W = \varprojlim_{i \geq 1} W_i.$$

The space W^{la} can be defined but as stated in [5] and as will be explained in Section 8, this space is too small in general for what we are interested in, and so we make the following definition, following [5, Def. 2.3].

Definition 3.2. If $W = \varprojlim_{i \geq 1} W_i$ is a Fréchet representation of G , then we say that a vector $w \in W$ is pro-analytic if its image $\pi_i(w)$ in W_i is locally analytic for all i . We let W^{pa} denote the set of all pro-analytic vectors of W , so that $W^{\text{pa}} = \varprojlim_{i \geq 1} W_i^{\text{la}}$.

We extend the definition of W^{la} and W^{pa} for LB and LF spaces respectively.

Proposition 3.3. *Let G be a p -adic Lie group which is a uniform pro- p -group, let B be a Banach G -ring and let W be a free B -module of finite rank, equipped with a compatible G -action. If the B -module W has a basis w_1, \dots, w_d in which $g \mapsto \text{Mat}(g)$ is a globally analytic function $G \rightarrow \text{GL}_d(B) \subset M_d(B)$, then*

- (1) $W^{H\text{-an}} = \bigoplus_{j=1}^d B^{H\text{-an}} \cdot w_j$ if H is a subgroup of G ;
- (2) $W^{\text{la}} = \bigoplus_{j=1}^d B^{\text{la}} \cdot w_j$.

Let G be a p -adic Lie group, let B be a Fréchet G -ring and let W be a free B -module of finite rank, equipped with a compatible G -action. If the B -module W has a basis w_1, \dots, w_d in which $g \mapsto \text{Mat}(g)$ is a pro-analytic function

$$G \rightarrow \text{GL}_d(B) \subset M_d(B),$$

then

$$W^{\text{pa}} = \bigoplus_{j=1}^d B^{\text{pa}} \cdot w_j.$$

Proof. The part for Banach ring is proven in [7, Prop. 2.3] and the one for Fréchet rings is proven in [5, Prop. 2.4]. ■

Note that the map $\log \chi_\pi: \Gamma_K \rightarrow \mathcal{O}_F$ induces isomorphisms $\Gamma_n \simeq \pi^n \mathcal{O}_F$ for $n \gg 0$, and endows Γ_K with an \mathcal{O}_F -analytic structure as a p -adic Lie group.

If W is an F -linear Banach representation of $\Gamma_K = \text{Gal}(K_\infty/K)$, and if $n \geq 1$, we say that $w \in W$ is F -analytic on $\Gamma_n = \text{Gal}(K_\infty/K_n)$ if there exists a sequence $\{w_k\}_{k \geq 0}$ of elements of W such that $\pi^{nk} w_k \rightarrow 0$ such that $g(w) = \sum_{k \geq 0} \log \chi_\pi(g)^k w_k$ for all $g \in \Gamma_n$. This means that w is a Γ_n -analytic vector, with Γ_n viewed as a p -adic Lie group defined over \mathcal{O}_F instead of \mathbf{Z}_p .

If W is a locally analytic representation of Γ_K , we can define operators $\nabla_\tau: W \rightarrow W$, for $\tau \in \Sigma_F := \text{Emb}(F, \bar{\mathbf{Q}}_p)$ in the following way, as in [5, §2].

Definition 3.4. Let L be a field that contains F^{Gal} . If $\tau \in \Sigma_F$, then we have the derivative in the direction τ , which is an element $\nabla_\tau \in L \otimes_{\mathbf{Q}_p} \text{Lie}(\Gamma_F)$. The L -vector space $\text{Hom}_{\mathbf{Q}_p}(F, L)$ is generated by the elements of Σ_F . If W is an L -linear Banach representation of Γ_F and if w is a \mathbf{Q}_p -locally analytic element of W and $g \in \Gamma_F$, then there exists elements $\{\nabla_\tau\}_{\tau \in \Sigma_F}$ of $F^{\text{Gal}} \otimes_{\mathbf{Q}_p} \text{Lie}(\Gamma_F)$ such that we can write

$$\log g(w) = \sum_{\tau \in \Sigma_F} \tau(\log \chi_\pi(g)) \cdot \nabla_\tau(w).$$

In particular, there exist $m \gg 0$ and elements $\{w_k\}_{k \in \mathbf{N}^{\Sigma_F}}$ such that if $g \in \Gamma_m$, then $g(w) = \sum_{k \in \mathbf{N}^{\Sigma_F}} \log \chi_\pi(g)^k w_k$, where $\log \chi_\pi(g)^k = \prod_{\tau \in \Sigma_F} \tau \circ \log \chi_\pi(g)^{k_\tau}$. We have $\nabla_\tau(w) = w_{\mathbf{1}_\tau}$ where $\mathbf{1}_\tau$ is the Σ_F -tuple whose entries are 0 except the τ -th one which is 1. If $\mathbf{k} \in \mathbf{N}^{\Sigma_K}$, and if we set $\nabla^{\mathbf{k}}(w) = \prod_{\tau \in \Sigma_F} \nabla_\tau^{k_\tau}(w)$, then $w_k = \nabla^{\mathbf{k}}(w)/\mathbf{k}!$

Remark 3.5. If w is an F -analytic element of W , so that there exists a sequence $\{w_k\}_{k \geq 0}$ of elements of W such that $\pi^{nk} w_k \rightarrow 0$ such that

$$g(w) = \sum_{k \geq 0} \log \chi_\pi(g)^k w_k \quad \text{for all } g \in \Gamma_n,$$

with $n \gg 0$, then $w_k = \nabla_{\text{id}}^k(w)/k!$.

The standard notations for the set of F -analytic elements of W is $W^{\Gamma_n\text{-an}, F\text{-la}}$, following the notations from [5, §2]. These are the Γ_n -analytic vectors when we treat Γ_n as a p -adic Lie group over \mathcal{O}_F instead of \mathbf{Z}_p . Since in this article we almost always consider the former case and in order to improve the readability and reduce the need for additional notations, we will still denote by $W^{\Gamma_n\text{-an}}$ the set of Γ_n F -analytic vectors of W in the Lubin–Tate setting. We also let $W^{\text{la}} = \bigcup_{n \geq 1} W^{\Gamma_n\text{-an}}$. The main advantage of this formalism is that the cyclotomic case is exactly the particular case in the Lubin–Tate setting where $F = \mathbf{Q}_p$ and $\pi = p$, so that the statements made in the “ F -analytic Lubin–Tate setting” also contain the statements regarding the cyclotomic case.

In the rare cases where we will consider \mathbf{Q}_p -locally analytic vectors of W in the Lubin–Tate setting, we’ll write $W^{\mathbf{Q}_p\text{-la}}$ for the set of \mathbf{Q}_p -locally analytic vectors of W . We also use the same formalism for pro-analytic vectors.

4. Sen theory by Berger–Colmez

Recall that to a p -adic representation V of \mathcal{G}_K , one can attach the K_∞ -vector space $\mathbf{D}_{\text{Sen}}(V)$ which is the set of elements of $W = (\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$ which belong to some finite-dimensional K -vector subspace of W which is stable by Γ_K . The K_∞ -vector space D_{Sen} comes equipped with an action of the Lie algebra of Γ_K and admits a canonical generator $\nabla = \lim_{\gamma \rightarrow 1} \frac{\gamma - 1}{\chi_{\text{cycl}}(\gamma) - 1}$ which is the operator of Sen, usually denoted by Θ_{Sen} and whose eigenvalues are called the generalized Hodge-Tate weights of the representation V .

Colmez has constructed in [12] a ring \mathbf{B}_{Sen} as follows.

Definition 4.1. Let u be a variable and $\mathbf{B}_{\text{Sen}}^n = \mathbf{C}_p\{\{u\}\}_n$ be the set of power series $\sum_{k \geq 0} a_k u^k$ with coefficients in \mathbf{C}_p such that the series $\sum_{k \geq 0} (p^n)^k a_k$ converges in \mathbf{C}_p and equip it with the natural topology and with an action of $\text{Gal}(\bar{K}/K(\mu_{p^n}))$ by setting

$$g\left(\sum_{k \geq 0} a_k u^k\right) = \sum_{k \geq 0} g(a_k)(u + \log \chi_{\text{cycl}}(g))^k.$$

Note that this makes sense since $\log \chi_{\text{cycl}}(g) \in p^n \mathbf{Z}_p$ if $g \in \text{Gal}(\bar{K}/K(\mu_{p^n}))$. Let $\mathbf{B}_{\text{Sen}} = \bigcup_{n \geq 0} \mathbf{B}_{\text{Sen}}^n$, endowed with the inductive limit topology.

We let ∇_u denote the \mathbf{C}_p -linear operator on \mathbf{B}_{Sen} given by $\nabla_u = -\frac{d}{du}$.

We now recall the following properties (for more details, see [12] and [7, §2.2]).

Proposition 4.2. *We have the following:*

- (1) $(\mathbf{B}_{\text{Sen}}^n)^{\mathcal{G}_{K_n}} = K_n$;
- (2) if V is a p -adic representation of \mathcal{G}_K and if n is an integer, let $\mathbf{D}'_{\text{Sen},n}(V) := (\mathbf{B}_{\text{Sen}}^n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ equipped with the operator ∇_u induced by the operator ∇_u on $\mathbf{B}_{\text{Sen}}^n$ (meaning that $(\nabla_u)_{\mathbf{D}'_{\text{Sen},n}(V)}$ acts by $\nabla_u \otimes 1$ on $\mathbf{B}_{\text{Sen}}^n \otimes_{\mathbf{Q}_p} V$) and let $\mathbf{D}'_{\text{Sen}}(V) := \bigcup_{n \geq 0} \mathbf{D}'_{\text{Sen},n}(V)$. Every element δ of $\mathbf{D}'_{\text{Sen},n}(V)$ can be written as $\delta^{(0)} + \delta^{(1)}u + \dots$ where the $\delta^{(i)}$ belong to $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V$. Then the map $\delta \mapsto \delta^{(0)}$ induces an isomorphism of K_∞ -vector spaces between $\mathbf{D}'_{\text{Sen}}(V)$ and $\mathbf{D}_{\text{Sen}}(V)$, and of K_n -vector spaces between $\mathbf{D}'_{\text{Sen},n}(V)$ and $\mathbf{D}_{\text{Sen},n}(V)$ for $n \gg 0$. Moreover, the image of ∇_u by this isomorphism is Θ_{Sen} .

Proof. Item (i) is [12, Thm. 2 (i)]. For item (ii), see [12, Thm. 2 (ii)] and [7, Prop. 2.8]. ■

When K_∞/K is any p -adic Lie extension with Galois group Γ_K (such that $\dim \Gamma_K \geq 2$ or such that K_∞/K is almost totally ramified), Berger and Colmez offer to replace classical Sen theory with the theory of locally analytic vectors, by considering the locally analytic vectors of semilinear \widehat{K}_∞ -representations of Γ_K .

Theorem 4.3. *If W is a \widehat{K}_∞ -semilinear representation of Γ_K , then the map*

$$\widehat{K}_\infty \otimes_{\widehat{K}_\infty^{\text{la}}} W^{\text{la}} \rightarrow W$$

is an isomorphism. Moreover, if K_∞/K is the cyclotomic extension of K , and if $W = (\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$ then $W^{\Gamma_n - \text{an}} = \mathbf{D}_{\text{Sen},n}(V)$.

Proof. The main claim is [7, Thm. 3.4], and the particular case for the cyclotomic extension follows from [7, Rem. 3.3]. ■

We also have in general a nice description of the structure of $\widehat{K_\infty}^{\text{la}}$: if K_∞/K is a p -adic Lie extension with Galois group $\Gamma_K = \text{Gal}(K_\infty/K)$, then by [7, Thm. 6.1 and Rem. 6.2 (ii)], if L is a subfield of \mathbf{C}_p , containing $K_\infty(\mu_{p^m})$ for $m \gg 0$, then $L \widehat{\otimes}_{K_n} \widehat{K_\infty}^{\Gamma_n-\text{an}}$ is isomorphic to the set $L\{\{X_1, \dots, X_{d-1}\}\}_n$ of power series with coefficients in L and radius of convergence $\geq p^{-n}$, where $K_n = K_\infty \cap \widehat{K_\infty}^{\Gamma_n-\text{an}}$.

Note that in the cyclotomic case, the map $\log \chi_{\text{cycl}}: \Gamma_K \rightarrow \mathbf{Z}_p$ induces isomorphisms between $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{C}_p)$ and $\mathbf{B}_{\text{Sen}}^n$, and by taking the inductive limit, between $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{C}_p)_1$, the stalk at the identity of the sheaf of locally analytic functions on Γ_K with coefficients in \mathbf{C}_p and \mathbf{B}_{Sen} , so that this formalism generalizes the construction of \mathbf{B}_{Sen} of Colmez. More generally, we make the following definition.

Definition 4.4. For any topological \mathbf{Q}_p -algebra \mathbf{B} which is an LF or LB space, equipped with a continuous action of \mathcal{G}_K , we denote by $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B})_1$ the stalk at the identity of the sheaf of locally analytic functions on Γ_K with coefficients in \mathbf{B} , which is the inductive limit of the rings $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{B})$, endowed with the inductive limit topology.

Using the rings $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p)$, we can recover Sen theory (and its generalization by Berger and Colmez).

Proposition 4.5. *Let K_∞/K be a p -adic Lie extension with Galois group*

$$\Gamma_K = \text{Gal}(K_\infty/K)$$

and let V be a p -adic representation of \mathcal{G}_K . Then we have

$$((\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K-\text{la}} = \bigcup_{n \geq 1} (\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{C}_p) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}},$$

where \mathcal{G}_{K_n} acts on $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{C}_p) \otimes_{\mathbf{Q}_p} V \simeq \mathcal{C}^{\text{an}}(\Gamma_n, V)$ through the $\Delta_{1,2}$ map defined in Section 3.

Note that this proposition is the consequence of the following proposition which is itself an immediate consequence of the definition of locally analytic vectors, as stated in [20, §3.3].

Proposition 4.6. *Let \mathbf{B} be a \mathbf{Q}_p -algebra which is an LF or LB space, endowed with an action of \mathcal{G}_K , and let V be a \mathbf{Q}_p -representation of \mathcal{G}_K . Then for any $n \geq 0$, we have*

$$\begin{aligned} (\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B})^{\mathcal{G}_{K_n}} &\simeq (\mathbf{B}^{H_K})^{\Gamma_n-\text{an}}, \\ ((\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} &\simeq (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K, \Gamma_n-\text{an}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \bigcup_{n \geq 1} (\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B})^{\mathcal{G}_{K_n}} &\simeq (\mathbf{B}^{H_K})^{\Gamma_K-\text{la}}, \\ \bigcup_{n \geq 1} ((\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} &\simeq (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K, \Gamma_K-\text{la}}. \end{aligned}$$

Proof. For LB spaces, this is tautological, since H_K acts trivially on $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p)$ and since the set of Γ_n -locally analytic vectors of $W := (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}$ is by definition the subset of elements of $\mathcal{C}^{\text{an}}(\Gamma_n, W) = \mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} W$ which are invariant by the action given by $\Delta_{1,2}$ following the notations of Section 3.

For LF spaces, the proof is almost the same because the set of Γ_n -analytic vectors of W is still the subset of elements of $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} W$ which are invariant by the action given by $\Delta_{1,2}$ by [20, Cor. 3.4.5].

The last two isomorphisms follow by taking the inductive limit. ■

In particular, if \mathbf{B} is a topological \mathbf{Q}_p -algebra which is an LF or LB space, endowed with an action of \mathcal{G}_K such that for V a p -adic representation of \mathcal{G}_K , $((\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K\text{-la}}$ is related to some module attached to V which appears in p -adic Hodge theory (e.g. its (φ, Γ) -modules), then $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} \mathbf{B}$ can be thought of as a ring of periods that computes those modules.

5. De Rham computations

In this section we compute locally analytic vectors and pro-analytic vectors in \mathbf{B}_{dR}^+ , both in the cyclotomic case and in the Lubin–Tate case, and we explain how to recover the module $\mathbf{D}_{\text{Dif}}^+(V)$ attached to a p -adic representation V thanks to the use of the locally analytic vectors. The fact that locally analytic vectors are able to recover $\mathbf{D}_{\text{Dif}}^+(V)$ has already been proven in [38, §6.1] but here we will also use Proposition 4.6 to produce a ring of periods which “computes” the functor $\mathbf{D}_{\text{Dif}}^+$.

5.1. Computations in \mathbf{B}_{dR}^+

We let $\mathbf{B}_{\text{dR},K}^+$ and $\mathbf{B}_{\text{dR},K}$ denote respectively $\mathbf{B}_{\text{dR}}^{H_K}$ and $\mathbf{B}_{\text{dR}}^{H_K}$. Recall (cf. [2, §2]) that there is a natural injective, \mathcal{G}_K -equivariant map $\widetilde{\mathbf{B}}_{\text{rig}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+$, which sends t_π to a generator of $\ker(\theta)$ in \mathbf{B}_{dR}^+ and we still denote the image of t_π through this map by t_π . For $\tau \in \Sigma_F$, we have a corresponding element $t_\tau \in \widetilde{\mathbf{B}}_{\text{rig}}^+$ defined in [5, §5] such that $g(t_\tau) = \tau(\chi_\pi(g)) \cdot t_\tau$, and we still denote the image of t_τ through the map $\widetilde{\mathbf{B}}_{\text{rig}}^+ \rightarrow \mathbf{B}_{\text{dR}}^+$ by t_τ . Note that $t_\tau \in (\mathbf{B}_{\text{dR},F^{\text{Gal}}}^+)^{\times}$ if $\tau \neq \text{id}$ (see for example item (2) of [8, Prop. 3.4]). We let $\partial_{\text{id}} = \frac{1}{t_\pi} \nabla_{\text{id}}$.

Lemma 5.1. *We have $\partial_{\text{id}}((\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}) \subset (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$.*

Proof. Let $x \in (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$. Then $\theta(x) \in \widehat{K_\infty}^{\text{la}}$. Since $\nabla_{\text{id}} = 0$ on $\widehat{K_\infty}^{\text{la}}$, we get that $\nabla_{\text{id}} \circ \theta(x) = 0 = \theta \circ \nabla_{\text{id}}(x)$ so that $\nabla_{\text{id}}(x) \in t_\pi \mathbf{B}_{\text{dR}}^+$. Therefore, $\partial_{\text{id}}(x) \in \mathbf{B}_{\text{dR}}^+$. Since t_π is a pro-analytic vector of $\mathbf{B}_{\text{dR},K}$ and since $x \in (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$, we obtain $\partial_{\text{id}}(x) \in (\mathbf{B}_{\text{dR},K})^{\text{pa}}$. In order to conclude, we need to prove that $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}} = (\mathbf{B}_{\text{dR},K})^{\text{pa}} \cap \mathbf{B}_{\text{dR},K}^+$. But this is straightforward, because if x is a pro-analytic vector of $(\mathbf{B}_{\text{dR},K})^{\text{pa}}$ which belongs to $\mathbf{B}_{\text{dR},K}^+$ then the $\frac{\nabla_{\text{id}}^k(x)}{k!}$ belong to $\mathbf{B}_{\text{dR},K}^+$ and thus $x \in (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$ by Remark 3.5. ■

Lemma 5.2. We have $(\mathbf{B}_{\text{dR},K})^{\text{pa}} = K_\infty((t_\pi))$ and $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}} = K_\infty[[t_\pi]]$.

Proof. See [37, Prop. 2.6]. ■

We let $(\mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}}$ denote the set of \mathbf{Q}_p -pro-analytic vectors of $\mathbf{B}_{\text{dR},K}^+$ which are killed by ∇_{id} .

Proposition 5.3. We have $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}} = \{\sum_{k \geq 0} a_k t_\pi^k, a_k \in (\mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}}\}$.

Proof. Let $x \in (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$. For $i \geq 0$, we let

$$x_i = \frac{1}{i!} \sum_{k \geq 0} (-1)^k \frac{\partial_{\text{id}}^{i+k}(x)}{k!} t_\pi^k.$$

By Lemma 5.1, we have that for any $i, k \geq 0$, $\partial_{\text{id}}^{i+k}(x)$ belongs to $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$ so that the sum $\frac{1}{i!} \sum_{k \geq 0} (-1)^k \frac{\partial_{\text{id}}^{i+k}(x)}{k!} t_\pi^k$ converges in $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$ to an element x_i such that $\partial_{\text{id}}(x_i) = 0$.

The sum $\sum_{i \geq 0} x_i t_\pi^i$ converges in $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$ and a simple computation shows that $x = \sum_{i \geq 0} x_i t_\pi^i$.

Conversely, it is easy to check that if $(a_k)_{k \geq 0}$ is a sequence of elements of $(\mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}}$, the sum $\sum_{k \geq 0} a_k t_\pi^k$ converges to an element of $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$. ■

Lemma 5.4. Let $x \in (\mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}}$ such that $t_\pi | x$ in \mathbf{B}_{dR}^+ . Then $x = 0$.

Proof. Let $x \in (\mathbf{B}_{\text{dR},F}^+)^{\Sigma_0-\text{pa}}$ such that $t_\pi | x$, and assume that $x \neq 0$. We can therefore write $x = t_\pi^k \alpha$ with $k \geq 1$, $\alpha \in \mathbf{B}_{\text{dR}}^+$ and t_π does not divide α in \mathbf{B}_{dR}^+ . Moreover, since t_π is pro-analytic for the action of Γ_K , we get that α is pro-analytic for the action of Γ_K .

By Proposition 5.3, we can write $\alpha = \sum_{j \geq 0} a_j t_\pi^j$ where the a_j are elements of $(\mathbf{B}_{\text{dR},K}^+)^{\text{pa}}$ killed by ∇_{id} . The fact that x is killed by ∇_{id} translates into

$$\sum_{j \geq 0} (k + j) a_j t_\pi^{k+j} = 0.$$

Applying ∂_{id}^k to this equality and reducing mod t_π , we obtain that $a_0 = 0 \pmod{t_\pi}$ and thus $t_\pi | \alpha$, which is not possible. ■

Corollary 5.5. For any $N \geq 1$, the map $\theta_N: (\mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}} \rightarrow (\mathbf{B}_{\text{dR},K}^+ / t_\pi^N \mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}}$ is injective.

Note that $(\mathbf{B}_{\text{dR},K}^+ / t_\pi \mathbf{B}_{\text{dR},K}^+)^{\Sigma_0-\text{pa}} = \widehat{K}_\infty^{\Sigma_0-\text{la}}$ and that $\widehat{K}_\infty^{\Sigma_0-\text{la}} = \widehat{K}_\infty \mathbf{Q}_p^{-\text{la}}$ by [5, Prop. 2.10]. Note that this also implies that for any $m \geq 0$, the natural map

$$\theta: (\mathbf{B}_{\text{dR},K}^+) \mathbf{Q}_p^{-\Gamma_m-\text{an}, \Sigma_0-\text{pa}} \rightarrow \widehat{K}_\infty \mathbf{Q}_p^{-\Gamma_m-\text{an}}$$

is injective. By the same argument as in the proof of the surjectivity in [38, Thm. 6.2], the map $\theta: (\mathbf{B}_{\text{dR},K}^+) \mathbf{Q}_p^{-\text{pa}} \rightarrow \widehat{K}_\infty \mathbf{Q}_p^{-\text{la}}$ is surjective. In particular, using Proposition 5.3, we get the following ‘‘description’’ of $(\mathbf{B}_{\text{dR},K}^+) \mathbf{Q}_p^{-\text{pa}}$.

Proposition 5.6. *The natural map*

$$x \in (\mathbf{B}_{\text{dR},K}^+)^{\mathbf{Q}_p\text{-pa}} \mapsto \sum_{i \geq 0} \theta(x_i) t_\pi^i,$$

where

$$x_i = \frac{1}{i!} \sum_{k \geq 0} (-1)^k \frac{\partial_{\text{id}}^{i+k}(x)}{k!} t_\pi^k,$$

induces a Γ_K -equivariant isomorphism from $(\mathbf{B}_{\text{dR},K}^+)^{\mathbf{Q}_p\text{-pa}}$ to $\widehat{K}_\infty^{\mathbf{Q}_p\text{-la}}[[t_\pi]]$.

Proof. We already know from the above that the map $x \in (\mathbf{B}_{\text{dR},K}^+)^{\mathbf{Q}_p\text{-pa}} \mapsto \sum_{i \geq 0} \theta(x_i) t_\pi^i$ is injective. To prove that it is surjective, recall that the map

$$\theta: (\mathbf{B}_{\text{dR},K}^+)^{\mathbf{Q}_p\text{-pa}} \rightarrow \widehat{K}_\infty^{\mathbf{Q}_p\text{-la}}$$

is surjective. If $y \in \widehat{K}_\infty^{\mathbf{Q}_p\text{-la}}$, let $x \in (\mathbf{B}_{\text{dR},K}^+)^{\mathbf{Q}_p\text{-pa}}$ such that $\theta(x) = y$. One can write $x = \sum_{i \geq 0} x_i t_\pi^i$ with $\partial_{\text{id}}(x_i) = 0$ for all i , and thus x_0 satisfies $\theta(x_0) = \theta(x) = y$ and $\partial(x_0) = 0$, so that the map above is injective. ■

Remark 5.7. We have $\mathbf{B}_{\text{dR},K}^+ \simeq \widehat{K}_\infty[[t_\pi]]$ noncanonically but this isomorphism is not Γ_K -equivariant. However, taking only the \mathbf{Q}_p -pro-analytic vectors on both sides gives us a canonical isomorphism which is Γ_K -equivariant.

5.2. The modules $\mathbf{D}_{\text{Dif}}^+(V)$

When K_∞/K is the cyclotomic extension of K , Fontaine has proven in [24] that the set of sub- $K_\infty[[t]]$ -modules free of finite type of $(\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K}$ and stable by the action of Γ_K admits a maximal element, usually denoted by $\mathbf{D}_{\text{Dif}}^+(V)$, and which is such that $\mathbf{B}_{\text{dR}}^+ \otimes_{K_\infty[[t]]} \mathbf{D}_{\text{Dif}}^+(V) = \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V$.

If $\gamma \in \Gamma_K$ is close enough to 1, then the power series defining $\log(\gamma)$ converges as a power series of \mathbf{Q}_p -linear operators of $\mathbf{D}_{\text{Dif}}^+(V)$, and the operator $\nabla_V = \frac{\log(\gamma)}{\log(\chi_{\text{cycl}}(\gamma))}$ does not depend on the choice of γ and satisfies the Leibniz rule $\nabla_V(\lambda \cdot x) = \lambda \nabla_V(x) + \nabla(\lambda)x$ for all $\lambda \in K_\infty[[t]]$ and $x \in \mathbf{D}_{\text{Dif}}^+(V)$. The map $\theta: \mathbf{B}_{\text{dR}}^+ \rightarrow \mathbf{C}_p$ induces a surjective morphism of modules with connections $(\mathbf{D}_{\text{Dif}}^+(V), \nabla_V) \rightarrow (\mathbf{D}_{\text{Sen}}(V), \Theta_V)$ (see for example [2, §5.3]).

The map $\iota_n: \widehat{\mathbf{B}}_{\text{rig}}^{\dagger,r_n} \rightarrow \mathbf{B}_{\text{dR}}^+$ sends $\mathbf{B}_K^{\dagger,r_n}$ into $K_n[[t]] \subset \mathbf{B}_{\text{dR}}^+$ and $\mathbf{D}^{\dagger,r_n}(V)$ in a sub- $K_n[[t]]$ -module of $\mathbf{D}_{\text{Dif}}^+(V)$, and we let

$$\mathbf{D}_{\text{Dif},n}^+(V) := K_n[[t]] \otimes_{\iota_n(\mathbf{B}_K^{\dagger,r_n})} \iota_n(\mathbf{D}^{\dagger,r_n}(V)).$$

Proposition 5.7 of [2] shows that $\mathbf{D}_{\text{Dif}}^+(V) = K_\infty[[t]] \otimes_{K_n[[t]]} \mathbf{D}_{\text{Dif},n}^+(V)$ for $n \gg 0$.

The fact that one could retrieve the modules $\mathbf{D}_{\text{Dif}}^+(V)$ and $\mathbf{D}_{\text{Dif},n}^+(V)$ using the theory of locally analytic vectors had already been noticed by Berger and Colmez [7, Rem. 3.3] and proven by Porat in [37, Prop. 3.3] and [38, Thm. 6.2] but we now explain how this incorporates into the setting laid out at the end of Section 4.

Note that \mathbf{B}_{dR}^+ , endowed with its natural topology, is not a Banach ring but a Fréchet ring, and as Berger points out in [5], locally analytic vectors in the setting of Fréchet spaces usually have to be replaced with the weaker notion of pro-analytic vectors, because the resulting objects are too small in general. However, in the setting of \mathbf{B}_{dR}^+ and $\mathbf{D}_{\text{Dif}}^+(V)$, locally analytic vectors are actually sufficient to recover the theory.

Lemma 5.8. *We have $(\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}} = K_n[[t_\pi]]$, $(\mathbf{B}_{\text{dR},K}^+)^{\text{la}} = \bigcup_n K_n[[t_\pi]]$.*

Proof. The second equality follows directly from the first one. For the first equality, take $x \in (\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}}$. We have $\theta(x) \in \widehat{K_\infty}^{\Gamma_n\text{-an}} = K_n$ by [7, Cor. 4.8], so that we can write $x = x_0 + t_\pi y$, with $x_0 \in K_n$ and $y \in \mathbf{B}_{\text{dR},K}^+$, and one checks that y is Γ_n -analytic because x, x_0 and t_π are. By induction, $x \in K_n[[t_\pi]]$. Because $K_n \subset (\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}}$ and because t_π is Γ_0 -analytic, we have $K_n[[t_\pi]] \subset (\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}}$, which finishes the proof. ■

Proposition 5.9. *For $n \gg 0$, we have $\mathbf{D}_{\text{Dif},n}^+(V) = ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n\text{-an}}$.*

Proof. Since $(\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}} = K_n[[t]]$, it suffices to prove that the elements of $\iota_n(\mathbf{D}^{\dagger,r_n}(V))$ are Γ_n -analytic for $n \gg 0$.

Let $m \geq 0$ be such that $\mathbf{D}^{\dagger,r_m}(V)$ has the right dimension, and let e_1, \dots, e_d be a basis of $\mathbf{D}^{\dagger,r_m}(V)$. We can see the elements of $\mathbf{D}^{\dagger,r_m}(V)$ as elements of $\mathbf{D}^{[r_m;r_m]}(V)$. By [31, §2.1], these elements are Γ_n -analytic for $n \gg m$ big enough. A direct consequence of [7, Lem. 2.2] shows that if we let $u_i = \varphi^{n-m}(e_i)$, $1 \leq i \leq d$, then the u_i are Γ_n -analytic as elements of $\mathbf{D}^{[r_n;r_n]}(V)$, and we know that it is a basis of $\mathbf{D}^{\dagger,r_n}(V)$ (since $\varphi^*(\mathbf{D}^\dagger(V)) \simeq \mathbf{D}^\dagger(V)$) and thus of $\mathbf{D}^{[r_n;r_n]}(V)$. Therefore, $(\iota_n(u_1), \dots, \iota_n(u_d))$ generates $\mathbf{D}_{\text{Dif},n}^+(V)$, and forms a basis of Γ_n -analytic elements of $\mathbf{D}_{\text{Dif},n}^+(V)$. ■

Proposition 5.10. *We have $\mathbf{D}_{\text{Dif}}^+(V) = ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\text{pa}}$.*

Proof. This is [37, Prop. 3.3] and also follows from the previous proposition: we know that $\mathbf{D}_{\text{Dif}}^+(V) = K_\infty[[t]] \otimes_{K_n[[t]]} \mathbf{D}_{\text{Dif},n}^+(V)$. Rewriting this using Lemma 5.8 and Proposition 5.9, we get:

$$\mathbf{D}_{\text{Dif}}^+(V) = (\mathbf{B}_{\text{dR},K}^+)^{\text{pa}} \otimes_{(\mathbf{B}_{\text{dR},K}^+)^{\Gamma_n\text{-an}}} ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n\text{-an}}$$

so that $\mathbf{D}_{\text{Dif}}^+(V) = ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\text{pa}}$ by Proposition 3.3. ■

We can use this result to generalize the theory to the Lubin–Tate case, as it has been done in [37, §3]: we define $\mathbf{D}_{\text{Dif}}^+(V)$ by the formula $\mathbf{D}_{\text{Dif}}^+(V) := ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\text{pa}}$, so that our object is indeed the classical $\mathbf{D}_{\text{Dif}}^+$ when $F = \mathbf{Q}_p$ by Proposition 5.10, and if V is an F -representation of \mathcal{G}_K then $\mathbf{D}_{\text{Dif}}^+(V)$ is a free $K_\infty[[t_\pi]]$ -module of rank $\dim_F(V)$, and the natural map $\mathbf{B}_{\text{dR}}^+ \otimes_{K_\infty[[t_\pi]]} \mathbf{D}_{\text{Dif}}^+(V) \rightarrow \mathbf{B}_{\text{dR}}^+ \otimes_F V$ is an isomorphism.

Note that when V is an F -analytic representation of \mathcal{G}_K , one can define a module $\mathbf{D}_{\text{Dif},n}^+(V)$ in the F -analytic Lubin–Tate case using the same arguments as given in the proof of Proposition 5.9: we can define $\mathbf{D}_{\text{Dif},n}^+(V)$ by $\mathbf{D}_{\text{Dif},n}^+(V) := ((\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n\text{-an}}$ for $n \gg 0$, and we have $\mathbf{D}_{\text{Dif}}^+(V) = K_\infty[[t_\pi]] \otimes_{K_n[[t_\pi]]} \mathbf{D}_{\text{Dif},n}^+(V)$ for $n \gg 0$.

In particular, the rings $\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{B}_{\text{dR}}^+)$ and $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B}_{\text{dR}}^+)_1$ allow us to compute the modules $\mathbf{D}_{\text{Dif}}^+(V)$ in the spirit of Fontaine’s strategy. Moreover, this shows that every p -adic representation of \mathcal{G}_K is “ $\mathcal{C}^{\text{la}}(\Gamma_K, \mathbf{B}_{\text{dR}}^+)_1$ -admissible”.

In general, when K_∞/K is any p -adic Lie extension, one could define a module $\mathbf{D}_{\text{Dif},K}^+(V)$ in the same manner, taking the pro-analytic vectors of $(\mathbf{B}_{\text{dR}}^+)^{H_K}$ for the action of Γ_K . The fact that this module has the same dimension as $\dim_{\mathbf{Q}_p} V$ follows from an unpublished result of Porat, and one could show in that case that the ring

$$\varprojlim_k \varinjlim_n (\mathcal{C}^{\text{an}}(\Gamma_n, \mathbf{Q}_p) \hat{\otimes}_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}^+/t^k)$$

computes the said module.

6. (φ, Γ) -modules

Computations made by Berger in [5, §4 and §8] show that classical cyclotomic (φ, Γ) -modules over the Robba ring $\mathbf{B}_{\text{rig},K}^\dagger$ can be recovered by using pro-analytic vectors, and [5, Thm. 10.1] shows that this can be extended to F -analytic representations. If V is an F -analytic representation of \mathcal{G}_K then we can attach a (φ_q, Γ_K) -module which we will denote $\mathbf{D}_{\text{rig}}^\dagger(V)$.

Given an F -analytic E -representation V of \mathcal{G}_K , we let $\tilde{\mathbf{D}}_{\text{rig},K}^{\dagger,r}(V) = (\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger,r} \otimes_E V)^{H_K}$. For $r > 0$ and $n \geq 1$ we let $\mathbf{B}_{\text{rig},K,n}^{\dagger,r} = \varphi_q^{-n}(\mathbf{B}_{\text{rig},K,n}^{\dagger,q^n r})$, and we let $\mathbf{B}_{\text{rig},K,\infty}^{\dagger,r} = \bigcup_{n \geq 1} \mathbf{B}_{\text{rig},K,\infty}^{\dagger,r}$ in $\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger,r}$, and $\mathbf{B}_{\text{rig},K,n}^\dagger = \bigcup_{r>0} \mathbf{B}_{\text{rig},K,n}^{\dagger,r}$, $\mathbf{B}_{\text{rig},K,\infty}^\dagger = \bigcup_{r>0} \mathbf{B}_{\text{rig},K,\infty}^{\dagger,r}$.

Proposition 6.1. *We have*

- (1) $(\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger,r_k})^{\text{pa}} = \mathbf{B}_{\text{rig},K,\infty}^{\dagger,r_k}$;
- (2) $\tilde{\mathbf{D}}_{\text{rig},K}^\dagger(V)^{\text{pa}} = \mathbf{B}_{\text{rig},K,\infty}^\dagger \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} \mathbf{D}_{\text{rig},K}^\dagger(V)$;
- (3) if \mathbf{D} is a (φ_q, Γ_K) -module over $\mathbf{B}_{\text{rig},K}^\dagger$ such that $\tilde{\mathbf{D}}_{\text{rig},K}^\dagger(V)^{\text{pa}} = \mathbf{B}_{\text{rig},K,\infty}^\dagger \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} \mathbf{D}$ then $\mathbf{D} = \mathbf{D}_{\text{rig},K}^{\dagger,r}(V)$.

Proof. The first item is item (3) of [5, Thm. 4.4]. The second item is item (2) of [5, Thm. 8.1] in the cyclotomic setting, or follows from the proof of [5, Thm. 10.1] in the Lubin–Tate setting. For the last item, let M denote the base change matrix and P_1, P_2 denote the matrices of φ on $\mathbf{D}, \mathbf{D}_{\text{rig},K}^{\dagger,r}(V)$ respectively. There exists $n \gg 0$ such that $M \in \text{GL}_d(\mathbf{B}_{\text{rig},K,n}^\dagger)$, and the equation $M = P_2^{-1} \varphi_q(M) P_1$ implies that $M \in \text{GL}_d(\mathbf{B}_{\text{rig},K}^\dagger)$. ■

In particular, taking the pro-analytic vectors of $\tilde{\mathbf{D}}_{\text{rig},K}^{\dagger,r}(V)$ allows us to recover the (φ_q, Γ_K) -module $\mathbf{D}_{\text{rig},K}^{\dagger,r}(V)$, either in the cyclotomic setting or in the F -analytic Lubin–Tate setting.

As in the constructions for \mathbf{B}_{dR} and \mathbf{C}_p , the rings $\mathcal{C}^{\text{la}}(\Gamma_n, \mathbf{B})_1$, for \mathbf{B} an LB or LF space, are not endowed with an action of Γ_K but only with an action of its Lie algebra, so that

if V is a p -adic representation of \mathcal{G}_K , the module $(\mathcal{C}^{\text{la}}(\Gamma_n, \mathbf{B})_1 \otimes V)^{H_K}$ is only endowed with an operator ∇ coming from the infinitesimal action of Γ_K , and from the operator ∇_{id} , which we still denote by ∇ , in the Lubin–Tate setting. In particular, the constructions laid out in this subsection can only allow us to recover the (φ, ∇) -module (or F -analytic (φ_q, ∇) -module) attached to a representation V .

Proposition 6.2. *Let V be an F -analytic representation of \mathcal{G}_K and let $r > 0$. The collection $(\bigcup_n (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^I) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}})_{\min(I) \geq r}$ equipped with natural transition maps*

$$\bigcup_n (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^I) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \rightarrow \bigcup_n (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^J) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$$

when $J \subset I$, and Frobenius maps

$$\varphi_q : \bigcup_n (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^I) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \rightarrow \bigcup_n (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^{qI}) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$$

defines a (φ_q, ∇) -module $\tilde{\mathbf{D}}$ over $\lim_{\leftarrow I} (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}^I))^{\mathcal{G}_{K_n}} \simeq (\tilde{\mathbf{B}}_{\text{rig}, K}^{\dagger, r})^{\text{pa}}$, and we have $\tilde{\mathbf{D}} \simeq \tilde{\mathbf{D}}_{\text{rig}}^{\dagger, r}(V)^{\text{pa}}$ as (φ_q, ∇) -modules.

Moreover, there exists a (φ_q, ∇) -module \mathbf{D} on $\mathbf{B}_{\text{rig}, K}^{\dagger}$ inside $\tilde{\mathbf{D}}$ such that

$$\tilde{\mathbf{D}} = (\tilde{\mathbf{B}}_{\text{rig}, K}^{\dagger})^{\text{pa}} \otimes_{\mathbf{B}_{\text{rig}, K}^{\dagger}} \mathbf{D},$$

and if \mathbf{D}' is a (φ_q, ∇) -module on $\mathbf{B}_{\text{rig}, K}^{\dagger}$ such that $\tilde{\mathbf{D}}_{\text{rig}, K}^{\dagger}(V)^{\text{pa}} = \mathbf{B}_{\text{rig}, K, \infty}^{\dagger} \otimes_{\mathbf{B}_{\text{rig}, K}^{\dagger}} \mathbf{D}'$ then $\mathbf{D} = \mathbf{D}'$. In particular, $\mathbf{D} \simeq \mathbf{D}_{\text{rig}, K}^{\dagger}(V)$.

Proof. The first part of the proposition follows from the definition of pro-analytic vectors, Propositions 4.6 and 6.1.

Since $(\tilde{\mathbf{B}}_{\text{rig}, K}^{\dagger, r})^{\text{pa}} = \mathbf{B}_{\text{rig}, K, \infty}^{\dagger, r}$, there exist elements v_1, \dots, v_d of $\tilde{\mathbf{D}}$ and $n \gg 0$ such that

$$\mathbf{D} := \bigoplus_{i=1}^d \mathbf{B}_{\text{rig}, K}^{\dagger, p^{nr}} \cdot \varphi^n(v_i)$$

generates $\tilde{\mathbf{D}}$. The unicity of \mathbf{D} follows from the same argument as in the proof of the last item of Proposition 6.1. ■

7. Generalization to other p -adic Lie extensions

7.1. General results when K_{∞} contains a cyclotomic extension

In what follows, K is a finite extension of \mathbf{Q}_p , K_{∞}/K is an infinite Galois p -adic Lie extension, with $\dim \Gamma_K \geq 2$, and such that K_{∞} contains a cyclotomic extension, in the sense that there exists an unramified character $\eta: \mathcal{G}_K \rightarrow \mathbf{Z}_p^{\times}$ such that $K_{\infty} \cap \overline{\mathbf{Q}}_p^{\eta \chi_{\text{cycl}}}$ is infinitely ramified. We let K_{∞}^{η} denote the extension $K_{\infty} \cap \overline{\mathbf{Q}}_p^{\eta \chi_{\text{cycl}}}$.

Recall that K_∞^η/K is the extension of K attached to $\eta\chi_{\text{cycl}}$. Let $\Gamma_{K,\eta} = \text{Gal}(K_\infty^\eta/K)$ and $H_{K,\eta} = \text{Gal}(\overline{\mathbf{Q}}_p/K_\infty^\eta)$. Let $\mathbf{B}_{K,\eta}^\dagger$, $\mathbf{B}_{K,\eta}^I$ and $\mathbf{B}_{\text{rig},K,\eta}^\dagger$ be as in [5, §8]. By the same arguments as in [5, §8], there is an equivalence of categories between étale $(\varphi, \Gamma_{K,\eta})$ -modules over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig},K,\eta}^\dagger$ (it is also true over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{K,\eta}^\dagger$) and E -representations of \mathcal{G}_K .

If V is a p -adic representation of \mathcal{G}_K , we let $\mathbf{D}_\eta^\dagger(V) := \bigcup_{r \gg 0} \mathbf{D}_\eta^{\dagger,r}(V)$, where $\mathbf{D}_\eta^{\dagger,r}(V) := (\mathbf{B}_\eta^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$. Let $\mathbf{D}_\eta^{[r;s]}$ and $\mathbf{D}_{\text{rig},\eta}^{\dagger,r}(V)$ denote the various completions of $\mathbf{D}_\eta^{\dagger,r}(V)$. We let $\tilde{\mathbf{D}}_\eta^{[r;s]}(V) = (\tilde{\mathbf{B}}^{[r;s]} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$ and $\tilde{\mathbf{D}}_{\text{rig},\eta}^{\dagger,r}(V) = (\tilde{\mathbf{B}}_{\text{rig}}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$. By the variant of the Cherbonnier–Colmez theorem for twisted cyclotomic extensions, we have that $\tilde{\mathbf{D}}_\eta^{[r;s]}(V) = \tilde{\mathbf{B}}_{K,\eta}^{[r;s]} \otimes_{\mathbf{B}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]}$ and $\tilde{\mathbf{D}}_{\text{rig},\eta}^{\dagger,r}(V) = \tilde{\mathbf{B}}_{\text{rig},K,\eta}^{\dagger,r} \otimes_{\mathbf{B}_{\text{rig},K,\eta}^{\dagger,r}} \mathbf{D}_{\text{rig},K,\eta}^{\dagger,r}(V)$.

Lemma 7.1. *Let $r \geq 0$ be such that $\mathbf{D}_\eta^{\dagger,r}(V)$ is free of rank $\dim_{\mathbf{Q}_p}(V)$ as a $\mathbf{B}_{\text{rig},K,\eta}^{\dagger,r}$ -module, and let $s \geq r$. Then the elements of $\mathbf{D}_\eta^{[r;s]}(V)$ are locally analytic for the action of $\text{Gal}(K_\infty^\eta/K)$.*

Proof. See the proof of [5, Thm. 8.1]. ■

Corollary 7.2. *Let V be a p -adic representation of \mathcal{G}_K which factors through Γ_K , and let $r > 0$ be such that $\mathbf{D}_\eta^{\dagger,r}(V)$ is free of rank $\dim_{\mathbf{Q}_p}(V)$ as a $\mathbf{B}_{\text{rig},K,\eta}^{\dagger,r}$ -module, so that*

$$\mathbf{B}_\eta^{\dagger,r} \otimes_{\mathbf{Q}_p} V \simeq \mathbf{B}_\eta^{\dagger,r} \otimes_{\mathbf{B}_{K,\eta}^{\dagger,r}} \mathbf{D}_\eta^{\dagger,r}(V).$$

Then the coefficients in $\mathbf{B}_\eta^{\dagger,r}$ of a base change matrix between V and $\mathbf{D}_\eta^{\dagger,r}(V)$ belong to $(\mathbf{B}_{K,\eta}^{[r;s]})^{\text{la}}$ for any $s \geq r$.

Proof. Let V be such a p -adic representation of \mathcal{G}_K . Since V factors through Γ_K and by Cartan’s theorem (see for example [20, Prop. 3.6.10]), the elements of $V = V^{H_K}$ are locally analytic vectors for the action of Γ_K . Now, we have

$$\tilde{\mathbf{D}}_\eta^{[r;s]}(V)^{\text{la}} = (\tilde{\mathbf{B}}_{K,\eta}^{[r;s]} \otimes_{\mathbf{Q}_p} V)^{\text{la}}$$

since V factors through Γ_K , and thus

$$\tilde{\mathbf{D}}_\eta^{[r;s]}(V)^{\text{la}} = (\tilde{\mathbf{B}}_{K,\eta}^{[r;s]})^{\text{la}} \otimes_{\mathbf{Q}_p} V \tag{7.1}$$

by Proposition 3.3.

Since K_∞ contains K_∞^η , the elements of $\mathbf{D}_\eta^{[r;s]}(V)$ are locally analytic for the action of Γ_K , as they are locally analytic for the action of $\Gamma_{K,\eta}$ by Lemma 7.1 and invariant by the action of $\text{Gal}(K_\infty/K_\infty^\eta)$ (which is just another way of saying that the action of Γ_K on $\mathbf{D}_\eta^{[r;s]}(V)$ factor through a locally analytic action of Γ_K^η).

By Proposition 3.3, this implies that

$$\tilde{\mathbf{D}}_K^{[r;s]}(V)^{\text{la}} = (\tilde{\mathbf{B}}_K^{[r;s]})^{\text{la}} \otimes_{\tilde{\mathbf{B}}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]}. \tag{7.2}$$

Putting equations (7.1) and (7.2) together, we obtain that

$$(\tilde{\mathbf{B}}_K^{[r;s]})^{\text{la}} \otimes_{\mathbf{B}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]} = (\tilde{\mathbf{B}}_K^{[r;s]})^{\text{la}} \otimes_{\mathbf{Q}_p} V.$$

In particular, this implies that the coefficients of the base change matrix in $GL_d(\mathbf{B}_\eta^{\dagger,r})$ belong to $(\tilde{\mathbf{B}}_{K,\eta}^{[r;s]})^{\text{la}}$, and thus to $(\mathbf{B}_{K,\eta}^{[r;s]})^{\text{la}}$. ■

This corollary will prove very useful in order to produce locally analytic vectors for Γ_K in the rings $(\tilde{\mathbf{B}}_K^I)$.

Remark 7.3. Note that the fact that K_∞ contains K_∞^η is crucial for the proof of Corollary 7.2 to work.

7.2. Higher locally analytic vectors

Let V be a Banach representation of a p -adic Lie group G , and assume that G is small in the sense of [38, §2.1], so that the set of G -analytic vectors of V is well defined.

The functor $V \mapsto V^{G\text{-an}}$ is left exact, and following [36, §2.2], [40] and [38, §2.3], we define right derived functors for $i \geq 0$:

$$R_{G\text{-an}}^i(V) = H^i(G, V \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\text{an}}(G, \mathbf{Q}_p)),$$

where we consider continuous cohomology on the right-hand side.

If G is a compact p -adic Lie group (without the smallness assumption) with subgroups $\{G_n\}_{n \geq 1}$ as in §2, taking the colimit over n , there are right derived functors for $V \mapsto V^{G\text{-la}}$ given by

$$R_{G\text{-la}}^i(V) = \varinjlim_n R_{G_n\text{-an}}^i(V) = \varinjlim_n H^i(G_n, V \widehat{\otimes}_{\mathbf{Q}_p} \mathcal{C}^{\text{an}}(G_n, \mathbf{Q}_p)).$$

Following [38, §2.3], we call these groups the higher locally analytic vectors of V .

Note that if

$$0 \rightarrow V \rightarrow W \rightarrow X \rightarrow 0$$

is a short exact sequence of G -Banach spaces, we then have a long exact sequence

$$0 \rightarrow V^{\text{la}} \rightarrow W^{\text{la}} \rightarrow X^{\text{la}} \rightarrow R_{G\text{-la}}^1(V) \rightarrow R_{G\text{-la}}^1(W) \rightarrow R_{G\text{-la}}^1(X) \rightarrow \dots$$

The fact that the functor $V \mapsto V^{\text{la}}$ is exact is thus equivalent to the vanishing of the higher locally analytic vectors $R_{G\text{-la}}^i(V)$ for $i \geq 1$.

We now prove several results and recall some results from [38] regarding locally analytic vectors attached to p -adic Lie extensions in the rings $\tilde{\mathbf{B}}^I$. We let K_∞/K be a general p -adic Lie extension with Galois group Γ_K , and we let $H_K = \text{Gal}(\bar{\mathbf{Q}}_p/K_\infty)$. We let $\tilde{\mathbf{B}}_K^I = (\tilde{\mathbf{B}}^I)^{H_K}$.

We recall the following result, which is item (ii) of [38, Cor. 5.6].

Proposition 7.4. *If I is a compact subinterval of $[\frac{p}{p-1}; +\infty[$, if K_∞ contains a cyclotomic extension K_∞^η , and if M is a finite free $\tilde{\mathbf{B}}_K^I$ -semilinear representation of Γ_K then the higher analytic vectors $R_{\text{la}}^i(M)$ are zero for $i \geq 1$.*

Lemma 7.5. *Let $I = [r_k; r_\ell]$. For any $m \in [k; \ell]$ integer, the kernel of $\theta \circ \varphi_q^{-m}: \tilde{\mathbf{A}}_K^I \rightarrow \mathcal{O}_{\widehat{K_\infty}}$ is a principal ideal.*

Proof. The same proof as in [45, Prop. 4.3.7] (we are in the same setting since K_∞/K is strictly arithmetically profinite by the main theorem of [42]) shows that the kernel of $\theta: \tilde{\mathbf{A}}_K^+ \rightarrow \mathcal{O}_{\widehat{K_\infty}}$ is principal, generated by some element y such that $v_E(\bar{y}) = v_p(\pi_K)$, where π_K is a uniformizer of \mathcal{O}_K .

The same proof as for item (1) of [5, Lem. 3.2] shows that $\frac{\varphi_q^m(y)}{\pi_K}$ is then a generator of the kernel of $\theta \circ \varphi_q^{-m}: \tilde{\mathbf{A}}_K^+ \rightarrow \mathcal{O}_{\widehat{K_\infty}}$. ■

Corollary 7.6. *Let $I = [r_k; r_\ell]$. If K_∞ contains a cyclotomic extension K_∞^η then for any $m \in [k; \ell]$ integer, the map $\theta \circ \varphi_q^{-m}: (\tilde{\mathbf{B}}_K^I)^{\text{la}} \rightarrow \widehat{K_\infty}^{\text{la}}$ is surjective.*

Proof. We have an exact sequence

$$0 \rightarrow \ker(\theta: \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty}) \rightarrow \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty} \rightarrow 0$$

which gives rise to the exact sequence

$$0 \rightarrow (\ker(\theta: \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty}))^{\text{la}} \rightarrow (\tilde{\mathbf{B}}_K^I)^{\text{la}} \rightarrow (\widehat{K_\infty})^{\text{la}} \rightarrow R_{\text{la}}^1(\ker(\theta: \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty})) \rightarrow \dots$$

By Lemma 7.5, the kernel of $\theta \circ \varphi_q^{-m}: \tilde{\mathbf{A}}_K^+ \rightarrow \mathcal{O}_{\widehat{K_\infty}}$ is a principal ideal so that it gives rise to a one-dimensional $\tilde{\mathbf{B}}_K^I$ -semilinear representation of Γ_K , and thus by Proposition 7.4 we have that $R_{\text{la}}^1(\ker(\theta: \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty})) = 0$, so that we get the exact sequence

$$0 \rightarrow (\ker(\theta: \tilde{\mathbf{B}}_K^I \rightarrow \widehat{K_\infty}))^{\text{la}} \rightarrow (\tilde{\mathbf{B}}_K^I)^{\text{la}} \rightarrow (\widehat{K_\infty})^{\text{la}} \rightarrow 0,$$

and thus the map $\theta \circ \varphi_q^{-m}: (\tilde{\mathbf{B}}_K^I)^{\text{la}} \rightarrow \widehat{K_\infty}^{\text{la}}$ is surjective. ■

Recall that by [7, Thm. 6.2], $\widehat{K_\infty}^{\text{la}}$ is a ring of power series in $d - 1$ variables. Since in the case we consider K_∞ contains a cyclotomic extension, Lemma 7.5 shows that $\ker(\theta \circ \varphi_q^{-m})$ is a principal ideal generated by a locally analytic vector of $\tilde{\mathbf{B}}_K^I$, and Corollary 7.6 shows that the map $\theta \circ \varphi_q^{-m}: (\tilde{\mathbf{B}}_K^I)^{\text{la}} \rightarrow \widehat{K_\infty}^{\text{la}}$ is surjective. This (and the computations of Section 7.4) makes us think that the following conjecture should hold.

Conjecture 7.7. *If K_∞/K contains a cyclotomic extension, then for $n \gg 0$, there exist d elements $x_{1,n}, \dots, x_{d,n}$ in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ such that $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$ is the set of power series $\sum_{i=(i_1, \dots, i_d) \in \mathbb{N}^d} a_i x_{j,n}^{i_j}$ in the variables $(x_{i,n})_{i \in \{1, \dots, d\}}$ with coefficients in K such that the series $\sum_{i=(i_1, \dots, i_d) \in \mathbb{N}^d} a_i x_{j,n}^{i_j}$ converge in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n-\text{an}}$.*

7.3. The anticyclotomic case

Berger and Colmez have proven in [7] that the theory of locally analytic vectors is the right object to consider in order to generalize classical Sen theory to arbitrary p -adic Lie extensions. With that in mind, and considering the results above that show that in the cyclotomic (and in the F -analytic Lubin–Tate) case one recovers classical (φ, Γ) -modules theory, it seems reasonable to assume that the theory of locally analytic vectors is the right object to consider in order to generalize (φ, Γ) -modules to arbitrary p -adic Lie extensions.

It has already been noticed that, even in the Lubin–Tate case, “one-dimensional (φ_q, Γ_K) -modules” do not behave well [26] and that the kind of objects one should consider are multivariable Lubin–Tate (φ_q, Γ_K) -modules [3] which arise from locally analytic vectors [5].

Therefore, in general, one should expect to use that theory for arbitrary p -adic Lie extensions to get a theory of (φ_q, Γ_K) -modules over $(\tilde{\mathbf{B}}_{\text{rig}, K}^+)^{\text{pa}}$, and such that the functor $V \mapsto ((V \otimes_{\mathbf{Q}_p} \tilde{\mathbf{B}}_{\text{rig}})^{H_K})^{\text{pa}}$ is a faithfully exact functor. We now give some insight as to why such a generalization does not seem to be true in general, using the anticyclotomic extension as a potential counterexample.

Let F/\mathbf{Q}_p be the unramified extension of \mathbf{Q}_p of degree 2. We take π to be equal to p in our Lubin–Tate setting. We let σ denote the Frobenius on F . By [5, §5] the element $y_\sigma = \varphi(u) \in \tilde{\mathbf{A}}^+$ is such that $g(y_\sigma) = [\chi_p(g)]^\sigma(y_\sigma)$ for $g \in \mathcal{G}_F$. Since $[p](T) \in \mathbf{Z}_p[T]$, the series $\log_{\text{LT}}(T)$ and $\exp_{\text{LT}}(T)$ have all their coefficients in \mathbf{Q}_p , so that $t_\sigma = \varphi(t_p) = \log_{\text{LT}}(y_\sigma)$.

Let $F_{\text{cycl}} = F(\mu_{p^\infty})$ denote the cyclotomic extension of F . We let F_{ac} be the anticyclotomic extension of F : it is the unique \mathbf{Z}_p extension of F , Galois over \mathbf{Q}_p , which is pro-dihedral: the Frobenius σ of $\text{Gal}(F/\mathbf{Q}_p)$ acts on $\text{Gal}(F_{\text{ac}}/F)$ by inversion. It is linearly disjoint from F_{cycl} over F , and the compositum $F_{\text{cycl}} \cdot F_{\text{ac}}$ is equal to the Lubin–Tate extension F_p^{LT} attached to p by local class field theory. The anticyclotomic extension is then the subfield of F_∞ fixed under $G_\sigma := \{g \in \text{Gal}(F_\infty/F) : \chi_p(g) = \sigma(\chi_p(g))\}$, and the cyclotomic extension of F is the one fixed by $G := \{g \in \text{Gal}(F_\infty/F) : \chi_p(g) = (\sigma(\chi_p(g)))^{-1}\}$. Note that $G \simeq \text{Gal}(F_{\text{ac}}/F)$ as F_∞/F is abelian, and we still write G for the Galois group of F_{ac}/F . We let $H_{F, \text{ac}}$ denote the group $\text{Gal}(\tilde{\mathbf{Q}}_p/F_{\text{ac}})$, and if B is a ring of periods we let $B_{F, \text{ac}}$ denote $B^{H_{F, \text{ac}}}$. We write t_1 for t_p and t_2 for t_σ .

Proposition 7.8. *We have $(\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}} = F_{\text{ac}}[\![\frac{t_1}{t_2}]\!]$.*

Proof. Clearly, if $z \in F_{\text{ac}}[\![\frac{t_1}{t_2}]\!]$, then the corresponding power series converges to an element of \mathbf{B}_{dR}^+ which is invariant by H_F and pro-analytic for the action of Γ_F .

Now if $z \in (\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}}$, we have $\theta(z) \in \widehat{F_{\text{ac}}}^{\text{la}} = F_{\text{ac}}$ by [7, Thm. 3.2]. We can therefore write $z = \theta(z) + t_1 \cdot z'$ in \mathbf{B}_{dR}^+ . Since $\frac{t_1}{t_2}$ belongs to $(\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}} \cap \text{Fil}^1 \mathbf{B}_{\text{dR}}$, we can write $z = \theta(z) + \frac{t_1}{t_2} z_2$ with z_2 in $(\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}}$. Now we can do the same thing for z_2 , and doing this inductively gives us the result. ■

If I is big enough, so that the corresponding annulus contains a zero of t_1 and t_2 , then the localization map at the zero of t_1 gives an embedding $(\tilde{\mathbf{B}}_{F, \text{ac}}^I)^{\text{la}} \rightarrow (\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}} = F_{\text{ac}}[\![\frac{t_1}{t_2}]\!]$, and it seems difficult for an element in $\tilde{\mathbf{B}}_{F, \text{ac}}^I$ to have an “essential singularity at a zero of t_2 ”, even if it is after a localization at a zero of t_1 . Moreover, it is easy to prove that the image of $(\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}}$ in \mathbf{B}_{dR}^+ does not intersect $K_\infty[\![\frac{t_1}{t_2}]\!] \setminus F$ as soon as I is such that the corresponding annulus contains a zero of t_2 . It seems therefore reasonable to expect that $(\mathbf{B}_{\text{dR}, F, \text{ac}}^+)^{\text{pa}} = F$, even though we do not have a proof of that statement, except for I a subinterval of $[0, \infty[$ containing 0, which we are now going to prove.

We let $P(T) = [p](T) = T^q + pT$.

Lemma 7.9. *We have $P^{\circ k}(\varphi_q^{-k}(u^p)) \rightarrow y_\sigma$ in $\tilde{\mathbf{A}}^+$ when $k \rightarrow +\infty$ for the p -adic topology.*

Proof. Let $s_k := P^{\circ k}(\varphi_q^{-k}(u^p)) \in \tilde{\mathbf{A}}^+$, for all $k \geq 0$. We therefore have $s_0 = u^p$, and $s_{k+1} = \varphi_q^{-1}(P(s_k))$.

Let us assume that $s_k - s_{k-1}$ belongs to $p^b \tilde{\mathbf{A}}^+$, with $b \geq 1$.

Then we have $s_{k+1} = \varphi_q^{-1}(P(s_k))$, and we can write

$$P(s_k) = P(s_{k-1}) + \sum_{j=1}^q P^{(j)}(s_{k-1}) \frac{(s_k - s_{k-1})^j}{j!}.$$

Since $b \geq 1$ and since $P^{(j)}(T) \in p\mathcal{O}_F[[T]]$, this means that

$$P(s_k) = P(s_{k-1}) + (s_k - s_{k-1})h_k,$$

with $h_k \in p\tilde{\mathbf{A}}^+$. But then this means that

$$s_{k+1} - s_k = \varphi_q^{-1}(s_k - s_{k-1})\varphi_q^{-1}(h_k)$$

and thus $s_{k+1} - s_k \in p^{b+1}\tilde{\mathbf{A}}^+$.

We already know that $s_1 - s_0 \in p\tilde{\mathbf{A}}^+$ (because $\overline{s_1} = \overline{s_0} = \overline{u^p} \pmod p$) so that the sequence (s_k) converges in $\tilde{\mathbf{A}}^+$ to an element that we will denote by s .

Because both φ and θ are continuous for the p -adic topology, we know that $\theta \circ \varphi_q^{-j}(s) = \lim_{k \rightarrow +\infty} P^{\circ k}(\theta \circ \varphi_q^{-k}(u^p)) = P^{\circ k}(u_{j+k}^p)$. Therefore by [4, Lem. 5.3], s is such that $\theta \circ \varphi_q^{-j}(s) = \theta \circ \varphi_q^{-j}(y_\sigma)$ for all $j \in \mathbf{N}$, so that $s = y_\sigma$. ■

In particular, in [5, Lem. 5.3], we can actually take x_n to be equal to $P^{\circ k}(\varphi_q^{-k}(u^p))$ for some $k \gg 0$. In what follows, we let $h_\ell(u) := P^{\circ \ell}(\varphi_q^{-\ell}(u^p))$.

Let $I = [0, r_0]$, let $m \geq 0$ and let $x \in (\tilde{\mathbf{B}}_F^I)^{\Gamma_m - \mathbf{Q}_p^{-\text{an}}}$. Then there exists $n \geq 0$ such that $\|\partial_\sigma(x)\|_{\Gamma_m} \leq p^{nk} \|x\|_{\Gamma_m}$. Moreover, by [7, Lem. 2.4], there exists $k_0 \geq m$ such that $\|x\|_{\Gamma_k} = \|x\|$ for all $k \geq k_0$. There exists $\ell \geq k_0$ such that $h_\ell(u) - y_\sigma \in p^n \tilde{\mathbf{A}}^I$, and there exists $m' \geq \ell$ such that $h_\ell(u), y_\sigma \in (\tilde{\mathbf{B}}_F^I)^{\Gamma_{m'} - \mathbf{Q}_p^{-\text{an}}}$ and such that $\|h_\ell(u) - y_\sigma\|_{\Gamma_s} \leq p^{-n}$ for all $s \geq m'$.

Then for $s \geq m'$, the series $x_i := \frac{1}{i!} \sum_{k \geq 0} (-1)^k \partial_\sigma^{i+k}(x) \frac{(y_\sigma - h_\ell(u))^k}{k!}$ converges in $(\tilde{\mathbf{B}}_F^I)^{\Gamma_s - \mathbf{Q}_p^{-\text{an}}}$, and we have

$$x = \sum_{i \geq 0} x_i (y_\sigma - h_\ell(u))^i$$

in $(\tilde{\mathbf{B}}_F^I)^{\Gamma_s - \mathbf{Q}_p^{-\text{an}}}$ (this is the same as the proof of [5, Thm. 5.4]).

Now let

$$X_{\ell,s} := \left\{ x \in (\tilde{\mathbf{B}}_F^I)^{\Gamma_\ell - \mathbf{Q}_p^{-\text{an}}}, x = \sum_{i \geq 0} x_i (y_\sigma - h_\ell(u))^i \text{ and } x_i \in (\tilde{\mathbf{B}}_F^I)^{\Gamma_s - \text{an}} \right\}.$$

The above shows that any $x \in (\tilde{\mathbf{B}}_F^I)^{\mathbf{Q}_p^{-\text{la}}}$ belongs to some $X_{\ell,s}$, $s \geq \ell \geq 0$.

We denote by $F[[T_1, T_2]]$ the set of power series with coefficients in F in two variables T_1 and T_2 , endowed with an F -linear action of $\text{Gal}(F_p^{\text{LT}}/F)$ given by $g(T_1) = \chi_p(g) \cdot T_1$ and $g(T_2) = \sigma(\chi_p(g)) \cdot T_2$.

Proposition 7.10. *There is an injective, Galois-equivariant F -linear map $\iota_{\ell,s} : X_{\ell,s} \rightarrow F[[T_1, T_2]]$.*

Proof. Let $x \in X_\ell$. We can write $x = \sum_{i \geq 0} x_i (y_\sigma - h_\ell(u))^i$, where $x_i \in (\tilde{\mathbf{B}}_F^I)^{\Gamma_s - \text{an}}$. Note that $(\tilde{\mathbf{B}}_F^I)^{\Gamma_s - \text{an}} \subset \mathbf{B}_{F,s}^I$ by [5, Thm. 4.4] so that we can write $x_i = f_i(\varphi_q^{-s}(u))$, with $f_i \in F[[u]]$.

We can write $\varphi_q^s(x) = \sum_{i \geq 0} f_i(u) (P^{\text{os}}(y_\sigma) - P^{\text{os}}(u^p))^i$, so that

$$\varphi_q^s(x) = \sum_{i \geq 0} f_i(u) \sum_{k=0}^i \binom{i}{k} (P^{\text{os}}(y_\sigma))^k (-P^{\text{os}}(u^p))^{i-k}$$

and this is equal to (if everything converges)

$$\sum_{k \geq 0} (P^{\text{os}}(y_\sigma))^k \sum_{j \geq 0} f_{j+k}(u) (-P^{\text{os}}(u^p))^j.$$

Let $A_k := \sum_{j \geq 0} f_{j+k}(u) (-P^{\text{os}}(u^p))^j \in F[[u]]$. This is a well-defined element of $F[[u]]$ since $P^{\text{os}}(u^p) \in u \cdot F[[u]]$ and since the $f_{j+k}(u)$ belong to $F[[u]]$. Since $P^{\text{os}}(y_\sigma) \in y_\sigma \cdot F[[y_\sigma]]$ (because $s \geq \ell$), the sum $\sum_{k \geq 0} (P^{\text{os}}(y_\sigma))^k A_k$ defines an element of $F[[y_\sigma, u]]$. Now because $t_\sigma \in y_\sigma \cdot F[[y_\sigma]]$ and $t_p \in F[[u]]$, this can be rewritten as an element of $F[[T_1, T_2]]$. It remains to check that the map we have just constructed is well defined relative to the Galois action, which is straightforward (because φ_q^{-s} is Γ_K -equivariant and then the rest is just rewriting power series in $F[[Y_1, Y_2]] = F[[T_1, T_2]]$).

In order to see that the map that we obtain this way is injective, we can see that at every step the operations we make are injective. To see that we indeed have defined a map this way, we have to prove that the x_i coming from x are uniquely defined, which follows from the formula given in the proof of [5, Thm. 5.4]. ■

Corollary 7.11. *We have $(\tilde{\mathbf{B}}_F^{[0,r_0]})_{\text{ac,la}} = F$.*

Proof. Let ∇_1, ∇_2 denote respectively the maps $T_1 \cdot \frac{d}{dT_1}$ and $T_2 \cdot \frac{d}{dT_2}$ on $F[[T_1, T_2]]$. It is clear from the definition of the Galois action of $\text{Gal}(F_p^{\text{LT}}/F)$ on $F[[T_1, T_2]]$ that this action is locally analytic, and that the corresponding operators ∇_{id} and ∇_σ coincide respectively with ∇_1 and ∇_2 .

By the previous proposition, it therefore suffices to prove that $F[[T_1, T_2]]^{\nabla_1 + \nabla_2 = 0} = K$. This is straightforward because

$$(\nabla_1 + \nabla_2) \left(\sum_{i,j} a_{ij} T_1^i T_2^j \right) = \sum_{i,j} (i + j) a_{ij} T_1^i T_2^j$$

which is equal to 0 if and only if $a_{ij} = 0$ for all $i, j \neq 0$. ■

Those results highlight that in general, the rings $(\tilde{\mathbf{B}}_K^I)^{\text{la}}$ could be really small, even if we restrict ourselves to the case of p -adic abelian extensions. However, if we assume that K_∞/K contains a cyclotomic extension, then most of those problems should disappear. Note that the case of the anticyclotomic extension is precisely a case where we removed the cyclotomic extension contained inside the Lubin–Tate extension.

It would be interesting to know if “containing a cyclotomic extension” is the key component for the theory to behave properly.

Question 7.12. *Are there Galois p -adic Lie extensions K_∞/K almost totally ramified, not containing any cyclotomic extension, such that for all compact subinterval I of $[0; +\infty[$, $(\tilde{\mathbf{B}}_K^I)^{\text{la}} \neq K$?*

Note that, if we do not assume that K_∞/K is Galois but that its ramification is infinite and its Galois closure $L_\infty := K_\infty^{\text{Gal}}$ is such that L_∞/K is a p -adic Lie extension with Galois group $\Gamma_L := \text{Gal}(L_\infty/K)$, there is still a way to define locally analytic vectors attached to the extension K_∞/K , in the following way: if W is a p -adic Banach representation of Γ_L , we define the “ K_∞ -locally analytic vectors of W ” by $W^{K_\infty\text{-la}} := (W^{\Gamma_L\text{-la}})^{\text{Gal}(L_\infty/K_\infty)}$. Kummer extensions are particular cases of this setting, and in this case the theory does behave properly [27]. It is therefore not clear what to expect if we generalize the theory to “non Galois p -adic Lie extensions”.

7.4. A particular case of the conjecture

We now explain how to prove Conjecture 7.7 in a very particular case, which is already nontrivial and is a generalization of the Kummer case.

In this section, we assume that K_∞/K is a p -adic Lie extension which is a successive extension of \mathbf{Z}_p -extensions: there exist $(K_{\infty,i})_{i \in \{0, \dots, d\}}$ such that for all i , $K_{\infty,i}/K$ is Galois, $K_\infty = K_{\infty,d}$, $K_{\infty,0} = K$, and $\text{Gal}(K_{\infty,i+1}/K_{\infty,i}) \simeq \mathbf{Z}_p$. We also assume that there exists $\eta: \mathcal{G}_K \rightarrow \mathbf{Z}_p^\times$ an unramified character such that $K_{\infty,1} = K_\infty^\eta$. In particular, this implies that Γ_K is isomorphic to a semidirect product $\mathbf{Z}_p \rtimes \dots \rtimes \mathbf{Z}_p$. We write $g \mapsto (c_d(g), \dots, c_1(g))$ for the isomorphism $\Gamma_K \simeq \mathbf{Z}_p \rtimes \dots \rtimes \mathbf{Z}_p$, where if $1 \leq j \leq d$ and $g \in \Gamma_K$, $(c_j(g), \dots, c_1(g))$ denotes the image of g in the quotient

$$\text{Gal}(K_{\infty,j}/K) \simeq \mathbf{Z}_p \rtimes \dots \rtimes \mathbf{Z}_p.$$

For any $i \in \{1, \dots, d\}$, we let $g_i \in \text{Gal}(K_\infty/K_{\infty,i-1})$ be such that $c_i(g_i) = 1$, so that its image in the quotient $\text{Gal}(K_{\infty,i}/K_{\infty,i-1})$ is a topological generator, and we let $\nabla_i \in \text{Lie}(\Gamma_K)$ denote the operator corresponding to $\log g_i$. Note that since $g_i \in \text{Gal}(K_\infty/K_{\infty,i-1})$, we have $c_j(g_i) = 0$ for $j < i$. Since it is clear that the g_i generate Γ_K topologically, the operators ∇_i define a basis of the Lie algebra of Γ_K . We also let $G_i = \text{Gal}(K_{\infty,i}/K)$.

Lemma 7.13. *If x is a locally analytic vector of a p -adic Banach representation of Γ_K such that there exists $j \geq 2$, such that for all $k \geq j$, $\nabla_k(x) = 0$, then for all $\ell < j$ and for all $k \geq j$, $\nabla_k \circ \nabla_\ell(x) = \nabla_\ell \circ \nabla_k(x) = 0$.*

Proof. Let W be a p -adic Banach representation of Γ_K . Let x be a locally analytic vector of W which is killed by ∇_k , for all $k \geq j$. By definition of the ∇_k operators, this implies that there exist f_d, \dots, f_j integers such that we have $g_k^{p^{f_k}}(x) = x$ for all $k \in \{j, \dots, d\}$. Therefore, x belongs to $W^{\text{Gal}(K_\infty/M)}$ for some finite extension M of $K_{\infty,j}$, which we can assume to be Galois over K . But then $g_\ell(x) \in W^{\text{Gal}(K_\infty/M)}$ for all $\ell < j$, so that $\nabla_k \circ \nabla_\ell(x) = 0$. ■

Proposition 7.14. *For any $i \in \{2, \dots, d\}$, there exists $r_i \geq 0$ and $b_i \in \mathbf{B}_{K_{\infty,i}}^{\dagger,r_i}$ such that $(g_i - 1)(b_i) = 1$ and the image of b_i in $\tilde{\mathbf{B}}_{K_{\infty,i}}^I$, for $\min(I) \geq r_i$, is a locally analytic vector of $\tilde{\mathbf{B}}_{K_{\infty,i}}^I$ for the action of G_i .*

Proof. We only prove it for $i = d$, the proof for $i < d$ is the same replacing Γ_K by G_i .

Let V denote the 2-dimensional p -adic representation of \mathcal{G}_K given by

$$g \mapsto \begin{pmatrix} 1 & c_d(g) \\ 0 & 1 \end{pmatrix}.$$

By the theorem of Cherbonnier–Colmez, the (φ, Γ) -module attached to V is overconvergent, so that it admits a basis on $(\mathbf{B}_K^\eta)^{\dagger,r}$. If (e_1, e_2) was the basis of V giving rise to the matrix representation above, we see that a basis of the attached (φ, Γ) -module on $(\mathbf{B}_K^\eta)^{\dagger,r}$ is given by $(e_1 \otimes 1, e_2 \otimes 1 - e_1 \otimes b)$ for some $b \in \mathbf{B}_K^{\dagger,r}$. The fact that this basis is invariant by the action of $\text{Gal}(\bar{\mathbf{Q}}_p/K_{\infty,1})$ means that it also is invariant by the action of $\text{Gal}(\bar{\mathbf{Q}}_p/K_{\infty,d-1})$ and thus we get that $g_d(b) = b + c_d(g_b) = b + 1$ by our choice of g_d . The fact that we can find such an element b which is a locally analytic vector of $\tilde{\mathbf{B}}_{K_{\infty,i}}^I$ follows from Corollary 7.2. ■

We let $r_b = \max(r_i)$ so that the (b_i) can all be seen as elements of $(\tilde{\mathbf{B}}_K^{\dagger,r_b})$.

Recall that if M_∞^η is a finite extension of K_∞^η then there corresponds a finite unramified extension $\mathbf{B}_{M,\eta}^\dagger/\mathbf{B}_{K,\eta}^\dagger$ of degree $[M_\infty^\eta : K_\infty^\eta]$, and there exists $r(M) > 0$ and elements x_1, \dots, x_f in $\mathbf{A}_{M,\eta}^{\dagger,r(M)}$ such that

$$\mathbf{A}_{M,\eta}^{\dagger,s} = \bigoplus_{i=1}^f \mathbf{A}_{K,\eta}^{\dagger,s} \cdot x_i \quad \text{for all } s \geq r(M).$$

Lemma 7.15. *Let $M_\infty^\eta \subset K_\infty^\eta$ be a finite extension of K_∞^η . If $r_\ell \geq r(M)$ then the x_i defined above belong to $(\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger,r_\ell})^{\text{pa}}$ and are killed by ∇_i for all $i > 1$.*

Proof. The fact that the x_i are pro-analytic is a consequence of the proof of item (2) of [5, Thm. 4.4]. The second part is straightforward as $M_\infty^\eta/K_\infty^\eta$ is finite. ■

If M_∞^η is a finite extension of K_∞^η , and if I is a compact subinterval of $[0; +\infty[$ such that $\min(I) \geq r(M)$, we let $\mathbf{A}_{M,\eta}^I$ be the completion of $\mathbf{A}_{M,\eta}^{\dagger,r(M)}$ for $V(\cdot, I)$, and we let $\mathbf{B}_{M,\eta}^I = \mathbf{A}_{M,\eta}^I[1/p]$.

Lemma 7.16. *If $x \in \mathbf{A}_K^{\dagger,r}$ and if $k, n \in \mathbf{N}$ then there exists M_∞^η a finite extension of K_∞^η , $m \geq 0$ and $y \in \varphi^{-m}(\mathbf{A}_{M,\eta}^{\dagger,p^m r})$ such that $x - y \in \pi^j \tilde{\mathbf{A}}^{\dagger,r} + u^k \tilde{\mathbf{A}}^+$.*

Proof. By reducing mod π , we obtain that $\bar{x} \in \mathbf{E}_K$. But $\mathbf{E}_K = \bigcup \mathbf{E}_M$ where M goes through the set of finite extensions of $K_{\infty,\eta}$ contained in K_∞ . In particular, there exists a finite extension M_0 of $K_{\infty,\eta}$, contained in K_∞ , and $y_0 \in \mathbf{A}_{M_0}$ such that $x - y_0 \in p\mathbf{A}_K$, since $\mathbf{A}_{M_0,\eta} \subset \mathbf{A}_K$. Since $\frac{x-y_0}{p} \in \mathbf{A}_K$, the same arguments show that there exists a finite extension M_1 of $K_{\infty,\eta}$, contained in K_∞ , and $y_1 \in \mathbf{A}_{M_1,\eta}$ such that $\frac{x-y_0}{p} - y_1 \in \pi\mathbf{A}_K$, so that $x - y_0 - py_1 \in \pi\mathbf{A}_K$, and we can without loss of generality assume that $M_0 \subset M_1$. By induction, we find y_0, y_1, \dots, y_n in $\mathbf{A}_{M_n,\eta}$, with M_n finite extension of $K_{\infty,\eta}$ contained in K_∞ , such that $x - y_0 - py_1 - \dots - p^n y_n \in \pi^{n+1}\mathbf{A}_K$. Let $z_n = y_0 + \dots + \pi^n y_n$. Let $\sum_{i \geq 0} p^i [x_i]$ be the way x is written in $\tilde{\mathbf{A}}_K = W(\tilde{\mathbf{E}}_K)$. Then $x^{(n)} := \sum_{i=0}^n p^i [x_i]$ is such that $x - x^{(n)} \in \pi^{n+1}\tilde{\mathbf{A}}_K$, and thus $x^{(n)} - z_n \in \pi^{n+1}\tilde{\mathbf{A}}_K$. In particular, since $z_n \in \tilde{\mathbf{A}}_{M_n,\eta}$ by construction, we deduce that the x_i all belong to $\tilde{\mathbf{E}}_{M_n,\eta}$ for $i \leq n$, and thus $x^{(n)} \in \mathbf{A}_{M_n,\eta}$. Since $x \in \tilde{\mathbf{A}}_K^{\dagger,r}$, we have in particular that $x^{(n)} \in \mathbf{A}_K^{\dagger,r}$. By [14, Cor. 8.11], $\mathbf{A}_{M_n,\infty,\eta}^{\dagger,r}$ is dense in $\tilde{\mathbf{A}}_{M_n,\eta}^{\dagger,r}$ for the topology induced by $V(\cdot, r)$, so that we can find

$$y \in \mathbf{A}_{M_n,\infty,\eta}^{\dagger,r}$$

such that $x^{(n)} - y \in \pi^n \tilde{\mathbf{A}}^{\dagger,r} + u^k \tilde{\mathbf{A}}^+$. We thus have $x - y = (x - x^{(n)}) + (x^{(n)} - y) \in \pi^n \tilde{\mathbf{A}}^{\dagger,r} + u^k \tilde{\mathbf{A}}^+$. ■

Lemma 7.16 shows that for any $I = [r; s]$ with $r \geq r_b$, and any integer n we can find elements b_n^ℓ such that $b_\ell - b_n^\ell \in p^n \tilde{\mathbf{A}}^I$ for all $\ell \in \{2, \dots, d\}$, which by Lemma 7.15 belong to $(\tilde{\mathbf{B}}_K^I)^{\text{la}}$ and are killed by ∇_j , for all $j \in \{2, \dots, d\}$. Since they are locally analytic vectors, we let $m = m(n, I)$ be such that all these elements, along with the elements b_ℓ , belong to $(\tilde{\mathbf{B}}_K^I)^{\Gamma_{m-\text{an}}}$.

Proposition 7.17. *Let $I = [r; s]$ with $r \geq r_b$ and let $m \geq m(n, I)$. Let $\ell \in \{2, \dots, d\}$ and let $x \in (\tilde{\mathbf{B}}_K^I)^{m-\text{an}}$ be such that for all $k > \ell$, $\nabla_k(x) = 0$. Then there exist $(x_j)_{j \in \mathbb{N}} \in (\tilde{\mathbf{B}}_K^I)^{\Gamma_{m-\text{an}}}$ such that $\|x_j p^{nj}\| \rightarrow 0$, for all $k \geq \ell$, $\nabla_k(x_j) = 0$ and $x = \sum_{j \geq 0} x_j (b_\ell - b_n^\ell)^j$.*

Proof. Let $x \in (\tilde{\mathbf{B}}_K^I)^{\Gamma_{m-\text{an}}}$. By [7, Lem. 2.6], there exists $n \geq 1$ such that for all $j \geq 1$, $\|\nabla_\ell^j(x)\|_{\Gamma_m} \leq p^{nj} \|x\|$ for all $\ell \in \{2, \dots, d\}$. Let

$$x_j = \frac{1}{j!} \sum_{k \geq 0} (-1)^k \frac{(b_\ell - b_n^\ell)^k}{k!} \nabla_\ell^j(x).$$

Similarly to the proof of [5, Thm. 5.4], the series converges in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_{m-\text{an}}}$ to an element x_j such that $\nabla_\ell(x_j) = 0$. Moreover, by construction of the b_ℓ and b_n^ℓ , we have

$$\nabla_k(b_\ell - b_n^\ell) = 0 \quad \text{for all } k > \ell,$$

and thus using Lemma 7.13, the x_j are killed by ∇_k , $k > \ell$. ■

Proposition 7.18. *Let $I = [r, s]$ and let $m \geq m(I, n)$. Then there exists M a finite extension of K_∞^n , depending only on m , and $k \geq 0$ depending only on s, m and M , such that*

$$(\tilde{\mathbf{B}}_K^I)^{\Gamma_{m-\text{an}}, \nabla_d = \dots = \nabla_2 = 0} \subset \varphi^{-k}(\mathbf{B}_{M,\eta}^{p^k I}).$$

Proof. Note that if $x \in (\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}}$ and is killed by ∇_i for some $i \in \{2, \dots, d\}$, then this means that the orbit map $g \mapsto g(x)$, from the p -adic Lie group $G_i := g_i^{\mathbf{Z}_p}$ to $\tilde{\mathbf{B}}_K^I$, is an analytic function on $G_i \cap \Gamma_m$ which becomes constant on some compact open subgroup of G_i since $\nabla_i(x) = 0$. It is therefore constant on $G_i \cap \Gamma_m$. It follows that if we let

$$M_\infty^\eta = (K_\infty)g_d^{p^m=1, \dots, g_2^{p^m=1},$$

then x is invariant by $\text{Gal}(K_\infty/M_\infty^\eta)$. Note that since Γ_K is topologically generated by the g_i for $i = 1 \cdots d$, M_∞^η is a finite extension of K_∞^η .

Now we can conclude using the same argument as in the proof of [27, Thm. 4.2.9]. We let $f = [M_\infty^\eta : K_\infty^\eta]$ and we let $r(M) > 0$ and x_1, \dots, x_f in $\mathbf{A}_{M,\eta}^{\dagger,r(M)}$ be such that

$$\mathbf{A}_{M,\eta}^{\dagger,s} = \bigoplus_{i=1}^f \mathbf{A}_{K,\eta}^{\dagger,s} \cdot x_i \quad \text{for all } s \geq r(M).$$

Note that the proof of item (2) of [5] shows that the x_i are locally analytic for the action of $\text{Gal}(K_\infty/K)$, so that there exists $k \geq m$ such that the x_i are all $\Gamma_{m'}$ -analytic. Then we have

$$\begin{aligned} (\tilde{\mathbf{B}}_K^I)^{\Gamma_k\text{-an}, \text{Gal}(K_\infty/M_\infty^\eta)} &= (\tilde{\mathbf{B}}_{M,\eta}^I)^{(\Gamma_{K,\eta})k\text{-an}} \\ &= \left(\bigoplus_{i=1}^f \tilde{\mathbf{B}}_{K,\eta}^I \cdot x_i \right)^{(\Gamma_{K,\eta})k\text{-an}} \\ &= \bigoplus_{i=1}^f (\tilde{\mathbf{B}}_{K,\eta}^I)^{(\Gamma_{K,\eta})k\text{-an}} \cdot x_i \\ &\subset \bigoplus_{i=1}^f \varphi^k(\mathbf{B}_{K,\eta}^{p^k I}) \cdot x_i, \end{aligned}$$

where the second equality follows from Proposition 3.3, and the last inclusion follows from the specialization of [5, Thm. 4.4] to the twisted cyclotomic case.

Since $k \geq m$, we obtain that

$$(\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}, \text{Gal}(K_\infty/M_\infty^\eta)=1} \subset \bigoplus_{i=1}^f \varphi^k(\mathbf{B}_{K,\eta}^{p^k I}) \cdot x_i \subset \varphi^k(\mathbf{B}_{M,\eta}^{p^k I}),$$

which is what we wanted. ■

We can now prove the conjecture.

Theorem 7.19. *Let K_∞/K be a p -adic Lie extension of rank d which is a successive extension of \mathbf{Z}_p -extensions over a cyclotomic extension. Then for $n \gg 0$, there exist d elements $x_{1,n}, \dots, x_{d,n}$ in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n\text{-an}}$ such that $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n\text{-an}}$ is the set of power series $\sum_{i=(i_1, \dots, i_d) \in \mathbf{N}^d} a_i x_{j,n}^{i_j}$ in the variables $(x_{i,n})_{i \in \{1, \dots, d\}}$ with coefficients in K such that the series $\sum_{i=(i_1, \dots, i_d) \in \mathbf{N}^d} a_i x_{j,n}^{i_j}$ converge in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_n\text{-an}}$.*

Proof. Let $I = [r, s]$ with $r \geq r_b$ and let $m \geq m(n, I)$. Let $x \in (\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}}$.

We start by applying Proposition 7.17 with $\ell = d$, so that there is no condition on the nabla operators. Therefore, there exist $(x_j)_{j \in \mathbb{N}} \in (\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}}$ such that $\|x_j p^{nj}\| \rightarrow 0$, $\nabla_d(x_j) = 0$ and $x = \sum_{j \geq 0} x_j (b_\ell - b_n^\ell)^j$.

Now each x_j belongs to $(\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}}$, and is such that $\nabla_d(x_j) = 0$ so that we can apply Proposition 7.17 to each x_j with $\ell = d - 1$, so that there exist $(x_{jk})_{j,k \in \mathbb{N}}$ in $(\tilde{\mathbf{B}}_K^I)^{\Gamma_m\text{-an}}$ such that $\|x_j p^{n(j+k)}\| \rightarrow 0$, for all $k \geq d - 1$, $\nabla_k(x_{jk}) = 0$ and $x_j = \sum_{k \geq 0} x_{jk} (b_{d-1} - b_n^{d-1})^k$.

We thus have $x = \sum_{k,j \in \mathbb{N}} x_{jk} (b_d - b_n^d)^j (b_{d-1} - b_n^{d-1})^k$, where $\|x_j p^{n(j+k)}\| \rightarrow 0$, for all $k \geq d - 1$, $\nabla_k(x_{jk}) = 0$. We can now apply Proposition 7.17 to each x_{jk} with $\ell = d - 2$.

Inductively, we find $(x_i)_{i \in \mathbb{N}\{2, \dots, d\}}$ such that $\|x_i^{n|\mathbf{i}|}\| \rightarrow 0$, where $|\mathbf{i}| = \sum_{j=2}^d i_j$, for all $k > 1$, $\nabla_k(x_i) = 0$ and

$$x = \sum_{\mathbf{i} \in \mathbb{N}\{2, \dots, d\}} x_{\mathbf{i}} \prod_{j=2}^d (b_j - b_n^j)^{i_j}.$$

By Proposition 7.18, the elements $x_{\mathbf{i}}$ all belong to $\varphi^{-k}(\mathbf{B}_{M,\eta}^{p^k I})$ for some finite extension M_∞^η of K_∞^η , depending only on m , and $k \geq 0$ depending only on s, m and M .

Proposition 7.5 of [14] and its analog in the twisted cyclotomic case show that the elements of $\mathbf{B}_{M,\eta}^J$ are power series in one variable over a subfield of K which converge on some annulus depending on J , and so we are done. ■

8. Locally analytic vectors for Robba rings

In this section we study what happens when we take the locally analytic vectors attached to Lubin–Tate extensions in the corresponding Robba rings $\tilde{\mathbf{B}}_{\text{rig}}^\dagger$. In particular, we show that those locally analytic vectors recover objects that were defined by Colmez using completely different methods in [16]. We then use this result to explain why locally analytic vectors are usually not the right object to consider when working with Fréchet rings, and recall some properties satisfied by the corresponding attached modules. Finally, we prove that if V is an F -analytic representation of V , then

$$(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes V)^{H_K, \Gamma\text{-la}} \simeq (\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes V)^{H_K, \Gamma\text{-la}}.$$

8.1. Locally analytic vectors in Robba rings

If T is a variable and L is a finite extension of \mathbf{Q}_p , we let $L\langle\langle T \rangle\rangle$ denote the set of power series in T with coefficients in L and with infinite radius of convergence.

Proposition 8.1. *We have $(\tilde{\mathbf{B}}_{\text{rig},K}^\dagger)^{\text{la}} = (\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{\text{la}} = K\langle\langle t_\pi \rangle\rangle$.*

Proof. Let $r \geq 0$ and let $z \in (\tilde{\mathbf{B}}_{\text{rig}}^{\dagger,r})^{\text{la}}$. It is therefore Γ_n -analytic for some $n \geq 0$, so that for any $s \geq r$, z is a Γ_n - F -analytic vector of $\tilde{\mathbf{B}}^{[r;s]}$. By applying enough times φ_q to item (1) of [5, Thm. 4.4], we have that the images of z in $\tilde{\mathbf{B}}^{[r;s]}$ all belong to $\mathbf{B}_K^{[r;s]}$ as long as

s is such that $r_n \leq s$, where $r_n = p^{nh-1}(p-1)$ was defined in Section 2.3. Taking the inverse limit, this implies that $z \in \mathbf{B}_{\text{rig},K}^{\dagger,r}$.

Since φ commutes with the Galois action, the reasoning above also applies to $\varphi_q^{-1}(z)$. Therefore, for all $k \geq 0$, $\varphi_q^{-k}(z) \in \mathbf{B}_{\text{rig},K}^{\dagger}$. This implies that z belongs to the ring $K\langle\langle t_\pi \rangle\rangle$. Indeed, in the cyclotomic case this is [16, Prop. 3.9], and for the general case this follows from the same arguments, using the dictionary developed by Colmez in the Lubin–Tate case in [17, §2].

To finish the proof, it suffices to notice that any element of $K\langle\langle t_\pi \rangle\rangle$ is indeed F -locally analytic (and is actually Γ_0 -analytic). ■

Proposition 8.1 already shows that the set of F -analytic vectors of $\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger}$ is really small compared with the set of F -pro-analytic vectors of $\tilde{\mathbf{B}}_{\text{rig},K}^{\dagger}$.

We now explain what is $\mathbf{D}_{\text{rig},K}^{\dagger}(V)^{\text{la}}$ and prove that its rank as an $E\langle\langle t_\pi \rangle\rangle$ -module is too small in general.

Given a (φ, Γ) -module \mathbf{D} over E (in the cyclotomic setting), Colmez has defined [16, §3.3] a module denoted by $\mathbf{D} \boxtimes \{0\}$ by $\bigcap_{n \geq 0} \varphi^n(\mathbf{D})$, which is a free (φ, Γ) -module over $E\langle\langle t \rangle\rangle$ of rank $\leq \dim(V)$ by [16, Thm. 3.20].

Proposition 8.2. *Let V be an F -analytic representation of \mathcal{G}_K . Then*

$$(\tilde{\mathbf{B}}_{\text{rig}}^{\dagger} \otimes_{\mathbf{Q}_p} V)^{H_K, \Gamma_K\text{-la}} = \mathbf{D}_{\text{rig},K}^{\dagger}(V)^{\text{la}} = \bigcap_{n \geq 0} \varphi_q^n(\mathbf{D}_{\text{rig},K}^{\dagger}(V)).$$

Proof. Let $r \geq 0$ be such that $\mathbf{D}_{\text{rig}}^{\dagger}(V)$ and all its structures are defined over $\mathbf{B}_{\text{rig},K}^{\dagger,r}$ and let I be a compact subinterval of $[r, +\infty[$ such that $I \cap qI \neq \emptyset$. Let $x \in (\mathbf{D}_{\text{rig},K}^{\dagger,r}(V))^{\Gamma_n\text{-an}}$.

Let $m \geq n$ be such that there exists a basis (b_1, \dots, b_d) of $\mathbf{D}^I(V)$ of Γ_m -analytic vectors of $\mathbf{D}^I(V)$ (this is possible by the same argument as in [31, §2.1]). Then by Proposition 3.3, this implies that

$$(\mathbf{D}^I(V))^{\Gamma_m\text{-an}} = \bigoplus_{i=1}^d (\mathbf{B}_K^I)^{\Gamma_m\text{-an}} \cdot b_i.$$

Since $\mathbf{D}_{\text{rig}}^{\dagger}(V)$ is a φ_q -module, the elements $\varphi_q(b_1), \dots, \varphi_q(b_d)$ form a basis of $\mathbf{D}^{qI}(V)$, and are Γ_m -analytic vectors of $\mathbf{D}^{qI}(V)$ since φ_q commutes with the Galois action. In particular, we get that

$$(\mathbf{D}^{qI}(V))^{\Gamma_m\text{-an}} = \bigoplus_{i=1}^d (\mathbf{B}_K^{qI})^{\Gamma_m\text{-an}} \cdot \varphi_q(b_i).$$

Applying inductively the same argument, we see that for $\ell \gg 0$, we have

$$(\mathbf{D}^{q^\ell I}(V))^{\Gamma_m\text{-an}} = \bigoplus_{i=1}^d (\mathbf{B}_K^{q^\ell I})^{\Gamma_m\text{-an}} \cdot \varphi_q^\ell(b_i).$$

But by applying φ_q enough times to item (1) of [5, Thm. 4.4], we see that for ℓ big enough,

$$(\mathbf{B}_K^{q^{\ell+1}I})^{\Gamma_m\text{-an}} = \varphi_q((\mathbf{B}_K^{q^\ell I})^{\Gamma_m\text{-an}}),$$

so that for $\ell \gg 0$,

$$(\mathbf{D}^{q^{\ell+1}I}(V))^{\Gamma_m\text{-an}} = \varphi_q(\mathbf{D}^{q^\ell I}(V))^{\Gamma_m\text{-an}}.$$

Therefore, the image of the element $x \in (\mathbf{D}_{\text{rig},K}^\dagger(V))^{\Gamma_n\text{-an}}$ in $(\mathbf{D}^{q^{\ell+1}I}(V))$ is also in $\varphi_q(\mathbf{D}^{q^\ell I}(V))$, for $\ell \gg 0$, and thus $x \in \varphi(\mathbf{D}_{\text{rig},K}^\dagger(V))^{\Gamma_n\text{-an}}$. This proves that

$$\mathbf{D}_{\text{rig},K}^\dagger(V)^{\text{la}} = \bigcap_{n \geq 0} \varphi_q^n((\mathbf{D}_{\text{rig},K}^\dagger(V))^{\text{la}}) \subset \bigcap_{n \geq 0} \varphi_q^n(\mathbf{D}_{\text{rig},K}^\dagger(V)).$$

Note that

$$(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{H_K} = \tilde{\mathbf{B}}_{\text{rig},K}^\dagger \otimes_{\mathbf{B}_{\text{rig},K}^\dagger} \mathbf{D}_{\text{rig},K}^\dagger(V).$$

Using item (3) of [5, Thm. 4.4] and the proof of [5, Thm. 10.4], we know that

$$(\tilde{\mathbf{D}}_{\text{rig},K}^\dagger(V))^{\Gamma_K\text{-pa}} = \bigcup_{m \geq 0} \varphi_q^{-m}(\mathbf{D}_{\text{rig},K}^\dagger(V)).$$

Since Γ_n -analytic vectors are in particular also pro-analytic vectors of Γ_K , this means that

$$(\tilde{\mathbf{D}}_{\text{rig},K}^\dagger(V))^{\Gamma_n\text{-an}} = \bigcup_{m \geq 0} (\varphi_q^{-m}(\mathbf{D}_{\text{rig},K}^\dagger(V)))^{\Gamma_n\text{-an}},$$

by taking the Γ_n -analytic vectors, and thus

$$(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{H_K, \Gamma_K\text{-la}} = \mathbf{D}_{\text{rig},K}^\dagger(V)^{\text{la}}$$

since the latter is stable by taking inverse powers of φ_q .

It remains to prove that $\mathbf{D}_{\text{rig},K}^\dagger(V)^{\text{la}} \supset \bigcap_{n \geq 0} \varphi_q^n(\mathbf{D}_{\text{rig},K}^\dagger(V))$.

If $x \in \bigcap_{n \geq 0} \varphi_q^n(\mathbf{D}_{\text{rig},K}^\dagger(V))$, then x belongs to a free $K\langle\langle t_\pi \rangle\rangle$ -module which is Γ_K -stable so that for $g \in \text{Gal}(K_\infty/K)$, $g(x) = \text{Mat}(g) \cdot x$ where $\text{Mat}(g) \in \text{GL}_d(K\langle\langle t_\pi \rangle\rangle)$, so that the Galois action on x is locally analytic. This finishes the proof. ■

In particular, the following result of Colmez shows that the module $\mathbf{D}_{\text{rig},K}^\dagger(V)^{\text{la}}$ is too small in general.

Proposition 8.3. *Let V be a two-dimensional irreducible representation of $\mathcal{G}_{\mathbf{Q}_p}$. If V is not trianguline then $\mathbf{D}_{\text{rig},K}^\dagger(V) \boxtimes \{0\} = 0$.*

Proof. This is item (i) of [16, Thm. 3.23]. ■

Remark 8.4. Theorem 3.23 of [16] also says that if V is a semistable, noncrystalline 2-dimensional representation, then $\mathbf{D}_{\text{rig},K}^\dagger(V) \boxtimes \{0\}$ is a (φ, Γ) -module of rank 1 over $E\langle\langle t \rangle\rangle$.

8.2. φ -modules on $L\langle\langle t_\pi \rangle\rangle$

By Proposition 8.2, to any E -representation V of \mathcal{G}_K we can attach a module on $E\langle\langle t_\pi \rangle\rangle$, which is endowed with a Frobenius φ_q and an operator ∇ coming from the action of the Lie algebra of Γ_K . Note that φ_q and Γ_K act on $E\langle\langle t_\pi \rangle\rangle$ by

$$\varphi_q(t_\pi) = \pi t_\pi, \quad g(t_\pi) = \chi_\pi(g)t_\pi.$$

We can also define an operator ∇ on $E\langle\langle t_\pi \rangle\rangle$ by $\nabla_u = t_\pi \frac{d}{dt_\pi}$.

As a matter of fact, φ -modules on $E\langle\langle t_\pi \rangle\rangle$ were already studied by Colmez in [16, §3.1] and the results proved by Colmez show that φ -modules on $E\langle\langle t_\pi \rangle\rangle$ are not as bad as one may think. Be careful that what we call $E\langle\langle t_\pi \rangle\rangle$ corresponds in the notations of Colmez to $E\{\{t_\pi\}\}$. In this section, we recall Colmez’s results on φ -modules on $E\langle\langle t_\pi \rangle\rangle$.

Definition 8.5. A (φ_q, Γ_K) -module on $E\langle\langle t_\pi \rangle\rangle$ is a finite free $E\langle\langle t_\pi \rangle\rangle$ -module, endowed with semilinear actions of φ_q and Γ_K which commute one to another and such that φ_q is an isomorphism.

A (φ_q, ∇) -module on $E\langle\langle t_\pi \rangle\rangle$ is a finite free $E\langle\langle t_\pi \rangle\rangle$ -module, endowed with semilinear actions of φ_q and ∇ which commute one to another and such that φ_q is an isomorphism.

A (φ_q, Γ_K) -module on $E\langle\langle t_\pi \rangle\rangle$ gives rise to a (φ_q, ∇) -module on $E\langle\langle t_\pi \rangle\rangle$ by taking the same φ -structure and taking ∇ to be the operator $\frac{\log(g)}{\log \chi_\pi(g)}$ for g close enough to 1.

The ring $E\langle\langle t_\pi \rangle\rangle$ can be interpreted *via* analytic functions, as it is the projective limit of the rings of analytic functions on the disks $v_p(x) \geq -ne$ for $n \in \mathbf{N}$. Those rings are principal Banach rings and therefore $E\langle\langle t_\pi \rangle\rangle$ is a Fréchet–Stein ring, which in particular implies that any closed submodule of a free module of rank d is free of rank $\leq d$ and that a submodule of finite type of a free finite type module is closed and thus free. Moreover, Newton polygons theory show that an element $f \in E\langle\langle t_\pi \rangle\rangle$ does not vanish if and only if $f \in E^\times$, so that $(E\langle\langle t_\pi \rangle\rangle)^\times = E^\times$.

Lemma 8.6. *Let M be a rank d φ_q -module on $E\langle\langle t_\pi \rangle\rangle$ and let $v \in M$ be such that there exists $\alpha \in E^\times$ such that $\varphi_q(v) = \alpha v$. Then there exists $k \in \mathbf{N}$ such that $t_\pi^{-k}v \in M$ and $M/E\langle\langle t_\pi \rangle\rangle t_\pi^{-k}v$ is free of rank $d - 1$ on $E\langle\langle t_\pi \rangle\rangle$.*

Proof. See [16, Lem. 3.4]. ■

Let M be a φ_q -module on $E\langle\langle t_\pi \rangle\rangle$ and let $\bar{M} = M/t_\pi M$. If $P \in E[X]$ is unitary of degree d and irreducible, then we let M_P (resp. \bar{M}_P) denote the set of elements $v \in M$ (resp. in \bar{M}) such that $P(\varphi_q)^n \cdot v = 0$ for $n \gg 0$ and if $k \in \mathbf{N}$, we let $P[k]$ be the polynomial $\pi^{kd} P(X/\pi^k)$.

Theorem 8.7. *If M is a φ_q -module of rank d on $E\langle\langle t_\pi \rangle\rangle$, there exists a basis e_1, \dots, e_d of M in which the matrix of φ is $A + N$, where $A \in \text{GL}_d(L)$ is semisimple and invertible, and N is nilpotent and commutes with A . Moreover, N splits into $N = N_0 + t_\pi N_1 + \dots$, where $N_i \in M_d(L)$ sends the kernel M_P of $P(A)$ into the one $M_{P[-i]}$ of $P(\pi^i A)$ for all P (and thus in particular the sum is finite).*

Proof. See [16, Thm. 3.6]. ■

Given a (φ_q, Γ_K) -module D on $E\langle\langle t_\pi \rangle\rangle$, we say that an element v of D is proper for the action of φ_q and Γ_K if there exists an F -analytic character $\delta: K^\times \rightarrow E^\times$ such that $\varphi(v) = \delta(\pi)v$ and $g(v) = \delta(\chi_\pi(g))v$ for all $g \in \Gamma_K$.

Lemma 8.8. *Given a (φ_q, Γ_K) -module D of rank 1 on $E\langle\langle t_\pi \rangle\rangle$, F -analytic, with basis e , then there exists an F -analytic character $\delta: K^\times \rightarrow E^\times$ such that e is proper for δ .*

Proof. This just follows from the fact that a rank 1 (φ_q, Γ_n) -module on $E\langle\langle t_\pi \rangle\rangle$ has a unique basis e , up to multiplication by an element of $(E\langle\langle t_\pi \rangle\rangle)^\times = E^\times$. ■

8.3. Frobenius regularity

We now explain how to use the fact that our rings are embedded with a Frobenius in order to show some regularity property. This section is in the same spirit as [2, §3.1 and §3.2].

Lemma 8.9. *Let h be a positive integer. Then*

$$\bigcap_{s=0}^{+\infty} \pi^{-hs} \tilde{\mathbf{A}}^{\dagger, q^{-s}r} = \tilde{\mathbf{A}}^+ \quad \text{and} \quad \bigcap_{s=0}^{+\infty} \pi^{-hs} \tilde{\mathbf{A}}^{\dagger, q^{-s}r} \subset \tilde{\mathbf{B}}_{\text{rig}}^+.$$

Proof. This is [2, Lem. 3.1] when $K = \mathbf{Q}_p$. The generalization when K is a finite extension of \mathbf{Q}_p is straightforward. ■

Proposition 8.10. *Let $r, v \in \mathbf{N}$, and let $A \in \mathbf{M}_{v \times r}(\tilde{\mathbf{B}}_{\text{rig}}^+)$. If $P \in \text{GL}_v(K\langle\langle t_\pi \rangle\rangle)$ is such that $P \in \mathbf{M}_v(K[t_\pi])$ and such that $A = P\varphi^{-1}(A)$ then $A \in \mathbf{M}_{v \times r}(\tilde{\mathbf{B}}_{\text{rig}}^+)$.*

Proof. Write A as (a_{ij}) and P as (p_{ij}) . Let h_0 be such that the $\pi^{h_0} p_{ik}$ belong to $\mathcal{O}_F[t_\pi]$, and let n be the highest degree of the p_{ik} as polynomials in t_π . The assumption on the relation between P and A can be translated as:

$$p_{i1}\varphi^{-1}(a_{1j}) + \dots + p_{iv}\varphi^{-1}(a_{vj}) = a_{ij} \quad \forall i \leq v, j \leq r.$$

Let $c > 0$ and $r \geq 0$ be such that the a_{ij} belong to $p^{-c} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r}$. Using the relation between P and A , this implies that the a_{ij} belong to $p^{-h_0-c} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r/q} \cdot \mathcal{O}_K[t_\pi]_n$, where $\mathcal{O}_K[t_\pi]$ denotes the ring of polynomials in t_π with degree $\leq n$. Note that if $t_\pi \in \pi^{-\beta} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r}$, then since $\varphi_q^{-1}(t_\pi) = \frac{1}{\pi} t_\pi$, we get that $t_\pi \in \pi^{1-\beta} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r/q}$ and thus $t_\pi \in \pi^{-\beta} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r/q^\ell}$ for all $\ell \geq 0$. We therefore have that the a_{ij} belong to $\pi^{-h_0-c-n\beta} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r/q}$, and applying the result inductively, the a_{ij} belong to $\pi^{-c-h_0\ell-n\beta\ell} \tilde{\mathbf{A}}_{\text{rig}}^{\dagger, r/q^\ell}$. We can thus apply Lemma 8.9 to the $p^c a_{ij}$, which shows that they belong to $\tilde{\mathbf{B}}_{\text{rig}}^+$, as we wanted. ■

Proposition 8.11. *Let V be an F -analytic E -representation of \mathcal{G}_K . Then the morphism*

$$(\tilde{\mathbf{B}}_{\text{rig}}^+ \otimes V)^{H_K, \Gamma_K - \text{la}} \rightarrow (\tilde{\mathbf{B}}_{\text{rig}}^+ \otimes V)^{H_K, \Gamma_K - \text{la}}$$

induced by the inclusion $\tilde{\mathbf{B}}_{\text{rig}}^+ \subset \tilde{\mathbf{B}}_{\text{rig}}^+$, is an isomorphism of (φ_q, ∇) -modules on $E\langle\langle t_\pi \rangle\rangle$.

Proof. Let (v_1, \dots, v_r) and (d_1, \dots, d_v) be respectively an E -basis of V and an $E\langle\langle t_\pi \rangle\rangle$ -basis of $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes_E V)^{\text{la}}$. There exists $A \in \mathbf{M}_{r \times v}(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)$ such that $(d_i) = A(v_i)$. Let $P \in \text{GL}_v(E\langle\langle t_\pi \rangle\rangle)$ be the matrix of φ_q in the basis (d_i) . By Theorem 8.7, we can assume that the basis (d_i) of $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes_E V)^{\text{la}}$ is such that $P \in \mathbf{M}_v(K[t_\pi])$. We then have $\varphi_q(A) = PA$ and thus $A = \varphi_q^{-1}(P)\varphi_q^{-1}(A)$. By Proposition 8.10, we have $A \in M_{r \times v}(\tilde{\mathbf{B}}_{\text{rig}}^+)$ and hence

$$(\tilde{\mathbf{B}}_{\text{rig}}^\dagger \otimes_E V)^{\text{la}} \subset (\tilde{\mathbf{B}}_{\text{rig}}^+ \otimes_E V)^{\text{la}}. \quad \blacksquare$$

9. Applications to trianguline representations

We now explain how some of the rings previously introduced provide some results towards the question of the existence of a ring of periods for trianguline representations. We will start by recalling the notions of trianguline representations and refinements.

In a previous version of this paper, we claimed that trianguline representations of $\mathcal{G}_{\mathbf{Q}_p}$ were admissible for the ring $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$ but there was a gap in the proof and the claim is actually not true. We do expect though that if such a ring exists then it has to be some intermediate ring \mathbf{B} between $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$ and the rings $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}^I)_1$, but it is not clear at all “how many periods we have to add” to $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$. These constructions naturally extend to the F -analytic Lubin–Tate case.

9.1. Trianguline representations and refinements

We start by recalling the definitions of trianguline representations and some associated properties. The notion of trianguline representations was introduced by Colmez in [15]. Here we choose to follow Berger’s and Chenevier’s definitions [6] instead of Colmez’s.

Definition 9.1. We say that an E -representation V of \mathcal{G}_K is split trianguline if $\mathbf{D}_{\text{rig}}^\dagger(V)$ is a successive extension of (φ, Γ_K) -modules of rank 1 over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$.

We say that an L -representation V of \mathcal{G}_K is trianguline if there exists a finite extension E of L such that the E -representation $E \otimes_L V$ is split trianguline.

We say that an E -representation V of \mathcal{G}_K is potentially split trianguline (resp. potentially trianguline) if there exists a finite extension K' of K such that $V|_{\mathcal{G}_{K'}}$ is split trianguline (resp. trianguline).

Definition 9.1 can be equivalently stated in terms of B -pairs. We quickly recall that a B -pair is a pair $W = (W_e, W_{\text{dR}}^+)$, where W_e is a free $\mathbf{B}_e := (\tilde{\mathbf{B}}_{\text{rig}}^+[1/t])^{\varphi=1}$ -module of finite rank endowed with a continuous semilinear action of \mathcal{G}_K , and W_{dR}^+ is a \mathcal{G}_K -stable lattice in $W_{\text{dR}} := \mathbf{B}_{\text{dR}} \otimes_{\mathbf{B}_e} W_e$. To a p -adic representation V of \mathcal{G}_K one can attach a B -pair $W(V)$ by $W(V) = (\mathbf{B}_e \otimes_{\mathbf{Q}_p} V, \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} V)$. If E is a finite extension of \mathbf{Q}_p , one extends the definition of B -pairs to E -linear objects, and one gets objects called $\mathbf{B}_{|K}^{\otimes E}$ -pairs in [6] or $E - B$ -pairs of \mathcal{G}_K in [35]. Those objects are pairs $W = (W_e, W_{\text{dR}}^+)$, where W_e is a free $\mathbf{B}_{e,E} := E \otimes_{\mathbf{Q}_p} \mathbf{B}_e$ -module of finite rank endowed with a continuous semilinear action of \mathcal{G}_K , and W_{dR}^+ is a \mathcal{G}_K -stable lattice in $W_{\text{dR}} := (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{dR}}) \otimes_{\mathbf{B}_{e,E}} W_e$.

The category of $\mathbf{B}_{|K}^{\otimes E}$ -pairs is equivalent to the one of (φ, Γ_K) -modules over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$, and thus an E -representation V of \mathcal{G}_K is split trianguline if the attached $\mathbf{B}_{|K}^{\otimes E}$ -pair is a successive extension of rank 1 $\mathbf{B}_{|K}^{\otimes E}$ -pairs.

Lemma 9.2. *Let V be an F -analytic representation. Then the following are equivalent:*

- (1) V is split trianguline.
- (2) The Lubin–Tate (φ_q, Γ_K) -module $\mathbf{D}_{\text{rig}}^\dagger(V)$ is a successive extension of F -analytic Lubin–Tate (φ_q, Γ_K) -modules of rank 1.

Proof. See [39, Thm. 4.11]. ■

For a $\mathbf{B}_{e,E}$ -representation, we say that it is split triangulable if it is a successive extension of rank 1 $\mathbf{B}_{e,E}$ -representations.

Lemma 9.3. *An E -representation V of \mathcal{G}_K is split trianguline if and only if the corresponding $\mathbf{B}_{e,E}$ -representation is split triangulable as a $\mathbf{B}_{e,E}$ -representation of \mathcal{G}_K .*

Proof. See [8, Cor. 3.2]. ■

Proposition 9.4. *The categories of split trianguline representations and of trianguline representations are stable by subobjects, quotients, direct sums and tensor products.*

Proof. The fact that it is stable by quotients and subobjects follows from [8, Prop. 3.3]. For direct sums and tensor products it is a straightforward consequence of Definition 9.1. ■

Let D be a (φ, Γ_K) -module of rank d over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$ and equipped with a strictly increasing filtration $(\text{Fil}_i(D))_{i=0 \dots d}$:

$$\text{Fil}_0(D) := \{0\} \subsetneq \text{Fil}_1(D) \subsetneq \dots \subsetneq \text{Fil}_i(D) \subsetneq \dots \subsetneq \text{Fil}_{d-1}(D) \subsetneq \text{Fil}_d(D) := D,$$

of (φ, Γ_K) -submodules which are direct summand as $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$ -modules. We call such a D a triangular (φ, Γ_K) -module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$, and the filtration $\mathcal{T} := (\text{Fil}_i(D))$ a triangulation of D over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$.

Let D be a triangular (φ, Γ_K) -module. By [15, Prop. 3.1], each

$$\text{gr}_i(D) := \text{Fil}_i(D)/\text{Fil}_{i-1}(D), \quad 1 \leq i \leq d,$$

is isomorphic to the (φ, Γ_K) -module on $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\text{rig}, K}^\dagger$ attached to a character δ_i for some unique $\delta_i: K^\times \rightarrow E^\times$. Following [1, §2.3.2], we define the parameter of the triangulation to be the continuous homomorphism

$$\delta := (\delta_i)_{i=1, \dots, d}: K^\times \rightarrow (E^\times)^d.$$

When $K = \mathbf{Q}_p$, the parameter of a triangular (φ, Γ_K) -module refines the data of its Sen polynomial.

Proposition 9.5. *Let D be a triangular (φ, Γ) -module over $E \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^\dagger$ and δ the parameter of a triangulation of D . Then the Sen polynomial of D is*

$$\prod_{i=1}^d (T - w(\delta_i)).$$

Proof. See [1, Prop. 2.3.3]. ■

We now recall the notion of refinements for crystalline trianguline representations of $\mathcal{G}_{\mathbb{Q}_p}$ as in [1, §2.4]. Let V be finite, d -dimensional, continuous, E -representation of $\mathcal{G}_{\mathbb{Q}_p}$. We will assume that V is crystalline and that the crystalline Frobenius φ acting on $\mathbf{D}_{\text{crys}}(V)$ has all its eigenvalues in E^\times .

By a refinement of V , using the definition of [34, §3], we mean the data of a full φ -stable E -filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0, \dots, d}$ of $\mathbf{D}_{\text{crys}}(V)$:

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_d = \mathbf{D}_{\text{crys}}(V).$$

As in [1, §2.4.1], we remark that any refinement \mathcal{F} determines two orderings:

- (1) It determines an ordering $(\varphi_1, \dots, \varphi_d)$ of the eigenvalues of φ , defined by the formula

$$\det(T - \varphi|_{\mathcal{F}_i}) = \prod_{j=1}^i (T - \varphi_j).$$

If all these eigenvalues are distinct then such an ordering conversely determines \mathcal{F} .

- (2) It also determines an ordering (s_1, \dots, s_d) on the set of Hodge–Tate weights of V , defined by the property that the jumps of the weight filtration of $\mathbf{D}_{\text{crys}}(V)$ induced on \mathcal{F}_i are (s_1, \dots, s_i) .

The theory of refinements has a simple interpretation in terms of (φ, Γ) -modules: let D be a crystalline (φ, Γ) -module as above and let \mathcal{F} be a refinement of D . We can construct from \mathcal{F} a filtration $(\text{Fil}_i(D))_{i=0, \dots, d}$ of D by setting

$$\text{Fil}_i(D) := (E \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^\dagger [1/t] \mathcal{F}_i) \cap D,$$

which is a finite type saturated $E \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^\dagger$ -submodule of D .

Proposition 9.6. *The map defined above $(\mathcal{F}_i) \mapsto (\text{Fil}_i(D))$ induces a bijection between the set of refinements of D and the set of triangulations of D , whose inverse is $\mathcal{F}_i := \text{Fil}_i(D)[1/t]^\Gamma$. In the bijection above, for $i = 1, \dots, d$, the graded piece $\text{Fil}_i(D)/\text{Fil}_{i-1}(D)$ is isomorphic to the (φ, Γ) -module on $E \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{rig}, \mathbb{Q}_p}^\dagger$ attached to δ_i where $\delta_i(p) = \varphi_i p^{-s_i}$ and $\delta_i|_\Gamma = \chi^{-s_i}$, where the φ_i and s_i are defined by items (1) and (2) above.*

Proof. See [1, Prop. 2.4.1]. ■

Remark 9.7. In particular, Proposition 9.6 shows that crystalline representations are trianguline, and that the set of their triangulations is in natural bijection with the set of their refinements.

We now finish this section with a result regarding trianguline representations that we were not able to find in the literature.

Proposition 9.8. *Let V be an L -representation of \mathcal{G}_K . Then V is trianguline if and only if the underlying \mathbf{Q}_p -representation of V is trianguline.*

Proof. Let V be an L -representation of \mathcal{G}_K and let E be a finite extension of L , containing all the images of the embeddings $\tau: L \rightarrow \bar{K}$ and such that $E \otimes_L V$ is split trianguline. Then $E \otimes_{\mathbf{Q}_p} V = (E \otimes_{\mathbf{Q}_p} L) \otimes_L V = \bigoplus_{\tau \in \Sigma} (E \otimes_L V)_\tau$ where $\Sigma = \text{Emb}(L, \bar{K})$.

In particular, $E \otimes_L V$ is a subrepresentation of $E \otimes_{\mathbf{Q}_p} V$ and this concludes the first half of the proof by Proposition 9.4. For the other direction, let $W = W_e(E \otimes_L V)$ the corresponding $\mathbf{B}_{e,E}$ -representation and let $W_0 = 0 \subset W_1 \subset \dots \subset W_d = W$ a triangulation of W . For $\tau \in \Sigma$, let $\mathbf{B}_{e,E,\tau} = E \otimes_{L,\tau} \mathbf{B}_{e,E}$. For $\tau \in \Sigma$ and $1 \leq i \leq d$, let $W_{i,\tau} = \mathbf{B}_{e,E,\tau} \otimes_{\mathbf{B}_{e,E}} W_i$. By construction

$$0 \subset W_{1,\tau} \subset \dots \subset W_{d,\tau}$$

is a triangulation of $W((E \otimes_L V)_\tau)$ and thus $E \otimes_{\mathbf{Q}_p} V$ is trianguline. ■

9.2. Discussion on a ring of periods for trianguline representations

By Proposition 9.4, we know that the category of (split) trianguline representations of \mathcal{G}_K is a Tannakian category. Because of this and because of Proposition 9.8, it appears reasonable to look for a ring \mathbf{B} such that trianguline representations are exactly the representations which are \mathbf{B} -admissible in the sense of Fontaine.

Recall that the notion of admissibility in the sense of Fontaine is defined for what he called regular rings and is as follows (we only recall the definitions of [23] in the particular case of \mathbf{Q}_p -representations because that is all we need here).

Let \mathbf{B} be a topological \mathbf{Q}_p -algebra endowed with an action of a group G . For any \mathbf{Q}_p -representation of G , we let $\mathbf{D}_{\mathbf{B}}(V) := (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^G$. We let $\alpha_{\mathbf{B}}(V)$ denote the \mathbf{B} -linear map $\mathbf{B} \otimes_{\mathbf{B}^G} \mathbf{D}_{\mathbf{B}}(V) \rightarrow \mathbf{B} \otimes_{\mathbf{Q}_p} V$ deduced from the inclusion $\mathbf{D}_{\mathbf{B}}(V) \subset \mathbf{B} \otimes_{\mathbf{Q}_p} V$ by extending the scalars to \mathbf{B} . The ring \mathbf{B} is said to be G -regular if the following hold:

- (1) \mathbf{B} is reduced;
- (2) for any p -adic representation V of G , the map α_V is injective;
- (3) any element b of \mathbf{B} which is nonzero and is such that the \mathbf{Q}_p -line generated by \mathbf{B} is G -stable is invertible.

The last condition implies in particular that \mathbf{B}^G is a field. If \mathbf{B} is G -regular, a representation V of G is said to be \mathbf{B} -admissible if $\alpha_{\mathbf{B}}(V)$ is an isomorphism, which is equivalent as saying that $\dim_{\mathbf{B}^G} \mathbf{D}_{\mathbf{B}}(V) = \dim_{\mathbf{Q}_p} V$.

Unfortunately, it seems to us that in the case we consider, the last condition is too strong and thus we extend the notion of G -regularity as follows: we say that \mathbf{B} is G -regular if the following conditions are met:

- (1) \mathbf{B} is reduced;
- (2) for any p -adic representation V of G , $\mathbf{D}_{\mathbf{B}}(V)$ is a free \mathbf{B}^G -module;
- (3) the map α_V is injective.

It is clear that G -regular rings in the sense of Fontaine are G -regular for us, but that the converse does not hold.

In the rest of the paper, G -regularity and admissibility are to be understood in our sense.

We now explain exactly what we mean by a ring of trianguline periods.

Definition 9.9. A \mathcal{G}_K -regular ring \mathbf{B} is said to be a trianguline periods ring for \mathcal{G}_K if trianguline representations of \mathcal{G}_K are \mathbf{B} -admissible, and if \mathbf{B} -admissible representations of \mathcal{G}_K are trianguline.

Proposition 9.10. *Let \mathbf{B} be a \mathcal{G}_K -regular ring and let V be an L -representation of \mathcal{G}_K . Then V is \mathbf{B} -admissible if and only if there exists a finite extension E of L such that $V \otimes_L E$ is \mathbf{B} -admissible.*

Proof. It is clear that if V is \mathbf{B} -admissible, then there exists a finite extension E of L such that $V \otimes_L E$ is \mathbf{B} -admissible. To show the reverse, first note that the admissibility of an E representation V does not depend on whether one considers it as a \mathbf{Q}_p -representation or as an E -representation (the $\mathbf{B}^{\mathcal{G}_K}$ -module $\mathbf{D}_{\mathbf{B}}(V) = (V \otimes_{\mathbf{Q}_p} \mathbf{B})^{\mathcal{G}_K}$ is always the same). Now because the category of \mathbf{B} -admissible representations is clearly stable by subobjects, it suffices to note that V is a sub- \mathbf{Q}_p -representation of $V \otimes_L E$. ■

Unlike in the crystalline or semistable case, if such a ring exists, it has to depend on K .

Proposition 9.11. *There is no ring \mathbf{B} satisfying the properties above such that, for any finite extension K of \mathbf{Q}_p , \mathbf{B} is a trianguline periods ring for \mathcal{G}_K .*

Proposition 9.11 is a consequence of the following result.

Proposition 9.12. *Let L/K be any finite extension. Then there exists a representation V of \mathcal{G}_K such that V is trianguline as a representation of \mathcal{G}_L but is not trianguline as a representation of \mathcal{G}_K .*

Proof. Let $\eta: \mathcal{G}_L \rightarrow L^\times$ be a character such that there exists $\tau_1 \neq \tau_2 \in \text{Emb}(L, \overline{\mathbf{Q}}_p)$ with $(\tau_1)|_K = (\tau_2)|_K$, and such that η is τ_1 -de Rham but not τ_2 -de Rham in the sense of [18]. Our claim is that such a character cannot possibly extend to \mathcal{G}_K and neither can any of its conjugate, i.e. there is no character $\rho: \mathcal{G}_K \rightarrow L^\times$ such that $\rho|_{\mathcal{G}_L} = \sigma(\eta)$ for some $\sigma \in \text{Emb}(L, \overline{\mathbf{Q}}_p)$. Indeed, if such a ρ existed, then the dimension of $\mathbf{D}_{\text{dR},\sigma}(\rho)$ would only depend on the dimension of $\mathbf{D}_{\text{dR},\sigma_K}(\rho)$, which is not the case because of the assumption on τ_1 and τ_2 .

We now let $V = \text{ind}_{\mathcal{G}_L}^{\mathcal{G}_K} \eta$. This is a p -adic representation of \mathcal{G}_K , whose restriction to \mathcal{G}_L is the sum of the conjugates of η , so that it clearly is trianguline as a representation of \mathcal{G}_L . Let us assume that it also is trianguline as a representation of \mathcal{G}_K . Let W be the $\mathbf{B}_{|K}^{\otimes L}$ -pair attached to V . As a $\mathbf{B}_{|L}^{\otimes L}$ -pair, we can write

$$W = \bigoplus_{\sigma} W(\sigma(\eta)).$$

Since we assumed that W is trianguline as a $\mathbf{B}_{|K}^{\otimes L}$ -pair, there exists $W_1 \subset W$ a direct

summand of rank 1. For $\tau \in \text{Emb}(L, \overline{\mathbf{Q}}_p)$, we have the following exact sequence

$$0 \rightarrow \bigoplus_{\tau \neq \sigma} W(\sigma(\eta)) \rightarrow W \rightarrow W(\tau(\eta)) \rightarrow 0$$

so that, since W_1 is a direct summand of rank 1 of W and by [8, Prop. 2.4], we either have $W_1 = W(\tau(\eta))$ or $W_1 \subset \bigoplus_{\tau \neq \sigma} W(\sigma(\eta))$. By induction, W_1 has to be equal to one of the $W(\tau(\eta))$. Therefore, one of the conjugates of η has to extend to \mathcal{G}_K , which we have proven is not possible. ■

We can now give a proof of Proposition 9.11.

Proof. By the results of Section 6, the periods of any representation live in $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}^I)_1$, for I any compact subinterval of $[r_0; +\infty[$, and in particular so do the periods of any trianguline representation V of \mathcal{G}_K , so that we can assume that $\mathbf{B} \subset \mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}^I)_1$. Since every unramified representation of \mathcal{G}_K is trianguline, we can assume that $\mathbf{B} \supset \widehat{\mathbf{Q}}_p^{\text{unr}}$. Moreover, if $L/K/\mathbf{Q}_p$ are unramified then it is easy to see that

$$(\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}^I)_1)^{\mathcal{G}_L} = L \otimes_K (\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}^I)_1)^{\mathcal{G}_K},$$

and thus we can assume that $\mathbf{B}^{\mathcal{G}_L} = L \otimes_K \mathbf{B}^{\mathcal{G}_K}$.

Assume that \mathbf{B} is a ring satisfying the properties, and such that for any finite extension K/\mathbf{Q}_p , \mathbf{B} is a trianguline periods ring for \mathcal{G}_K . Let K be a finite unramified extension of \mathbf{Q}_p , let L be a finite unramified extension of K . Let V be a p -adic representation of \mathcal{G}_K which is not trianguline as a representation of \mathcal{G}_K but becomes trianguline over \mathcal{G}_L , which exists by the previous proposition. Let $\mathbf{D}_L = (\mathbf{B} \otimes V)^{\mathcal{G}_L}$. It is a $\mathbf{B}^{\mathcal{G}_L}$ -module, endowed with a semilinear action of $\text{Gal}(L/K)$. Let $\mathbf{D} = \mathbf{D}_L^{\text{Gal}(L/K)}$. By Speiser’s lemma, $\mathbf{D}_L \simeq L \otimes_K \mathbf{D}$ and thus $\mathbf{D}_L = \mathbf{B}^{\mathcal{G}_L} \otimes_{\mathbf{B}^{\mathcal{G}_K}} \mathbf{D}$. Thus, V is \mathbf{B} -admissible as a representation of \mathcal{G}_K . ■

9.3. F -analytic $\mathbf{B}_{\text{tri},K}^{\text{an}}$ -admissible representations

Since we are now thinking of rings of periods for trianguline representations, we let $\mathbf{B}_{\text{tri},K}^{\text{an}} = \mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$. We now explain why $\mathbf{B}_{\text{tri},K}^{\text{an}}$ is a good starting candidate as a ring of trianguline periods. Note that by Proposition 8.2 and Remark 8.4 we already know that there are F -analytic trianguline representations of \mathcal{G}_K which are not $\mathbf{B}_{\text{tri},K}^{\text{an}}$ -admissible. To put some emphasis on the point of view of rings of periods, we write $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ for $(\mathbf{D}_{\text{rig}}^\dagger(V))^{\text{la}}$ and $\mathbf{D}_{\text{tri},K}^{n,\text{an}}(V)$ for $(\mathbf{D}_{\text{rig}}^\dagger(V))^{\Gamma_n - \text{an}} = (\mathcal{C}^{\text{an}}(\Gamma_n, \tilde{\mathbf{B}}_{\text{rig}}^+) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$.

Proposition 9.13. *The ring $\mathbf{B}_{\text{tri},K}^{\text{an}}$ is \mathcal{G}_K -regular for F -analytic representations.*

Proof. By Proposition 8.11, it suffices to prove that the ring $\mathcal{C}^{\text{la}}(\Gamma_K, \tilde{\mathbf{B}}_{\text{rig}}^+)_1$ is \mathcal{G}_K -regular for F -analytic representations. But now this follows from Proposition 8.2 and in the cyclotomic case from [16, Lem. 3.19 and Thm. 3.20]. The proof of those results extend to the Lubin–Tate case verbatim. ■

We now define a notion of refinements for F -analytic representations of \mathcal{G}_K which are $\mathbf{B}_{\text{tri},K}^{\text{an}}$ -admissible. We let V be a $\mathbf{B}_{\text{tri},K}^{\text{an}}$ -admissible L -representation of dimension d of \mathcal{G}_K .

By a refinement of V , we mean the data of a full φ_q - and Γ_K -stable $L\langle\langle t_\pi \rangle\rangle$ -filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0,\dots,d}$ of $(\mathbf{D}_{\text{rig}}^\dagger(V))^{\text{la}}$:

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_d = (\mathbf{D}_{\text{rig}}^\dagger(V))^{\text{la}}.$$

Note that, as in the crystalline case studied in [1], the theory of refinements has a simple interpretation in terms of (φ_q, Γ_K) -modules: let $\mathbf{D}_{\text{rig}}^\dagger(V)$ be the triangulable (φ_q, Γ_K) -module over $L \otimes_K \mathbf{B}_{\text{rig},K}^\dagger$ attached to V and let \mathcal{F} be a refinement of $\mathbf{D}_{\text{rig}}^\dagger(V) \boxtimes \{0\}$. We can construct from \mathcal{F} a filtration $(\text{Fil}_i(D))_{i=0,\dots,d}$ of $\mathbf{D}_{\text{rig}}^\dagger(V)$ by setting

$$\text{Fil}_i(\mathbf{D}_{\text{rig}}^\dagger(V)) := ((L \otimes_K \mathbf{B}_{\text{rig},K}^\dagger) \otimes_{L\langle\langle t_\pi \rangle\rangle} \mathcal{F}_i)$$

which is a finite type saturated $E \otimes_K \mathbf{B}_{\text{rig},K}^\dagger$ -submodule of $\mathbf{D}_{\text{rig}}^\dagger(V)$.

Proposition 9.14. *The map defined above $(\mathcal{F}_i) \mapsto (\text{Fil}_i(\mathbf{D}_{\text{rig}}^\dagger(V)))$ induces a bijection between the set of refinements of V and the set of triangulations of $\mathbf{D}_{\text{rig}}^\dagger(V)$, whose inverse is $\mathcal{F}_i := ((\text{Fil}_i(\mathbf{D}_{\text{rig}}^\dagger(V)))^{\text{la}})$.*

Proof. This is exactly as in the crystalline case. ■

Proposition 9.15. *Let M be a (φ_q, Γ_K) -module of rank d on $L\langle\langle t_\pi \rangle\rangle$. Then, up to extending the scalars to some finite extension E of L , there exists a filtration*

$$M_0 = 0 \subsetneq M_1 \subsetneq \dots \subsetneq M_d = M$$

of M by saturated sub- (φ_q, Γ_K) -modules.

Proof. We prove the result by induction. If $d = 1$ there is nothing to prove. Assume now that $d \geq 2$ and that the result holds for $d - 1$.

By Theorem 8.7 and Lemma 8.6, up to replacing L by a finite extension E' of L , there exists e_1 proper for the action of φ and Γ_K such that $E'\langle\langle t_\pi \rangle\rangle \cdot e_1$ is saturated in M . By induction, $M/(E'\langle\langle t_\pi \rangle\rangle \cdot e_1)$ admits a full (φ_q, Γ_K) -stable filtration $(\mathcal{F}_i)_{i=1}^{d-1}$. We let M_{i+1} be a lift of \mathcal{F}_i containing $E'\langle\langle t_\pi \rangle\rangle \cdot e_1$ and we put $M_1 = E'\langle\langle t_\pi \rangle\rangle \cdot e_1$. We then have that $(M_i)_{i=1}^d$ is a full (φ_q, Γ_K) -stable filtration of M . ■

Theorem 9.16. *Let V be a $\mathbf{B}_{\text{tri},K}^{\text{an}}$ -admissible F -analytic p -adic representation V . Then V is trianguline.*

Proof. By Propositions 9.15 and 9.14, there exists a finite extension L of K such that $\mathbf{D}_{\text{rig}}^\dagger(V \otimes_K L)$ is a triangulable (φ_q, Γ_K) -module over $L \otimes_K \mathbf{B}_{\text{rig},K}^\dagger$.

Moreover, we see from Lemma 8.8 that the characters appearing in the triangulation are F -analytic. ■

Lemma 9.17. *Let V be an F -analytic E -representation of \mathcal{G}_K such that the attached (φ_q, Γ_K) -module $\mathbf{D}_{\text{rig}}^\dagger(V)$ is triangulable, and let $\delta: (K^\times)^d \rightarrow (E^\times)^d$ be the parameter*

of a triangulation of $\mathbf{D}_{\text{rig}}^\dagger(V)$. Then in an adapted basis for the refinement of $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ corresponding to δ by Proposition 9.14, the matrices of ∇ and φ_q are respectively of the form:

$$\begin{pmatrix} w(\delta_1) & * & \cdots & * \\ 0 & w(\delta_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\delta_d) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta_1(\pi) & * & \cdots & * \\ 0 & \delta_2(\pi) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_d(\pi) \end{pmatrix}.$$

Proof. We prove it by induction on d . For $d = 1$, by [15, Prop. 3.1] there exists a basis e_δ of $\mathbf{D}_{\text{rig}}^\dagger(V)$ in which $g(e_\delta) = \delta(\chi_\pi(g))e_\delta$ and $\varphi_q(e_\delta) = \delta(\pi)e_\delta$, and the action is F -analytic, so that e_δ is a basis of $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ which satisfies the result of the lemma. To see that it is unique note that since $(L\langle\langle t_\pi \rangle\rangle)^\times = L^\times$, the matrices of ∇ and φ_q in an other basis of $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ would be the same.

Let us assume now that $d \geq 2$ is such that the result holds for $d - 1$ and let $(\text{Fil}_i(\mathbf{D}_{\text{rig}}^\dagger(V)))_{i=0,\dots,d}$ be the filtration of $\mathbf{D}_{\text{rig}}^\dagger(V)$ corresponding to the triangulation. Since our constructions are stable by saturated sub-objects, we get by induction that in an adapted basis for the refinement of $\mathbf{D}_{\text{tri},K}^{\text{an}}(\text{Fil}_{d-1}(\mathbf{D}_{\text{rig}}^\dagger(V)))$, the matrices of ∇_u and φ are respectively of the form:

$$\begin{pmatrix} w(\delta_1) & * & \cdots & * \\ 0 & w(\delta_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\delta_{d-1}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta_1(\pi) & * & \cdots & * \\ 0 & \delta_2(\pi) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{d-1}(\pi) \end{pmatrix}.$$

Since our constructions are also stable by quotients by saturated sub-objects and using the proof in the rank 1 case, we know that the matrices of ∇ and φ_q in a basis of

$$\mathbf{D}_{\text{rig}}^\dagger(V)/\text{Fil}_{d-1}(\mathbf{D}_{\text{rig}}^\dagger(V)) \simeq E \otimes_K \mathbf{B}_{\text{rig},K}^\dagger(\delta_d)$$

are respectively of the form $(w(\delta_d))$ and $(\delta_d(\pi))$. Therefore, in an adapted basis for the refinement of $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ corresponding to δ , the matrices of φ_q and ∇_u are as we wanted. ■

In particular, as in the crystalline case, a refinement defines an ordering on both the eigenvalues of φ_q and on the set of Hodge–Tate weights of V , and encodes the data of the Hodge–Tate weights of its parameter.

Finally, we just remark that given the construction of our rings of periods, it is quite obvious that the modules $\mathbf{D}_{\text{tri},K}^{\text{an}}(V)$ attached to F -analytic p -adic representations of \mathcal{E}_K contain its crystalline periods.

Proposition 9.18. *Let V be an F -analytic p -adic representation. Then*

- (1) $\mathbf{D}_{\text{crys}}^+(V) \subset \mathbf{D}_{\text{tri},K}^{\text{an}}(V)^{\nabla=0}$;
- (2) $\mathbf{D}_{\text{crys}}(V) \subset (\mathbf{D}_{\text{tri},K}^{\text{an}}(V)[1/t_\pi])^{\nabla=0}$.

Proof. This just follows from the fact that $\mathbf{D}_{\text{crys}}^+(V) = (\tilde{\mathbf{B}}_{\text{rig}}^+ \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ and that $\mathbf{D}_{\text{crys}}(V) = (\tilde{\mathbf{B}}_{\text{rig}}^+[1/t_\pi] \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ by [37, Lem. 3.8]. ■

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Léo Poyeton

Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351, cours de la Libération,
33405 Talence, France; leo.poyeton@math.u-bordeaux.fr