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Prismatic F-crystals and Lubin–Tate (φ_q, Γ)-modules

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Abstract. Let $L/\mathbb{Q}_p$ be a finite extension. We introduce *L-typical prisms*, a mild generalization of prisms. Following ideas of Bhatt, Scholze, and Wu, we show that certain vector bundles, called Laurent F-crystals, on the L-typical prismatic site of a formal scheme X over $\operatorname{Spf} \mathcal{O}_L$ are equivalent to $\mathcal{O}_L$-linear local systems on the generic fiber X_η. We also give comparison theorems for computing the étale cohomology of a local system in terms of the cohomology of its corresponding Laurent F-crystal. In the case $X = \operatorname{Spf} \mathcal{O}_K$ for K/L a p-adic field, we show that this recovers the Kisin–Ren equivalence between Lubin–Tate (φ_q, Γ)-modules and $\mathcal{O}_L$-linear representations of G_K, as well as the results of Kupferer and Venjakob for computing Galois cohomology in terms of Herr complexes of (φ_q, Γ)-modules. We can thus regard Laurent F-crystals on the L-typical prismatic site as providing a suitable notion of relative (φ_q, Γ)-modules.

1. Introduction

Let $K/\mathbb{Q}_p$ be a p-adic field, let K_∞ be the p-adic completion of the infinite cyclotomic extension $K(\zeta_{p^\infty})$, and let $\Gamma_K = \operatorname{Gal}(K_\infty/K)$. In this setting, Fontaine's theory of (φ, Γ)-modules [17] gives an equivalence of categories

$$\operatorname{Mod}_{\mathbf{A}_K}^{\varphi, \Gamma_K, \mathrm{et}} \simeq \operatorname{Mod}_{W(K_\infty^\flat)}^{\varphi, \Gamma_K, \mathrm{et}} \simeq \operatorname{Rep}_{\mathbb{Z}_p}(G_K)$$

between – on the representation theoretic side – the category of finite free $\mathbb{Z}_p$-linear representations of the absolute Galois group $G_K = \operatorname{Gal}(\overline{K}/K)$ and – on the semilinear algebraic side – categories of (φ, Γ)-modules over the *perfect* period ring $W(K_\infty^\flat)$ or a certain *deperfected* period ring $\mathbf{A}_K \subseteq W(K_\infty^\flat)$. Here, the word "deperfected" refers to the fact that the imperfect sub-$\mathbb{F}_p$-algebra $\mathbf{E}_K = \mathbf{A}_K/p \subseteq K_\infty^\flat = W(K_\infty^\flat)/p$ becomes $K_\infty^\flat$ under completed perfection.

Following the discussion in [28, Section 0.2], we distinguish between two ways one might hope to relativize the theory of (φ, Γ)-modules. First, one might hope for a *geometric* relativization. On the representation theoretic side, this means replacing $\operatorname{Rep}_{\mathbb{Z}_p}(G_K)$ with étale local systems $\operatorname{Loc}_{\mathbb{Z}_p}(X_\eta)$ on the generic fiber of a formal scheme $X/\mathbb{Z}_p$. One then hopes to have a corresponding semilinear algebraic category of objects which can be thought of as (φ, Γ)-modules varying over the base X. The most satisfactory candidate

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here is the *Laurent F-crystals* of [9]. Recall that these are vector bundles

$$\mathcal{M} \in \mathrm{Vect}\left(X_{\Delta}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(p)}\right)^{\phi=1}$$

over a certain structure sheaf on the prismatic site of X equipped with an isomorphism $\phi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$. Bhatt–Scholze's key theorem is as follows.

Theorem 1.1 ([9, Corollary 3.8]). *Let X be a bounded formal scheme adic over* $\mathrm{Spf}\,\mathbb{Z}_p$ *with adic generic fiber X_η. Then there is an equivalence*

$$\mathrm{Vect}\left(X_{\Delta}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(p)}\right)^{\phi=1} \simeq \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta).$$

In the case $X = \mathrm{Spf}\,\mathcal{O}_K$ for $K/\mathbb{Q}_p$ a p-adic field, work of Wu [46] shows that

$$\mathrm{Vect}\left((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(p)}\right)^{\phi=1} \simeq \mathrm{Mod}^{\varphi,\Gamma_K,\mathrm{et}}_{\mathbf{A}_K} \simeq \mathrm{Mod}^{\varphi,\Gamma_K,\mathrm{et}}_{W(K^\flat_\infty)},$$

recovering Fontaine's original theory.

Remark 1.2. Due to obstructions related to the fact that Cohen rings can be formed functorially for perfect fields (via the Witt vector construction) but not for arbitrary characteristic p fields, it is significantly easier to give a relative construction of (φ,Γ)-modules over the perfect period ring $W(K^\flat_\infty)$; for example, relative (φ,Γ)-modules over a perfect period sheaf $W(\mathcal{O}^\flat_X)$ are defined in work of Kedlaya and Liu [28]. In follow-up work, Kedlaya and Liu [29] attempt to define satisfactory imperfect period sheaves via an axiomatic approach, but these axioms fail to attain in the important Lubin–Tate case discussed below [39]. On the other hand, the Bhatt–Scholze approach to relative (φ,Γ)-modules circumvents this difficulty using the theory of prisms [8], which can be viewed as deperfections of perfectoid rings.

Alternatively, one might also want *arithmetic* relativizations of the theory of (φ,Γ)-modules. On the representation theory side, this means replacing the $\mathbb{Z}_p$ in $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ with affinoid algebras over $\mathbb{Z}_p$, as in [1, 3, 30]. The simplest such case is to study $\mathrm{Rep}_{\mathcal{O}_L}(G_K)$ for L a finite subextension of $K/\mathbb{Q}_p$. A key goal of this paper is to extend Bhatt–Scholze's prismatic approach to relative (φ,Γ)-modules to this case. We do this by introducing a mild generalization of prisms, which we call L-typical prisms, and the L-typical prismatic site X_{Δ_L} of a formal scheme X over $\mathrm{Spf}\,\mathcal{O}_L$. This done, we show the following.

Theorem 1.3. *Let $L/\mathbb{Q}_p$ be a finite extension with uniformizer π, and let X be a bounded formal scheme adic over* $\mathrm{Spf}\,\mathcal{O}_L$ *with adic generic fiber X_η.*

(1) *There is an equivalence of categories*

$$\mathrm{Vect}\left(X_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{I}\right]^{\wedge}_{(\pi)}\right)^{\phi=1} \simeq \mathrm{Loc}_{\mathcal{O}_L}(X_\eta)$$

between Laurent F-crystals on X_{Δ_L} and $\mathcal{O}_L$-local systems on X_η.

(2) *If $\mathcal{M} \in \mathrm{Vect}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{I}]^{\wedge}_{(\pi)})^{\phi=1}$ and $T \in \mathrm{Loc}_{\mathcal{O}_L}(X_\eta)$ correspond under the equivalence above, then there is an isomorphism*

$$R\Gamma(X_{\Delta_L}, \mathcal{M})^{\phi=1} \cong R\Gamma(X_{\eta,\mathrm{et}}, T).$$

Note that Theorem 1.3 (2) is an étale comparison generalizing [20, Theorem 1.10 (i)], itself a generalization of the Bhatt–Scholze étale comparison [8, Theorem 1.8 (4)]. Here and throughout the paper, if E is a complex in a derived category with an endomorphism ϕ, then $E^{\phi=1} := \mathrm{Cone}(\phi - \mathrm{id})[-1]$ is the mapping cocone of $\phi - \mathrm{id}$.

Before going on, we say a few words about L-typical prisms, which were independently defined by Ito and called "$\mathcal{O}_L$-prisms" in his concurrent work [22]. The category of L-typical prisms is a mild generalization of the category of prisms. It arises by replacing δ-rings with what we call δ_L-algebras. In the same way that p-complete δ-rings relate to $\mathbb{Z}_p$-algebras with a lift of Frobenius, δ_L-algebras relate to $\mathcal{O}_L$-algebras with a lift of q-Frobenius. And just as the category of prisms has a subcategory of perfect prisms, which is equivalent to the category of (integral) perfectoid rings (this is what we mean when we say that prisms can be viewed as "deperfections of perfectoid rings"), we will show the following.

Theorem 1.4. *Let $L/\mathbb{Q}_p$ be a finite extension. The categories of perfect L-typical prisms and perfectoid $\mathcal{O}_L$-algebras (i.e. integral perfectoid rings which are also $\mathcal{O}_L$-algebras) are equivalent.*

Remark 1.5. The notion of δ_L-algebras defined here coincides with Borger's notion of a π-typical $\Lambda_{\mathcal{O}_L}$-ring [11]. More generally, following a suggestion of Kisin, the author suspected that Borger's Λ-rings were the right formalism for arithmetically relativizing (φ, Γ)-modules in general. We hope that this work – which carries out this relativization in the simplest case beyond $\mathbb{Z}_p$-coefficients – provides evidence that the same techniques will be useful more generally.

Fix now a Lubin–Tate formal $\mathcal{O}_L$-module $\mathcal{G}$ corresponding to the uniformizer π of $\mathcal{O}_L$. If K/L is a p-adic field, then we let K_∞ be the p-adic completion of the infinite extension $K(\mathcal{G}[\pi^\infty])$ formed by adjoining the π-power torsion points of $\mathcal{G}$. In this case, one can use the periods of $\mathcal{G}$ to construct an element $\omega \in W(K_\infty^\flat) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_L$ and a period ring

$$\mathbf{A}_{K/L} \subseteq W(K_\infty^\flat) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_L.$$

(If $L = \mathbb{Q}_p$ and $\mathcal{G} = \mu_{p^\infty}$, then $\mathbf{A}_{K/L}$ coincides with the $\mathbf{A}_K$ discussed above.) One also gets a category $\mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q, \Gamma_K}$ of *Lubin–Tate* (φ_q, Γ)*-modules*, first studied by Kisin and Ren [31] following ideas of Fontaine, and recently a subject of significant interest in the context of explicit reciprocity laws, p-adic local Langlands, and Iwasawa theory [2, 3, 18, 38, 40].

In Section 3.3 we give general constructions for producing interesting subprisms of a perfect L-typical prisms. When applied with inputs derived from periods of $\mathcal{G}$ and the perfect L-typical prism $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$ corresponding via Theorem 1.4 to the perfectoid $\mathcal{O}_L$-algebra $\mathcal{O}_{K_\infty}$, we show that this construction produces a prism $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$ such that

$$\mathbf{A}_{K/L} = \mathbf{A}_{K/L}^+\Big[\tfrac{1}{q_n(\omega)}\Big]^\wedge_{(\pi)}.$$

This period ring has an interesting dependence on the Lubin–Tate formal group $\mathcal{G}$; for example, we construct a prismatic logarithm map $T\mathcal{G} \to \mathbf{A}_{L/L}^+\{1\}$ to the Breuil–Kisin twist, as in [6]. Using the prism $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$, we show that Theorem 1.3 recovers both the Kisin–Ren equivalence

$$\mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\Gamma_K,\mathrm{et}} \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_K)$$

and the computation of Galois cohomology in terms of φ-Herr complexes from [32].

Theorem 1.6. *Let $L/\mathbb{Q}_p$ be a finite extension with uniformer π, and let K/L be a p-adic field.*

(1) *There are equivalences of categories*

$$\mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\mathrm{et}} \simeq \mathrm{Mod}_{W_L(K_\infty^\flat)}^{\varphi_q,\mathrm{et}} \simeq \mathrm{Vect}\left((\mathcal{O}_{K_\infty})_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^\wedge_{(\pi)}\right)^{\phi=1} \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty}),$$
$$\mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\Gamma_K,\mathrm{et}} \simeq \mathrm{Mod}_{W_L(K_\infty^\flat)}^{\varphi_q,\Gamma_K,\mathrm{et}} \simeq \mathrm{Vect}\left((\mathcal{O}_{K})_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^\wedge_{(\pi)}\right)^{\phi=1} \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_{K}).$$

(Here $W_L(K_\infty^\flat) = A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})[\frac{1}{\ker\theta}]^\wedge_{(\pi)}$ is the period ring corresponding to the perfect L-typical prism $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$.)

(2) *If $M \in \mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\mathrm{et}}$ corresponds to $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty})$ under the above equivalence, then*

$$R\Gamma(K_{\infty,\mathrm{et}}, T) \cong \left(M \xrightarrow{\phi-1} M\right)$$

where the complex on the right is concentrated in degrees 0 *and* 1.

(3) *If $M \in \mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\Gamma_K,\mathrm{et}}$ corresponds to $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_K)$, then*

$$R\Gamma(K_{\mathrm{et}}, T) \cong C^\bullet_{\mathrm{cont}}(\Gamma_K, M)^{\phi=1}$$

where $C^\bullet_{\mathrm{cont}}(\Gamma_K, M)$ denotes the continuous cochain complex of Γ_K with values in M.

1.1. Explicit reciprocity laws and Iwasawa theory

A key motivation for this work is explicit reciprocity laws in Iwasawa theory. Let $K_n = \mathbb{Q}_p(\zeta_{p^n})$ and $K = \mathbb{Q}_p$. In the most classical case, Iwasawa's explicit reciprocity law [23] computes, for a system $u = (u_n)_n \in \varprojlim K_n^\times$ of p-power compatible units and $m \geq 1$, the image of u under the composition

$$\lambda_m \colon \varprojlim K_n^\times \xrightarrow{\kappa} \varprojlim H^1\big(K_n, \mathbb{Z}_p(1)\big) \cong \varprojlim H^1\big(K_n, \mathbb{Z}_p(k)\big) \xrightarrow{\mathrm{Tr}_{K_n/K_m}} H^1(K_m, \mathbb{Z}_p) \xrightarrow{\exp^*} K_m$$

where κ is the Kummer map, the isomorphism is a Soulé twist,[1] and $\exp^*$ is the Bloch–Kato dual exponential map [24, Section II.1.2]. Explicitly,

$$\lambda_m(u) = p^{-m} u_m (\mathrm{dlog}\,\theta_u)(u_m - 1)$$

[1]Concretely, using the isomorphism $\varprojlim H^1(K_n, \mathbb{Z}_p(1)) \cong H^1(K, \mathbb{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))$, the Soulé twist arises from the isomorphism $\mathbb{Z}_p[\![\Gamma_K]\!] \to \mathbb{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ of G_K-modules given by $\gamma \mapsto \gamma \otimes \gamma e$ corresponding to a choice of basis e of $\mathbb{Z}_p(1)$.

where $\theta_u \in \mathbb{Z}_p[\![T]\!]^\times$ is the Coleman power series for u and $\operatorname{dlog}\theta = \frac{\theta'(T)}{\theta(T)}$. The Iwasawa cohomology group

$$H^1_{Iw}\big(K_\infty/K, \mathbb{Z}_p(1)\big) := \varprojlim H^1\big(K_n/K, \mathbb{Z}_p(1)\big)$$

is important, in part, because its contains as an element the Euler system of cyclotomic units. This formula for λ_m thereby allows one to relate this Euler system to zeta values.

More generally, let $L/\mathbb{Q}_p$ be a finite extension with uniformizer π, let $\mathscr{G}$ be a Lubin–Tate formal $\mathcal{O}_L$-module corresponding to π, let $L_n = L(\mathscr{G}[\pi^n])$, and let $T\mathscr{G} \in \operatorname{Rep}_{\mathcal{O}_L}(G_L)$ be the Tate module of $\mathscr{G}$. Then for each $m \geq 1$ and $k \in \mathbb{Z}$ there is a map

$$\begin{aligned}\lambda_{m,k}\colon \varprojlim L_n^\times &\xrightarrow{\kappa} H^1_{Iw}\big(L_\infty/L, \mathbb{Z}_p(1)\big)\\ &\cong H^1_{Iw}\big(L_\infty/L, T\mathscr{G}^{\otimes -k}(1)\big) \xrightarrow{\mathrm{Tr}} H^1\big(L_m, T\mathscr{G}^{\otimes -r}(1)\big) \xrightarrow{\exp^*} L_m t_{\mathscr{G}}^k t_{\mathrm{cycl}}^{-1}\end{aligned}$$

where $t_{\mathscr{G}} \in D_{\mathrm{dR}}(T\mathscr{G}^{\otimes -1})$ and $t_{\mathrm{cycl}} \in D_{\mathrm{dR}}(\mathcal{O}_L(-1))$ are the usual de Rham periods. Work of Bloch and Kato [10] gives the explicit reciprocity law

$$\lambda_{m,k}(u) = \frac{1}{k!}\pi^{-mk}(\partial_{\mathscr{G}}^k \log\theta_u)(u_m) t_{\mathscr{G}}^k t_{\mathrm{cycl}}^{-1}$$

for $k \geq 1$, where $\theta_u \in \mathcal{O}_L[\![T]\!]$ is again a Coleman power series and $\partial_{\mathscr{G}}(f(T)) := \frac{1}{g(T)} f'(T)$ with $g(T)dT$ being the invariant differential for $\mathscr{G}$.

Intuitively speaking, for a fixed $k \geq 1$, the above explicit reciprocity law for $\lambda_{m,k}$ extracts information from the system $(u_n)_{n\geq 1}$ related to the special value of a p-adic L-function at $s = k$. On the other hand, work of Perrin-Riou, Colmez, and Cherbonnier [15, 36] in the cyclotomic case $\mathscr{G} = \mu_{p^\infty}$ and Schneider and Venjakob [38] in the general case shows how to interpolate all of the above "little" explicit reciprocity laws into one "big" explicit reciprocity law which sees the entire p-adic L-function at once. More precisely, if $M \in \operatorname{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{L/L}}$ corresponds to $T\mathscr{G} \in \operatorname{Rep}_{\mathcal{O}_L}(G_L)$ under Theorem 1.6, then there is a big dual exponential map [38, Section 5]

$$\operatorname{Exp}^*\colon H^1_{Iw}\big(L_\infty/L, \mathcal{O}_L(1)\big) \xrightarrow{\sim} M^{\psi=1}$$

where ψ is a certain endomorphism of M. Moreover, we have $M \cong \Omega^1_{\mathcal{O}_{\mathscr{G}}/\mathcal{O}_L} \cong \Omega^1_{\mathcal{O}_L[\![T]\!]/\mathcal{O}_L}$ and the big explicit reciprocity law

$$(\operatorname{Exp}^* \circ \kappa)(u) = \operatorname{dlog}\theta_u.$$

Intuitively, this shows how to relate a p-adic L-function corresponding to a system $(u_n)_n$ of units to a function $\theta_u \in \mathcal{O}_{\mathscr{G}} \cong \mathbf{A}_{L/L} \cong \mathcal{O}_L[\![T]\!]$ on the Lubin–Tate group $\mathscr{G}$.

Two ingredients were essential for the above big explicit reciprocity law to be formulated and proved. First, there is a map $\mathcal{O}_G \to \mathbf{A}_{L/L}$ from the ring of functions on $\mathscr{G}$ to the period ring for the φ-modules. Second, the period ring $\mathbf{A}_{L/L}$ is *imperfect*; indeed, the corresponding perfect period ring $W_L(L_\infty^\flat)$ has $\Omega^1_{W_L(L_\infty^\flat)/\mathcal{O}_L} = 0$, presenting a fundamental obstruction to a big explicit reciprocity law like the one above. Moreover, in [38],

ψ is shown to be related to the endomorphism ϕ of $\mathbf{A}_{L/L}$ via Pontryagin duality using an argument that makes use of local Tate duality and a *residue* pairing

$$\mathbf{A}_{L/L} \otimes_{\mathbf{A}_{L/L}} \Omega^1_{\mathbf{A}_{L/L}/\mathcal{O}_L} \xrightarrow{\text{res}} \mathcal{O}_L,$$

which suggests it is crucial that $\mathbf{A}_{L/L}$ not be too much larger than $\mathcal{O}_{\mathcal{G}}$.

In settings beyond the case of Lubin–Tate formal groups, there are families of little explicit reciprocity laws which lack big explicit reciprocity laws. For instance Kato's generalized explicit reciprocity law [25], a key technical ingredient to Kato's work [26] on Iwasawa main conjectures for modular forms, is used to relate special values of L-functions with special values of derivatives of logarithms of *Siegel units*, which are certain functions on the p-divisible group of an elliptic curve.

The author suspects that the path forward in formulating and proving big explicit reciprocity laws in this setting involves constructing certain imperfect prisms (A, I) over (the ordinary locus of) a modular curve X such that the p-divisible group $\mathcal{E}[p^\infty]$ of the universal elliptic curve $\mathcal{E} \to X$ has a map $\mathcal{O}_{\mathcal{E}[p^\infty]} \to A$. Some partial progress is presented in Example 3.26: Given an ordinary elliptic curve over a p-complete ring R equipped with a compatible system of sections $\operatorname{Spf} R_n \to \ker F^n$ of the subgroups $\ker F^n$ over étale R-algebras, the general constructions given in Section 3.3 produce a map

$$\mathcal{O}_{\varinjlim \ker F^n} \to W\big((\varinjlim R_n)^\flat\big).$$

(If $(\varinjlim R_n)^\wedge_{(p)}$ is perfectoid, then $(W((\varinjlim R_n)^\flat), \ker\theta) \in R_{\Delta}$ is a perfect prism.)

1.2. Overview of the proofs

We briefly outline the key ideas in the proofs of Theorems 1.3 and 1.6. When $X = \operatorname{Spf} R$ for a perfectoid $\mathcal{O}_L$-algebra R,

$$\operatorname{Vect}\big(R_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{I}]^\wedge_{(\pi)}\big)^{\phi=1} \simeq \operatorname{Mod}^{\varphi_q,\text{et}}_{W(R[\frac{1}{\pi}]^\flat)},$$

and Theorem 1.3 is shown via standard arguments (due originally to Katz and Fontaine [17, 27]) for relating étale φ-modules and local systems. The general case then follows via descent from the perfectoid case. This descent crucially relies on the existence of a perfection functor $(A, I) \mapsto (A, I)_{\text{perf}}$ which induces an equivalence on the corresponding categories of étale φ_q-modules.

Theorem 1.7 (cf. [46, Theorem 4.6] for the $\mathbb{Q}_p$-typical case). *Let (A, I) be a bounded L-typical prism with perfection $(A_{\text{perf}}, IA_{\text{perf}})$. Then base change induces an equivalence*

$$\begin{aligned} \operatorname{Mod}^{\phi,\text{et}}_{(A,I)} &\xrightarrow{\sim} \operatorname{Mod}^{\phi,\text{et}}_{(A,I)_{\text{perf}}} \\ M &\mapsto M \otimes_{A[\frac{1}{I}]^\wedge_{(\pi)}} A_{\text{perf}}[\tfrac{1}{I}]^\wedge_{(\pi)} \end{aligned}$$

between the categories of étale φ_q-modules over (A, I) and $(A, I)_{\text{perf}}$.

The $X = \operatorname{Spf} \mathcal{O}_{K_\infty}$ part of Theorem 1.6 follows nearly immediately from Theorem 1.3. Intuitively, one would like to conclude the $X = \operatorname{Spf} \mathcal{O}_K$ part by descending along $Y = \operatorname{Spf} \mathcal{O}_{K_\infty} \to X = \operatorname{Spf} \mathcal{O}_K$ and picking up a semilinear action of $\Gamma_K = \operatorname{Gal}(K_\infty/K)$ in the process. However, instead of using this angle of attack, we will use a more delicate descent argument along the Čech nerve $(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta)^\bullet$ in the perfect prismatic site $(\mathcal{O}_K)_{\mathbb{\Delta}_L}^{\mathrm{perf}}$. This argument allows us to recover a Laurent F-crystal $\mathcal{M}$ over $(\mathcal{O}_K)_{\mathbb{\Delta}_L}$ from the data of $M = \mathcal{M}(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta)$ and a semilinear action of $\operatorname{Aut}_{(\mathcal{O}_K)_{\mathbb{\Delta}_L}}(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta) \cong \Gamma_K$, and to compute $R\Gamma((\mathcal{O}_K)_{\mathbb{\Delta}_L}, \mathcal{M}) \cong C^\bullet_{\mathrm{cont}}(\Gamma_K, M)$.

1.3. Structure of the paper

In Section 2 we introduce δ_L-algebras, review ramified Witt vectors, and develop basic results about distinguished elements and perfect δ_L-algebras. In Section 3 we then introduce L-typical prisms, with perfectoid $\mathcal{O}_L$-algebras, the proof of Theorem 1.4, and the perfection functor appearing in Section 3.2. In Section 3.3 we describe two general constructions which – given an L-typical prism (A, I), a perfectoid $\mathcal{O}_L$-algebra R, and a ϕ-compatible system of maps $(\iota_n\colon A \to R)_n$ – produce a map $(A, I) \to (A_{\mathrm{inf},L}(R), \ker\theta)$ to the perfect L-typical prism corresponding to R; Example 3.26 discussed above, which involves constructing a map from a sub-p-divisible group of the p-divisible group of an elliptic curve to $W((\varinjlim R_n)^\flat)$, is also sited here.

Starting in Section 4, we will take $\mathcal{G}$ to be a Lubin–Tate formal $\mathcal{O}_L$-module corresponding to a uniformizer π of L. We explain in Section 4.1 how to equip $\mathcal{O}_{\mathcal{G}} \cong \mathcal{O}_L[\![T]\!]$ with ideals $(q_n(T))$ which turn it into an L-typical prism; furthermore, the constructions from Section 3.3 allow us to, given a choice of basis e for the rank one $\mathcal{O}_L$-module $T\mathcal{G}$, produce an embedding $(\mathcal{O}_{\mathcal{G}}, (q_n(T))) \hookrightarrow (W_L(\mathcal{O}_{L_\infty}^\flat), \ker\theta)$ into a perfect prism. Given a p-adic field K/L, we extend this construction in Section 4.2 to give a prism $(\mathbf{A}_{K/L}^+, (q_n(\omega))) \in (\mathcal{O}_K)_{\mathbb{\Delta}_L}$ with perfection $(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta)$. In Section 4.3 we review the basics of the theory of Lubin–Tate (φ_q, Γ)-modules and the Γ_K-action on $\mathbf{A}_{K/L}$. Then Section 4.4 contains discussion of the prismatic logarithm for $\mathcal{G}$; we included this section because we believed the construction was interesting, but it plays no further role in this paper.

Finally, Section 5 is the technical heart of the paper. In Section 5.1 we define φ_q-modules over L-typical prisms and prove Theorem 1.7. Then Section 5.2 defines Laurent F-crystals and proves Theorem 1.3, with Theorem 4.14 following in Section 5.3.

2. δ_L-algebras and ramified Witt vectors

Recall that a δ-ring is a ring A together with a map $\delta\colon A \to A$ of sets satisfying certain properties which guarantee that

$$\begin{aligned} \phi\colon A &\to A \\ x &\mapsto x^p + p\delta(x) \end{aligned}$$

is a ring homomorphism lifting the Frobenius endomorphism $x \mapsto x^p$ of A/p. In this section, we will recall a mild generalization of the theory of δ-rings which applies in the following context.

Let $L/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_L$, uniformizer π, and residue field $\mathcal{O}_L/\pi$ of size q. Then a δ_L-algebra will be an $\mathcal{O}_L$-algebra A equipped with a map $\delta_L\colon A \to A$ of sets satisfying certain properties which guarantee that

$$\phi(x) = x^q + \pi\delta_L(x)$$

is a ring homomorphism lifting the q-Frobenius $\varphi_q(x) = x^q$ of A/π.

Remark 2.1. By a theorem of Wilkerson [44], δ-rings are the same as p-typical λ-rings, a notion generalized by the Λ-rings of Borger [11]. The results of this section are obtained as special cases of Borger's theory of Λ-rings over $\mathcal{O}_L$ in the π-typical setting.

2.1. Basic theory

Definition 2.2. We define a δ_L-algebra and related notions as follows.

(1) A δ_L*-algebra* is an $\mathcal{O}_L$-algebra A equipped with a map $\delta_L\colon A \to A$ of sets satisfying the identities

$$\begin{aligned} \delta_L(\alpha) &= \frac{\alpha - \alpha^q}{\pi} && \text{for } \alpha \in \mathcal{O}_L, \\ \delta_L(xy) &= \delta_L(x)y^q + x^q\delta_L(y) + \pi\delta_L(x)\delta_L(y) && \text{for } x, y \in A, \\ \delta_L(x+y) &= \delta_L(x) + \delta_L(y) + \frac{x^q + y^q - (x+y)^q}{\pi} && \text{for } x, y \in A, \end{aligned} \tag{2.1}$$

where in (2.1) the expression $\frac{x^q+y^q-(x+y)^q}{\pi}$ is shorthand for

$$-\sum_{i=1}^{q-1} \frac{1}{\pi}\binom{q}{i} x^i y^{q-i}$$

which makes sense even when A has π-torsion. If A is an $\mathcal{O}_L$-algebra then by a δ_L*-structure* on A we mean a choice of map $\delta_L\colon A \to A$ as above making A into a δ_L-algebra.

(2) There is an evident category Alg_{δ_L} of δ_L-algebras, with maps being $\mathcal{O}_L$-algebra maps which commute with the δ_L-structures.

(3) If A is a δ_L-algebra, then we have a map $\phi_{A,\delta_L}\colon A \to A$ given by $\phi_{A,\delta_L}(x) = x^q + \pi\delta_L(x)$ which lifts the q-Frobenius φ_q on A/π. Using the assumed identities on δ_L, one verifies that ϕ_{A,δ_L} is an $\mathcal{O}_L$-algebra homomorphism. Usually A and δ_L will be clear from context and we will simply write ϕ_A or ϕ for ϕ_{A,δ_L}.

Remark 2.3. We note some basic properties of δ_L-algebras.

(1) If A is a π-torsion-free $\mathcal{O}_L$-algebra and ϕ is an endomorphism lifting φ_q, then we obtain a δ_L-structure on A by

$$\delta_L(x) = \frac{\phi(x) - x^q}{\pi}.$$

This is easily seen to give a one-to-one correspondence between δ_L-structures on A and lifts of φ_q to A. When A has π-torsion, having a δ_L-structure is stronger than having a lift of φ_q.

(2) The properties defining the map δ_L evidently depend on the choice of uniformizer π, so one might worry that δ_L algebra structures on an $\mathcal{O}_L$-algebra A might depend on the choice of π as well. Fortunately, there is no essential dependence: if A has a δ_L-structure with respect to π and $\pi' = u\pi$ for $u \in \mathcal{O}_L^\times$ is another uniformizer, then $\alpha \mapsto u^{-1}\delta_L(\alpha)$ is a δ_L-structure with respect to π'.

(3) See [22, Remark 2.2.7] (generalizing [8, Remark 2.4]) for an alternative characterization of δ_L-structures on A in terms of $\mathcal{O}_L$-algebra sections of the length 2 ramified Witt vectors $W_{L,2}(A)$. In particular, this characterization immediately implies that the category of δ_L-algebras admits all limits and colimits, and that they commute with the forgetful functor to $\mathcal{O}_L$-algebras.

(4) The category of δ_L-algebras is also closed with respect to classical I-adic completion with respect to an ideal $I \subseteq A$ containing π (cf. [22, Lemma 2.2.10] or the proof of [8, Lemma 2.17]).

A key fact about δ_L-algebras is that the forgetful functor $\mathrm{Alg}_{\delta_L} \to \mathrm{Alg}_{\mathcal{O}_L}$ has a right adjoint W_L, which is identified with Hazewinkel's ramified Witt vector functor [21]. Explicitly, for $n \geq 0$, let

$$\begin{aligned} w_n(X_0, \dots, X_n) &= X_0^{q^n} + \pi X_1^{q^{n-1}} + \cdots + \pi^{n-1} X_{n-1}^q + \pi^n X_n \in \mathcal{O}_L[X_0, \dots, X_n] \\ &\subseteq \mathcal{O}_L[X_0, X_1, \dots] \end{aligned}$$

be the nth ghost component polynomial. For any $\mathcal{O}_L$-algebra R, let $W_L(R) = R^{\mathbb{N}}$ as sets, and let

$$\begin{aligned} w_R : W_L(R) &\to R^{\mathbb{N}} \\ x = (x_0, x_1, \dots) &\mapsto \big(w_0(x), w_1(x), \dots\big) \end{aligned}$$

be the ghost component map. Since w_R is a bijection when R is π-torsion-free and any $\mathcal{O}_L$-algebra is a quotient of a free $\mathcal{O}_L$-algebra, there is a unique choice of $\mathcal{O}_L$-algebra structure on $W_L(R)$ such that w_R is a map of $\mathcal{O}_L$-algebras *and* W_L is a functor $\mathrm{Alg}_{\mathcal{O}_L} \to \mathrm{Alg}_{\mathcal{O}_L}$; equip $W_L(R)$ with this $\mathcal{O}_L$-algebra structure. One also checks that the projection map

$$W_L(R) = R^{\mathbb{N}} \twoheadrightarrow R$$

onto the first factor is an $\mathcal{O}_L$-algebra homomorphism.

The above paragraph explains the $\mathcal{O}_L$-algebra structure on $W_L(R)$; we now explain the δ_L-structure. In the case that R is π-torsion-free, $W_L(R)$ is π-torsion-free as well, so giving a δ_L-structure is the same as giving a lift of q-Frobenius. This is provided by the canonical Witt vector Frobenius.

Proposition 2.4. *If R is an $\mathcal{O}_L$-algebra, then there are endomorphisms F_R and V_R of $W_L(R)$, natural in R, such that for $x, y \in W_L(R)$ we have*

$$\begin{aligned} F_R(x) &\equiv x^q \mod \pi W_L(R), \\ F_R\big(V_R(x)\big) &= \pi x, \\ V_R\big(x F_R(y)\big) &= V_R(x) y, \end{aligned}$$

and the diagrams

$$\begin{tikzcd} W_L(R) \arrow[r, "w_R"] \arrow[d, "F_R"] & R^{\mathbb{N}} \arrow[d, "(w_0,w_1,\dots)\mapsto(w_1,w_2,\dots)"] \\ W_L(R) \arrow[r, "w_R"] & R^{\mathbb{N}} \end{tikzcd} \qquad \begin{tikzcd} W_L(R) \arrow[r, "w_R"] \arrow[d, "V_R"] & R^{\mathbb{N}} \arrow[d, "(w_0,w_1,\dots)\mapsto(0,\pi w_0,\pi w_1,\dots)"] \\ W_L(R) \arrow[r, "w_R"] & R^{\mathbb{N}} \end{tikzcd} \tag{2.2}$$

commute.

Proof. This uses the same arguments as for p-typical Witt vectors; see [37, p. 14] for details. ∎

In fact, $W_L(R)$ has a δ_L-structure even when R is not π-torsion-free.

Lemma 2.5. *W_L extends to a functor $\mathrm{Alg}_{\mathcal{O}_L} \to \mathrm{Alg}_{\delta_L}$ which is right adjoint to the forgetful functor. Explicitly, this means that if A is a δ_L-algebra then any $\mathcal{O}_L$-algebra map $A \to R$ lifts to a unique δ_L-algebra map $A \to W_L(R)$ making the following diagram commute.*

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[dr] & W_L(R) \arrow[d, two heads] \\ & R \end{tikzcd}$$

Proof. See [11]. ∎

Remark 2.6. We note that since the ghost components w_n depend on π, so does the $\mathcal{O}_L$-algebra structure on $W_L(R)$ and the endomorphisms F_R and V_R.

We will make use of two distinct sections of $W_L(R) \twoheadrightarrow R$. One is the usual *Teichmüller* lift $r \mapsto [r]$, a multiplicative section which exists for any $\mathcal{O}_L$-algebra R. In the case that R also has a δ_L-structure, another section exists which is moreover a δ_L-algebra map.

Proposition 2.7. *We have sections of $W_L(R) \twoheadrightarrow R$ as follows.*

(1) *If A is a δ_L-algebra, then there is a unique map $s_A\colon A \to W_L(A)$ of δ_L-algebras which is a section of $W_L(A) \twoheadrightarrow A$. It satisfies*

$$w_n\big(s_A(\alpha)\big) = \phi_A^n(\alpha)$$

for all $n \geq 0$ and $\alpha \in A$.

(2) *If R is a $\mathcal{O}_L$-algebra, then the map*

$$\begin{aligned}[-] : R &\to W_L(R) \\ r &\mapsto (r, 0, 0, \dots)\end{aligned}$$

is a multiplicative section of $W_L(R) \twoheadrightarrow R$.

Note that if R is π-torsion-free, then the formula in (1) uniquely determines the map s_R.

Proof. For part (1), s_A is the unit of the adjunction from Lemma 2.5 (i.e. apply the lemma to $\mathrm{id} : A \to A$). The formula for $w_n(s_A(\alpha))$ follows from the left diagram in (2.2) and the defining property of s_A as

$$w_n\big(s_A(\alpha)\big) = w_0\big(F^n_{W_L(A)} s_A(\alpha)\big) = w_0\big(s_A(\phi^n_A \alpha)\big) = \phi^n_A \alpha.$$

Part (2) is clear, as one only needs to check that formula given defines a multiplicative map. But let us explain the relationship to part (1): let R^{mm} denote R viewed as a multiplicative monoid. Then the free $\mathcal{O}_L$-algebra $\mathcal{O}_L[R^{mm}]$ has a lift of q-Frobenius induced by $r \mapsto r^q$. Thus applying Lemma 2.5 to the canonical map $\mathcal{O}_L[R^{mm}] \to R$ gives a δ_L-algebra map $\mathcal{O}_L[R^{mm}] \to W_L(R)$, and the Teichmüller map is the composite

$$R^{mm} \to \mathcal{O}_L[R^{mm}] \to W_L(R).$$

To get the formula $[r] = (r, 0, 0, \dots)$, one uses the same reasoning as in part (1) to show that this formula holds when R is π-torsion-free, from which it follows in general. ∎

2.2. Distinguished elements and perfect δ_L-algebras

This section develops results about distinguished elements and perfect δ_L-algebras analogous to those in [8, Sections 2.3 and 2.4].

Definition 2.8. Let A be a δ_L-algebra. An element $d \in A$ is *distinguished* if $\delta_L(d)$ is a unit of A.

Remark 2.9. The following remarks motivate the definition of distinguished elements.

(1) As $\delta_L(\pi) = 1 - \pi^{q-1}$, we have that π is distinguished in any δ_L-algebra.

(2) The significance of distinguished elements is that if (A, I) is a L-typical prism (to be introduced in Section 3), then I is locally generated by distinguished elements (see condition (iii) in the following lemma). As such, we are interested in the case that A is d-adically complete; more generally we will typically assume that $d \in \mathrm{Rad}(A)$ is in the Jacobson radical of A.

Lemma 2.10. *Let A be a δ_L-algebra, and let $d \in$ Rad(A). The following are equivalent:*

(i) *d is distinguished.*

(ii) *The ideal (d) contains a distinguished element.*

(iii) $\pi \in (d^q, \phi(d))$.

(iv) $\pi \in (d, \phi(d))$.

Proof. Clearly (i)⇒(ii). Conversely, suppose we have $d' = \alpha d$ for some $\alpha, d' \in A$ with d' distinguished. Applying δ_L and working mod (π, d), we have

$$\delta_L(d') \equiv \alpha^q \delta_L(d) \pmod{(\pi, d)},$$

which shows that $\delta_L(d)$ is a unit in $A/(\pi, d)$. As $\pi, d \in \mathrm{Rad}(A)$, we have that $\delta_L(d) \in A^\times$ as well.

We now show (i)⇒(iii)⇒(iv)⇒(i). The first implication follows directly from the formula $\phi(d) = d^q + \pi\delta_L(d)$, and the second implication is clear. For the last implication, suppose that $\pi = \alpha d + \beta\phi(d)$ for some $\alpha, \beta \in A$. Applying δ_L to this formula and working mod (π, d) we get

$$\delta_L(\pi) \equiv \delta_L(d)\big(\alpha^q + \beta^q \delta_L(d)^{q-1}\big) \pmod{(\pi, d)}.$$

Then since π is distinguished in any δ_L-algebra, we conclude that $\delta_L(d)$ is a unit in $A/(\pi, d)$ and thus in A as well. ∎

Definition 2.11. A δ_L-algebra A is *perfect* if ϕ_A is an isomorphism.

Lemma 2.12 (See [8, Lemma 2.28]). *Let A be a δ_L-algebra. Then if $\alpha \in A$ is π-torsion, we have $\phi(\alpha) = 0$. In particular, if A is perfect then A is π-torsion-free.*

Proof. Applying δ_L to $\pi\alpha = 0$ gives

$$0 = \pi^q \delta_L(\alpha) + \delta_L(\pi)\alpha^q + \pi\delta_L(\pi)\delta_L(\alpha) = \pi^q\delta_L(\alpha) + \delta_L(\pi)\phi(\alpha).$$

As $\delta_L(\pi) = 1 - \pi^{q-1}$ is a unit and

$$\pi^q \delta_L(\alpha) = \phi(\pi^{q-1}\alpha) - \pi^{q-1}\alpha^q = 0$$

we are done. ∎

Lemma 2.13. *If A is a perfect and π-adically complete δ_L-algebra, and d is distinguished, then d is a nonzerodivisor.*

Proof. Suppose that $d\alpha = 0$ and suppose towards a contradiction that $\alpha \neq 0$. Since A is π-torsion-free by Lemma 2.12 and π-adically complete, we can further assume that $\pi \nmid \alpha$. Applying δ_L to $d\alpha = 0$ gives

$$\alpha^q \delta_L(d) + \delta_L(\alpha)\phi(d) = 0.$$

Multiplying by $\phi(\alpha)$ and using that d is distinguished then implies $\alpha^q\phi(\alpha) = 0$. Thus

$$\alpha^{2q} \equiv 0 \pmod{\pi}.$$

But as ϕ is a bijection, φ_q is injective, so $\pi | \alpha$, a contradiction. ∎

A key fact about perfect δ_L-algebras is the following.

Proposition 2.14 (See [8, Corollary 2.31]). *The following functors are equivalences of categories.*

$$\begin{tikzcd} \left\{\begin{matrix}\pi\text{-adically complete}\\ \text{perfect } \delta_L\text{-algebras}\end{matrix}\right\} \arrow[r, "\text{forget}"] & \left\{\begin{matrix}\pi\text{-adically complete}\\ \pi\text{-torsion-free}\\ \mathcal{O}_L\text{-algebras } A\\ \text{with } A/\pi \text{ perfect}\end{matrix}\right\} \arrow[r, "A\mapsto A/\pi"] & \{\text{perfect } \mathbb{F}_q\text{-algebras}\}. \arrow[ll, bend left, "W_L(R)\mapsfrom R"] \end{tikzcd}$$

Proof. By Lemma 2.12, the forgetful functor has image in π-torsion-free rings. By the vanishing of the cotangent complex $\mathbb{L}_{R/\mathbb{F}_q}$ for a perfect $\mathbb{F}_q$-algebra R and deformation theory, there is a unique π-adically complete and π-torsion-free $\mathcal{O}_L$-algebra $\widetilde{R}$ such that $\widetilde{R}/\pi \cong R$. Since $R \mapsto \widetilde{R}$ is clearly quasi-inverse to $A \mapsto A/\pi$, it suffices to show that $\widetilde{R}$ is naturally isomorphic to forget$(W_L(R))$.

Since $R \mapsto \widetilde{R}$ is a functor, $\widetilde{R}$ comes equipped with a canonical lift of q-Frobenius and thus by Lemma 2.5 a canonical map $s_{\widetilde{R}}\colon \widetilde{R} \to W_L(R)$ lifting $\widetilde{R} \twoheadrightarrow \widetilde{R}/\pi = R$. By [37, Proposition 1.1.18], $W_L(R)$ is π-adically complete, so it suffices to show that $s_{\widetilde{R}}$ induces an isomorphism $R \to W_L(R)/\pi$. But this is clear since an inverse is given by the map $W_L(R)/\pi \to R/\pi = R$ induced by $W_L(R) \twoheadrightarrow R$. ∎

Corollary 2.15. *If R is a perfect $\mathbb{F}_q$ algebra, then $W_L(R) \cong W(R) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_L$ where W denotes the p-typical Witt vectors. In particular, $W_L(\mathbb{F}_q) = \mathcal{O}_L$.*

3. L-typical prisms

As before, let $L/\mathbb{Q}_p$ be a finite extension with uniformizer π and residue field $\mathbb{F}_q$. In this section, we introduce L-typical prisms, which are a mild generalization of prisms obtained by replacing δ-rings with δ_L-algebras. In Section 3.1 we define L-typical prisms and the L-typical prismatic site of a formal scheme X over Spf $\mathcal{O}_L$.

Prisms as defined can be viewed as "deperfections" of perfectoid rings, in the sense that the subcategory of perfect prisms is equivalent to the category of perfectoid rings. Similarly, in Section 3.2 we show that the category of L-typical prisms has a subcategory of perfect L-typical prisms, which are equivalent to *perfectoid $\mathcal{O}_L$-algebras* (i.e. perfectoid rings with an $\mathcal{O}_L$-algebra structure). We also show that there is a perfection functor for L-typical prisms.

In Section 3.3, we give two constructions which – given an L-typical prism (A, I), a perfectoid $\mathcal{O}_L$-algebra R, and a system of ϕ-compatible maps $(\iota_n\colon A \to R)_n$ – produce a map $(A, I) \to (A_{\mathrm{inf},L}(R), \ker\theta)$ to the perfect L-typical prism corresponding to R. These constructions will play a crucial role in Section 4, where they are used to embed an L-typical prism coming from a Lubin–Tate formal $\mathcal{O}_L$-module inside a perfect L-typical prism.

3.1. Basic theory

Definition 3.1. We define L-typical prisms and related notions as follows.

(1) An *L-typical prism* is a pair (A, I) where A is a δ_L-algebra and $I \subseteq A$ is an ideal defining a Cartier divisor on $\mathrm{Spec}(A)$ such that A is derived (π, I)-complete and $\pi \in (I, \phi(I))$. A morphism $(A, I) \to (B, J)$ of prisms is a δ_L-algebra morphism $f: A \to B$ such that $f(I) \subseteq J$.

(2) An L-typical prism (A, I) is *perfect* if A is a perfect δ_L-algebra. It is *bounded* if A/I has bounded π^∞-torsion, i.e. $A/I[\pi^\infty] = A/I[\pi^n]$ for some $n \geq 0$.

(3) If X is a formal scheme over $\mathrm{Spf}\, \mathcal{O}_L$ then the (absolute) L-typical prismatic site X_{Δ_L} has
   - objects: bounded L-typical prisms (A, I) together with a map of formal schemes $\mathrm{Spf}(A/I) \to X$ over $\mathrm{Spf}\, \mathcal{O}_L$;
   - morphisms: maps of L-typical prisms compatible with the structure map to X;
   - covers: morphisms $(A, I) \to (B, J)$ such that $A \to B$ is (π, I)-completely faithfully flat.

   If $X = \mathrm{Spf}(R)$ then we write R_{Δ_L} for X_{Δ_L}.

The same definition was independently given in the concurrent work of Ito [22], where L-typical prisms are called $\mathcal{O}_L$-prisms in Ito's terminology.

Remark 3.2. For the notions of derived I-completeness and I-complete faithful flatness, see [8, Section 1.2]. Note that [46] omits the word "faithfully" in the definition of a cover.

As suggested by the definition of the prismatic site, we will only be interested in bounded prisms. In this case, we need not worry about the word "derived" in the definition of a L-typical prism.

Lemma 3.3. *If (A, I) is a bounded L-typical prism, then A is classically (π, I) complete.*

Proof. This is the same as in [8, Lemma 3.7]. In more detail, we may suppose that $I = (d)$ for a nonzerodivisor d. Then by the derived (π, d)-completeness of A, the fact that A/d^m has bounded π-torsion for all m (by devissage), and [43, Tag 091X], we have

$$\begin{aligned} A &\cong R\lim_m R\lim_n \left(A \otimes^L_{\mathbb{Z}[d]} \mathbb{Z}[d]/(d^m)\right) \otimes^L_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi]/(\pi^n) \\ &\cong R\lim_m R\lim_n A/(d^m) \otimes^L_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi]/(\pi^n) \cong \lim_m \lim_n A/(d^m, \pi^n) \end{aligned}$$

as desired. ■

If (A, I) is a L-typical prism with I *principal*, then Lemma 2.10 shows that the condition $\pi \in (I, \phi(I))$ is equivalent to I being generated by a distinguished element. Under the weaker assumption that I is Zariski-locally principal, the condition $\pi \in (I, \phi(I))$ is equivalent to I being ind-Zariski-locally generated by a distinguished element. (The "ind-" is necessary because after passing to a Zariski open, we may no longer have $(\pi, I) \subseteq \mathrm{Rad}(A)$, which necessitates passing to a further localization along (π, I); see [8, Footnote 8] for more details.)

Lemma 3.4. *Let A be a δ_L-algebra and $I \subseteq A$ a Zariski-locally principal ideal such that $(\pi, I) \subseteq \mathrm{Rad}(A)$. Then the following are equivalent:*

(i) $\pi \in (I^q, \phi(I))$.

(ii) $\pi \in (I, \phi(I))$.

(iii) *There is a faithfully flat map of δ_L-algebras $A \to A'$ with A' an ind-(Zariski localization) of A such that IA' is generated by a distinguished element d and* $(\pi, d) \in \mathrm{Rad}(A')$.

Proof. We follow [8, Lemma 3.1]. Clearly (i)⇒(ii). For (ii)⇒(iii), since I is locally principal we can select $f_1, \dots, f_n \in A$ generating the unit ideal in A such that each $IA[1/f_i]$ is principal. Take $A' = (\prod A[1/f_i])_{(\pi,I)}$, where the subscript denotes Zariski localization along $V((\pi, I))$. Then A' has a unique δ_L-structure by the L-typical analogue of [8, Remark 2.16], $A \to A'$ is a faithfully flat of δ_L-algebras, and $I' = IA'$ is principal with $\pi \in (I', \phi(I'))$. By Lemma 2.10, any generator of I' is distinguished.

For (iii)⇒(i), we would like to check that $\pi = 0$ in $A/(I^q, \phi(I))$. This can be checked after faithfully flat extension to A', in which case it follows from Lemma 2.10. ∎

Even though I is only assumed locally principal, $\phi(I)$ is always principal.

Lemma 3.5. *The ideal $\phi(I)$ is principal and generated by a distinguished element for any L-typical prism* (A, I).

Proof. By Lemma 3.4, we can pick $a \in I^q, b \in \phi(I)$ so that $\pi = a + b$. We will show that b generated $\phi(I)$. This can be checked after passing to the ind-Zariski-localization A' of Lemma 3.4. Let d be a distinguished generator of IA' so that $a = \alpha d^q$ and $b = \beta\phi(d)$ in A'; it suffices to show that β is a unit. Indeed, applying δ_L to the equation $\pi = \alpha d^q + \beta\phi(d)$ and working mod (π, d) gives

$$\delta_L(\pi) \equiv \beta^q \delta_L(d)^q \pmod{(\pi, d)},$$

which implies that β is a unit in $A'/(\pi, d)$ and hence in A', as desired. ∎

3.2. Perfect L-typical prisms and perfectoid $\mathcal{O}_L$-algebras

It is shown in [8, Theorem 3.10] that the functor $(A, I) \mapsto A/I$ is (one half of) an equivalence of categories between perfect prisms and perfectoid rings. In fact, this can be taken as the definition of a perfectoid ring, as is done in [4, Lecture IV]. We take the same perspective here, initially *defining* perfectoid $\mathcal{O}_L$-algebras as those $\mathcal{O}_L$-algebras which come from perfect L-typical prisms. We will later show (Theorem 3.18) that one can equivalently define perfectoid $\mathcal{O}_L$-algebras as perfectoid rings equipped with an $\mathcal{O}_L$-algebra structure.

Definition 3.6. An $\mathcal{O}_L$-algebra R is a *perfectoid $\mathcal{O}_L$-algebra* if it is isomorphic to A/I for some perfect L-typical prism (A, I).

The functor from perfectoid rings to perfect prisms is $R \mapsto (A_{\mathrm{inf},L}(R), \ker\theta)$. To generalize this functor to the present setting, recall that the *tilt* of a ring R is $R^\flat = \varprojlim_{\varphi_p} R/p$. If R is an $\mathcal{O}_L$-algebra, then we have an isomorphism of rings

$$R^\flat \cong \varprojlim_{\varphi_q} R/\pi$$

so that $R^\flat$ is in fact a perfect $\mathbb{F}_q$-algebra. If R is moreover π-adically complete then we have an isomorphism of multiplicative monoids $R^\flat \xrightarrow{\sim} \varprojlim_{x\mapsto x^q} R$; by composing this with projection onto the first factor of the inverse limit, we get a multiplicative map $\sharp\colon R^\flat \to R$ explicitly given by

$$x^\sharp = \lim_{n\to\infty} \widehat{x_n}^{q^n} \quad \text{where } x = (\dots, x_1, x_0) \in \varprojlim_{\varphi_q} R/\pi = R^\flat$$

and where each $\widehat{x_n} \in R$ is an arbitrary lift of $x_n \in R/\pi$.

Definition 3.7. If R is a π-adically complete $\mathcal{O}_L$-algebra, then let $A_{\mathrm{inf},L}(R) = W_L(R^\flat)$ and $\theta\colon A_{\mathrm{inf},L}(R) \to R$ be the map given in Witt coordinates by

$$(x_0, x_1, \dots) \mapsto \sum_{n\geq 0} (x_n^{1/q^n})^\sharp \pi^n.$$

By Corollary 2.15 we have $W_L(R^\flat) \cong W(R^\flat) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_L$, and θ is a ring homomorphism coinciding with the base change to $\mathcal{O}_L$ of the usual map $\theta\colon W(R^\flat) \to R$ of p-adic Hodge theory.

Remark 3.8. If $L/\mathbb{Q}_p$ is unramified and $\pi = p$, then $A_{\mathrm{inf},L}(R)$ is the ring typically denoted $A_{\mathrm{inf}}(R)$ and θ is the usual map of p-adic Hodge theory.

Lemma 3.9. *Let R be a π-adically complete $\mathcal{O}_L$-algebra with $\varphi_q\colon R/\pi \to R/\pi$ surjective.*

(1) *The map $\theta\colon A_{\mathrm{inf},L}(R) \to R$ is surjective.*

(2) *$A_{\mathrm{inf},L}(R)$ is $(\pi, \ker\theta)$-adically complete.*

Proof. First note that $A_{\mathrm{inf},L}(R)$ is π-adically complete by [37, Proposition 1.1.18]. Thus part (1) reduces to showing that $R^\flat \to R/\pi$ is surjective, which follows from the assumption that φ_q is surjective.

For (2), using again the π-completeness of $A_{\mathrm{inf},L}(R)$, it suffices to check that $R^\flat$ is complete with respect to the ideal $J = \ker(R^\flat \to R/\pi)$ which is the mod π reduction of $\ker\theta$. Indeed, we have

$$R^\flat = \varprojlim_{\varphi_q} R/\pi \cong \varprojlim_{n} R^\flat/J^{q^n}$$

via the isomorphisms $R/\pi = R^\flat/J \xrightarrow{\varphi_q^n} R^\flat/J^{q^n}$. ∎

The following properties make perfect L-typical prisms especially well behaved.

Lemma 3.10. *Let (A, I) be a perfect L-typical prism.*

(1) *I is principal and generated by a distinguished element.*

(2) *(A, I) is bounded.*

(3) *A/I is π-adically complete.*

Proof. (1) follows from Lemma 3.5. Let $d \in A$ be the distinguished generator of I.

For (2), we will in fact show that $A/d[\pi^2] = A/d[\pi]$. Suppose that $\alpha \in A/d[\pi^2]$ so that there is some $\beta \in A$ with $\pi^2\alpha = \beta d$. Applying δ_L and working mod π, we get that

$$d^q\delta_L(\beta) + \beta^q\delta_L(d) \equiv 0 \pmod{\pi}.$$

Multiplying by β^q and using that $\delta_L(d) \in A^\times$ then gives that $\pi | \beta^{2q}$. This implies $\pi | \phi^2(\beta)$, so that $\pi | \beta$ since ϕ is an $\mathcal{O}_L$-linear isomorphism. Thus we have

$$\pi^2\alpha = \pi\beta' d \quad \text{for some } \beta' \in A$$

so that $d | \pi\alpha$ by Lemma 2.12.

(3) follows from [43, Tag 091X] since A/d is derived π-complete with bounded π-power torsion. ■

Proposition 3.11. *We have an equivalence of categories*

$$\begin{tikzcd} \left\{\begin{matrix} \textit{perfect} \\ L\textit{-typical prisms}\end{matrix}\right\} \arrow[r, bend left, "(A,I)\mapsto A/I"] & \left\{\begin{matrix} \textit{perfectoid} \\ \mathcal{O}_L\textit{-algebras}\end{matrix}\right\} \arrow[l, bend left, "(A_{\mathrm{inf},L}(R),\ker\theta)\leftarrow R"] \end{tikzcd}$$

Proof. Let $R = A/I$ be a perfectoid $\mathcal{O}_L$-algebra coming from a perfect L-typical prism (A, I). Since R is π-adically complete by Proposition 3.10 and $\varphi_q : R/\pi \to R/\pi$ is surjective (as it is the mod (π, I) reduction of $\phi : A \to A$), Lemma 3.9 (1) implies that

$$\theta : A_{\mathrm{inf},L}(R) \to R$$

is surjective. Thus to prove the proposition, it suffices to show that $A_{\mathrm{inf},L}(R)$ identifies with A in such a way that

$$\begin{tikzcd} A \arrow[rr, "\sim"] \arrow[dr, two heads] & & A_{\mathrm{inf},L}(R) \arrow[dl, two heads, "\theta"] \\ & A/I = R & \end{tikzcd}$$

commutes (thereby identifying I with $\ker\theta$). Since $A_{\mathrm{inf},L}(R)$ and A are π-adically complete perfect δ_L-algebras, by Proposition 2.14 it suffices to show that A/π identifies with

$R^\flat$ compatibly with the maps to $A/(\pi, I) = R/\pi$. Indeed, we have a commutative diagram

$$\begin{tikzcd} A/\pi \arrow[r, no head, "\sim"] \arrow[dr, two heads] & \varprojlim_{\varphi_q} A/(\pi, I) \arrow[d, two heads] \arrow[r, no head, "\sim"] & R^\flat \arrow[dl, two heads] \\ & A/(\pi, I) & \end{tikzcd}$$

via the I-adic completeness of A/π. ∎

Lemma 3.12. *A map $R \to S$ of perfectoid $\mathcal{O}_L$-algebras is π-completely (faithfully) flat if and only if the corresponding map$A_{\mathrm{inf},L}(R) \to A_{\mathrm{inf},L}(S)$ is $(\pi, \ker\theta)$-completely (faithfully) flat.*

Proof. It is easy to show that

$$A_{\mathrm{inf},L}(S) \otimes^L_{A_{\mathrm{inf},L}(R)} \frac{A_{\mathrm{inf},L}(R)}{\ker\theta_{A_{\mathrm{inf},L}(R)}} \cong S \otimes^L_R R$$

using either the L-typical analogue of the rigidity result [8, Lemma 3.5] or the fact that a distinguished element can only factor as a unit times another distinguished element [8, Lemma 2.24]. Thus $R \to S$ being π-completely (faithfully) flat and $A_{\mathrm{inf},L}(R) \to A_{\mathrm{inf},L}(S)$ being $(\pi, \ker\theta)$-completely (faithfully) flat are both equivalent to $R/\pi \to S \otimes^L_R R/\pi$ being (faithfully) flat. ∎

Given a L-typical prism (A, I) we can form its perfection.

Definition 3.13. If (A, I) is a L-typical prism, then we write

$$A_{\mathrm{perf}} = \big(\varinjlim_{\phi} A\big)^\wedge_{(\pi,I)}$$

for the (classical) (π, I)-completion of the naive perfection $\varinjlim_\phi A$. We call $(A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ the *perfection* of (A, I).

By Remarks 3 (3)–(4), A_{perf} is a perfect δ_L-algebra. We now show that $(A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ is the initial L-typical prism over (A, I).

Proposition 3.14 (cf. [8, Lemma 3.9]). *Let (A, I) be a L-typical prism.*

(1) *The derived (π, I)-adic completion of $\varinjlim_\phi A$ coincides with the classical (π, I)-adic completion (and thus with A_{perf}).*

(2) *The map $(A, I) \to (A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ is initial among maps from (A, I) to a perfect L-typical prism.*

Proof. (1) clearly implies (2). To show (1) first note that by construction $\varinjlim_\phi A$ is a perfect δ_L-algebra. Thus by Lemma 2.12 A is π-torsion-free, so that the derived and classical π-adic completions agree. As $A \to (\varinjlim_\phi A)^\wedge_{(\pi)}$ factors through $\phi\colon A \to A$, Lemma 3.5 implies that $I(\varinjlim_\phi A)^\wedge_{(\pi)}$ is principal and generated by a distinguished element d. By Lemma 2.10 it thus suffices to show that d is a nonzerodivisor.

For this, suppose that $fd = 0$ for some $0 \neq f \in (\varinjlim_\phi A)^\wedge_{(\pi)}$; since this ring is π-torsion-free and classically π-adically complete (and thus π-adically separated), we can suppose that $\pi \nmid f$ by dividing out powers of π. Applying δ_L and working mod π we get

$$f^q \delta_L(d) + d^q \delta_L(f) \equiv 0 \pmod{\pi}.$$

Multiplying by f^q and using that $\delta_L(d)$ is a unit then shows that $\pi | f^{2q}$. Thus $\pi | \phi^2(f)$, which implies that $\pi | f$ since ϕ is a $\mathcal{O}_L$-linear isomorphism. But this is a contradiction, so d must be a nonzerodivisor. ∎

Finally, we will show that perfectoid $\mathcal{O}_L$-algebras coincide with perfectoid rings (in the sense of [7, Definition 3.5]) equipped with an $\mathcal{O}_L$-algebra structure. We begin by establishing a more intrinsic criterion for being a perfectoid $\mathcal{O}_L$-algebra; indeed the following proposition is the L-typical version of [4, Proposition IV.2.10], in which p is replaced by π and the Frobenius is replaced by the q-Frobenius.

Proposition 3.15. *Let R be an $\mathcal{O}_L$-algebra. Then R is a perfectoid $\mathcal{O}_L$-algebra if and only if*

(1) *R is π-adically complete,*

(2) *there exists some $\varpi \in R$ such that $\varpi^q = \pi u$ for some $u \in R^\times$,*

(3) *$\varphi_q \colon R/\pi \to R/\pi$ is surjective, and*

(4) *the kernel of $\theta \colon A_{\mathrm{inf},L}(R) \to R$ is principal.*

If R is assumed π-torsion-free, then the above remains true with (4) replaced by

(4') *if $x \in R[1/\pi]$ with $x^q \in R$, then $x \in R$.*

Remark 3.16. Note that once a qth root of πu as in (2) exists, we get a full q-power-compatible system of roots (ϖ^{1/q^n}) by letting $\varpi^\flat \in R^\flat$ be any lift of ϖ along the map $R^\flat \to R/\pi$ (which is surjective by assumption (3)) and then taking (ϖ^{1/q^n}) to be the image of $\varpi^\flat$ under the bijection $R^\flat \cong \varprojlim_{x \mapsto x^q} R$ (which exists by assumption (1)).

Proof of Proposition 3.15. Suppose that $R = A/I$ is a perfectoid $\mathcal{O}_L$-algebra coming from a perfect L-typical prism (A, I); using Proposition 3.11 we can identify $(A, I) \cong (A_{\mathrm{inf},L}(R), \ker\theta)$. (1) and (4) follow from Proposition 3.10, and (3) follows from the surjectivity of $\phi \colon A \to A$. For (2), let $d \in A_{\mathrm{inf},L}(R)$ be a distinguished generator of $\ker\theta$ (which exists by Proposition 3.10). Then we can take $\varpi = \theta(\phi^{-1}(d))$ since

$$\varpi^q = \theta\big(\phi^{-1}(d^q)\big) = \theta\big(d - \pi\phi^{-1}\big(\delta_L(d)\big)\big) = -\pi\theta\big(\phi^{-1}\big(\delta_L(d)\big)\big),$$

with $u = -\theta(\phi^{-1}(\delta_L(d))) \in R^\times$.

Remark 3.17. Using the diagram

$$\begin{array}{ccc} A_{\mathrm{inf},L}(R)/(\pi, d) & \xrightarrow[\theta]{\sim} & R/\pi \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \varphi_q} \\ A_{\mathrm{inf},L}(R)/(\pi, d) & \xrightarrow[\theta]{\sim} & R/\pi \end{array}$$

we note that the mod π reduction of the element ϖ constructed above is the kernel of $\varphi_q\colon R/\pi \to R/\pi$. Thus the surjective map $\varphi_q\colon R/\pi \to R/\pi = R/\varpi^q$ factors through an isomorphism $R/\varpi \xrightarrow{\sim} R/\varpi^q$. This fact will be used later in the proof.

For the converse, suppose that R is an $\mathcal{O}_L$-algebra satisfying (1)–(4); we want to show that $(A_{\mathrm{inf},L}(R), \ker\theta)$ is a perfect L-typical prism. $A_{\mathrm{inf},L}(R) = W_L(R^\flat)$ is a perfect δ_L-algebra by Proposition 2.14 and is $(\pi, \ker\theta)$-adically complete by Lemma 3.9. By assumption $\ker\theta = (d)$ for some $d \in A_{\mathrm{inf},L}(R)$. Thus by Lemma 2.10 it suffices to show that d is distinguished.

Let $\varpi, u \in R$ be as in (2), and let $\omega, v \in A_{\mathrm{inf},L}(R)$ be lifts along θ. Then $\omega^q - \pi v \in \ker\theta$, so we can write

$$\omega^q - \pi v = \alpha d$$

for some $\alpha \in A_{\mathrm{inf},L}(R)$. Applying δ_L to this equation and working mod (π, d) gives

$$-v\delta_L(\pi) \equiv \alpha^q \delta_L(d) \pmod{(\pi, d)}.$$

As $-v\delta_L(\pi) \in (A_{\mathrm{inf},L}(R)/(\pi,d))^\times$, this shows that $\delta_L(d) \in (A_{\mathrm{inf},L}(R)/(\pi,d))^\times$ as well, and thus $\delta_L(d) \in A_{\mathrm{inf},L}(R)^\times$ by (π, d)-completeness.

Assume now that R is π-torsion-free. Supposing R is a perfectoid $\mathcal{O}_L$-algebra, we prove (4'). Suppose that $x \in R[1/\pi]$ with $x^q \in R$. Let $\varpi \in R$ be the element satisfying (2) constructed earlier in this proof; by Remark 3.17 we have that the q-power map $R/\varpi \to R/\varpi^q$ is bijective. Let $n \geq 0$ be minimal such that $\varpi^n x \in R$ (such an n exists since $\varpi^q | \pi$), and suppose towards a contradiction that $n \geq 1$. Then

$$(\varpi^n x)^q = \varpi^{nq} x^q \in \varpi^{nq} R \subseteq \varpi^q R,$$

which implies that $\varpi^n x \in \varpi R$. As R is ϖ-torsion-free, this implies that $\varpi^{n-1} x \in R$, giving the contradiction.

Finally, suppose that R is a π-torsion-free $\mathcal{O}_L$-algebra satisfying (1)–(3) and (4'); we will prove (4). Let $\varpi \in R$ be as in (2), and let (ϖ^{1/q^n}) be a system of q-power roots of ϖ, which exists by Remark 3.16 (which uses only (1)–(3)). We have that ϖ^{1/q^n} mod π generates $\ker(\varphi_q^n\colon R/\pi \to R/\pi)$: if $x \in R$ with $\pi | x^{q^n}$ then the q^n-th power of $x/\varpi^{1/q^n} \in R[1/\pi]$ is in R, so that $x/\varpi^{1/q^n} \in R$ as well by assumption. It follows that the element

$$(\ldots, \overline{\varpi^{1/q^2}}, \overline{\varpi^{1/q}}, \overline{\varpi}, 0) \in R^\flat$$

formed from the mod π reductions of the ϖ^{1/q^n} generates $\ker(R^\flat \to R/\pi) = \varprojlim \ker(\varphi_q^n)$. But since R is π-torsion-free and $\ker\theta$ is π-adically complete with mod π reduction $\ker(R^\flat \to R/\pi)$, this implies that $\ker\theta$ is principal as well. ∎

Theorem 3.18. *Let R be a ring. Then R is a perfectoid $\mathcal{O}_L$-algebra if and only if R is a perfectoid ring and an $\mathcal{O}_L$-algebra.*

Proof. First, suppose that R is a perfectoid ring and an $\mathcal{O}_L$-algebra. To show that R is a perfectoid $\mathcal{O}_L$-algebra, it suffices to show that $(A_{\mathrm{inf},L}(R), \ker\theta)$ is an L-typical prism (it is automatically perfect by Proposition 2.14). This is done in [22, Lemma 2.4.3]; we briefly sketch the argument here. First, one shows that $\ker(\theta\colon A_{\mathrm{inf},L}(R^\flat) \to R)$ is generated by an element of the form $\xi = \pi - [\varpi^\flat]b$, where $\varpi \in R$ is such that R is ϖ-adically complete and $\varpi^p | p$, the element $\varpi^\flat \in R^\flat$ satisfies $(\varpi^\flat)^\sharp = \varpi$, and $b \in A_{\mathrm{inf},L}(R)$. Since any generator of $\ker(W(R^\flat) \to R)$ is a nonzerodivisor and $W(\mathbb{F}_q) \to \mathcal{O}_L$ is flat, any generator of $\ker\theta$ ia a nonzerodivisor. It is easy to show that $A_{\mathrm{inf},L}(R)$ is (π, ξ)-adically complete. And ξ is distinguished as $\delta_L(\xi) \equiv 1 - \pi^{q-1} \pmod{(\pi, \xi)}$ is a unit in R.

Conversely, suppose that R is a perfectoid $\mathcal{O}_L$-algebra; we want to show that R is a perfectoid ring. By Lemma 3.19 below, we have that R can be written as a fiber product $\overline{R} \times_{\overline{S}} S$ where $\overline{R} \to \overline{S}$ is a surjection of perfect $\mathbb{F}_q$-algebras, and S is a π-torsion-free perfectoid $\mathcal{O}_L$-algebra. Once we show that $\overline{R}$, $\overline{S}$, and S are perfectoid rings, we may conclude that R is a perfectoid ring as well by [14, Proposition 2.1.4]. Thus we may assume that R is a perfect $\mathbb{F}_q$-algebra or π-torsion-free. In the former case, the result is clear since perfect $\mathbb{F}_p$-algebras are perfectoid rings.

Suppose now that R is π-torsion-free. By [5, Lemma 3.6] (cf. also [4, Proposition IV.2.10]) it suffices to show that R satisfies the "p-analogues" of properties (1)–(3), (4') in Proposition 3.15:

- (1_{p}) R is p-adically complete,
- (2_{p}) there exists some $\varpi' \in R$ such that $(\varpi')^p = pu'$ for some $u' \in R^\times$,
- (3_{p}) $\varphi\colon R/p \to R/p$ is surjective, and
- $(4'_{\mathrm{p}})$ if $x \in R[1/p]$ with $x^p \in R$, then $x \in R$.

(1_{p}) and $(4'_{\mathrm{p}})$ follow immediately from (1) and (4'), respectively. Taking $\varpi, u \in R$ as in (2) and letting e be the ramification index of $L/\mathbb{Q}_p$, we conclude (2_{p}) by taking $\varpi' = \varpi^{qe/p}$ and $u' = u^e$. To show (3_{p}), it suffices to show the q-Frobenius $\varphi_q\colon R/p \to R/p$ is surjective. So let $\alpha \in R$; we will successively approximate $\alpha^{1/q}$ modulo higher powers of π. Indeed, by (3) there is $\beta_1 \in R$ so that $\beta_1^q \equiv \alpha \pmod{\pi}$. Thus $\frac{\alpha - \beta_1^q}{u\pi} \in R$, where $u \in R^\times$ is as in (2). Again by (3), there is $\gamma_1 \in R$ so that $\gamma_1^q \equiv \frac{\alpha - \beta_1^q}{u\pi} \pmod{\pi}$. Then let $\beta_2 = \beta_1 + \gamma_1\varpi$ with ϖ as in (2). We see that $\beta_2^q \equiv \beta_1^q + \gamma_1^q u\pi = \alpha \pmod{\pi^2}$ (so long as the ramification index $e \geq 2$; if $e = 1$ then $\pi = p$, so we were already done). Repeating this process, we get for $1 \leq n \leq e$ elements $\beta_n \in R$ satisfying $\beta_n^q \equiv \alpha \pmod{\pi^n}$. We then get that $\beta_e^q \equiv \alpha \pmod{p}$ as desired. ∎

Lemma 3.19 (cf. [4, Proposition IV.3.2] or [33, Remark 8.8]). *Let R be a perfectoid $\mathcal{O}_L$-algebra. Then the $\mathcal{O}_L$-algebras*

$$\overline{R} = R/\sqrt{\pi R}, \quad S = R/R[\sqrt{\pi R}], \quad \text{and} \quad \overline{S} = S/\sqrt{\pi S}$$

are perfectoid $\mathcal{O}_L$-algebras and the commutative square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \overline{R} & \longrightarrow & \overline{S} \end{array}$$

is Cartesian.

Proof. This is proved the same way as in [4], so we only sketch the argument here. As in the proof of Proposition 3.15, we can write a distinguished generator of

$$\ker\big(\theta\colon A_{\mathrm{inf},L}(R) \to R\big)$$

as $d = [a_0] - \pi u$ for $a_0 \in R^\flat$ such that $R^\flat$ is a_0-adically complete and $a_0^\sharp = \pi$, and $u \in A_{\mathrm{inf},L}(R)^\times$ (concretely $a_0 = (\varpi^\flat)^q$ for $\varpi^\flat$ as in Remark 3.16). Let $I = (a_0^{1/q^\infty}) \subseteq R^\flat$ and $J = R^\flat[I]$. Then the commutative square

$$\begin{tikzcd} W_L(R^\flat) \arrow[r] \arrow[d] & W_L(R^\flat/J) \arrow[d] \\ W_L(R^\flat/I) \arrow[r] & W_L\big(R^\flat/(I+J)\big) \end{tikzcd}$$

is a homotopy fiber square by [33, Lemma 8.1] and devissage. Since d is a nonzerodivisor in all of these perfect δ_L-algebras by Lemma 2.13, the square remains a homotopy fiber square upon application of $- \otimes^L_{W_L(R^\flat)} R$.

$$\begin{tikzcd} W_L(R^\flat)/(d) \arrow[r] \arrow[d] & W_L(R^\flat/J)/(d) \arrow[d] \\ W_L(R^\flat/I)/(d) \arrow[r] & W_L(R^\flat/(I+J))/(d) \end{tikzcd}$$

It is easy to see that the rings in this square are all perfectoid $\mathcal{O}_L$-algebras. Finally, one shows that this cartesian square identifies with the one in the statement of the lemma; see [4, Lecture IV] or [33, Section 8] for the details. ∎

3.3. Constructing maps of L-typical prisms

In Section 4 we will construct inclusions of L-typical prisms coming from Lubin–Tate formal groups into perfect L-typical prisms. The construction used can be understood in at least three different ways, one of which is specific to the scenario in Section 4, and two of which are general constructions for producing maps between L-typical prisms. Here we explain the two general constructions.

Construction 3.20. Let R be an $\mathcal{O}_L$-algebra, and let A be a δ_L-algebra with a sequence of ϕ-compatible $\mathcal{O}_L$-algebra maps $\iota_n\colon A \to R$ for $n \geq 0$, i.e. a sequence of maps ι_n making the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\iota_0"] & A \arrow[r, "\phi"] \arrow[dl, "\iota_1"] & A \arrow[r, "\phi"] \arrow[dll, bend left, "\iota_2"] & \cdots \\ R & & & \end{tikzcd}$$

commute. We will construct from this data a map $\iota\colon A \to W_L(R^\flat)$ of δ_L-algebras.

Indeed, using that φ_q commutes with maps of $\mathbb{F}_q$-algebras, we can form a map

$$\begin{aligned} \bar{\iota} : A/\pi &\to \varprojlim_{\varphi_q} R/\pi = R^\flat \\ a &\mapsto \big(\overline{\iota_n}(a)\big)_n \end{aligned}$$

in characteristic p, where $\overline{\iota_n}\colon A/\pi \to R/\pi$ denotes the mod π reduction. Then applying the universal property of W_L (Lemma 2.5) to the $\mathcal{O}_L$-algebra map $A \twoheadrightarrow A/\pi \xrightarrow{\bar{\iota}} R^\flat$, we get a δ_L-algebra map $\iota\colon A \to W_L(R^\flat)$. Note that this construction is purely δ_L-algebraic; it uses nothing from the theory of prisms.

Remark 3.21. Intuitively, we think of the system $(\iota_n)_n$ as giving a way to extract ϕ-power roots in the perfect δ_L-algebra $W_L(R^\flat)$. More precisely, it is easy to show that $\phi^{-m}(\iota(a))$ coincides with $\iota^{\to m}(a)$, where $\iota^{\to m}$ denotes the map produced by applying the above construction to the right-shifted system $(\iota_{n+m})_n$.

If R is π-adically complete then we additionally have a map $\theta\colon W_L(R^\flat) \to R$. The following lemma computes the composite $A \xrightarrow{\iota} W_L(R^\flat) \xrightarrow{\theta} R$ (possibly with a ϕ-twist).

Proposition 3.22. *Fix all notation as above, with R being a π-adically complete $\mathcal{O}_L$-algebra. Then for any $n \geq 0$, we have*

$$\theta \circ \phi^{-n} \circ \iota = \iota_n.$$

Proof. By Remark 3.21, it suffices to prove this for $n = 0$; thus we will show that $\theta \circ \iota = \iota_0$. The proof is by direct computation. Fixing some $a \in A$, it suffices to show that $\theta(\iota(a)) \equiv \iota_0(a) \pmod{\pi^{k+1}}$ for all $k \geq 0$.

We can factor the map ι as

$$A \xrightarrow{s_A} W_L(A) \xrightarrow{W_L(\bar{\iota})} W_L(R^\flat),$$

where s_A is the section of Proposition 2.7 (1). Writing $(s_0, s_1, \ldots) = s_A(a) \in W_L(A)$, we find

$$\begin{aligned} \theta\big(\iota(a)\big) &= \sum_{n=0}^{\infty} \big(\bar{\iota}(\overline{s_n})^{1/q^n}\big)^{\#} \pi^n = \sum_{n=0}^{\infty} \lim_{m\to\infty} \big(\bar{\iota}(\overline{s_n})_m\big)^{\wedge, q^{m-n}} \pi^n \\ &\equiv \lim_{m\to\infty} \sum_{n=0}^{k} \big(\bar{\iota}(\overline{s_n})_{k+m}\big)^{\wedge, q^{k+m-n}} \pi^n \pmod{\pi^{k+1}} \\ &= \lim_{m\to\infty} w_k\Big(\big(\bar{\iota}(\overline{s_0})_{k+m}\big)^{\wedge, q^m}, \ldots, \big(\bar{\iota}(\overline{s_k})_{k+m}\big)^{\wedge, q^m}\Big). \end{aligned} \tag{3.1}$$

Here we have written $\overline{s_n}$ for the mod π reduction of $s_n \in A$, and for $r = (\ldots, r_1, r_0) \in R^\flat$, we have written $(r_n)^\wedge$ for an arbitrary lift of $r_n \in R/\pi$ to R. Since

$$\big(\bar{\iota}(\overline{s_n})_{k+m}\big)^{q^m} = \bar{\iota}(\overline{s_n})_k \equiv \iota_k(s_n) \pmod{\pi},$$

Lemma 3.23 below allows us to continue (3.1):

$$\begin{aligned}\theta\big(\iota(a)\big) &\equiv w_k\big(\iota_k(s_0),\ldots,\iota_k(s_k)\big) \pmod{\pi^{k+1}}\\ &= \iota_k\big(w_k(s_0,\ldots,s_k)\big) = \iota_k\big(\phi^k(a)\big) = \iota_0(a)\end{aligned}$$

which is what we wanted. Here we have used that ι_k is an $\mathcal{O}_L$-algebra map (and thus commutes with w_k) and the fact that $w_k \circ s_A = \phi^k$ from the second part of Proposition 2.7 (1). ■

Lemma 3.23. *If R is an $\mathcal{O}_L$-algebra, $a_0,\ldots,a_k,b_0,\ldots,b_k \in R$, and $a_n \equiv b_n \pmod{\pi^s}$ for $n = 0,\ldots,k$, then*

$$w_k(a_0,\ldots,a_k) \equiv w_k(b_0,\ldots,b_k) \pmod{\pi^{s+k}}.$$

Proof. See [37, Lemma 1.1.2]. ■

Proposition 3.22 suggests viewing the map $\iota\colon A \to W_L(R^\flat)$ as a map of prisms when doing so makes sense, i.e. when R is a perfectoid $\mathcal{O}_L$-algebra and (A,I) is a prism with $I \subseteq \ker\iota_0$. This is the perspective taken in the following construction.

Construction 3.24. Let (A,I) be a L-typical prism, let R be a perfectoid $\mathcal{O}_L$-algebra, and suppose given a map $\iota_0\colon A/I \to R$. If (A,I) were assumed perfect, then Proposition 3.11 would allow us to lift ι_0 into a map $\iota\colon (A,I) \to (A_{\mathrm{inf},L}(R), \ker\theta)$, but we do not make this assumption. Instead, we further assume given a collection of ϕ-compatible $\mathcal{O}_L$-algebra maps $\iota_n\colon A \to R$ as above with $I \subseteq \ker\iota_0$; this will allow us to construct such a lift ι.

Using the ι_n, we can factor $\iota_0\colon A/I \to R$ through

$$\big(\varinjlim_{\phi} A\big)/I \xrightarrow{(\iota_n)_n} R,$$

and then, after π-adically completing, through the map of perfectoid $\mathcal{O}_L$-algebras

$$A_{\mathrm{perf}}/IA_{\mathrm{perf}} = \big(\varinjlim_{\phi} A\big)^{\wedge}_{(\pi)}/I \to R.$$

Applying Proposition 3.11 to the map $A_{\mathrm{perf}}/IA_{\mathrm{perf}} \to R$ gives a map of prisms fitting into the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & A_{\mathrm{perf}} \arrow[r, "\text{using prop. 3.11}"] \arrow[d, two heads] & A_{\mathrm{inf},L}(R) \arrow[d, two heads] \\ A/I \arrow[r] & A_{\mathrm{perf}}/IA_{\mathrm{perf}} \arrow[r, "\text{using } (\iota_n)_n"] & R \end{tikzcd}$$

We take ι to be the composite along the top row, which a map of L-typical prisms $(A,I) \to (A_{\mathrm{inf},L}(R), \ker\theta)$ by construction.

Proposition 3.25. *Let X be an adic space over $\operatorname{Spf} \mathcal{O}_L$, let $(A, I) \in X_{\Delta_L}$, and let R be a perfectoid $\mathcal{O}_L$-algebra with a structure map $\operatorname{Spf} R \to X$ over $\operatorname{Spf} \mathcal{O}_L$. Suppose we have a ϕ-compatible direct system of $\mathcal{O}_L$-algebra maps $\iota_n \colon A \to R$ such that $I \subseteq \ker \iota_0$ and the map $\iota_0 : A/I \to R$ is an X-morphism. Then there is morphism*

$$\iota \colon (A, I) \to \big(W_L(R^\flat), \ker \theta\big)$$

in X_{Δ_L} reducing to $\iota_0 \colon A/I \to R$. Moreover, the map $A \to A_{\mathrm{inf},L}(R) = W_L(R^\flat)$ of δ_L-algebras obtained this way coincides with that of Construction 3.20.

Proof. The map ι of the proposition is given by Construction 3.24; it is immediate that the morphism constructed this way respects the structure maps to X. To show that this coincides with Construction 3.20, it suffices to show that the maps $\iota^\delta, \iota^\Delta \colon A_{\mathrm{perf}} \to W_L(R^\flat)$ induced by constructions 3.20 and 3.24, respectively, coincide. By the definition of the $A_{\mathrm{inf},L}$ functor, ι^Δ is W_L of the tilt of $A_{\mathrm{perf}}/I \to R$, so by Proposition 2.14 it suffices to show that taking λ^δ mod π gives

$$(A_{\mathrm{perf}}/I)^\flat \to R^\flat.$$

This is easy to check using the identification $(A_{\mathrm{perf}}/I)^\flat \cong A_{\mathrm{perf}}/\pi$ from the proof of Proposition 3.11. ∎

Example 3.26. Let R be a p-adically complete ring, and let E be an ordinary elliptic curve over R, and let $E[p^\infty] = \varinjlim E[p^n]$ denote the p-divisible group of E. Then by the theory of the canonical subgroup, there are lifts $F \colon E \to E^{(p)}$ of the relative Frobenius $E/p \to (E/p)^{(p)}$ and $V \colon E^{(p)} \to E$ of the Verschiebung with $VF = [p]$. It follows that we have maps

$$\cdots \to \ker F^3 \xrightarrow{[p]} \ker F^2 \xrightarrow{[p]} \ker F.$$

Note that $A = \varprojlim \mathcal{O}_{\ker F^n}$ (inverse limit taken with respect to the inclusion maps $\ker F^n \hookrightarrow \ker F^{n+1}$) has a lift of Frobenius given by $\phi = [p]^*$.

For $n \geq 1$, suppose R_n are étale R-algebras with sections $e_n \colon \operatorname{Spf} R_n \to \ker F^n$. Then, setting $R_\infty = (\varinjlim R_n)^\wedge_{(p)}$, we have that the maps

$$\iota_n \colon A \twoheadrightarrow \ker F^n \xrightarrow{e_n^*} R_n \hookrightarrow R_\infty$$

form a ϕ-compatible system. Thus Construction 3.20 gives a map $A \to W(R_\infty^\flat)$.

4. Lubin–Tate (φ_q, Γ)-modules

In this section, we introduce the key objects involved in Kisin–Ren's theory of (φ_q, Γ)-modules. Pleasantly, much of this theory can be succinctly stated in the prismatic language developed in Section 3.

As before, let $L/\mathbb{Q}_p$ be a finite extension with uniformizer π, and let $q = |\mathcal{O}_L/\pi|$. Let $\mathcal{G}$ be a Lubin–Tate formal $\mathcal{O}_L$-module corresponding to π.

By a *p-adic field* K/L we will mean an algebraic extension such that $\mathcal{O}_K$ is a discrete valuation ring with perfect residue field; equivalently this means that K has a perfect residue field k and $K/W_L(k)[1/p]$ is finite. If K/L is a p-adic field and $n \geq 0$ we write

$$K_n = K\big(\mathcal{G}[\pi^n]\big)$$

for the extension given by adjoining the π^n-torsion points of $\mathcal{G}$. We also write K_∞ for the p-adic completion of $\bigcup K_n$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. The action of the absolute Galois group G_K on the free $\mathcal{O}_L$-rank one Tate module $T\mathcal{G}$ factors through Γ_K and gives an injective character $\chi_{\mathcal{G}}\colon \Gamma_K \to \mathcal{O}_L^\times$. If $K = L$ then by local class field theory $\chi_{\mathcal{G}}$ is an isomorphism $\Gamma_L \xrightarrow{\sim} \mathcal{O}_L^\times$.

Throughout this section, we fix once and for all

- a coordinate T on $\mathcal{G}$, so that the action of $\mathcal{O}_L$ on $\mathcal{G} \cong \mathrm{Spf}(\mathcal{O}_L[\![T]\!])$ is given by power series $[a](T) \in \mathcal{O}_L[\![T]\!]$ for $a \in \mathcal{O}_L$;
- a basis $e = (e_n)_{n\geq 0}$ of the free $\mathcal{O}_L$-module $T\mathcal{G}$, viewed as a sequence of $e_n \in \mathcal{O}_{\bar{K}}$ such that $[\pi](e_n) = e_{n-1}$, $e_0 = 0$, and $e_1 \neq 0$.

Note that $[\pi](T) \equiv T^q \pmod{\pi}$ and $K_n = K(e_n)$.

In Section 4.1, we will see that $\mathcal{O}_{\mathcal{G}} \otimes_{\mathcal{O}_L} W_L(k) \cong W_L(k)[\![T]\!]$ is a L-typical prism in $(W_L(k))_{\mathbb{\Delta}_L}$, and that our choice of basis $e \in T\mathcal{G}$ allows us to naturally view this prism inside of the perfect prism $A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ via the constructions in Section 3.3. Though the map $W_L(k)[\![T]\!] \to A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ depends on the choices of T and e, its image does not, and we denote this image by $\mathfrak{S}_{K/L}$. In Section 4.2, we extend $\mathfrak{S}_{K/L}$ to a prism $\mathbf{A}^+_{K/L}$ in $(\mathcal{O}_K)_{\mathbb{\Delta}_L}$ using the theory of imperfect norm fields. In Section 4.3 we recall the definition of Lubin–Tate (φ_q, Γ)-modules. Later, in Section 5.3 we will see that $\mathbf{A}^+_{K/L}$ and $A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ are "large enough" that Laurent F-crystals over $(\mathcal{O}_K)_{\mathbb{\Delta}_L}$ are equivalent to (φ_q, Γ_K)-modules over $\mathbf{A}_{K/L}$ or $W_L(K_\infty^\flat)$.

Remark 4.1. We note that the period rings $\mathfrak{S}_{K/L}, \mathbf{E}_{K/L}, \mathbf{A}^+_{K/L}$, etc. in this section depend not only on the field extension K/L but also on the choice of Lubin–Tate formal group $\mathcal{G}$. However, to avoid further cluttering notation, we choose to leave this dependence tacit.

4.1. Construction of $\mathfrak{S}_{K/L}$

In this subsection we construct prisms $(\mathfrak{S}_{K/L}, (q_n(\omega))) \in (W_L(k))_{\mathbb{\Delta}_L}$ for $n \geq 1$; when $n = 1$, this prism plays the same role that the q-de Rham prism $(\mathbb{Z}_p[\![q-1]\!], \frac{q^p-1}{q-1})$ plays in the cyclotomic theory. By construction we will have that $\mathfrak{S}_{K/L}$ is a sub-δ_L-algebra of $W_L(\mathcal{O}_{K_\infty}^\flat)$, so that the map $\phi^{-1}\colon \mathfrak{S}_{K/L} \to W_L(\mathcal{O}_{K_\infty}^\flat)$ makes sense, for $\phi = \phi_{W_L(\mathcal{O}_{K_\infty}^\flat)}$. We will also show that $\mathcal{O}_{K_\infty}$ is a perfectoid $\mathcal{O}_L$-algebra and that the map

$$\phi^{-n}\colon \big(\mathfrak{S}_{K/L}, \big(q_n(\omega)\big)\big) \to \big(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta\big)$$

is a map of prisms.

Fix all notation as at the start of this section, and equip

$$W_L(k)[\![T]\!] \cong W_L(k) \otimes_{\mathcal{O}_L} \mathcal{O}_L[\![T]\!]$$

with the lift ϕ of q-Frobenius coming from the natural ϕ-actions on $W_L(k)$ and $\mathcal{O}_L[\![T]\!] \cong \mathcal{O}_{\mathcal{G}}$; explicitly this is given by

$$f(T) \mapsto f^{\phi_{W_L(k)}}\big([\pi](T)\big)$$

where $f^{\phi_{W_L(k)}}$ denotes the coefficient-wise action on $f \in W_L(k)[\![T]\!]$. For $n \geq 1$ set

$$q_n(T) = \frac{[\pi^n](T)}{[\pi^{n-1}](T)} \in \mathcal{O}_L[\![T]\!].$$

Lemma 4.2. *$(W_L(k)[\![T]\!], (q_n(T)))$ is a L-typical prism in $(W_L(k))_{\Delta_L}$ for every $n \geq 1$.*

Proof. For $(\pi, q_n(T))$-adic completeness use that $q_n(T) \equiv T^{q^n - q^{n-1}} \pmod{\pi}$ and the (π, T)-adic completeness of $W_L(k)[\![T]\!]$. To see that

$$\pi \in \big(q_n(T), \phi\big(q_n(T)\big)\big) = \big(q_n(T), q_{n+1}(T)\big),$$

note that $q_1(T) = \frac{[\pi](T)}{T} \equiv \pi \pmod{T}$ so that

$$\pi = q_1(T) + Tf(T)$$

for some $f(T) \in \mathcal{O}_L[\![T]\!]$; applying ϕ^n then gives

$$\begin{aligned}\pi &= q_{n+1}(T) + [\pi^n](T) f\big([\pi^n](T)\big) \\ &= q_{n+1}(T) + q_n(T)[\pi^{n-1}](T) f\big([\pi^n](T)\big) \in \big(q_n(T), q_{n+1}(T)\big).\end{aligned}$$

∎

We now invoke Construction 3.20 to produce a map $W_L(k)[\![T]\!] \to W_L(\mathcal{O}_{K_\infty}^\flat)$ of δ_L-algebras. Since we have not yet shown that $\mathcal{O}_{K_\infty}$ is a perfectoid $\mathcal{O}_L$-algebra, we cannot yet use Construction 3.24, but once we have shown that $\mathcal{O}_{K_\infty}$ is perfectoid the two constructions amount to the same thing.

Indeed, apply Construction 3.20 with $A = W_L(k)[\![T]\!]$, $R = \mathcal{O}_{K_\infty}$, and $\iota_n \colon W_L(k)[\![T]\!] \to R$ defined by sending $f(T)$ to $f^{\phi^{-n}_{W_L(k)}}(e_n)$. Note that the system $(\iota_n)_n$ is ϕ-compatible since

$$\iota_{n+1}\big(\phi(f)\big) = \iota_{n+1}\big(f^{\phi_{W_L(k)}}\big([\pi](T)\big)\big) = f^{\phi^{-n}_{W_L(k)}}\big([\pi](e_{n+1})\big) = \iota_n(f),$$

where we have used that ϕ acts as the identity on $\mathcal{O}_L$ and $[\pi](e_{n+1}) = e_n$. This gives us a map δ_L-algebra map $\iota \colon W_L(k)[\![T]\!] \to W_L(\mathcal{O}_{K_\infty}^\flat)$ lifting the map $k[\![T]\!] \to \mathcal{O}_K^\flat$ given by

$$T \mapsto \overline{\omega} := (\ldots, \overline{e_2}, \overline{e_1}, 0)$$

where $\overline{e_n} \in \mathcal{O}_{K_\infty}/\pi$ is the mod π reduction of $e_n \in \mathcal{O}_{K_n}$ for $n \geq 0$. Let $\omega = \iota(T) \in W_L(\mathcal{O}_{K_\infty}^\flat)$ denote the given lift of $\overline{\omega}$, and write $\mathfrak{S}_{K/L} = W_L(k)[\![\omega]\!] \subseteq W_L(\mathcal{O}_{K_\infty})$ for the image of ι. By Lemma 4.2, $(\mathfrak{S}_{K/L}, q_n(\omega))$ is a L-typical prism for every $n \geq 1$.

Remark 4.3. When $n = 1$ and $\mathscr{G} = \mu_{p^\infty}$, we see that $q_1(T) = \frac{(1+T)^p-1}{T}$. After a change of variables $T \mapsto T - 1$, we thus get the q-de Rham prism.

Remark 4.4. A different choice of coordinate on $\mathscr{G}$ amounts to changing T by a unit in $\mathcal{O}_L[\![T]\!]$, and a different choice of basis for the Tate module of $\mathscr{G}$ corresponds to multiplying e by a unit in $\mathcal{O}_L$. Hence changing T and e results in changing ω by a unit but does not change the image $\mathfrak{S}_{K/L}$ of $W_L(k)[\![T]\!]$ in $A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$.

Lemma 4.5. *$\mathcal{O}_{K_\infty}$ is a perfectoid $\mathcal{O}_L$-algebra, and*

$$\big(\mathfrak{S}_{K/L}, \big(q_n(\omega)\big)\big) \xrightarrow{\phi^{-n}} \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big)$$

is a map of prisms for every $n \geq 1$.

Proof. Note first that by Proposition 3.22, we have $\phi^{-n}(q_n(\omega)) \in \ker\theta$ since

$$\theta\big(\phi^{-n}\big(q_n(\omega)\big)\big) = (\theta \circ \phi^{-n} \circ \iota)\big(q_n(T)\big) = \iota_n\big(q_n(T)\big) = \frac{[\pi^n](e_n)}{[\pi^{n-1}](e_n)} = 0.$$

Thus, we will be done if we show that $(W_L(\mathcal{O}_{K_\infty}^\flat), \ker\theta)$ is a prism (equivalently, that $\mathcal{O}_{K_\infty}$ is a perfectoid $\mathcal{O}_L$-algebra). One way to proceed would be to use a rigidity result like [8, Lemma 3.6]. Instead, we will use Proposition 3.15; O_{K_∞} clearly satisfies conditions (1), (3), and (4'), so it suffices to show that it satisfies condition (2) as well.

Let $d = \phi^{-n}(q_n(\omega)) \in \ker\theta$. Following the proof of Proposition 3.15, we guess that $\xi = \theta(\phi^{-1}(d)) \in \mathcal{O}_{K_\infty}$ satisfies $\xi^d = \pi u$ for a unit $u \in \mathcal{O}_{K_\infty}^\times$. Indeed, we have

$$\xi^d = \theta\big(\phi^{-q}(d^q)\big) = \theta\big(d - \pi\delta_L\big(\phi^{-1}(d)\big)\big) = \pi\theta\big(-\delta_L\big(\phi^{-1}(d)\big)\big),$$

and since $q_n(T) \in \mathfrak{S}_{K/L}$ is distinguished, so is

$$\phi^{-1}(d) = \phi^{-n-1}\iota\big(q_n(T)\big) \in W_L(\mathcal{O}_{K_\infty}^\flat). \qquad \blacksquare$$

Remark 4.6. Though we will not use this fact, we note that in this setting, there is an analytic way to construct the map $\iota\colon W_L(k)[\![T]\!] \to W_L(\mathcal{O}_{K_\infty}^\flat)$. Namely, following [31, Lemma 1.2], we let $\hat{\omega}$ be any lift of $\bar{\omega} = (\ldots, \overline{e_2}, \overline{e_1}, 0) \in \mathcal{O}_{K_\infty}^\flat$, and set

$$\omega = \lim_{n\to\infty} [\pi^n]\big(\phi^{-n}_{W_L(\mathcal{O}_{K_\infty}^\flat)}(\hat{\omega})\big).$$

Then $\phi(\omega) = [\pi](\omega)$, so that

$$\begin{aligned} W_L(k)[\![T]\!] &\to W_L(\mathcal{O}_{K_\infty}^\flat) \\ T &\mapsto \omega \end{aligned}$$

is a δ_L-algebra map lifting the $\mathcal{O}_L$-algebra map $W_L(k)[\![T]\!] \to \mathcal{O}_{K_\infty}^\flat$ via $T \mapsto \bar{\omega}$, hence it coincides with ι by Lemma 2.5.

4.2. Extension to $\mathbf{A}^+_{K/L}$

The prisms $(\mathfrak{S}_{K/L}, (q_n(\omega)))$ from Section 4.1 can be viewed as objects in $(W_L(k)[e_n])_{\Delta_L}$. However, when there is ramification in K/L outside of the ramification in L_∞/L (i.e. $\mathcal{O}_K \not\subseteq \bigcup_{n\geq 0} W_L(k)[e_n]$) then $(\mathfrak{S}_{K/L}, (q_n(\omega)))$ will never be a prism over $\operatorname{Spf} \mathcal{O}_K$. In this section, we will extend $\mathfrak{S}_{K/L}$ to a larger sub-δ_L-algebra $\mathbf{A}^+_{K/L}$ of $A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ which is a prism over $\operatorname{Spf} \mathcal{O}_K$. The key point is that the formation of $\mathfrak{S}_{K/L}$ is insensitive to taking ramified extensions of K; we capture ramification coming from the tower L_∞/L by our choice of n (since $\mathfrak{S}_{K/L}/q_n(\omega) \cong W_L(k)[e_n]$), but capturing the rest of the ramification in $K/W_L(k)[1/\pi]$ requires Fontaine and Wintenberger's theory of imperfect norm fields.

Let

$$\mathbf{E}^+_{K/L} = \left\{(\alpha_n)_n \in \varprojlim_{\varphi_q} \mathcal{O}_{K_\infty}/e_1 = \mathcal{O}^\flat_{K_\infty} : \alpha_n \in \mathcal{O}_{K_n}/e_1 \text{ for } n \gg 0\right\} \subseteq \mathcal{O}^\flat_{K_\infty},$$

so that $\overline{\omega} = (\dots, \overline{e_2}, \overline{e_1}, 0) \in \mathbf{E}^+_{K/L}$. We recall some facts from the theory of norm fields [45].

Proposition 4.7. *The following properties of $\mathbf{E}^+_{K/L}$ hold.*

(1) $\mathbf{E}^+_{K/L}$ *is a complete discrete valuation ring with fraction field*

$$\mathbf{E}_{K/L} := \mathbf{E}^+_{K/L}[1/\overline{\omega}] \subseteq K^\flat_\infty.$$

(2) *If K/L is unramified, then $\mathbf{E}^+_{K/L} = k[\![\overline{\omega}]\!]$. In general, $\mathbf{E}_{K/L}$ is a totally ramified extension of $\mathbf{E}_{W_L(k)[1/\pi]/L}$ of degree $[K_n : W_L(k)[e_n][1/\pi]]$ for n large enough.*

(3) *The completed perfection $(\varinjlim_{\varphi_q} \mathbf{E}^+_{K/L})^\wedge_{(\overline{\omega})}$ of $\mathbf{E}^+_{K/L}$ is $\mathcal{O}^\flat_{K_\infty}$.*

(4) *There is an equivalence of Galois categories*

$$\left\{\text{finite extensions of } \bigcup_{n\geq 1} L_n \text{ in } \overline{L}\right\} \simeq \{\text{finite separable extensions of } \mathbf{E}_{L/L} \text{ in } K^\flat_\infty\}$$

where, given a finite subextension $M/\bigcup_{n\geq 1} L_n$ of $\overline{L}$, the functor from the left to the right is given by selecting any finite M'/L with $\bigcup_n M'_n = M$ and sending M to $\mathbf{E}_{M'/L}$.

Proof. See [13] or [12, Sections 13.3 and 13.4]. ∎

We would like to form Cohen rings $\mathbf{A}_{K/L}$ for the fields $\mathbf{E}_{K/L}$ in characteristic p. For $K = W_L(k)[1/\pi]$ unramified over L, we write

$$\mathbf{A}_{K/L} = \mathfrak{S}_{K/L}[1/\omega]^\wedge_{(\pi)} \subseteq W_L(K^\flat_\infty)$$

for the π-adic completion of $\mathfrak{S}_{K/L}[1/\omega] \cong W_L(k)[\![T]\!][1/T]$. Then $\mathbf{A}_{K/L}$ is a complete discrete valuation ring in characteristic zero with uniformizer π, and by Proposition 4.7 $\mathbf{A}_{K/L}$ has residue field $\mathbf{E}_{K/L}$. When K/L is possibly ramified, Hensel's lemma allows us to lift the extension $\mathbf{E}_{K/L}$ of $\mathbf{E}_{W_L(k)[1/\pi]/L} \cong k((T))$ to an unramified extension $\mathbf{A}_{K/L}$ of

$\mathbf{A}_{W_L(k)[1/\pi]/L} \cong W_L(k)[\![T]\!][1/T]^\wedge$ inside of $W_L(K_\infty^\flat)$:

$$\begin{tikzcd}
W_L(k)[\![T]\!][1/T]^\wedge \arrow[r, "\iota", "\sim"'] \arrow[d, two heads] & \mathbf{A}_{W_L(k)[1/\pi]/L} \arrow[r, hook] \arrow[d, two heads] & \mathbf{A}_{K/L} \arrow[r, hook] \arrow[d, two heads] & W_L(K_\infty^\flat) \arrow[d, two heads] \\
k((T)) \arrow[r, "\bar{\iota}", "\sim"'] & \mathbf{E}_{W_L(k)[1/\pi]/L} \arrow[r, hook] & \mathbf{E}_{K/L} \arrow[r, hook] & K_\infty^\flat
\end{tikzcd}$$

By construction, $\mathbf{A}_{K/L}$ is stable under $\phi_{W_L(K_\infty^\flat)}$ (since $\phi(a) \mod \pi = \bar{a}^q \in \mathbf{E}_{K/L}$ for any $a \in \mathbf{A}_{K/L}$). To form a subring of $\mathbf{A}_{K/L}$ lifting $\mathbf{E}_{K/L}^+$, we set $\mathbf{A}_{K/L}^+$ to be the π-adic completion of the integral closure of $\mathfrak{S}_{K/L}$ in $\mathbf{A}_{K/L}$. Since $\mathbf{E}_{K/L}^+$ is the ring of integers of $\mathbf{E}_{K/L}$ by Proposition 4.7 (1), we have that $\mathbf{A}_{K/L}^+/\pi \cong \mathbf{E}_{K/L}^+$. Moreover, as $\mathbf{A}_{K/L}^+$ is ϕ-stable and π-torsion-free, it has a δ_L-algebra structure compatible with that on $\mathbf{A}_{K/L}$.

Remark 4.8. Instead of forming $\mathbf{A}_{K/L}$ by lifting the extension $\mathbf{E}_{K/L}/\mathbf{E}_{W_L(k)[1/\pi]/L}$ over $\mathbf{A}_{W_L(k)[1/\pi]/L}$, we could have instead lifted the extension $\mathbf{E}_{K/L}/\mathbf{E}_{L/L}$ over $\mathbf{A}_{L/L}$. This would have amounted to the same thing. We also note that the ϕ-action on $\mathbf{A}_{K/L}$ above clearly coincides with the one induced by lifting $\varphi_q\colon \mathbf{E}_{K/L} \to \mathbf{E}_{K/L}$ via Hensel's lemma (using that $\mathbf{A}_{W_L(k)[1/\pi]/L}$ is ϕ-stable by construction).

Remark 4.9. Note that $\mathbf{A}_{K/L} = \mathbf{A}_{K/L}^+[1/q_n(\omega)]_{(\pi)}^\wedge$ since $q_n(\omega) \equiv \omega^{q^{n-1}(q-1)} \pmod{\pi}$, so that after π-adically completing, inverting ω has the same effect as inverting $q_n(\omega)$.

Lemma 4.10. *We have the following.*

(1) *If $A \to B$ is a map of π-adically complete π-torsion-free rings with A noetherian and $A/\pi \to B/\pi$ is flat, then $A \to B$ is flat as well.*

(2) *The maps*

$$\mathfrak{S}_{K/L} \hookrightarrow \mathbf{A}_{K/L}^+, \quad \mathbf{A}_{K/L}^+ \hookrightarrow A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \quad \textit{and} \quad \phi\colon \mathbf{A}_{K/L}^+ \to \mathbf{A}_{K/L}^+$$

are all faithfully flat.

(2) *$\mathfrak{S}_{K/L}/q_n(\omega)$ and $\mathbf{A}_{K/L}^+/q_n(\omega)$ are π-torsion-free. $(\mathfrak{S}_{K/L}, (q_n(\omega)))$ and $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$ are bounded.*

Proof. Part (1) is [7, Remark 4.31] with p replaced by π; the proof remains the same. The flatness in (2) follows from (1) since the mod π reductions of the given maps are

$$k[\![T]\!] \hookrightarrow \mathbf{E}_{K/L}^+, \quad \mathbf{E}_{K/L}^+ \hookrightarrow \mathcal{O}_{K_\infty}^\flat, \quad \text{and} \quad \varphi_q\colon \mathbf{E}_{K/L}^+ \to \mathbf{E}_{K/L}^+$$

which are injective maps from discrete valuation rings to integral domains, hence flat; note that $\mathbf{A}_{K/L}^+$ is noetherian by [43, Tags 05GH, 0DYC]. Faithful flatness follows since ϖ is not a unit in $\mathbf{E}_{K/L}^+$ or $\mathcal{O}_{K_\infty}^\flat$. For (3), we have that $\mathfrak{S}_{K/L}/q_n(\omega) \cong W_L(k)[\![T]\!]/q_n(T) \cong W_L(k)[e_n]$ is an integral domain hence π-torsion-free. By part (2) we have that

$$\mathfrak{S}_{K/L}/q_n(\omega) \to \mathbf{A}_{K/L}^+/q_n(\omega)$$

is flat, so $\mathbf{A}_{K/L}^+/q_n(\omega)$ is π-torsion-free as well. ∎

It follows immediately from Lemma 4.2 that $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$ is a L-typical prism for every $n \geq 1$. Moreover, just as $\mathbf{E}_{K/L}^+$ can be viewed as a deperfection of $\mathcal{O}_{K_\infty}^\flat$, we have that the prism $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$ can be viewed as a deperfection of the perfect prism $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$.

Proposition 4.11. *Let $(A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ be the perfection of $(\mathbf{A}_{K/L}^+, (q_n(\omega)))$ as in Proposition 3.14. Then $A_{\mathrm{perf}} \cong A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$. The natural map $\mathbf{A}_{K/L}^+ \to A_{\mathrm{perf}} \cong A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ is the usual inclusion, i.e. the map on the left in the following commutative diagram.*

$$\begin{tikzcd}[column sep=small]
\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \big(q_n(\omega)\big)\big) \arrow[rr, "\phi^{-n}", "\sim"'] & & \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \\
& \big(\mathbf{A}_{K/L}^+, \big(q_n(\omega)\big)\big) \arrow[ul, "\subseteq"] \arrow[ur, "\phi^{-n}"'] &
\end{tikzcd}$$

Proof. By Proposition 2.14, it suffices to show that $A_{\mathrm{perf}}/\pi \cong \mathcal{O}_{K_\infty}^\flat$. Indeed, we have

$$A_{\mathrm{perf}}/\pi = \big(\varinjlim_{\phi} \mathbf{A}_{K/L}^+\big)^\wedge_{(q_n(\omega))}/\pi \cong \big(\varinjlim_{\varphi_q} \mathbf{E}_{K/L}^+\big)^\wedge_{(\bar\omega)} = \mathcal{O}_{K_\infty}^\flat$$

since $q_n(\omega) \equiv \omega^{q^n - q^{n-1}} \pmod{\pi}$ and modding out by π commutes with the colimit and $(\pi, q_n(\omega))$-adic completion. ■

Corollary 4.12. *For $n \gg 0$ we have structure maps $\mathcal{O}_K \to \mathbf{A}_{K/L}^+/q_n(\omega)$ such that the maps*

$$\big(\mathbf{A}_{K/L}^+, \big(q_n(\omega)\big)\big) \xrightarrow{\phi^{-n}} \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big)$$

are morphisms in $(\mathcal{O}_K)_{\Delta_\pi}$.

We will give two proofs, the first an abstract argument following [46, Proposition 2.19] and the second a more concrete argument involving norm fields.

Proof 1. By Proposition 4.11 we have an isomorphism

$$\Big(\varinjlim_{\phi} \mathbf{A}_{K/L}^+\Big)^\wedge_{(\pi, q_1(\omega))}/q_1(\omega) \xrightarrow{\phi^{-1}} A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})/\ker\theta = \mathcal{O}_{K_\infty}.$$

This isomorphism can be rewritten as

$$\Big(\varinjlim_{\phi} \mathbf{A}_{K/L}^+/q_n(\omega)\Big)^\wedge_{(p)} \xrightarrow{\sim} \Big(\bigcup_{n\geq 1} \mathcal{O}_{K_n}\Big)^\wedge_{(p)}.$$

Using that $\varinjlim \mathbf{A}_{K/L}^+/q_n(\omega)$ and $\bigcup \mathcal{O}_{K_n}$ are integral over $W_L(k)$ and that there are no integral extensions between $\bigcup \mathcal{O}_{K_n}$ and its completion $\mathcal{O}_{K_\infty}$ (by Krasner's lemma applied to the Henselian ring $\bigcup \mathcal{O}_{K_n}$), we conclude that there is a short exact sequence

$$0 \to \varinjlim_{\phi} \mathbf{A}_{K/L}^+/q_n(\omega) \to \bigcup \mathcal{O}_{K_n} \to M \to 0$$

with M π-torsion and $M^{\wedge}_{(\pi)} = 0$. Moreover, since $\varinjlim \mathbf{A}^+_{K/L}/q_n(\omega)$ contains the subring

$$\varinjlim \mathfrak{S}_{K/L}/q_n(\omega) \cong \bigcup W_L(k)[e_n]$$

over which $\bigcup \mathcal{O}_{K_n}$ is finite, we have that M is π-adically complete, so that $M = M^{\wedge} = 0$.

Moreover, since $\mathfrak{S}_{K/L}/q_n(\omega) \xrightarrow{\phi} \mathfrak{S}_{K/L}/q_{n+1}(\omega)$ identifies with the inclusion

$$W_L(k)[e_n] \hookrightarrow W_L(k)[e_{n+1}]$$

and $\mathfrak{S}_{K/L} \hookrightarrow \mathbf{A}^+_{K/L}$ is flat by Lemma 4.10, the transition maps in the direct limit are injective as well. All together, this gives

$$\bigcup_{n \geq 1} \mathbf{A}^+_{K/L}/q_n(\omega) \cong \bigcup_{n \geq 1} \mathcal{O}_{K_n} \supseteq \mathcal{O}_K.$$

As $\mathcal{O}_K$ is finite over $W_L(k)$ and the left-hand side is an increasing union of $W_L(k)$-modules, we get maps

$$\mathcal{O}_K \to \mathbf{A}^+_{K/L}/q_n(\omega)$$

for $n \gg 0$. These maps commute with $\phi^{-n}\colon \mathbf{A}^+_{K/L} \to A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})$ by construction. ∎

Proof 2. To simplify notation, set $F = W_L(k)[1/\pi]$. Let

$$\overline{\omega}_K = (\overline{\pi}_n)_n \in \varprojlim_{\varphi_q} \mathcal{O}_{K_\infty}/e_1 = \mathcal{O}^\flat_{K_\infty}$$

be a uniformizer of $\mathbf{E}^+_{K/L}$, so that $\overline{\pi}_n \in \mathcal{O}_{K_n}/e_1$ for $n \gg 0$ and $\mathbf{E}^+_{K/L} = k[\![\overline{\omega}_K]\!]$. Let $P(W,T) \in k[\![W]\!][T]$ so that $P(\overline{\omega},T) \in k[\![\overline{\omega}]\!][T] = \mathbf{E}^+_{F/L}[T]$ is the minimal polynomial of $\overline{\omega}_K$ over $\mathbf{E}_{F/L}$. As explained above, $\mathbf{E}_{K/L}/\mathbf{E}_{F/L}$ is a totally ramified extension of degree $d = [K_\infty : F_\infty]$, so $P(\overline{\omega},T)$ is a degree d Eisenstein polynomial. Since $\overline{\omega} = (\overline{e_n})_n$, it follows that $P(\overline{e_n}, \overline{\pi_n}) = 0 \in \mathcal{O}_{K_n}/e_1$ for $n \gg 0$.

Let $\widehat{P}(W,T) \in \mathcal{O}_F[\![W]\!][T]$ be a lift of P. Using an argument involving Lang's refinement of Hensel's lemma, it is shown in [45, 3.2.5] (or see also [12, Proof of Proposition 13.4.4]) that $P(e_n,T)$ has d distinct roots $\{\pi_{n,1},\ldots,\pi_{n,d}\}$ in $\mathcal{O}_{\overline{K}}$; one of these roots, call it π_n, is a lift of $\overline{\pi}_n$. Moreover, using that the roots of $P(\overline{\omega},T)$ in $\mathcal{O}^\flat_{K_\infty}$ are distinct, one can show that $\pi_{n,1},\ldots,\pi_{n,d}$ are distinct mod e_1 for $n \gg 0$. On the other hand, since $\pi^q_{n+1} \equiv \pi_n \pmod{e_1}$ for $n \gg 0$, Krasner's lemma shows that for $n \gg 0$ we have $\pi_n \in F_n(\pi_{n+1})$.

Now, set $K'_n = F_n(\pi_n)$. For some $N \gg 0$ and all $n \geq N$, we have that $K'_n \subseteq K'_{n+1}$. This gives us the following diagram of field extensions, with degrees indicated:

$$\begin{tikzcd}
& K'_{n+1} \arrow[dd, dash] \\
F_{n+1} \arrow[ur, dash, "d"] \arrow[dd, dash, "q"'] & \\
& K'_n \\
F_n \arrow[ur, dash, "d"] &
\end{tikzcd}$$

This implies that $K'_{n+1} = K'_n F_{n+1}$, and thus that $K'_n = K'_N F_n$ for all $n \geq N$. We also have that $\mathbf{E}_{K'_N/L} = \mathbf{E}_{K/L}$, since they are both degree $d = [K_\infty : F_\infty] = [K'_{N,\infty} : F_\infty]$ extensions of $\mathbf{E}_{F/L}$ and $\mathbf{E}_{K/L} = k((\bar{\omega}_K)) \subseteq \mathbf{E}_{K'/L}$ by construction. Thus by Proposition 4.7 (4), we get that $\bigcup_n K'_{N,n} = \bigcup_n K_n$, so that for $n \gg 0$, we have $K \subseteq K'_{N,n}$. Hence for $n \gg 0$,

$$\mathcal{O}_K \subseteq \mathcal{O}_{K'_{N,n}} = \mathcal{O}_F[e_n][\pi_n] = \mathcal{O}_F[\![\omega]\!][T]/\big(q_n(\omega), \hat{P}(\omega, T)\big) = \mathbf{A}_{K/L}/\big(q_n(\omega)\big)$$

as desired.

Using the formula $\theta(\phi^{-n}(\omega)) = e_n$ (which follows from Proposition 3.22), we can trace the inclusion above through the map $\phi^{-n}\colon \mathbf{A}_{K/L}/(q_n(\omega)) \to A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})/\ker\theta$ to find that it coincides with $\mathcal{O}_K \subseteq \mathcal{O}_{K_\infty} \cong A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})/\ker\theta$. ∎

4.3. Γ_K-actions and étale (φ_q, Γ)-modules

In this section we summarize the main results in the theory of Lubin–Tate (φ_q, Γ)-modules. These results will be recovered as special cases of the results in Section 5.

Definition 4.13. *A φ_q-module over $\mathbf{A}_{K/L}$* is a finite flat $\mathbf{A}_{K/L}$ module M equipped with a $\phi_{\mathbf{A}_{K/L}}$-semilinear endomorphism $\phi_M\colon M \to M$. It is called *étale* if the $\mathbf{A}_{K/L}$-linear map

$$\begin{aligned} \phi^* M := \mathbf{A}_{K/L} \otimes_{\phi, \mathbf{A}_{K/L}} M &\to M \\ a \otimes m &\mapsto a\phi_M(m) \end{aligned}$$

is an isomorphism. When equipped with $\mathbf{A}_{K/L}$-module maps that commute with the ϕ_M's, these form categories $\mathrm{Mod}^{\varphi_q}_{\mathbf{A}_{K/L}}$ and $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}}$. We similarly define φ_q-modules over $W_L(K^\flat_\infty)$ and the categories $\mathrm{Mod}^{\varphi_q}_{W_L(K^\flat_\infty)}$ and $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K^\flat_\infty)}$.

By a result of Kisin–Ren [31] (and Fontaine [17] in the cyclotomic case), we have that étale φ_q-modules are equivalent to the category $\mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty})$ of continuous finite free $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$-representations over $\mathcal{O}_L$. In more detail, Proposition 4.7 (4) implies that $\mathbf{E} := \bigcup_{K/L} \mathbf{E}_{K/L}$ is the separable closure of $\mathbf{E}_{L/L}$, and that $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_{K/L}) = G_{K_\infty}$. It follows that $\mathbf{A} := (\bigcup_{K/L} \mathbf{A}_{K/L})^\wedge$ is the completion of the maximal unramified extension of $\mathbf{A}_{L/L}$; $\mathbf{A}$ thus inherits a $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_{L/L}) = G_{L_\infty}$-action with $\mathbf{A}^{G_{K_\infty}} = \mathbf{A}_{K/L}$.[2] Moreover, $\mathbf{A} \subseteq W_L(K^\flat_\infty)$ has a ϕ-action. The key theorem is as follows.

Theorem 4.14 (cf. [31, Theorem 1.6]). *The functors*

$$\begin{tikzcd} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}} \arrow[r, bend left, "M \mapsto (M \otimes_{\mathbf{A}_{K/L}} \mathbf{A})^{\phi=1}"] & \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty}) \arrow[l, bend left, "(T \otimes_{\mathcal{O}_L} \mathbf{A})^{G_{K_\infty}} \leftarrow\!\shortmid T"] \end{tikzcd}$$

form an equivalence of exact tensor categories.

[2] We warn that despite the notation, $\mathbf{E}$ and $\mathbf{A}$ still depend on the choice of L.

We make two observations about Theorem 4.14. First, that the base of the φ_q-modules is $\mathbf{A}_{K/L}$. In fact, this is a red herring: base change induces an equivalence of categories

$$\begin{aligned} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}} &\xrightarrow{\sim} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K^\flat_\infty)} \\ M &\mapsto M \otimes_{\mathbf{A}_{K/L}} W_L(K^\flat_\infty) \end{aligned}$$

so that theorem would remain true with $W_L(K^\flat_\infty)$ replacing $\mathbf{A}_{K/L}$ (and $W_L(\bar{K}^\flat)$ replacing $\mathbf{A}$). (This result is due to Fontaine [17] in the cyclotomic case, but, as far as the author is aware, the general case has not yet appeared in the literature; it will follow from Proposition 5.4 below.)

Our second observation is that we would like to descend the equivalence to the full category $\mathrm{Rep}_{\mathcal{O}_L}(G_K)$ of continuous finite free G_K-representations over $\mathcal{O}_L$. Indeed, this is not hard to do, and involves picking up a semilinear action of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Before stating the result, we first explain the Γ_K actions on the rings $\mathfrak{S}_{K/L}$, $\mathbf{A}_{K/L}$, and $W_L(K^\flat_\infty)$.

Equip $W_L(k)[\![T]\!]$ with the Γ_K-action where $\sigma \in \Gamma_K$ acts by $f(T) \mapsto f([\chi_{\mathcal{G}}(\sigma)](T))$, and equip $W_L(\mathcal{O}^\flat_{K_\infty})$ with the natural Γ_K-action (coming from the Γ_K-action on $K^\flat_\infty$ and the functoriality of W_L). By the definition of the Lubin–Tate character $\chi_{\mathcal{G}}$ we have

$$[\chi_{\mathcal{G}}(\sigma)](\bar{\omega}) = (\overline{e_n^\sigma})_n = \bar{\omega}^\sigma$$

so that $W_L(k)[\![T]\!] \twoheadrightarrow k[\![T]\!] \xrightarrow{\bar{\iota}} \mathcal{O}^\flat_{K_\infty}$ is Γ_K-equivariant. Thus ι is Γ_K-equivariant as well by naturality, and the Γ_K-actions on $\mathfrak{S}_{K/L}$ induced by ι and $W_L(K^\flat_\infty)$ coincide. By the uniqueness of lifts given by Hensel's lemma, we further have that the Γ_K-action on $\mathbf{A}_{K/L}$ induced by viewing it as a subring of $W_L(K^\flat_\infty)$ coincides with the Γ_K-action defined by lifting the Γ_K-action on $\mathbf{E}_{K/L}$. Note also that all of these Γ_K-actions commute with the ϕ-actions, because this is true for $W_L(K^\flat_\infty)$. This can also be seen directly for $W_L(k)[\![T]\!]$ using properties of Lubin–Tate formal $\mathcal{O}_L$-modules:

$$\phi(f)^\sigma(T) = f\big([\pi\chi_{\mathcal{G}}(\sigma)](T)\big) = f\big([\chi_{\mathcal{G}}(\sigma)\pi](T)\big) = \phi(f^\sigma)(T).$$

We can also view Γ_K as acting on the corresponding prisms.

Proposition 4.15. *Γ_K acts via automorphisms on the L-typical prisms $(\mathbf{A}^+_{K/L}, (q_n(\omega)))$ and $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$. Moreover, we have that*

$$\mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(\mathbf{A}^+_{K/L}, (q_n(\omega))\big) \cong \mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \cong \Gamma_K$$

if n is large enough that $(\mathbf{A}^+_{K/L}, (q_n(\omega))) \in (\mathcal{O}_K)_{\Delta_L}$.

Proof. Since any $\sigma \in \Gamma_K$ commutes with ϕ, we know that σ gives a map of δ_L-algebras. Additionally, since $q_n(\omega)^\sigma = [\chi_{\mathcal{G}}(\sigma)](q_n(\omega))$ and

$$[\chi_{\mathcal{G}}(\sigma)](T) = \chi_{\mathcal{G}}(\sigma)T + \text{higher order terms},$$

we have that any $\sigma \in \Gamma_K$ preserves $(q_n(\omega))$, and thus gives an automorphism of $(\mathbf{A}^+_{K/L}, (q_n(\omega)))$. If n is large enough that $(\mathbf{A}^+_{K/L}, (q_n(\omega))) \in (\mathcal{O}_K)_{\Delta_L}$ then σ respects the structure map $\mathcal{O}_K \to \mathbf{A}^+_{K/L}/q_n(\omega)$ as well, so that

$$\Gamma_K \hookrightarrow \mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(\mathbf{A}^+_{K/L}, (q_n(\omega))\big).$$

Moreover, we see that any automorphism of $(\mathbf{A}^+_{K/L}, (q_n(\omega)))$ is automatically $(\pi, q_n(\omega))$-adically continuous and ϕ-equivariant, hence extends to an automorphism of the perfection $(\mathbf{A}^+_{K/L}, (q_n(\omega)))_{\mathrm{perf}} \cong (A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$ by Proposition 4.11. But Proposition 3.11, we have that

$$\mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \cong \mathrm{Aut}(\mathcal{O}_{K_\infty}/\mathcal{O}_K) = \Gamma_K.$$

Thus we have shown

$$\Gamma_K \hookrightarrow \mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(\mathbf{A}^+_{K/L}, (q_n(\omega))\big) \hookrightarrow \mathrm{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \cong \Gamma_K$$

which gives the result. ∎

Descending from $\mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty})$ to $\mathrm{Rep}_{\mathcal{O}_L}(G_K)$ involves picking up a Γ_K-action.

Definition 4.16. A (φ_q, Γ)-*module over* $\mathbf{A}_{K/L}$ is a φ_q-module M over $\mathbf{A}_{K/L}$ with a semilinear Γ_K-action which commutes with the ϕ-action. It is *étale* if M is étale as a φ_q module. These form categories $\mathrm{Mod}^{\varphi_q,\Gamma}_{\mathbf{A}_{K/L}}$ and $\mathrm{Mod}^{\varphi_q,\Gamma,\mathrm{et}}_{\mathbf{A}_{K/L}}$. We similarly define (φ_q, Γ)-modules over $W_L(K^\flat_\infty)$.

The equivalence of Theorem 4.14 extends to (φ_q, Γ)-modules. So in summary, we have the following inclusions and equivalences among exact tensor categories.

$$\begin{array}{ccccc}
\mathrm{Rep}_{\mathbb{Z}_p}(G_K) & \xrightarrow{\cong} & \mathrm{Mod}^{\varphi,\Gamma,\mathrm{et}}_{\mathbf{A}_{K/L}} & \xrightarrow{\cong} & \mathrm{Mod}^{\varphi,\Gamma,\mathrm{et}}_{W(K^\flat_\infty)} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty}) & \xrightarrow{\cong} & \mathrm{Mod}^{\varphi,\mathrm{et}}_{\mathbf{A}_{K/L}} & \xrightarrow{\cong} & \mathrm{Mod}^{\varphi,\mathrm{et}}_{W(K^\flat_\infty)}
\end{array}$$

4.4. The prismatic logarithm for $\mathfrak{S}_{L/L}$

For convenience, throughout this section let

$$(A, I) = \big(\mathbf{A}^+_{L/L}, (q_n(\omega))\big) = \big(\mathfrak{S}_{L/L}, (q_n(\omega))\big) \cong \big(\mathcal{O}_L[\![T]\!], (q_n(T))\big)$$

be the prism of Section 4.1. We will construct a map $\log_\Delta$ from a certain subset $I_{\phi=[\pi]}$ of I to the Breuil–Kisin twist $A\{1\}$ of A. Heuristically, we can think of $\log_\Delta$ as being given by $\log_\Delta(u) =$ "$\lim_{n\to\infty} \frac{[\pi^n](u)}{\pi^n}$". We will further see that $\log_\Delta$ is $\mathcal{O}_L$-linear, where $I_{\phi=[\pi]}$ is viewed as an $\mathcal{O}_L$-module via the Lubin–Tate formal group law $\mathcal{G}$.

Remark 4.17. In the cyclotomic case $\mathcal{G} = \mu_{p^\infty}$, our $\log_\Delta$ coincides with the map $u \mapsto \log_\Delta(1+u)$ of [6, Section 2]. In that setting, $\log_\Delta(1+u) =$ "$\lim_{n\to\infty} \frac{u^{p^n}-1}{p^n}$", which is analogous to the classical formula $\log(1+x) = \lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha}$.

For this paragraph only, let (A, I) be an arbitrary bounded L-typical prism. Informally, we define

$$A\{1\} = \bigotimes_{n=0}^{\infty} (\phi^n)^* I.$$

More precisely, for $n \geq 1$ set I_n to be the product $\prod_{i=0}^{n-1} \phi^i(I)$ as an ideal of A. Note that $I_n \equiv I^{\frac{q^n-1}{q-1}} \pmod{\pi}$. Thus, since A is bounded and (π, I)-adically complete we have $\mathrm{Pic}(A) \simeq \lim_n \mathrm{Pic}(A/I_n)$, and we let $A\{1\} \in \mathrm{Pic}(A)$ correspond to $((\phi^n)^* I \otimes_A A/I_n)n \geq 0$. See [16, Section 4.9] for additional details, or [6, Section 2] for a more explicit construction bootstrapping from the case where A/I is π-torsion-free.

Taking $(A, I) = (\mathfrak{S}_{L/L}, (q_n(\omega)))$ once more, we give also a more explicit definition. We can define $A\{1\}$ by

$$\varprojlim \left(\cdots \xrightarrow{1/\pi} I_3/I_3^2 \xrightarrow{1/\pi} I_2/I_2^2 \xrightarrow{1/\pi} I_1/I_1^2 \right).$$

Here

$$I_m = \left(q_n(\omega) q_{n+1}(\omega) \cdots q_{n+m-1}(\omega)\right) = \left(\frac{[\pi^{n+m-1}](\omega)}{[\pi^{n-1}](\omega)} \right)$$

and the transition maps $I_{m+1}/I_{m+1}^2 \to I_m/I_m^2$ are quotienting by I_m^2 followed with dividing by π; these are well defined and surjective as

$$\frac{[\pi^{n+m}](\omega)}{[\pi^{n-1}](\omega)} \equiv \pi \frac{[\pi^{n+m-1}](\omega)}{[\pi^{n-1}](\omega)} \mod \left(\frac{[\pi^{n+m-1}](\omega)}{[\pi^{n-1}](\omega)} \right)^2$$

since $[\pi^{n+m}](\omega) = [\pi]([\pi^{n+m-1}](\omega))$ and $[\pi](T) = \pi T +$ (higher order terms).

Lemma 4.18. *The I_m satisfy the following properties.*

(1) *A/I_m is π-torsion-free for all $m \geq 1$.*

(2) *We have $I_m = \bigcap_{i=0}^{m-1} \phi^i(I) = \bigcap_{i=0}^{m} (q_{n+i}(\omega))$.*

Proof. We prove part (1) by induction. The result is clear for $m = 1$, and for $m \geq 2$ we have an exact sequence

$$0 \to I_m \otimes_A A/\phi^m(I) \cong I_m/I_{m+1} \to A/I_{m+1} \to A/I_m \to 0$$

where the first and third terms are π-torsion-free.

For part (2) we follow [6, Lemmas 2.2.8 and 2.2.9]. First, we show that the natural map $f : \phi^m(I)/I_{m+1} \to A/I_m$ is injective. As above, using the identity $[\pi](T) = \pi T +$ (higher order terms) one shows that the f has image containing (π, I_m); by part (1), f therefore factors as $f = \pi f_0$. We show that f_0 is an isomorphism; as the domain and codomain are invertible A/I_m modules, it suffices to show surjectivity. One shows by induction over $m \geq 1$ that if $\alpha \in I$ then $f_0(\phi^m(\alpha)) \mod (\pi, I)$ is a unit in $A/(\pi, I)$. Then by (π, I)-adic completeness and the inclusion $I_m \subseteq (\pi, I)$ we conclude that the image of $I \xrightarrow{\phi^m} \phi^m(I)/I_m \xrightarrow{f_0} A/I_m$ is the unit ideal as desired.

We now prove the statement in the lemma by induction on $m \geq 0$, with $m = 0$ being interpreted as the equality $(1) = (1)$ of unit ideals. For $m \geq 1$, let $\alpha \in \bigcap_{i=0}^{m} \phi^i(I)$. By induction, we have $\alpha \in I_m \cap \phi^m(I)$. Thus α is in the kernel of $\phi^m(I) \to A/I_m$, which factors as

$$\phi^m(I) \to \phi^m(I)/I_{m+1} \xrightarrow{f} A/I_m.$$

Since we showed that f is injective, we have that $\alpha \in I_{m+1}$ as desired. ∎

We now define $\log_{\Delta}$. Let $I_{\phi=[\pi]}$ denote the subset of $\alpha \in I$ such that $\phi(\alpha) = [\pi](\alpha)$. For example, we have that

$$[\pi^n](\omega) \in I_{\phi=[\pi]}$$

since $\phi([\pi^n](\omega)) = [\pi^n]([\pi](\omega)) = [\pi]([\pi^n](\omega))$.

Lemma 4.19. *If $\alpha \in I_{\phi=[\pi]}$ and $m \geq 1$ then $[\pi^m](\alpha) \in I_{m+1}$ and $[\pi^m](\alpha) \equiv \pi \cdot [\pi^{m-1}](\alpha)$ $\pmod{I_m^2}$.*

Proof. The second part of the lemma is clear from $[\pi](T) = \pi T +$ (higher order terms). For the first part, for each $0 \leq i \leq m$ we have $[\pi^m](\alpha) = [\pi^{m-i}]([\pi^i](\alpha)) \in \phi^i(I)$. Thus $[\pi^m](\alpha) \in I_{m+1}$ by Lemma 4.18 (2). ∎

Definition 4.20. Let $\log_{\Delta}\colon I_{\phi=[\pi]} \to \mathfrak{S}_{L/L}\{1\}$ be defined by

$$\log_{\Delta}(\alpha) = \big([\pi^{m-1}](\alpha)\big)_{m\geq 1} = \big(\phi^{m-1}(\alpha)\big)_{m\geq 1} \in \varprojlim_{1/\pi} I_m/I_m^2 = \mathfrak{S}_{L/L}\{1\}.$$

Recall that the Lubin–Tate formal $\mathcal{O}_L$-module $\mathcal{G}$ comes with a formal group law $X +_{\mathcal{G}} Y \in \mathcal{O}_L[\![X, Y]\!]$ satisfying

$$X +_{\mathcal{G}} Y = X + Y + (\text{degree} \geq 2 \text{ terms}), \tag{4.1}$$

$$[a](X +_{\mathcal{G}} Y) = [a](X) +_{\mathcal{G}} [a](Y) \quad \text{for } a \in \mathcal{O}_L. \tag{4.2}$$

This second condition with $a = \pi$ implies that if $\alpha, \beta \in I_{\phi=[\pi]}$ then $\alpha +_{\mathcal{G}} \beta \in I_{\phi=[\pi]}$ as well. Similarly, we have that if $\alpha \in I_{\phi=[\pi]}$ and $a \in \mathcal{O}_L$ then $[a](\alpha) \in I_{\phi=[\pi]}$. Thus $I_{\phi=[\pi]}$ can be viewed as an $\mathcal{O}_L$-module. We show that $\log_{\Delta}$ is an $\mathcal{O}_L$-module homomorphism.

Proposition 4.21. *For $\alpha, \beta \in I_{\phi=[\pi]}$ and $a \in \mathcal{O}_L$ we have $\log_{\Delta}(\alpha +_{\mathcal{G}} \beta) = \log_{\Delta}(\alpha) + \log_{\Delta}(\beta)$ and $\log_{\Delta}([a](\alpha)) = a \log_{\Delta}(\alpha)$.*

Proof. We have

$$\begin{aligned}\log_{\Delta}(\alpha +_{\mathcal{G}} \beta) &= \big([\pi^{m-1}](\alpha +_{\mathcal{G}} \beta)\big)_{m\geq 1} = \big([\pi^{m-1}](\alpha) +_{\mathcal{G}} [\pi^{m-1}](\beta)\big)_{m\geq 1} \\ &= \big([\pi^{m-1}](\alpha) + [\pi^{m-1}](\beta)\big)_{m\geq 1} = \log_{\Delta}(\alpha) + \log_{\Delta}(\beta)\end{aligned}$$

where the penultimate equality uses that $X +_{\mathcal{G}} Y = X + Y +$ (degree ≥ 2 terms) and that $[\pi^{m-1}](\alpha), [\pi^{m-1}](\beta) \in I_m$ by Lemma 4.19. The identity $\log_{\Delta}([a](\alpha)) = a \log_{\Delta}(\alpha)$ is shown similarly. ∎

Remark 4.22. Recall that $\mathfrak{S}_{L/L}$ was defined by applying Construction 3.20 to an element $e \in T\mathcal{G}$ of the Tate module of $\mathcal{G}$; this gave a map $\iota\colon \mathcal{O}_L[\![T]\!] \to W_L(\mathcal{O}^\flat_{L_\infty})$ with image $\mathfrak{S}_{L/L}$ and the element $\omega := \iota(T)$. As in Remark 4.4, applying the same construction with the element $e' = ae$ for some $a \in \mathcal{O}_L$ results in the element $\omega' = [a](\omega)$ still in $\mathfrak{S}_{L/L}$. We thus get a natural $\mathcal{O}_L$-module map

$$\begin{aligned} \rho : T\mathcal{G} &\to I_{\phi=[\pi]} \\ ae &\mapsto [a\pi^n](\omega) \end{aligned}$$

and by composition an $\mathcal{O}_L$-module map $T\mathcal{G} \to \mathfrak{S}_{L/L}\{1\}$.

5. Laurent F-crystals

In this section, we introduce étale φ_q-modules over L-typical prisms and Laurent F-crystals, and we prove Theorem 1.3. In Section 5.1, we show that the equivalence

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}} \simeq \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K^\flat_\infty)}$$

is in fact a special case of an equivalence

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)} \simeq \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)_{\mathrm{perf}}}$$

between categories of étale φ_q-modules which make sense for any L-typical prism (A, I). In Section 5.2, we define Laurent F-crystals in the L-typical prismatic setting; these are objects which serve as relativizations of étale φ_q-modules over a base formal scheme X over $\mathrm{Spf}\,\mathcal{O}_L$. We go on to show that the category of Laurent F-crystals over X is equivalent to the category of lisse local systems on the adic generic fiber X_η with coefficients in $\mathcal{O}_L$. Finally, in Section 5.3 we use this theory to recover the Kisin–Ren equivalence between Lubin–Tate (φ_q, Γ)-modules and continuous G_K representations over $\mathcal{O}_L$.

5.1. Étale φ_q-modules over L-typical prisms

Given a p-adic field K/L and a Lubin–Tate formal $\mathcal{O}_L$-module, we described in Section 4 prisms $(\mathbf{A}^+_{K/L}, (q_n(\omega)))$ with perfection $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$. We also saw that the categories of étale φ_q-modules over

$$\mathbf{A}_{K/L} = \mathbf{A}^+_{K/L}[1/q_n(\omega)]^\wedge_{(\pi)} \quad \text{and} \quad W_L(K^\flat_\infty) = A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty})[1/\ker\theta]^\wedge_{(\pi)}$$

were equivalent. In fact, this reflects a general fact about categories of φ_q-modules over L-typical prisms, which we prove here.

The definition of φ_q-modules in this setting is as follows.

Definition 5.1. We define étale φ_q modules and related notions as follows.

(1) Let $\mathcal{A}$ be a ring together with a ring homomorphism $\varphi\colon \mathcal{A} \to \mathcal{A}$. An étale φ-module over $\mathcal{A}$ is a finite projective $\mathcal{A}$-module M equipped with an isomorphism

$$\varphi_M\colon \varphi^* M := \mathcal{A} \otimes_{\varphi,\mathcal{A}} M \xrightarrow{\sim} M.$$

This gives us a φ-semilinear map $M \to M$ via $m \mapsto \varphi_M(1 \otimes m)$; we will abuse notation and write φ_M also for this map. Equipped with $\mathcal{A}$-module endomorphisms commuting with the φ_M's, étale φ-modules over $\mathcal{A}$ form a category $\mathrm{Mod}_{\mathcal{A}}^{\varphi,\mathrm{et}}$.

(2) Let (A, I) be a bounded L-typical prism. Then an étale φ_q-module over (A, I) is an étale $\varphi = \phi_{\mathcal{A}}$-module over $\mathcal{A} = A[\frac{1}{I}]^{\wedge}_{(\pi)}$ in the sense of (1). In other words, it is a finite projective $\mathcal{A}$-module M with an isomorphism $\varphi_M \colon \varphi^* M \xrightarrow{\sim} M$ (which we also view as a φ-semilinear endomorphism of M). We denote the resulting category by $\mathrm{Mod}_{(A,I)}^{\varphi_q,\mathrm{et}} = \mathrm{Mod}_{\mathcal{A}}^{\phi_{\mathcal{A}},\mathrm{et}}$.

(3) For the corresponding category of derived objects, let $D_{\mathrm{perf}}(\mathcal{A})$ denote the category of perfect complexes in modules over the ring $\mathcal{A}$, i.e. objects in the derived category of $\mathcal{A}$-modules quasi-isomorphic to a bounded complex of finite projective $\mathcal{A}$-modules. If $\mathcal{A}$ has an endomorphism φ then we write $D_{\mathrm{perf}}(\mathcal{A})^{\varphi=1}$ for the category of pairs (E, φ_E) where $E \in D_{\mathrm{perf}}(\mathcal{A})$ and $\varphi_E \colon \varphi^* E \xrightarrow{\sim} E$.

On the representation-theory side, the appropriate generalization of G_{K_∞}-representations on finite free $\mathbb{Z}_p$-modules is $\mathcal{O}_L$-local systems on $\mathrm{Spec}(\mathcal{A}/\pi)$. Recall that this means the following.

Definition 5.2 (cf. [41, Definition 8.1]). Let X be a scheme, formal scheme, or adic space, and denote by X_{et} the étale site of X.

(1) For $n \geq 1$, an $\mathcal{O}_L/\pi^n$-local system on X_{et} is a sheaf of flat $\mathcal{O}_L/\pi^n$-modules on X_{et} which is locally a constant sheaf associated to a finitely generated $\mathcal{O}_L/\pi^n$-module. We denote this category by $\mathrm{Loc}_{\mathcal{O}_L/\pi^n}(X)$.

(2) An $\mathcal{O}_L$-local system on X_{et} is an inverse system $(\mathbb{L}_n)_{n \geq 1}$ of $\mathcal{O}_L/\pi^n$-local systems on X_{et} in which the transition maps induce isomorphisms $\mathbb{L}_{n+1}/\pi^n \xrightarrow{\sim} \mathbb{L}_n$. We denote this category by $\mathrm{Loc}_{\mathcal{O}_L}(X)$. This identifies with the category of lisse $\widehat{\mathcal{O}}_L$-sheaves on the pro-étale site X_{proet}.

(3) Let $D^b_{\mathrm{lisse}}(X, \mathcal{O}_L)$ be the subcategory of the derived category of $\widehat{\mathcal{O}}_L$-modules on X_{proet} spanned by objects T which are locally bounded, derived π-complete, and have $H^i(X_{\mathrm{proet}}, T/\pi)$ locally constant with finitely generated stalks.

When $X = \mathrm{Spec}\, R$ is affine, we simplify notation by writing $\mathrm{Loc}_{\mathcal{O}_L}(R)$ for $\mathrm{Loc}_{\mathcal{O}_L}(\mathrm{Spec}\, R)$ and similarly for D^b_{lisse}.

Remark 5.3. For a field K we have equivalences $\mathrm{Loc}_{\mathcal{O}_L/\pi^n}(K) \cong \mathrm{Rep}_{\mathcal{O}_L/\pi^n}(G_K)$ and $\mathrm{Loc}_{\mathcal{O}_L}(K) \cong \mathrm{Rep}_{\mathcal{O}_L}(G_K)$ with the categories of continuous G_K-representations on finite free $\mathcal{O}_L/\pi^n$- or $\mathcal{O}_L$-modules

The main result of this section is as follows.

Proposition 5.4. *Let (A, I) be a bounded L-typical prism. Let $(A_{\mathrm{perf}}, IA_{\mathrm{perf}})$ be the perfection of (A, I) as in Proposition 3.14. Then base change gives an equivalence*

$$\mathrm{Mod}_{(A,I)}^{\varphi_q,\mathrm{et}} \to \mathrm{Mod}_{(A_{\mathrm{perf}},IA_{\mathrm{perf}})}^{\varphi_q,\mathrm{et}}$$
$$M \mapsto A_{\mathrm{perf}}[\tfrac{1}{I}]^{\wedge} \otimes_{A[\frac{1}{I}]^{\wedge}} M.$$

Both of these categories are in turn equivalent to $\mathrm{Loc}_{\mathcal{O}_L}(A[\frac{1}{I}]/\pi)$*. We similarly have equivalences*

$$D_{\mathrm{perf}}\big(A\big[\tfrac{1}{I}\big]^{\wedge}_{(\pi)}\big)^{\phi=1} \simeq D_{\mathrm{perf}}\big(A_{\mathrm{perf}}\big[\tfrac{1}{I}\big]^{\wedge}_{(\pi)}\big)^{\phi=1} \simeq D^b_{\mathrm{lisse}}\big(A\big[\tfrac{1}{I}\big]/\pi, \mathcal{O}_L\big).$$

Remark 5.5. If $(A, I) = (W_L(R^\flat), \ker\theta)$ is a perfect L-typical prism, then the equivalence of the theorem is given by

$$\begin{aligned} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)} &\simeq \mathrm{Loc}_{\mathcal{O}_L}\big(R\big[\tfrac{1}{\pi}\big]\big) \\ M &\mapsto \Big(R\big[\tfrac{1}{\pi}\big]_{\mathrm{et}} \ni S \mapsto \big(W_L(S^\flat) \otimes_{W_L(R[\frac{1}{\pi}]^\flat)} M/\pi^n\big)^{\phi=1}\Big)_{n\geq 1}, \end{aligned}$$

where $(-)^{\phi=1}$ denotes taking fixed points for $\phi = \phi_{W_L(S^\flat)} \otimes \phi_M$. The same formula holds for the derived categories, with the tensor replaced by $\otimes^L$ and with the inverse system replaced with $R\lim$ of the inverse system.

Remark 5.6. Theorem 4.14 follows from Proposition 5.4: taking $(A, I) = (\mathbf{A}^+_{K/L}, (q_n(\omega)))$, we get

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}} \simeq \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K^\flat_\infty)} \simeq \mathrm{Loc}_{\mathcal{O}_L}(K^\flat_\infty) \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty}).$$

We will discuss this point further in Section 5.3.

The key input to the proof of Proposition 5.4 is the following comparison between π-torsion φ_q-modules and $\mathbb{F}_q$-local systems.

Lemma 5.7. *Let R be a $\mathbb{F}_q$-algebra. Then there is an equivalence of categories*

$$\begin{aligned} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_R &\simeq \mathrm{Loc}_{\mathbb{F}_q}(R) \\ M &\mapsto (R_{\mathrm{et}} \ni S \mapsto S \otimes_R M)^{\varphi_q=1} \\ (\mathcal{O}_{R,\mathrm{et}} \otimes_{\mathbb{F}_q} T)(R) &\leftarrow\!\shortmid T. \end{aligned}$$

The corresponding derived statement $D_{\mathrm{perf}}(R)^{\varphi_q=1} \simeq D^b_{\mathrm{lisse}}(R, \mathbb{F}_q)$ *also holds.*

Proof. Using the same argument in [9, Proposition 3.6], we reduced the derived statement to the statement $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_R \simeq \mathrm{Loc}_{\mathbb{F}_q}(R)$. But this is well known and due originally to Katz [27, Proposition 4.1.1]. ∎

Proof of Proposition 5.4. We explain the proof for $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$, with the derived version being identical. First, we show that the base change functor is an equivalence. By the π-adic completeness of $A[\frac{1}{I}]^{\wedge}_{(\pi)}$ and dévissage, we reduce to the π-torsion case, i.e. to showing that base change gives an equivalence

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{A[\frac{1}{I}]/\pi} \xrightarrow{\sim} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{A_{\mathrm{perf}}[\frac{1}{I}]/\pi}.$$

Applying Lemma 5.7 with $R = A[\frac{1}{I}]/\pi$, we are reduced to showing that base change gives an equivalence

$$\mathrm{Loc}_{\mathbb{F}_q}\big(A/\pi\big[\tfrac{1}{I}\big]\big) \simeq \mathrm{Loc}_{\mathbb{F}_q}\big(A_{\mathrm{perf}}/\pi\big[\tfrac{1}{I}\big]\big).$$

As I is a Cartier divisor, we may assume that I is generated by a nonzerodivisor $d \in A$. Then this equivalence holds since the maps

$$A/\pi\big[\tfrac{1}{d}\big] \to \big(\varinjlim_{\varphi_q} A/\pi\big)\big[\tfrac{1}{d}\big] \to \big(\varinjlim_{\varphi_q} A/\pi\big)^{\wedge}_{(d)}\big[\tfrac{1}{d}\big]$$

induce equivalences of étale sites (the first by topological invariance of the étale site and the second by [19, Proposition 5.4.53]).

For the identification with $\mathrm{Loc}_{\mathcal{O}_L}(A[\frac{1}{I}])$, note that as $A_{\mathrm{perf}}[\frac{1}{I}]^{\wedge}_{(\pi)}$ is a π-adically complete perfect δ_L-algebra, we have $A_{\mathrm{perf}}[\frac{1}{I}]^{\wedge} = W_L(A_{\mathrm{perf}}[\frac{1}{I}]/\pi)$ by Proposition 2.14. Thus by π-adic completeness and Lemma 5.7 we get

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A_{\mathrm{perf}},I)} \simeq \mathrm{Loc}_{\mathcal{O}_L}\big(A_{\mathrm{perf}}\big[\tfrac{1}{I}\big]/\pi\big)$$

which identifies in turn with $\mathrm{Loc}_{\mathcal{O}_L}(A[\frac{1}{I}]/\pi)$ by the same argument as above. ∎

As a corollary of Proposition 5.4, we get that the equivalence $D_{\mathrm{perf}}(A[\frac{1}{I}]^{\wedge}_{(\pi)})^{\phi=1} \simeq D_{\mathrm{perf}}(A_{\mathrm{perf}}[\frac{1}{I}]^{\wedge}_{(\pi)})^{\phi=1}$ also holds "on the level of objects."

Corollary 5.8. *Let (A, I) be a bounded L-typical prism, and let $M \in D_{\mathrm{perf}}(A[\frac{1}{I}]^{\wedge}_{(\pi)})^{\phi=1}$. Then the canonical map*

$$M^{\phi=1} \to \big(A_{\mathrm{perf}}\big[\tfrac{1}{I}\big]^{\wedge}_{(\pi)} \otimes_{A[\frac{1}{I}]^{\wedge}_{(\pi)}} M\big)^{\phi=1}$$

is an isomorphism.

Proof. Our proof will follow [20, Lemma 6.3]. First we recall how $M^{\phi=1}$ is defined. In general, let B be a ring with an endomorphism φ and let $B[F]$ be the noncommutative polynomial ring with relation $Fb = \varphi(b)F$. Then we get a fully faithful embedding $D_{\mathrm{perf}}(B)^{\varphi=1} \hookrightarrow D(B[F])$ into the derived category by sending $(N, \varphi_N \colon \varphi^*N \xrightarrow{\sim} N) \in D_{\mathrm{perf}}(B)^{\varphi=1}$ to the B-algebra N with F-action given by $N \to (\varphi_N)_*N$ (this is the normal way of seeing an element of $D_{\mathrm{perf}}(B)^{\varphi=1}$ as being a B-module with a φ-semilinear endomorphism). Then $N^{\varphi=1}$ is defined by

$$N^{\varphi=1} := R\,\mathrm{Hom}\big(B[F]/(1-F)B[F], N\big).$$

Thus, setting $\mathcal{A} = A[\frac{1}{I}]^{\wedge}_{(\pi)}$ and $\mathcal{A}_{\mathrm{perf}} = A_{\mathrm{perf}}[\frac{1}{I}]^{\wedge}_{(\pi)}$ to simplify notation, our goal is to show that

$$R\,\mathrm{Hom}\big(\mathcal{A}[F]/(1-F), M\big) \to R\,\mathrm{Hom}\big(\mathcal{A}_{\mathrm{perf}}[F]/(1-F), \mathcal{A}_{\mathrm{perf}} \otimes_{\mathcal{A}} M\big)$$

is an isomorphism. As this can be checked on cohomology and $D_{\mathrm{perf}}(\mathcal{A})$ is closed under shifting, it thus suffices to show that

$$\mathrm{Hom}\big(\mathcal{A}[F]/(1-F), M\big) \to \mathrm{Hom}\big(\mathcal{A}_{\mathrm{perf}}[F]/(1-F), \mathcal{A}_{\mathrm{perf}} \otimes_{\mathcal{A}} M\big)$$

is an isomorphism. But, up to fully faithful embedding, the hom-set on the right comes from the one of the left by applying the functor $M \mapsto \mathcal{A}_{\mathrm{perf}} \otimes_{\mathcal{A}} M$, which is an equivalence by Proposition 5.4. Thus the hom-sets are isomorphic as desired. ∎

5.2. Laurent F-crystals

For a bounded formal scheme X adic over $\mathrm{Spf}\,\mathcal{O}_L$, denote by $\mathcal{O}_{\Delta}$ the presheaf $(A, I) \mapsto A$ on the L-typical prismatic site X_{Δ_L}. By (π, I)-completely faithfully flat descent (see [8, Corollary 3.12]), $\mathcal{O}_{\Delta}$ is a sheaf, which we take as the structure sheaf for X_{Δ_L}. It has a natural endomorphism ϕ lifting φ_q on $\mathcal{O}_{\Delta}/\pi$ and an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\Delta}$ given by $(A, I) \mapsto I$. We will also make use of the sheaf $\mathcal{O}_{\Delta,\mathrm{perf}}$, which sends $(A, I) \mapsto A_{\mathrm{perf}}$.

Denote by $\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}$ the π-adic completion of the localization of $\mathcal{O}_{\Delta}$ away from $\mathcal{I}$ (i.e. locally inverting a generator of $\mathcal{I}$; recall that if (A, I) is a prism then I is a Cartier divisor hence locally principal).

Definition 5.9. Let X be a bounded formal scheme adic over $\mathrm{Spf}\,\mathcal{O}_L$.

(1) A Laurent F-crystal is a finite locally free $\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}$-module $\mathcal{M}$ over X_{Δ_L} equipped with an isomorphism
$$F \colon \phi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}.$$
As before, we abusively write $\phi_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M}$ also for the resulting ϕ-semilinear endomorphism.

(2) Write $\mathrm{Vect}(\mathcal{X}, \mathcal{O})$ for the category of vector bundles on a ringed topos $(\mathcal{X}, \mathcal{O})$, we can describe the category of Laurent F-crystals over X_{Δ_L} as
$$\mathrm{Vect}\left(X_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{\mathcal{I}}]^{\wedge}_{\pi}\right)^{\phi=1},$$
the ϕ-fixed objects of $\mathrm{Vect}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})$.

(3) Similarly, write $D_{\mathrm{perf}}(\mathcal{X}, \mathcal{O})$ for the category of perfect complexes on $(\mathcal{X}, \mathcal{O})$, i.e. objects E in the derived category of $\mathcal{O}$-modules over $\mathcal{X}$ such that there is a cover $\{U_i\}$ of $\mathcal{X}$ with each $E|_{U_i}$ a perfect complex of $\mathcal{O}(U_i)$-modules. Let $D_{\mathrm{perf}}(\mathcal{X}, \mathcal{O})^{\phi=1}$ denote corresponding category of ϕ-fixed objects.

Given a Laurent F-crystal $\mathcal{M}$ and an object $(A, I) \in X_{\Delta_L}$, we have that $\mathcal{M}(A, I) \in \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$ is an étale φ_q-module. We further have the following.

Lemma 5.10. *There is an equivalence*
$$\mathrm{Vect}\left(X_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}\right)^{\phi=1} \xrightarrow{\sim} \lim_{(A,I)\in X_{\Delta_L}} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$$
$$\mathcal{M} \mapsto \left(\mathcal{M}(A, I)\right)_{(A,I)\in X_{\Delta_L}}.$$
Similarly, $D_{\mathrm{perf}}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})^{\phi=1} \simeq \lim_{(A,I)\in X_{\Delta_L}} D_{\mathrm{perf}}(A[\frac{1}{I}]^{\wedge}_{(\pi)})^{\phi=1}$. *A similar result holds with* $\mathcal{O}_{\Delta,\mathrm{perf}}$ *replacing* $\mathcal{O}_{\Delta}$ *(and* $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)_{\mathrm{perf}}}$ *replacing* $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$*).*

Proof. The proof is the same as [9, Proposition 2.7]: one can reduce via devissage to the π-torsion case, where the result follows from the descent results in [35, Theorem 5.8]. ∎

We regard Laurent F-crystals as (geometrically) relativizing étale φ_q-modules over the base formal scheme X. We then have the following analogues of Proposition 5.4 and Corollary 5.8 (except without the local systems, which will appear in Theorem 5.15).

Theorem 5.11. *Let X be a bounded formal scheme adic over* $\operatorname{Spf} \mathcal{O}_L$.

(1) *Base change induces an equivalence of categories*

$$\begin{aligned} \operatorname{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}\big[\tfrac{1}{\mathcal{I}}\big]^{\wedge}_{(\pi)}\big)^{\phi=1} &\xrightarrow{\sim} \operatorname{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta,\mathrm{perf}}\big[\tfrac{1}{\mathcal{I}}\big]^{\wedge}_{(\pi)}\big)^{\phi=1} \\ \mathcal{M} &\mapsto \mathcal{O}_{\Delta,\mathrm{perf}}\big[\tfrac{1}{\mathcal{I}}\big]^{\wedge}_{(\pi)} \otimes_{\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}} \mathcal{M} \end{aligned}$$

and the same holds with D_{perf} replacing Vect.

(2) *For $\mathcal{M} \in D_{\mathrm{perf}}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})^{\phi=1}$, the canonical map*

$$\mathcal{M}^{\phi=1} \to \big(\mathcal{O}_{\Delta,\mathrm{perf}}\big[\tfrac{1}{\mathcal{I}}\big]^{\wedge}_{(\pi)} \otimes_{\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}} \mathcal{M}\big)^{\phi=1}$$

is an isomorphism.

Proof. For part (1), we have the following commutative diagram.

$$\begin{tikzcd} \operatorname{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}\big)^{\phi=1} \arrow[rr, dashed] \arrow[ddd] & & \operatorname{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta,\mathrm{perf}}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}\big)^{\phi=1} \arrow[ddd] \\ \mathcal{M} \arrow[r, mapsto, dashed] \arrow[d, mapsto] & \mathcal{O}_{\Delta,\mathrm{perf}}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)} \otimes_{\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)}} \mathcal{M} \arrow[d, mapsto] & \\ \big(\mathcal{M}(A,I)\big)_{(A,I)\in X_{\Delta_L}} \arrow[r, mapsto] & \big(A_{\mathrm{perf}}[\frac{1}{I}]^{\wedge}_{(\pi)} \otimes_{A[\frac{1}{I}]^{\wedge}_{(\pi)}} \mathcal{M}(A,I)\big)_{(A,I)\in X_{\Delta_L}} & \\ \lim\limits_{(A,I)\in X_{\Delta_L}} \operatorname{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)} \arrow[rr] & & \lim\limits_{(A,I)\in X^{\mathrm{perf}}_{\Delta_L}} \operatorname{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)_{\mathrm{perf}}} \end{tikzcd}$$

By Lemma 5.10 the vertical arrows are equivalences of categories, and the bottom horizontal arrow is an equivalence by Proposition 5.4. The same holds replacing Vect with D_{perf} and $\operatorname{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$ with $D_{\mathrm{perf}}(A[\frac{1}{I}]^{\wedge}_{(\pi)})$. For part (2), we can again check on individual prisms $(A, I) \in X_{\Delta_L}$, in which case the result follows from Corollary 5.8. ■

Let $X^{\mathrm{perf}}_{\Delta_L}$ denote the subsite of X_{Δ_L} consisting of perfect L-typical prisms.

Corollary 5.12. *For X a bounded formal scheme adic over* $\operatorname{Spf} \mathcal{O}_L$ *we have*

$$\operatorname{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}\big[\tfrac{1}{\mathcal{I}}\big]^{\wedge}_{(\pi)}\big)^{\phi=1} \simeq \lim_{(A,I)\in X^{\mathrm{perf}}_{\Delta_L}} \operatorname{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)}$$

and similarly for D_{perf}.

Proof. This follows from Theorem 5.11 (1), Lemma 5.10, and the fact that for $\mathcal{M} \in D_{\mathrm{perf}}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})^{\phi=1}$ and $(A, I) \in X_{\Delta_L}$ we have

$$\mathcal{M}\big((A,I)_{\mathrm{perf}}\big) \cong A_{\mathrm{perf}}\big[\tfrac{1}{I}\big]^{\wedge}_{(\pi)} \otimes_{A[\frac{1}{I}]^{\wedge}_{(\pi)}} \mathcal{M}(A,I).$$ ■

We now globalize the relationship between étale φ_q-modules and local systems from Proposition 5.4. We have essentially already shown this in the case that $X = \mathrm{Spf}(R)$ for a perfectoid $\mathcal{O}_L$-algebra.

Proposition 5.13. *If R is a perfectoid $\mathcal{O}_L$-algebra, there are equivalences*

$$\mathrm{Vect}\left(R_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(\pi)}\right)^{\phi=1} \simeq \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A_{\mathrm{inf},L}(R),\ker\theta)} \simeq \mathrm{Loc}_{\mathcal{O}_L}\left(R\left[\tfrac{1}{\pi}\right]\right)$$

and

$$D_{\mathrm{perf}}\left(R_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(\pi)}\right)^{\phi=1} \simeq D_{\mathrm{perf}}\left(W_L\left(R\left[\tfrac{1}{\pi}\right]^{\flat}\right)\right)^{\phi=1} \simeq D^b_{\mathrm{lisse}}\left(R\left[\tfrac{1}{\pi}\right], \mathcal{O}_L\right).$$

Proof. By Proposition 3.14 (2), $R^{\mathrm{perf}}_{\Delta_L}$ has an initial object $(A_{\mathrm{inf},L}(R), \ker\theta)$. By Corollary 5.12 and Proposition 5.4, we then have that

$$\mathrm{Vect}\left(R_{\Delta_L}, \mathcal{O}_{\Delta}\left[\tfrac{1}{\mathcal{I}}\right]^{\wedge}_{(\pi)}\right)^{\phi=1} \simeq \mathrm{Loc}_{\mathcal{O}_L}\left(A_{\mathrm{inf},L}(R)\left[\tfrac{1}{\ker\theta}\right]/\pi\right).$$

In the proof of Proposition 3.15, we showed that $\ker\theta$ has a generator of the form $d = [a_0] - \pi u$ for $a_0 \in R^{\flat}$ such that $R^{\flat}$ is a_0-adically complete and $a_0^{\sharp} = \pi$. Thus

$$A_{\mathrm{inf},L}\left[\tfrac{1}{\ker\theta}\right]/\pi \cong R^{\flat}\left[\tfrac{1}{a_0}\right],$$

and we conclude by the tilting equivalence. ∎

Corollary 5.14. *If R is perfectoid and $\mathcal{M} \in D_{\mathrm{perf}}(R_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})^{\phi=1}$ corresponds to $T \in D^b_{\mathrm{lisse}}(R[\frac{1}{\pi}]_{\mathrm{et}}, \mathcal{O}_L)$ under the equivalence of Proposition 5.13, then there is an isomorphism*

$$R\Gamma(R_{\Delta_L}, \mathcal{M})^{\phi=1} \cong R\Gamma\left(R\left[\tfrac{1}{\pi}\right]_{\mathrm{proet}}, T\right).$$

Proof. Since the map $D_{\mathrm{perf}}(R_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)})^{\phi=1} \to D_{\mathrm{perf}}(W_L(R[\frac{1}{\pi}]^{\flat}))^{\phi=1}$ is given by $\mathcal{M} \mapsto R\Gamma(R_{\Delta_L}, \mathcal{M})$, this follows from the description of the map

$$D_{\mathrm{perf}}\left(W_L\left(R\left[\tfrac{1}{\pi}\right]^{\flat}\right)\right)^{\phi=1} \to D^b_{\mathrm{lisse}}\left(R\left[\tfrac{1}{\pi}\right], \mathcal{O}_L\right)$$

given in Remark 5.5. ∎

The following theorem globalizes Proposition 5.13 by descent from the affine perfectoid case. This generalizes [9, Corollary 3.8]. More specifically, we will use v-descent: by [42, Section 15], X_η can be viewed as a locally spatial diamond, so that the categories $\mathrm{Loc}_{\mathcal{O}_L}(X_\eta)$ and $D^b_{\mathrm{lisse}}(X_\eta, \mathcal{O}_L)$ satisfy v-descent with respect to v-covers of X_η (i.e. covers by surjective maps of v-sheaves; see [34] and especially [34, Theorem 3.11] for the relationship between local systems on the diamondification of X_η and pro-étale local systems on X_η). By [42, Lemma 15.3], any analytic adic space has a v-cover by generic fibers of perfectoid rings; by Theorem 3.18 this also gives a v-cover by perfectoid $\mathcal{O}_L$-algebras.

For now this globalization will result in losing the étale φ_q-module part of the result; that part will be restored in the special case $X = \mathrm{Spf}\,\mathcal{O}_K$ in Section 5.3.

Theorem 5.15. *Let X be a formal scheme adic over* $\mathrm{Spf}\,\mathcal{O}_L$ *with adic generic fiber X_η over* $\mathrm{Spa}(L, \mathcal{O}_L)$.

(1) *There are equivalence of categories*

$$\mathrm{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{\mathcal{I}}]^{\wedge}\big)^{\phi=1} \simeq \mathrm{Loc}_{\mathcal{O}_L}(X_\eta),$$
$$D_{\mathrm{perf}}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{\mathcal{I}}]^{\wedge}\big)^{\phi=1} \simeq D^b_{\mathrm{lisse}}(X_\eta, \mathcal{O}_L).$$

(2) *Let $\mathcal{M} \in D_{\mathrm{perf}}(X_{\Delta_L}, \mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge})^{\phi=1}$ and $T \in D^b_{\mathrm{lisse}}(X_\eta, \mathcal{O}_L)$ correspond under the above equivalence. Then there is an isomorphism*

$$R\Gamma(X_{\Delta_L}, \mathcal{M})^{\phi=1} \cong R\Gamma(X_{\eta,\mathrm{proet}}, T).$$

Note that if $\pi = 0$ on X, then the theorem is trivial: $X_\eta = \mathrm{Spf}\,0$ and $\mathcal{O}_{\Delta}[\frac{1}{\mathcal{I}}]^{\wedge}_{(\pi)} = 0$ (since $\mathcal{I}$ is the ideal sheaf generated by π).

Proof. For part (1), we have

$$\begin{aligned}\mathrm{Vect}\big(X_{\Delta_L}, \mathcal{O}_{\Delta}[\tfrac{1}{\mathcal{I}}]^{\wedge}\big)^{\phi=1} &\simeq \lim_{(A,I)\in X^{\mathrm{perf}}_{\Delta_L}} \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{(A,I)} \\ &\simeq \lim_{\substack{\mathrm{Spf}\,R\to X\\ R\text{ perfd }\mathcal{O}_L\text{-alg}}} \mathrm{Loc}_{\mathcal{O}_L}\big(R[1/\pi]\big) \simeq \mathrm{Loc}_{\mathcal{O}_L}(X_\eta)\end{aligned}$$

where the first equivalence is Corollary 5.12, the second is Proposition 5.13 and Proposition 3.11, and the final equivalence is by v-descent. The same argument works for the derived categories.

The proof of part (2) is formally identical:

$$\begin{aligned}R\Gamma(X_{\Delta_L}, \mathcal{M})^{\phi=1} &\cong \lim_{(A,I)\in X^{\mathrm{perf}}_{\Delta_L}} R\Gamma\big((X/A)_{\Delta_L}, \mathcal{M}\big)^{\phi=1} \\ &\cong \lim_{\substack{\mathrm{Spf}\,R\to X\\ R\text{ perfd }\mathcal{O}_L\text{-alg}}} R\Gamma\big(R[\tfrac{1}{\pi}]_{\mathrm{proet}}, T\big) \cong R\Gamma(X_{\eta,\mathrm{proet}}, T)\end{aligned}$$

where $(X/A)_{\Delta_L}$ denotes the relative prismatic site of $(B, J) \in X_{\Delta_L}$ with a map from (A, I) compatible with the maps $\mathrm{Spf}(A/J), \mathrm{Spf}(B/J) \to X$, and we are now using Corollary 5.14 instead of Proposition 5.13. ∎

5.3. Lubin–Tate étale (φ_q, Γ)-modules and Laurent F-crystals

Let K/L be a p-adic field. Recall that work of Kisin–Ren [31] gives the solid equivalences in the following diagram.

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Z}_p}(G_K) \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Mod}^{\varphi,\Gamma,\mathrm{et}}_{\mathbf{A}_{K/L}} \arrow[r, dashed, "\cong"] \arrow[d, hook] & \mathrm{Mod}^{\varphi,\Gamma,\mathrm{et}}_{W(K^\flat_\infty)} \arrow[d, hook] \\
\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty}) \arrow[r, "\cong"] & \mathrm{Mod}^{\varphi,\mathrm{et}}_{\mathbf{A}_{K/L}} \arrow[r, dashed, "\cong"] & \mathrm{Mod}^{\varphi,\mathrm{et}}_{W(K^\flat_\infty)}
\end{tikzcd}$$

In this section, we show that Theorem 5.15 (1) specializes to the top row of this diagram when $X = \mathrm{Spf}(\mathcal{O}_{K_\infty})$ and the bottom row when $X = \mathrm{Spf}(\mathcal{O}_K)$. We will further find that the comparison morphism in Theorem 5.15 (2) recovers the results on φ_q-Herr complexes from [32, Theorem A].

We begin with the case $X = \mathrm{Spf}(\mathcal{O}_{K_\infty})$.

Theorem 5.16. *Let K/L be a p-adic field.*

(1) *There are equivalences of categories*

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}} \simeq \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K_\infty^\flat)} \simeq \mathrm{Vect}\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{O}_{\mathbb{\Delta}}[\tfrac{1}{\mathcal{I}}]^\wedge_{(\pi)}\big)^{\phi=1} \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty}).$$

For the derived category, we similarly have

$$\begin{aligned} D_{\mathrm{perf}}(\mathbf{A}_{K/L})^{\phi=1} \simeq D_{\mathrm{perf}}\big(W_L(K_\infty^\flat)\big)^{\phi=1} &\simeq D_{\mathrm{perf}}\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{O}_{\mathbb{\Delta}}[\tfrac{1}{\mathcal{I}}]^\wedge_{(\pi)}\big)^{\phi=1} \\ &\simeq D^b_{\mathrm{lisse}}(K_{\infty,\mathrm{proet}}, \mathcal{O}_L). \end{aligned}$$

(2) *For $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_{K_\infty})$ corresponding to $M \in \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{\mathbf{A}_{K/L}}$ or $\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K_\infty^\flat)}$ under the equivalence from (1), we have that $R\Gamma(K_{\infty,\mathrm{proet}}, T)$ is isomorphic to the complex*

$$M \xrightarrow{\phi-1} M$$

concentrated in degrees 0 *and* 1.

Proof. By Proposition 3.14, $(\mathcal{O}_{K_\infty})^{\mathrm{perf}}_{\mathbb{\Delta}_L}$ has an initial object given by $(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta)$. Thus by Corollary 5.12 we get the equivalence

$$\mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K_\infty^\flat)} \simeq \mathrm{Vect}\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{O}_{\mathbb{\Delta}}[\tfrac{1}{\mathcal{I}}]^\wedge_{(\pi)}\big)^{\phi=1}.$$

Then Proposition 5.4 and Theorem 5.15 give the first part of part (1). The argument for the derived categories is identical.

For part (2) note that, viewing M as a complex concentrated in degree 0, we have

$$M^{\phi=1} := \mathrm{Cone}(\phi_M - 1)[-1] = (M \xrightarrow{\phi-1} M).$$

Thus by Corollary 5.8, it suffices to prove part (2) for $M \in \mathrm{Mod}^{\varphi_q,\mathrm{et}}_{W_L(K_\infty^\flat)}$ corresponding to T. Letting $\mathcal{M} \in \mathrm{Vect}((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{O}_{\mathbb{\Delta}}[\frac{1}{\mathcal{I}}]^\wedge_{(\pi)})^{\phi=1}$ correspond to T and M, we have by Theorem 5.15 (2) that

$$R\Gamma\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{M}\big)^{\phi=1} \cong R\Gamma(K_{\infty,\mathrm{proet}}, T).$$

Thus it suffices to show that $R\Gamma((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{M}) \cong M$; this is given by the following lemma. ∎

Lemma 5.17. *If $\mathcal{M} \in \mathrm{Vect}((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{O}_{\mathbb{\Delta}}[\frac{1}{\mathcal{I}}]^\wedge_{(\pi)})^{\phi=1}$ then*

$$R\Gamma\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{M}\big) \cong \Gamma\big((\mathcal{O}_{K_\infty})_{\mathbb{\Delta}_L}, \mathcal{M}\big).$$

Proof. We want to show that

$$H^i\big((\mathcal{O}_{K_\infty})_{\Delta_L}, \mathcal{M}\big) = 0 \quad \text{for } i \geq 1.$$

Indeed, by derived π-completeness and derived Nakayama [43, Tag 0G1U], it suffices to show this upon replacing $\mathcal{M}$ with $\mathcal{M}/\pi$. By Corollary 5.12 we can compute cohomology on the site $(\mathcal{O}_{K_\infty})^{\mathrm{perf}}_{\Delta_L}$, which identifies with the category of perfectoid $\mathcal{O}_L$-algebras over $\mathcal{O}_{K_\infty}$ by Proposition 3.11. Under this identification, $\mathcal{M}/\pi$ is the sheaf which sends a perfectoid $\mathcal{O}_L$-algebra S over $\mathcal{O}_{K_\infty}$ to

$$\mathcal{M}\big(A_{\mathrm{inf},L}(S), \ker\theta\big)/\pi = S\big[\tfrac{1}{\pi}\big]^\flat \otimes_{K^\flat_\infty} \mathcal{M}\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big)/\pi.$$

Thus it suffices to show that the sheaf which sends a perfectoid $\mathcal{O}_L$-algebra S over $\mathcal{O}_{K_\infty}$ to $S[\frac{1}{\pi}]^\flat$ has vanishing higher cohomology. But, via the tilting equivalence, this is just the basic fact about Galois cohomology that $H^i(K^\flat_\infty, \bar{K}^\flat) = 0$ for $i \geq 1$. ■

Naively, we might hope to deduce the corresponding result for $X = \operatorname{Spf}\mathcal{O}_K$ by descent along $\operatorname{Spf}\mathcal{O}_{K_\infty} \to \operatorname{Spf}\mathcal{O}_K$. However, instead of using this angle of attack, we will use a more delicate descent argument along the Čech nerve $(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta)^\bullet$ in the perfect prismatic site $(\mathcal{O}_K)^{\mathrm{perf}}_{\Delta_L}$. This approach, which is the same as the one in [46, Proof of Theorem 5.2], allows us to recover a Laurent F-crystal $\mathcal{M}$ over $(\mathcal{O}_K)_{\Delta_L}$ from the data of $\mathcal{M}(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta)$ and a semilinear action of

$$\operatorname{Aut}_{(\mathcal{O}_K)_{\Delta_L}}\big(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta\big) \cong \Gamma_K$$

(by Proposition 4.15).

Lemma 5.18. $(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta)$ *is a cover of the final object of the topos* $\operatorname{Shv}((\mathcal{O}_K)_{\Delta_L})$.

Proof. We want to show that for any $(A, I) \in (\mathcal{O}_K)_{\Delta_L}$, there is a cover (B, J) of (A, I) with a map

$$\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \to (B, J).$$

Let $(A_{\mathrm{inf},L}(R), \ker\theta) = (A, I)_{\mathrm{perf}}$, using Proposition 3.11. As $\mathcal{O}_K \to \mathcal{O}_{K_\infty}$ is π-completely faithfully flat, so is

$$R \to S := R\,\widehat{\otimes}^L_{\mathcal{O}_K}\,\mathcal{O}_{K_\infty},$$

where S is the derived π-completion of the derived tensor product. Using the same argument as in [4, IV, Proposition 2.11] or [9, Example 2.6], we have that S is a perfectoid $\mathcal{O}_L$-algebra. Thus by Proposition 3.14 and Lemma 3.12, we have that the composite

$$(A, I) \to \big(A_{\mathrm{inf},L}(R), \ker\theta\big) \to \big(A_{\mathrm{inf},L}(S), \ker\theta\big)$$

is a cover in $(\mathcal{O}_K)_{\Delta_L}$. But also from the map $\mathcal{O}_{K_\infty} \to S$, we get a morphism

$$\big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \to \big(A_{\mathrm{inf},L}(S), \ker\theta\big)$$

as desired. ■

Lemma 5.19. *Let $n \geq 1$ and let*

$$(B, J) = \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big)^{(n)} := \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big) \times \cdots \times \big(A_{\mathrm{inf},L}(\mathcal{O}_{K_\infty}), \ker\theta\big)$$

be the $(n+1)$-times iterated self-product in $(\mathcal{O}_K)^{\mathrm{perf}}_{\Delta_L}$. Then

$$B = \mathrm{Hom}_{\mathrm{cont}}\big(\Gamma_K^n, W_L(\mathcal{O}_{K_\infty}^\flat)\big),$$
$$B\big[\tfrac{1}{J}\big]^\wedge_{(\pi)} = \mathrm{Hom}_{\mathrm{cont}}\big(\Gamma_K^n, W_L(K_\infty^\flat)\big).$$

Proof. We first compute B. By Proposition 3.11, we are interested in the self-product of $\mathcal{O}_{K_\infty}$ in the category of perfectoid $\mathcal{O}_L$-algebras over $\mathcal{O}_K$. To compute this, let $U = \varprojlim \mathrm{Spa}(K_m, \mathcal{O}_{K_m})$ be the element of the pro-étale site X_{proet} for $X = \mathrm{Spa}(K, \mathcal{O}_K)$. By [41, Lemma 4.10], the self-product we are looking for can be computed as $H^0(U^{(n)}, \widehat{\mathcal{O}}_X^+)$ where $U^{(n)} = U \times_X \cdots \times_X U$ (iterated $n+1$ times). As $U \to X$ is Galois with Galois group Γ_K, we have $U^{(n)} = U \times \Gamma_K^n$ where Γ_K^n is viewed in X_{proet} as a profinite set with trivial Galois action (cf. [41, Proof of Lemma 5.6]). But then [41, Theorem 4.9] and [41, Lemma 3.16] imply that

$$H^0(U \times \Gamma_K^n, \widehat{\mathcal{O}}_X^+) = \mathrm{Hom}_{\mathrm{cont}}\big(\Gamma_K^n, H^0(U, \widehat{\mathcal{O}}_X^+)\big) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma_K^n, \mathcal{O}_{K_\infty}).$$

It is easy to verify that tilting and taking $W_L(-)$ commutes with $\mathrm{Hom}_{\mathrm{cont}}(\Gamma_K^n, -)$, giving the first part of the result.

Since B/J is a perfectoid $\mathcal{O}_L$-algebra, we have $B[\frac{1}{J}]^\wedge_{(\pi)} = W_L(B/J[\frac{1}{\pi}]^\flat)$. We thus have

$$B\big[\tfrac{1}{J}\big]^\wedge_{(\pi)} = W_L\big(\mathrm{Hom}_{\mathrm{cont}}(\Gamma_K^n, \mathcal{O}_{K_\infty})\big[\tfrac{1}{\pi}\big]^\flat\big) = \mathrm{Hom}_{\mathrm{cont}}\big(\Gamma_K^n, W_L(K_\infty^\flat)\big)$$

as desired. ∎

Theorem 5.20. *Let K/L be a p-adic field.*

(1) *There are equivalences of categories*

$$\mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\Gamma_K,\mathrm{et}} \simeq \mathrm{Mod}_{W_L(K_\infty^\flat)}^{\varphi_q,\Gamma_K,\mathrm{et}} \simeq \mathrm{Vect}\big((\mathcal{O}_K)_{\Delta_L}, \mathcal{O}_\Delta\big[\tfrac{1}{I}\big]^\wedge_{(\pi)}\big)^{\phi=1} \simeq \mathrm{Rep}_{\mathcal{O}_L}(G_K)$$

and similarly for the corresponding derived categories.

(2) *Let $T \in \mathrm{Rep}_{\mathcal{O}_L}(G_K)$ correspond to $M \in \mathrm{Mod}_{\mathbf{A}_{K/L}}^{\varphi_q,\Gamma_K,\mathrm{et}}$ or $\mathrm{Mod}_{W_L(K_\infty^\flat)}^{\varphi_q,\Gamma_K,\mathrm{et}}$ under the equivalence from (1). Let $C^\bullet_{\mathrm{cont}}(\Gamma_K, M)$ denote the continuous cochain complex of Γ_K with values in M. Then $R\Gamma(K_{\mathrm{proet}}, T)$ is isomorphic to $C^\bullet_{\mathrm{cont}}(\Gamma_K, M)^{\phi=1}$.*

Proof of Theorem 5.20. The first and last equivalences in the theorem follow from Proposition 5.4 and Theorem 5.15, so we focus on the equivalence

$$\mathrm{Mod}_{W_L(K_\infty^\flat)}^{\phi,\Gamma_K,\mathrm{et}} \simeq \mathrm{Vect}\big((O_K)_{\Delta_L}, \mathcal{O}_\Delta\big[\tfrac{1}{I}\big]^\wedge_{(\pi)}\big)^{\phi=1}.$$

Since $(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta)$ is a cover of the final object $*$ of $\mathrm{Shv}((\mathcal{O}_K)_{\Delta_L})$ by Lemma 5.18, we have that

$$\begin{aligned}&\mathrm{Vect}\left((\mathcal{O}_K)_{\Delta_L}, \mathcal{O}_\Delta[\tfrac{1}{I}]^\wedge_{(\pi)}\right)^{\phi=1}\\ &\quad\simeq \varprojlim\left(\mathrm{Mod}^{\phi,\mathrm{et}}_{(W_L(\mathcal{O}^\flat_{K_\infty}),\ker\theta)} \rightrightarrows \mathrm{Mod}^{\phi,\mathrm{et}}_{(W_L(\mathcal{O}^\flat_{K_\infty}),\ker\theta)^{(1)}} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right),\end{aligned}$$

where

$$(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta)^{(1)} := \left(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta\right) \times_* \left(W_L(\mathcal{O}^\flat_{K_\infty}), \ker\theta\right)$$

denotes the self product in $(\mathcal{O}_K)^{\mathrm{perf}}_{\Delta_L}$ (here we have used a general fact about recovering a vector bundle from its value on the Čech nerve of a cover of the final object; see [8, Footnote 10] or [46, Section 3] for more details). By Lemma 5.19 we then get

$$\begin{aligned}&\mathrm{Vect}\left((\mathcal{O}_K)_{\Delta_L}, \mathcal{O}_\Delta[\tfrac{1}{I}]^\wedge_{(\pi)}\right)^{\phi=1}\\ &\quad\simeq \varprojlim\left(\mathrm{Mod}^{\phi,\mathrm{et}}_{W_L(K^\flat_\infty)} \rightrightarrows \mathrm{Mod}^{\phi,\mathrm{et}}_{\mathrm{Hom}_{\mathrm{cont}}(\Gamma_K, W_L(K^\flat_\infty))} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right).\end{aligned}$$

By the same argument as for usual Galois descent, this identifies

$$\mathrm{Vect}\left((\mathcal{O}_K)_{\Delta_L}, \mathcal{O}_\Delta[\tfrac{1}{I}]^\wedge_{(\pi)}\right)^{\phi=1}$$

with the category of étale φ_q-modules over $W_L(K^\flat_\infty)$ with a semilinear action of Γ_K which also commutes with ϕ. But this is exactly the definition of the category $\mathrm{Mod}^{\phi,\Gamma_K,\mathrm{et}}_{W_L(K^\flat_\infty)}$, giving part (1).

For part (2), we can again focus on the case $M \in \mathrm{Mod}^{\varphi_q,\Gamma_K,\mathrm{et}}_{W_L(K^\flat_\infty)}$ by Corollary 5.8. For

$$\mathcal{M} \in \mathrm{Vect}\left((\mathcal{O}_K)_{\Delta_L}, \mathcal{O}_\Delta[\tfrac{1}{I}]^\wedge_{(\pi)}\right)^{\phi=1}$$

corresponding to T and M, we get by the same computation as above that

$$R\Gamma\left((\mathcal{O}_K)_{\Delta_L}, \mathcal{M}\right) \simeq C^\bullet_{\mathrm{cont}}(\Gamma_K, M).$$

We then conclude by Theorem 5.15 (2). ∎

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