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Dynamics of continued fractions and distribution of modular symbols

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Abstract. We formulate a dynamical approach to the study of distribution of modular symbols, motivated by the work of Baladi–Vallée. We introduce the modular partition functions of continued fractions and observe that the modular symbols are special cases of modular partition functions. We prove the limit Gaussian distribution and residual equidistribution for modular partition functions as random variables on the set of rationals whose denominators are up to a fixed positive integer, by studying the spectral properties of the transfer operator associated to the underlying dynamics. The approach leads to a few applications. We show an average version of a conjecture of Mazur–Rubin on statistics for modular symbols of rational elliptic curves. We further observe that the equidistribution of mod p values of modular symbols leads to a mod p non-vanishing result for special modular L-values twisted by Dirichlet characters.

Keywords: mod p non-vanishing of special L-values, modular symbols, Mazur–Rubin conjecture, continued fractions, skewed Gauss map, transfer operators.

Contents

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1. Introduction and statements of results

The statistics of continued fractions has been a rich source of research. For instance, it is a longstanding conjecture that the distribution of the length of continued fractions over the rational numbers follows the Gaussian distribution. More precisely, for a rational number $r \in (0,1)$, write $[0; m_1, \ldots, m_\ell]$ for the continued fraction expansion of r where $m_1, \ldots, m_{\ell-1}$ are integers greater than 0 and m_ℓ is an integer greater than 1, and $\ell = \ell(r)$ is the length of the expansion. We consider the set $\Sigma_M := \{a/M \mid 1 \le a < M, (a, M) = 1\}$. One can regard Σ_M as a probability space with the uniform distribution and ℓ as a random variable on Σ_M. The unsettled conjecture is that the variable ℓ follows asymptotically the Gaussian distribution as $M \to \infty$.

The first prominent result goes back to Hensley [16]. He obtained a partial result on the problem in an average setting; in other words, instead of Σ_M, he proved the conjecture for a larger probability space

$$\Omega_M = \bigcup_{n \le M} \Sigma_n.$$

Later, Baladi–Vallée [3] showed the average version in full generality with an optimal error based on the dynamical analysis of the Euclidean algorithm.

In this paper, we study the statistics of generalizations of the variable ℓ, so-called modular partition functions.

1.1. Modular partition functions

Let P_i/Q_i be the i-th convergent of r, i.e.,

$$\frac{P_i}{Q_i} = [0; m_1, \ldots, m_i], \quad P_0 = 0,\ Q_0 = 1.$$

For $1 \le i \le \ell$, we define 2×2 integral matrices

$$g_i(r) := \left[\begin{smallmatrix} P_{i-1} & P_i \\ Q_{i-1} & Q_i \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z}) \quad \text{and} \quad g(r) := g_\ell(r).$$

The matrices satisfy the recurrence relation $g_{i+1}(r) = g_i(r)\left[\begin{smallmatrix} 0 & 1 \\ 1 & m_{i+1} \end{smallmatrix}\right]$.

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For a right coset $u \in \Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ and a rational $r \in (0,1)$, a natural quantity to consider is $\#\{1 \le i \le \ell \mid \Gamma g_i(r) \in u\}$. We observe that it can be written as $\sum_{i=1}^{\ell} \mathbb{I}_u(\Gamma g_i(r))$ where $\mathbb{I}_u(v)$ is 1 if $u = v$ and 0 otherwise. Extending it to a function ψ on $\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$, let us define a more general quantity

$$\mathfrak{a}_\psi(r) := \sum_{i=1}^{\ell} \psi(\Gamma g_i(r)).$$

In order to define an SL_2-version, let us introduce

$$\mathrm{j} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this paper, we assume that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] \text{ is finite and } \Gamma \text{ is normalized by j.}$$

For $g \in \mathrm{GL}_2(\mathbb{Z})$, we define

$$\widehat{g} := \begin{cases} g & \text{if } \det g = 1, \\ \mathrm{j}g & \text{otherwise,} \end{cases} \qquad \widetilde{g} := \begin{cases} g & \text{if } \det g = 1, \\ g\mathrm{j} & \text{otherwise.} \end{cases}$$

For a function ψ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$, we define

$$\mathfrak{b}_\psi(r) := \sum_{i=1}^{\ell} \psi(\Gamma\widehat{g}_i(r)) \quad \text{and} \quad \mathfrak{c}_\psi(r) := \sum_{i=1}^{\ell} \psi(\Gamma\widetilde{g}_i(r)).$$

The functions $\mathfrak{a}_\psi$, $\mathfrak{b}_\psi$, and $\mathfrak{c}_\psi$ are called the *modular partition functions* or *modular cost functions*.

One of the main goals in the present paper is to determine the moment generating functions of the random variables $\mathfrak{b}_\psi$ and $\mathfrak{c}_\psi$ on Ω_M. In particular, there are two applications: the recent conjecture of Mazur–Rubin on the distribution of modular symbols and the non-vanishing modulo p of special L-values of modular forms.

1.2. Main results

Let I be the interval $[0, 1]$. For the later applications, we study more general probability spaces. For a map φ on the right cosets of Γ, denote the functions on $I \cap \mathbb{Q}$ given by

$$r \mapsto \varphi(\Gamma g(r)), \quad r \mapsto \varphi(\Gamma\widehat{g}(r)), \quad \text{or} \quad r \mapsto \varphi(\Gamma\widetilde{g}(r))$$

by the same symbol φ according to the context, unless any confusion arises. For an open subinterval $J \subseteq I$ and a non-trivial non-negative function φ, let

$$\Omega_{M,\varphi,J}$$

be the probability space $\Omega_M \cap J$ with a density function $(\sum_{r\in\Omega_M\cap J} \varphi(r))^{-1}\varphi$ as long as the denominator is non-zero. For a random variable $\mathfrak{g}$ on a probability space X, we denote by $\mathbb{P}[\mathfrak{g} \mid X]$, $\mathbb{E}[\mathfrak{g} \mid X]$, and $\mathbb{V}[\mathfrak{g} \mid X]$ the probability, mean, and variance of $\mathfrak{g}$ on X, respectively.

In order to state the main results we also need the following:

Definition 1.1. Let $\Bbbk$ be an abelian group and $\psi : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \Bbbk$.

(1) If there exists a $\Bbbk$-valued function β on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ such that

$$\psi(u) = \beta(u) - \beta\left(\mathrm{j}u\begin{bmatrix} -m & 1 \\ 1 & 0 \end{bmatrix}\right)$$

for all $u \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ and integers $m \geq 1$, then ψ is called a $\mathfrak{b}$-*coboundary* over $\Bbbk$, *associated with* β. Let $\mathcal{B}_{\mathfrak{b}}(\Gamma, \Bbbk)$ be the abelian group of all $\mathfrak{b}$-coboundaries over $\Bbbk$.

(2) If there exists a $\Bbbk$-valued function β on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ such that

$$\psi(u)+\psi\left(u\begin{bmatrix}-n & 1\\ 1 & 0\end{bmatrix}\mathrm{j}\right)=\beta(u)-\beta\left(u\begin{bmatrix}-n & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}-m & 1\\ 1 & 0\end{bmatrix}\right)$$

for all $u\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ and integers $m,n\geq 1$, then ψ is called a $\mathfrak{c}$-*coboundary* over $\Bbbk$, *associated with* β. Let $\mathcal{B}_{\mathfrak{c}}(\Gamma,\Bbbk)$ be the abelian group of all $\mathfrak{c}$-coboundaries over $\Bbbk$.

Example 1.2. For a prime p and $\Gamma=\Gamma_0(p)$, we set

$$u_1=\Gamma\begin{bmatrix}* & *\\ 1 & 0\end{bmatrix},\quad u_2=\Gamma\begin{bmatrix}* & *\\ 0 & 1\end{bmatrix}=\Gamma.$$

One can show that $\mathbb{I}_{u_1}$ is neither a $\mathfrak{b}$- nor a $\mathfrak{c}$-coboundary over $\mathbb{R}$. Observe that $\mathbb{I}_{u_1}\big(\mathrm{j}v\big[\begin{smallmatrix}-m & 1\\ 1 & 0\end{smallmatrix}\big]\big)=\mathbb{I}_{u_2}(v)$ for all $v\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ and $m\in\mathbb{Z}$. Hence, $\psi=\mathbb{I}_{u_1}-\mathbb{I}_{u_2}$ is a $\mathfrak{b}$-coboundary associated with $\mathbb{I}_{u_1}$. As $\psi(u\mathrm{j})=\psi(\mathrm{j}u)$ for all u, we can also show that ψ is a $\mathfrak{c}$-coboundary associated with $\mathbb{I}_{u_1}$. Hence, $\phi=\mathbb{I}_{u_1}-\frac{1}{2}\mathbb{I}_{u_2}$ is neither a $\mathfrak{b}$- nor a $\mathfrak{c}$-coboundary over $\mathbb{R}$. A numerical example is presented in Figure 1.

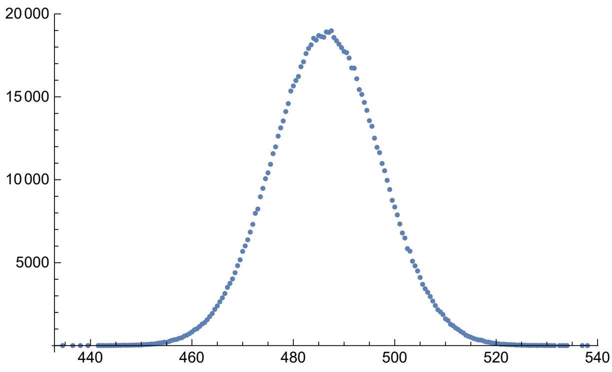

Fig. 1. Distribution of $\mathfrak{b}_\phi$ in Example 1.2 on the random samples of size 10^6 chosen from $\Omega_{2^{10000}}$ for $\Gamma=\Gamma_0(5)$.

1.2.1. Joint Gaussian distribution. One of our main results is that a vector of modular partition functions follows the Gaussian distribution asymptotically.

Theorem A. *Let J be a non-empty open subinterval of* $(0,1)$*, φ a non-trivial non-negative function on* $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$*, and* $\mathfrak{g}=\mathfrak{b}$ *or* $\mathfrak{c}$.

(1) *Let $\boldsymbol{\psi}:\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})\to\mathbb{R}^d$ with $\boldsymbol{\psi}=(\psi_1,\dots,\psi_d)$. Set $\mathfrak{g}_{\boldsymbol{\psi}}:=(\mathfrak{g}_{\psi_1},\dots,\mathfrak{g}_{\psi_d})$. For each $\boldsymbol{\psi}$, there exists $\mathrm{H}_{\boldsymbol{\psi}}\in M_d(\mathbb{R})$ (see Section 3.1 for definition) such that:*

   (a) *$\mathrm{H}_{\boldsymbol{\psi}}$ is non-singular if and only if $\psi_1,\dots,\psi_d$ are $\mathbb{R}$-linearly independent modulo $\mathcal{B}_{\mathfrak{g}}(\Gamma,\mathbb{R})$.*

   (b) *When $\mathrm{H}_{\boldsymbol{\psi}}$ is non-singular, the distribution of $\mathfrak{g}_{\boldsymbol{\psi}}$ on $\Omega_{M,\varphi,J}$ is asymptotically Gaussian as $M\to\infty$. More precisely, there exists $\mu_{\boldsymbol{\psi}}\in\mathbb{R}^d$ such that for* *any* $\mathbf{x} \in \mathbb{R}^d$,

$$\mathbb{P}\left[\frac{\mathfrak{g}_{\boldsymbol{\psi}} - \mu_{\boldsymbol{\psi}} \log M}{\sqrt{\log M}} \leq \mathbf{x} \;\middle|\; \Omega_{M,\varphi,J}\right] = \frac{1}{(2\pi)^{d/2}\sqrt{\det \mathrm{H}_{\boldsymbol{\psi}}}} \int_{\mathbf{t}\leq\mathbf{x}} \exp\left(-\frac{1}{2}\mathbf{t}^T \mathrm{H}_{\boldsymbol{\psi}}^{-1}\mathbf{t}\right) d\mathbf{t} + O\left(\frac{1}{\sqrt{\log M}}\right)$$

where $\mathbf{t} \leq \mathbf{x}$ *means* $t_j \leq x_j$ *for all* $1 \leq j \leq d$ *and the implicit constant is uniform in* $\mathbf{x}$.

(2) *Let* $d = 1$. *For* $\psi : \Gamma\backslash \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{R}$ *and* $C_\psi = \mathrm{H}_\psi$, *there exists* $D_{\psi,\varphi,J}$ *such that the variance satisfies*

$$\mathbb{V}[\mathfrak{g}_\psi \mid \Omega_{M,\varphi,J}] = C_\psi \log M + D_{\psi,\varphi,J} + O(M^{-\gamma})$$

for some $\gamma > 0$. *In particular,* $C_\psi = 0$ *if and only if* ψ *is a* $\mathfrak{g}$*-coboundary over* $\mathbb{R}$.

(3) *Let* $d = 1$ *and* $k \geq 3$. *There exists a polynomial* $Q_{J,\varphi,k}$ *of degree at most* k *such that*

$$\mathbb{E}[\mathfrak{g}_\psi^k \mid \Omega_{M,\varphi,J}] = Q_{J,\varphi,k}(\log M) + O((\log M)^k M^{-\gamma}).$$

Example 1.3. For ψ in Example 1.2, note that

$$\mathfrak{b}_\psi(r) = \sum_{i=1}^{\ell} \left(\mathbb{I}_{u_1}(\Gamma\widehat{g}_i) - \mathbb{I}_{u_1}\left(\mathrm{j}\Gamma\widehat{g}_i\left[\begin{smallmatrix} -m_i & 1 \\ 1 & 0\end{smallmatrix}\right]\right)\right),$$

which is equal to $\mathbb{I}_{u_1}(\Gamma\widehat{g}_\ell) - \mathbb{I}_{u_1}(\Gamma)$. Since $\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in u_1$ if and only if $d \equiv 0 \pmod p$, we conclude that

$$\mathfrak{b}_\psi(r) = \mathbb{I}_{u_1}(\Gamma\widehat{g}_\ell) = \begin{cases} 0 & \text{if } Q(r) \not\equiv 0 \pmod p, \\ 1 & \text{if } Q(r) \equiv 0 \pmod p. \end{cases}$$

In particular, $\mathfrak{b}_\psi$ does not follow the Gaussian distribution asymptotically.

Remark 1.4. Numerical evidence suggests that an analogue of Theorem A for the variable $\mathfrak{a}$ is plausible. However, due to the fact that the relevant transfer operator for $\mathfrak{a}$ is not topologically mixing (see Remark 5.2), the arguments in the present paper do not work for $\mathfrak{a}$. The second-named author plans to provide an approach to deal with this problem in future work.

1.2.2. Residual equidistribution. Another main result is the equidistribution of integer valued modular partition functions in residue classes of a fixed modulus. Let $\Omega_{M,J} := \Omega_M \cap J$.

Theorem B. *Let* $\mathfrak{g} = \mathfrak{b}$ *or* $\mathfrak{c}$ *and* $Q \geq 3$ *be an integer.*

(1) *Let* J *be a non-empty open subinterval of* $(0, 1)$ *and* $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}^d$. *If* $\psi_1, \ldots, \psi_d \pmod Q$ *are* $\mathbb{Z}/Q\mathbb{Z}$*-linearly independent modulo* $\mathcal{B}_\mathfrak{g}(\Gamma, \mathbb{Z}/Q\mathbb{Z})$, *then for all* $\mathbf{g} \in (\mathbb{Z}/Q\mathbb{Z})^d$ *we have*

$$\mathbb{P}[\mathfrak{g}_{\boldsymbol{\psi}} \equiv \mathbf{g} \pmod Q \mid \Omega_{M,J}] = Q^{-d} + O(M^{-\delta})$$

for some $\delta > 0$.

(2) *Let Q be relatively prime to $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ and $\psi : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}$. If there is a non-empty open subinterval J that*

$$\lim_{M\to\infty} \mathbb{P}[\mathfrak{g}_\psi \equiv g \pmod Q \mid \Omega_{M,J}] = Q^{-1}$$

for all $g \in \mathbb{Z}/Q\mathbb{Z}$, then $\psi \pmod q$ is a $\mathfrak{g}$-coboundary over $\mathbb{Z}/q\mathbb{Z}$ for no prime divisor q of Q.

Remark 1.5. From Example 1.3, we see that the random vectors $(\mathfrak{b}_{\mathbb{I}_u})_{u\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})}$ and $(\mathfrak{c}_{\mathbb{I}_u})_{u\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})}$ are not equidistributed modulo Q for $Q \geq 3$.

A specialization $d = 1$ and $\psi \equiv 1$ gives us a new result that the length ℓ of the continued fractions on $\Omega_{M,J}$ is residually equidistributed.

Corollary 1.6. *For $g \in \mathbb{Z}/Q\mathbb{Z}$, we have*

$$\mathbb{P}[\ell \equiv g \pmod Q \mid \Omega_{M,J}] = Q^{-1} + o(1).$$

1.3. Applications of main results

First, we introduce an application of Theorem A.

1.3.1. Conjecture of Mazur–Rubin. In order to understand the growth of the Mordell–Weil ranks of a rational elliptic curve in large abelian extensions, Mazur and Rubin [29] described heuristically the behavior of special values of twisted modular L-functions by presenting the conjecture on statistics for modular symbols based on numerical computations.

Let $\Gamma_0(N) = \mathrm{SL}_2(\mathbb{Z}) \cap \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right]$. Let f be a newform for $\Gamma_0(N)$ and of weight 2 with Fourier coefficients $a_f(n)$. Let χ be a Dirichlet character of conductor M. We denote by $L(s, f, \chi)$ the twisted modular L-function, which is given as the meromorphic continuation of the Dirichlet series with the coefficients $a_f(n)\chi(n)$. Let $\mathbb{Q}_f$ be the field generated by the coefficients $a_f(n)$ over $\mathbb{Q}$. It is known that $\mathbb{Q}_f$ is real. There are suitable periods $\Omega_f^\pm$ such that the following normalized special L-values are algebraic:

$$L_f(\chi) := \frac{G(\overline{\chi})L(1, f, \chi)}{\Omega_f^\pm} \in \mathbb{Q}_f(\chi)$$

where $G(\overline{\chi})$ denotes the Gauss sum and $\pm$ corresponds to the sign $\chi(-1) = \pm 1$.

The *modular symbols* are period integrals of the form

$$\mathfrak{m}_f^\pm(r) := \frac{1}{\Omega_f^\pm}\left\{\int_r^{i\infty} f(z)\,dz \pm \int_{-r}^{i\infty} f(z)\,dz\right\} \in \mathbb{Q}_f$$

for $r \in \mathbb{Q}$. We regard $\mathfrak{m}_f^\pm$ as a random variable. Set $\mathfrak{m}_E^\pm = \mathfrak{m}_{f_E}^\pm$ for the newform f_E corresponding to an elliptic curve E over $\mathbb{Q}$. The periods $\Omega_{f_E}^\pm$ can be chosen as the Néron periods $\Omega_E^\pm$. Mazur–Rubin [28] proposed the following conjecture.

Conjecture A (Mazur–Rubin). *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N.*

(1) *The random variable $\mathfrak{m}_E^{\pm}$ on Σ_M follows the asymptotic Gaussian distribution as $M \to \infty$.*

(2) *For a divisor d of N, there exist two constants $C_E^{\pm}$ and $D_{E,d}^{\pm}$, called the* variance slope *and the* variance shift *respectively, such that*

$$\lim_{\substack{M\to\infty \\ (M,N)=d}} (\mathbb{V}[\mathfrak{m}_E^{\pm} \mid \Sigma_M] - C_E^{\pm} \log M) = D_{E,d}^{\pm}.$$

Petridis–Risager [34] obtained the Ω_M-version of statement (1) for general cuspforms f of cofinite Fuchsian groups and statement (2) for the congruence subgroup $\Gamma_0(N)$ with a square-free integer N. They gave an explicit formula for the constant $C_f^{\pm}$ as well as $D_{f,d}^{\pm}$ in terms of special values of a symmetric square L-function of f. They further established an interval version of (1), that is, for any interval $J \subseteq I$, the variable $\mathfrak{m}_f^{\pm}$ on $\Omega_M \cap J$ follows the Gaussian distribution asymptotically. Their approach is based on the sophisticated theory of non-holomorphic Eisenstein series twisted by moments of modular symbols. Their work has been generalized to arbitrary weights by Nordentoft [32].

In this paper, we present another proof of the average version of Conjecture A for a newform of weight 2 for $\Gamma_0(N)$ and an arbitrary N as a specialization of the result (Theorem A) on modular partition functions.

Theorem C. *Let f be a newform for $\Gamma_0(N)$ and of weight 2.*

(1) *The random variable $\mathfrak{m}_f^{\pm}$ on $\Omega_{M,\varphi,J}$ follows the asymptotic Gaussian distribution as $M \to \infty$. More precisely, there exist $\sigma_f^{\pm}$ and $C_f^{\pm} > 0$ such that*

$$\mathbb{P}\left[\frac{\mathfrak{m}_f^{\pm} - \sigma_f^{\pm}\log M}{\sqrt{C_f^{\pm}\log M}} \leq x \;\middle|\; \Omega_{M,\varphi,J}\right] = \frac{1}{2\pi}\int_{-\infty}^{x} \exp\left(-\frac{1}{2}t^2\right) dt + O\left(\frac{1}{\sqrt{\log M}}\right).$$

Here the implicit constant is independent of x.

(2) *The* variance slope *$C_f^{\pm}$ is independent of φ and there exists a* variance shift *$D_{f,\varphi,J}^{\pm}$ such that*

$$\mathbb{V}[\mathfrak{m}_f^{\pm} \mid \Omega_{M,\varphi,J}] = C_f^{\pm}\log M + D_{f,\varphi,J}^{\pm} + O(M^{-\gamma}).$$

(3) *Let $k \geq 3$. There exists a polynomial $Q_{J,\varphi,k}$ of degree at most k such that*

$$\mathbb{E}[(\mathfrak{m}_f^{\pm})^k \mid \Omega_{M,\varphi,J}] = Q_{J,\varphi,k}(\log M) + O((\log M)^k M^{-\gamma}).$$

Theorem C directly implies the result of Petridis–Risager or the average version of Conjecture A with specific choices of φ. For (1), we take $\varphi = 1$. For a divisor d of N, we define $\varphi_d\left(\left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right]\right)$ to be 1 when $(\delta, N) = d$ and 0 otherwise. Note that φ_d is well-defined on $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. The particular choice $\varphi = \varphi_d$ shows that the Ω_M-version of Conjecture A (2) is a special case of our result.

Theorem C is a specialization of Theorem A: More precisely, there is a function $\psi_f^{\pm}$ on $\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ such that it is not a coboundary over $\mathbb{R}$ and $\mathfrak{m}_f^{\pm}$ follows the distributions of $\mathfrak{b}_{\psi_f^+}$ and $\mathfrak{c}_{\psi_f^-}$ (see Sections 4.3 and 4.4).

Remark 1.7. From the specializations $\mathfrak{b}_{\psi_f^+}$ and $\mathfrak{c}_{\psi_f^-}$, we observe that the asymptotic normality of modular symbols comes essentially not from the modularity of f, but from the dynamics of continued fractions. The modularity in our paper plays a role only in showing that $\psi_f^{\pm}$ is not a coboundary (see Section 4.3); it is also a crucial ingredient in calculating the mean (Diamantis et al. [11], Sun [41]), the variance slope and shift (Petridis–Risager [34], Blomer et al. [5]).

Remark 1.8. One may wonder if $\mathfrak{b}_{\psi_f^-}$ can be used to study the modular symbols instead of $\mathfrak{c}_{\psi_f^-}$. In fact, the answer is negative. It is the action of $\mathfrak{j}$ that prohibits $\mathfrak{m}_f^-$ from being expressed in terms of $\mathfrak{b}_{\psi_f^-}$. We refer to Remarks 4.1 and 4.2.

Remark 1.9. Bettin–Drappeau [4] proved the asymptotic Gaussian distribution of modular symbols for level 1 and arbitrary higher weights (see Remark 1.12). We speculate that by adopting their arguments, our work can be extended to arbitrary weights.

Remark 1.10. Even though computable in polynomial time (Lhote [24]), no closed forms for the variance slope and shift for the length ℓ are known from the dynamical approach. It is an interesting question whether the expressions of Petridis–Risager for $C_f^{\pm}$ and $D_{f,\varphi}^{\pm}$ are hints for this open problem.

In the next section, we discuss an application of Theorem B.

1.3.2. Non-vanishing mod p of modular L-values. Non-vanishing of twisted L-values seems to genuinely rely on the equidistribution or density results for special algebraic cycles (see Vatsal [45]). The first prominent example goes back to Ferrero–Washington [13] and Washington [46] for mod p non-vanishing of special Dirichlet L-values. A key lemma used in their proof comes precisely from a p-adic analogue of the classical density result due to Kronecker in ergodic theory. One of the main motivations of the present paper is to suggest a new dynamical approach towards the study of modular L-values with Dirichlet twists.

We can choose a suitable period $\Omega_f^{\pm}$ so that the corresponding algebraic parts $L_f(\chi)$ are p-integral with minimum p-adic valuation when, for example, the mod p Galois representation $\overline{\rho}_{f,p}$ is irreducible, p does not divide $2N$, and $N \geq 3$ (see Section 4.2). In these circumstances, the p-integral L-values are expected to be generically non-vanishing modulo p. One also obtains the p-integrality of $\mathfrak{m}_E^{\pm}$ when the residual Galois representation $\overline{\rho}_{E,p}$ of E is irreducible, and E has good and ordinary reduction at p.

For a Dirichlet character χ of modulus M, we define a variant of the special L-value $L_f(\chi)$ by

$$\Lambda_f(\chi) := \sum_{a \in (\mathbb{Z}/M\mathbb{Z})^\times} \overline{\chi}(a) \cdot \mathfrak{m}_f^{\pm}\left(\frac{a}{M}\right).$$

This L-value is closely related to the special L-value: They can differ by an Euler-like product over the prime divisors of the conductor of χ. In particular, when χ is primitive, one has $L_f(\chi) = \Lambda_f(\chi)$. We obtain a version of the mod p non-vanishing result from our dynamical setup.

Theorem D. *Let $N \geq 3$ and $p \nmid 2N$. Let f be an elliptic newform for $\Gamma_0(N)$ such that $\overline{\rho}_{f,p}$ is irreducible. Then*

$$\# \bigcup_{n \leq M} \{\chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})^\times} \mid \Lambda_f(\chi) \not\equiv 0 \ (\mathrm{mod}\ \mathfrak{p}^{1+v_{\mathfrak{p}}(\phi(n))})\} \gg M$$

where $\mathfrak{p}$ is a prime over p in $\overline{\mathbb{Q}}_p$ and $v_{\mathfrak{p}}(\phi(n))$ is the $\mathfrak{p}$-adic valuation of the Euler totient $\phi(n)$.

A similar quantitative mod p non-vanishing of Dirichlet L-values was studied by Burungale–Sun [7]: Let λ be a Dirichlet character of modulus N and $(p, NM) = 1$ with $(N, M) = 1$. Removing the condition $p \nmid \phi(M)$, their result can be formulated as follows:

$$\#\{\chi \in \widehat{(\mathbb{Z}/M\mathbb{Z})^\times} \mid L(0, \lambda\chi) \not\equiv 0 \ (\mathrm{mod}\ \mathfrak{p}^{1+v_{\mathfrak{p}}(\phi(M))})\} \gg M^{1/2-\varepsilon}.$$

Let us remark that even though Theorem D is not as strong as the result of Burungale–Sun, as far as we know it is the first result of this type for modular L-values with Dirichlet twists. In fact, this non-vanishing result is a consequence of one of our main results on another Mazur–Rubin conjecture [28].

Conjecture B (Mazur–Rubin). *Assume that $\overline{\rho}_{E,p}$ is irreducible and E has good and ordinary reduction at p. Then, for any integer a,*

$$\lim_{M \to \infty} \mathbb{P}[\mathfrak{m}_E^{\pm} \equiv a \ (\mathrm{mod}\ p) \mid \Sigma_M] = \frac{1}{p}.$$

Here is our result on the residual equidistribution of modular symbols.

Theorem E. *Assume that $\overline{\rho}_{E,p}$ is irreducible and E has good and ordinary reduction at p. Then, for any $e \geq 1$ and any integer a,*

$$\mathbb{P}[\mathfrak{m}_E^{\pm} \equiv a \ (\mathrm{mod}\ p^e) \mid \Omega_{M,J}] = \frac{1}{p^e} + O(M^{-\delta})$$

for some $\delta > 0$.

This is a specialization of Theorem B. More precisely, there are integer valued functions $\zeta_E^{\pm}$ on $\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ such that their reductions modulo p^e are not coboundaries over $\mathbb{Z}/p^e\mathbb{Z}$ and $\mathfrak{m}_E^{\pm}$ (mod p^e) follow the distributions of $\mathfrak{b}_{\zeta_E^+}$ and $\mathfrak{c}_{\zeta_E^-}$ (see Section 4.5).

Remark 1.11. Constantinescu–Nordentoft [10] obtained a discrete version of the result of Petridis–Risager [34], implying Theorem E.

Mazur, in a private communication, raised the question of whether the Gaussian (or Archimedean) and residual distributions of the modular symbols are correlated or not. We answer this question in Theorem 4.7, which is a consequence of a more general discussion in Section 3.3.

1.4. Dynamics of continued fractions: Work of Baladi–Vallée

We now describe our approach. It is deeply motivated by the work of Baladi–Vallée [3] on dynamics of continued fractions. Let us briefly outline their result and strategy for the proof.

Baladi–Vallée established the quasi-power behavior of the moment generating function $\mathbb{E}[\exp(w\ell) \mid \Omega_M]$, which ensures the asymptotic Gaussian distribution of ℓ (see Theorem 3.7). More precisely, they studied a Dirichlet series whose coefficients are essentially given by the moment generating function $\mathbb{E}[\exp(w\ell) \mid \Sigma_n]$:

$$L(s,w) = \sum_{n \geq 1} \frac{c_n(w)}{n^s}, \quad c_n(w) = \sum_{r \in \Sigma_n} \exp(w\ell(r)),$$

for two complex variables s, w with $\Re s > 1$ and $|w|$ being sufficiently small. The desired estimate then follows from the Tauberian argument on $L(s,w)$. To this end, they established the analytic properties of the poles of the Dirichlet series $L(s,w)$ and uniform estimates on its growth in a vertical strip. Their crucial observation is that the weighted transfer operator plays a central role in ensuring the necessary properties of $L(s,w)$.

Let $T : I \to I$ denote the Gauss map given by $T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ for $x \neq 0$ and $T(0) = 0$. They considered the weighted transfer operator associated with the Gauss dynamical system (I, T), defined by

$$H_{s,w} f(x) := \sum_{y : T(y) = x} \frac{\exp(w)}{|T'(y)|^s} \cdot f(y)$$

for two complex variables s and w. A key relation they established is that

$$L(2s, w) = F_{s,w}(Id - H_{s,w})^{-1} 1(0) \tag{1.1}$$

where Id is the identity operator and $F_{s,w}$ is the final operator defined like $H_{s,w}$ with summation restricted to $y(x) = \frac{1}{m+x}$ with $m \geq 2$. The properties of Dirichlet series that are crucial for the Tauberian argument thus directly follow from the spectral properties of the transfer operator. In particular, the estimate on the growth of $L(s,w)$ in a vertical strip comes from the Dolgopyat–Baladi–Vallée estimate on the operator norm of $H^n_{s,w}$, $n \geq 1$.

Our idea is to follow their framework by finding a certain dynamical system and the corresponding transfer operator that naturally describe the analytic properties of Dirichlet series associated to modular partition functions.

Remark 1.12. Bettin–Drappeau [4] generalized the work of Baladi–Vallée in a different direction and obtained distributional results for crucial examples of quantum modular forms.

1.5. Dynamical system for modular partition functions

Let us describe the dynamics and transfer operators for modular partition functions.

Let φ be a function on the right cosets of Γ and J a non-empty open subinterval of I. To study the moment generating function of $\mathfrak{g}_{\boldsymbol{\psi}}$ on $\Omega_{M,\varphi,J}$, for $\mathbf{w} \in \mathbb{C}^d$ we set

$$c_n(\mathbf{w}) := \sum_{r \in \Sigma_n \cap J} \varphi(r) \exp(\mathbf{w} \cdot \mathfrak{g}_{\boldsymbol{\psi}}(r)).$$

Obviously,

$$\mathbb{E}[\exp(\mathbf{w} \cdot \mathfrak{g}_{\boldsymbol{\psi}}) \mid \Omega_{M,\varphi,J}] = \frac{\sum_{n \le M} c_n(\mathbf{w})}{\sum_{n \le M} c_n(\mathbf{0})}.$$

In order to study $c_n(\mathbf{w})$, we consider the generating function, namely a Dirichlet series: For $s \in \mathbb{C}$, set $L^{\mathfrak{g}}(s, \mathbf{w}) := \sum_{n \ge 1} \frac{c_n(\mathbf{w})}{n^s}$. The strategy is to apply Tauberian arguments to $L^{\mathfrak{g}}(s, \mathbf{w})$ using their behaviors in a critical strip of $\mathbb{C}$, which are expected to be consequences of the dynamical analysis of the modular partition functions $\mathfrak{g}_{\boldsymbol{\psi}}$. For $\psi = \mathbf{w} \cdot \boldsymbol{\psi}$, we get $\mathbf{w} \cdot \mathfrak{g}_{\boldsymbol{\psi}} = \mathfrak{g}_{\psi}$. Hence, for the dynamical analysis, we consider the transfer operators with parameter ψ instead of $\mathbf{w}$.

1.5.1. Random variable $\mathfrak{a}$. Let us define an operator $\mathbf{T}$ on $I \times \Gamma\backslash \mathrm{GL}_2(\mathbb{Z})$ by

$$\mathbf{T}(x, v) := \left(T(x), v \begin{bmatrix} -m_1(x) & 1 \\ 1 & 0 \end{bmatrix} \right)$$

where $m_1(x)$ denotes the first digit of the continued fraction expansion of x. We call $\mathbf{T}$ the *skewed Gauss map*.

Let Ψ be a bounded function on the product $I \times \Gamma\backslash \mathrm{GL}_2(\mathbb{Z})$. For $s \in \mathbb{C}$ and $\psi : \Gamma\backslash \mathrm{GL}_2(\mathbb{Z}) \to \mathbb{C}$, we consider the weighted *transfer operator* associated to the dynamical system $(I \times \Gamma\backslash \mathrm{GL}_2(\mathbb{Z}), \mathbf{T})$ defined by

$$\mathcal{L}_{s,\psi} \Psi(x, u) := \sum_{(y,v) \in \mathbf{T}^{-1}(x,u)} \frac{\exp[\psi(v)]}{|T'(y)|^s} \cdot \Psi(y, v).$$

Let $\mathcal{F}_{s,\psi}$ be the *final operator* defined like $\mathcal{L}_{s,\psi}$ with summation restricted to $(y, v) = \left(\frac{1}{m+x}, u\left[\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}\right]\right)$ with $m \ge 2$. To study the space $\Omega_{M,\varphi,J}$, we also introduce interval operators $\mathcal{D}^J_{s,\psi}$ (see Section 6.4). Our crucial observation is that the Dirichlet series for $\mathfrak{a}$ admits an alternative expression in terms of weighted transfer operators (see Theorem 6.10): The quasi-inverse $(\mathcal{I} - \mathcal{L}_{s,\psi})^{-1}$ is well-defined when $(\Re s, \Re \mathbf{w})$ is close to $(1, \mathbf{0})$ (See Theorem 8.5). Then, for an interval $J \subset I$, we have

$$L^{\mathfrak{a}}(2s, \mathbf{w}) = \mathcal{B}^J_{s,\psi}(1 \otimes \varphi)(0, \Gamma) + \mathcal{D}^J_{s,\psi}(\mathcal{I} - \mathcal{L}_{s,\psi})^{-1} \mathcal{F}_{s,\psi}(1 \otimes \varphi)(0, \Gamma) \tag{1.2}$$

for $\psi = \mathbf{w} \cdot \boldsymbol{\psi}$ and an auxiliary analytic operator $\mathcal{B}^J_{s,\psi}$.

Remark 1.13. When $J = (0, 1)$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and $\varphi = 1$, the expression (1.1) can be recovered from the above expression of $L^{\mathfrak{a}}(s, \mathbf{w})$.

1.5.2. Random variable $\mathfrak{b}$. First of all, note that there is a natural right action of $\mathrm{GL}_2(\mathbb{Z})$ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ given by

$$(\Gamma h)\cdot g := \Gamma\,\widehat{hg}.$$

With this right action, we consider the spaces

$$I_\Gamma := I \times \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}).$$

Let us define the *skewed Gauss map* $\widehat{\mathbf{T}}$ on I_Γ by

$$\widehat{\mathbf{T}}(x,v) := \left(T(x), v\cdot\begin{bmatrix}-m_1(x) & 1\\ 1 & 0\end{bmatrix}\right).$$

As in the case of $\mathfrak{a}$, for a function φ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ we define the weighted *transfer operator* $\widehat{\mathcal{L}}_{s,\varphi}$ associated to the dynamical system $(I_\Gamma, \widehat{\mathbf{T}})$, the *final operator* $\widehat{\mathcal{F}}_{s,\varphi}$, and the *interval operator* $\widehat{\mathcal{D}}^J_{s,\varphi}$ (see Section 6.4). We are also able to obtain a version of (1.2), i.e., an analogous expression for $L^{\mathfrak{b}}(s,\mathbf{w})$ in terms of $\widehat{\mathcal{L}}_{s,\mathbf{w}\cdot\boldsymbol{\psi}}$, $\widehat{\mathcal{F}}_{s,\mathbf{w}\cdot\boldsymbol{\psi}}$, $\widehat{\mathcal{D}}^J_{s,\mathbf{w}\cdot\boldsymbol{\psi}}$, and $\widehat{\mathcal{B}}^J_{s,\mathbf{w}\cdot\boldsymbol{\psi}}$ (see Theorem 6.10), which is partly a consequence of the existence of the right action.

Remark 1.14. This type of skew-product Gauss map has already been discussed by Manin–Marcolli [26] in a different context, to study the Gauss–Kuz'min operator and the limiting behavior of modular symbols. We refer to Remark 4.2.

1.5.3. Random variable $\mathfrak{c}$. Unlike $g\mapsto\widehat{g}$, the map $g\mapsto\widetilde{g}$ does not induce a right action of $\mathrm{GL}_2(\mathbb{Z})$ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Even though one can easily define $\mathfrak{c}$-analogues of $\widehat{\mathbf{T}}$ and $\widehat{\mathcal{L}}_{s,\mathbf{w}}$, say $\widetilde{\mathbf{T}}$ and $\widetilde{\mathcal{L}}_{s,\mathbf{w}}$, the Dirichlet series $L^{\mathfrak{c}}(s,\mathbf{w})$ no longer admits an expression similar to (1.2), especially in terms of $\widetilde{\mathcal{L}}_{s,\mathbf{w}}$, mainly due to the absence of a suitable right action of $\mathrm{GL}_2(\mathbb{Z})$.

Instead, we first observe that the maps $\widehat{\mathbf{T}}^2$ and $\widetilde{\mathbf{T}}^2$ are the same as $\mathbf{T}^2|_{I_\Gamma}$, whose second component is now the canonical right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Then one can define another weighted transfer operator $\mathcal{M}_{s,\varphi}$ associated to the system $(I_\Gamma, \widetilde{\mathbf{T}}^2)$. After defining analogues of the previous operators, namely the final operator $\widetilde{\mathcal{F}}_{s,\varphi}$, the interval operator $\widetilde{\mathcal{D}}^J_{s,\varphi}$, and the auxiliary operator $\widetilde{\mathcal{B}}^J_{s,\varphi}$, we are able to express the Dirichlet series $L^{\mathfrak{c}}(s,\mathbf{w})$ in terms of those operators as before (see Theorem 6.10).

Remark 1.15. Note that a function $\psi : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \Bbbk$ is a $\mathfrak{b}$-coboundary over $\Bbbk$ if and only if there exists a $\Bbbk$-valued function β on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ such that $\psi = \beta - \beta\circ\pi_2\widehat{\mathbf{T}}$. And ψ is a $\mathfrak{c}$-coboundary if and only if $\psi + \psi\circ\pi_2\widetilde{\mathbf{T}} = \beta - \beta\circ\pi_2\mathbf{T}^2$ for some β.

1.6. Spectral analysis of transfer operators

For (s,ψ) with real part $(\Re s, \Re\psi)$ close to $(1,\mathbf{0})$, the transfer operators for $\mathfrak{b}$ and $\mathfrak{c}$ act boundedly on $C^1(I_\Gamma)$ and admit a spectral gap with dominant eigenvalues $\lambda_{s,\psi}$. Then by analogues of the identity (1.2), the poles s of the Dirichlet series with a fixed ψ in a certain vertical strip are in a bijection with the values s with $\lambda_{s,\psi} = 1$. Hence the analytic properties of $L^{\mathfrak{g}}(s,\mathbf{w})$ (Proposition 3.2) that are necessary to apply the Tauberian theorem follow from the spectral properties of the transfer operator: For general Γ, the dominant eigenvalue of the transfer operator is simple. The topological mixing property of $\widehat{\mathbf{T}}$ ensures the uniqueness of the eigenvalue. See Section 5.1 for more details.

Structure of the paper

In Section 2, we collect several group-theoretic results relevant to topological mixing of $\widehat{\mathbf{T}}$ and the coboundary condition for modular partition functions. In Sections 3 and 4, a series of number-theoretic results on the distribution of modular partition functions are deduced by a Tauberian argument from the behaviors of Dirichlet series. Their proofs will be presented in Section 10. Two transitional sections 5 and 6 are devoted to transforming the number-theoretic assertions to dynamical ones. In Sections 7–9, dynamical analyses of the corresponding transfer operators are presented.

2. $\mathrm{GL}_2(\mathbb{Z})$-action on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$

Throughout, we fix a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.

2.1. *Right action of* $\mathrm{GL}_2(\mathbb{Z})$

Set $\mathrm{J} := \langle \mathrm{j} \rangle$ and let $\mathrm{G}\Gamma := \langle \Gamma, \mathrm{j} \rangle$ be the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by Γ and j. Both the right cosets $\mathrm{G}\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ and the double cosets $\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})/\mathrm{J}$ are identified with $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ by the maps

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{G}\Gamma\backslash\mathrm{GL}_2(\mathbb{Z}), \quad u = \Gamma h \mapsto \mathrm{G}u := \mathrm{G}\Gamma h, \tag{2.1}$$
$$\Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \simeq \Gamma\backslash\mathrm{GL}_2(\mathbb{Z})/\mathrm{J}, \quad v = \Gamma h \mapsto v\mathrm{J} := \Gamma h\mathrm{J}. \tag{2.2}$$

The right action of $\mathrm{GL}_2(\mathbb{Z})$ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ discussed in Section 1.5.2 actually comes from the natural action on $\mathrm{G}\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ via the identification (2.1).

Remark 2.1. On the other hand, no right action of $\mathrm{GL}_2(\mathbb{Z})$ originates from (2.2). We can still observe that the map $\Gamma h\mathrm{J} \mapsto \Gamma hg\mathrm{J}$ is a permutation of $\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})/\mathrm{J}$ for a $g \in \mathrm{GL}_2(\mathbb{Z})$, in other words, the map $t_g : \Gamma h \mapsto \Gamma\widetilde{hg}$ is a permutation of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Further, for $u \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$, it can be observed that

$$t_{g_2}(t_{g_1}(u)) = ug_1g_2 \quad \text{if } g_1 \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.3}$$

Let u be a right coset of Γ in $\mathrm{SL}_2(\mathbb{Z})$ and $g \in \mathrm{GL}_2(\mathbb{Z})$. It is easy to see that $\widehat{g} \in u$ if and only if $\mathrm{G}\Gamma g = \mathrm{G}u$ and that $\widetilde{g} \in u$ if and only $\Gamma g\mathrm{J} = u\mathrm{J}$. We extend a function ψ on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ to $\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ such that $\widehat{\psi} := \psi \circ \mathrm{G}^{-1} \circ p_1$ and $\widetilde{\psi} := \psi \circ \mathrm{J}^{-1} \circ p_2$ where p_i are the canonical surjections $p_1 : \Gamma\backslash\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{G}\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ and $p_2 : \Gamma\backslash\mathrm{GL}_2(\mathbb{Z}) \to \Gamma\backslash\mathrm{GL}_2(\mathbb{Z})/\mathrm{J}$. It is easy to see that

$$\widehat{\psi}(\Gamma g) = \psi(\Gamma\widehat{g}) \quad \text{and} \quad \widetilde{\psi}(\Gamma g) = \psi(\Gamma\widetilde{g}).$$

Hence, from the definition of $\mathfrak{b}_\psi$ and $\mathfrak{c}_\psi$, one obtains

$$\mathfrak{b}_\psi(r) = \sum_{i=1}^{\ell} \widehat{\psi}(\Gamma g_i(r)) \quad \text{and} \quad \mathfrak{c}_\psi(r) = \sum_{i=1}^{\ell} \widetilde{\psi}(\Gamma g_i(r)). \tag{2.4}$$

Let us first prove several preliminary results on the special linear group.

2.2. *T-mixing*

In this section, a matrix of the form $\left[\begin{smallmatrix} -m & \pm 1 \\ 1 & 0 \end{smallmatrix}\right]$ is called a *digit matrix*. The following lemma and proposition are useful when we discuss the topological properties of $\widehat{\mathbf{T}}$.

Lemma 2.2. *Let $\varepsilon = \pm 1$ be fixed.*

(1) *For any $v \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$, we have*

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) = \left\{ v \cdot \begin{bmatrix} -m_1 & \varepsilon \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -m_2 & \varepsilon \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -m_\ell & \varepsilon \\ 1 & 0 \end{bmatrix} \,\middle|\, \ell \geq 0,\ m_i \in \mathbb{Z}_{\geq 1} \right\} \tag{2.5}$$

where the element for $\ell = 0$ corresponds to v.

(2) *There exists $K \geq 1$ such that for each integer $k \geq K$, we can find integers $m_1, \ldots, m_k \geq 1$ such that*

$$\Gamma \cdot \begin{bmatrix} -m_1 & \varepsilon \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -m_2 & \varepsilon \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -m_k & \varepsilon \\ 1 & 0 \end{bmatrix} = \Gamma.$$

Proof. Consider first the case of $\varepsilon = 1$. Let us denote the RHS of (2.5) by S. Let

$$a = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \quad \text{and} \quad b = \left[\begin{smallmatrix} -2 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

Let $u \in S$. As Γ is of finite index, there exist integers $p, q \geq 1$ such that for all u, we have $u \cdot a^p = u$ and $u \cdot b^q = u$, and hence $u \cdot a^{-1}, u \cdot b^{-1} \in S$. In sum, we conclude that for any $g \in \mathrm{GL}_2(\mathbb{Z})$ generated by a and b, we have $u \cdot g \in S$. On the other hand, observe that

$$ab^{-1} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], \quad a^{-1}b = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \quad ab^{-1}a = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right].$$

It is well-known that $\mathrm{GL}_2(\mathbb{Z})$ is generated by these three elements, hence by a and b. Since $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) = \Gamma \cdot \mathrm{GL}_2(\mathbb{Z})$, we obtain statement (1).

For the second statement, observe that $ab^{-1}a^2b^{-1}a = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $b^{-1}a^3b^{-1} = \mathrm{j}$. When applied to a coset, the first product above can be regarded as the product of $2(q-1)+4$ digit matrices. The second one is the product of $2(q-1)+3$ digit matrices. Then there exists a number K such that any integer $k \geq K$ can be written as $k = (2(q-1)+4)s + (2(q-1)+3)t$ with $s, t \geq 1$. Then

$$\Gamma \cdot (ab^{-1}a^2b^{-1}a)^s(b^{-1}a^3b^{-1})^t = \Gamma \cdot \mathrm{j}^t = \Gamma$$

since $\Gamma \cdot \mathrm{j} = \Gamma$.

For the case of $\varepsilon = -1$, set $c = \left[\begin{smallmatrix} -Q & -1 \\ 1 & 0 \end{smallmatrix}\right]$ where Q is an integer > 0 such that $u\left[\begin{smallmatrix} 1 & -Q \\ 0 & 1 \end{smallmatrix}\right] = u$ for all u. Then $uc = u\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ for each u. We also set $d = \left[\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $e = \left[\begin{smallmatrix} -2 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then $de^{-1} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, we obtain the first statement by a similar argument to the one above. Note that $u = uc^4$ and $u = ud^3$ for all u. As before, we obtain the second statement. ∎

Remark 2.3. A version of (1) can also be found in Manin–Marcolli [26, Theorem 0.2.1].

Proposition 2.4. *Fix $\varepsilon = \pm 1$. There exists an $M > 0$ such that for any $u \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ and any $\ell \geq M$,*

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) = \left\{ u \cdot \begin{bmatrix} -m_1 & \varepsilon \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} -m_\ell & \varepsilon \\ 1 & 0 \end{bmatrix} \,\middle|\, m_i \in \mathbb{Z}_{\geq 1} \right\}.$$

Proof. Let Γ_0 be the kernel of the homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to the permutation group on $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ induced from the right action. Then Γ_0 is normal, of finite index, and is normalized by j. Since the statement for Γ follows from one for Γ_0, we may assume that Γ is normal.

First fix representations of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ in (2.5), i.e., product representations by digit matrices. For a right coset u, we can find a product $\mathrm{m}(u)$ of digit matrices such that $u = \Gamma \cdot \mathrm{m}(u)$. Let $\ell(u)$ be the number of digit matrices that form $\mathrm{m}(u)$ and $L := \max_u \ell(u)$. We claim that for any $n \geq L + K$ and any two right cosets u, v of Γ, there are n digit matrices whose product, say m_{uv}, satisfies $u = v \cdot \mathrm{m}_{uv}$.

First of all, we show the claim for $v = \Gamma$. Let w_k be the product of k digit matrices in Lemma 2.2 (2). For any $n \geq L + K$, set $\mathrm{m}_n[u] := \mathrm{w}_{n-\ell(u)}\mathrm{m}(u)$. Observe that $u = \Gamma \cdot \mathrm{m}_n[u]$ and $\mathrm{m}_n[u]$ is the product of n digit matrices.

Let v be a general right coset. Set $u = \Gamma g$ and $v = \Gamma h$ for $g, h \in \mathrm{SL}_2(\mathbb{Z})$. Let $n \geq L + K$. Then

$$\Gamma g \cdot \mathrm{m}_n[\Gamma g^{-1} h] = g\Gamma \cdot \mathrm{m}_n[\Gamma g^{-1} h] = g\Gamma g^{-1} h = \Gamma h$$

as Γ is normal in $\mathrm{SL}_2(\mathbb{Z})$. ∎

2.3. *Coboundary functions*

Let $\Bbbk$ be an abelian group. In this section, we characterize all the coboundary functions over $\Bbbk$. We fix $\beta : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \Bbbk$ corresponding to a coboundary ψ, i.e., $\psi(u) = \beta(u) - \beta(u \cdot \left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right])$ if $\mathfrak{g} = \mathfrak{b}$ and $\psi(u) + \psi\big(u\left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right]\mathrm{j}\big) = \beta(u) - \beta\big(u\left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} -n & 1 \\ 1 & 0 \end{smallmatrix}\right]\big)$ if $\mathfrak{g} = \mathfrak{c}$. Let

$$\mathrm{L} := \begin{bmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{bmatrix}.$$

Since $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ is a finite set, the natural right action of L on the cosets $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ factors through $\left[\begin{smallmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 1 & 0 \\ \mathbb{Z}/Q\mathbb{Z} & 1 \end{smallmatrix}\right]$ for an integer $Q > 1$.

We collect several properties of β:

Proposition 2.5. *Let $\psi : \Gamma\backslash \mathrm{SL}_2(\mathbb{Z}) \to \Bbbk$ be a function.*

(1) *If ψ is a $\mathfrak{g}$-coboundary, then β is* L*-invariant. In particular,*

$$\beta\big(u\cdot\big[\begin{smallmatrix} m & 1\\ 1 & 0\end{smallmatrix}\big]\big)=\beta\big(u\cdot\big[\begin{smallmatrix} n & 1\\ 1 & 0\end{smallmatrix}\big]\big)\quad \textit{for all } m,n\in\mathbb{Z}. \tag{2.6}$$

(2) *Let ψ be a $\mathfrak{b}$-coboundary. If $\psi(u)=\psi(-u)$ and $\psi(u\mathrm{j})=\psi(\mathrm{j}u)$ for all u, then $\beta(u\mathrm{j})=\beta(\mathrm{j}u)$ for all u.*

Proof. (1) Consider $\mathfrak{g}=\mathfrak{b}$. Then $\psi(u)=\beta(u)-\beta\big(u\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big)$ for all $m\in\mathbb{Z}$ and u. Observe also that $\big[\begin{smallmatrix} \mathbb{Z} & 1\\ 1 & 0\end{smallmatrix}\big]=\big[\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\big]\big[\begin{smallmatrix} 1 & 0\\ \mathbb{Z} & 1\end{smallmatrix}\big]$. Then the coboundary condition on ψ implies that β is invariant under L.

Let ψ be a $\mathfrak{c}$-coboundary. As above, for all $m,n\in\mathbb{Z}$, we get $\psi(u)+\psi\big(u\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\mathrm{j}\big)=\beta(u)-\beta\big(u\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big[\begin{smallmatrix} -n & 1\\ 1 & 0\end{smallmatrix}\big]\big)$. Setting $m=0$, we get $\beta\big(u\big[\begin{smallmatrix} 1 & 0\\ -n & 1\end{smallmatrix}\big]\big)=\beta(u)-\psi(u)-\psi(u\iota)$. Hence, β is L-invariant.

(2) Setting $\alpha(u):=\beta(u)-\beta(-u)$, we get $\alpha(u)=\alpha\big(u\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big)$ for all u and m. By Proposition 2.4, we know α is constant, in particular $\alpha(u)=\alpha(-u)$. But $\alpha(-u)=-\alpha(u)$. Hence, $\alpha=\mathbf{0}$, i.e.,

$$\beta(u)=\beta(-u)\quad \text{for all } u. \tag{2.7}$$

The expression for ψ can be written as $\beta(u\mathrm{j})-\beta(\mathrm{j}u)=\beta\big(u\mathrm{j}\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big)-\beta\big(\mathrm{j}u\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big)$. Using $\mathrm{j}\big[\begin{smallmatrix} m & 1\\ 1 & 0\end{smallmatrix}\big]\mathrm{j}=-\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]$ and (2.7) with (2.6), the last expression equals $\beta(u\mathrm{j})-\beta(\mathrm{j}u)=\beta\big(u\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\mathrm{j}\big)-\beta\big(\mathrm{j}u\cdot\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\big)$. Hence, $u\mapsto\beta(u\mathrm{j})-\beta(\mathrm{j}u)$ is constant. Considering $\mathrm{j}u\mathrm{j}$, we obtain the statement. ∎

The following is crucial for determining $\mathcal{B}_{\mathfrak{g}}(\Gamma,\Bbbk)$.

Proposition 2.6. *Let ψ be a $\mathfrak{g}$-coboundary over $\Bbbk$ for an* L*-invariant β. Then ψ is zero if and only if β is a constant. In this case, β can be chosen to be zero.*

Proof. First let $\mathfrak{g}=\mathfrak{b}$. If ψ is zero, then by Proposition 2.4, we can find $m_1,\dots,m_\ell$ for a sufficiently large ℓ and v such that $u\cdot\big[\begin{smallmatrix} -m_1 & 1\\ 1 & 0\end{smallmatrix}\big]\cdots\big[\begin{smallmatrix} -m_k & 1\\ 1 & 0\end{smallmatrix}\big]=v$. Hence, β is a constant. The converse is trivial.

Let $\mathfrak{g}=\mathfrak{c}$. Let β be a constant function. Then $\psi(u)+\psi\big(u\big[\begin{smallmatrix} -m & 1\\ 1 & 0\end{smallmatrix}\big]\mathrm{j}\big)=0$ for all m, i.e., $\psi(u)=-\psi\big(u\big[\begin{smallmatrix} -m & -1\\ 1 & 0\end{smallmatrix}\big]\big)=0$ for all $m\geq 1$. By Proposition 2.4, we can find $m_1,\dots,m_\ell$ for a sufficiently large odd ℓ such that $u\big[\begin{smallmatrix} -m_1 & -1\\ 1 & 0\end{smallmatrix}\big]\cdots\big[\begin{smallmatrix} -m_\ell & -1\\ 1 & 0\end{smallmatrix}\big]=u$. So, we get $\psi=0$. Conversely, suppose that ψ is zero. Then, as above, we can show that β is a constant. ∎

Now, the boundary functions are completely characterized.

Corollary 2.7. (1) *Let $U=\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})/\mathrm{L}\setminus\{\Gamma\}$. There is an isomorphism*

$$\mathcal{B}_{\mathfrak{g}}(\Gamma,\Bbbk)\simeq\Bbbk^U.$$

(2) *For every integer $Q>1$, the Q-torsion subgroup of $\mathcal{B}_{\mathfrak{g}}(\Gamma,\mathbb{R}/2\pi\mathbb{Z})$ is equal to $\mathcal{B}_{\mathfrak{g}}(\Gamma,2\pi Q^{-1}\mathbb{Z}/2\pi\mathbb{Z})$, which is isomorphic to $\mathcal{B}_{\mathfrak{g}}(\Gamma,\mathbb{Z}/Q\mathbb{Z})$ by the map $\psi\mapsto\frac{Q}{2\pi}\psi \pmod{Q}$.*

Proof. The map $\psi \mapsto \beta|_U - \beta(\Gamma)$ is an isomorphism. For the second statement, observe that the Q-torsion subgroup of $(\mathbb{R}/2\pi\mathbb{Z})^U$ is just $(2\pi Q^{-1}\mathbb{Z}/2\pi\mathbb{Z})^U$. ∎

3. Modular partition functions of continued fractions

In this section, we prove the limit joint Gaussian distribution and the residual equidistribution of modular partition functions of $\mathfrak{b}$ and $\mathfrak{c}$ over $\Omega_{M,\varphi,J}$. We consider the setting of Section 1.5. Throughout, we fix a non-empty open interval $J \subseteq I$ and a non-trivial function $\varphi : \Gamma\backslash \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{R}_{\geq 0}$ unless explicitly mentioned otherwise. Recall that $\mathfrak{j}$ is assumed to normalize Γ. Set

$$\mathfrak{g} := \mathfrak{b} \text{ or } \mathfrak{c}.$$

We define a map on Σ_M by

$$r = [0; m_1, \ldots, m_\ell] \mapsto r^* := [0; m_\ell, \ldots, m_1].$$

For a function Ψ on I_Γ, we denote the two functions on $\mathbb{Q} \cap I$,

$$r \mapsto \Psi(r^*, \Gamma\widehat{g}(r)) \quad \text{and} \quad r \mapsto \Psi(r^*, \Gamma\widetilde{g}(r)),$$

by the same symbol Ψ^* according to the choice of $\mathfrak{g}$. We define the Dirichlet series associated to $\mathfrak{g}$ as

$$L_{\Psi,J}(s, \mathbf{w}) := \sum_{n\geq 1} \frac{d_n(\mathbf{w})}{n^s} \quad \text{with} \quad d_n(\mathbf{w}) = \sum_{r\in\Sigma_n\cap J} \Psi^*(r) \exp(\mathbf{w}\cdot \mathfrak{g}_{\boldsymbol{\psi}}(r))$$

for $s \in \mathbb{C}$, $\mathbf{w} \in \mathbb{C}^d$, and $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d)$. The average of the coefficients $d_n(\mathbf{w})$ can be studied using the following truncated Perron formula.

Theorem 3.1 (Perron's formula, Titchmarsh [42, Lemma 3.12]). *Suppose a_n is a sequence and $A(x)$ is a non-decreasing function such that $|a_n| = O(A(n))$. Let $F(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ for $\sigma := \Re s > \sigma_a$, the abscissa of absolute convergence of $F(s)$. Then for all $D > \sigma_a$ and $T > 0$,*

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{D-iT}^{D+iT} F(s)\frac{x^s}{s}\,ds + O\left(\frac{x^D|F|(D)}{T}\right) + O\left(\frac{A(2x)x\log x}{T}\right) + O\left(A(N)\min\left\{\frac{x}{T|x-N|}, 1\right\}\right)$$

where $|F|(\sigma) = \sum_{n\geq 1} \frac{|a_n|}{n^\sigma}$ for $\sigma > \sigma_a$ and N is the nearest integer to x.

In order to shift the contour, we use the following properties of Dirichlet series in the vertical strip. We say an $\mathbb{R}$-valued function is a $\mathfrak{g}$-*coboundary over* $\mathbb{R}/2\pi\mathbb{Z}$ if its composition with the surjection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ is a $\mathfrak{g}$-coboundary.

Proposition 3.2. *Let $\mathbf{v} \in \mathbb{R}^d$. There exists $0 < \alpha_1 \leq 1/2$ such that for any $\widehat{\alpha}_1$ with $0 < \widehat{\alpha}_1 < \alpha_1$, there exists a neighborhood W of $i\mathbf{v}$ in $\mathbb{C}^d$ such that:*

(1) *If* $\mathbf{v}\cdot\boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z})$, *then* $L_{\Psi,J}(2s, \mathbf{w})$ *has a unique simple pole at* $s = s(\mathbf{w})$ *in the strip* $|\Re s - 1| \le \alpha_1$ *for each* $\mathbf{w} \in W$ *with the following properties:*
   (a) $s(\mathbf{w})$ *is analytic in* W *and* $s(i\mathbf{v}) = 1$.
   (b) $\Re s(\mathbf{w}) > 1 - (\alpha_1 - \hat{\alpha}_1)$.
   (c) *The Hessian of* $s(\mathbf{w})$ *is non-singular at* $\mathbf{w} = i\mathbf{v}$ *if and only if* $\psi_1, \ldots, \psi_d$ *are* $\mathbb{R}$*-linearly independent modulo* $\mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R})$.
   (d) *The residue* $E_{\mathbf{v}}(\mathbf{w})$ *at* $s(\mathbf{w})$ *is analytic on* W *with*

$$E_{\mathbf{v}}(i\mathbf{v}) = \frac{e^{i\beta(\Gamma)} 6|J|}{\pi^2 \log 2} \int_{(0,1/2)\times\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} e^{-i\beta}\Psi\, dm$$

   where β *is associated with* $\mathbf{v}\cdot\boldsymbol{\psi}$. *Here* $|J|$ *is the length of* J.

(2) *If* $\mathbf{v}\cdot\boldsymbol{\psi} \notin \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z})$, *then* $L_{\Psi,J}(2s, \mathbf{w})$ *is analytic in the strip* $|\Re s - 1| \le \alpha_1$ *for all* $\mathbf{w} \in W$.

(3) *For* $0 < \xi < 1/5$, *there exist* $0 < \alpha_0 \le \alpha_1$, $0 < \rho < 1$, *and a neighborhood* B *of* $\mathbf{0}$ *in* $\mathbb{R}^d$ *such that for any* $\Psi \in C^1(I_\Gamma)$ *and all* $\mathbf{w} \in \mathbb{C}^d$ *with* $\Re\mathbf{w} \in B$, *we have*

$$|L_{\Psi,J}(2s, \mathbf{w})| \ll \max(1, |\Im s|^{\xi})$$

   when $|\Re s - 1| \le \alpha_0$ *with* $|\Im s| \ge 1/\rho^2$ *or* $\Re s = 1 \pm \alpha_0$ *with* $|\Im s| \le 1/\rho^2$.

Remark 3.3. Proposition 3.2 is a combination of Lemmas 8 and 9 of Baladi–Vallée [3], which correspond to the cases (1) $d = 1$, $\mathbf{v} = 0$ and (2) $d = 1$, $\mathbf{w} = i\mathbf{v}$ with $\mathbf{v} \neq 0$, respectively. See also Remarks 3.10 and 9.6.

We postpone the proof of Proposition 3.2 to the end of the paper after introducing the skewed Gauss map and the associated transfer operator on $C^1(I_\Gamma)$, and establishing an explicit relation between the resolvent of the operator and Dirichlet series associated to the modular partition functions in Sections 5–9.

The following is one of our main results which leads to both the asymptotic Gaussian behavior and residual equidistribution of the variable $\mathfrak{g}_{\boldsymbol{\psi}}$.

Proposition 3.4. *Let* $\mathbf{v} \in \mathbb{R}^d$. *There exist a constant* $0 < \delta < 2$ *and a neighborhood* W *of* $i\mathbf{v}$ *in* $\mathbb{C}^d$ *such that for* $\Psi \in C^1(I_\Gamma)$ *and* $\mathbf{w} \in W$,

$$\sum_{n \le M} d_n(\mathbf{w}) = R_{M,\mathbf{v}}(\mathbf{w}) + O(M^{\delta}) \tag{3.1}$$

where

$$R_{M,\mathbf{v}}(\mathbf{w}) := \begin{cases} \frac{E_{\mathbf{v}}(\mathbf{w})}{s(\mathbf{w})} M^{2s(\mathbf{w})} & \text{if } \mathbf{v}\cdot\boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z}), \\ 0 & \text{otherwise.} \end{cases}$$

The implicit constant and δ *are independent of* $\mathbf{w}$.

Proof. Proposition 3.2 enables us to do the contour integration using Cauchy's residue theorem

$$\frac{1}{2\pi i} \int_{\mathcal{U}_T(\mathbf{w})} L_{\Psi,J}(2s, \mathbf{w}) \frac{M^{2s}}{2s} d(2s) = R_{M,\mathbf{v}}(\mathbf{w})$$

where $\mathcal{U}_T(\mathbf{w})$ denotes the rectangle with vertices $1+\alpha_0+iT$, $1-\alpha_0+iT$, $1-\alpha_0-iT$, and $1+\alpha_0-iT$ with positive orientation.

Applying the Perron formula from Theorem 3.1 to $L_{\Psi,J}(2s,\mathbf{w})$ for s along the vertical line $1+\alpha_0 \pm iT$, we have

$$\begin{aligned}\sum_{n\le M} d_n(\mathbf{w}) &= R_{M,\mathbf{v}}(\mathbf{w}) + O\left(\frac{M^{2(1+\alpha_0)}}{T}\right) + O(A(M)) \\ &\quad + O\left(\frac{A(2M)M\log M}{T}\right) + O\left(\int_{1-\alpha_0-iT}^{1-\alpha_0+iT} |L_{\Psi,J}(2s,\mathbf{w})| \frac{M^{2(1-\alpha_0)}}{|s|}\, ds\right) \\ &\quad + O\left(\int_{1-\alpha_0\pm iT}^{1+\alpha_0\pm iT} |L_{\Psi,J}(2s,\mathbf{w})| \frac{M^{2\Re s}}{T}\, ds\right).\end{aligned}$$

Note that the last two error terms come from the contour integral and each of them corresponds to the left vertical line and horizontal lines of the rectangle $\mathcal{U}_T$ respectively. Let us write this as

$$\sum_{n\le M} d_n(\mathbf{w}) = R_{M,\mathbf{v}}(\mathbf{w}) + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.$$

We choose $\widehat{\alpha}_0$ with $\frac{8}{59}\alpha_0 < \widehat{\alpha}_0 < \alpha_0$ and set

$$T = M^{2\alpha_0+4\widehat{\alpha}_0}.$$

Notice that $E_{\mathbf{v}}(\mathbf{w})/s(\mathbf{w})$ is bounded in the neighborhood W since $s(i\mathbf{v}) = 1$. Then the error terms are bounded as follows.

The error term I is $O(M^{2(1-2\widehat{\alpha}_0)})$ and by Proposition 3.2, the exponent of M satisfies $2(1-2\widehat{\alpha}_0) < 2$.

Set $|\mathbf{x}| := \max_i |x_i|$ for $\mathbf{x} = \Re\mathbf{w}$. Since $\mathbf{x}\cdot\boldsymbol{\psi}(r) \ll |\mathbf{x}|\ell(r)$ and $\ell(r) \ll \log n$ for $r \in \Sigma_n$, for some $c > 0$ we obtain

$$d_n(\mathbf{w}) \ll n^{1+c|\mathbf{x}|}. \tag{3.2}$$

By (3.2), for any $0 < \varepsilon < \widehat{\alpha}_0/2$, we can take W from Proposition 3.2 small enough to have $c|\mathbf{x}| < \varepsilon/2$ so that $A(M) = O(M^{1+\varepsilon/2})$ and $\log M \ll M^{\varepsilon/2}$. Hence, the exponent of M in the error term III is equal to

$$1 + (1 + c|\mathbf{x}|) + \frac{\varepsilon}{2} - (2\alpha_0 + 4\widehat{\alpha}_0) \le 2 - \frac{23}{4}\widehat{\alpha}_0 < 2.$$

Similarly the error term II is $O(M^{1+\varepsilon/2})$, so the exponent satisfies

$$1 + \frac{\varepsilon}{2} < 1 + \frac{1}{4}\widehat{\alpha}_0 < 2.$$

Also for $0 < \xi < 1/5$, we have $|L_{\Psi,J}(2s,\mathbf{w})| \ll |\Im s|^{\xi}$ by Proposition 3.2. Hence, the error term IV is $O(M^{2(1-\alpha_0)}T^{\xi})$ and the exponent of M is equal to

$$2(1-\alpha_0) + (2\alpha_0 + 4\widehat{\alpha}_0)\xi < 2 - \frac{4}{5}(2\alpha_0 - \widehat{\alpha}_0) < 2.$$

The last term V is $O(T^{\xi-1} \cdot M^{2(1+\alpha_0)}(\log M)^{-1})$, hence the exponent of M satisfies

$$(2\alpha_0 + 4\widehat{\alpha}_0)(\xi - 1) + 2(1+\alpha_0) - \frac{\varepsilon}{2} < 2 - \left(-\frac{2}{5}\alpha_0 + \frac{59}{20}\widehat{\alpha}_0\right) < 2.$$

In total, setting

$$\delta = \max\left(2 - \frac{23}{4}\widehat{\alpha}_0, 1 + \frac{1}{4}\widehat{\alpha}_0, 2 - \frac{4}{5}(2\alpha_0 - \widehat{\alpha}_0), 2 - \left(-\frac{2}{5}\alpha_0 + \frac{59}{20}\widehat{\alpha}_0\right)\right)$$

concludes the proof. ∎

Remark 3.5. For Proposition 3.4, we have used a version of Perron's formula which is different from the one used by Baladi–Vallée [3]. The current version directly leads us to the desired estimate for the moment generating function of spaces smaller than Ω_M, namely $\Sigma_M(\varepsilon)$, without using the extra smoothing process of Baladi–Vallée. See Lee–Sun [23] for the relevant discussion concerning the length of continued fractions.

Observe that by Proposition 3.4 with $\Psi \equiv 1$ and $\mathbf{v} = \mathbf{0}$,

$$|\Omega_{M,J}| = E_{\mathbf{0}}(\mathbf{0})M^2 + O(M^{\delta}) \quad \text{with} \quad E_{\mathbf{0}}(\mathbf{0}) = \frac{3|J|}{\pi^2 \log 2}. \tag{3.3}$$

3.1. Joint Gaussian distribution: Proof of Theorem A

In this subsection, we obtain an explicit quasi-power behavior for the moment generating function of modular partition functions and show the limit joint Gaussian distribution.

Theorem 3.6. *There exist a neighborhood W of $\mathbf{0}$, an analytic function $B_{\varphi,J}$ on W, and a constant $0 < \gamma < \alpha_1$ with α_1 from Proposition 3.2, such that $B_{\varphi,J}$ is non-vanishing on W and*

$$\mathbb{E}[\exp(\mathbf{w}\cdot \mathfrak{g}_{\boldsymbol{\psi}}) \mid \Omega_{M,\varphi,J}] = \frac{B_{\varphi,J}(\mathbf{w})}{B_{\varphi,J}(\mathbf{0})} M^{2(s(\mathbf{w})-s(\mathbf{0}))}(1 + O(M^{-\gamma}))$$

with $s(\mathbf{w})$ from Proposition 3.2 (1) with $\mathbf{v} = \mathbf{0}$ and $\Psi = 1 \otimes \varphi$. The implicit constant and the constant γ are independent of $\mathbf{w} \in W$.

Proof. Setting $B_{\varphi,J}(\mathbf{w}) := E_{\mathbf{0}}(\mathbf{w})/s(\mathbf{w})$, we obtain the conclusion from Proposition 3.4 with $\mathbf{v} = \mathbf{0}$ and $\Psi = 1 \otimes \varphi$. ∎

The following probabilistic result ensures that the asymptotic normality of a sequence of random vectors comes from the quasi-power behavior of their moment generating functions.

Theorem 3.7 (Heuberger–Kropf [17], Hwang [3]). *Suppose that the moment generating function for a sequence $\mathbf{X}_N$ of m-dimensional real random vectors on spaces Ξ_N has a quasi-power expression*

$$\mathbb{E}[\exp(\mathbf{w}\cdot\mathbf{X}_N) \mid \Xi_N] = \exp(\beta_N U(\mathbf{w}) + V(\mathbf{w}))(1 + O(\kappa_N^{-1}))$$

with $\beta_N, \kappa_N \to \infty$ as $N \to \infty$, and $U(\mathbf{w}), V(\mathbf{w})$ analytic for $\mathbf{w} = (w_i) \in \mathbb{C}^m$ with $|\mathbf{w}|$ sufficiently small. Assume that the Hessian $\mathbf{H}_U(\mathbf{0})$ of U at $\mathbf{0}$ is non-singular.

(1) *The distribution of $\mathbf{X}_N$ is asymptotically normal with the speed of convergence $O(\kappa_N^{-1} + \beta_N^{-1/2})$. In other words, for any $\mathbf{x} \in \mathbb{R}^m$,*

$$\mathbb{P}\left[\frac{\mathbf{X}_N - \nabla U(\mathbf{0})\beta_N}{\sqrt{\beta_N}} \leq \mathbf{x} \,\middle|\, \Xi_N\right] = \frac{1}{(2\pi)^{m/2}\sqrt{\det \mathbf{H}_U(\mathbf{0})}} \int_{\mathbf{t}\leq\mathbf{x}} \exp\left(-\frac{1}{2}\mathbf{t}^T \mathbf{H}_U(\mathbf{0})^{-1}\mathbf{t}\right) d\mathbf{t} + O\left(\frac{1}{\kappa_N} + \frac{1}{\sqrt{\beta_N}}\right)$$

where $\mathbf{t} \leq \mathbf{x}$ means $t_j \leq x_j$ for all $1 \leq j \leq k$ and the O-term is uniform in $\mathbf{x}$.

(2) *Let $m = 1$. The moments of X_N satisfy*

$$\mathbb{E}[X_N \mid \Xi_N] = \beta_N U'(0) + V'(0) + O(\kappa_N^{-1}),$$
$$\mathbb{V}[X_N \mid \Xi_N] = \beta_N U''(0) + V''(0) + O(\kappa_N^{-1}),$$
$$\mathbb{E}[X_N^k \mid \Xi_N] = P_k(\beta_N) + O(\beta_N^k \kappa_N^{-1}),$$

for some polynomials P_k of degree at most $k \geq 3$.

Proof of Theorem A. Let $U(\mathbf{w}) = 2(s(\mathbf{w}) - s(\mathbf{0}))$ and $V(\mathbf{w}) = \log \frac{B_{\varphi,J}(\mathbf{w})}{B_{\varphi,J}(\mathbf{0})}$ with s and $B_{\varphi,J}$ from Theorem 3.6. By Proposition 3.2 with $\mathbf{v} = \mathbf{0}$, both U and V are independent of M and analytic for sufficiently small $\mathbf{w}$, the Hessian of U at $\mathbf{0}$ is equal to the Hessian of $s(\mathbf{w})$ at $\mathbf{0}$, and it is non-singular if and only if ψ_i are $\mathbb{R}$-linearly independent modulo $\mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R})$. Letting $\mathrm{H}_{\boldsymbol{\psi}}$ be the Hessian of U, Theorem 3.7 enables us to finish the proof. ∎

3.2. *Residual equidistribution: Proof of Theorem B*

In this subsection, we give a proof for the residual equidistribution of modular partition functions. First we need:

Theorem 3.8. *Let $\mathbf{v} \in \mathbb{R}^d$ and $\boldsymbol{\psi} : \Gamma\backslash\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{R}^d$. There exists $\gamma_1 > 0$ such that*

$$\mathbb{E}[\exp(i\mathbf{v}\cdot \mathfrak{g}_{\boldsymbol{\psi}}) \mid \Omega_{M,J}] = R(\mathbf{v}) + O(M^{-\gamma_1})$$

with

$$R(\mathbf{v}) = \begin{cases} \dfrac{\sum_v \exp[i(\beta(\Gamma) - \beta(v))]}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} & \text{if } \mathbf{v}\cdot\boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z}), \\ 0 & \text{otherwise}. \end{cases}$$

Here in the first case, $\mathbf{v}\cdot\boldsymbol{\psi}$ is associated with β.

Proof. Note that if $\mathbf{v}\cdot\boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z})$, then by Proposition 3.2 with $\Psi \equiv 1$,

$$E_{\mathbf{v}}(i\mathbf{v}) = \frac{3|J| \sum_v \exp[i(\beta(\Gamma) - \beta(v))]}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]\pi^2 \log 2}.$$

Setting $R(\mathbf{v}) = \frac{E_{\mathbf{v}}(i\mathbf{v})}{E_{\mathbf{0}}(\mathbf{0})}$, by Proposition 3.4 and (3.3) we obtain the statement with $\gamma_1 = 2 - \delta$. ∎

Proof of Theorem B. Recall that $\mathfrak{g}_{\boldsymbol{\psi}}(r) \in \mathbb{Z}^d$ as r varies over Ω_M. For $\mathbf{g} \in (\mathbb{Z}/Q\mathbb{Z})^d$, it is easy to see that

$$\mathbb{P}[\mathfrak{g}_{\boldsymbol{\psi}} \equiv \mathbf{g} \ (\mathrm{mod}\ Q) \mid \Omega_{M,J}] = \frac{1}{Q^d} \sum_{\mathbf{s} \in (\mathbb{Z}/Q\mathbb{Z})^d} e^{-\frac{2\pi i}{Q}\mathbf{s}\cdot\mathbf{g}} \cdot \mathbb{E}\left[\exp\left(\frac{2\pi i}{Q}\mathbf{s}\cdot\mathfrak{g}_{\boldsymbol{\psi}}\right) \,\middle|\, \Omega_{M,J}\right].$$

We then split the summation into two parts: $\mathbf{s} = \mathbf{0}$ and $\mathbf{s} \neq \mathbf{0}$. The term corresponding to $\mathbf{s} = \mathbf{0}$ is the main term which is Q^{-d}. For the sum over $\mathbf{s} \neq \mathbf{0}$, we assert that

$$\frac{2\pi}{Q}\mathbf{s} \cdot \boldsymbol{\psi} \notin \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z}). \tag{3.4}$$

Note that the condition is independent of any choice of a lift of $\mathbf{s}$ to $\mathbb{Z}^d$. Assume the contrary of (3.4). Then, by Corollary 2.7 (2) we get $\frac{2\pi}{Q}\mathbf{s} \cdot \boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, 2\pi Q^{-1}\mathbb{Z}/2\pi\mathbb{Z})$ and hence $\mathbf{s} \cdot \boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{Z}/Q\mathbb{Z})$. This contradicts the condition on $\boldsymbol{\psi}$. Hence, from Theorem 3.8, we obtain

$$\mathbb{E}\left[\exp\left(\frac{2\pi i \mathbf{s}}{Q}\cdot\mathfrak{g}_{\boldsymbol{\psi}}\right) \,\middle|\, \Omega_{M,J}\right] \ll M^{-\gamma_1}.$$

This yields the first statement.

For the second one, suppose that ψ is a $\mathfrak{g}$-coboundary over $\mathbb{Z}/q\mathbb{Z}$ for a prime $q \mid Q$, associated with β. First, we have

$$\mathbb{P}[\mathfrak{g}_{\psi} \equiv a \ (\mathrm{mod}\ q) \mid \Omega_{M,J}] = \frac{1}{q} \sum_{t \in \mathbb{Z}/q\mathbb{Z}} e^{-2\pi i a t/q} \mathbb{E}\left[\exp\left(\frac{2\pi i}{q} t \mathfrak{g}_{\psi}\right) \,\middle|\, \Omega_{M,J}\right].$$

Note that $\frac{2\pi i t}{q}\psi$ is also a coboundary associated with $\frac{2\pi i t}{q}\beta$. From Theorem 3.8, the last expression equals

$$\frac{1}{q[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]} \sum_{t} e^{-2\pi i a t/q} \sum_{v \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})} \exp\left[\frac{2\pi i t}{q}(\beta(\Gamma) - \beta(v))\right] + o(1).$$

This equals $c_a[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]^{-1} + o(1)$ where $c_a = \#\{v \mid \beta(v) \equiv \beta(\Gamma) - a \ (\mathrm{mod}\ q)\}$. On the other hand,

$$\mathbb{P}[\mathfrak{g}_{\psi} \equiv a \ (\mathrm{mod}\ q) \mid \Omega_{M,J}] = \sum_{\substack{g \equiv a (q) \\ g \in \mathbb{Z}/Q\mathbb{Z}}} \mathbb{P}[\mathfrak{g}_{\psi} \equiv g \ (\mathrm{mod}\ Q) \mid \Omega_{M,J}] = \frac{Q}{q} \cdot \frac{1}{Q} + o(1)$$

for each a. Hence, $c_a[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]^{-1}$ are all q^{-1}. This is possible only when q is a divisor of $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, which is a contradiction. Hence, we obtain the statement. ■

3.3. Weak correlation between Archimedean and residual distributions

In this subsection, we show that the Gaussian distribution and residual distribution of modular partition functions are weakly correlated, i.e., non-correlated asymptotically.

Let $\psi : \Gamma\backslash \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}$ and $Q > 1$ be an integer. For a $g \in \mathbb{Z}/Q\mathbb{Z}$, let $\Omega_{M,J}^g$ be the probability space $\{r \in \Omega_{M,J} \mid \mathfrak{g}_\psi(r) \equiv g \pmod{Q}\}$ with the uniform density. Let $\underline{\mathfrak{g}}_\psi$ be the normalization of $\mathfrak{g}_\psi$, i.e.,

$$\underline{\mathfrak{g}}_\psi := \frac{\mathfrak{g}_\psi - \mu_\psi \log M}{\sqrt{C_\psi \log M}}$$

where μ_ψ and C_ψ are given in Theorem A. The following result shows that the two distributions of $\mathfrak{g}_\psi$ on $\Omega_{M,J}$ are asymptotically non-correlated for a residual non-coboundary ψ:

Theorem 3.9. *Assume that $\psi \pmod{q} \notin \mathcal{B}_{\mathfrak{q}}(\Gamma, \mathbb{Z}/q\mathbb{Z})$ for each prime $q \mid Q$. For each $x \in \mathbb{R}$, as $M \to \infty$,*

(1) $\mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x \mid \Omega_{M,J}^g] = \mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x \mid \Omega_{M,J}] + o(1)$;

(2) $\mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x,\ \mathfrak{g}_\psi \equiv g \pmod{Q} \mid \Omega_{M,J}] = \mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x \mid \Omega_{M,J}] \cdot \mathbb{P}[\mathfrak{g}_\psi \equiv g \pmod{Q} \mid \Omega_{M,J}] + o(1)$.

Proof. For (1), it suffices to show that there exists $\gamma_2 > 0$ such that for all $w \in \mathbb{C}$ sufficiently close to 0, $\mathbb{E}[\exp(w\mathfrak{g}_\psi) \mid \Omega_{M,J}^g] = \mathbb{E}[\exp(w\mathfrak{g}_\psi) \mid \Omega_{M,J}] + O(M^{-\gamma_2})$. Set

$$R(w) := \sum_{r \in \Omega_{M,J}^g} \exp(w\mathfrak{g}_\psi(r)).$$

Note that $R(w)/R(0) = \mathbb{E}[\exp(w\mathfrak{g}_\psi) \mid \Omega_{M,J}^g]$ by definition.

Using the orthogonality of the additive character $t \mapsto \exp(2\pi i t/Q)$, we have

$$R(w) = \frac{1}{Q} \sum_{t \in \mathbb{Z}/Q\mathbb{Z}} e^{-\frac{2\pi i}{Q} tg} \sum_{r \in \Omega_{M,J}} \exp\left(\left(\frac{2\pi i}{Q} t + w\right)\mathfrak{g}_\psi(r)\right).$$

Split the sum over t into two parts, $t = 0$ and $t \not\equiv 0 \pmod{Q}$. As in the proof of Theorem B, from $t \not\equiv 0 \pmod{Q}$ and the hypothesis on ψ, we get $\frac{2\pi t}{Q}\psi \notin \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R}/2\pi\mathbb{Z})$. Since w is near 0, from Proposition 3.4, for some $0 < \delta < 2$, we find that

$$\sum_{r \in \Omega_{M,J}} \exp\left(\left(\frac{2\pi t}{Q} i + w\right)\mathfrak{g}_\psi(r)\right) = O(M^\delta) \quad \text{if } t \not\equiv 0 \pmod{Q}.$$

In sum, $R(w) = \frac{1}{Q} \sum_{r \in \Omega_{M,J}} \exp(w\mathfrak{g}_\psi(r)) + O(M^\delta)$, and hence

$$\frac{R(w)}{R(0)} = \mathbb{E}[\exp(w\mathfrak{g}_\psi) \mid \Omega_{M,J}] + O\left(\frac{M^\delta}{|\Omega_{M,J}|}\right).$$

Since $|\Omega_{M,J}| \gg M^2$ by (3.3), we obtain the statement with $\gamma_2 = 2 - \delta$.

For (2), a simple calculation gives

$$\mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x,\ \mathfrak{g}_\psi \equiv g \pmod{Q} \mid \Omega_{M,J}] = \mathbb{P}[\underline{\mathfrak{g}}_\psi \leq x \mid \Omega_{M,J}^g] \cdot \frac{|\Omega_{M,J}^g|}{|\Omega_{M,J}|}.$$

Using (1), we conclude the proof. ∎

In particular, we obtain the weak correlation for Archimedean and residual distributions of the length of continued fractions for $\Omega_{M,J}$.

Remark 3.10. Two special cases of Proposition 3.4, namely (1) any $\mathbf{w}$ near $\mathbf{v} = \mathbf{0}$ in Section 3.1 and (2) $\mathbf{w} = i\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$ in Section 3.2, are sufficient for our main ends of the present paper. Nevertheless, we still need the luxury of generality (any $\mathbf{w}$ near $i\mathbf{v}$) as it is indispensable for the proof of Theorem 3.9.

4. Distribution of modular symbols

In this section, we show that modular symbols are non-degenerate specializations of modular partition functions in both zero and positive characteristics. Using this, we deduce distribution results for modular symbols from those for modular partition functions.

4.1. Involution on de Rham cohomology

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \Im z > 0\}$ be the upper-half plane, $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$, and $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $X_\Gamma := \Gamma\backslash\mathbb{H}^*$ the corresponding modular curve. For two cusps r, s in $\mathbb{P}^1(\mathbb{Q})$, we write $\{r, s\}_\Gamma$ for the relative homology class corresponding to the projection to X_Γ of the geodesic on $\mathbb{H}^*$ connecting r to s. For $\Gamma = \Gamma_1(N)$, set $\{r, s\}_N := \{r, s\}_{\Gamma_1(N)}$.

Let $H^1_{\mathrm{dR}}(X_\Gamma)$ denote the first de Rham cohomology of X_Γ. We define an operator ι on $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}^*$ by

$$\gamma^\iota := \mathrm{j}\gamma\mathrm{j} \in \mathrm{SL}_2(\mathbb{Z}) \quad \text{and} \quad z^\iota := -\bar{z} \in \mathbb{H}^*.$$

As Γ is assumed to be normalized by j (e.g. $\Gamma = \Gamma_1(N)$), the action of ι yields a well-defined involution on X_Γ. Let $S_2(\Gamma)$ be the space of cuspforms of weight 2 for Γ. The involution ι then acts on $H^1_{\mathrm{dR}}(X_\Gamma) \simeq S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$ by interchanging $S_2(\Gamma)$ and $\overline{S_2(\Gamma)}$. Moreover, ι is normal with respect to the cap product

$$\cap : H_1(X_\Gamma, \mathbb{Z}) \times H^1_{\mathrm{dR}}(X_\Gamma) \to \mathbb{C}, \quad (\xi, \omega) \mapsto \xi \cap \omega = \int_\xi \omega. \tag{4.1}$$

The cap product can be interpreted as follows. For $f \in S_2(\Gamma)$, $g \in \overline{S_2(\Gamma)}$, and $\{r, s\}_\Gamma \in H_1(X_\Gamma, \mathbb{Q})$, set

$$\langle \{r, s\}_\Gamma, (f, g) \rangle = \int_r^s f(z)\, dz + \int_r^s g(z)\, dz^\iota. \tag{4.2}$$

Then it is known that the pairing $\langle \cdot\, , \cdot \rangle$ is non-degenerate (see Merel [30]). Note that the modular symbol $\mathfrak{m}_f^{\pm}(r)$ can be understood as the above pairing (4.2) between a relative homology class $\{r, i\infty\}_N$ and the de Rham cohomology class $(f, \mp f^\iota)$, respectively.

4.2. Optimal periods

We discuss some preliminary results to study the residual distribution of modular symbols.

Let f be a newform of level N and weight 2. Let $\mathbf{m}$ be a maximal ideal of the Hecke algebra $\mathbb{T}_N$ such that the characteristic of $\mathbb{T}_N/\mathbf{m}$ is p and corresponds to f. There exists a Galois representation $\rho_{\mathbf{m}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_N/\mathbf{m})$.

Let $C_1(N)$ be the set of cusps on $X_1(N)$. Consider the $(X_1(N), C_1(N))$-relative homology sequence

$$0 \to H_1(X_1(N), \mathbb{Z}) \to H_1(X_1(N), C_1(N), \mathbb{Z}) \to H_0(C_1(N), \mathbb{Z}) \to \mathbb{Z} \to 0. \quad (4.3)$$

For a prime q with $q \equiv 1 \pmod{Np}$, let $D_q = \mathrm{T}_q - q\langle q\rangle - 1$. It was observed by Greenberg–Stevens [14] that the operator D_q annihilates $H_0(C_1(N), \mathbb{Z})$ in (4.3). Let $\mathbb{T}_{N,\mathbf{m}}$ denote the completion of the Hecke algebra $\mathbb{T}_N$ at $\mathbf{m}$. Since D_q is a unit in $\mathbb{T}_{N,\mathbf{m}}$ if $\rho_{\mathbf{m}}$ is irreducible, we conclude that $H_1(X_1(N), \mathbb{Z})_{\mathbf{m}}$ is isomorphic to $H_1(X_1(N), C_1(N), \mathbb{Z})_{\mathbf{m}}$. For a $\mathbb{Z}_p$-algebra R with the trivial action of $\mathbb{T}_N$, we have a perfect pairing

$$H_1(X_1(N), R)_{\mathbf{m}} \times H^1(\Gamma_1(N), R)_{\mathbf{m}} \to R. \quad (4.4)$$

When R is given by $\mathbb{C}$, the pairing is realized as the Poincaré pairing under the isomorphism $\mathbb{C}_p \simeq \mathbb{C}$.

Let $\mathcal{O}$ be an integral extension of $\mathbb{Z}_p$ including the Fourier coefficients of f. Assume that $N \geq 3$, $p \nmid 2N$, and $\rho_{\mathbf{m}}$ is irreducible. Then there is a Hecke equivariant isomorphism

$$\delta^{\pm} : S_2(\Gamma_1(N), \mathcal{O})_{\mathbf{m}} \cong H^1(\Gamma_1(N), \mathcal{O})^{\pm}_{\mathbf{m}}. \quad (4.5)$$

It is the isomorphism mentioned in Vatsal [43].

Let $\omega_f \in H_1(\Gamma_1(N), \mathbb{C})$ be the cohomology class corresponding to $2\pi i f(z)dz$. Using the isomorphism (4.5) and the theorem of strong multiplicity 1, the periods $\Omega_f^{\pm} \in \mathbb{C}_p$ can be chosen (see Vatsal [43]) so that

$$\Omega_f^{\pm}\delta^{\pm}(f) = \omega_f \pm \omega_f^{\iota}. \quad (4.6)$$

It is known that for a newform f_E corresponding to an elliptic curve E over $\mathbb{Q}$, the periods $\Omega_{f_E}^{\pm}$ can be chosen as the Néron periods $\Omega_E^{\pm}$ of E.

4.3. Modular symbol as a modular partition function: Manin's trick

We describe how the statistics of continued fractions enters our discussion on the distribution of modular symbols.

Let f be a newform for $\Gamma_0(N)$ of weight 2, i.e., a cuspform for $\Gamma_1(N)$ with the trivial Nebentypus. Manin [25] noticed that the period integral can be written as

$$\int_0^r f(z)\,dz = \sum_{i=1}^{\ell} \int_{P_{i-1}/Q_{i-1}}^{P_i/Q_i} f(z)\,dz = -\sum_{i=1}^{\ell} \int_{\tilde{g}_i(r)\cdot 0}^{\tilde{g}_i(r)\cdot\infty} f(z)\,dz$$

with $\widetilde{g}_i(r) = \left[\begin{smallmatrix} P_{i-1} & -P_i \\ Q_{i-1} & -Q_i \end{smallmatrix}\right]$ if $\det g_i(r) = -1$. Setting $g_i = g_i(r)$, we also get

$$\int_0^{-r} f(z)\,dz = \sum_{i=1}^{\ell} \int_{-P_{i-1}/Q_{i-1}}^{-P_i/Q_i} f(z)\,dz = -\sum_{i=1}^{\ell} \int_{\mathrm{j}\widetilde{g}_i\mathrm{j}\cdot 0}^{\mathrm{j}\widetilde{g}_i\mathrm{j}\cdot\infty} f(z)\,dz.$$

For $u \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$, define

$$\psi_f^{\pm}(u) := \frac{1}{\Omega_f^{\pm}}\left(\int_{u\cdot 0}^{u\cdot\infty} f(z)\,dz \pm \int_{\mathrm{j}u\mathrm{j}\cdot 0}^{\mathrm{j}u\mathrm{j}\cdot\infty} f(z)\,dz\right) \in \mathbb{Q}_f.$$

By the definition of $\mathfrak{c}_{\psi_f^-}$, the modular symbols are expressed as

$$\mathfrak{m}_f^-(r) = -\mathfrak{c}_{\psi_f^-}(r). \tag{4.7}$$

We observe that $\psi_f^{\pm}(u\mathrm{j}) = \pm\psi_f^{\pm}(\mathrm{j}u)$. Hence, we also get $\psi_f^+(\Gamma\widetilde{g}_i) = \psi_f^+(\Gamma\widehat{g}_i)$ and

$$\mathfrak{m}_f^+(r) = \frac{2L(1,f)}{\Omega_f^+} - \mathfrak{b}_{\psi_f^+}(r). \tag{4.8}$$

Let us use the optimal periods of Section 4.2 with the same notation $\Omega_f^{\pm}$ when we study the residual equidistribution of modular symbols. By the previous discussion, one obtains $\mathfrak{m}_f^{\pm}(r) \in \mathcal{O}$ for each r. We define $\zeta_f^{\pm} : \Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z}) \to \mathcal{O}$ by

$$\zeta_f^{\pm}(u) := \{u\cdot 0, u\cdot\infty\}_N \cap \delta^{\pm}(f).$$

Note that

$$\mathfrak{m}_f^+(r) = \frac{2L(1,f)}{\Omega_f^+} - \mathfrak{b}_{\zeta_f^+}(r) \quad \text{and} \quad \mathfrak{m}_f^-(r) = -\mathfrak{c}_{\zeta_f^-}(r). \tag{4.9}$$

Remark 4.1. The representation (4.7) is no longer true for $\widehat{g}_i(r)$. In fact,

$$\int_{\widehat{g}_i(r)\cdot 0}^{\widehat{g}_i(r)\cdot\infty} f(z)\,dz = -\int_{\pm P_{i-1}/Q_{i-1}}^{\pm P_i/Q_i} f(z)\,dz$$

as $\widehat{g}_i = \left[\begin{smallmatrix} \pm P_{i-1} & \pm P_i \\ Q_{i-1} & Q_i \end{smallmatrix}\right]$ for $\det g_i(r) = \pm 1$ and hence

$$\int_0^r f(z)\,dz - \int_0^{-r} f(z)\,dz = \sum_{i=1}^{\ell} (-1)^i \psi_f^-(\Gamma\widehat{g}_i),$$

which is not equal to $\mathfrak{b}_{\psi_f^-}(r)$ in general.

Remark 4.2. As far as we understand, Manin–Marcolli seemed to assert that the modular symbols are expressible in terms of $\mathfrak{b}_{\psi_f^{\pm}}$. As discussed in Remark 4.1, it is doubtful that there is such an expression for ψ_f^-. A relevant mistake is that they regarded $g_j(r)$ as an element of $\mathrm{PSL}_2(\mathbb{Z})$, which is not the case if $\det g_j(r) = -1$ (see Manin–Marcolli [26, p. 6, line 12]).

4.4. Gaussian distribution: Proof of Theorem C

In this subsection, we prove the limit Gaussian distribution of modular symbols $\mathfrak{m}_f^{\pm}$ on $\Omega_{M,\varphi,J}$.

Proposition 4.3. *For any non-trivial $f \in S_2(\Gamma_0(N))$, the function ψ_f^+ (ψ_f^-, resp.) is not a $\mathfrak{b}$-coboundary ($\mathfrak{c}$-coboundary, resp.) over $\mathbb{R}$.*

Proof. First assume that ψ_f^- is a $\mathfrak{c}$-coboundary over $\mathbb{R}$, so there exists $\beta \in \mathbb{R}^{\langle\Gamma\rangle}$ such that $\psi_f^-(u) + \psi_f^-(u\left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right]\mathrm{j}) = \beta(u) - \beta(u\left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} -n & 1 \\ 1 & 0 \end{smallmatrix}\right])$ for all u and $m, n \in \mathbb{Z}$. Taking $m = n = 0$, we get $\psi_f^-(u) = -\psi_f^-(u\iota)$ for each u and $\iota = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Furthermore, we have $u\iota \cdot \infty = -u \cdot 0$ and $u\iota \cdot 0 = -u \cdot \infty$. Hence, $\psi_f^-(u\iota) = \psi_f^-(u)$. In sum, $2\psi_f^-$ is the zero function. On the other hand, Manin's trick implies that the set of Manin symbols $\{u \cdot 0, u \cdot \infty\}_{\Gamma_0(N)}$ for $u \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ generates the first homology group of $X_0(N)$. Since the pairing (4.1) is non-degenerate, we conclude that $\psi_f^{\pm}$ are not trivially zero as long as f is non-trivial. This is a contradiction and hence ψ_f^- is not a $\mathfrak{c}$-coboundary.

Assume that ψ_f^+ is a $\mathfrak{b}$-coboundary over $\mathbb{R}$, i.e., $\psi_f^+(u) = \beta(u) - \beta(u \cdot \left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right])$ for all m and u. Let $h : \Gamma_0(N) \to H_1(X_0(N), \mathbb{Z})$ by given by $h(\gamma) := \{0, \gamma \cdot 0\}_{\Gamma_0(N)}$. Let $\gamma \in \Gamma_0(N)$. Note that $\{0, \gamma \cdot 0\}_{\Gamma_0(N)} = \{\infty, \gamma \cdot \infty\}_{\Gamma_0(N)}$. Since $\left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma_0(N)$ for each $m \in \mathbb{Z}$, we get

$$\{\infty, \gamma \cdot \infty\}_{\Gamma_0(N)} = \left\{\left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right] \cdot \infty, \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]\gamma \cdot \infty\right\}_{\Gamma_0(N)} = \left\{\infty, \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]\gamma \cdot \infty\right\}_{\Gamma_0(N)}.$$

Therefore we have $h(\gamma) = \left\{0, \left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]\gamma \cdot 0\right\}_{\Gamma_0(N)} = h\left(\left[\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right]\gamma\right)$ for each $m \in \mathbb{Z}$ and hence may assume that $0 < \gamma \cdot 0 < 1$. Let $\gamma \cdot 0 = [0; m_1, \ldots, m_\ell]$ and $g_i = g_i(\gamma \cdot 0)$. Note that $g_i\left[\begin{smallmatrix} -m_i & 1 \\ 1 & 0 \end{smallmatrix}\right] = g_{i-1}$ with $g_0 = I$. Then

$$\begin{aligned} h(\gamma) \cap (\omega_f + \omega_f^{\iota}) &= \int_0^{\gamma \cdot 0} f(z)\, dz + \int_0^{-\gamma \cdot 0} f(z)\, dz = \sum_{i=1}^{\ell} \psi_f^+(\Gamma_0(N)\widehat{g}_i) \\ &= \sum_{i=1}^{\ell} \left(\beta(\Gamma_0(N) \cdot g_i) - \beta\left(\Gamma_0(N) \cdot g_i\left[\begin{smallmatrix} -m_i & 1 \\ 1 & 0 \end{smallmatrix}\right]\right)\right) = \beta(\Gamma_0(N)\widehat{g}_\ell) - \beta(\Gamma_0(N)). \end{aligned}$$

Observe that $\psi_f^+(u) = \psi_f^+(-u)$ and $\psi_f^+(\mathrm{j}u) = \psi_f^+(u\mathrm{j})$ for all u. By Proposition 2.5 (2), we know $\beta(\mathrm{j}u) = \beta(u\mathrm{j})$ for all u. In particular, $\beta(\Gamma_0(N)\widehat{g}_\ell) = \beta(\Gamma_0(N)\widetilde{g}_\ell)$. Note that $\gamma \cdot 0 = \widetilde{g}_\ell \cdot 0$, i.e., $\widetilde{g}_\ell \in \gamma \mathrm{L}$. Since β is L-invariant, we get $\beta(\Gamma_0(N)\widetilde{g}_\ell) = \beta(\Gamma_0(N)\gamma) = \beta(\Gamma_0(N))$. In sum, $h(\gamma) \cap (\omega_f + \omega_f^{\iota}) = 0$ for all $\gamma \in \Gamma_0(N)$. Since it is well-known that h is surjective, we conclude from the non-degeneracy of the pairing (4.1) that $f = 0$, which is a contradiction. ∎

Proof of Theorem C. By the expressions (4.7) and (4.8), the distribution of modular symbols follows the ones of the modular partition functions $\mathfrak{b}_{\psi_f^+}$ and $\mathfrak{c}_{\psi_f^-}$. Now Theorem C follows from Theorem A and Proposition 4.3. ∎

4.5. *Residual distribution: Proofs of Theorems E and D*

In this subsection, we establish the residual equidistribution of the integral random variable $\mathfrak{m}_E^{\pm}$ on $\Omega_{M,\varphi,J}$ and non-vanishing of special L-values.

First, we need the following:

Proposition 4.4. *Let ϖ be a uniformizer of $\mathcal{O}$. Assume that $N \geq 3$, $p \nmid 2N$, and $\rho_{\mathfrak{m}}$ is irreducible. Let $f \not\equiv 0 \pmod{\varpi}$. Then $\zeta_f^+ \pmod{\varpi}$ ($\zeta_f^- \pmod{\varpi}$, resp.) is not a $\mathfrak{b}$-coboundary ($\mathfrak{c}$-coboundary, resp.) over $\mathcal{O}/(\varpi)$.*

Proof. First assume that $\zeta_f^- \pmod{\varpi}$ with $\zeta_f^-(u) = \{u \cdot 0, u \cdot i\infty\}_N \cap \delta^{\pm}(f)$ is a $\mathfrak{c}$-coboundary over $\mathcal{O}/(\varpi)$. As in the proof of Proposition 4.3, for each u, we get $\zeta_f^-(u) = -\zeta_f^-(u\iota)$ with the action of ι on $\delta^{\pm}(f)$. Moreover, we have $\zeta_f^-(u\iota) = \zeta_f^-(u)$ using the action of ι on 0 and $i\infty$. In sum, we get $2\zeta_f^- \equiv 0 \pmod{\varpi}$. On the other hand, the Manin symbols generate the first homology group of $X_1(N)$. Therefore due to the perfectness of the pairing (4.4), the congruence $\zeta_f^- \equiv 0 \pmod{\varpi}$ implies that $\delta^-(f) \equiv 0 \pmod{\pi}$. However, this is forbidden by the hypothesis using (4.5), and we conclude that ζ_f^- is not a $\mathfrak{c}$-coboundary over $\mathcal{O}/(\varpi)$.

Assume that $\zeta_f^+ \pmod{\varpi}$ is a $\mathfrak{b}$-coboundary over $\mathcal{O}/(\varpi)$, i.e., there exists a function $\beta \in (\mathcal{O}/(\varpi))^{\langle \Gamma_1(N) \rangle}$ such that $\zeta_f^+(u) \equiv \beta(u) - \beta(u \cdot \left[\begin{smallmatrix} -m & 1 \\ 1 & 0 \end{smallmatrix}\right]) \pmod{\varpi}$ for all m and $u \in \Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Set $k(\gamma) := \{\infty, \gamma \cdot \infty\}_N \cap \delta^+(f) \in \mathcal{O}$ for $\gamma \in \Gamma_1(N)$. As in the previous proof, from the observation that $\zeta_f^+(u) = \zeta_f^+(-u)$ and $\zeta_f^+(\mathrm{j}u) = \zeta_f^+(u\mathrm{j})$ for all u, we conclude that $k(\gamma) \equiv 0 \pmod{\varpi}$ for all $\gamma \in \Gamma_1(N)$. Using the non-degeneracy of the pairing (4.4) and the isomorphism (4.6), we obtain $f \equiv 0 \pmod{\varpi}$, which is a contradiction. This finishes the proof. ∎

Proof of Theorem E. By the expressions (4.9), the residual distribution of modular symbols follows the ones of the modular partition functions $\mathfrak{b}_{\zeta_f^+}$ and $\mathfrak{c}_{\zeta_f^-}$. Now Theorem E follows from Theorem B and Proposition 4.4. ∎

Proof of Theorem D. Let $c < 1 - \sqrt{1 - \frac{6}{\pi^2}(1 - \frac{1}{p})}$. Let $\widehat{(\mathbb{Z}/n\mathbb{Z})}_{\pm}^{\times}$ be the set of Dirichlet characters modulo n that are even or odd according to the parity $\pm$. Set

$$T_M^{\pm} := \{1 < n \leq M \mid \exists \chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})}_{\pm}^{\times},\ \Lambda_E(\chi) \not\equiv 0 \pmod{\mathfrak{p}^{1+v_{\mathfrak{p}}(\phi(n))}}\}.$$

The statement follows once we verify that $\#T_M^{\pm} \geq cM$ for all sufficiently large M. Assume the contrary, i.e., $\#T_M^{\pm} < cM$ for infinitely many M. Then for each $n \notin T_M^{\pm}$ with $1 < n \leq M$ and $m \in (\mathbb{Z}/n\mathbb{Z})^{\times}$, we obtain

$$\sum_{\chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})}_{\pm}^{\times}} \overline{\chi}(m) \Lambda_E(\chi) \equiv 0 \pmod{\mathfrak{p}^{1+v_{\mathfrak{p}}(\phi(n))}}.$$

Then for all $r \in \Sigma_n$ with $n \notin T_M^{\pm}$, we obtain $\mathfrak{m}_E^{\pm}(r) \equiv 0 \pmod{p}$. Hence,

$$\sum_{\substack{1<n\leq M \\ n \notin T_M^{\pm}}} \phi(n) < \frac{1}{p} \sum_{1<n\leq M} \phi(n), \quad \text{i.e.,} \quad \sum_{n \in T_M^{\pm}} \phi(n) > \left(1 - \frac{1}{p}\right) \sum_{1<n\leq M} \phi(n).$$

Since $\#T_M^{\pm} < cM$, the LHS is smaller than or equal to

$$\sum_{M-cM<n\leq M} n \leq \frac{1}{2}(1-(1-c)^2)M^2.$$

Note that $\lim_{M\to\infty} \frac{1}{M^2}\sum_{n\leq M}\phi(n) = \frac{3}{\pi^2}$. Hence, $(1-c)^2 \leq 1-\frac{6}{\pi^2}(1-\frac{1}{p})$, contradicting the choice of c. ■

Remark 4.5. It seems that it is currently not doable to deduce an estimate on

$$\# \bigcup_{n\leq M,\, p\nmid\phi(n)} \{\chi \in \widehat{(\mathbb{Z}/n\mathbb{Z})^\times} \mid \Lambda_E(\chi) \not\equiv 0 \pmod{\mathfrak{p}},\ \chi(-1) = \pm 1\}$$

from the previous proof or a similar argument since the set $\{n \leq M \mid p \nmid \phi(n)\}$ is too thin as its size is asymptotic to $M(\log M)^{-1/(p-1)}$ (see Spearman–Williams [39]).

Remark 4.6. It is worth mentioning previous research on residual non-vanishing of L-values. The ergodic approach to Dirichlet L-values has been extensively generalized to the study of anti-cyclotomic twists (for example, see Hida [18], Burungale–Hida [8], and Vatsal [44]). Meanwhile, until now, there has been no notable analogous progress for modular L-values with cyclotomic or Dirichlet twists except a few cases. The first non-vanishing result goes back to Ash–Stevens [1] and Stevens [40] for a large class of characters. Kim–Sun [21] recently obtained a non-vanishing result for a positive proportion of characters χ of ℓ-power conductors with a prime $\ell \neq p$. However, all of these results are based on the classical arguments and their improvements. It is also worth mentioning another ergodic approach to Dirichlet L-values proposed recently by Lee–Palvannan [22].

Using Theorem 3.9, we present an answer to Mazur's question on weak correlation between Archimedean and residual distributions of modular symbols:

Theorem 4.7. *Assume that $\overline{\rho}_{E,p}$ is irreducible and $p \nmid N_E$. For $x \in \mathbb{R}$ and $a \in \mathbb{Z}/p^e\mathbb{Z}$, as $M \to \infty$, we get*

$$\mathbb{P}[\underline{\mathfrak{m}}_E^{\pm} \leq x,\ \mathfrak{m}_E^{\pm} \equiv a \pmod{p^e} \mid \Omega_{M,J}] = \mathbb{P}[\underline{\mathfrak{m}}_E^{\pm} \leq x \mid \Omega_{M,J}] \cdot \mathbb{P}[\mathfrak{m}_E^{\pm} \equiv a \pmod{p^e} \mid \Omega_{M,J}] + o(1).$$

5. Skewed Gauss dynamical systems

The remaining part of this paper will be devoted to explaining in detail how Proposition 3.2 can be obtained. We first present an underlying dynamical description for modular partition functions motivated by the work of Baladi–Vallée [3].

5.1. Skewed Gauss map

Recall that the skewed Gauss map $\mathbf{T}$ on $I \times \Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$ is given by

$$\mathbf{T}(x,v) = \big(T(x), v\big[\begin{smallmatrix} -m_1(x) & 1 \\ 1 & 0 \end{smallmatrix}\big]\big)$$

and the skewed Gauss map $\widehat{\mathbf{T}}$ on $I_\Gamma = I \times \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\widehat{\mathbf{T}}(x, v) = \big(T(x), v \cdot \left[\begin{smallmatrix} -m_1(x) & 1 \\ 1 & 0 \end{smallmatrix}\right]\big).$$

Let $K^\circ(m_1, \dots, m_\ell)$ be the open fundamental interval associated with the digits m_i, in other words,

$$K^\circ(m_1, \dots, m_\ell) := \{[0; m_1, \dots, m_\ell + x] \mid 0 < x < 1\}.$$

An easy observation is that

$$\widehat{\mathbf{T}}^\ell(K^\circ(m_1, \dots, m_\ell) \times \{v\}) = (0, 1) \times \big\{v \cdot \left[\begin{smallmatrix} -m_1 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} -m_2 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} -m_\ell & 1 \\ 1 & 0 \end{smallmatrix}\right]\big\}. \tag{5.1}$$

It can be easily seen that $\mathbf{T}$ and $\widehat{\mathbf{T}}$ are measure-preserving and in fact ergodic with respect to the product measure of the Gauss measure and the counting measure on the skewed Gauss dynamical systems $(I \times \Gamma\backslash\mathrm{GL}_2(\mathbb{Z}), \mathbf{T})$ and $(I_\Gamma, \widehat{\mathbf{T}})$, respectively. However, measure-theoretic properties will not be investigated in this paper as we restrict our attention to topological properties.

For a dynamical system (X, f), the map f is called *topologically transitive* if for any non-empty open subsets U and V in X, there exists a positive integer L such that $f^L(U) \cap V \neq \varnothing$; and *topologically mixing* if $f^n(U) \cap V \neq \varnothing$ for all $n \geq L$. Notice that if f is topologically mixing, then it is topologically transitive.

Proposition 5.1. (1) *The map $\widehat{\mathbf{T}}$ on I_Γ is topologically mixing.*
(2) *For any sequence (x_n, v_n) in I_Γ, the set $\bigcup_{n\geq 1} \widehat{\mathbf{T}}^{-n}(x_n, v_n)$ is dense in I_Γ.*

Proof. (1) Take any non-empty open sets U and V in I_Γ. One can assume that U is of the form $(a, b) \times \{u\}$ for some $0 < a < b < 1$ and $u \in \Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$. Since the Gauss map T has the strong Markov property, i.e., $T([\frac{1}{m+1}, \frac{1}{m})) = [0, 1)$ for all $m \geq 1$, we have $T^n(a, b) = I$ for all sufficiently large n. Once we have the full image on the first coordinate, we obtain all the elements in the skewed component at all sufficiently many iterations as well using Proposition 2.4. Hence we conclude that $\widehat{\mathbf{T}}^n(U) \cap V = I_\Gamma \cap V \neq \varnothing$ for all sufficiently large n.

(2) Let V be an open subset of I_Γ. We may assume $V = K^\circ(a_1, \dots, a_k) \times \{u\}$. By Proposition 2.4, there exists $L \geq 1$ such that for all $\ell \geq L$,

$$u \cdot \left[\begin{smallmatrix} -a_1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} -a_k & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} -m_1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \cdots \left[\begin{smallmatrix} -m_\ell & 1 \\ 1 & 0 \end{smallmatrix}\right] = v_n$$

for some $m_1, \dots, m_\ell$. Then, by (5.1), we have $(x_n, v_n) \in \widehat{\mathbf{T}}^n(V)$ when $n \geq k + L$. Hence, we get the second statement. ∎

Remark 5.2. In a similar way, one can show that $\mathbf{T}$ is also transitive. However, it is easy to see that $\mathbf{T}$ is not topologically mixing for any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$.

5.2. *Inverse branches*

Let $\mathbf{Q}$ be the set of inverse branches of $\mathbf{T}$, that is,

$$\mathbf{Q} := \{\mathbf{q}_m \mid m \in \mathbb{Z}_{\geq 1}\}$$

where an inverse branch $\mathbf{q}_m : I \times \Gamma\backslash \mathrm{GL}_2(\mathbb{Z}) \to I \times \Gamma\backslash \mathrm{GL}_2(\mathbb{Z})$ is given by

$$\mathbf{q}_m(x, v) = \left(\frac{1}{m+x}, v \begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix} \right). \tag{5.2}$$

Let $\mathbf{F} \subseteq \mathbf{Q}$ be the final set that consists of the branches corresponding to the final digits of continued fractions. In other words,

$$\mathbf{F} := \{\mathbf{q}_m \mid m \geq 2\}.$$

5.2.1. Basic setting. For $n \geq 1$, denote by $\mathbf{Q}^{\circ n}$ the set of inverse branches of the n-th iterate $\mathbf{T}^n$,

$$\mathbf{Q}^{\circ n} := \mathbf{Q} \circ \cdots \circ \mathbf{Q} = \{\mathbf{q}_{m_n} \circ \mathbf{q}_{m_{n-1}} \circ \cdots \circ \mathbf{q}_{m_1} \mid m_1, \ldots, m_n \geq 1\}$$

and $\mathbf{Q}^{\circ 0} := \{\mathbf{id}_{I_\Gamma}\}$. Also, set

$$\mathbf{Q}^{\infty} := \bigcup_{n \geq 0} \mathbf{Q}^{\circ n}.$$

The index n is called the *depth* of inverse branches. For an inverse branch $\mathbf{q} = \mathbf{q}_{m_n} \circ \cdots \circ \mathbf{q}_{m_2} \circ \mathbf{q}_{m_1}$ of depth n and $i \leq n$, define the *i-th part* $\mathbf{q}^{(i)}$ of $\mathbf{q}$ to be

$$\mathbf{q}^{(i)} := \mathbf{q}_{m_i} \circ \cdots \circ \mathbf{q}_{m_2} \circ \mathbf{q}_{m_1} = \mathbf{T}^{n-i} \circ \mathbf{q} \in \mathbf{Q}^{\circ i}.$$

For a branch $\mathbf{q}(x,u) = (y(x), ug)$, let $\pi_i \mathbf{q}$ be the i-th component of $\mathbf{q}$, i.e.,

$$\pi_1 \mathbf{q}(x,u) := y(x) \quad \text{and} \quad \pi_2 \mathbf{q}(x,u) := ug.$$

5.2.2. Definitions for $\mathfrak{b}$ *and* $\mathfrak{c}$. Let $\mathbf{q} \in \mathbf{Q}^{\infty}$ be given by $\mathbf{q}(x,u) = (y(x), ug)$ for some $y(x)$ and $g \in \mathrm{GL}_2(\mathbb{Z})$. Then for $v \in \Gamma\backslash \mathrm{SL}_2(\mathbb{Z})$, define

$$\widehat{\mathbf{q}}(x, v) := (y(x), v\widehat{g}) = (y(x), v \cdot g).$$

These functions constitute the set of inverse branches of $\widehat{\mathbf{T}}$, denoted by $\widehat{\mathbf{Q}}$. We also set

$$\widetilde{\mathbf{q}}(x, v) := (y(x), v\widetilde{g}).$$

It can be easily checked from the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})$ that for all $\mathbf{p}, \mathbf{q} \in \mathbf{Q}^{\infty}$,

$$\widehat{\mathbf{p} \circ \mathbf{q}} = \widehat{\mathbf{p}} \circ \widehat{\mathbf{q}} \quad \text{and} \quad \widehat{\mathbf{q}}^{(i)} = \widehat{\mathbf{q}^{(i)}} = \widehat{\mathbf{T}}^{n-i} \circ \widehat{\mathbf{q}}. \tag{5.3}$$

For $\mathbf{q} \in \mathbf{Q} \circ \mathbf{Q}$, the map $\pi_2 \mathbf{q}$ is now just the right action of $\mathrm{SL}_2(\mathbb{Z})$ by the relation (2.3). Therefore, if $\mathbf{q}$ is of even depth, then $\widehat{\mathbf{q}} = \widetilde{\mathbf{q}} = \mathbf{q}|_{I_\Gamma}$ and for all $\mathbf{p} \in \mathbf{Q}^{\infty}$ we obtain

$$\widetilde{\mathbf{p} \circ \mathbf{q}} = \widetilde{\mathbf{p}} \circ \widetilde{\mathbf{q}}. \tag{5.4}$$

In particular, for $\mathbf{q} \in \mathbf{Q}^{\circ 2n}$ and $1 \leq i \leq n$,

$$\widetilde{\mathbf{q}}^{(2i)} = \widetilde{\mathbf{q}^{(2i)}} = (\widehat{\mathbf{T}}^2)^{n-i} \circ \widetilde{\mathbf{q}} \quad \text{and} \quad \widetilde{\mathbf{q}^{(2i-1)}} = \widetilde{\mathbf{T}} \circ (\widehat{\mathbf{T}}^2)^{n-i} \circ \widetilde{\mathbf{q}}.$$

5.2.3. Specialization. It can be easily seen that there is a one-to-one correspondence between $\mathbb{Q} \cap (0,1)$ and $\mathbf{F} \circ \mathbf{Q}^{\infty}$ given by

$$r = [0; m_1, \dots, m_\ell] \mapsto \mathbf{q}_r := \mathbf{q}_{m_\ell} \circ \cdots \circ \mathbf{q}_{m_1}$$

with $m_1, \dots, m_{\ell-1} \geq 1$ and $m_\ell \geq 2$.

Proposition 5.3. *For each $r \in \mathbb{Q} \cap (0,1)$,*

$$\mathbf{q}_r(0,\Gamma) = (r^*, \Gamma g(r)), \quad \widehat{\mathbf{q}}_r(0,\Gamma) = (r^*, \Gamma \widehat{g}(r)), \quad \widetilde{\mathbf{q}}_r(0,\Gamma) = (r^*, \Gamma \widetilde{g}(r)). \tag{5.5}$$

Proof. We get $\pi_2 \mathbf{q}_r^{(i)}(\Gamma) = \Gamma g_i(r)$ from the expression

$$\mathbf{q}_{m_n} \circ \mathbf{q}_{m_{n-1}} \circ \cdots \circ \mathbf{q}_{m_1}(0,\Gamma) = \left(\frac{Q_{n-1}}{Q_n}, \Gamma g([0; m_1, \dots, m_n])\right)$$

where $P_n/Q_n = [0; m_1, \dots, m_n]$ and $Q_{n-1}/Q_n = [0; m_n, m_{n-1}, \dots, m_1]$. This finishes the proof. ∎

5.3. Branch analogues of modular partition functions

In this section, we introduce the branch versions of modular partition functions, which liaise between the Dirichlet series and the corresponding transfer operators in Section 6.

For a function φ on $\Gamma \backslash \mathrm{GL}_2(\mathbb{Z})$, let us abuse the notation $\mathfrak{a}_\varphi$ to define a branch analogue of $\mathfrak{a}_\varphi(r)$ in an inductive way by $\mathfrak{a}_\varphi(\mathbf{q}) := \mathfrak{a}_\varphi(\mathbf{q}^{(n-1)}) + \varphi \circ \pi_2 \mathbf{q}$ for each $\mathbf{q} \in \mathbf{Q}^{\circ n}$, $n \geq 1$, and $\mathfrak{a}_\varphi(\mathbf{id}_{I_\Gamma}) := \mathbf{0}$. Similarly, we define

$$\mathfrak{b}_\psi(\mathbf{q}) := \mathfrak{b}_\psi(\mathbf{q}^{(n-1)}) + \widehat{\psi} \circ \pi_2 \mathbf{q},$$
$$\mathfrak{c}_\psi(\mathbf{q}) := \mathfrak{c}_\psi(\mathbf{q}^{(n-1)}) + \widetilde{\psi} \circ \pi_2 \mathbf{q}$$

for $\mathbf{q} \in \mathbf{Q}^{\circ n}$, $n \geq 1$, and a function ψ on $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. We also set $\mathfrak{b}_\psi(\mathbf{id}_{I_\Gamma}) = \mathfrak{c}_\psi(\mathbf{id}_{I_\Gamma}) = \mathbf{0}$.

Proposition 5.4. *For $r \in \mathbb{Q} \cap (0,1)$, we have $\mathfrak{a}_\varphi(r) = \mathfrak{a}_\varphi(\mathbf{q}_r)(0,\Gamma)$, $\mathfrak{b}_\psi(r) = \mathfrak{b}_\psi(\mathbf{q}_r)(0,\Gamma)$, and $\mathfrak{c}_\psi(r) = \mathfrak{c}_\psi(\mathbf{q}_r)(0,\Gamma)$.*

Proof. It is immediate from the definitions that $\mathfrak{a}_\varphi(\mathbf{q}) = \sum_{i=1}^n \varphi \circ \pi_2 \mathbf{q}^{(i)}$, $\mathfrak{b}_\psi(\mathbf{q}) = \sum_{i=1}^n \widehat{\psi} \circ \pi_2 \mathbf{q}^{(i)}$, and $\mathfrak{c}_\psi(\mathbf{q}) = \sum_{i=1}^n \widetilde{\psi} \circ \pi_2 \mathbf{q}^{(i)}$ for $\mathbf{q} \in \mathbf{Q}^{\circ n}$. From (2.4) and (5.5), we obtain the statement. ∎

Remark 5.5. Since $\mathbf{q} \in \mathbf{Q}^{\infty}$ is completely determined by $\pi_1 \mathbf{q}$, the value $\mathfrak{b}_\psi(\mathbf{q})$ also depends on $\widehat{\mathbf{q}}$. In fact, one can define $\widehat{\mathbf{q}}^{(i)}$ as a product of $\widehat{\mathbf{q}}_m$'s and hence define $\mathfrak{b}(\widehat{\mathbf{q}})$ analogously. Since $\mathrm{G}[\pi_2 \mathbf{q}^{(i)}(v)] = \mathrm{G}u$ is equivalent to $\pi_2 \widehat{\mathbf{q}^{(i)}}(v) \in u$, one can conclude that $\mathfrak{b}_\psi(\mathbf{q}) = \mathfrak{b}_\psi(\widehat{\mathbf{q}})$.

Remark 5.6. One may want to define $\widetilde{\mathbf{q}}^{(i)}$ as a product of $\widetilde{\mathbf{q}}_m$'s and hence to define $\mathfrak{c}(\widetilde{\mathbf{q}})$ analogously. However, due to the absence of an analogue of (5.3), the i-th part $\widetilde{\mathbf{q}}^{(i)}$ is not equal to $\widetilde{\mathbf{q}^{(i)}}$ in general. Instead, using (5.4), we can give a new definition for $\mathfrak{c}(\widetilde{\mathbf{q}})$, which is equal to $\mathfrak{c}(\mathbf{q})$. Since the variable $\mathfrak{c}(\mathbf{q})$ is enough for our discussion, we are not going to pursue this direction. Note that $\mathfrak{c}_\psi(\mathbf{q})$ is also completely determined by $\widetilde{\mathbf{q}}$. Hence, we also set $\mathfrak{c}_\psi(\widetilde{\mathbf{q}}) := \mathfrak{c}_\psi(\mathbf{q})$ for $\mathbf{q} \in \mathbf{Q}^\infty$.

6. Transfer operators

The transfer operator is one of the main tools for studying the statistical properties of trajectories of a dynamical system. Ruelle [35] first made a deep observation that the behavior of trajectories of dynamics can be well explained by the spectral properties of the transfer operator. In this section, we define weighted transfer operators corresponding to modular partition functions and several miscellaneous operators necessary to obtain the desired relations between Dirichlet series and operators.

We use the notation

$$\langle\Gamma\rangle := \Gamma\backslash \mathrm{GL}_2(\mathbb{Z}) \text{ or } \Gamma\backslash \mathrm{SL}_2(\mathbb{Z}),$$

according to the symbol $\mathfrak{a}$, $\mathfrak{b}$, or $\mathfrak{c}$ under discussion. We also set

$$X = X_\Gamma := I \times \langle\Gamma\rangle.$$

For a set A, let $A^{\langle\Gamma\rangle}$ be the set of all maps from $\langle\Gamma\rangle$ to A. For a $\psi \in \mathbb{C}^{\langle\Gamma\rangle}$, we set

$$\psi = \eta + i\zeta \quad \text{for } \eta, \zeta \in \mathbb{R}^{\langle\Gamma\rangle}.$$

6.1. *Branch operators*

In order to represent Dirichlet series, we first obtain expressions for the iterations of the transfer operator by studying a component of the operator, which corresponds to the continued fraction expansion of a rational number in $(0, 1)$.

For $s \in \mathbb{C}$ and $\psi \in \mathbb{C}^{\langle\Gamma\rangle}$, the *branch operators* for $\mathfrak{g}$ are defined by

$$\mathcal{B}^{\mathbf{q}}_{s,\psi}\Psi := \exp[\mathfrak{a}_\psi(\mathbf{q})]|\partial\pi_1\mathbf{q}|^s\,\Psi\circ\mathbf{q},$$
$$\widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Psi := \exp[\mathfrak{b}_\psi(\mathbf{q})]|\partial\pi_1\mathbf{q}|^s\,\Psi\circ\widehat{\mathbf{q}},$$
$$\widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Psi := \exp[\mathfrak{c}_\psi(\mathbf{q})]|\partial\pi_1\mathbf{q}|^s\,\Psi\circ\widetilde{\mathbf{q}}$$

for $\mathbf{q} \in \mathbf{Q}^\infty$ and $\Psi \in L^\infty(X)$. Here $\partial\pi_1\mathbf{q}$ is the derivative of the first component of $\mathbf{q}$. We have a multiplicative property:

Proposition 6.1. (1) *For* $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{Q}^\infty$ *and* $\mathbf{q} = \mathbf{p}_n \circ \cdots \circ \mathbf{p}_2 \circ \mathbf{p}_1$, *we have*

$$\mathcal{B}^{\mathbf{q}}_{s,\psi} = \mathcal{B}^{\mathbf{p}_1}_{s,\psi} \circ \cdots \circ \mathcal{B}^{\mathbf{p}_n}_{s,\psi} \quad \text{and} \quad \widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi} = \widehat{\mathcal{B}}^{\mathbf{p}_1}_{s,\psi} \circ \cdots \circ \widehat{\mathcal{B}}^{\mathbf{p}_n}_{s,\psi}.$$

(2) *For* $\mathbf{p}, \mathbf{q} \in \mathbf{Q}^\infty$ *with* $\mathbf{q}$ *of even depth, we have*

$$\widetilde{\mathcal{B}}^{\mathbf{q}\circ\mathbf{p}}_{s,\psi} = \widetilde{\mathcal{B}}^{\mathbf{p}}_{s,\psi} \circ \widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}.$$

In particular, for $\mathbf{q} = \mathbf{p}_n \circ \mathbf{p}_{n-1} \circ \cdots \circ \mathbf{p}_1$ *with* $\mathbf{p}_i \in \mathbf{Q}^\infty$ *of even depth for* $1 \le i \le n-1$ *and* $\mathbf{p}_n \in \mathbf{Q} \sqcup \mathbf{Q}^{\circ 2}$, *we have*

$$\widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi} = \widetilde{\mathcal{B}}^{\mathbf{p}_1}_{s,\psi} \circ \cdots \circ \widetilde{\mathcal{B}}^{\mathbf{p}_n}_{s,\psi}.$$

Proof. For the first statement it suffices to show that $\mathcal{B}^{\mathbf{p}}_{s,\psi} \circ \mathcal{B}^{\mathbf{q}}_{s,\psi}\Psi = \mathcal{B}^{\mathbf{q}\circ\mathbf{p}}_{s,\psi}\Psi$ for all $\mathbf{p}, \mathbf{q} \in \mathbf{Q}^\infty$. This is just a consequence of the identity $\mathfrak{a}_\psi(\mathbf{p}) + \psi \circ \pi_2 \mathbf{q} \circ \mathbf{p} = \mathfrak{a}_\psi(\mathbf{q} \circ \mathbf{p})$ and the chain rule. Similarly, we obtain the statements for $\mathfrak{b}$ and $\mathfrak{c}$ from (5.3) and (5.4). ■

From Proposition 5.4, we obtain the relation between a term in a Dirichlet series and an evaluation of a branch operator, both of which correspond to a rational number:

Corollary 6.2. *For* $r \in \mathbb{Q} \cap (0,1)$, *we have*

$$\begin{aligned}
\mathcal{B}^{\mathbf{q}_r}_{s,\psi}\Psi(0,\Gamma) &= \exp[\mathfrak{a}_\psi(r)]Q(r)^{-2s}\Psi(r^*, \Gamma g(r)),\\
\widehat{\mathcal{B}}^{\mathbf{q}_r}_{s,\psi}\Phi(0,\Gamma) &= \exp[\mathfrak{b}_\psi(r)]Q(r)^{-2s}\Phi(r^*, \Gamma \widehat{g}(r)),\\
\widetilde{\mathcal{B}}^{\mathbf{q}_r}_{s,\psi}\Phi(0,\Gamma) &= \exp[\mathfrak{c}_\psi(r)]Q(r)^{-2s}\Phi(r^*, \Gamma \widetilde{g}(r)).
\end{aligned}$$

For later use, we also record:

Proposition 6.3. *Setting* $\widehat{\Upsilon}_{s,\psi}(x,v) := s\log|x| + \psi(v)$, *for* $\mathbf{q} \in \mathbf{Q}^{\circ n}$ *we have*

$$\widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Psi = \widehat{\mathcal{B}}^{\mathbf{q}}_{1,0}\Big[\exp\Big(\sum_{i=0}^{n-1} \Upsilon_{2s-2,\psi} \circ \widehat{\mathbf{T}}^i\Big)\Psi\Big].$$

Setting $\widetilde{\Upsilon}_{s,\psi}(x,v) := s\log|x| + \psi(v) + \psi(\pi_2\widetilde{\mathbf{T}}(x,v))$, *for* $\mathbf{q} \in \mathbf{Q}^{\circ 2n}$ *we have*

$$\widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Psi = \widetilde{\mathcal{B}}^{\mathbf{q}}_{1,0}\Big[\exp\Big(\sum_{i=0}^{n-1} \widetilde{\Upsilon}_{2s-2,\psi} \circ \mathbf{T}^{2i}\Big)\Psi\Big].$$

Proof. The first statement follows from the chain rule and the expression $\mathfrak{b}_\psi(\mathbf{q}) = \sum_{i=0}^{n-1} \psi \circ \pi_2(\widehat{\mathbf{T}}^i \circ \widehat{\mathbf{q}})$ for $\mathbf{q} \in \mathbf{Q}^{\circ n}$. The second one follows from

$$\mathfrak{c}_\psi(\mathbf{q}) = \sum_{i=1}^{2n} \psi \circ \pi_2 \widetilde{\mathbf{q}^{(i)}} = \sum_{i=0}^{n-1} (\psi + \psi \circ \pi_2 \widetilde{\mathbf{T}}) \circ (\mathbf{T}^{2i} \circ \widehat{\mathbf{q}})$$

for $\mathbf{q} \in \mathbf{Q}^{\circ 2n}$. ■

6.2. Transfer operators

In this section, to put more emphasis on iterations, we express transfer operators in terms of branch operators rather than, as in traditional interpretations, the so-called density transformers associated to dynamical systems.

For $s \in \mathbb{C}$ and $\psi \in \mathbb{C}^{\langle\Gamma\rangle}$, the transfer operator for the variable $\mathfrak{a}$ can be written as $\mathcal{L}_{s,\psi} = \sum_{\mathbf{q}\in\mathbf{Q}} \mathcal{B}^{\mathbf{q}}_{s,\psi}$. It is a weighted transfer operator associated to the skewed Gauss map $\mathbf{T}$. Using (5.2), we can rewrite the operator in a more explicit way as

$$\mathcal{L}_{s,\psi}\Psi(x,v) = \sum_{m\geq 1} \frac{\exp[\psi(v[\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}])]}{(m+x)^{2s}} \Psi\left(\frac{1}{m+x}, v\begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}\right)$$

for $\Psi \in L^\infty(X)$. It can be easily observed that this series converges absolutely for $\Re s > 1/2$. The transfer operator for the variable $\mathfrak{b}$ is defined as

$$\widehat{\mathcal{L}}_{s,\psi} := \sum_{\mathbf{q}\in\mathbf{Q}} \widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}.$$

It is a weighted transfer operator associated to $\widehat{\mathbf{T}}$, which can be written as

$$\widehat{\mathcal{L}}_{s,\psi}\Psi(x,v) = \sum_{m\geq 1} \frac{\exp[\widehat{\psi}(v[\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}])]}{(m+x)^{2s}} \Psi\left(\frac{1}{m+x}, v\cdot\begin{bmatrix} 0 & 1 \\ 1 & m \end{bmatrix}\right).$$

Our discussions in Sections 5.2 and 6.1, especially the expression (5.4) and Proposition 6.1, lead us to define

$$\mathcal{M}_{s,\psi} := \sum_{\mathbf{p}\in\mathbf{Q}\circ\mathbf{Q}} \widetilde{\mathcal{B}}^{\mathbf{p}}_{s,\psi}.$$

It is a weighted transfer operator associated to $\widehat{\mathbf{T}}^2 = \widetilde{\mathbf{T}}^2$ which equals the restriction of $\mathbf{T}^2$ to I_Γ. Explicitly, we have

$$\mathcal{M}_{s,\psi}\Phi(x,v)$$
$$= \sum_{m,n\geq 1} \frac{\exp[\psi(v[\begin{smallmatrix} 0 & 1 \\ 1 & m \end{smallmatrix}]\mathrm{j}) + \psi(v[\begin{smallmatrix} 1 & n \\ m & 1+mn \end{smallmatrix}])]}{(1+mn+mx)^s} \Phi\left(\frac{1}{n+\frac{1}{m+x}}, v\begin{bmatrix} 1 & n \\ m & 1+mn \end{bmatrix}\right).$$

It can also be easily checked that the series converges absolutely for $\Re s > 1/2$.

Remark 6.4. For the variable $\mathfrak{c}$, one may want to define the transfer operator for $\mathfrak{c}$ such that $\widetilde{\mathcal{L}}_{s,\psi} := \sum_{\mathbf{q}\in\mathbf{Q}} \widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}$. However, due to the absence of a $\mathfrak{c}$-analogue of (5.3), it seems unlikely that $L^{\mathfrak{c}}_{\Psi,J}(s,\psi)$ is expressible in terms of $\widetilde{\mathcal{L}}_{s,\psi}$.

Remark 6.5. Note that $\mathcal{M}_{s,\psi}$ is not equal to $\widetilde{\mathcal{L}}^2_{s,\psi}$, but we have $\mathcal{M}_{s,\mathbf{0}} = \widehat{\mathcal{L}}^2_{s,\mathbf{0}}$.

6.3. *Final operators*

The *final operator for* $\mathfrak{a}$ is defined as

$$\mathcal{F}_{s,\psi} := \sum_{\mathbf{q}\in\mathbf{F}} \mathcal{B}^{\mathbf{q}}_{s,\psi}.$$

We can also define the *final operator for* $\mathfrak{b}$ by

$$\widehat{\mathcal{F}}_{s,\psi} := \sum_{\mathbf{q}\in\mathbf{F}} \widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}$$

and the *final operator for* c by

$$\widetilde{\mathcal{F}}_{s,\psi} := \sum_{\mathbf{p}\in(\mathbf{F}\circ\mathbf{Q})\sqcup\mathbf{F}} \widetilde{\mathcal{B}}^{\mathbf{p}}_{s,\psi}.$$

6.4. Interval and auxiliary operators

To deal with distributions over intervals, we need an operator, called an interval operator, which corresponds to most of rational numbers in the interval; to deal with the missing rational points avoided by the interval operator, we devise an auxiliary operator.

We first give a preliminary result on structures of intervals. Recall that

$$K^\circ(m_1,\dots,m_n) = \{[0;m_1,\dots,m_n+x] \mid 0 < x < 1\}.$$

Observe that

$$K^\circ(m_1,\dots,m_n) = \bigsqcup_{k=1}^{\infty} K^\circ(m_1,\dots,m_n,k) \sqcup \{[0;m_1,\dots,m_n,k] \mid k > 1\}. \tag{6.1}$$

For an integer $n \geq 1$, we define a collection A'_n of open fundamental intervals inductively as follows:

(1) Let A'_1 be the collection of (consecutive) open fundamental intervals of depth 1 that are included in J.

(2) Suppose A'_j has been defined for $1 \leq j \leq n$. Then A'_{n+1} is the collection of open fundamental intervals of depth $n+1$ that are included in $J \setminus \bigcup_{j=1}^{n} \bigcup_{K\in \mathrm{A}'_j} K$.

Obviously $\mathrm{A}'_n \neq \varnothing$ for some n.

The following is useful when we discuss the convergence of interval and auxiliary operators.

Proposition 6.6. *Let* $J = (a,b) \subseteq (0,1)$. *Let* $a = [0;u_1,u_2,\dots]$ *and* $b = [0;v_1,v_2,\dots]$ *be the (possibly finite) continued fraction expansions. When n is even,*

$$\mathrm{A}'_n \subseteq \{K^\circ(u_1,\dots,u_{n-1},k) \mid k \geq u_n+1\} \cup \{K^\circ(v_1,\dots,v_{n-1},k) \mid 1 \leq k \leq v_n\}.$$

When n is odd,

$$\mathrm{A}'_n \subseteq \{K^\circ(u_1,\dots,u_{n-1},k) \mid 1 \leq k \leq u_n\} \cup \{K^\circ(v_1,\dots,v_{n-1},k) \mid k \geq v_n+1\}.$$

Proof. Suppose that $c \in (0,1)$ has a (possibly finite) continued fraction expansion $c = [0;m_1,m_2,\dots]$ and let P_n/Q_n be the n-th convergent of c. It is well-known that for any $n,m \geq 1$ we have $P_{2n}/Q_{2n} \leq c \leq P_{2m-1}/Q_{2m-1}$.

Let $n > 1$ be an even integer. The leftmost open fundamental intervals of depths $n-1$ and n that are included in an interval $(c,1)$ are respectively

$$K^\circ(m_1,\dots,m_{n-1}) \quad \text{and} \quad K^\circ(m_1,\dots,m_{n-1},m_n+1).$$

Note that the left end points of these intervals are P_{n-1}/Q_{n-1} and $\frac{P_n+P_{n-1}}{Q_n+Q_{n-1}}$, respectively.

Then it can be easily seen that the open fundamental intervals of depth n that are included in an interval $(c, P_{n-1}/Q_{n-1})$ are $K^\circ(m_1, \dots, m_{n-1}, k)$ for $k \geq m_n + 1$.

The rightmost open fundamental intervals of depths $n-1$ and n that are included in an interval $(0, c)$ are respectively

$$K^\circ(m_1, \dots, m_{n-2}, m_{n-1} + 1) \quad \text{and} \quad K^\circ(m_1, \dots, m_n).$$

Note that the right end points of these intervals are $\frac{P_{n-1}+P_{n-2}}{Q_{n-1}+Q_{n-2}}$ and P_n/Q_n, respectively. Then the open fundamental intervals of depth n that are included in an interval $(\frac{P_{n-1}+P_{n-2}}{Q_{n-1}+Q_{n-2}}, c)$ are $K^\circ(m_1, \dots, m_{n-1}, k)$ for $1 \leq k \leq m_n$. In sum, we obtain the desired statements.

The arguments for odd n are similar. ∎

For an open fundamental interval $K = K^\circ(m_1, \dots, m_n)$, we define

$$\mathbf{q}_K := \mathbf{q}_{m_n} \circ \cdots \circ \mathbf{q}_{m_1}.$$

Later, we shall only need fundamental intervals of even length (see Remark 6.8). Having in mind the identity (6.1), set

$$\mathrm{A}_n := \begin{cases} \mathrm{A}'_n & \text{if } n \text{ is even,} \\ \{(K, k) \mid K \in \mathrm{A}'_n,\ k \geq 1\} & \text{if } n \text{ is odd,} \end{cases}$$

where $(K, k) := K^\circ(m_1, \dots, m_n, k)$ for $K = K^\circ(m_1, \dots, m_n)$. Then set

$$\mathbf{Q}_J := \left\{ \mathbf{q}_K \;\middle|\; K \in \bigcup_{n \geq 1} \mathrm{A}_n \right\}.$$

In particular, $\mathbf{Q}_K = \{\mathbf{q}_K\}$ for an open fundamental interval K if K is of even depth, and $\mathbf{Q}_K = \{\mathbf{q}_k \circ \mathbf{q}_K \mid k \geq 1\} = \mathbf{Q} \circ \mathbf{q}_K$ if K is of odd depth. Therefore, all the branches in $\mathbf{Q}_J$ are of even depth.

Set

$$U_J := \bigcup_{n \geq 1} \bigcup_{K \in \mathrm{A}_n} K \quad \text{and} \quad V_J := J \setminus U_J.$$

Note that V_J is a countable set of rational numbers that consists of the endpoints of all $K \in \bigcup_{n \geq 1} \mathrm{A}_n$ except the boundaries of J. Set

$$\partial \mathbf{Q}_J := \{\mathbf{q}_r \mid r \in V_J\}.$$

Let $s \in \mathbb{C}$ and $\eta \in \mathbb{C}^{\langle \Gamma \rangle}$ with sufficiently large $\Re s$ and small $\max |\eta|$. We define an *interval operator* and an *auxiliary operator for* $\mathfrak{a}$ as

$$\mathcal{D}^J_{s,\psi} := \sum_{\mathbf{q} \in \mathbf{Q}_J} \mathcal{B}^{\mathbf{q}}_{s,\psi} \quad \text{and} \quad \mathcal{K}^J_{s,\psi} := \sum_{\mathbf{q} \in \partial \mathbf{Q}_J} \mathcal{B}^{\mathbf{q}}_{s,\psi}, \tag{6.2}$$

respectively. Similarly, we define an *interval operator* and an *auxiliary operator for* $\mathfrak{b}$ as

$$\widehat{\mathcal{D}}^J_{s,\psi} := \sum_{\mathbf{q} \in \mathbf{Q}_J} \widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi} \quad \text{and} \quad \widehat{\mathcal{K}}^J_{s,\psi} := \sum_{\mathbf{q} \in \partial \mathbf{Q}_J} \widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}. \tag{6.3}$$

We also define an *interval operator* and an *auxiliary operator for* $\mathfrak{c}$ as

$$\widetilde{\mathcal{D}}^{J}_{s,\psi} := \sum_{\mathbf{q}\in \mathbf{Q}_J} \widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi} \quad \text{and} \quad \widetilde{\mathcal{K}}^{J}_{s,\psi} := \sum_{\mathbf{q}\in \partial\mathbf{Q}_J} \widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}. \tag{6.4}$$

These operators are well-defined for $\Re s > 1/2$:

Proposition 6.7. *The series in* (6.2)–(6.4) *are uniformly convergent for* $\Re s \geq \sigma_0$ *for any* $\sigma_0 > 1/2$*. Hence, they are analytic in the region* $\Re s > 1/2$*.*

Proof. We first consider the interval operator in (6.2). Let $J = (a, b)$. Let p_n/q_n and P_n/Q_n be the convergents of a and b, respectively. From Proposition 6.6, for $\Re s = \sigma > 1/2$ and a bounded function Ψ we obtain

$$\begin{aligned}\sum_{K\in \mathrm{A}_n} \mathcal{B}^{\mathbf{q}_K}_{s,\psi}\Psi &\ll \sum_{k=1}^{\infty}\left(\frac{1}{(q_{n-1}k+q_{n-2})^{2\sigma}}+\frac{1}{(Q_n k+Q_{n-1})^{2\sigma}}\right)\\ &\ll \frac{1}{2\sigma-1}\left(\frac{1}{q_n^{2\sigma}}+\frac{1}{Q_n^{2\sigma}}\right).\end{aligned}$$

Hence,

$$\mathcal{D}^{J}_{s,\psi}\Psi \ll \frac{1}{2\sigma-1}\sum_{n}\left(\frac{1}{q_n^{2\sigma}}+\frac{1}{Q_n^{2\sigma}}\right).$$

The latter sum is either finite or a convergent series for $\sigma > 1/2$ since $q_n, Q_n \geq n$.

For (6.3), it suffices to observe that for a bounded function Φ and a branch $\mathbf{q}$, we have

$$\|\widehat{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Phi\|_0 \leq \|\mathcal{B}^{\mathbf{q}}_{\sigma,\eta_0}\mathbf{1}\|_0 \cdot \|\Phi\|_0$$

where η_0 is the constant function $\max|\Re\psi|$ and $\mathbf{1} = 1 \otimes 1$. By following the previous calculation for $\mathcal{D}^{J}_{s,\psi}$, we obtain the statement. A similar argument is applied to (6.4).

For the auxiliary operators, we observe that for $r \in V_J$, there are at most two K, K' in $\bigcup_n \mathrm{A}_n$ such that r is their common endpoint. Then $\mathbf{q}_r = \mathbf{q}_K$ or $\mathbf{q}_{K'}$. Hence, we can obtain the statement from the observation that $\|\mathcal{K}^{J}_{s,\psi}\Psi\|_0 \leq 2\|\mathcal{D}^{J}_{\sigma,\eta}|\Psi|\,\|_0$ for a bounded function Ψ. The discussions for the other operators are similar. ∎

Remark 6.8. One can define the operators $\mathcal{D}^{J}_{s,\psi}$, $\widehat{\mathcal{D}}^{J}_{s,\psi}$, $\mathcal{K}^{J}_{s,\psi}$, $\widehat{\mathcal{K}}^{J}_{s,\psi}$ by using A'_n as well. However, we have no choice but to use A_n to define the operators for $\mathfrak{c}$ since Proposition 6.1 holds only for branches of even depth.

6.5. Key relations for Dirichlet series

In this subsection, we present an underlying connection between the transfer operators of the skewed Gauss dynamical systems and Dirichlet series.

Recall $\psi = \eta + i\zeta$. Observe that for $\Re s > 1 + \frac{c\max|\eta|}{2}$ (for the definition of c, see (3.2)), the Dirichlet series for $\mathfrak{a}$ can be written as

$$L^{\mathfrak{a}}_{\Psi,J}(s,\psi) = \sum_{r\in \mathbb{Q}\cap J} \frac{\Psi(r^*, \Gamma g(r))\exp(\mathfrak{a}_\psi(r))}{Q(r)^s}. \tag{6.5}$$

One can easily obtain similar expressions for $\mathfrak{b}$ and $\mathfrak{c}$.

In Theorem 8.5, it will be shown that the quasi-inverses $(\mathcal{I}-\mathcal{L}_{s,\psi})^{-1}$, $(\mathcal{I}-\widehat{\mathcal{L}}_{s,\psi})^{-1}$, and $(\mathcal{I}-\mathcal{M}_{s,\psi})^{-1}$ are well-defined as geometric series of operators when (σ,η) is close enough to $(1,\mathbf{0})$; in the remaining part of this section, we assume this. Then a portion of the sum (6.5) can be described as follows:

Proposition 6.9. *For an open fundamental interval K and a bounded function Ψ on $I\times\Gamma\backslash \mathrm{GL}_2(\mathbb{Z})$, we have*

$$\mathcal{D}^K_{s,\psi}(\mathcal{I}-\mathcal{L}_{s,\psi})^{-1}\mathcal{F}_{s,\psi}\Psi(0,\Gamma)=\sum_{r\in\mathbb{Q}\cap K}\frac{\Psi(r^*,\Gamma g(r))\exp(\mathfrak{a}_\psi(r))}{Q(r)^{2s}}.$$

For a bounded function Φ on $I\times\Gamma\backslash \mathrm{SL}_2(\mathbb{Z})$, we also have

$$\widehat{\mathcal{D}}^K_{s,\psi}(\mathcal{I}-\widehat{\mathcal{L}}_{s,\psi})^{-1}\widehat{\mathcal{F}}_{s,\psi}\Phi(0,\Gamma)=\sum_{r\in\mathbb{Q}\cap K}\frac{\Phi(r^*,\Gamma \widehat{g}(r))\exp(\mathfrak{b}_\psi(r))}{Q(r)^{2s}},$$

$$\widetilde{\mathcal{D}}^K_{s,\psi}(\mathcal{I}-\mathcal{M}_{s,\psi})^{-1}\widetilde{\mathcal{F}}_{s,\psi}\Phi(0,\Gamma)=\sum_{r\in\mathbb{Q}\cap K}\frac{\Phi(r^*,\Gamma \widetilde{g}(r))\exp(\mathfrak{c}_\psi(r))}{Q(r)^{2s}}.$$

Proof. Let $n\geq 0$ be an integer. From Proposition 6.1, we obtain

$$\mathcal{D}^K_{s,\psi}\mathcal{L}^n_{s,\psi}\mathcal{F}_{s,\psi}\Psi(0,\Gamma)=\sum_{\mathbf{q}\in\mathbf{F}\circ\mathbf{Q}^{\circ n}\circ\mathbf{Q}_K}\mathcal{B}^{\mathbf{q}}_{s,\psi}\Psi(0,\Gamma).$$

Since $r\mapsto\mathbf{q}_r$ is a one-to-one correspondence between $K\cap\mathbb{Q}$ and $\mathbf{F}\circ\mathbf{Q}^\infty\circ\mathbf{Q}_K$, and $Q(r)=Q(r^*)$, we obtain the first statement by Corollary 6.2. The proof for the second statement is similar. For the third statement, with Proposition 6.1, we have

$$\widetilde{\mathcal{D}}^K_{s,\psi}\mathcal{M}^n_{s,\psi}\widetilde{\mathcal{F}}_{s,\psi}\Phi(0,\Gamma)=\sum_{\mathbf{q}}\widetilde{\mathcal{B}}^{\mathbf{q}}_{s,\psi}\Phi(0,\Gamma)$$

where $\sum_{\mathbf{q}}$ means summation over $\mathbf{q}\in\mathbf{F}\circ(\mathbf{Q}^{\circ 2n}\circ\mathbf{Q}_K)\sqcup(\mathbf{F}\circ\mathbf{Q})\circ(\mathbf{Q}^{\circ 2n}\circ\mathbf{Q}_K)$. By the correspondence $\mathbb{Q}\cap K\to\mathbf{F}\circ\mathbf{Q}^\infty\circ\mathbf{Q}_K$, we obtain the statement from Corollary 6.2. ∎

Finally, we establish the following explicit expressions for Dirichlet series:

Theorem 6.10. *Let Ψ be a bounded function on $I\times\Gamma\backslash\mathrm{GL}_2(\mathbb{Z})$, Φ a bounded function on I_Γ, and $\Re s>1+\frac{c\max|\eta|}{2}$. Then*

$$L^{\mathfrak{a}}_{\Psi,J}(2s;\varphi)=\mathcal{K}^J_{s,\psi}\Psi(0,\Gamma)+\mathcal{D}^J_{s,\psi}(\mathcal{I}-\mathcal{L}_{s,\psi})^{-1}\mathcal{F}_{s,\psi}\Psi(0,\Gamma),$$
$$L^{\mathfrak{b}}_{\Phi,J}(2s;\psi)=\widehat{\mathcal{K}}^J_{s,\psi}\Phi(0,\Gamma)+\widehat{\mathcal{D}}^J_{s,\psi}(\mathcal{I}-\widehat{\mathcal{L}}_{s,\psi})^{-1}\widehat{\mathcal{F}}_{s,\psi}\Phi(0,\Gamma),$$
$$L^{\mathfrak{c}}_{\Phi,J}(2s;\psi)=\widetilde{\mathcal{K}}^J_{s,\psi}\Phi(0,\Gamma)+\widetilde{\mathcal{D}}^J_{s,\psi}(\mathcal{I}-\mathcal{M}_{s,\psi})^{-1}\widetilde{\mathcal{F}}_{s,\psi}\Phi(0,\Gamma).$$

Proof. From the correspondence $V_J\to\partial\mathbf{Q}_J$, as in the last proof, we get

$$\mathcal{K}^J_{s,\psi}\Psi(0,\Gamma)=\sum_{r\in V_J}\frac{\Psi(r^*,\Gamma g(r))\exp(\mathfrak{a}_\psi(r))}{Q(r)^{2s}}.$$

We also have similar expressions for $\mathfrak{b}$ and $\mathfrak{c}$. Note that there are one-to-one correspondences $\mathbb{Q}\cap U_J\to\mathbf{F}\circ\mathbf{Q}^\infty\circ\mathbf{Q}_J$ and $V_J\to\partial\mathbf{Q}_J$, given by $r\mapsto\mathbf{q}_r$. Now Proposition 6.9 and the disjoint union $J=U_J\sqcup V_J$ enable us to conclude the proof. ∎

Remark 6.11. Instead of using the interval operator, one might want to choose for Ψ a product of a smooth approximation of J and the function φ to study $\Omega_{M,\varphi,J}$. A problem is that no relation is known between $\mathfrak{c}_\psi(r)$ and $\mathfrak{c}_\psi(r^*)$, in general. However, there is a relation between $\mathfrak{m}_f^{\pm}(r)$ and $\mathfrak{m}_f^{\pm}(r^*)$, called the Atkin–Lehner relation (see Mazur–Rubin [29]). We tried this direction and were only able to obtain a partial and unsatisfactory result. One advantage of introducing the interval operator is that the Atkin–Lehner relation is unnecessary.

7. Spectral analysis of the transfer operator

In this section, we present a dynamical analysis of the transfer operator associated to the skewed Gauss dynamical system.

7.1. Basic settings and properties

For the remaining part of the paper, we write $s := \sigma + it \in \mathbb{C}$ with $\sigma, t \in \mathbb{R}$. In order to discuss all the modular partition functions simultaneously, we set

$$\mathfrak{g} := \mathfrak{a}, \mathfrak{b}, \text{ or } \mathfrak{c}.$$

We use the symbol

$$\mathcal{H}_{s,\psi} := \mathcal{L}_{s,\psi}, \widehat{\mathcal{L}}_{s,\psi}, \text{or } \mathcal{M}_{s,\psi}$$

according to the choice of $\mathfrak{g}$. Since $\mathfrak{b}_\psi$ and $\mathfrak{c}_\psi$ are also functions on the inverse branches of $\widehat{\mathbf{T}}$ and $\widetilde{\mathbf{T}}^2$, respectively (Remarks 5.5 and 5.6), let us use $\mathbf{B}$ to represent the branches:

$$\mathbf{B} := \mathbf{Q}, \widehat{\mathbf{Q}}, \text{ or } \widetilde{\mathbf{Q}^{\circ 2}}$$

according to the choice of $\mathfrak{g}$. Note that for $\Psi \in C^1(X)$, the transfer operator can be written as

$$\mathcal{H}_{s,\psi}\Psi = \sum_{\mathbf{p}\in\mathbf{B}} \exp[\mathfrak{g}_\psi(\mathbf{p})] |\partial \pi_1 \mathbf{p}|^s \Psi \circ \mathbf{p}.$$

Here $\mathfrak{a}_\psi(\mathbf{q}) = \psi \circ \pi_2 \mathbf{q}$ and $\mathfrak{b}_\psi(\mathbf{q}) = \psi \circ \pi_2 \widehat{\mathbf{q}}$ for $\mathbf{q} \in \mathbf{Q}$. For $\mathbf{p} = \mathbf{p}_2 \circ \mathbf{p}_1 \in \widetilde{\mathbf{Q}^{\circ 2}}$, we get $\mathfrak{c}_\psi(\mathbf{p}) = \psi \circ \pi_2 \widetilde{\mathbf{p}}_1 + \psi \circ \pi_2 \widetilde{\mathbf{p}}$.

Recall that $\mathcal{H}_{s,\psi}$ acts on the space $C^1(X) = \{\Psi : X \to \mathbb{C} \mid \Psi \text{ and } \partial\Psi \text{ are continuous}\}$ where ∂ is the partial derivative with respect to the first coordinate, $\partial\Psi(x, v) := \frac{\partial}{\partial x}\Psi(x, v)$. The space $C^1(X)$ is just a finite union of $C^1(I)$'s and its elements are linear combinations of tensor type $(f \otimes g)(x, v) := f(x)g(v)$ for a function f on I and a function g on the set of right cosets of Γ. It is a Banach space with the norm

$$\|\Psi\|_1 = \|\Psi\|_0 + \|\partial\Psi\|_0.$$

It is easy to show that the operator acts boundedly: For $\Psi \in C^1(X)$ and (σ, η) in a small compact real neighborhood $B \subseteq \mathbb{R} \times \mathbb{R}^{\langle\Gamma\rangle}$ of $(1, \mathbf{0})$, we have

$$\|\mathcal{H}_{\sigma,\eta}\Psi\|_1 \ll_B \|\Psi\|_1. \tag{7.1}$$

7.2. Geometric properties of the skewed Gauss dynamical system

We study the spectrum of our transfer operator in a later section. This will be done by applying the metric properties of the set $\mathbf{Q}$ of inverse branches of $\mathbf{T}$ based on the following geometric properties of the Gauss dynamical system.

We define the *contraction ratio* as

$$\rho := \begin{cases} 1/2 & \text{if } \mathfrak{g} = \mathfrak{a} \text{ or } \mathfrak{b}, \\ 1/4 & \text{if } \mathfrak{g} = \mathfrak{c}, \end{cases}$$

We remark that the ρ from Proposition 3.2 is given by the contraction ratio.

Proposition 7.1 (Baladi–Vallée [3]). *For any branch* $\mathbf{q} \in \mathbf{B}^{\circ n}$ *for* $n \geq 1$, *we have:*

(1) (Uniform contraction)

$$\|\partial \pi_1 \mathbf{q}\|_0 \ll \rho^n.$$

(2) (Bounded distortion)

$$\left\| \frac{\partial^2 \pi_1 \mathbf{q}}{\partial \pi_1 \mathbf{q}} \right\|_0 \ll 1.$$

Proof. This is a mere translation of the results of Baladi–Vallée [3, Section 2.2] or Naud [31, Lemma 3.5] in our notation. ∎

We recall the UNI property for the Gauss dynamical system established by Baladi–Vallée [3]. For any $n \geq 1$ and for two inverse branches $\mathbf{p}$ and $\mathbf{q}$ of $\mathbf{T}^n$, their temporal distance is defined by

$$\Delta(\mathbf{p}, \mathbf{q}) := \inf_X |\partial \Pi_{\mathbf{p},\mathbf{q}}|$$

where the map $\Pi_{\mathbf{p},\mathbf{q}}$ on X is given by

$$\Pi_{\mathbf{p},\mathbf{q}} := \log \frac{|\partial \pi_1 \mathbf{p}|}{|\partial \pi_1 \mathbf{q}|}.$$

Proposition 7.2 (Baladi–Vallée [3, Lemma 6]). *The skewed Gauss dynamical systems satisfy the UNI condition:*

(1) *Let* m *be the product of Lebesgue measure on* I *and the counting measure on the right cosets of* Γ. *For* $0 < a < 1$, $n \geq 1$, *and* $\mathbf{p} \in \mathbf{Q}^{\circ n}$, *we have*

$$m\Big(\bigcup_{\substack{\mathbf{q} \in \mathbf{Q}^{\circ n} \\ \Delta(\mathbf{p},\mathbf{q}) \leq \rho^{an}}} \mathbf{q}(X) \Big) \ll \rho^{an}$$

where the implicit constant is independent of a, n, *and* $\mathbf{p}$.

(2) *We have*

$$\sup_{\mathbf{p},\mathbf{q} \in \mathbf{Q}^{\infty}} \|\partial^2 \Pi_{\mathbf{p},\mathbf{q}}\|_0 < \infty.$$

7.3. Dominant eigenvalue and spectral gap of the positive transfer operator

We describe the spectrum of the positive transfer operator $\mathcal{H}_{\sigma,\eta}$ acting on $C^1(X)$. We begin by stating the following sufficient condition for quasi-compactness, due to Hennion.

Theorem 7.3 (Hennion [15]). *Let $\mathcal{H}$ be a bounded operator on a Banach space X, endowed with two norms $\|\cdot\|$ and $\|\cdot\|'$, and suppose that $\mathcal{H}(\{\phi \in X : \|\phi\| \leq 1\})$ is conditionally compact in $(X, \|\cdot\|')$. Suppose that there exist two sequences of real numbers r_n and t_n such that for any $n \geq 1$ and $\phi \in X$,*

$$\|\mathcal{H}^n \phi\| \leq t_n \|\phi\|' + r_n \|\phi\|. \tag{7.2}$$

Then the essential spectral radius of $\mathcal{H}$ is at most $\liminf_{n\to\infty} r_n^{1/n}$.

In the theory of dynamical systems, inequalities of the form (7.2) are often said to be of *Lasota–Yorke type*. They enable us not only to show quasi-compactness of $\mathcal{H}_{\sigma,\eta}$ on $C^1(X)$, but also to obtain an explicit estimate for the iterates, which is a crucial ingredient for the uniform spectral bound in Section 8.

The main estimate for norms of our transfer operators is controlled by geometric behavior of inverse branches of the Gauss map. For example, we have the following result.

Proposition 7.4. *Let B be the neighborhood of $(1, \mathbf{0})$ as in (7.1). If B is small enough, then for $(\sigma, \eta) \in B$ the following hold:*

(1) *For any $n \geq 1$ and $\Psi \in C^1(X)$, we have*

$$\|\partial \mathcal{H}^n_{\sigma,\eta} \Psi\|_0 \ll_B |\sigma|\, \|\Psi\|_0 + \rho^n \|\partial \Psi\|_0.$$

(2) *The operator $\mathcal{H}_{\sigma,\eta}$ on $C^1(X)$ is quasi-compact.*

Proof. By Proposition 6.1, for $n \geq 1$, the iteration is of the form

$$\mathcal{H}^n_{\sigma,\eta} \Psi = \sum_{\mathbf{p} \in \mathbf{B}^{\circ n}} \exp[\mathfrak{g}_\eta(\mathbf{p})] |\partial \pi_1 \mathbf{p}|^\sigma \Psi \circ \mathbf{p}.$$

Then differentiation gives

$$\partial \mathcal{H}^n_{\sigma,\eta} \Psi = \sum_{\mathbf{p}} \exp[\mathfrak{g}_\eta(\mathbf{p})] \left(\sigma |\partial \pi_1 \mathbf{p}|^\sigma \frac{|\partial^2 \pi_1 \mathbf{p}|}{|\partial \pi_1 \mathbf{p}|} \cdot \Psi \circ \mathbf{p} + |\partial \pi_1 \mathbf{p}|^\sigma \partial \pi_1 \mathbf{p} \cdot \partial \Psi \circ \mathbf{p} \right).$$

By uniform contraction in Proposition 7.1, we obtain the first statement.

Notice that the embedding of $(C^1(X), \|\cdot\|_1)$ into $(C^1(X), \|\cdot\|_0)$ is a compact operator, since Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$. Hence by Theorem 7.3 and the first statement, we have the second one. ∎

We collect the spectral properties of $\mathcal{H}_{\sigma,\eta}$. We mainly refer to Baladi [2, Theorem 1.5] for the general theory.

Proposition 7.5. *Let B be as before. For $(\sigma, \eta) \in B$, set*

$$\lambda_{\sigma,\eta} := \lim_{n\to\infty} \|\mathcal{H}^n_{\sigma,\eta} \mathbf{1}\|_0^{1/n}.$$

Then:

(1) *The value $\lambda_{\sigma,\eta}$ is the spectral radius of $\mathcal{H}_{\sigma,\eta}$ on $C^1(X)$ with $\|\cdot\|_1$.*

(2) *The operator* $\mathcal{H}_{\sigma,\eta}$ *has a positive eigenfunction* $\Phi_{\sigma,\eta}$ *with eigenvalue* $\lambda_{\sigma,\eta}$. *In particular,* $\Phi_{1,\mathbf{0}}(x,v) = \frac{1}{(\log 2)(x+1)}$ *and* $\lambda_{1,\mathbf{0}} = 1$.

(3) *The eigenvalue* $\lambda_{\sigma,\eta}$ *is of maximal modulus, positive, and simple.*

(4) *There is an eigenmeasure* $\mu_{\sigma,\eta}$ *of the adjoint of* $\mathcal{H}_{\sigma,\eta}$ *that is a Borel probability measure with* $\int_X \Phi_{\sigma,\eta}\, d\mu_{\sigma,\eta} = 1$ *after normalizing* $\Phi_{\sigma,\eta}$ *suitably. In particular,* $\mu_{1,\mathbf{0}}$ *is equivalent to Lebesgue measure.*

Proof. Even though the proof is almost the same as in [2, Theorem 1.5], we sketch it for the reader's convenience. See also Baladi–Vallée [3] and Parry–Pollicott [33] for more details.

(1) Using Proposition 7.4 (1), it can be shown that the spectral radius of $\mathcal{H}_{\sigma,\eta}$ on $C^1(X)$ with $\|\cdot\|_1$ is less than or equal to $\lambda_{\sigma,\eta}$. We also have

$$\lim_{n\to\infty} \|\mathcal{H}^n_{\sigma,\eta}\|_1^{1/n} \geq \lim_{n\to\infty} \|\mathcal{H}^n_{\sigma,\eta}\mathbf{1}\|_1^{1/n} \geq \lim_{n\to\infty} \|\mathcal{H}^n_{\sigma,\eta}\mathbf{1}\|_0^{1/n},$$

which implies the statement.

(2) Let $\lambda_0 = \lambda_{\sigma,\eta}, \lambda_1, \ldots, \lambda_\ell$ be the distinct eigenvalues of maximal modulus. Then, by spectral projection, there exist $\Psi, \Psi_j \in C^1(X)$ such that $\mathbf{1} = \Psi + \sum_{j=0}^{\ell} \Psi_j$, $\|\mathcal{H}^n_{\sigma,\eta}\Psi\|_1 = o(\lambda_0^n)$, and Ψ_j is in the generalized eigenspace for λ_j. By analyzing the Jordan normal form of $\mathcal{H}_{\sigma,\eta}$ on the generalized eigenspace, it can be shown that there exists an integer $k > 0$ such that for each $j = 0, \ldots, \ell$, the following limits exist:

$$\lim_{n\to\infty} \frac{1}{\lambda_j^n n^k} \mathcal{H}^n_{\sigma,\eta}\Psi_j =: \Phi_j.$$

Then $\mathcal{H}_{\sigma,\eta}\Phi_j = \lambda_j \Phi_j$ for each $j = 0, \ldots, \ell$ and at least one Φ_j is not trivial. In sum,

$$0 \leq \frac{\mathcal{H}^n_{\sigma,\eta}\mathbf{1}}{\lambda_0^n n^k} = o(1) + \sum_j \left(\frac{\lambda_j}{\lambda_0}\right)^n \frac{\mathcal{H}^n_{\sigma,\eta}\Psi_j}{\lambda_j^n n^k}.$$

From this inequality, using a version of the orthogonality relation:

$$\frac{1}{M}\sum_{n=1}^{M}\left(\frac{\lambda_j}{\lambda_0}\right)^n = \begin{cases} 1 & \text{if } \lambda_j = \lambda_0, \\ o(1) & \text{otherwise,} \end{cases}$$

we deduce that the function Φ_0 is non-negative and non-trivial.

Suppose that $\Phi_0(x_0, v_0) = 0$ for some $(x_0, v_0) \in X$. For each $n \geq 1$, we have

$$0 = \mathcal{H}^n_{\sigma,\eta}\Phi_0(x_0, v_0) = \sum_{\mathbf{p}\in\mathbf{B}^{\circ n}} \exp[\mathfrak{g}_\eta(\mathbf{p})]|\partial\pi_1\mathbf{p}(x_0, v_0)|^\sigma \Phi_0(\mathbf{p}(x_0, v_0)).$$

Since the weights $\exp[\mathfrak{g}_\eta(\mathbf{p})]|\partial\pi_1\mathbf{p}(x_0, v_0)|^\sigma$ are positive, $\Phi_0(\mathbf{p}(x_0, v_0)) = 0$ for all $\mathbf{p} \in \mathbf{B}^{\circ n}$. The density result of Proposition 5.1 (2) and continuity of Φ result in a contradiction. Hence, we obtain the statement with $\Phi_{\sigma,\eta} := \Phi_0$.

It is a classical result that $\Phi_{1,\mathbf{0}}(x,v) = \frac{1}{\log(2(x+1))}$ is an eigenfunction of $\mathcal{L}_{1,\mathbf{0}}$ and $\widehat{\mathcal{L}}_{1,\mathbf{0}}$ with eigenvalue 1 of maximal modulus. For $\mathcal{M}_{s,\mathbf{w}}$, recall that $\mathcal{M}_{1,\mathbf{0}} = \widehat{\mathcal{L}}^2_{1,\mathbf{0}}$.

(3) The first two claims come from the definition of $\lambda_{\sigma,\eta}$. For geometric simplicity, let Ψ be a eigenfunction for λ_0 and set $t := \min\{\frac{\Psi(x,v)}{\Phi_0(x,v)} \mid (x,v) \in X\}$. Since $t = \frac{\Psi(x_0,v_0)}{\Phi_0(x_0,v_0)}$ for some $(x_0, v_0) \in X$ by continuity, we conclude that $\Psi = t\,\Phi_0$ using a density argument.

For algebraic simplicity, assume that for a non-trivial $\Psi \in C^1(X)$, one has $(\mathcal{H}_{\sigma,\eta} - \lambda_0 \mathcal{I})^2\Psi = \mathbf{0}$ and $\Phi := (\mathcal{H}_{\sigma,\eta} - \lambda_0 \mathcal{I})\Psi \neq \mathbf{0}$. Then $\mathcal{H}^n_{\sigma,\eta}\Psi = \lambda_0^n\Psi + n\lambda_0^{n-1}\Phi$, and we deduce a contradiction from

$$\|\mathcal{H}^n_{\sigma,\eta}\Psi\|_0 \le \|\mathcal{H}^n_{\sigma,\eta}\Phi_{\sigma,\eta}\|_0\|\Phi^{-1}_{\sigma,\eta}\Psi\|_0 = \lambda_0^n\|\Phi^{-1}_{\sigma,\eta}\Psi\|_0.$$

(4) Extending the functionals on the eigenspace for $\lambda_{\sigma,\eta}$ to $C^1(X)$ using a spectral projection, we obtain a positive Radon eigenmeasure of the adjoint, which corresponds to a Borel probability measure on X. Normalizing suitably, we obtain the statement. ■

We show the uniqueness of the eigenvalue of maximal modulus.

Proposition 7.6. *Let $\mathfrak{g} = \mathfrak{b}$ or $\mathfrak{c}$. Then, for $(\sigma,\eta) \in B$, the eigenvalue $\lambda_{\sigma,\eta}$ is unique, i.e., $\mathcal{H}_{\sigma,\eta}$ has no other eigenvalue on the circle of radius $\lambda_{\sigma,\eta}$.*

Proof. Since the proof is almost the same as in [2, Theorem 1.5 (5)], we only sketch it. First we need to show that

$$\lim_{n\to\infty}\left\| \frac{1}{\lambda^n_{\sigma,\eta}}\mathcal{H}^n_{\sigma,\eta}\Psi - \Phi_{\sigma,\eta}\int_X \Psi\, d\mu_{\sigma,\eta}\right\|_0 = 0 \tag{7.3}$$

for all $\Psi \in C^1(X)$. Our version of the density result in Proposition 5.1 (2) together with Proposition 2.4 enables us to show that a continuous accumulation point of the sequence $\lambda^{-n}_{\sigma,\eta}\mathcal{H}_{\sigma,\eta}\Psi$ is actually the function $\Phi_{\sigma,\eta}\int_X \Psi\, d\mu_{\sigma,\eta}$. The limit is verified by applying the Arzelà–Ascoli theorem to the equicontinuous family $\{\lambda^{-n}_{\sigma,\eta}\mathcal{H}^n_{\sigma,\eta}\Psi \mid n \ge 1\}$. Then the statement of the proposition follows easily from (7.3). ■

8. Dolgopyat–Baladi–Vallée bound in a vertical strip

The main objective of this section is a uniform polynomial bound for the iterations of $\mathcal{H}_{s,\psi}$, namely the Dolgopyat–Baladi–Vallée estimate. As a consequence, along with the results from Section 7, we complete the proof of Proposition 3.2 at the end of this section.

Dolgopyat [12] first established a result of this type for the plain transfer operators associated to certain Anosov systems with a finite Markov partition, which depends on a single complex parameter s. Let us roughly overview his ideas for the proof:

(1) Due to the spectral properties of the transfer operator, the main estimate can be reduced to an L^2-norm estimate, which involves a sum of oscillatory integrals over pairs of inverse branches. This sum is divided into two parts.

(2) Relatively *separated pairs* of inverse branches form one part, in which the oscillatory integrals can be easily dealt with, using the van der Corput lemma.

(3) In order to control the other part that consists of *close pairs*, the dynamical system must satisfy the Uniform Non-Integrability (UNI) condition, which explains why there are few such pairs.

This groundbreaking work has been generalized to other dynamical systems. In particular, Baladi–Vallée [3] modified the UNI condition to obtain a Dolgopyat-type estimate for the weighted transfer operator associated to the Gauss map with countably many inverse branches. Our proof is similar. In fact, focusing on classical continued fractions, we mainly follow the more concise exposition of Naud [31].

8.1. Reduction to L^2-estimates

Consider the normalized operator defined by

$$\underline{\mathcal{H}}_{s,\psi}\Psi := \lambda_{\sigma,\eta}^{-1}\Phi_{\sigma,\eta}^{-1}\mathcal{H}_{s,\psi}(\Phi_{\sigma,\eta}\cdot\Psi) \tag{8.1}$$

for $\Psi \in C^1(X)$. Then $\underline{\mathcal{H}}_{\sigma,\eta}$ on $C^1(X)$ has spectral radius 1 and fixes the constant function $\mathbf{1}$, i.e., $\underline{\mathcal{H}}_{\sigma,\eta}\mathbf{1} = \mathbf{1}$.

For $t \neq 0$, we use the norm

$$\|\Psi\|_{(t)} := \|\Psi\|_0 + \frac{1}{|t|}\|\Psi\|_0 \quad \text{for } \Psi \in C^1(X),$$

which is equivalent to $\|\cdot\|_1$. One of the main aims of this section is to estimate $\|\mathcal{H}_{s,\psi}^n\|_{(t)}$ or equivalently $\|\underline{\mathcal{H}}_{s,\psi}^n\|_{(t)}$.

We start the calculation to obtain a Dolgopyat–Baladi–Vallée bound by reducing our main estimate to an L^2-type estimate. For the reduction we need the following result.

Lemma 8.1. *Let* $(\sigma,\eta) \in B$ *where* B *is chosen small enough so that* $\sigma > 3/4$. *For all* $n \geq 1$, *we have*

$$\|\underline{\mathcal{H}}_{\sigma,\eta}^n\Psi\|_0^2 \ll_B A_{\sigma,\eta}^{2n}\cdot\left\|\underline{\mathcal{H}}_{1,\mathbf{0}}^n|\Psi|^2\right\|_0$$

for $A_{\sigma,\eta} = \lambda_{\sigma,\eta}^{-1}\sqrt{\lambda_{2\sigma-1,2\eta}} > 0$.

Proof. By the Cauchy–Schwarz inequality, we have

$$\|\underline{\mathcal{H}}_{\sigma,\eta}^n\Psi\|_0^2 \leq \lambda_{\sigma,\eta}^{-2n}\|\mathcal{H}_{2\sigma-1,2\eta}^n\Phi_{2\sigma-1,2\eta}\|_0\cdot\left\|\mathcal{H}_{1,\mathbf{0}}^n|\Psi|^2\right\|_0.$$

The desired result comes from

$$\mathcal{H}_{1,\mathbf{0}}^n|\Psi|^2 \ll_B \underline{\mathcal{H}}_{1,\mathbf{0}}^n|\Psi|^2 \quad\text{and}\quad \mathcal{H}_{2\sigma-1,2\eta}^n\Phi_{2\sigma-1,2\eta} \ll_B \lambda_{2\sigma-1,2\eta}^n. \qquad \blacksquare$$

A crucial observation based on the spectral gap is that the projection operator $\underline{\mathcal{P}}_{1,\mathbf{0}}$ associated with the dominant eigenvalue 1 satisfies $\underline{\mathcal{H}}_{1,\mathbf{0}} = \underline{\mathcal{P}}_{1,\mathbf{0}} + \underline{\mathcal{N}}_{1,\mathbf{0}}$ and the subdominant spectral radius R_1 of $\underline{\mathcal{N}}_{1,\mathbf{0}}$ is strictly less than 1. In particular, $\underline{\mathcal{P}}_{1,\mathbf{0}}\Psi = \int_X \Psi\, dm$ and hence $\underline{\mathcal{H}}_{1,\mathbf{0}}^n\Psi = \int_X \Psi\, dm + O(R_1^n)$. In sum, from Lemma 8.1, we get

$$\|\underline{\mathcal{H}}_{s,\psi}^{n+k}\Psi\|_0^2 \ll_B A_{\sigma,\eta}^{2n}\left(\int_X |\underline{\mathcal{H}}_{s,\psi}^k\Psi|^2 dm + O(R_1^n)|t|\,\|\Psi\|_{(t)}^2\right). \tag{8.2}$$

8.2. Estimating L^2-norms

The normalized operator satisfies the Lasota–Yorke inequality, which comes from a direct computation similar to the proof of Proposition 7.4.

Proposition 8.2. *Let B be as before. For (s, ψ) with $(\sigma, \eta) \in B$ and all $n \geq 1$, we have*

$$\|\underline{\mathcal{H}}_{s,\psi}^n \Psi\|_1 \ll_B |s| \, \|\Psi\|_0 + \rho^n \|\Psi\|_1.$$

The following L^2-estimate is the heart of Section 8, in which the UNI property of Proposition 7.2 plays an essential role together with the Lasota–Yorke inequality.

Proposition 8.3. *For suitable constants $\alpha, \beta > 0$, for large $|t| \geq 1/\rho^2$, and for (s, ψ) with $(\sigma, \eta) \in B$, we have*

$$\int_X |\underline{\mathcal{H}}_{s,\psi}^{\lceil \alpha \log |t| \rceil} \Psi|^2 \, dm \ll_B \rho^{\beta \lceil \alpha \log |t| \rceil} \|\Psi\|_{(t)}^2.$$

Proof. First we express the integrand as

$$|\underline{\mathcal{H}}_{s,\psi}^n \Psi|^2 = \frac{1}{\lambda_{\sigma,\eta}^{2n}} \sum_{(\mathbf{p},\mathbf{q}) \in \mathbf{B}^{\circ n} \times \mathbf{B}^{\circ n}} |\partial \pi_1 \mathbf{p}|^{it} |\partial \pi_1 \mathbf{q}|^{-it} \cdot R_{\mathbf{p},\mathbf{q}}^{\sigma}$$

where we set

$$\begin{aligned} g_\psi(\mathbf{p}, \mathbf{q}) &:= \exp[\mathfrak{g}_\psi(\mathbf{p}) + \mathfrak{g}_{\overline{\psi}}(\mathbf{q})], \\ R_{\mathbf{p},\mathbf{q}}^{\sigma} &:= \Phi_{\sigma,\eta}^{-2} \cdot g_\psi(\mathbf{p}, \mathbf{q}) |\partial \pi_1 \mathbf{p}|^{\sigma} |\partial \pi_1 \mathbf{q}|^{\sigma} \cdot (\Phi_{\sigma,\eta} \Psi) \circ \mathbf{p} \cdot (\Phi_{\sigma,\eta} \overline{\Psi}) \circ \mathbf{q} \end{aligned}$$

in order to simplify the notation. Thus we have

$$\int_X |\underline{\mathcal{H}}_{s,\psi}^n \Psi|^2 \, dm = \frac{1}{\lambda_{\sigma,\eta}^{2n}} \sum_{(\mathbf{p},\mathbf{q})} \int_X \exp[it \Pi_{\mathbf{p},\mathbf{q}}] R_{\mathbf{p},\mathbf{q}}^{\sigma} \, dm. \tag{8.3}$$

Here recall that $\Pi_{\mathbf{p},\mathbf{q}} = \log |\partial \pi_1 \mathbf{p}| - \log |\partial \pi_1 \mathbf{q}|$.

Since the $R_{\mathbf{p},\mathbf{q}}^{\sigma}$ are bounded, the sum is dominated by the oscillatory integrals which are controlled by the behavior of the phase function $\Pi_{\mathbf{p},\mathbf{q}}$, hence essentially by the geometric properties of the skewed Gauss map. We divide the sum (8.3) into two parts: one with close pairs, i.e., with small $\Delta(\mathbf{p}, \mathbf{q})$, and the other with relatively separated pairs, i.e., with relatively large $\Delta(\mathbf{p}, \mathbf{q})$. In other words, $\int_X |\underline{\mathcal{H}}_{s,\psi}^n \Psi|^2 \, dm = I^{(1)} + I^{(2)}$, where

$$\begin{aligned} I^{(1)} &:= \frac{1}{\lambda_{\sigma,\eta}^{2n}} \sum_{\Delta(\mathbf{p},\mathbf{q}) \leq \varepsilon} \int_X e^{it \Pi_{\mathbf{p},\mathbf{q}}} R_{\mathbf{p},\mathbf{q}}^{\sigma} \, dm, \\ I^{(2)} &:= \frac{1}{\lambda_{\sigma,\eta}^{2n}} \sum_{\Delta(\mathbf{p},\mathbf{q}) > \varepsilon} \int_X e^{it \Pi_{\mathbf{p},\mathbf{q}}} R_{\mathbf{p},\mathbf{q}}^{\sigma} \, dm. \end{aligned}$$

Let us consider the integral $I^{(1)}$. We set $\nu_{\sigma,\eta} := \Phi_{\sigma,\eta} \mu_{\sigma,\eta}$, which is fixed by the normalized adjoint operator. We need the following results that are mere reformulations of Naud's [31, Lemma 4.2].

Lemma 8.4. (1) *For all* $\mathbf{p} \in \mathbf{B}^{\circ n}$, *we have* $\|\partial\pi_1 \mathbf{p}\|_0^\sigma / \lambda_{\sigma,\eta}^n \asymp_B \nu_{\sigma,\eta}(\mathbf{p}(X))$.

(2) *Let* $\mathbf{A}$ *be a subset of* $\mathbf{B}^{\circ n}$ *and* $Y = \bigcup_{\mathbf{q}\in\mathbf{A}} \mathbf{q}(X)$. *Then* $\nu_{\sigma,\eta}(Y) \ll_B A_{\sigma,\eta}^{2n} m(Y)^{1/2}$.

Obviously,

$$I^{(1)} \ll_B \frac{\|\Psi\|_0^2}{\lambda_{\sigma,\eta}^{2n}} \sum_{\Delta(\mathbf{p},\mathbf{q})\le\varepsilon} \|\partial\pi_1\mathbf{p}\|_0^\sigma \|\partial\pi_1\mathbf{q}\|_0^\sigma \int_X g_\eta(\mathbf{p},\mathbf{q})\, dm. \tag{8.4}$$

Hence, by Lemma 8.4,

$$\begin{aligned} I^{(1)} &\ll_B \|\Psi\|_0^2 \sum_{\Delta(\mathbf{p},\mathbf{q})\le\varepsilon} \nu_{\sigma,\eta}(\mathbf{p}(X))\nu_{\sigma,\eta}(\mathbf{q}(X)) \int_X g_\eta(\mathbf{p},\mathbf{q})\, dm \\ &\ll_B \|\Psi\|_0^2 \sum_{\mathbf{p}\in\mathbf{B}^{\circ n}} \nu_{\sigma,\eta}(\mathbf{p}(X)) \Big(\sum_{\substack{\mathbf{q}\in\mathbf{B}^{\circ n}\\ \Delta(\mathbf{p},\mathbf{q})\le\varepsilon}} \nu_{\sigma,\eta}(\mathbf{q}(X)) \Big). \end{aligned}$$

For any $0 < a < 1$, taking $\varepsilon = \rho^{an}$, we finally have

$$|I^{(1)}| \ll_B \|\Psi\|_0^2 \lambda_{\sigma,\eta}^{-n} \|\mathcal{H}_{\sigma,\mathbf{0}}^n \mathbf{1}\|_0 \rho^{an/2} \ll_B \rho^{an/2} A_{\sigma,\eta}^{2n} \|\Psi\|_0^2$$

by the UNI condition of Proposition 7.2 (1) and Lemma 8.4 (1).

The main point in estimating $I^{(2)}$ is to deal with the oscillatory integrals with the phase function $\Pi_{\mathbf{p},\mathbf{q}}$. By Proposition 7.2 (2) and a version of the van der Corput lemma (see Baladi–Vallée [3, p. 359]), we obtain

$$|I^{(2)}| \ll \sum_{\Delta(\mathbf{p},\mathbf{q})>\varepsilon} \frac{\|R_{\mathbf{p},\mathbf{q}}^\sigma\|_1}{|t|} \Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\Big) \ll_B \|\Psi\|_{(t)}^2 \frac{1+\rho^n|t|}{|t|} \Big(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\Big)$$

by applying the Lasota–Yorke type estimate for $R_{\mathbf{p},\mathbf{q}}^\sigma$. Then again choosing the scale $\varepsilon = \rho^{an}$ with $n = \lceil \alpha \log|t| \rceil$ for some α and a satisfying $|t| \le \rho^{-an}$, we have $|I^{(2)}| \ll \rho^{(1-2a)n} \|\Psi\|_{(t)}^2$.

Hence by the above choices of ε, n, a and α, we finally obtain an estimate for $I^{(1)} + I^{(2)}$ with a constant $\beta = 1 - 2a > 0$. ■

8.3. *Uniform polynomial growth*

Finally, the following Dolgopyat–Baladi–Vallée estimate can be deduced from the L^2-type estimate of Proposition 8.3.

Theorem 8.5. *For* $0 < \xi < 1/5$, *there is a small neighborhood* $B' \subseteq B$ *such that for all complex pairs* (s,ψ) *with* $(\sigma,\eta) \in B'$, *an integer* $n \ge 1$, *and* $|t| \ge 1/\rho^2$, *we have*

$$\|\mathcal{H}_{s,\psi}^n\|_{(t)} \ll_{B,\xi} (r\lambda_{\sigma,\eta})^n |t|^\xi$$

for some $0 < r < 1$. *In particular, the quasi-inverse* $(\mathcal{I} - \mathcal{H}_{s,\psi})^{-1}$ *is well-defined and analytic when* $(\sigma,\eta) \in B'$.

Proof. Set $n_0 = n_0(t) := \lceil \alpha \log |t| \rceil$. From (8.2), for $n_1 = n_1(t) \geq n_0$, we have

$$\|\underline{\mathcal{H}}^{n_1}_{s,\psi} \Psi\|_0^2 \ll_B A^{2(n_1-n_0)}_{\sigma,\eta} \left(\int_X |\underline{\mathcal{H}}^{n_0}_{s,\psi} \Psi|^2 \, dm + R_1^{n_1-n_0} |t| \, \|\Psi\|_{(t)}^2 \right)$$
$$\ll_B A^{2(n_1-n_0)}_{\sigma,\eta} (\rho^{\beta n_0} + R_1^{n_1-n_0}|t|) \|\Psi\|_{(t)}^2 .$$

We take $n_1 = \lceil \widetilde{\alpha} n_0 \rceil$ for some $\widetilde{\alpha} > 1$ large enough to have $R_1^{n_1-n_0}|t| = O(\rho^{\beta n_0})$ and choose B' small enough that $A^{n_1-n_0}_{\sigma,\eta} \ll_{B'} \rho^{-\beta n_0/2}$. Then

$$\|\underline{\mathcal{H}}^{n_1}_{s,\psi} \Psi\|_0 \ll_{B'} \rho^{\widetilde{\beta} n_1} \|\Psi\|_{(t)}$$

for a suitable $\widetilde{\beta} > 0$. Repeated application of the Lasota–Yorke inequality from Proposition 8.2 enables us to write

$$\|\underline{\mathcal{H}}^{2n_1}_{s,\psi} \Psi\|_1 \ll |s| \, \|\underline{\mathcal{H}}^{n_1}_{s,\psi} \Psi\|_0 + \rho^{n_1} \|\underline{\mathcal{H}}^{n_1}_{s,\psi} \Psi\|_1 \ll \rho^{\widetilde{\beta} n_1} |t| \, \|\Psi\|_{(t)}$$

and hence $\|\underline{\mathcal{H}}^{2n_1}_{s,\psi}\|_{(t)} \ll \rho^{\widetilde{\beta} n_1}$. For a fixed t with $|t| \geq 1/\rho^2$, writing any integer n as $(2n_1)q + m$ with $m < 2n_1$, we obtain

$$\|\underline{\mathcal{H}}^{n}_{s,\psi}\|_{(t)} \leq \|\underline{\mathcal{H}}^{m}_{s,\psi}\|_{(t)} \|\underline{\mathcal{H}}^{2n_1}_{s,\psi}\|^q_{(t)} \ll \rho^{\widetilde{\beta} q n_1} \leq \rho^{\widetilde{\beta} n/2} \rho^{-\widetilde{\beta} n_1}$$

since for large $|t|$, we have $\|\underline{\mathcal{H}}^{m}_{s,\psi}\|_{(t)} \ll 1$. This leads to the assertion by choosing $\xi = \widetilde{\beta}\widetilde{\alpha}$ and $r = \rho^{\widetilde{\beta}/2}$.

The more detailed computation of Baladi–Vallée [3, Section 3.3, (3.21)–(3.23)] gives closed forms of $\widetilde{\alpha}$ and $\widetilde{\beta}$, which show that the constant ξ can be taken to be any value between 0 and 1/5. ∎

9. Coboundary conditions

First we collect some preliminary results to prove the main steps of Proposition 3.2. We follow a similar argument of Baladi–Vallée [3, Proposition 1].

From [3, Proposition 0.6a]) together with Remark 6.5, we get

Proposition 9.1.

$$\left. \frac{\partial \lambda_{s,\mathbf{0}}}{\partial s} \right|_{s=1} = -\frac{\pi^2}{12 \rho \log 2} .$$

For a fixed real number h, define a piecewise differentiable cost function $\Upsilon \in L^1(X)$ by

$$\Upsilon(x, v) := \begin{cases} 2h \log |x| + \psi(v) & \text{if } \mathfrak{g} = \mathfrak{b}, \\ 2h \log |x| + \psi(v) + \psi\left(v\left[\begin{smallmatrix} -m_1(x) & 1 \\ 1 & 0 \end{smallmatrix}\right]\mathrm{j}\right) & \text{if } \mathfrak{g} = \mathfrak{c}. \end{cases}$$

From now on, set

$$\mathbf{S} := \widehat{\mathbf{T}} \text{ or } \widetilde{\mathbf{T}}^2$$

according to the choice of $\mathfrak{g} = \mathfrak{b}$ or $\mathfrak{c}$. The following two results will be useful when discussing the pole $s(\mathbf{w})$.

Proposition 9.2. *For $\psi \in \mathbb{C}^{\langle \Gamma \rangle}$, we have*

$$\frac{d^2}{dw^2}\lambda_{1+hw,w\psi}\Big|_{w=0} = \lim_{n\to\infty}\frac{1}{n}\int_X \Big(\sum_{k=0}^{n-1} \Upsilon \circ \mathbf{S}^k\Big)^2 \Phi_{1,0}\, dm.$$

Proof. We set $\kappa(w) := \lambda_{1+hw,w\psi}$ and $\Psi(w) := \Phi_{1+hw,w\psi}$. Note that $\kappa(0) = 1$ and $\kappa'(0) = 0$. From Proposition 6.3, we obtain

$$\kappa(w)^n \Psi(w) = \mathcal{H}^n_{1+hw,w\psi}\Psi(w) = \mathcal{H}^n_{1,0}\Big[\exp\Big[w\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big]\Psi(w)\Big].$$

Differentiating this twice and setting $w = 0$, we have

$$n\kappa''(0)\Psi(0) + \Psi''(0) = \mathcal{H}^n_{1,0}\Big[\Big(\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)^2\Psi(0) + 2\Big(\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)\Psi'(0) + \Psi''(0)\Big].$$

Hence,

$$\kappa''(0) = \frac{1}{n}\int_X \Big(\Big(\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)^2\Psi(0) + 2\Big(\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)\Psi'(0)\Big)\, dm.$$

We will show that the second term satisfies

$$\int_X \Big(\frac{1}{n}\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)\Psi'(0)\, dm = o(1). \tag{9.1}$$

Indeed,

$$\int_X \Big(\frac{1}{n}\sum_{k=0}^{n-1}\Upsilon\circ\mathbf{S}^k\Big)\Psi'(0)\, dm = \int_X \Upsilon\Big(\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{H}^k_{1,0}[\Psi'(0)]\Big)\, dm,$$

and since the spectral radius of $\mathcal{N}_{1,0}$ is strictly less than 1, we get

$$\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{H}^k_{1,0}[\Psi'(0)] = \frac{1}{n}\sum_{k=0}^{n-1}\mathcal{P}^k_{1,0}[\Psi'(0)] + o(1) = C\Phi_{1,0} + o(1)$$

for a constant C. Hence the LHS of (9.1) equals

$$C\int_X \Upsilon\Phi_{1,0}\, dm + o(1) = C\kappa'(0) + o(1) = o(1).$$

This finishes the proof. ∎

Proposition 9.3. *There is $\Theta \in C^1(X)$ such that for $\widetilde{\Upsilon} := \Upsilon + \Theta\circ\mathbf{S} - \Theta$, we have*

$$\frac{d^2}{dw^2}\lambda_{1+hw,w\psi}\Big|_{w=0} = \int_X \widetilde{\Upsilon}^2\Phi_{1,0}\, dm.$$

In particular, the last quantity is zero if and only if $h = 0$ and ψ is a coboundary over $\mathbb{R}$.

Proof. Recall that $\int_X \Upsilon \Phi_{1,0}\, dm = 0$. Hence, $\|\mathcal{H}^n_{1,0} \Upsilon \Phi_{1,\mathbf{0}}\|_0 \ll R_1^n$ for the subdominant eigenvalue $R_1 < 1$ as $\mathcal{H}_{1,\mathbf{0}}[\Upsilon \Phi_{1,\mathbf{0}}] \in C^1(X)$ and we obtain a function

$$\Theta := \Phi_{1,\mathbf{0}}^{-1}(\mathcal{I} - \mathcal{H}_{1,\mathbf{0}})^{-1}\mathcal{H}_{1,\mathbf{0}}[\Upsilon \Phi_{1,\mathbf{0}}]$$

that is well-defined in $C^1(X)$. Like (9.1), it can be shown that

$$\int_X \left(\frac{1}{n}\sum_{k=0}^{n-1} \widetilde{\Upsilon} \circ \mathbf{S}^k\right)\Phi_{1,\mathbf{0}}\, dm = o(1).$$

Since $\Theta \circ \mathbf{S} - \Theta$ is bounded, we conclude that

$$\frac{d^2\lambda}{dw^2}(1,0) = \lim_{n\to\infty}\frac{1}{n}\int_X \Big(\sum_{k=0}^{n-1} \widetilde{\Upsilon} \circ \mathbf{S}^k\Big)^2 \Phi_{1,\mathbf{0}}\, dm.$$

Since $\mathcal{H}_{1,0}[\Psi_1\Psi_2 \circ \mathbf{S}] = \mathcal{H}_{1,0}[\Psi_1]\Psi_2$ for any $\Psi_1, \Psi_2 \in C^1(X)$, one can show that $\mathcal{H}_{1,\mathbf{0}}[\widetilde{\Upsilon}\Phi_{1,\mathbf{0}}] = 0$. Hence, for $k > j$,

$$\int_X \widetilde{\Upsilon} \circ \mathbf{S}^k \widetilde{\Upsilon} \circ \mathbf{S}^j \Phi_{1,\mathbf{0}}\, dm = \int_X \mathcal{H}^{k-j-1}_{1,\mathbf{0}}\big[\widetilde{\Upsilon} \circ \mathbf{S}^{k-j-1}\mathcal{H}_{1,\mathbf{0}}[\widetilde{\Upsilon}\Phi_{1,\mathbf{0}}]\big]\, dm = 0,$$

which yields the first statement.

For the second statement, observe that the integral is zero if and only if $\widetilde{\Upsilon} = 0$, i.e., $\Upsilon = \Theta - \Theta \circ \mathbf{S}$. Since Θ is bounded, the statement is equivalent to the conditions that $h = 0$ and ψ is a coboundary over $\mathbb{R}$. ■

Remark 9.4. Observe that ψ is a $\mathfrak{g}$-coboundary over $\Bbbk$ if and only if there exists a $\beta \in \Bbbk^{\langle \Gamma \rangle}$ such that $\mathfrak{g}_\psi(\mathbf{p}) = \beta - \beta \circ \pi_2\mathbf{p}$ for any $\mathbf{p} \in \mathbf{B}$.

The following result will be one of the crucial ingredients in the next section.

Proposition 9.5. *Let $\mathcal{H}_{s,\psi} = \widehat{\mathcal{L}}_{s,\psi}$ or $\mathcal{M}_{s,\psi}$. Let t be a real number. Then 1 is an eigenvalue of $\mathcal{H}_{1+it,i\zeta}$ if and only if $t = 0$ and ζ is a $\mathfrak{g}$-coboundary over $\mathbb{R}/2\pi\mathbb{Z}$.*

Proof. First assume that there is a $\Psi \in C^1(X)$ with $\|\Psi\|_0 = 1$ such that

$$\mathcal{H}_{1+it,i\zeta}\Phi_{1,\mathbf{0}}\Psi = \Phi_{1,\mathbf{0}}\Psi.$$

Suppose that $|\Psi|$ attains a maximum at (x_0, v_0). Setting

$$a_{\mathbf{q}} := \frac{1}{\Phi_{1,\mathbf{0}}(x_0, v_0)}|\partial\pi_1\mathbf{q}(x_0)|\Phi_{1,0} \circ \mathbf{q}(x_0, v_0),$$
$$b_{\mathbf{q}} := \frac{1}{\Psi(x_0, v_0)}\Psi \circ \mathbf{q}(x_0, v_0)\exp[i\,\mathfrak{g}_\zeta(\mathbf{q})(v_0)]|\partial\pi_1\mathbf{q}(x_0)|^{it},$$

we have $\sum_{\mathbf{q}\in\mathbf{B}^{\circ n}} a_{\mathbf{q}} b_{\mathbf{q}} = 1$ for all $n \geq 1$. Since $\sum_{\mathbf{q}\in\mathbf{B}^{\circ n}} a_{\mathbf{q}} = 1$ and $|b_{\mathbf{q}}| \leq 1$, we obtain $b_{\mathbf{q}} = 1$ for all $\mathbf{q}$. In other words,

$$\exp[i\,\mathfrak{g}_\zeta(\mathbf{q})(v_0)]|\partial\pi_1\mathbf{q}(x_0)|^{it}\Psi \circ \mathbf{q}(x_0, v_0) = \Psi(x_0, v_0)$$

for any $\mathbf{q} \in \mathbf{B}^{\infty}$. Proposition 5.1 enables us to show $|\Psi| \equiv 1$, the constant function. Then we repeat the above process for any $(x, v) \in X$ to conclude that

$$\exp[i\,\mathfrak{g}_{\zeta}(\mathbf{q})]|\partial\pi_1\mathbf{q}|^{it}\Psi \circ \mathbf{q} = \Psi \tag{9.2}$$

for all $\mathbf{q} \in \mathbf{B}^{\infty}$.

From (9.2), we have

$$|t| \cdot \|\partial\Pi_{\mathbf{p},\mathbf{q}}\|_0 = \|\partial\pi_1\mathbf{p} \cdot \partial(\log\Psi) \circ \mathbf{p} - \partial\pi_1\mathbf{q} \cdot \partial(\log\Psi) \circ \mathbf{q}\|_0$$

for all $\mathbf{p}, \mathbf{q} \in \mathbf{B}^{\circ n}$ and $n \geq 1$. By uniform contraction in Proposition 7.1, we find that $|t| \cdot \|\partial\Pi_{\mathbf{p},\mathbf{q}}\|_0 \ll \rho^n \leq \rho^{an}$ for all $0 < a < 1$ and $\mathbf{p}, \mathbf{q} \in \mathbf{B}^{\infty}$. Hence, $t = 0$, because otherwise the last estimate violates the UNI property (a) in Proposition 7.2. In sum, there exists a $\Psi \in C^1(X)$ such that $|\Psi| \equiv 1$ and

$$\exp[i\,\mathfrak{g}_{\zeta}(\mathbf{p})] = \frac{\Psi}{\Psi \circ \mathbf{p}} \quad \text{for } \mathbf{p} \in \mathbf{B}. \tag{9.3}$$

Since $\mathfrak{g}_{\zeta}(\mathbf{p})$ is independent of x, differentiating both sides of (9.3) with respect to x we get $|\partial\pi_1\mathbf{p}| \cdot |\partial\Psi \circ \mathbf{p}| = |\partial\Psi|$ for any $\mathbf{p}$. As $\|\partial\pi_1\mathbf{p}\|_0$ can be arbitrarily small, we get $\partial\Psi \equiv 0$, i.e., Ψ is a function only on $\langle\Gamma\rangle$. This implies $\Psi = \exp[i\beta]$ for a $\beta \in (\mathbb{R}/2\pi\mathbb{Z})^{\langle\Gamma\rangle}$. From Remark 9.4, we conclude that ζ is a coboundary over $\mathbb{R}/2\pi\mathbb{Z}$.

Conversely, assume (9.3). Then it can be easily seen that $\Phi_{1,\mathbf{0}}\Psi$ is an eigenfunction for $\mathcal{H}_{1,i\zeta}$, which finishes the proof. ∎

Remark 9.6. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Then $\psi : \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ is just a variable $\psi = w$ and $\zeta = \tau \in \mathbb{R}$. Hence, ζ is a coboundary over $\mathbb{R}/2\pi\mathbb{Z}$ if and only if τ is zero in $\mathbb{R}/2\pi\mathbb{Z}$ if and only if τ is an integral multiple of 2π. This is the result of [3, Proposition 0] with $L = 1$.

10. Proof of Proposition 3.2

Combining all the previous results, we now prove Proposition 3.2. The proofs are quite similar to those of Lemmas 8 and 9 of Baladi–Vallée [3]. For $\mathbf{w} \in \mathbb{C}^d$, set

$$\mathcal{H}_{s,\mathbf{w}} := \mathcal{H}_{s,\mathbf{w}\cdot\boldsymbol{\psi}} \quad \text{and} \quad \lambda(s, \mathbf{w}) := \lambda_{s,\mathbf{w}\cdot\boldsymbol{\psi}}.$$

For each statement, we specify regions W_1, W_2, and W_3 around $\mathbf{0}$ and in the end take the intersection to get the desired W.

10.1. Statement (1)

We first handle the case $\mathbf{v} = \mathbf{0}$. By Theorem 6.10, it is enough to discuss the behavior of $(\mathcal{I} - \mathcal{H}_{s,\mathbf{w}})^{-1}$ since all the interval and auxiliary operators are analytic by Proposition 6.7; and so are the final operators. To obtain statements (a) and (b), following the proof of [3, Lemma 8], we split the region into three pieces I, II, and III according to $t = \Im s$.

(I) When $|t|$ is small: Let $|\mathbf{w}_0| = 1$ be fixed. As discussed by Baladi-Vallée [3], Kato [20], and Sarig [36], when (s, w) is subjected to a small perturbation near $(1, 0) \in \mathbb{C}^2$, one can show that the operators $\mathcal{H}_{s,w\mathbf{w}_0}$ and (6.2)–(6.4) with $\psi = w\mathbf{w}_0 \cdot \boldsymbol{\psi}$ are all analytic. Furthermore, the properties of the spectral gap, uniqueness, and simplicity of the eigenvalue in Propositions 7.5 and 7.6 extend to a complex parameter family $\mathcal{H}_{s,w\mathbf{w}_0}$. As $\mathbf{w}_0$ is arbitrary, a standard argument in the theory of several complex variables ensures that all those operators are analytic for the general variable $\mathbf{w} \in \mathbb{C}^d$ instead of $w\mathbf{w}_0$. Similarly we also have the following result.

Proposition 10.1. *There exists a complex neighborhood U of $(1, \mathbf{0})$ such that for all $(s, \mathbf{w}) \in U$, the operator $\mathcal{H}_{s,\psi}$ has a spectral gap with the decomposition $\mathcal{H}_{s,\mathbf{w}} = \lambda_{s,\mathbf{w}}\mathcal{P}_{s,\mathbf{w}} + \mathcal{N}_{s,\mathbf{w}}$, where $\lambda_{s,\mathbf{w}}$, $\mathcal{P}_{s,\mathbf{w}}$, $\mathcal{N}_{s,\mathbf{w}}$ are analytic on U and $R(\mathcal{N}_{s,\mathbf{w}}) < |\lambda_{s,\mathbf{w}}|$. Further, the corresponding eigenfunction $\Phi_{s,\mathbf{w}}$ and its derivative $\partial\Phi_{s,\mathbf{w}}$ are well-defined and analytic on U.*

Note that $\frac{\partial}{\partial s}\lambda_{s,\mathbf{0}}\big|_{s=1} \neq 0$ by Proposition 9.1. By the implicit function theorem, we have an analytic map s from a neighborhood W_1 of $\mathbf{0}$ to $\mathbb{C}$ such that for some $\delta_1 > 0$ and $t_0 > 0$, $\lambda(s(\mathbf{w}), \mathbf{w}) = 1$ with $|\Re s(\mathbf{w}) - 1| \leq \delta_1$ and $|\Im s(\mathbf{w})| < t_0$ for all $\mathbf{w} \in W_1$. Obviously, $s(\mathbf{0}) = 1$.

(II) When $t_0 \leq |t| \leq 1/\rho^2$: As in [3, Lemma 8], one can conclude with the help of Proposition 9.5 that there exists $\delta_2 > 0$ and a neighborhood W_2 of $\mathbf{0}$ such that the distance between 1 and the spectrum of $\mathcal{H}_{s,\mathbf{w}}$ is positive in the region $|\Re s - 1| \leq \delta_2$ for all $\mathbf{w} \in W_2$. Hence, $(\mathcal{I} - \mathcal{H}_{s,\mathbf{w}})^{-1}$ is analytic and bounded on that region.

(III) When $|t| \geq 1/\rho^2$: Using Theorem 8.5, we can find $\delta_3 > 0$ and a neighborhood W_3 of $\mathbf{0}$ such that $(\mathcal{I} - \mathcal{H}_{s,\mathbf{w}})^{-1}$ is analytic on the region $|\Re s - 1| \leq \delta_3$ with $|\Im s| \geq 1/\rho^2$ for all $\mathbf{w} \in W_3$.

Now take α_1 to be the minimum of δ_1, δ_2, and δ_3. For any $0 < \widehat{\alpha}_1 < \alpha_1$, choose a neighborhood W of $\mathbf{0}$ small enough that $W \subseteq W_1 \cap W_2 \cap W_3$ and $\Re s(\mathbf{w}) > 1 - (\alpha_1 - \widehat{\alpha}_1)$ for all $w \in W$.

To obtain (c), fix $\mathbf{w}_0 \neq \mathbf{0}$ and set $s(w) := s(w\mathbf{w}_0)$ and $\psi_0 = \mathbf{w}_0 \cdot \boldsymbol{\psi}$. Also set $\mathcal{H}_{s,w} := \mathcal{H}_{s,w\psi_0}$ and $\lambda(s, w) := \lambda_{s,w\psi_0}$ for $w \in \mathbb{C}$.

Since $\lambda(s(w), w) = 1$ for small $|w|$, we have

$$s'(0) = -\frac{\partial\lambda}{\partial w}(1,0)\Big/\frac{\partial\lambda}{\partial s}(1,0). \tag{10.1}$$

We also note that

$$\frac{\partial\lambda}{\partial s}(1,0)s''(0) = \frac{d^2}{dw^2}\lambda(1 + s'(0)w, w)\bigg|_{w=0}. \tag{10.2}$$

From Propositions 9.1 and 9.3, we find that ψ_0 is not a $\mathfrak{g}$-coboundary over $\mathbb{R}$ if and only if $s''(0) \neq 0$. Since $\mathbf{w}_0$ is arbitrary, we conclude that the Hessian of $s(\mathbf{w})$ at $\mathbf{w} = \mathbf{0}$ is non-singular if and only if the ψ_i are linearly independent over $\mathbb{R}$ modulo $\mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R})$.

Let us consider statement (d). Let $\mathcal{R}_{\mathbf{w}}$ be the residue operator of the quasi-inverse at $s = s(\mathbf{w})$. Since $\mathcal{R}_{\mathbf{w}}$ is the residue operator of $(1 - \lambda_{s,\mathbf{w}})^{-1} \mathcal{P}_{s,\mathbf{w}}$ at $s = s(\mathbf{w})$, by Theorem 6.10 the residue of the Dirichlet series is

$$\mathcal{E}^{J}_{s,\mathbf{w}} \mathcal{R}_{\mathbf{w}} \mathcal{G}_{s,\mathbf{w}} \Psi(0, \Gamma) = -\frac{\mathcal{E}^{J}_{s,\mathbf{w}} \Phi_{s,\mathbf{w}}(0,\Gamma)}{\frac{\partial}{\partial s} \lambda_{s,\mathbf{w}}} \left(\int_X \mathcal{G}_{s,\mathbf{w}} \Psi \, d\mu_{s,\mathbf{w}} \right) \tag{10.3}$$

where $\mathcal{E}^{J}_{s,\mathbf{w}} := \widehat{D}^{J}_{s,\mathbf{w}}$ or $\widetilde{D}^{J}_{s,\mathbf{w}}$ and $\mathcal{G}_{s,\mathbf{w}} = \widehat{\mathcal{F}}_{s,\mathbf{w}}$ or $\widetilde{\mathcal{F}}_{s,\mathbf{w}}$ according as $\mathfrak{g} = \mathfrak{b}$ or $\mathfrak{c}$. Let $\mathbf{w} = \mathbf{0}$. To evaluate the integral in (10.3) at $\mathbf{w} = \mathbf{0}$, observe first that $\int_X \Psi \Phi^{-1}_{1,\mathbf{0}} \, d\mu_{1,\mathbf{0}} = \int_X \Psi \, dm$ since $\mu_{1,\mathbf{0}} = dm$. For $\mathfrak{g} = \mathfrak{c}$, note also that $\int_X \widetilde{\mathcal{F}}_{1,\mathbf{0}} \Psi \, d\mu_{1,\mathbf{0}} = 2 \int_X \widehat{\mathcal{F}}_{1,\mathbf{0}} \Psi \, d\mu_{1,\mathbf{0}}$. Hence the integral in (10.3) equals

$$\frac{1}{2\rho \log 2} \sum_{m \geq 2} \int_X \frac{1}{(m+x)^2} \Psi\left(\frac{1}{m+x}, v \right) dx \, dv = \frac{1}{2\rho \log 2} \int_{(0,1/2) \times \Gamma \backslash \mathrm{SL}_2(\mathbb{Z})} \Psi \, dm.$$

Note also that for an open fundamental interval $K = K^{\circ}(m_1, \ldots, m_\ell)$, we obtain

$$\mathcal{E}^{K}_{1,\mathbf{0}} \Phi_{1,\mathbf{0}}(0, \Gamma) = \frac{[0; 1, m_\ell, \ldots, m_1]}{Q([0; m_1, \ldots, m_\ell])^2 \log 2} = \frac{|K|}{\log 2}.$$

Hence we also obtain the same expression for an interval J. In total, we obtain the desired expression for the residue.

Now consider a general $\mathbf{v}$. Let $\mathbf{w}$ be written as $\mathbf{u} + i\mathbf{v}$ for $\mathbf{u}$ in W, a neighborhood of $\mathbf{0}$. With Remark 9.4, one can easily show that $\mathcal{H}_{s,\mathbf{w}} \Psi = e^{i\beta} \mathcal{H}_{s,\mathbf{u}}[e^{-i\beta} \Psi]$ for all Ψ, and the same expressions for the operators $\mathcal{E}^{J}_{s,\mathbf{w}}$ and $\mathcal{G}_{s,\mathbf{w}}$. Therefore, $L_{\Psi,J}(s, \mathbf{w}) = e^{i\beta(\Gamma)} L_{e^{-i\beta}\Psi,J}(s, \mathbf{u})$. Hence, all the necessary properties of $L_{\Psi,J}(s, \mathbf{w})$ follow from those of $L_{e^{-i\beta}\Psi,J}(s, \mathbf{u})$. This concludes the proof of statement (1).

10.2. Statement (2)

For a given $\mathbf{v} \neq \mathbf{0}$, choose a neighborhood W_1 of $i\mathbf{v}$ small enough that $\mathbf{w} \cdot \boldsymbol{\psi}$ is not a $\mathfrak{g}$-coboundary over $\mathbb{R}/2\pi\mathbb{Z}$ for all $\mathbf{w} \in W_1$. Then, using Proposition 9.5, the proof goes exactly as the proof for (a) and (b).

10.3. Statement (3)

We split the region into three pieces I, II, and III as before. In region I, whether $\mathbf{v} \cdot \boldsymbol{\psi} \in \mathcal{B}_{\mathfrak{g}}(\Gamma, \mathbb{R})$ or not, i.e., whether the series is meromorphic or not, $L_{\Psi,J}(s, \mathbf{w})$ is bounded on $\Re s = 1 \pm \alpha_1$. In region II, the series is bounded as it is analytic. In region III, we apply the Dolgopyat–Baladi–Vallée bound.

This finishes the proof of Proposition 3.2. ∎

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