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Corrigendum to “The Klein–Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps”

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Abstract. We fix an erroneous step in the earlier work [J. Eur. Math. Soc. **22**, 1703–1757 (2020)] on solutions to the $(1 + 1)$ -dimensional Klein–Gordon equation having Fourier transforms supported on one branch of the hyperbola associated with the Klein–Gordon equation.

Keywords: Fourier uniqueness, transfer operator, completeness, Klein–Gordon equation.

1. Background

In [3], uniqueness problems for solutions to the Klein–Gordon equation in one temporal and one spatial dimensions were studied. Follow-up work in [4, 5] further sharpened the results and methods, which were based on a certain reduction to ergodic theory. A further direction opened up recently with the interpolation methods of [1], which connect with the Fourier interpolation methods employed in, e.g., [6, 7] (cf. [2]).

2. The issue

It was pointed out to me a couple of years ago by Rajesh Srivastava that the proof of Theorem 1.6.1 in [4] contains an error. Here we fix that error.

2.1. Fixing the erroneous equivalence

We begin with the following assertion.

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Lemma 2.1. *Suppose that $g \in L^1(\mathbb{R}_+)$, that is, $g \in L^1(\mathbb{R})$ and $g(t) = 0$ for a.e. $t \in \mathbb{R}_-$. Then we have the equivalence*

$$\int_{\mathbb{R}_+} e^{i2\pi mt} g(t) dt = 0 \quad \forall m \in \mathbb{Z} \Leftrightarrow \sum_{j=0}^{+\infty} g(t+j) = 0 \quad \text{a.e. on } [0, 1].$$

Proof. The calculation

$$\begin{aligned} \int_{\mathbb{R}_+} e^{i2\pi mt} g(t) dt &= \sum_{j=0}^{+\infty} \int_{[j, j+1]} e^{i2\pi mt} g(t) dt = \sum_{j=0}^{+\infty} \int_{[0, 1]} e^{i2\pi mt} g(t+j) dt \\ &= \int_{[0, 1]} e^{i2\pi mt} \sum_{j=0}^{+\infty} g(t+j) dt \end{aligned}$$

combined with the uniqueness theorem for Fourier series gives the claimed equivalence. $\blacksquare$

We remark that the assertion of Lemma 2.1 replaces equation (6.1.5) in [4] which is erroneous, as it claims that the sum in the lemma should vanish on all of $\mathbb{R}_+$, which need not be true.

We proceed to indicate how the proof of Theorem 1.6.1 in [4] runs with the indicated change. It is given that $f \in L^1(\mathbb{R}_+)$, and that the relations

$$\int_{\mathbb{R}_+} e^{i2\pi mt} f(t) dt = \int_{\mathbb{R}_+} e^{i2\pi \gamma n/t} f(t) dt = 0, \quad m, n \in \mathbb{Z}, \quad (2.1)$$

hold for a given real parameter γ with $0 < \gamma \leq 1$. We may apply Lemma 2.1 to the vanishing of the left-hand side expression in (2.1) with $g = f$ to conclude that

$$\sum_{j=0}^{+\infty} f(t+j) = 0 \quad \text{a.e. on } [0, 1],$$

which is the same as the relation

$$f(t) = - \sum_{j=1}^{+\infty} f(t+j) \quad \text{a.e. on } [0, 1]. \quad (2.2)$$

As for the vanishing of the second integral in (2.1), we apply a change of variables in the integral, which gives that

$$\int_{\mathbb{R}_+} e^{i2\pi \gamma n/t} f(t) dt = \int_{\mathbb{R}_+} e^{i2\pi nt} f\left(\frac{\gamma}{t}\right) \frac{dt}{t^2} = 0, \quad m, n \in \mathbb{Z}. \quad (2.3)$$

Next, we apply Lemma 2.1 with $g \in L^1(\mathbb{R}_+)$ given by $g(t) = t^{-2} f(\gamma/t)$, and find an equivalent formulation for the vanishing of the second expression in (2.3):

$$\sum_{j=0}^{+\infty} \frac{1}{(t+j)^2} f\left(\frac{\gamma}{t+j}\right) = 0 \quad \text{a.e. on } [0, 1]. \quad (2.4)$$

We single out the first term in the sum, and rewrite (2.4) further,

$$\frac{1}{t^2} f\left(\frac{\gamma}{t}\right) = - \sum_{j=1}^{+\infty} \frac{1}{(t+j)^2} f\left(\frac{\gamma}{t+j}\right), \quad (2.5)$$

a.e. on $[0, 1]$. After the change of variables $t \mapsto \gamma/t$, (2.5) becomes

$$f(t) = - \sum_{j=1}^{+\infty} \frac{\gamma^2}{(\gamma+jt)^2} f\left(\frac{\gamma t}{\gamma+jt}\right), \quad (2.6)$$

now a.e. on $[\gamma, +\infty[$. By combining the equality in (2.2) on $[0, 1]$ with the equality in (2.6) on $[\gamma, +\infty[$, using that if $j \geq 1$ and $t \in [0, 1]$, we get $t+j \geq 1 \geq \gamma$, and that

$$f(t) = \sum_{j,l=1}^{+\infty} \frac{\gamma^2}{[\gamma+l(j+t)]^2} f\left(\frac{\gamma(t+j)}{\gamma+l(t+j)}\right) \quad \text{a.e. on } [0, 1]. \quad (2.7)$$

Finally, since $I_1^+ =]0, 1[$, condition (2.7) amounts to having

$$f = \mathbf{S}_\gamma^2 f \quad \text{a.e. on } I_1^+,$$

where $\mathbf{S}_\gamma$ is the subtransfer operator used in [4]. The rest of the proof of Theorem 1.6.1 remains unchanged.

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References

- [1] Bakan, A., Hedenmalm, H., Montes-Rodríguez, A.: Hyperbolic Fourier series and the Klein–Gordon equation. arXiv:2401.06871v2 (2024)
- [2] Bakan, A., Hedenmalm, H., Montes-Rodríguez, A., Radchenko, D., Viazovska, M.: [Fourier uniqueness in even dimensions](#). Proc. Natl. Acad. Sci. USA **118**, article no. 2023227118 (2021) Zbl [1556.42039](#) MR [4294062](#)
- [3] Hedenmalm, H., Montes-Rodríguez, A.: [Heisenberg uniqueness pairs and the Klein–Gordon equation](#). Ann. of Math. (2) **173**, 1507–1527 (2011) Zbl [1227.42002](#) MR [2800719](#)
- [4] Hedenmalm, H., Montes-Rodríguez, A.: [The Klein–Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps](#). J. Eur. Math. Soc. (JEMS) **22**, 1703–1757 (2020) Zbl [1437.42010](#) MR [4092897](#)
- [5] Hedenmalm, H., Montes-Rodríguez, A.: [The Klein–Gordon equation, the Hilbert transform and Gauss-type maps: \$H^\infty\$ approximation](#). J. Anal. Math. **144**, 119–190 (2021) Zbl [1481.42010](#) MR [4361892](#)
- [6] Radchenko, D., Viazovska, M.: [Fourier interpolation on the real line](#). Publ. Math. Inst. Hautes Études Sci. **129**, 51–81 (2019) Zbl [1455.11075](#) MR [3949027](#)
- [7] Viazovska, M. S.: [The sphere packing problem in dimension 8](#). Ann. of Math. (2) **185**, 991–1015 (2017) Zbl [1373.52025](#) MR [3664816](#)