Ann. Inst. H. Poincaré D
Comb. Phys. Interact. 12 (2025), 445–461
DOI 10.4171/AIHPD/192

Faces of cosmological polytopes

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Abstract. A cosmological polytope is a lattice polytope introduced by Arkani-Hamed, Benincasa, and Postnikov in their study of the wavefunction of the universe in a class of cosmological models. More concretely, they construct a cosmological polytope for any Feynman diagram, i.e., an undirected graph. In this paper, we initiate a combinatorial study of these polytopes. We give a complete description of their faces, identify minimal faces that are not simplices and compute the number of faces in specific instances. In particular, we give a recursive description of the f-vector of cosmological polytopes of trees.

1. Introduction

Arkani-Hamed, Benincasa, and Postnikov defined a cosmological polytope P_G for every undirected graph $G = (V, E)$, that is $V = \{v_1, \ldots, v_k\}$ is a finite set of *vertices* and $E = \{e_1, \ldots, e_n\}$ a finite set of *edges* with $e_i = \{v_{j_1}, v_{j_2}\}$ for some $1 \leq j_1, j_2 \leq k$. Throughout this article, we work in the space $\mathbb{R}^{|V|+|E|}$ with standard basis vectors $\mathbf{x}_{v_i}$, $\mathbf{y}_{e_j}$ for $1 \leq i \leq k$ and $1 \leq j \leq n$.

Definition 1.1 ([2]). The cosmological polytope P_G associated with a graph $G = (V, E)$ is the convex hull of the following $3|E| + |V|$ vertices:

$$P_G = \operatorname{conv}\Big(\bigcup_{e=\{v,w\}\in E} \{\mathbf{y}_e + \mathbf{x}_v - \mathbf{x}_w, \mathbf{y}_e - \mathbf{x}_v + \mathbf{x}_w, -\mathbf{y}_e + \mathbf{x}_v + \mathbf{x}_w\} \cup \bigcup_{v\in V} \mathbf{x}_v\Big).$$

For an edge $e = \{v, w\}$, we will denote the above points in $\mathbb{R}^{n+k}$ by

$$p_e = -\mathbf{y}_e + \mathbf{x}_v + \mathbf{x}_w, \quad p_{e,w} = \mathbf{y}_e - \mathbf{x}_v + \mathbf{x}_w, \quad p_{e,v} = \mathbf{y}_e + \mathbf{x}_v - \mathbf{x}_w.$$

Remark 1.2. Definition 1.1 is slightly different from the standard definition of cosmological polytopes in [2], where a cosmological polytope is defined as a convex hull of vertices p_e, $p_{e,v}$, $p_{e,w}$ only. The two definitions coincide in the case of graphs with no isolated vertices since $\mathbf{x}_v \in \operatorname{conv}(p_e, p_{e,v}, p_{e,w})$ for any edge $e = \{v, w\}$

Mathematics Subject Classification 2020: 52B05 (primary); 52B12, 83F05 (secondary).
Keywords: lattice polytopes, positive geometry, wavefunction of the universe.

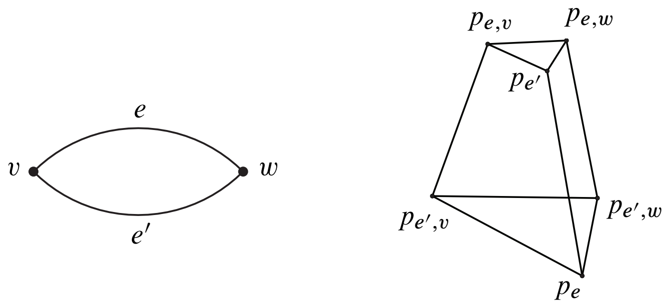

Figure 1. The graph on the left corresponds to the cosmological polytope on the right.

of G. However, Definition 1.1 works better for graphs with isolated vertices, which might appear in the recursion in Section 4.

Example 1.3. The cosmological polytope of a graph consisting of two parallel edges e, e' between two vertices v, w is a prism over a triangle which is depicted in Figure 1.

1.1. Physical perspective

In recent years, a connection between the physics of scattering amplitudes and a class of mathematical objects called *positive geometries* was discovered [1]. Positive geometries can be thought of as a vast generalization of convex polytopes, they encompass objects such as polytopes, the positive Grassmannian, and tree and loop amplituhedra [4, 5]. The connection of positive geometries to physics is usually via a top-dimensional form uniquely determined by the condition that it has logarithmic singularities (only) along all boundary components of a positive geometry. Thus, computing the canonical form is the central goal in the studies of positive geometries.

There are two standard ways to compute the canonical form of a positive geometry. The first method is to find a subdivision of a positive geometry X into simpler positive geometries $Y_1, \ldots, Y_k$. In this case, the canonical form $\boldsymbol{\Omega}_X$ of X is given as the sum

$$\boldsymbol{\Omega}_X = \boldsymbol{\Omega}_{Y_1} + \cdots + \boldsymbol{\Omega}_{Y_k}.$$

This strategy provides a direct way to get an expression for the canonical form and was successfully applied in a number of situations [8, 11, 14, 17, 18, 20].

The disadvantage of this method is that one does not obtain a closed formula for the canonical form. Thus, it is sometimes more convenient to compute the canonical form directly from its definition. More concretely, let $D_1, \ldots, D_r$ be the boundary components of a positive geometry X defined by polynomials $f_1, \ldots, f_r$, respectively. Then, the condition on the singularities of the canonical form guarantees that it can be written as

$$\boldsymbol{\Omega}_X = \frac{g}{f_1 \cdot \cdots \cdot f_r}\omega,$$

where g is some polynomial and ω is a regular form on X. Thus, the problem of computing $\mathbf{\Omega}_X$ boils down to the computation of the numerator polynomial g. Moreover, the polynomial g is determined by the condition that it should cancel on the poles of $1/(f_1 \cdot \dots \cdot f_r)$ outside of X, i.e., g should vanish along the intersections of the D_i's outside of X [3]. This observation leads to explicit formulas for the canonical form. One particular example is [16], where the numerator of the canonical form of a plane positive geometry was identified as the adjoint curve to the boundary.

In [2], it was noticed that the connection between physics and positive geometries extends further to cosmology. More concretely, the cosmological polytope is constructed in [2] as the positive geometric counterpart to the physics of cosmological time evolution and the wavefunction of the universe. This motivates the study of subdivisions and the face structure of cosmological polytopes as their facets are the components $D_1, \dots, D_r$ that are relevant to the computation of $\mathbf{\Omega}_X$ as discussed above.

1.2. Combinatorial perspective

There are several constructions of polytopes arising from graphs. The most relevant to cosmological polytopes are *symmetric edge polytopes* which recently gained considerable attention [9, 10, 15]. In particular, the symmetric edge polytope is the image of a linear projection of a facet of the cosmological polytope (the scattering facet) of the same graph. Moreover, this projection sends the vertices of the scattering facet to the vertices of the symmetric edge polytope. Thus, the information on the faces of cosmological polytopes can be used to study coherent subdivisions of symmetric edge polytopes.

1.3. Our contribution

In this paper, we start a comprehensive study of the faces of cosmological polytopes. Concretely, we give a criterion for a subset of vertices of P_G to form a face in Section 3.1. This criterion can be checked easily by considering basic properties of the graph G. As explained above, knowledge of the faces of P_G is relevant to determine both the numerator and denumerator of the canonical form of the polytope.

Subsequently, in Section 3.2, we describe two special families of faces of P_G, corresponding to the vertices and cycles of G, respectively. Our general face criterion yields that the faces of these two families are exactly the minimal faces in P_G that are not simplices.

For the special case of a tree T, we present a recursive way to compute the f-vector of the cosmological polytope P_T via the f-vectors of smaller trees in Section 4. Such a recursive relation is based on the geometric realization of the cosmological polytope $P_{G'}$ as a pyramid over a bipyramid over a cosmological polytope P_G if G' obtained from G by adding a leaf. As a byproduct this yields that the normalized volume of the cosmological polytope of any tree with e edges is 4^e. This geometric construction also lies behind the recursive formulae for the wavefunction of the universe obtained in [2] via the frequency representation of the propagators. For example, for a path graph Π_n on n nodes, we obtain the following recursion for the f-polynomial $f_{\Pi_n}(t)$ of the polytope P_{Π_n}:

$$f_{\Pi_{n+2}}(t) = (1+t)((1+2t)f_{\Pi_{n+1}}(t) - t^2(1+t)f_{\Pi_n}(t)).$$

We close in Section 5 by applying our methods to counting specific classes of faces of cosmological polytopes. Theorem 5.1 gives exact formulae for the number of 1- and 2-dimensional faces of cosmological polytopes. Subsequently, we count simplex faces of cosmological polytopes of graphs with one cycle in Section 5.2. For example, the total number of simplex faces of the cycle graph on n nodes is $5^n - 2^{n+1}$.

2. Preliminaries

In this section, we recall standard definitions and previous results on cosmological polytopes. We refer to [21] for an in-depth introduction to polytopes and to [2,6] for a detailed introduction to cosmological polytopes.

A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. A *face* $F \subseteq P$ is the set of points in the polytope P that maximizes a linear functional $\phi : \mathbb{R}^d \to \mathbb{R}$. The dimension of a face is the dimension of the affine space spanned by its points. Faces of dimension $\dim(P) - 1$ are called *facets*. Each polytope has finitely many faces and their numbers are counted in the f*-vector* $f(P)$ of P which is $f(P) = (f_{-1}, f_0, \ldots, f_{\dim(P)})$ where f_i is the number i-dimensional faces of P, and we set $f_{-1} = f_{\dim(P)} = 1$.

Our analysis of the facial structure of cosmological polytopes relies on the following characterization of the facets of a cosmological polytope proved by Arkani-Hamed, Benincasa, and Postnikov.

Theorem 2.1 ([2]). *Facets of P_G are in bijection with connected subgraphs $H = (V_H, E_H)$ of G. Under this bijection, a subgraph H corresponds to the facet F_H with all vertices of P_G except*

- *p_e for an edge $e \in E_H$;*
- *$p_{e,v}$ for an edge $e = \{v, w\}$ of G with $v \in V_H$ and $e \notin E_H$.*

So, in particular, if an edge $e = \{v, w\}$ is not in E_H but both v and w are in V_H both vertices $p_{e,v}$ and $p_{e,w}$ are not part of the facet F_H.

Moreover, the facet F_H is the intersection of P_G with the following hyperplane:

$$\sum_{v \in V_H} x_v + \sum_{\substack{e=\{v,w\},\\ v \in V_H, w \notin V_H}} y_e + \sum_{\substack{e=\{v,w\} \notin E_H,\\ v \in V_H, w \in V_H}} 2y_e = 0.$$

The facet F_G associated with the entire graph is called the *scattering facet*.

3. Face structure of cosmological polytopes

3.1. General faces

We start by giving a criterion that characterizes the faces of cosmological polytopes.

Theorem 3.1. *Let $G = (V, E)$ be an undirected graph. A set of vertices $X \subseteq V(P_G)$ defines a face of P_G if and only if X satisfies both of the following two conditions.*

(i) *If for a node $v \in V$ the set X contains the vertices p_e and $p_{e,v}$ for an edge $e = \{v, w\} \in E$ then X contains the vertices $p_{e'}$ and $p_{e',v}$ for all edges $e' = \{v, w'\} \in E$.*

(ii) *If X contains a subset $\{p_{e_1,v_1}, p_{e_2,v_2} \dots, p_{e_k,v_k}\}$ for a cycle*

$$e_1 = \{v_1, v_2\}, \dots, e_k = \{v_k, v_1\}$$

in G, then X also contains the subset $\{p_{e_1,v_2}, p_{e_2,v_3}, \dots, p_{e_k,v_1}\}$.

Proof. First, we show that every set of vertices X satisfying the conditions (i) and (ii) defines a face of P_G. We do this by proving that X is an intersection of facets of P_G. Let y be a vertex of P_G with $y \notin X$. It suffices to find a facet F_H with $X \subseteq F_H$ and $y \notin F_H$. There are two cases: $y = p_e$ or $y = p_{e,v}$ for some $e \in E$ and $v \in V$.

Case 1. Suppose $y = p_e$. Let us define $H = (V_H, E_H)$ to be the connected component of the edge induced subgraph $\{e \mid p_e \notin X\}$ containing the edge e. Then, the facet F_H contains all vertices of P_G except the vertices $p_{e'}$ with $e' \in E_H$ and the vertices $p_{e'',v''}$ with $e'' = \{v'', w''\} \notin E_H$ and $v'' \in V_H$. Since $e \in H$, the facet F_H does not contain y. Moreover, by construction, F_H contains all vertices in X of the form $p_{e'}$ for some $e' \in E$.

Secondly, consider a vertex $p_{e'',v''} \in X$ for some $e'' = \{v'', w''\}$. Assume for a contradiction that $p_{e'',v''} \notin F_H$. Thus, by Theorem 2.1 this means that $v'' \in V_H$, but $e'' \notin E_H$ and thus $p_{e''} \in X$. Since H is connected, there exists an edge $f \in E_H$ that is adjacent to v''. By condition (i), the vertex $p_f \in X$ which contradicts the construction of H.

Case 2. Let $y = p_{e,v}$ with $e = \{v, w\}$. We define a connected subgraph H inductively. Let $H_0 = v$ be the vertex v itself. The subgraph H_{i+1} is defined from H_i in the following way:

$$H_{i+1} := H_i \cup \left\{e' = \{v', w'\} \mid v' \in H_i, e' \notin H_i \text{ and } p_{e',v'} \in X\right\}.$$

Since $H_i \subseteq H_{i+1}$ and G is a finite graph, the sequence $(H_i)_{i\in\mathbb{N}}$ stabilizes. We define H to be the limit of the sequence $(H_i)_{i\in\mathbb{N}}$, by construction H is connected. We need to show the following three cases.

(1) Let $X_1 := \{p_{e'} \mid p_{e'} \in X\}$. We need to show $X_1 \subseteq F_H$. We inductively show $X_1 \subseteq F_{H_i}$ for all $i \geq 0$ which implies the claim. The case of F_{H_0} is trivial as this facet contains all vertices of the form $p_{e'}$ in P_G. Next, we show $X_1 \subseteq F_{H_1}$. Consider the edge $e' = \{v, w'\} \in H_1$ and assume that $p_{e'} \in X$. By construction of H_1, we have $p_{e',v} \in X$; hence, by condition (i), the vertex $y = p_{e,v}$ must also be in X which contradicts the assumption that $y \notin X$. Hence, $p_{e'} \notin X$ and $X_1 \subseteq F_{H_1}$.
So now assume that $X_1 \subseteq F_{H_i}$ for some $i \geq 1$. Consider again an edge $e' = \{v', w'\} \in H_{i+1} \setminus H_i$ with $v' \in H_i$. Then, there exists an edge $e'' = \{v', w''\} \in H_i$ since H_i is connected and $i \geq 1$. By induction, this implies that $p_{e''} \notin X$ and by construction of H_{i+1} the vertex $p_{e',v'} \in X$. Hence, $p_{e'} \notin X$ by condition (i) and thus $X_1 \subseteq F_{H_{i+1}}$.

(2) Let $X_2 := X \setminus X_1$. Secondly, we show that $X_2 \subseteq F_H$. This follows from the construction of H as if there is a vertex $p_{e',v'} \in X \setminus F_H$ we would add the edge e' to H which contradicts the definition of H.

(3) Lastly, we need to show $y \notin F_H$. Assume that $y = p_{e,v} \in F_H$. This means that $e \in H$. Therefore, there exists a cycle in H of the form $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, …, $e_r = \{v_r, v_1\}$ with $e = e_r$, $v = v_1$, and $e_i \in H_i \setminus H_{i-1}$. Thus, $p_{e_i,v_i} \in X$ for all $i = 1, \ldots, r$ by construction of H_i. By condition (ii) this implies that $p_{e_i,v_{i+1}} \in X$ for all $i \in \mathbb{Z}/r\mathbb{Z}$. In particular, $y = p_{e_r,v_1} \in X$ which contradicts the assumption on y. Therefore, $y \notin F_H$.

For the converse, we need to show that every face of P_G satisfies the condition (i) and (ii). First, notice that if two subsets X and Y satisfy both conditions so does their intersection $X \cap Y$. Therefore, it suffices to show conditions (i) and (ii) for the facets of P_G.

Let $H = (V_H, E_H)$ be a connected subgraph of G and F_H its associated facet. Consider a node $v \in V$ and assume that p_e and $p_{e,v}$ are in F_H, hence E_H does not contain the edge e as well as any other edge adjacent to the node v. Hence, by definition of F_H, the vertices $p_{e'}, p_{e',v}$ are in F_H for all edges e' adjacent to the node v. So, F_H satisfies the condition (i).

For the condition (ii), let F_H contain a subset $\{p_{e_1,v_1}, p_{e_2,v_2}, \dots, p_{e_k,v_k}\}$ for a cycle $e_1 = \{v_1, v_2\}, \dots, e_k = \{v_k, v_1\}$ in G. First, assume that $\{e_1, \dots, e_k\} \subseteq E_H$. In this case, by construction of F_H it contains all the vertices $\{p_{e_1,v_2}, p_{e_2,v_3}, \dots, p_{e_k,v_1}\}$ as well, so the condition (ii) is satisfied.

In the case when $\{e_1, \dots, e_k\} \not\subseteq E_H$, any edge adjacent to the nodes $v_1, \dots, v_k$ cannot be contained in H as otherwise one of the vertices p_{e_i,v_i} would be excluded from F_H. But then, by construction of F_H vertices $\{p_{e_1,v_2}, p_{e_2,v_3}, \dots, p_{e_k,v_1}\}$ are contained in the facet F_H, so the condition (ii) is satisfied. ■

An immediate corollary of this theorem yields a complete description of the edge graph Γ_G of P_G.

Corollary 3.2. *The edge graph Γ_G of P_G is a complete graph on the vertices of P_G with the following edges removed:*

(1) $\{p_e, p_{e,v}\}$ *for any edge e and a non-leaf node v of G,*

(2) $\{p_{e,v_1}, p_{e',v_2}\}$ *for a pair of parallel edges e, e' between the nodes v_1 and v_2.*

A more general statement provides a description of all simplex faces of P_G.

Theorem 3.3. *Let $G = (V, E)$ be an undirected graph. A set of vertices $X \subseteq V(P_G)$ defines a simplex face of P_G if and only if*

(i) *the induced subgraph of the edge graph Γ_G to the vertex set X is a complete graph,*

(ii) *X does not contain a subset $\{p_{e_1,v_1}, \dots, p_{e_k,v_k}\}$ for a cycle $e_1 = \{v_1, v_2\}, \dots, e_k = \{v_k, v_1\}$ in G.*

Proof. By Corollary 3.2, if the subgraph of Γ_G induced by the vertices X is a complete graph, X must satisfy property (i) in Theorem 3.1. Hence, a subset X satisfying the properties (i) and (ii) is a face of P_G by Theorem 3.1. Moreover, the same properties (i) and (ii) are satisfied for every subset of vertices of the set X. Hence, any subset of vertices of X defines a face of P_G, so the vertices of X form a simplex face.

For the converse, if X defines a simplex face, then the induced subgraph of Γ_G to X is a complete graph and X satisfies (ii) by Theorem 3.1. ■

Remark 3.4. The face structure of cosmological polytopes was also studied in [7, 8] from a slightly different perspective. More concretely, these works study which collections of facets of P_G intersect in a face of expected codimension. The main tool in this analysis is the connection to the residues of the canonical form $\boldsymbol{\Omega}_{P_G}$.

3.2. Special faces

In this subsection, we will discuss two types of special faces appearing in cosmological polytopes which are the minimal non-simplex faces of cosmological polytopes (see Corollary 3.8). We call the ones described in Proposition 3.5 *vertex faces* and the ones described in Proposition 3.6 *cycle faces*.

Proposition 3.5. *Let $v \in V$ be a vertex of G of degree d with the adjacent edges $e_1 = \{v, w_1\}, \dots, e_d = \{v, w_d\}$. Then, the cosmological polytope P_G has a face of dimension d with the $2d$ vertices given by*

$$p_{e_1}, p_{e_1,v}, \dots, p_{e_d}, p_{e_d,v}.$$

We call this face a vertex face F_v.

Moreover, the face F_v is combinatorially equivalent to a d-dimensional cross-polytope.

Proof. The first statement follows directly from Theorem 3.1 as the set of vertices

$$\{p_{e_1}, p_{e_1,v}, \dots, p_{e_d}, p_{e_d,v}\}$$

clearly satisfies conditions (i) and (ii).

For the proof of the second statement notice that a shift of the face F_v by the vector $-\mathbf{x}_v$ is the convex hull of the points

$$\mathbf{y}_{e_1} - \mathbf{x}_{w_1}, \mathbf{x}_{w_1} - \mathbf{y}_{e_1}, \dots, \mathbf{y}_{e_d} - \mathbf{x}_{w_d}, \mathbf{x}_{w_d} - \mathbf{y}_{e_d}.$$

Since the vectors $\mathbf{y}_{e_1} - \mathbf{x}_{w_1}, \dots, \mathbf{y}_{e_d} - \mathbf{x}_{w_d}$ are linearly independent, the vertex face F_v is affinely (and in particularly combinatorially) equivalent to a d-dimensional cross-polytope. ∎

Recall that the d-dimensional *cyclic polytope* $C(n, d)$ with n vertices is the convex hull of $x(t_1), \dots, x(t_n)$ where $t_1 < t_2 < \dots < t_n$ are real numbers and $x : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, t^3, \dots, t^d)$ is a parametrization of the moment curve. It is known that cyclic polytopes are simplicial, i.e., all its proper faces are simplices (see, for example, [12]). By Gale's evenness condition (see [21, Theorem 0.7]), for a set of indices $I \subset [n]$ of size d, the corresponding set of vertices $\{x(t_i)\}_{i \in I}$ form a facet of $C(n, d)$ if and only if any two elements in $[n] \setminus I$ are separated by an even number of elements from $[n]$.

Proposition 3.6. *Let $e_1 = \{v_1, v_2\}, \dots, e_d = \{v_d, v_1\}$ be a cycle σ of length d in G with $v_i \neq v_j$ for $1 \leq i < j \leq d$ and $d > 1$. Then, the cosmological polytope P_G has a face of dimension $2d - 2$ with the $2d$ vertices:*

$$p_{e_1,v_1}, p_{e_1,v_2}, \dots, p_{e_d,v_d}, p_{e_d,v_1}.$$

We call this face a cycle face F_σ.

Moreover, the face F_σ is combinatorially equivalent to a cyclic polytope of dimension $2d - 2$ with $2d$ vertices.

Proof. The first statement follows directly from Theorem 3.1 as the set of vertices

$$X = \{p_{e_1,v_1}, p_{e_1,v_2}, \dots, p_{e_d,v_d}, p_{e_d,v_1}\}$$

clearly satisfies conditions (i) and (ii) of this theorem.

For the second statement, we will show that analogously to $C(2d, 2d-2)$, facets of F_σ are described by Gale's evenness condition which implies that the polytopes are combinatorially equivalent.

First, notice that if a subset $Y \subseteq X$ defines facet of F_σ, then $|Y| \geq 2d-2$. Moreover, by condition (ii) of Theorem 3.1, to define a proper face of F_σ, the set Y should not contain at least one of the points of type p_{e_i,v_i} and $p_{e_j,v_{j+1}}$ for $i, j \in \mathbb{Z}/d\mathbb{Z}$. Therefore,

$$|Y| = 2d-2;$$

i.e., every facet of F_σ is a simplex. Finally, notice that the condition that

$$X \setminus Y = \{p_{e_i,v_i}, p_{e_j,v_{j+1}}\}$$

for some $i, j \in \mathbb{Z}/d\mathbb{Z}$ is equivalent to Gale's evenness condition if we order the elements of X in the following way:

$$(p_{e_1,v_1}, p_{e_1,v_2}, \dots, p_{e_d,v_d}, p_{e_d,v_1}). \qquad \blacksquare$$

Remark 3.7. Note that the cycle face F_σ of P_G corresponding to a cycle σ of G coincides with the scattering facet of P_σ. More generally, for any subgraph $H \subset G$, the scattering facet of P_H appears as a face of P_G.

We can now characterize the minimal non-simplex faces of a cosmological polytope, i.e., the faces that are combinatorially a simplicial polytope but not a simplex.

Corollary 3.8. *A minimal non-simplex face F of a cosmological polytope is either a vertex face or a cycle face.*

Proof. The above propositions imply that the vertex faces are cross-polytopes and the cycle faces are cyclic polytopes which are both simplicial polytopes but not simplices.

So, assume that F is a simplicial polytope that does not contain vertices of a vertex face or a cycle face. Condition (i) in Theorem 3.1 together with the assumption that F does not contain a vertex face now implies that for every non-leaf edge $e = \{v, w\}$ the face F can contain at most one of the vertices p_e, $p_{e,v}$, and $p_{e,w}$. The assumption that F does not contain a cycle face implies that for any pair of parallel edges e, e' between the nodes v_1, v_2 the face F can only contain at most one of the vertices p_{e,v_1} and p_{e',v_2}. By Corollary 3.2 this implies that the edge graph of F is a complete graph on its vertices. Moreover, the same assertion is true for every subset of vertices of the face F which implies that F is combinatorially a simplex. ∎

4. Trees

In this section, we investigate the cosmological polytopes associated to the trees. Our main tool is the following proposition which describes how the cosmological polytope P_G changes after adding a leaf to the graph G.

Proposition 4.1. *Let G be a graph and let G' be the graph that arises from G by adding an vertex v and an edge $e = \{v, w\}$ for some vertex w of G. Then, the following statements hold.*

(i) *The cosmological polytope $P_{G'}$ has a facet F containing all vertices except of $p_{e,v}$. In particular, $P_{G'}$ is a pyramid over F with apex $p_{e,v}$.*

(ii) *The facet F is a bipyramid over P_G with apices p_e and $p_{e,w}$ with the interval between $p_e, p_{e,w}$ intersecting P_G in the interior of the vertex face F_w defined in Proposition 3.5.*

Proof. (i) By Theorem 2.1 the facet F corresponding to the subgraph $\{v\}$ in G' contains all vertices of $P_{G'}$ except $p_{e,v}$. Thus, $P_{G'}$ is a pyramid over F with apex $p_{e,v}$.

(ii) Filtered by the x_e coordinate, the vertices of F come in three layers: The layer $x_e = -1$ contains the vertex p_e, the layer $x_e = 0$ the vertices in P_G and the layer $x_e = 1$ the vertex $p_{e,w}$. The interval between p_e and $p_{e,w}$ intersects the $x_e = 0$ layer in the point $\mathbf{x}_w$ which is in the interior of the face F_w. This implies the claim. ∎

One corollary of Proposition 4.1 is the computation of the volume of the cosmological polytopes of trees.

Corollary 4.2. *Let G, G' be as before, then one has*

$$\operatorname{Vol}(P_{G'}) = 4 \operatorname{Vol}(P_G),$$

where Vol *is the normalized volume. In particular, for any tree T with e edges, the normalized volume of the cosmological polytope P_T equals 4^e.*

Proof. One can check that the facet F is a union of two pyramids of lattice height 1 over P_G which shows that $\operatorname{Vol}(F) = 2 \operatorname{Vol}(P_G)$. Moreover, the cosmological polytope $P_{G'}$ is a pyramid of lattice height 2 over F, so the normalized volume of $P_{G'}$ is computed as follows:

$$\operatorname{Vol}(P_{G'}) = 2 \operatorname{Vol}(F) = 4 \operatorname{Vol}(P_G).$$ ∎

Proposition 4.3. *For a cosmological polytope P_G and a vertex $w \in V(G)$ it holds that $f_{P_{G \setminus w}}(t)$ equals the "upper f-vector" of the vertex face F_w in P_G, that is the f-vector of faces containing F_w.*

Proof. Consider the partition of the vertices of P_G by coordinate $\mathbf{x}_w$.

Claim 1. There is a bijection ϕ of facets containing F_w in P_G and facets of $P_{G\setminus w}$. Specifically, the facets of P_G which contain F_w are determined by connected subgraphs H of G which do not contain w. Such subgraphs are in bijection with the connected subgraphs of $G \setminus w$ which in turn determines facets of $P_{G\setminus w}$. The map ϕ maps a facet of P_G given by a connected subgraph of G that avoids w to the facet of $P_{G\setminus w}$ given by the corresponding connected subgraph of $G \setminus w$.

Claim 2. This bijection extends to a bijection between the faces containing F_w in P_G and all faces of $P_{G\setminus w}$. Specifically, we consider the following map:

$$\phi : \{\sigma \supseteq F_w \mid \sigma \text{ a face of } P_G\} \to \{\tau \mid \tau \text{ a face of } P_{G\setminus w}\},$$
$$\sigma \mapsto \tau = \bigcap_{\sigma \subset F, F \text{ is a facet of } P_G} \phi(F).$$

This map respects the dimension of the faces, that is $\dim(\phi(\sigma)) = \dim(\sigma) - \dim(F_w)$.

Claim 3. This implies that the upper f-vector of F_w in P_G equals the (shifted) f-vector of $P_{G\setminus w}$. ∎

Definition 4.4. For a polytope P, with f-vector $(f_{-1}, \ldots, f_{\dim P})$ we will define its f-polynomial $f_P(t)$ to be

$$f_P(t) = \sum_{i=-1}^{\dim P} f_i t^{i+1}.$$

We can use Proposition 4.1 to get a recursive relation for the f-polynomial of cosmological polytopes of the graphs G and G'.

Theorem 4.5. *The f-polynomials of P_G and $P_{G'}$ are related in the following way:*

$$f_F(t) = (1+2t) f_{P_G}(t) - t^{\deg(w)+1}(1+t) f_{P_{G\setminus w}}(t),$$
$$f_{P_{G'}} = (1+t) f_F(t).$$

Proof. Indeed, the f-polynomial of a pyramid P with base F is given by $f_P = (1+t) f_F(t)$ as every face of F of dimension d produces two faces of P one of dimensions d and another of dimension $d+1$. The f-polynomial of a the generic bipyramid over a polytope P_G can be computed as $(1+2t) f_{P_G}(t)$. The description of faces of non-generic bipyramid follows, for example, from [19, Proposition 2.3] as it is a particular example of subdirect sum: $F = (I, I) \oplus (F_v, P_G)$, where I is an interval, and F_v is a vertex face corresponding to the node v of G. The face count involves the correction of $(1+2t) f_{P_G}(t)$ by the generating polynomial of the number of faces of P_G containing F_v. Using Proposition 4.3, we obtain

$$f_F(t) = (1+2t) f_{P_G}(t) - t^{\deg(w)+1}(1+t) f_{P_{G\setminus w}}(t),$$

which completes the proof. ∎

This gives an inductive way of computing the f-vector of cosmological polytopes of trees. In the case of paths this yields the following recursion for their f-polynomials.

Corollary 4.6. *Let Π_n be the path graph on n vertices. Then, we have the following recursion for the f-vector $f_{\Pi_n}(t)$ of the cosmological polytopes P_{Π_n}:*

$$f_{\Pi_{n+2}}(t) = (1+t)((1+2t)f_{\Pi_{n+1}}(t) - t^2(1+t)f_{\Pi_n}(t))$$

and

$$f_{\Pi_1}(t) = t + 1, \quad f_{\Pi_2}(t) = t^3 + 3t^2 + 3t + 1.$$

The number of all faces of P_{Π_n} is the evaluation of this recursion at $t = 1$ which is the sequence A154626 in the Online Encyclopedia of Integer Sequences (OEIS).

5. Counting faces

In this section, we use our general description of faces of cosmological polytopes to compute their number in certain examples. To simplify the exposition and obtain closed formulae for the face numbers, we assume that the graph G does not have loops throughout this section.

5.1. Low-dimensional faces

The main result of this subsection are formulas for the number of the edges and 2-dimensional faces of cosmological polytopes.

Theorem 5.1. *Let $G = (V, E)$ be an undirected graph where e is the number of edges, l the number of leaves, v_2 the number of vertices of degree 2, and Δ_i the number of cycles of length i in G. The characterization of faces then yields the following.*

(1) *The number of edges of the cosmological polytope P_G is*

$$f_1(P_G) = \binom{3e}{2} - 2e + l - 2\Delta_2.$$

(2) *For a simple graph G, the number of 2-dimensional faces of the cosmological polytope P_G is*

$$f_2(P_G) = 27\binom{e}{3} + 3(e+l)(e-1) + v_2 - 2\Delta_3.$$

Proof. The first part follows directly from Corollary 3.2. Indeed, the number of edges of the complete graph on $3e$ vertices is $\binom{3e}{2}$; the number of removed edges of type

$\{p_{e,v}, p_e\}$ for each non-leaf node v is $2e - l$ and the number of edges p_{e,v_1}, p_{e',v_2} for a pair of parallel edges e, e' between the nodes v_1 and v_2 is $2\Delta_2$.

The second part is deduced similarly. By Corollary 3.8 and the discussion thereafter, there are only two types of faces of dimension 2: triangles and quadrilaterals. First, let us count the number of triangular faces of P_G. For this we first count the number of complete subgraphs of size 3 of the edge graph Γ_G of P_G. Since G is simple, from the description of Corollary 3.2 it follows that the number of complete subgraphs of size 3 of Γ_G is

$$27\binom{e}{3} + 3(e+l)(e-1).$$

Now, for every cycle of length 3 in G, there are exactly 2 complete subgraphs in Γ_G which do not satisfy condition (ii) of Theorem 3.3. So, the final count of triangles in a simple graph is given by

$$27\binom{e}{3} + 3(e+l)(e-1) - 2\Delta_3.$$

On the other hand, each quadrilateral is either a vertex face of a node of degree 2, or a cycle face for a cycle of length 2. However, in a simple graph there is no cycle of length 2, hence the total number of 2-dimensional faces is given by

$$27\binom{e}{3} + 3(e+l)(e-1) - 2\Delta_3 + v_2. \qquad \blacksquare$$

It is possible to deduce a formula for the number of 2-dimensional faces of P_G for a general graph G. For this one has to take into account contributions coming from the banana subgraphs (the graph on two vertices with multiple parallel edges). In particular, one has to compute the number of 2-dimensional faces of banana graphs which we do in Example 5.2. It amounts to careful bookkeeping to deduce the general formula for the number of 2-faces of P_G from the general description of simplex faces in Theorem 3.3 and the count in Example 5.2.

Example 5.2. Let B_k be the banana graph consisting of two vertices and k parallel edges between them for $k \geq 1$. The cosmological polytopes P_{B_1} and P_{B_2} have one and five 2-dimensional faces, respectively. In general, for $k \geq 3$, we have

$$f_2(P_{B_k}) = 15\binom{k}{3} + 3\binom{k}{2}.$$

One way to prove this formula is as follows. Using, for example, `polymake` [13] one can compute that P_{B_2} has two triangles and P_{B_3} has 21 triangles. Since every triangle can involve vertices corresponding to at most three edges of B_k this immediately yields that P_{B_k} has

$$15\binom{k}{3} + 2\binom{k}{2}$$

many triangles for $k \geq 2$. By Theorem 3.1, we obtain that every quadrilateral of P_{B_k} for $k \geq 3$ is a cycle face stemming from a cycle of length two in B_k. The general formula now follows from the fact that there are $\binom{k}{2}$ many such cycles in B_k.

5.2. Simplex faces

In this subsection, we count simplex faces in particular graphs. For a polytope P we denote by $f_k^{\Delta}(P)$ the number of k-dimensional simplex faces of P.

Proposition 5.3. *Let C_n be the cycle graph on n nodes. Then, for $1 \leq k \leq 2n$ the cosmological polytope P_{C_n} has*

$$f_{k-1}^{\Delta}(P_{C_n}) = -2\binom{n}{k-n} + \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{i}\binom{n-i}{k-2i} 3^{k-2i}. \tag{1}$$

In total, P_{C_n} has $5^n - 2^{n+1}$ many simplex faces.

Proof. Say the cycle graph C_n has nodes $v_1, \dots, v_n$ and edges $e_1, \dots, e_n$, where

$$e_i = \{v_i, v_{i+1}\} \quad \text{for } i \in \mathbb{Z}/n\mathbb{Z}.$$

The edge graph $\Gamma_{P_{C_n}}$ of P_{C_n} consists of n triples of vertices p_{e_i}, p_{e_i,v_i}, and $p_{e_i,v_{i+1}}$ where within a cluster only the last two vertices are joined by an edge in $\Gamma_{P_{C_n}}$ and all pairs of vertices between different clusters are joined by an edge. Therefore,

$$\binom{n}{i}\binom{n-i}{k-2i} 3^{k-2i}$$

is the number of complete subgraphs of $\Gamma_{P_{C_n}}$ on k vertices with exactly i edges within an edge cluster as described above for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$. Hence, the sum in equation (1) is the number of all complete subgraphs of $\Gamma_{P_{C_n}}$ on k vertices.

By Theorem 3.3 we need to exclude the complete subgraphs of $\Gamma_{P_{C_n}}$ that contain either all vertices p_{e_i,v_i} or all vertices $p_{e_i,v_{i+1}}$ for $i \in \mathbb{Z}/n\mathbb{Z}$. For $n \leq k \leq 2n$ there are exactly $2\binom{n}{k-n}$ such complete subgraphs which is the term we subtract in equation (1).

For the last statement, note that when we want to count all complete subgraphs of $\Gamma_{P_{C_n}}$ we have five choices for each of the n vertex clusters: no vertex, one of the vertices p_{e_i}, p_{e_i,v_i}, and $p_{e_i,v_{i+1}}$ or both of the vertices p_{e_i,v_i} and $p_{e_i,v_{i+1}}$. We just need to exclude the choice of no vertices at all, which yields $5^n - 1$ complete subgraphs in total. We claim that for the complete subgraphs we need to exclude by the cycle condition exactly $2^{n+1} - 1$ choices. Indeed, once a complete subgraphs contains all vertices p_{e_i,v_i} for $i \in \mathbb{Z}/n\mathbb{Z}$ then there are two choices in every cluster: the complete subgraph can contain the vertex $p_{e_i,v_{i+1}}$ or not. After counting the analogous possibilities for the complete subgraphs containing all vertices $p_{e_i,v_{i+1}}$ for $i \in \mathbb{Z}/n\mathbb{Z}$, we obtain the term $2^{n+1} - 1$ we need to subtract since there is one configuration that appears in both of these versions. ■

A similar argument yields the following generalization.

Proposition 5.4. *Let $G = (V, E)$ be a graph with exactly one cycle. Say this cycle is of length d and assume $d > 2$. Let $e = |E|$ and l be the number of leaves of G. Then, the number simplex faces of P_G is*

$$6^l \cdot 5^{e-l-d} \cdot (5^d - 2^{d+1} + 1) - 1.$$

Acknowledgments. We would like to thank Bernd Sturmfels for introducing us to the topic of cosmological polytopes. Furthermore, we are very grateful to Paolo Benincasa for numerous insightful discussions about these polytopes and their physical background. Moreover, we thank Martina Juhnke and Justus Bruckamp for drawing our attention to a mistaken recitation of Theorem 2.1 in a previous version of this article. Last but not least, we would like to thank the anonymous referees for their careful reading which indeed helped to improve the exposition.

Funding. The first author is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB-TRR 358/1 2023 – 491392403.

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Communicated by Adrian Tanasă

Received 26 September 2022; revised 31 January 2024.

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