Ann. Inst. H. Poincaré D
Comb. Phys. Interact. 12 (2025), 517–544
DOI 10.4171/AIHPD/180

On the escape rate of subshifts of finite type and 2-multiplicative integer systems on $\mathbb{N}^d$

Jung-Chao Ban, Wen-Guei Hu, and Guan-Yu Lai

Abstract. In this article, we establish the escape rate formula for an $\mathbb{N}^d$ subshift of finite type X by means of the horizontal transition matrices of the strip shifts induced from X. This extends the previous result of Haritha and Agarwal (2019) to $\mathbb{N}^d$, $d \geq 1$. The concept of the escape rate for $\mathbb{N}^d$ 2-multiplicative integer systems is also introduced, and we give a similar result for the estimate of the escape rate for such systems. Finally, the continuity property for both systems is presented.

1. Introduction

The open dynamical system (or dynamical system with a hole) problem, which was initiated by Piangiani and Yorke [39], can be put into the following frame. Let (X, d) be a compact metric space and $T\colon X \to X$ be a continuous map. For $U \subset X$, we consider the set of points which under forward iteration do not enter U, i.e.,

$$X_U = \{x \in X\colon T^k(x) \notin U \text{ for all } k\}. \tag{1.1}$$

Or specifically, consider $U = B_\varepsilon(z) := \{y \in X\colon d(y,z) < \varepsilon\}$ for $z \in X$ and $\varepsilon > 0$. Let μ be a T-invariant probability measure, the *escape rate* of μ through U is defined below. Let $(X \backslash U)^n = \{x \in X\colon T^i(x) \notin U \text{ for } 0 \leq i < n\}$,

$$\rho_\mu(U, X) = -\lim_{n\to\infty} \frac{1}{n} \log \mu((X \backslash U)^n),$$

whenever the limit exists. The limit $\rho_\mu(U, X)$ is known as the escape rate of μ through U. Some interesting and significant research topics for open systems are listed below:

- Find a formula for the escape rate $\rho_\mu(U, X)$.
- Find a relationship between the escape rate $\rho_\mu(U, X)$ and the dimension (Hausdorff or Minkowski) or the entropy of the open map $(X_U, T|_{X_U})$ (cf. [3, 19, 20]).

2020 Mathematics Subject Classification. Primary 37B10; Secondary 37A35, 37B51.
Keywords. Multiplicative integer systems, escape rate, strip shift, shift entropy.

- Discuss the continuity of the map $\varepsilon \mapsto \rho_\mu(B_\varepsilon(z), X)$, for each $z \in X$.
- The dependence of the escape rate $\rho_\mu(B_\varepsilon(z), X)$ on the position of the hole, that is, the point $z \in X$ for each $\varepsilon > 0$ (cf. [1, 2, 12, 21, 23]).

No attempt has been made here to provide a comprehensive bibliography for the open system problem, and we refer the reader to [15] for a general survey. We emphasize that the limit defining $\rho_\mu(U, X)$ may not exist in general. However, the same questions can be asked with respect to the *upper and lower escape rate*; namely,

$$\overline{\rho}_\mu(U, X) = -\liminf_{n\to\infty} \frac{1}{n} \log \mu((X\backslash U)^n)$$

and

$$\underline{\rho}_\mu(U, X) = -\limsup_{n\to\infty} \frac{1}{n} \log \mu((X\backslash U)^n).$$

The purpose of this study is to investigate both problems for two systems; namely, the '$\mathbb{N}^d$ subshifts of finite type (SFTs) for $d \geq 2$' and the 'multiplicative integer systems' (MISs). The motivation behind this study and the results are presented below.

Let $\mathcal{A}$ be an alphabet with $|\mathcal{A}| = k$ and $\mathcal{F} \subseteq \mathcal{A}^* := \bigcup_{n\geq 1} \mathcal{A}^n$ is a finite collection of words from $\mathcal{A}^*$ that are forbidden. Suppose $(\Sigma_{\mathcal{F}}, \sigma)$ be an irreducible SFT on $\mathbb{N}$ and σ is the usual shift map, i.e., $\sigma((x_i)_{i=0}^\infty) = (x_{i+1})_{i=0}^\infty$. It is known that $(\Sigma_{\mathcal{F}}, \sigma)$ admits a unique Parry measure, for the case $\mathcal{F} \subseteq \mathcal{A} \cup \mathcal{A}^2$, we may assume that $A = [a_{ij}] \in \{0, 1\}^{k\times k}$ is the associated transition matrix of $(\Sigma_{\mathcal{F}}, \sigma)$ and $v = (v_0, v_1, \ldots, v_{k-1})^t$, $u = (u_0, u_1, \ldots, u_{k-1})$ are normalized right and left eigenvectors of A with respect to the largest eigenvalue λ_A such that $uv = 1$. The *Parry measure* on each admissible cylinder set of $w = i_1 \ldots i_n \in \mathcal{A}^n$ is defined as

$$\mu([w]) = \frac{u_{i_1} v_{i_n}}{\lambda_A^{n-1}} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{n-1} i_n}$$

(general cases are defined in Section 2.2). Let $\mathcal{F}_1$ be another finite collection of words from $\mathcal{A}^*$ with $\mathcal{F} \cap \mathcal{F}_1 = \emptyset$. Then the escape rate of μ through the hole $\Sigma_{\mathcal{F}} \setminus \Sigma_{\mathcal{F}\cup\mathcal{F}_1}$, denoted by $\rho_\mu(\mathcal{F}_1, \Sigma_{\mathcal{F}})$, can be characterized by the topological entropies of $h_{\text{top}}(\Sigma_{\mathcal{F}})$ and $h_{\text{top}}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})$. Precisely, Haritha and Agarwal prove that [22, Theorem 3.1]

$$\rho_\mu(\mathcal{F}_1, \Sigma_{\mathcal{F}}) = h_{\text{top}}(\Sigma_{\mathcal{F}}) - h_{\text{top}}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}), \tag{1.2}$$

where $h_{\text{top}}(X)$ stands for the topological entropy of X.

A natural question arises: is it possible to extend formula (1.2) to $\mathbb{N}^d$ SFTs? To this end, we begin by introducing some necessary concepts first. It is known that some $\mathbb{N}^d$ SFTs are the classical models in statistical physics, e.g., Ice model [28] and Hard square system [11].

A *shape* F is a finite subset of $\mathbb{N}^d$. A *pattern* f on a shape F is a function $f : F \to \mathcal{A}$. Given a finite list $\mathcal{F}$ of patterns, put

$$\Sigma_{\mathcal{F}}^{[d]} = \{x = (x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d} \in \mathcal{A}^{\mathbb{N}^d} : \sigma^{\mathbf{j}}(x)_F \notin \mathcal{F} \text{ for all } \mathbf{j} \in \mathbb{N}^d \text{ and for all shapes } F\},$$

where $(\sigma^{\mathbf{j}}(x))_{\mathbf{i}} = x_{\mathbf{i}+\mathbf{j}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbb{N}^d$, and x_F is the projection of x on the shape F. We note that the calculation of the topological entropy of $\Sigma_{\mathcal{F}}^{[d]}$ is a difficult job (cf. [8, 10, 31, 32, 35, 36]), and there is a deep connection between the exact value of the topological entropy and the pressure in statistical physics (cf. [11, 28, 34]). For $d = 2$, a systematic method for generating possible $\mathbb{N}_{m\times n}$ size patterns $\mathcal{B}$, i.e., $\mathcal{B} \subseteq \mathcal{A}^{\mathbb{N}_{m\times n}}$, where $\mathbb{N}_{m\times n} = [1, m] \times [1, n]$[1] using the sequence of 'horizontal transition matrices' say $\{\mathbf{T}_n\}_{n=1}^{\infty}$, is established in [10]. The authors prove that $h_{\text{top}}(\Sigma_{\mathcal{F}}^{[2]}) = \lim_{n\to\infty} \frac{\log \lambda_{\mathbf{T}_n}}{n}$ with similar results also found in [32, 36]. Fortunately, the method in [10] also leads us to establish the formula and calculate the explicit value of escape rate $\rho(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{[2]})$. Precisely, if we put $\mathcal{F}$ and $\mathcal{F}_1$ in the same size, say $\mathbb{N}_{m\times n}$, then let $\{\mathbf{T}_k\}_{k\geq n}$ be the sequence of horizontal transition matrices and $\{\mu_k\}_{k\geq n}$ be the associated sequence of Parry measures for $k \geq n$. Define the upper and lower escape rate $\overline{\rho}^{[m,n]}(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{[2]})$ and $\underline{\rho}^{[m,n]}(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{[2]})$ as in (2.1) and (2.2). That is, the upper and lower growth rate of $\rho_{\mu_k}(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{[2]})$ on $\mathbb{N} \times [1, k]$. An analogous result as [22, Theorem 3.1] for $\mathbb{N}^2$ SFTs is presented in Theorem 2.1. Specifically, the rigorous formula for the lower escape rate is derived, and the error function between the lower and upper escape rate is also presented. This provides us a sufficient condition for the existence of the escape rate, and it is worth pointing out that such a condition is strongly related to the mixing properties of the $\mathbb{N}^2$ SFTs. Furthermore, we prove that $\rho_\mu(w|_{[1,n]^d}, \Sigma_{\mathcal{F}}^{[d]}) \to 0$ as $n \to \infty$ $\forall w \in \Sigma_{\mathcal{F}}^{[d]}$ for $d \geq 1$ (cf. Theorem 4.2). This establishes the continuity of $\varepsilon \mapsto \rho_\mu(B_\varepsilon(z))$ using $w = z$, where $w|_{[1,n]^d}$ stands for the term $B_\varepsilon(z)$. As mentioned, in $\mathbb{N}^2$ SFT, it is extremely difficult to calculate the explicit value (or existence) of the escape rate by computing the exact value of the topological entropy. However, the method we used here enables us to compute the exact value of the escape rate and the topological entropy for a class of $\mathbb{N}^2$ SFTs with local forbidden sets possessing a symmetric structure (cf. Theorem 2.6). Finally, similar results for $\mathbb{N}^d$ SFTs with $d > 2$ are presented in Theorem 2.4 (escape rate formula) and Theorem 4.2 (continuity).

The second part of this article motivated from the multifractal analysis of the multiple ergodic average, e.g., for Hausdorff and Minkowski dimensions of 1-d multiple subshifts [16, 17, 37], for Hausdorff and Minkowski dimensions of multidimensional multiple subshifts [4, 6], for large deviation principle of 1-d multiple Ising models [13], for thermodynamics formalism of multiple subshifts [14]. We study the open

[1] $[1, m] \times [1, n] = \{1, \ldots, m\} \times \{1, \ldots, n\}$.

system problem on the 2-multiplicative integer systems (2-MISs). Suppose $\mathcal{F} \subseteq \mathcal{A}^2$, let

$$\Sigma_{\mathcal{F}}^{(q)} = \{x = (x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x_k x_{qk} \notin \mathcal{F} \ \forall k \in \mathbb{N}\}. \tag{1.3}$$

Roughly speaking, $\Sigma_{\mathcal{F}}^{(q)}$ may be considered as X_U in (1.1) by substituting $T^k(x)$ with $(T^{(q)})^k(x)$, where $(T^{(q)}(x))_k = x_{qk}$ for all $k \in \mathbb{N}$, and $U = \mathcal{A}^{\mathbb{N}} \setminus \Sigma_{\mathcal{F}}$. We are not aware of any further research on the escape rate of $\Sigma_{\mathcal{F}}^{(q)}$. The reason for the study (1.3) comes from the particular case where $\mathcal{A} = \{0, 1\}$, $q = 2$ and $\mathcal{F} = \{11\}$. Under the circumstances,

$$\begin{aligned}\Sigma_{\mathcal{F}}^{(2)} &= \{(x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x_k x_{2k} \neq 11 \ \forall k \in \mathbb{N}\} \\ &= \{(x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x_k \cdot x_{2k} = 0 \ \forall k \in \mathbb{N}\},\end{aligned}$$

where $a \cdot b$ is the usual product of a and b. The study of $\Sigma_{\mathcal{F}}^{(2)}$ is inspired by Furstenberg on his proof of the Szemerédi's theorem [17, 25]. The Minkowski dimension is computed in [17] and it is shown that (cf. [26, 38])

$$\dim_H \Sigma_{\mathcal{F}}^{(2)} = \dim_H \left\{(x_k) \in \mathcal{A}^{\mathbb{N}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k \cdot x_{2k} = 0\right\}.$$

Hence, the study of $\Sigma_{\mathcal{F}}^{(2)}$ is strongly related to the multifractal analysis problem of the multiple ergodic average $\frac{1}{n} \sum_{k=1}^{n} x_k \cdot x_{2k}$. Kenyon et al. [24] study the general form of $\Sigma_{\mathcal{F}}^{(q)}$, namely, let $\Omega \subseteq \{0, \ldots, m-1\}^{\mathbb{N}}$ be a subshift, define

$$\Sigma_{\Omega}^{(q)} = \{(x_k)_{k=1}^{\infty} \in \{0, \ldots, m-1\}^{\mathbb{N}} : (x_{iq^l})_{l=0}^{\infty} \in \Omega \text{ for all } i \in \mathbb{N}\}, \tag{1.4}$$

and call such shifts *multiplicative shifts* since those systems are invariant under the action of multiplicative integers, i.e., $(x_k)_{k=1}^{\infty} \in \Sigma_{\Omega}^{(q)}$ implies $(x_{rk})_{k=1}^{\infty} \in \Sigma_{\Omega}^{(q)}$ $\forall r \in \mathbb{N}$. There has been plenty of research on multiplicative shifts since then (cf. [5, 13, 14, 18, 26, 27]). We also refer the reader to [16] for a nice survey of this subject.

To choose a good measure for the escape rate, we are reminded of the concept of the *telescope measure* built in [24] to calculate the Hausdorff and Minkowski dimensions of $\Sigma_{\mathcal{F}}^{(q)}$. It is known that the first stage of the study of $\Sigma_{\mathcal{F}}^{(q)}$ is to decompose the lattice into independence lattice (according to q) for which the rule in each independent lattices behaves as the usual 1-step SFT. Suppose A is the associated transition matrix of such 1-step SFT, and μ is the corresponding Parry measure on it. The *telescope measure* $\mathbb{P}_{\mu}$, which is defined as the product of measures of cylinders on each independence lattice, is used to compute the Hausdorff and Minkowski dimension of $\Sigma_{\mathcal{F}}^{(q)}$. In (1.4), if $\Omega = \Sigma_{\mathcal{F}}$, we simply write

$$\Sigma_{\mathcal{F}}^{(q)} = \Sigma_{\Sigma_{\mathcal{F}}}^{(q)} = \Sigma_{\Omega}^{(q)}.$$

We study the escape rate of $\mathbb{P}_\mu$ through the hole $\Sigma_{\mathcal{F}}^{(q)} \setminus \Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)}$,

$$\rho(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{(q)}) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\mu(B_n(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)})),$$

where $B_n(X)$ is the set of all n-block patterns of X. In Theorem 3.1, we estimate the values of $\overline{\rho}(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{(q)})$ and $\underline{\rho}(\mathcal{F}_1, \Sigma_{\mathcal{F}}^{(q)})$ with the difference of the entropies

$$K = h_{\text{top}}(\Sigma_{\mathcal{F}}^{(q)}) - h_{\text{top}}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)}),$$

and prove the continuity of escape rate (Theorem 4.7), i.e.,

$$\lim_{n\to\infty} \rho(w|_{[1,n]}, \Sigma_{\mathcal{F}}^{(q)}) = 0 \quad \forall w \in \Sigma_{\mathcal{F}}^{(q)}.$$

Finally, we extend these results to multidimensional multiplicative shifts (see Theorems 3.2 and 4.7).

There are two possible extensions for these two systems. For $\mathbb{N}^d$ SFTs, one can consider the same problem on $\mathbb{N}^d$ sofic shifts, i.e., the factor of an $\mathbb{N}^d$ SFT. The difficulty in studying the escape rate problem of an $\mathbb{N}^d$ sofic shift is to construct the appropriate measure and observe how the measure is affected by the factor. On the other hand, the problem for 2-MIS can be extended to k-MIS for $k \geq 3$. In this situation, new ideas are necessary because our method for k-MIS ($k \geq 3$) is no longer valid.

2. Escape rate of $\mathbb{N}^d$ subshifts of finite type

In this section, we would like to find the formula of the escape rate for $\mathbb{N}^d$ SFTs for $d \geq 2$ by using the method established in [10]. Such a method could lead us to find the rigorous value of the escape rate for $\mathbb{N}^d$ SFTs with some specific symmetrical structure.

2.1. Strip shifts and their transition matrices

In a one-dimensional SFT $\Sigma_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{N}}$ with a finite forbidden set $\mathcal{F} \subseteq \mathcal{A}^n$ for some $n \geq 1$, the associated transition matrix $\mathbf{T}(\mathcal{F})$ is usually introduced to study the concepts about topological entropy, maximal measures, dynamical zeta functions and so on [29]. Clearly, let

$$\mathcal{B} = \mathcal{A}^n \setminus \mathcal{F}$$

be the corresponding *basic set* of admissible local patterns, $\Sigma_{\mathcal{F}}$ (resp. $\mathbf{T}(\mathcal{F})$) can also be determined by $\mathcal{B}$ and is presented by $\Sigma_{\mathcal{A}^n\setminus\mathcal{B}}$ (resp. $\mathbf{T}(\mathcal{A}^n \setminus \mathcal{B})$). Similarly, for an $\mathbb{N}^d$ SFT $\Sigma_{\mathcal{F}}^{[d]} \subseteq \mathcal{A}^{\mathbb{N}^d}$ with forbidden set $\mathcal{F} \subseteq \mathcal{A}^{\mathbb{N}_{\ell \times k_1 \times \cdots \times k_{d-1}}}$, $d \geq 2$, the basic set of local patterns $\mathcal{B}$ is equal to the set $\mathcal{A}^{\mathbb{N}_{\ell \times k_1 \times \cdots \times k_{d-1}}} \setminus \mathcal{F}$. However, to the best of our knowledge, there is no such transition matrix for an $\mathbb{N}^d$ SFT. For simplicity, we only focus on $d = 2$ below. Instead of a transition matrix for an $\mathbb{N}^2$ SFT, Markly and Paul [33] introduced a sequence of transition matrices $\{\mathbf{T}_n\}_{n=1}^{\infty}$ on the *horizontal strip shift* $H_n(\Sigma_{\mathcal{F}}^{[2]})$ to study maximal measures and entropy, where $H_n(\Sigma_{\mathcal{F}}^{[2]})$ is the set of all patterns on $\mathbb{N} \times [1, n]$ that contains no forbidden patterns in $\mathcal{F}$ and it can be regarded as a one-dimensional SFT. It is well known that

$$h(\Sigma_{\mathcal{F}}^{[2]}) = \lim_{n\to\infty} \frac{\log \lambda_{\mathbf{T}_n}}{n},$$

where $\lambda_{\mathbf{T}_n}$ is the maximal eigenvalue of $\mathbf{T}_n$. By considering the measure-theoretic method on $H_n(\Sigma_{\mathcal{F}}^{[2]})$, Pavlov [36] approximated the hard square entropy constant.

Ban and Lin [10] introduced the ordering matrices $\mathbf{X}_n$ (or $\mathbf{Y}_n$), $n \geq 2$, to arrange systematically all patterns in $\{0, 1\}^{\mathbb{N}_{2\times n}}$ (or $\{0, 1\}^{\mathbb{N}_{n\times 2}}$) as follows.

For $n \geq 1$, let the nth-order counting function

$$\psi \equiv \psi_n \colon \{0, 1\}^n \to \{j : 1 \leq j \leq 2^n\}$$

be

$$\psi(\beta_1\beta_2\cdots\beta_n) = 1 + \sum_{j=1}^{n} \beta_j 2^{(n-j)}.$$

The horizontal ordering matrix $\mathbf{X}_n$ of patterns in $\{0, 1\}^{\mathbb{N}_{2\times n}}$ and vertical ordering matrix $\mathbf{Y}_n$ of patterns in $\{0, 1\}^{\mathbb{N}_{n\times 2}}$, $n \geq 2$, can be defined by

$$\mathbf{X}_n = [x_{n;ij}] = \begin{bmatrix} \beta_{n1} & \beta_{n2} \\ \vdots & \vdots \\ \beta_{21} & \beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix}_{2^n \times 2^n}$$

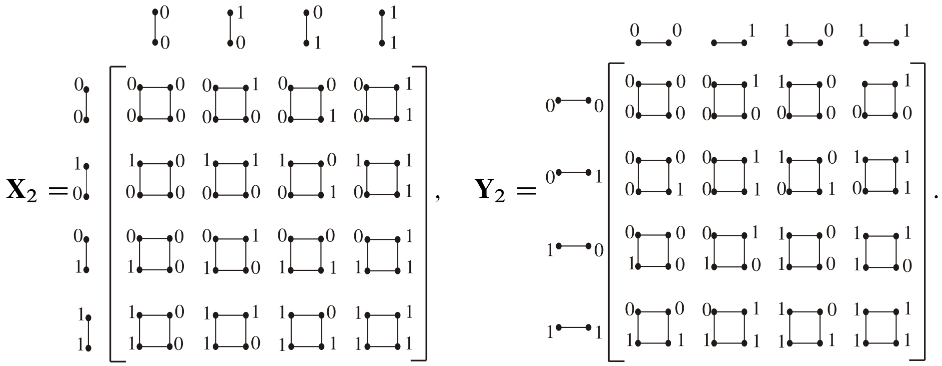
and

$$\mathbf{Y}_n = [y_{n;ij}] = \begin{bmatrix} \beta_{12} & \beta_{22} & \cdots & \beta_{n2} \\ \beta_{11} & \beta_{21} & \cdots & \beta_{n1} \end{bmatrix}_{2^n \times 2^n},$$

where

$$\begin{cases} i = \psi(\beta_{11}\beta_{21}\cdots\beta_{n1}), \\ j = \psi(\beta_{12}\beta_{22}\cdots\beta_{n2}). \end{cases}$$

In particular,



Given a basic set $\mathscr{B} \subseteq \{0,1\}^{\mathbb{N}_{2\times 2}}$, by the ordering matrices $\mathbf{X}_n$ and $\mathbf{Y}_n$, the corresponding transition matrices $\mathbf{H}_n$ and $\mathbf{V}_n$ can be defined, and their recursive formulae are obtained as follows. For $n \geq 2$, the associated horizontal and vertical transition matrices $\mathbf{H}_n = \mathbf{H}_n(\mathscr{B}) = [h_{n;ij}]_{2^n\times 2^n}$ and $\mathbf{V}_n = \mathbf{V}_n(\mathscr{B}) = [v_{n;ij}]_{2^n\times 2^n}$ are defined by

$$h_{n;ij} = \begin{cases} 1 & \text{if } x_{n;ij} \text{ can be generated by } \mathscr{B}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$v_{n;ij} = \begin{cases} 1 & \text{if } y_{n;ij} \text{ can be generated by } \mathscr{B}, \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Furthermore, the recursive formulae for generating $\mathbf{H}_n$ and $\mathbf{V}_n$ is also presented as follows. For $n \geq 2$, if

$$\mathbf{H}_n = \begin{bmatrix} \mathbf{H}_{n;1} & \mathbf{H}_{n;2} \\ \mathbf{H}_{n;3} & \mathbf{H}_{n;4} \end{bmatrix}_{2^n\times 2^n},$$

where $H_{n;j}$ is a $2^{n-1}\times 2^{n-1}$ matrix, then

$$\mathbf{H}_{n+1} = \left[\begin{array}{cc|cc} h_{2;11}\mathbf{H}_{n;1} & h_{2;12}\mathbf{H}_{n;2} & h_{2;13}\mathbf{H}_{n;1} & h_{2;14}\mathbf{H}_{n;2} \\ h_{2;21}\mathbf{H}_{n;3} & h_{2;22}\mathbf{H}_{n;4} & h_{2;23}\mathbf{H}_{n;3} & h_{2;24}\mathbf{H}_{n;4} \\ \hline h_{2;31}\mathbf{H}_{n;1} & h_{2;32}\mathbf{H}_{n;2} & h_{2;33}\mathbf{H}_{n;1} & h_{2;34}\mathbf{H}_{n;2} \\ h_{2;41}\mathbf{H}_{n;3} & h_{2;42}\mathbf{H}_{n;4} & h_{2;43}\mathbf{H}_{n;3} & h_{2;44}\mathbf{H}_{n;4} \end{array}\right].$$

The recursive formula of $\mathbf{V}_n$ is similar to that of $\mathbf{H}_n$ and is omitted here. It is noteworthy that Pierce [40] also obtained the same formula and applied it to study the entropy problems for $\mathbb{N}^2$ SFTs.

For example, consider the hard square model, whose rule is that 1 cannot be next to 1 in both horizontal and vertical directions. We have

$$\mathbf{H}_2(=\mathbf{V}_2)=\begin{bmatrix}\mathbf{H}_{2;1} & \mathbf{H}_{2;2}\\ \mathbf{H}_{2;3} & \mathbf{H}_{2;4}\end{bmatrix}=\begin{bmatrix}1&1&1&0\\1&0&1&0\\1&1&0&0\\0&0&0&0\end{bmatrix}.$$

Then,

$$\mathbf{H}_3(=\mathbf{V}_3)=\begin{bmatrix}1\cdot\begin{bmatrix}1&1\\1&0\end{bmatrix} & 1\cdot\begin{bmatrix}1&0\\1&0\end{bmatrix} & 1\cdot\begin{bmatrix}1&1\\1&0\end{bmatrix} & 0\cdot\begin{bmatrix}1&0\\1&0\end{bmatrix}\\ 1\cdot\begin{bmatrix}1&1\\0&0\end{bmatrix} & 0\cdot\begin{bmatrix}0&0\\0&0\end{bmatrix} & 1\cdot\begin{bmatrix}1&1\\0&0\end{bmatrix} & 0\cdot\begin{bmatrix}0&0\\0&0\end{bmatrix}\\ 1\cdot\begin{bmatrix}1&1\\1&0\end{bmatrix} & 1\cdot\begin{bmatrix}1&0\\1&0\end{bmatrix} & 0\cdot\begin{bmatrix}1&1\\1&0\end{bmatrix} & 0\cdot\begin{bmatrix}1&0\\1&0\end{bmatrix}\\ 0\cdot\begin{bmatrix}1&1\\0&0\end{bmatrix} & 0\cdot\begin{bmatrix}0&0\\0&0\end{bmatrix} & 0\cdot\begin{bmatrix}1&1\\0&0\end{bmatrix} & 0\cdot\begin{bmatrix}0&0\\0&0\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1&0&1&1&0&0\\1&0&1&0&1&0&0&0\\1&1&0&0&1&1&0&0\\0&0&0&0&0&0&0&0\\1&1&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{bmatrix}_{8\times 8}.$$

In certain cases, the topological entropy of $\Sigma_{\mathcal{B}}^{[2]}$ can be computed explicitly by using the recursive formula [10] (the results are listed in Remark 2.8 (1)). In addition, different kinds of topological mixing properties for $\mathbb{N}^2$ SFTs can be characterized by $\{\mathbf{H}_n\}_{n=2}^{\infty}$ and $\{\mathbf{V}_n\}_{n=2}^{\infty}$ [9], which also leads us to study the dynamical zeta functions for $\mathbb{N}^d$ SFTs [8]. It is noteworthy that the idea of ordering matrices also can be considered for general $\mathcal{B}\subseteq\mathcal{A}^{\mathbb{N}_{m\times n}}$ and for $\mathbb{N}^d$, $d\geq 2$.

2.2. Parry measure and escape rate

Let $\mathcal{B}\subseteq\{0,1\}^{\mathbb{N}_{m\times n}}$ be a basic set of admissible patterns which is a set of $m\times n$ patterns over $\{0,1\}$. For $m\geq 2$, $n\geq 1$ and $k\geq n$, the (m,k)-counting function $\psi_{m,k}$ is a bijection from $\{0,1\}^{\mathbb{N}_{(m-1)\times k}}$ to $\{1,\ldots,2^{(m-1)k}\}$. The matrices $\{\mathbf{T}_k^{[m,n]}\}_{k=n}^{\infty}$ are horizontal transition matrices of $\Sigma_{\infty\times k;\mathcal{F}}$ and are indexed by $\psi_{m,k}$, $k\geq n$, respectively, where $\Sigma_{\infty\times k;\mathcal{F}}$ is the SFT on $\mathbb{N}_{\infty\times k}$ with $\mathcal{F}=\{0,1\}^{\mathbb{N}_{m\times n}}\setminus\mathcal{B}$. For convenience, we denote $\mathbf{T}_k^{[m,n]}$ by $\mathbf{T}_k$. For each $k \geq n$, let λ_k be the maximum eigenvalue of $\mathbf{T}_k$ and $(u_k)^t$, v_k be the normalized left and right eigenvectors of $\mathbf{T}_k$ with respect to λ_k with $(u_k)^t v_k = 1$. It can be shown that the measure

$$\mu_k(U_{\ell\times k}) = \frac{(u_k)_{i_1}(v_k)_{i_{\ell-m+2}}}{\lambda_k^{\ell-m+1}}(\mathbf{T}_k)_{i_1 i_2}(\mathbf{T}_k)_{i_2 i_3}\cdots(\mathbf{T}_k)_{i_{\ell-m+1} i_{\ell-m+2}}$$

is an equilibrium measure on $\Sigma_{\infty\times k;\mathcal{F}}$, where $U_{\ell\times k} = [j_1 j_2 \cdots j_\ell]$ is an $\ell \times k$ cylinder set of $\Sigma_{\infty\times k;\mathcal{F}}$ and $i_r = \psi_{m,k}(j_r j_{r+1}\cdots j_{r+m-2})$ for all $1 \leq r \leq \ell - m + 2$.

Assume the transition matrices of $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{F}_1$ are same sizes for each $k \geq n$, say $\{\mathbf{T}_k\}_{k=n}^{\infty}$ and $\{\widehat{\mathbf{T}}_k\}_{k=n}^{\infty}$, respectively, and define the upper and lower escape rates of $\mathbb{N}^2$ action $\sigma|_{\Sigma_{\mathcal{F}}^{[2]}}$ to the hole $H_{\mathcal{F}_1}^{[2]}$ by

$$\widehat{\rho}^{[m,n]}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[2]}) = -\liminf_{\ell,k\to\infty}\frac{1}{\ell k}\log\mu_k(B_{\ell\times k}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[2]})), \tag{2.1}$$

and

$$\underline{\rho}^{[m,n]}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[2]}) = -\limsup_{\ell,k\to\infty}\frac{1}{\ell k}\log\mu_k(B_{\ell\times k}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[2]})), \tag{2.2}$$

respectively, where $B_{\ell\times k}(X)$ is the set of all $\ell \times k$ patterns of X. If the above limit exists, then we denote

$$\rho^{[m,n]} = \widehat{\rho}^{[m,n]} = \underline{\rho}^{[m,n]}.$$

For convenience, we denote $\rho^{[m,n]}$, $\widehat{\rho}^{[m,n]}$ and $\underline{\rho}^{[m,n]}$ by ρ, $\widehat{\rho}$ and $\underline{\rho}$, respectively.

Theorem 2.1. *For $2 \leq m, n \in \mathbb{N}$, we have the following assertions:*

$$K \leq \underline{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[2]}) \leq \widehat{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[2]}) \leq K + \limsup_{\ell,k\to\infty}\frac{\log\frac{1}{\alpha_k}}{\ell k},$$

where

$$K = h(\Sigma_{\mathcal{F}}^{[2]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[2]})$$

and $\{\alpha_k\}_{k=n}^{\infty}$ is a sequence such that $(u_k)_i(v_k)_j \geq \alpha_k \geq 0$ for all $k \geq n$ and for all $1 \leq i, j \leq 2^{mk}$. Moreover, if

$$\limsup_{\ell,k\to\infty}\frac{\log\frac{1}{\alpha_k}}{\ell k} = 0, \tag{2.3}$$

then $\underline{\rho} = \widehat{\rho}$ and

$$\rho(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[2]}) = h(\Sigma_{\mathcal{F}}^{[2]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[2]}).$$

Proof. The proof is a case of Theorem 2.4, which we have omitted here. ■

In order to consider the relation between condition (2.3) and mixing property of an $\mathbb{N}^2$ SFT, we introduce the *block gluing* as follows. An $\mathbb{N}^d$ subshift X is called *block gluing* if there exists a constant $N > 0$ such that for any two blocks U and V of X with $d(s(U), s(V)) \geq N$, there exists $x \in X$ such that

$$x|_{s(U)} = U \quad \text{and} \quad x|_{s(V)} = V,$$

where $s(W)$ is the support of W.

Corollary 2.2. *Let X be an $\mathbb{N}^2$ SFT with horizontal transition matrices $\{\mathbf{T}_k\}$. If X is block gluing, then $\underline{\rho} = \widehat{\rho}$ and*

$$\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{[2]}) = h(\Sigma_{\mathcal{F}}^{[2]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[2]}).$$

Proof. We claim that if X is block gluing, then condition (2.3) holds. Then, applying above theorem, the proof is complete.

Now, since X is block gluing, we have that the horizontal transition matrices $\{\mathbf{T}_k\}$ are primitive with common primitive number $m + N$ for all $k \geq n$, where N is the block gluing constant and m is the width of patterns in $\mathcal{B}$.

For any $k \geq n$, since $\mathbf{T}_k$ is primitive, by [29, Theorem 4.5.12], for all $p \geq 1$ and for all $1 \leq i, j \leq 2^{mk}$

$$[(\mathbf{T}_k)^p]_{i,j} = [(u_k)_i(v_k)_j + r_{k;i,j}(p)](\lambda_k)^p, \tag{2.4}$$

where $r_{k;i,j}(p) \to 0$ as $p \to \infty$. This implies that

$$(u_k)_i(v_k)_j = \frac{[(\mathbf{T}_k)^p]_{i,j}}{(\lambda_k)^p} - r_{k;i,j}(p).$$

Then, by primitivity of $\mathbf{T}_k$, it can be shown that

$$\min_{1\leq i,j\leq 2^{mk}} [(\mathbf{T}_k)^p]_{i,j} \geq [(\mathbf{T}_k)^{p-2(m+N)}]_{i',j'}$$

for any $1 \leq i', j' \leq 2^{mk}$ and for any $p \geq 2(m+N)$.

Hence,

$$\begin{aligned}\frac{\max_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j}{\min_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j} &= \frac{\max_{1\leq i,j\leq 2^{mk}}[(\mathbf{T}_k)^p]_{i,j} - r_{k;i,j}(p)(\lambda_k)^p}{\min_{1\leq i,j\leq 2^{mk}}[(\mathbf{T}_k)^p]_{i,j} - r_{k;i,j}(p)(\lambda_k)^p} \\ &\leq \frac{[(\mathbf{T}_k)^p]_{I,J} - r_{k;I,J}(p)(\lambda_k)^p}{[(\mathbf{T}_k)^{p-2(m+N)}]_{I,J} - r_{k;I',J'}(p)(\lambda_k)^p},\end{aligned} \tag{2.5}$$

where I, J are the indices for which $\max_{1\leq i,j\leq 2^{mk}}[(\mathbf{T}_k)^p]_{i,j} - r_{k;i,j}(p)(\lambda_k)^p$ is attained, and I', J' are the indices for which $\min_{1\leq i,j\leq 2^{mk}}[(\mathbf{T}_k)^p]_{i,j} - r_{k;i,j}(p)(\lambda_k)^p$ is attained.

Combining (2.4) and (2.5), we have that for $p \geq 2(m+N)$,

$$\begin{aligned}
&\frac{\max_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j}{\min_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j} \\
&\quad\leq \frac{[(u_k)_I(v_k)_J + r_{k;I,J}(p)](\lambda_k)^p - r_{k;I,J}(p)(\lambda_k)^p}{[(u_k)_I(v_k)_J + r_{k;I,J}(p-2(m+N))](\lambda_k)^{p-2(m+N)} - r_{k;i,j}(p)(\lambda_k)^p} \\
&\quad\xrightarrow{p\to\infty} (\lambda_k)^{2(m+N)}.
\end{aligned}$$

Since

$$\max_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j = \max_{1\leq i\leq 2^{mk}}(u_k)_i \max_{1\leq j\leq 2^{mk}}(v_k)_j$$

and

$$\min_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j = \min_{1\leq i\leq 2^{mk}}(u_k)_i \min_{1\leq j\leq 2^{mk}}(v_k)_j,$$

it is clear that

$$1 \leq \frac{\max_{1\leq i\leq 2^{mk}}(u_k)_i}{\min_{1\leq i\leq 2^{mk}}(u_k)_i}, \frac{\max_{1\leq j\leq 2^{mk}}(v_k)_j}{\min_{1\leq j\leq 2^{mk}}(v_k)_j} \leq (\lambda_k)^{2(m+N)}.$$

Since u_k and v_k are normalized vectors, we have that the smallest possible value of $\min_{1\leq j\leq 2^{mk}}(v_k)_j$ and $\min_{1\leq i\leq 2^{mk}}(u_k)_i$ is

$$\sqrt{\frac{\frac{1}{(\lambda_k)^{4(m+N)}}}{2^{mk}-1+\frac{1}{(\lambda_k)^{4(m+N)}}}} = \sqrt{\frac{1}{(\lambda_k)^{4(m+N)}(2^{mk}-1)+1}},$$

which yields

$$\alpha_k := \frac{1}{(\lambda_k)^{4(m+N)}2^{mk}} \leq \frac{1}{(\lambda_k)^{4(m+N)}(2^{mk}-1)+1} \leq \min_{1\leq i,j\leq 2^{mk}}(u_k)_i(v_k)_j.$$

Therefore, we obtain that

$$\begin{aligned}
\limsup_{\ell,k\to\infty}\frac{\log\frac{1}{\alpha_k}}{\ell k} &= \limsup_{\ell,k\to\infty}\frac{\log((\lambda_k)^{4(m+N)}2^{mk})}{\ell k} \\
&= \limsup_{\ell,k\to\infty}\frac{4(m+N)\log(\lambda_k)+mk\log 2}{\ell k} \\
&= 0.
\end{aligned}$$

The last equality is due to the fact $\lim_{k\to\infty}\frac{1}{k}\log\lambda_k = h(X)$ (the topological entropy of X). The proof is complete. ∎

Example 2.3. We should note that the primitive assumption of $\mathbf{T}_k$ reflects the existence of lower bounds of α_k such that $\alpha_k > 0$ for all $k \geq n$. For example, when $\mathcal{F}$ is empty, then $\alpha_k = \frac{1}{2^{mk}}$ for all $k \geq n$. This implies

$$\limsup_{\ell,k\to\infty} \frac{\log \frac{1}{\alpha_k}}{\ell k} = \limsup_{\ell,k\to\infty} \frac{\log 2^{mk}}{\ell k} = 0,$$

and so the limit of escape rate exists.

2.3. Setup and results for $\mathbb{N}^d$ subshifts of finite type

The following theorem is the $\mathbb{N}^d$ version of Theorem 2.1. Let $\mathcal{B} \subseteq \{0,1\}^{\mathbb{N}_{n_1\times\cdots\times n_d}}$ be a basic set of admissible patterns. For $n_1 \geq 2, n_2, \ldots, n_d \geq 1$ and $k_1 \geq n_2, \ldots, k_{d-1} \geq n_d$, the $(n_1, k_1, \ldots, k_{d-1})$-counting function $\psi_{n_1,k_1,\ldots,k_{d-1}}$ is a bijection from $\{0,1\}^{\mathbb{N}_{(n_1-1)\times k_1\times\cdots\times k_{d-1}}}$ to $\{1, \ldots, 2^{(n_1-1)k_1\cdots k_{d-1}}\}$. The matrices

$$\{\mathbf{T}^{[n_1,\ldots,n_d]}_{k_1,\ldots,k_{d-1}}\}_{k_1=n_2,\ldots,k_{d-1}=n_d}^{\infty}$$

are horizontal transition matrices of $\Sigma_{\infty\times k_1\times\cdots\times k_{d-1};\mathcal{F}}$, and they are indexed by $\psi_{n_1,k_1,\ldots,k_{d-1}}, k_1 \geq n_2, \ldots, k_{d-1} \geq n_d$, respectively, where $\mathcal{F} = \{0,1\}^{\mathbb{N}_{n_1\times\cdots\times n_d}} \setminus \mathcal{B}$. For convenience, denote $\mathbf{T}^{[n_1,\ldots,n_d]}_{k_1,\ldots,k_{d-1}}$ by $\mathbf{T}_{k_1,\ldots,k_{d-1}}$. For each $k_1 \geq n_2, \ldots, k_{d-1} \geq n_d$, let $\lambda_{k_1,\ldots,k_{d-1}}$ be the maximum eigenvalue of $\mathbf{T}_{k_1,\ldots,k_{d-1}}$ and $u_{k_1,\ldots,k_{d-1}}, v_{k_1,\ldots,k_{d-1}}$ are the normalized left and right eigenvectors of $\mathbf{T}_{k_1,\ldots,k_{d-1}}$ with respect to $\lambda_{k_1,\ldots,k_{d-1}}$ with $(u_{k_1,\ldots,k_{d-1}})^t v_{k_1,\ldots,k_{d-1}} = 1$. It can be shown that the measure

$$\begin{aligned}\mu_{k_1,\ldots,k_{d-1}}(U_{\ell\times k_1\times\cdots\times k_{d-1}}) = {}& \frac{(u_{k_1,\ldots,k_{d-1}})_{i_1}(v_{k_1,\ldots,k_{d-1}})_{i_{\ell-n_1+2}}}{\lambda^{\ell-n_1+1}_{k_1,\ldots,k_{d-1}}}(\mathbf{T}_{k_1,\ldots,k_{d-1}})_{i_1 i_2}\\ &\times (\mathbf{T}_{k_1,\ldots,k_{d-1}})_{i_2 i_3}\cdots(\mathbf{T}_{k_1,\ldots,k_{d-1}})_{i_{\ell-m+1} i_{\ell-n_1+2}}\end{aligned}$$

is an equilibrium measure on $\Sigma_{\infty\times k_1\times\cdots\times k_{d-1};\mathcal{F}}$, where $U_{\ell\times k_1\times\cdots\times k_{d-1}} = [j_1 j_2 \cdots j_\ell]$ is an $\ell \times k_1 \times \cdots \times k_{d-1}$ cylinder set of $\Sigma_{\infty\times k_1\times\cdots\times k_{d-1};\mathcal{F}}$ and

$$i_r = \psi_{n_1,k_1,\ldots,k_{d-1}}(j_r j_{r+1} \cdots j_{r+n_1-2})$$

for all $1 \leq r \leq \ell - n_1 + 2$.

Assume that the transition matrices of $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{F}_1$ have the same sizes for each $k_1 \geq n_2, \ldots, k_{d-1} \geq n_d$, say, for example, $\{\mathbf{T}_{k_1,\ldots,k_{d-1}}\}_{k_1=n_2,\ldots,k_{d-1}=n_d}^{\infty}$ and $\{\widehat{\mathbf{T}}_{k_1,\ldots,k_{d-1}}\}_{k_1=n_2,\ldots,k_{d-1}=n_d}^{\infty}$, respectively, and define the upper and lower escape rates of $\mathbb{N}^d$ action $\sigma|_{\Sigma^{[d]}_{\mathcal{F}}}$ to the hole $H^{[d]}_{\mathcal{F}_1}$ by

$$\begin{aligned}\hat{\rho}^{[n_1,\ldots,n_d]}(\mathcal{F}_1; \Sigma^{[d]}_{\mathcal{F}}) = -\liminf_{\ell,k_1,\ldots,k_{d-1}\to\infty} &\frac{1}{\ell k_1\cdots k_{d-1}}\\ &\times \log \mu_{k_1,\ldots,k_{d-1}}(B_{\ell,k_1,\ldots,k_{d-1}}(\Sigma^{[d]}_{\mathcal{F}\cup\mathcal{F}_1})),\end{aligned}$$

and

$$\underline{\rho}^{[n_1,\dots,n_d]}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) = -\limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \times \log \mu_{k_1,\dots,k_{d-1}}(B_{\ell,k_1,\dots,k_{d-1}}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]})),$$

respectively, where $B_{\ell,k_1,\dots,k_{d-1}}(X)$ is the set of all $\ell\times k_1\times\cdots\times k_{d-1}$ patterns of X. If the above limit exists, then we denote $\rho^{[n_1,\dots,n_d]} = \widehat{\rho}^{[n_1,\dots,n_d]} = \underline{\rho}^{[n_1,\dots,n_d]}$. For convenience, we denote $\rho^{[n_1,\dots,n_d]}$, $\widehat{\rho}^{[n_1,\dots,n_d]}$ and $\underline{\rho}^{[n_1,\dots,n_d]}$ by ρ, $\widehat{\rho}$ and $\underline{\rho}$, respectively.

Theorem 2.4. *For $2\le n_1,\dots,n_d\in\mathbb{N}$, we have the following assertions:*

$$K \le \underline{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) \le \widehat{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) \le K + \limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log\frac{1}{\alpha_{k_1,\dots,k_{d-1}}}}{\ell k_1\cdots k_{d-1}}, \tag{2.6}$$

where

$$K = h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]})$$

and $\{\alpha_{k_1,\dots,k_{d-1}}\}_{k_1=n_2,\dots,k_{d-1}=n_d}^{\infty}$ is a sequence such that

$$(u_{k_1,\dots,k_{d-1}})_i(v_{k_1,\dots,k_{d-1}})_j \ge \alpha_{k_1,\dots,k_{d-1}} \ge 0$$

for all $k_1\ge n_2,\dots,k_{d-1}\ge n_d$ and for all $1\le i,j\le 2^{n_1k_1\cdots k_{d-1}}$. Moreover, if

$$\limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log\frac{1}{\alpha_{k_1,\dots,k_{d-1}}}}{\ell k_1\cdots k_{d-1}} = 0, \tag{2.7}$$

then $\underline{\rho}=\widehat{\rho}$ and

$$\rho(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) = h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]}). \tag{2.8}$$

Proof. We first prove the most left inequality of (2.6). It can be shown that

$$\begin{aligned}
\underline{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) &= -\limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \log \mu_{k_1,\dots,k_{d-1}}(B_{\ell,k_1,\dots,k_{d-1}}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]}))\\
&= -\limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}}\\
&\quad \times \log \frac{\sum_{i,j=1}^{2^{n_1k_1\cdots k_{d-1}}} (u_{k_1,\dots,k_{d-1}})_i (v_{k_1,\dots,k_{d-1}})_j ((\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1})_{ij}}{\lambda_{k_1,\dots,k_{d-1}}^{\ell-n_1+1}}\\
&\ge -\limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \log \frac{\sum_{i,j=1}^{2^{n_1k_1\cdots k_{d-1}}} ((\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1})_{ij}}{\lambda_{k_1,\dots,k_{d-1}}^{\ell-n_1+1}}\\
&= -\lim_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \log \frac{|(\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1}|}{\lambda_{k_1,\dots,k_{d-1}}^{\ell-n_1+1}}\\
&= h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]}).
\end{aligned}$$

The middle inequality of (2.6) is due to the definitions of $\underline{\rho}$ and $\widehat{\rho}$. We are now in a position to prove the most right inequality of (2.6). It is straightforward to verify that

$$\begin{aligned}
\widehat{\rho}(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) &= \limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \\
&\quad \times \log \frac{\lambda_{k_1,\dots,k_{d-1}}^{\ell-n_1+1}}{\sum_{i,j=1}^{2^{n_1k_1\cdots k_{d-1}}}(u_{k_1,\dots,k_{d-1}})_i(v_{k_1,\dots,k_{d-1}})_j((\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1})_{ij}} \\
&\leq \limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{1}{\ell k_1\cdots k_{d-1}} \\
&\quad \times \log \frac{\lambda_{k_1,\dots,k_{d-1}}^{\ell-n_1+1}}{\sum_{i,j=1}^{2^{n_1k_1\cdots k_{d-1}}}\alpha_{k_1,\dots,k_{d-1}}((\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1})_{ij}} \\
&= \lim_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log \lambda_{k_1,\dots,k_{d-1}}}{k_1\cdots k_{d-1}} \\
&\quad - \lim_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log|(\widehat{\mathbf{T}}_{k_1,\dots,k_{d-1}})^{\ell-n_1+1}|}{\ell k_1\cdots k_{d-1}} \\
&\quad + \limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log\frac{1}{\alpha_{k_1,\dots,k_{d-1}}}}{\ell k_1\cdots k_{d-1}} \\
&= h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]}) + \limsup_{\ell,k_1,\dots,k_{d-1}\to\infty} \frac{\log\frac{1}{\alpha_{k_1,\dots,k_{d-1}}}}{\ell k_1\cdots k_{d-1}}.
\end{aligned}$$

Finally, combining (2.6) with (2.7) yields (2.8). The proof is thus completed. ■

Using similar argument to Corollary 2.2, we have the following result.

Corollary 2.5. *Let X be an $\mathbb{N}^d$ SFT with horizontal transition matrices $\{\mathbf{T}_{k_1,\dots,k_{d-1}}\}$. If X is block gluing, then $\underline{\rho} = \widehat{\rho}$ and*

$$\rho(\mathcal{F}_1;\Sigma_{\mathcal{F}}^{[d]}) = h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{[d]}).$$

2.4. $\mathbb{N}^2$ subshifts of finite type where forbidden patterns are of size 2 × 2 with 2 symbols

Let $\mathcal{B} \subseteq \{0,1\}^{\mathbb{N}_{2\times 2}}$ be a basic set. The matrices $\{\mathbf{H}_n\}_{n=2}^{\infty}$ are horizontal transition matrices of $\Sigma_{\infty\times n;\mathcal{F}}$, $n\geq 2$, respectively, where $\mathcal{F} = \{0,1\}^{\mathbb{N}_{2\times 2}} \setminus \mathcal{B}$. For each $n \geq 2$, let λ_n be the maximum eigenvalue of $\mathbf{H}_n$, and let u_n, v_n be the normalized left and right eigenvectors of $\mathbf{H}_n$ with respect to λ_n with $u_n^t v_n = 1$. It can be shown that the measure

$$\mu_n(U_{m\times n}) = \frac{(u_n)_{i_1}(v_n)_{i_m}}{\lambda_n^m}(\mathbf{H}_n)_{i_1i_2}(\mathbf{H}_n)_{i_2i_3}\cdots(\mathbf{H}_n)_{i_{m-1}i_m}$$

is an equilibrium measure on $\Sigma_{\infty\times n;\mathcal{F}}$, where $U_{m\times n} = [i_1 i_2 \cdots i_m]$ is an $m \times n$ cylinder set of $\Sigma_{\infty\times n;\mathcal{F}}$.

Assume the transition matrices of $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{F}_1$ are same sizes for each $n \geq 2$, say $\{\mathbf{H}_n\}_{n=2}^{\infty}$ and $\{\overline{\mathbf{H}}_n\}_{n=2}^{\infty}$, respectively. If $\mathbf{H}_2$ and $\overline{\mathbf{H}}_2$ are of the form

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \quad \text{with } A = \begin{bmatrix} a & a_2 \\ a_3 & a \end{bmatrix} \text{ and } B = \begin{bmatrix} b & b_2 \\ b_3 & b \end{bmatrix},$$

by applying [10], we have the following results.

Theorem 2.6. *If $\mathbf{H}_2$ and $\overline{\mathbf{H}}_2$ are of the form above, then we have*

$$K \leq \underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{[2]}) \leq \hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{[2]}) \leq K + \limsup_{m,n\to\infty} \frac{\log \frac{1}{\alpha_n}}{mn},$$

where $K = h(\mathbf{H}_2) - h(\overline{\mathbf{H}}_2)$ and $\{\alpha_n\}_{n=2}^{\infty}$ is a sequence such that $(u_n)_i(v_n)_j \geq \alpha_n \geq 0$ for all $n \geq 2$ and for all $1 \leq i, j \leq 2^n$. Moreover, if

$$\limsup_{m,n\to\infty} \frac{\log \frac{1}{\alpha_n}}{mn} = 0,$$

then

$$\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{[2]}) = h(\mathbf{H}_2) - h(\overline{\mathbf{H}}_2). \tag{2.9}$$

Proof. The proof can be obtained using Theorem 2.1 and the table in [10] (see Remark 2.8 (2)). ∎

Example 2.7. Let

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \overline{\mathbf{H}}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Then the Example 2.3 gives us the existence of the escape rate, and (2.9) of Theorem 2.6 provides the escape rate which equals

$$\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{[2]}) = \log 2 - \log\left(\frac{1+\sqrt{5}}{2}\right) = \log\left(\frac{4}{1+\sqrt{5}}\right) \approx 0.2119.$$

Remark 2.8. (1) In [9], by using certain conditions on $\mathbf{H}_n(\mathbf{V}_m)$, the sufficient conditions for block gluing is provided.

(2) Table of A, B, $P(\lambda)$ the polynomial of recurrence relation of the maximal eigenvalue of $\mathbf{H}_n$, λ_* is the largest root of $P(\lambda)$ and $\log \lambda_* = h(\mathbf{H}_2)$ in [10].

A	**B**	$P(\lambda)$	λ_*
$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\lambda-2$	2
$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$	$\lambda^3-2\lambda^2+\lambda-1$	≈ 1.75488
$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\lambda^2-\lambda-1$	≈ 1.61803
$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$	$\lambda^2-\lambda-1$	≈ 1.61803
$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\lambda^2-\lambda-1$	≈ 1.61803
$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$	$\lambda^3-\lambda^2-1$	≈ 1.46557
$\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$	$\lambda^3-\lambda^2-1$	≈ 1.46557
$\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\lambda^3-\lambda-1$	≈ 1.32472
$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]$	$\lambda^4-\lambda-1$	≈ 1.22074

Let X_1 and X_2 be two $\mathbb{N}$ subshifts. The *axial product* of X_1 and X_2, which is introduced by Louidor, Marcus and Pavlov [30], is defined by

$$X = X_1 * X_2 = \{(x_{i,j})_{i,j=1}^{\infty} \colon (x_{i,j})_{i=1}^{\infty} \in X_1 \text{ and } (x_{i,j})_{j=1}^{\infty} \in X_2 \text{ for all } i, j \in \mathbb{N}\}.$$

Many interesting and important $\mathbb{N}^2$ shifts in statistical physics can be formed by means of this product; for example, the hard square model. Theorem 2.9 reveals the relation between the axial product and the horizontal transition matrix $\mathbf{H}_2$. The formula allows us to use the aforementioned formula of the escape rate to the systems derived from the axial product of two shifts. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then the *tensor product* of A and B is defined by $A \otimes B = [a_{ij} B]$, and the *Hadamard product* of A and B is defined by $A \circ B = [a_{ij} b_{ij}]$.

Theorem 2.9. *If X is the axial product of two 1-step SFTs X_1 and X_2, then the corresponding two-dimensional transition matrix $\mathbf{H}_2$ equals*

$$\mathbf{H}_2 = (A_1 \otimes A_1) \circ (A_2 \otimes A_2)^Z,$$

where A_1 and A_2 are transition matrices of X_1 and X_2, respectively. Here $\otimes$ denotes the tensor product, $\circ$ is the Hadamard product, and A^Z is the Z-shape transformation *of A, that is,*

$$A^Z = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^Z = \begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{13} & a_{14} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{41} & a_{42} \\ a_{33} & a_{34} & a_{43} & a_{44} \end{bmatrix}.$$

Proof. It is easy to see that the admissibility of the horizontal and vertical directions of a 2×2 pattern are determined by $A_1 \otimes A_1$ and $(A_2 \otimes A_2)^Z$, respectively. Thus the proof is complete. ■

Example 2.10. (1) Table of A_1 and A_2.

A_1	A_2	$\mathbf{H}_2$	$h(\mathbf{H}_2)$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\log 2$
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$	$\log g$
$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	> 0

The first classis full $\mathbb{N}^2$ shift, the second class is full $\mathbb{N}$ shift axial golden mean shift, the third class contains a safe pattern.

(2) In case $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the transition matrices $\mathbf{V}_n = \otimes^n G$; this implies

$$(u_n)_i (v_n)_j \geq \frac{1}{(1+g^2)^n}$$

for all $1 \leq i, j \leq 2^n$. Thus,

$$\limsup_{m,n\to\infty} \frac{\log \frac{1}{\alpha_n}}{mn} = \limsup_{m,n\to\infty} \frac{\log(1+g^2)^n}{mn} = 0,$$

and so the escape rate exists.

(3) In case $A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, it can be verified that X is block gluing in [9]. Thus, the escape rate exists.

3. Escape rate of 2-multiplicative integer systems on $\mathbb{N}^d$

In this section, we discuss the escape rate for the 2-MISs on $\mathbb{N}^d$. Precisely, we consider the following set:

$$\Sigma_{\mathcal{F}}^{(2)} = \{(x_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : x_i x_{2i} \notin \mathcal{F} \text{ for all } i \in \mathbb{N}\}.$$

The constraint $\mathcal{F}_1 \subseteq \mathcal{A}^*$ is imposed as a hole, and then the following associated set is considered:

$$\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(2)} = \{(x_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : x_i x_{2i} \notin \mathcal{F} \cup \mathcal{F}_1 \text{ for all } i \in \mathbb{N}\}.$$

3.1. 2-multiplicative integer systems on $\mathbb{N}$

Let $\mathcal{A} = \{0, 1, \ldots, m-1\}$ be a finite alphabet,

$$\Sigma_{\mathcal{F}}^{(q)} = \{(x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : (x_{iq^\ell})_{\ell=0}^{\infty} \in \Sigma_{\mathcal{F}} \text{ for all } i, q \nmid i\},$$

where $\Sigma_{\mathcal{F}}$ is a shift of finite type and A is the associated transition matrix (assume A is a $k \times k$ matrix). The topological entropy of $\Sigma_{\mathcal{F}}^{(q)}$ is $h(\Sigma_{\mathcal{F}}^{(q)}) = \frac{(q-1)^2}{q} \sum_{n=1}^{\infty} \frac{\log |A^{n-1}|}{q^n}$ (see [7]), where $|A|$ of a matrix A is the sum of all entries in A.

Let $\mathbb{P}_\mu(x) = \prod_{q \nmid i} \mu((x_{iq^\ell})_{\ell=1}^{\infty})$. The lower and upper escape rates of multiplicative integer action on $\Sigma_{\mathcal{F}}^{(q)}$ into the hole $\mathcal{F}_1$ are defined by

$$\hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(q)}) = -\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\mu(B_n(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)}))$$

and

$$\underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(q)}) = -\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\mu(B_n(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)})),$$

respectively. Assume the sizes of transition matrices of $\Sigma_{\mathcal{F}\cup\mathcal{F}_1}$ and $\Sigma_{\mathcal{F}}$ are equal, then we have the following estimate of the escape rate for the 2-MIS on $\mathbb{N}$.

Theorem 3.1. *Let $\Sigma_{\mathcal{F}}^{(q)}$ be a 2-MIS on $\mathbb{N}$ and $\mathcal{F}_1 \subseteq \mathcal{A}^*$ be a hole. If the transition matrix A of the SFT $\Sigma_{\mathcal{F}}$ is primitive, then for $q \geq 2$,*

$$K - \frac{q-1}{q} \log C_2 \leq \underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(q)}) \leq \hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(q)}) \leq K - \frac{q-1}{q} \log(\alpha C_1),$$

where $K = h(\Sigma_{\mathcal{F}}^{(q)}) - h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(q)})$, and for all $n \in \mathbb{N}$, $C_1\lambda_A^n \leq |A^n| \leq C_2\lambda_A^n$ with λ_A is the maximum eigenvalue of A and the corresponding normalized eigenvectors (left and right) are u and v satisfying $u^t v = 1$ and $0 < \alpha \leq u_i v_j \leq 1$ for all $1 \leq i, j \leq k$.

Proof. The proof is a direct consequence of Theorem 3.2 and is omitted. ∎

3.2. 2-multiplicative integer systems on $\mathbb{N}^d$

For $d \geq 2$, the d-dimensional upper and lower escape rates of multiplicative integer action on $\Sigma_{\mathcal{F}}^{(\mathbf{q})}$ into the hole $\mathcal{F}_1$ are defined by

$$\hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) = -\liminf_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d} \log \mathbb{P}_\mu(B_{n_1,\ldots,n_d}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(\mathbf{q})}))$$

and

$$\underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) = -\limsup_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d} \log \mathbb{P}_\mu(B_{n_1,\ldots,n_d}(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}^{(\mathbf{q})})),$$

respectively, where $\mathbf{q} = (q_1, \ldots, q_d)$ and[2]

$$\Sigma^{(\mathbf{q})}_{\mathcal{F}\cup\mathcal{F}_1} = \{(x_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in \mathcal{A}^{\mathbb{N}^d} : (x_{\mathbf{i}\cdot\mathbf{q}^\ell})_{\ell=0}^{\infty} \in \Sigma_{\mathcal{F}\cup\mathcal{F}_1} \text{ for all } \mathbf{i} \in \mathcal{I}_{\mathbf{q}}\}$$

with

$$\mathcal{I}_{\mathbf{q}} = \{(i_1, \ldots, i_d) \in \mathbb{N}^d : q_j \nmid i_j \text{ for some } 1 \le j \le d\}.$$

Denote $\rho = \widehat{\rho} = \underline{\rho}$ if the above limit exists. We have the following theorem for the relationship between the escape rate and topological entropy.

Theorem 3.2. *Let $\Sigma^{(\mathbf{q})}_{\mathcal{F}}$ be a 2-MIS on $\mathbb{N}^d$ and $\mathcal{F}_1 \subseteq \mathcal{A}^*$ be a hole. If the transition matrix A of the SFT $\Sigma_{\mathcal{F}}$ is primitive, then for $q_1, \ldots, q_d \ge 2$,*

$$\begin{aligned} K - \frac{q_1\cdots q_d - 1}{q_1\cdots q_d}\log C_2 \le \underline{\rho}(\mathcal{F}_1; \Sigma^{(\mathbf{q})}_{\mathcal{F}}) &\le \widehat{\rho}(\mathcal{F}_1; \Sigma^{(\mathbf{q})}_{\mathcal{F}}) \\ &\le K - \frac{q_1\cdots q_d - 1}{q_1\cdots q_d}\log(\alpha C_1), \end{aligned}$$

where

$$K = h(\Sigma^{(\mathbf{q})}_{\mathcal{F}}) - h(\Sigma^{(\mathbf{q})}_{\mathcal{F}\cup\mathcal{F}_1}),$$

and for all $n \in \mathbb{N}$,

$$C_1\lambda_A^n \le |A^n| \le C_2\lambda_A^n, \tag{3.1}$$

where λ_A is maximum eigenvalue of A, and the corresponding normalized eigenvectors (left and right) are u and v satisfying $u^t v = 1$ and $0 < \alpha \le u_i v_j \le 1$ for all $1 \le i, j \le k$.

Proof. Let A and B be the associated transition matrices of $\Sigma_{\mathcal{F}}$ and $\Sigma_{\mathcal{F}\cup\mathcal{F}_1}$, respectively (assume A and B are $k \times k$ matrices). Then we have

$$\begin{aligned} \widehat{\rho}(\mathcal{F}_1; \Sigma^{(\mathbf{q})}_{\mathcal{F}}) &= -\liminf_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d}\log \mathbb{P}_\mu(B_{n_1,\ldots,n_d}(\Sigma^{(\mathbf{q})}_{\mathcal{F}\cup\mathcal{F}_1})) \\ &= -\liminf_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d}\log \prod_{n=1}^{n_1\cdots n_d} \mu(B_n(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}))^{R_{n;n_1,\ldots,n_d}} \\ &= -\liminf_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d}\log \prod_{n=1}^{n_1\cdots n_d} \left(\sum_{i,j=1}^{k} \frac{u_i v_j}{\lambda_A^{n-1}}(B^{n-1})_{ij}\right)^{R_{n;n_1,\ldots,n_d}}, \end{aligned}$$

and

$$\underline{\rho}(\mathcal{F}_1; \Sigma^{(\mathbf{q})}_{\mathcal{F}}) = -\limsup_{n_1,\ldots,n_d\to\infty} \frac{1}{n_1\cdots n_d}\log \prod_{n=1}^{n_1\cdots n_d} \left(\sum_{i,j=1}^{k} \frac{u_i v_j}{\lambda_A^{n-1}}(B^{n-1})_{ij}\right)^{R_{n;n_1,\ldots,n_d}},$$

---

[2] $\mathbf{q}^\ell = (q_1^\ell, \ldots, q_d^\ell)$.

where

$$R_{n;n_1,\ldots,n_d} = \Big(1 - \frac{1}{q_1 \cdots q_d}\Big)\Big(\prod_{k=1}^{d} \Big\lfloor \frac{n_k}{q_k^{n-1}} \Big\rfloor - \prod_{k=1}^{d} \Big\lfloor \frac{n_k}{q_k^{n}} \Big\rfloor\Big),$$

$u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ are the normalized left and right eigenvectors corresponding to the maximum eigenvalue of A, λ_A, that is $u^t v = 1$.

If $\lambda_A > 0$, there is an $\alpha > 0$ such that $0 < \alpha \leq u_i v_j \leq 1$, then we have

$$\begin{aligned} -\frac{(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{n=1}^{\infty} \frac{\log \frac{|B^{n-1}|}{\lambda_A^{n-1}}}{(q_1 \cdots q_d)^n} &\leq \underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \leq \hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\ &\leq -\frac{(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{n=1}^{\infty} \frac{\log(\alpha \frac{|B^{n-1}|}{\lambda_A^{n-1}})}{(q_1 \cdots q_d)^n}. \end{aligned}$$

Then by condition (3.1), we have

$$\begin{aligned} -\frac{(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{n=1}^{\infty} \frac{\log(C_2 \frac{|B^{n-1}|}{|A^{n-1}|})}{(q_1 \cdots q_d)^n} &\leq \underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \leq \hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\ &\leq -\frac{(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{n=1}^{\infty} \frac{\log(\alpha C_1 \frac{|B^{n-1}|}{|A^{n-1}|})}{(q_1 \cdots q_d)^n}. \end{aligned}$$

This implies

$$\begin{aligned} K - \frac{q_1 \cdots q_d - 1}{q_1 \cdots q_d} \log C_2 &\leq \underline{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \leq \hat{\rho}(\mathcal{F}_1; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\ &\leq K - \frac{q_1 \cdots q_d - 1}{q_1 \cdots q_d} \log(\alpha C_1), \end{aligned}$$

where

$$K = h(\Sigma_{\mathcal{F}}^{(\mathbf{q})}) - h(\Sigma_{\mathcal{F} \cup \mathcal{F}_1}^{(\mathbf{q})}).$$

The proof is complete. ∎

4. Continuities of the escape rates of subshifts of finite type and 2-multiplicative integer systems on $\mathbb{N}^d$

4.1. $\mathbb{N}^d$ subshifts of finite type

We apply the result in [35], which gives the continuity of entropy of $\mathbb{N}^d$ SFTs.

Theorem 4.1. *For any $d > 1$ and any strongly irreducible $\mathbb{N}^d$ SFT X on an alphabet $\mathcal{A}$ with $|\mathcal{A}| > 1$, there exist constants $C_X, D_X > 0$, A_X, B_X and $N_X \in \mathbb{N}$ such that* *for any* $n > N_X$ *and any pattern* $w \in \mathcal{A}^{[1,n]^d}$ *which appears as a subpattern of some point of* X,

$$\frac{C_X}{e^{h(X)(n+A_X)^d}} < h(X) - h(X_w) < \frac{D_X}{e^{h(X)(n+B_X)^d}},$$

where $X_w = \{x \in X : w \text{ not appear as a subpattern of } x\}$.

Using Theorem 4.1 with the escape rate formula (Theorem 2.4), the following result for the continuity of an $\mathbb{N}^d$ SFT is immediate.

Theorem 4.2. *If* $\Sigma_{\mathcal{F}}$ *is strongly irreducible and satisfies* (2.7), *there exist constants* $C_{\mathcal{F}}, D_{\mathcal{F}} > 0$, $A_{\mathcal{F}}, B_{\mathcal{F}}$ *and* $N_{\mathcal{F}} \in \mathbb{N}$ *such that for any* $n > N_{\mathcal{F}}$ *and any pattern* $w \in \{0,1\}^{[1,n]^d}$ *which appears as a subpattern of some point of* $\Sigma_{\mathcal{F}}^{[d]}$,

$$\frac{C_{\mathcal{F}}}{e^{h(\Sigma_{\mathcal{F}}^{[d]})(n+A_{\mathcal{F}})^d}} < \rho(w; \Sigma_{\mathcal{F}}^{[d]}) < \frac{D_{\mathcal{F}}}{e^{h(\Sigma_{\mathcal{F}}^{[d]})(n+B_{\mathcal{F}})^d}}.$$

Moreover, for any $w \in \Sigma_{\mathcal{F}}^{[d]}$, $\lim_{n\to\infty} \rho(w|_{[1,n]^d}; \Sigma_{\mathcal{F}}^{[d]}) = 0$.

Proof. Theorem 2.4 gives the existence of the escape rate

$$\rho(w; \Sigma_{\mathcal{F}}^{[d]}) = h(\Sigma_{\mathcal{F}}^{[d]}) - h(\Sigma_{\mathcal{F}\cup w}^{[d]}),$$

and the proof is thus complete by applying Theorem 4.1. ∎

Remark 4.3. We remark that the continuity of the case $d = 1$ is obtained by combining [22, Theorem 3.1] and Theorem 4.6.

4.2. 2-multiplicative integer systems on $\mathbb{N}^d$

Theorem 4.4. *If* $\Sigma_{\mathcal{F}}$ *is* n*-step SFT and* $w \in \{0,1\}^n$ *is any pattern which appears as a subpattern of some point of* $\Sigma_{\mathcal{F}}$, *then*

$$\begin{aligned} K - {}&\frac{(q_1 \cdots q_d - 1)\log(C_2)}{(q_1 \cdots q_d)^{n+1}} \\ &\le \underline{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \le \hat{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\ &\le K + \frac{(q_1 \cdots q_d - 1)[(n-1)\log 2 - \log(C_1\alpha)] + \log 2}{(q_1 \cdots q_d)^{n+1}}, \end{aligned}$$

where

$$K = h(\Sigma_{\mathcal{F}}^{(\mathbf{q})}) - h(\Sigma_{\mathcal{F}\cup w}^{(\mathbf{q})}).$$

In particular, for $w \in \Sigma_{\mathcal{F}}$,

$$\lim_{n\to\infty} \rho(w|_{[1,n]}, \Sigma_{\mathcal{F}}^{(\mathbf{q})}) = K.$$

Proof. Our goal is to estimate α, C_1, C_2 in Theorem 3.2. For simplicity, we assume that $\Sigma_{\mathcal{F}}$ is a 2-step SFT with the transition matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and let λ_A be the largest eigenvalue of A with the corresponding eigenvector $v = (v_1, v_2)^t$ such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This implies

$$\begin{cases} a_{11}v_1 + a_{12}v_2 = \lambda_A v_1, \\ a_{21}v_1 + a_{22}v_2 = \lambda_A v_2. \end{cases}$$

For 2-block representation of A, say $A^{[2]}$, is defined by

$$A^{[2]} = \begin{bmatrix} a_{11}a_{11} & a_{11}a_{12} & 0 & 0 \\ 0 & 0 & a_{12}a_{21} & a_{12}a_{22} \\ a_{21}a_{11} & a_{21}a_{12} & 0 & 0 \\ 0 & 0 & a_{22}a_{21} & a_{22}a_{22} \end{bmatrix}.$$

It is easy to see that the vector $v^{[2]} = (v_1, v_2, v_1, v_2)^t$ is an eigenvector corresponding to the eigenvalue λ_A (after deleting the zero columns and rows of $A^{[2]}$). Similarly, the n-block representation of A, say $A^{[n]}$, has λ_A as the largest eigenvalue with corresponding eigenvector

$$v^{[n]} = [(\otimes^{n-1}(1,1)) \otimes (v_1, v_2)]^t$$

(after deleting the zero columns and rows of $A^{[n]}$).

Let

$$\alpha = \frac{\min\{v_1^2, v_2^2\}}{v_1^2 + v_2^2} \quad \text{and} \quad \alpha_n = \min_{i,j} v_i^{[n]} v_j^{[n]},$$

then we have

$$\alpha \le 2^{n-1}\alpha_n. \tag{4.1}$$

If A is primitive, then $A^{[n]}$ is primitive (after deleting the zero columns and rows) for all $n \ge 2$. Then the Perron–Frobenius theorem provides that for $m \ge 1$,

$$C_1 \lambda_A^{m+n} \le |(A^{[n]})^m| = |A^{n+m}| \le C_2 \lambda_A^{m+n}, \tag{4.2}$$

where C_1 and C_2 are defined in (3.1).

Applying (4.2), we have

$$
\begin{aligned}
&-\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=1}^{\infty}\frac{\log(C_2\frac{|(B^{[n]})^{i-1}|}{|(A^{[n]})^{i-1}|})}{(q_1\cdots q_d)^{n+i}}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le\hat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le -\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=1}^{\infty}\frac{\log(\alpha_n C_1\frac{|(B^{[n]})^{i-1}|}{|(A^{[n]})^{i-1}|})}{(q_1\cdots q_d)^{n+i}}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&K-\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=1}^{\infty}\frac{\log(C_2)}{(q_1\cdots q_d)^{n+i}}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le\hat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le K-\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=1}^{\infty}\frac{\log(\alpha_n C_1)}{(q_1\cdots q_d)^{n+i}},
\end{aligned}
$$

where $K=h(\Sigma_{\mathcal{F}}^{(\mathbf{q})})-h(\Sigma_{\mathcal{F}\cup w}^{(\mathbf{q})})$.

Then by (4.1), we have

$$
\begin{aligned}
&K-\frac{(q_1\cdots q_d-1)\log(C_2)}{(q_1\cdots q_d)^{n+1}}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le\hat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\\
&\quad\le K+\frac{(q_1\cdots q_d-1)[(n-1)\log 2-\log(C_1\alpha)]+\log 2}{(q_1\cdots q_d)^{n+1}}.
\end{aligned}
$$

The proof is thus complete. ∎

Remark 4.5. Since $\rho(\mathcal{F}_1;\Sigma_{\mathcal{F}})=-\lim_{n\to\infty}\frac{1}{n}\log\mu(B_n(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}))$ and by [22, Theorem 3.1], $\rho(\mathcal{F}_1;\Sigma_{\mathcal{F}})=h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})$, we can see that there is a sequence of positive real numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ such that

$$
\begin{aligned}
n(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})-\varepsilon_n)&\le-\log\mu(B_n(X_{\Sigma_{\mathcal{F}\cup\mathcal{F}_1}}))\\
&\le n(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})+\varepsilon_n)
\end{aligned}
$$

for all $n\in\mathbb{N}$. Then we have

$$
\begin{aligned}
&\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{n=1}^{\infty}\frac{n(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})-\varepsilon_n)}{(q_1\cdots q_d)^n}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le\hat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\\
&\quad\le\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{n=1}^{\infty}\frac{n(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1})+\varepsilon_n)}{(q_1\cdots q_d)^n}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&(q_1\cdots q_d-1)(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}))-\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{n=1}^{\infty}\frac{n\varepsilon_n}{(q_1\cdots q_d)^n}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le \widehat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\\
&\quad\le (q_1\cdots q_d-1)(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup\mathcal{F}_1}))+\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{n=1}^{\infty}\frac{n\varepsilon_n}{(q_1\cdots q_d)^n}.
\end{aligned}
$$

We need the following result [35, Theorem 1.1].

Theorem 4.6. *For any irreducible $\mathbb{N}$ SFT X on an alphabet $\mathcal{A}$ with positive entropy, there exist constants $C_X, D_X > 0$, and $N_X \in \mathbb{N}$ such that for any $n > N_X$ and any pattern $w \in \mathcal{A}^n$ which appears as a subpattern of some point of X,*

$$
\frac{C_X}{e^{h(X)n}} < h(X)-h(X_w) < \frac{D_X}{e^{h(X)n}}.
$$

Then we have the continuity of the escape rate of 2-MISs on $\mathbb{N}^d$.

Theorem 4.7. *If $\Sigma_{\mathcal{F}}$ is irreducible SFT on $\mathbb{N}$ with positive entropy, there exist constants $C_{\mathcal{F}}, D_{\mathcal{F}} > 0$, and $N_{\mathcal{F}} \in \mathbb{N}$ such that for any $n > N_{\mathcal{F}}$ and any pattern $w \in \mathcal{A}^n$ which appears as a subpattern of some point of $\Sigma_{\mathcal{F}}$,*

$$
\begin{aligned}
\frac{(q_1\cdots q_d-1)C_{\mathcal{F}}}{e^{h(\Sigma_{\mathcal{F}})n}}-C_n &\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le \widehat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\\
&\le \frac{(q_1\cdots q_d-1)D_{\mathcal{F}}}{e^{h(\Sigma_{\mathcal{F}})n}}+C_n,
\end{aligned}
$$

where

$$
C_n = C\frac{(q_1\cdots q_d-1)n+1}{(q_1\cdots q_d)^n}.
$$

Moreover, for any $w \in \Sigma_{\mathcal{F}}$,

$$
\lim_{n\to\infty}\rho(w|_{[1,n]};\Sigma_{\mathcal{F}}^{(\mathbf{q})}) = 0.
$$

Proof. By Remark 4.5, we have

$$
\begin{aligned}
&(q_1\cdots q_d-1)(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup w}))-\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=n}^{\infty}\frac{i\varepsilon_i}{(q_1\cdots q_d)^i}\\
&\quad\le \underline{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\le \widehat{\rho}(w;\Sigma_{\mathcal{F}}^{(\mathbf{q})})\\
&\quad\le (q_1\cdots q_d-1)(h(\Sigma_{\mathcal{F}})-h(\Sigma_{\mathcal{F}\cup w}))+\frac{(q_1\cdots q_d-1)^2}{q_1\cdots q_d}\sum_{i=n}^{\infty}\frac{i\varepsilon_i}{(q_1\cdots q_d)^i}.
\end{aligned}
$$

Since $\varepsilon_i \to 0$ as $i \to \infty$, there is a constant C such that $|\varepsilon_i| \leq C$ for all $n \geq 1$. This implies

$$
\begin{aligned}
&(q_1 \cdots q_d - 1)(h(\Sigma_{\mathcal{F}}) - h(\Sigma_{\mathcal{F}\cup w})) - \frac{C(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{i=n}^{\infty} \frac{i}{(q_1 \cdots q_d)^i} \\
&\quad \leq \underline{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \leq \hat{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\
&\quad \leq (q_1 \cdots q_d - 1)(h(\Sigma_{\mathcal{F}}) - h(\Sigma_{\mathcal{F}\cup w})) + \frac{C(q_1 \cdots q_d - 1)^2}{q_1 \cdots q_d} \sum_{i=n}^{\infty} \frac{i}{(q_1 \cdots q_d)^i}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(q_1 \cdots q_d - 1)(h(\Sigma_{\mathcal{F}}) - h(\Sigma_{\mathcal{F}\cup w})) - C\frac{(q_1 \cdots q_d - 1)n + 1}{(q_1 \cdots q_d)^n} \\
&\quad \leq \underline{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \leq \hat{\rho}(w; \Sigma_{\mathcal{F}}^{(\mathbf{q})}) \\
&\quad \leq (q_1 \cdots q_d - 1)(h(\Sigma_{\mathcal{F}}) - h(\Sigma_{\mathcal{F}\cup w})) + C\frac{(q_1 \cdots q_d - 1)n + 1}{(q_1 \cdots q_d)^n}.
\end{aligned}
$$

The proof is thus completed by Theorem 4.6. ■

Acknowledgments. We would like to sincerely thank the anonymous referee for providing helpful suggestions for the first draft of this article. These improve the readability of the paper.

Funding. Jung-Chao Ban is partially supported by the Ministry of Science and Technology, ROC (Contract MOST 111-2115-M-004-005-MY3). Wen-Guei Hu is partially supported by the National Natural Science Foundation of China (Grant 12271381). Guan-Yu Lai is partially supported by the Ministry of Science and Technology, ROC (Contract MOST 111-2811-M-004-002-MY2).

References

[1] V. S. Afraimovich and L. A. Bunimovich, Which hole is leaking the most: a topological approach to study open systems. *Nonlinearity* **23** (2010), no. 3, 643–656 Zbl 1186.37015 MR 2593912

[2] H. Attarchi and L. A. Bunimovich, Why escape is faster than expected. *J. Phys. A* **53** (2020), no. 43, article no 435002 Zbl 07645584 MR 4177063

[3] W. Bahsoun, C. Bose, and G. Froyland, Open dynamical systems: Ergodic theory, probabilistic methods and applications. 2012, https://www.birs.ca/workshops/2012/12w5050/report12w5050.pdf, visited on 18 December 2023

[4] J.-C. Ban, W.-G. Hu, and G.-Y. Lai, On the entropy of multidimensional multiplicative integer subshifts. *J. Stat. Phys.* **182** (2021), no. 2, article no. 31 MR 4209079

[5] J.-C. Ban, W.-G. Hu, and G.-Y. Lai, Large deviation principle of multidimensional multiple averages on $\mathbb{N}^d$. *Indag. Math. (N.S.)* **33** (2022), no. 2, 450–471 Zbl 1497.37010 MR 4383121

[6] J.-C. Ban, W.-G. Hu, and G.-Y. Lai, Hausdorff dimension of multidimensional multiplicative subshifts. *Ergodic Theory Dynam. Systems* (2023)

[7] J.-C. Ban, W.-G. Hu, and S.-S. Lin, Pattern generation problems arising in multiplicative integer systems. *Ergodic Theory Dynam. Systems* **39** (2019), no. 5, 1234–1260 Zbl 1512.37008 MR 3928614

[8] J.-C. Ban, W.-G. Hu, S.-S. Lin, and Y.-H. Lin, Zeta functions for two-dimensional shifts of finite type. *Mem. Amer. Math. Soc.* **221** (2013), no. 1037, vi+60 pp. Zbl 1329.37015 MR 3025123

[9] J.-C. Ban, W.-G. Hu, S.-S. Lin, and Y.-H. Lin, Verification of mixing properties in two-dimensional shifts of finite type. *J. Math. Phys.* **62** (2021), no. 7, article no. 072703 Zbl 1476.37019 MR 4281219

[10] J.-C. Ban and S.-S. Lin, Patterns generation and transition matrices in multi-dimensional lattice models. *Discrete Contin. Dyn. Syst.* **13** (2005), no. 3, 637–658 Zbl 1090.37054 MR 2152335

[11] R. J. Baxter, Solvable models in statistical mechanics: From Ising to chiral Potts. In *Yang–Baxter systems, nonlinear models and their applications*, pp. 1–13, World Scientific, Singapore, 2000

[12] L. A. Bunimovich and A. Yurchenko, Where to place a hole to achieve a maximal escape rate. *Israel J. Math.* **182** (2011), 229–252 Zbl 1236.37004 MR 2783972

[13] G. Carinci, J.-R. Chazottes, C. Giardinà, and F. Redig, Nonconventional averages along arithmetic progressions and lattice spin systems. *Indag. Math. (N.S.)* **23** (2012), no. 3, 589–602 Zbl 1250.82006 MR 2948646

[14] J.-R. Chazottes and F. Redig, Thermodynamic formalism and large deviations for multiplication-invariant potentials on lattice spin systems. *Electron. J. Probab.* **19** (2014), article no. 39 Zbl 1290.82004 MR 3194738

[15] M. F. Demers and L.-S. Young, Escape rates and conditionally invariant measures. *Nonlinearity* **19** (2006), no. 2, 377–397 Zbl 1134.37322 MR 2199394

[16] A.-H. Fan, Some aspects of multifractal analysis. In *Geometry and analysis of fractals*, pp. 115–145, Springer Proc. Math. Stat. 88, Springer, Heidelberg, 2014 Zbl 1371.37039 MR 3276001

[17] A.-H. Fan, L. Liao, and J.-H. Ma, Level sets of multiple ergodic averages. *Monatsh. Math.* **168** (2012), no. 1, 17–26 Zbl 1263.37041 MR 2971737

[18] A.-H. Fan, J. Schmeling, and M. Wu, Multifractal analysis of some multiple ergodic averages. *Adv. Math.* **295** (2016), 271–333 Zbl 1358.37016 MR 3488037

[19] A. Ferguson, T. Jordan, and M. Rams, Dimension of self-affine sets with holes. *Ann. Fenn. Math.* **40** (2015), no. 1, 63–88 Zbl 1352.37066 MR 3310073

[20] A. Ferguson and M. Pollicott, Escape rates for Gibbs measures. *Ergodic Theory Dynam. Systems* **32** (2012), no. 3, 961–988 Zbl 1263.37004 MR 2995652

[21] G. Froyland and O. Stancevic, Escape rates and Perron–Frobenius operators: Open and closed dynamical systems. *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010), no. 2, 457–472 Zbl 1213.37012 MR 2660868

[22] C. Haritha and N. Agarwal, Product of expansive Markov maps with hole. *Discrete Contin. Dyn. Syst.* **39** (2019), no. 10, 5743–5774 Zbl 1436.37048 MR 4027010

[23] G. Keller and C. Liverani, Rare events, escape rates and quasistationarity: Some exact formulae. *J. Stat. Phys.* **135** (2009), no. 3, 519–534 Zbl 1179.37010 MR 2535206

[24] R. Kenyon, Y. Peres, and B. Solomyak, Hausdorff dimension for fractals invariant under multiplicative integers. *Ergodic Theory Dynam. Systems* **32** (2012), no. 5, 1567–1584 Zbl 1277.37023 MR 2974210

[25] Y. Kifer, Nonconventional limit theorems. *Probab. Theory Related Fields* **148** (2010), no. 1–2, 71–106 Zbl 1205.60047 MR 2653222

[26] L. Liao and M. Rams, Multifractal analysis of some multiple ergodic averages for the systems with non-constant Lyapunov exponents. *Real Anal. Exchange* **39** (2014), no. 1, 1–14 Zbl 1323.28015 MR 3261895

[27] L. Liao and M. Rams, Normal sequences with given limits of multiple ergodic averages. *Publ. Mat.* **65** (2021), no. 1, 271–290 Zbl 1480.37012 MR 4185833

[28] E. H. Lieb, Exact solution of the problem of the entropy of two-dimensional ice. *Phys. Rev. Lett.* **18** (1967), no. 17, 692–694

[29] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*. Cambridge University Press, Cambridge, 1995 Zbl 1106.37301 MR 1369092

[30] E. Louidor, B. Marcus, and R. Pavlov, Independence entropy of $\mathbb{Z}^d$-shift spaces. *Acta Appl. Math.* **126** (2013), no. 1, 297–317 Zbl 1329.37021 MR 3077954

[31] B. Marcus and R. Pavlov, Approximating entropy for a class of $\mathbb{Z}^2$ Markov random fields and pressure for a class of functions on $\mathbb{Z}^2$ shifts of finite type. *Ergodic Theory Dynam. Systems* **33** (2013), no. 1, 186–220 Zbl 1276.37010 MR 3009110

[32] B. Marcus and R. Pavlov, Computing bounds for entropy of stationary $\mathbb{Z}^d$ Markov random fields. *SIAM J. Discrete Math.* **27** (2013), no. 3, 1544–1558 Zbl 1315.37023 MR 3103245

[33] N. G. Markley and M. E. Paul, Maximal measures and entropy for $\mathbb{Z}^\nu$ subshifts of finite type. In *Classical mechanics and dynamical systems (Medford, Mass., 1979)*, pp. 135–157, Lect. Notes Pure Appl. Math. 70, Dekker, New York, 1981 Zbl 0475.58010 MR 640123

[34] L. Onsager, Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)* **65** (1944), no. 3–4, 117–149 Zbl 0060.46001 MR 10315

[35] R. Pavlov, Perturbations of multidimensional shifts of finite type. *Ergodic Theory Dynam. Systems* **31** (2011), no. 2, 483–526 Zbl 1234.37016 MR 2776386

[36] R. Pavlov, Approximating the hard square entropy constant with probabilistic methods. *Ann. Probab.* **40** (2012), no. 6, 2362–2399 Zbl 1418.37032 MR 3050506

[37] Y. Peres, J. Schmeling, S. Seuret, and B. Solomyak, Dimensions of some fractals defined via the semigroup generated by 2 and 3. *Israel J. Math.* **199** (2014), no. 2, 687–709 Zbl 1295.28012 MR 3219554

[38] Y. Peres and B. Solomyak, Dimension spectrum for a nonconventional ergodic average. *Real Anal. Exchange* **37** (2012), no. 2, 375–387 Zbl 1287.37015 MR 3080599

[39] G. Pianigiani and J. A. Yorke, Expanding maps on sets which are almost invariant. Decay and chaos. *Trans. Amer. Math. Soc.* **252** (1979), 351–366 Zbl 0417.28010 MR 534126

[40] L. A. Pierce, II, *Computing entropy for $\mathbb{Z}^2$-actions*. Ph.D. thesis, 2008, Oregon State University MR 2712112

Communicated by Adrian Tanasă

Received 23 November 2022.

Jung-Chao Ban
Department of Mathematical Sciences, National Chengchi University, No. 64, Sec. 2, ZhiNan Rd., 116302 Taipei; Mathematical Division, National Center for Theoretical Science, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., 10617 Taipei, Taiwan; jcban@nccu.edu.tw

Wen-Guei Hu
College of Mathematics, Sichuan University, No. 24, South Section 1, Yihuan Road, 610064 Chengdu, P. R. China; wghu@scu.edu.cn

Guan-Yu Lai
Department of Mathematical Sciences, National Chengchi University, No. 64, Sec. 2, ZhiNan Rd., 116302 Taipei, Taiwan; guanyu.am04g@g2.nctu.edu.tw