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Note on the density of ISE and a related diffusion

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Abstract. The integrated super-Brownian excursion (ISE) is the occupation measure of the spatial component of the head of the Brownian snake with lifetime process the normalized Brownian excursion. It is a random probability measure on $\mathbb{R}$, and it is known to describe the continuum limit of the distribution of labels in various models of random discrete labelled trees. We show that f_{ISE}, its (random) density, has almost surely a derivative f'_{ISE} which is continuous and $(\frac{1}{2}-\varepsilon)$-Hölder for any $\varepsilon > 0$ but for no $\varepsilon < 0$ (proving a conjecture of Bousquet-Mélou and Janson). We conjecture that f_{ISE} can be represented as a second-order diffusion of the form

$$df'_{\text{ISE}}(t) = 2\sqrt{f_{\text{ISE}}(t)}\,dB_t + g\left(f'_{\text{ISE}}(t), f_{\text{ISE}}(t), \int_{-\infty}^{t} f_{\text{ISE}}(s)\,ds\right)dt,$$

for some continuous function g, for $t > 0$, and we give a number of remarks and questions in that direction. The proof of regularity is based on a moment estimate coming from a discrete model of trees, while the heuristic of the diffusion comes from an analogous statement in the discrete setting, which is a reformulation of explicit product formulas of Bousquet-Mélou and the first author (2012).

1. Introduction

1.1. The integrated super-Brownian excursion

In this note, we are interested in a random probability measure called the integrated super-Brownian excursion, or ISE. We give here its most intrinsic definition via the Brownian snake. Our combinatorially inclined readers may prefer to think about it in terms of continuum limits of random trees (for this, see Proposition 2.2 below, which can be taken as a definition).

Recall that a Brownian snake $(W, c) = ((W_s, c_s), s \in [0,1])$ with lifetime process $c = 2\mathsf{e}$, where e is the normalized Brownian excursion, is a family of standard Brownian motions indexed by a continuum random tree with contour process c, namely:

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- for each $s \in [0,1]$, the spatial component W_s is a standard Brownian motion indexed by $[0, c_s]$,
- for $0 \le s \le t \le 1$, W_s and W_t coincide on $[0, T_{s,t}]$ for $T_{s,t} = \min\{c_u : s \le u \le t\}$,
- the two Brownian motions $(W_s(T_{s,t} + u) - W_s(T_{s,t}), 0 \le u \le c_s - T)$ and $(W_t(T_{s,t} + u) - W_t(T_{s,t}), 0 \le u \le c_t - T_{s,t})$ are independent.

The lifetime process $2\mathsf{e}$ encodes a random tree $T_{2\mathsf{e}}$, usually called Aldous' continuum random tree, or CRT. The Brownian snake can be viewed as a Brownian motion indexed by $T_{2\mathsf{e}}$. Each real number s in $[0,1]$ encodes a node in the tree, at depth $2\mathsf{e}(s)$, and the Brownian motion W_s is the spatial component function: for $t \in [0, 2\mathsf{e}(s)]$, $W_s(t)$ is simply the spatial position of the node of the branch from the root to the node s, which is at depth t.

The head of the (spatial component of the) Brownian snake is the process $H = (H_s, s \in [0,1])$ of terminal points of the Brownian snake, and it is defined on $[0,1]$ by $H_s := W_s(c_s)$. Conditional on c, it is a centred Gaussian process with covariance function $\mathsf{Cov}(H_s, H_t) = \min\{c_u : u \in [s \wedge t, s \vee t]\}$ for $0 \le s, t \le 1$.

The *integrated super-Brownian excursion*, or *ISE*, is the random probability measure μ_{ISE} defined as the occupation measure of H for all continuous functions g: $\mathbb{R} \to \mathbb{R}$ with compact support

$$\int_{-\infty}^{+\infty} g \, d\mu_{\text{ISE}} = \int_0^1 g(H_s)\, ds.$$

We refer to Le Gall [17], or [14, 21] for more information on the subject.

It is known that μ_{ISE} has almost surely (a.s.) a (random) compact support: this is a consequence of the a.s. continuity of H (see [3, 8] for the distribution of the support). Moreover, as shown by Bousquet-Mélou and Janson [5], μ_{ISE} is a.s. absolutely continuous with respect to the Lebesgue measure and its random density has a continuous version denoted by f_{ISE}: the study of the random process f_{ISE} is the aim of this paper.

1.2. Main contributions of this note

The main point of this note is to convey the intuition that not only μ_{ISE} has a continuous density f_{ISE}, but this density has a continuous derivative f'_{ISE}, and we expect f'_{ISE} to behave, in some sense, as a diffusion. In particular, we expect the 3D-process made by f_{ISE}, its derivative f'_{ISE}, and its cumulative integral $t \mapsto \int_{-\infty}^{t} f_{\text{ISE}}(s)\, ds$, to be Markovian – which as far as we know has never been suggested before.

Technically, our main contributions are the following:

- We prove (Theorem 2.3 in Section 2.2) that indeed f'_{ISE} exists and is continuous and even $(\frac{1}{2} - \varepsilon)$-Hölder for any $\varepsilon > 0$. However, this is not true for any $\varepsilon < 0$, and, in particular, f_{ISE} has no second derivative. This proves the main conjecture in the paper by Bousquet-Mélou and Janson [5, Conjecture 2.3].

- We introduce a 3D-valued process, noted ζ, made by f_{ISE}, its derivative, and its integral. The main message of this note is that this 3D-process opens the way to an understanding of *ISE* in a Markovian way.
- We observe (Proposition 2.6 in Section 2.4) that, in the discrete model of uniform random binary trees, whose convergence to ISE is known, the discrete analogue of the triple process ζ can be represented as a Markov process, when conditioned to its values at the *two* boundaries of a discrete interval. This is a direct consequence, which seems to have been unnoticed before, of a theorem of [4].
- We prove that this discrete Markov process indeed converges to a diffusion (Proposition 2.7 in Section 2.5), when conditioned to its value at only *one* boundary of a given interval, and properly tamed near 0.
- We conjecture (Section 2.6) that f'_{ISE} (more precisely, ζ) can be represented by a diffusion, with or without conditioning at boundaries. We give a number of questions related to this.

We hope to stimulate efforts by researchers best acquainted with the subject, with the hope that someone can give a rigorous meaning (or several) to this statement, and prove it.

2. Convergence results and related questions

2.1. Discrete approach to f_{ISE}

From the discrete perspective, the ISE is the limit in distribution of several natural random probability measures, in particular, the distribution profile of distances to a random vertex in a random planar map [7], and most directly, of the distribution of labels in various models of random spatial trees [2, 9, 14, 20, 21]. The model of "spatial trees" involved are discrete branching random walks (sometimes viewed as discrete snakes): some random values (called spatial displacements) are attached to the edges of a discrete random tree, and the label/abscissa attached to a node u, is obtained by summing the values on the path going from the root of the tree to u. They are discrete analogues to the Brownian snake (indexed by the CRT).

Hence, to study ISE, a possible way consists in choosing a simple discrete model having the Brownian snake as a limit (or simply, having a spatial occupation measure converging to ISE).

For the purpose of this note, we will use the fact that the ISE is the weak limit of a discrete model for which some useful formulas have been obtained [4] (additional details are given in the proof of Proposition 2.2).

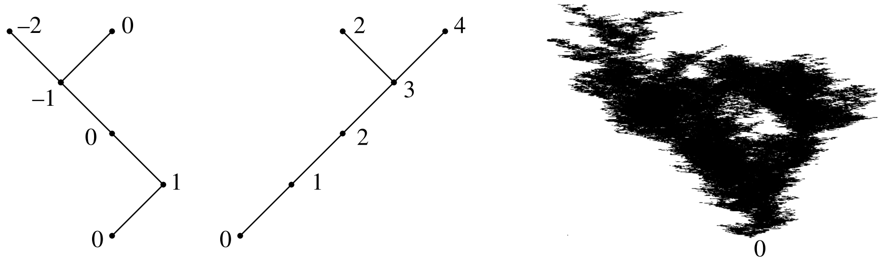

Figure 1. Left: Two different binary trees with $n = 6$ vertices. The abscissa of each vertex is displayed next to it. These two trees have respective vertical profiles $(1, 1; 3, 1)$ and $(; 1, 1, 2, 1, 1)$. Right: A uniformly random binary tree T of size n with $n \approx 10^5$; the projection of the uniform measure of points on the horizontal axis is the measure $\mu_T(n)$ – and, properly normalized, it is a good approximation of μ_{ISE}.

Binary trees. For us, a binary tree is a rooted plane tree, in which each vertex has a (possibly empty) *left* subtree and a (possibly empty) *right* subtree. See Figure 1.

Remark 2.1. Some authors would call these objects *incomplete* binary trees. Replacing each empty subtree by a single leaf gives a *complete* binary tree (a plane tree with only vertices of arity 2 and 0).

The number of binary trees with n vertices is the n-th Catalan number,

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}.$$

We define the *abscissa* [3, 4] of a vertex v as the number of right steps minus the number of left steps on the path from the root to v. The uniform distribution on the set $\mathcal{B}_n$ of binary trees with n vertices is denoted by $\mathbb{U}_n$. A binary tree, equipped with the labelling of its vertices by their abscissas, gives a model of a spatial tree whose distribution of labels, as we will see, converges after an appropriate rescaling to ISE.

The *vertical profile* of the tree T is the tuple

$$M(T) = (M_{\ell(T)}(T), \dots, M_{-1}(T); M_0(T), \dots, M_{r(T)}(T)),$$

where $\ell = \ell(T)$, $r = r(T)$ are the smallest and largest abscissa (ℓ and r stand for left and right), respectively, and where for each integer i, $M_i(T)$ is the number of vertices of T of abscissa i. For probabilistic applications, it is most natural to turn this object into a probability measure. Introducing a suitable normalization, we thus define the probability measure

$$\mu_T(n) := \frac{1}{n} \sum_{k=\ell(T)}^{r(T)} M_k(T)\, \mathsf{Dirac}_{k/(2n)^{1/4}},$$

where Dirac_x is the Dirac mass at x. Note that this carries all the information of the profile, and in fact, it is the measure $\mu_T(n)$ that we will refer to as *the profile of* T. In addition to $M(T)$, we introduce two processes $\Delta(T) := (\Delta_i(T), i \in \mathbb{Z})$ and $S(T) := (S_i(T), i \in \mathbb{Z})$, which can be thought of as the discrete derivative and discrete integral of $M(T)$, respectively,

$$\Delta_i(T) = M_i(T) - M_{i-1}(T), \quad S_i(T) = \sum_{j \leq i} M_j(T), \quad i \in \mathbb{Z}.$$

Finally, we define the triple

$$Z_i(T) = (\Delta_i(T), M_i(T), S_i(T)), \quad i \in \mathbb{Z}. \tag{2.1}$$

Note that each of $\Delta(T)$, $M(T)$ and $S(T)$ determines

$$Z(T) := (\Delta(T), M(T), S(T)),$$

and observe also that the knowledge of (Δ_i, M_i) is equivalent to that of (M_{i-1}, M_i).

The following proposition, essentially due to Bousquet-Mélou and Janson [5], states that μ_{ISE} is nothing but the limit in distribution of the profile of a random element of $\mathbb{U}_n$.

Proposition 2.2. *If* $\mathbf{T}_n$ *is picked according to* $\mathbb{U}_n$*, then* $\mu_{\mathbf{T}_n}(n)$ *converges in distribution to* μ_{ISE}*, for the topology of weak convergence of probability measures. Moreover,* $(\frac{1}{n^{3/4}} M_{n^{1/4}x}(\mathbf{T}_n), x \in \mathbb{R})$ *converges in distribution to* $(\gamma f_{\mathrm{ISE}}(\gamma x), x \in \mathbb{R})$ *in* C_0 *(for* $\gamma = 2^{-1/4}$*), the space of continuous functions with 0 limit at* $\pm\infty$*, equipped with the topology of uniform convergence.*

Moreover,

$$(\ell(\mathbf{T}_n), r(\mathbf{T}_n))n^{-1/4} \xrightarrow[n]{(d)} (L, R), \tag{2.2}$$

for some non-trivial random variables (L, R) *such that, a.s.,* $L < 0 < R$ *(and* $[L, R]$ *is distributed as the a.s. finite support of* $(f_{\mathrm{ISE}}(\gamma x), x \in \mathbb{R})$*).*

In the proposition, $M(\mathbf{T}_n)$ is the continuous process which interpolates $(M_k(\mathbf{T}_n), k \in \mathbb{Z})$ linearly between integer points. The same notation will be used for all integer-indexed processes encountered in the paper.

This result implies that f_{ISE} is almost surely continuous (which can be stated also as follows: μ_{ISE} possesses a.s. a density g_{ISE} with respect to the Lebesgue measure on $\mathbb{R}$, and the process g_{ISE} admits a continuous version f_{ISE}). These authors also show that f_{ISE} has a derivative a.e. f'_{ISE}, and they conjecture that f'_{ISE} possesses a continuous version, but no second derivative (see Figure 2 for a simulation of the processes Δ, M, S of a large random tree, on which one can guess, among other things, the regularity of the limiting processes).

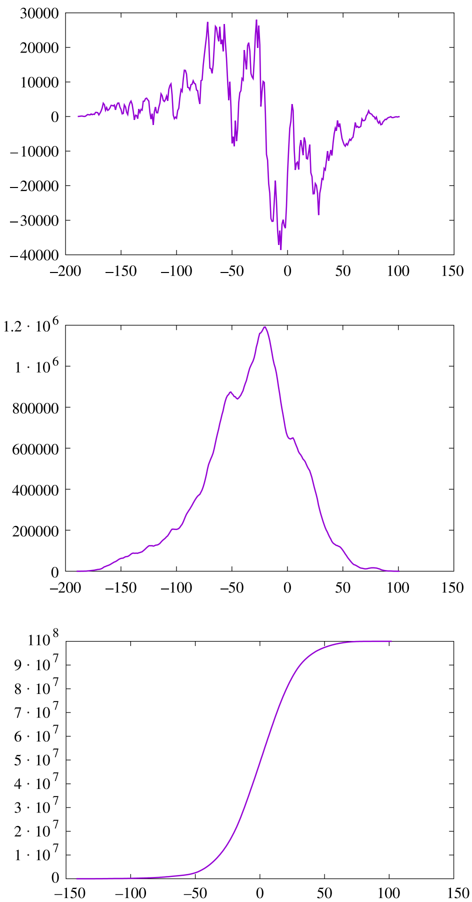

Figure 2. Uniform plane tree with 50 millions of vertices, spatial increments i.i.d. ± 1 with probability $1/2$. From top to bottom, representation of the process $\Delta(T)$, $M(T)$, $S(T)$. Up to a normalization, they are discrete approximations of the continuum processes $f'_{\mathrm{ISE}}(t)$, $f_{\mathrm{ISE}}(t)$, $\int_{-\infty}^{t} f_{\mathrm{ISE}}(s)\,ds$.

Proof. The result concerning the convergence of $(\frac{1}{n^{3/4}} M_{n^{1/4}x}(\mathbf{T}_n), x \in \mathbb{R})$ is due to Bousquet-Mélou and Janson [5, Theorem 3.1].

The first and last assertions of the proposition are byproducts of the proof of convergence of the so-called discrete snakes (see [14,21]). Let us give some details. A discrete snake is a sequence of trajectories associated with a branching random walk. This later is a random structure made by two levels of randomness: first, a Galton–Watson tree conditioned by its total population is picked (its offspring distribution has mean 1 and satisfies some moment conditions, notably, a finite variance). Each edge e of the tree is associated with a spatial displacement X_e, which is a centered random variable, with a finite moment of order > 4 (the X_e are independent, and independent of the tree). A label $\ell(u)$ is then associated to each node u of the tree: $\ell(u)$ is the sum of the variables X_e of each edge e on the shorter path from u to the root.

A branching random walk built on a tree with size $n+1$ (having n edges) is then encoded using two dependent processes (C_n, R_n): the contour process $C_n = (C_n(k), 0 \leq k \leq 2n)$ encodes the tree, and the label process

$$R_n = (R_n(k), 0 \leq k \leq 2n)$$

encodes the successive labels of nodes encountered when turning around the tree ($R_n(k)$ is the label visited at time k when doing the contour process). The pair $(C_n(2n.)/n^{1/2}, R_n(2n.)/n^{1/4})$ converges in $(C[0,1], \mathbb{R})^2$, the limit being the so-called contour process of the Brownian snake (up to some additional factors depending on the Galton–Watson offspring distribution variance and the variance of X_e). As a consequence, the analogue of (2.2) holds for branching random walks: the rescaled pair $(\min_k R_n(k), \max_k R_n(k))n^{-1/4}$ converges in distribution (up to some constant factor) to the support of μ_{ISE}, which is the occupation measure of the limit label process.

In our particular case, the spatial displacements are not i.i.d. but correspond to the following branching random walk: the offspring distribution is $p_0 = 1/4$, $p_1 = 1/2$, $p_2 = 1/4$. It has mean 1 and variance $\sigma^2 = 1/2$. In the case where a node u has a unique child v, the spatial displacement associated with the edge (u, v) is $+1$ or -1 with probability $1/2$, if there are two children, the spatial displacement is $(-1, +1)$, that is, -1 for the left child, $+1$ for the right one. However, one sees that in the two children case, if one replaces the spatial displacement by $(-1, +1)$ with probability $1/2$ and $(+1, -1)$ with probability $1/2$, one preserves the distribution of $(\min R_n, \max R_n)$ (and much more than that), since this amounts to exchanging the left and right subtrees with probability $1/2$, independently at each internal nodes (which preserves the law of this kind of trees): this trick is used in [19], for the case of binary trees. This modified model falls into those allowed in [14,21] (which allows dependence between the displacements of siblings, as long as they are centered), and can also be seen as globally centered snake (in the sense of [20]). This concludes the proof. ∎

To be complete, we mention that another proof of the tightness of (L_n, R_n) can be obtained by observing that our model of binary trees corresponds to the set of internal nodes of standard complete binary trees. For this model, the results of [19] imply also the convergence of (L_n, R_n) after rescaling to the range of the label process of the Brownian snake, from which the wanted result follows. Moreover, for the interested reader, we point out that Devroye and Janson [9] obtained analogues of Proposition 2.2 for general models of discrete snakes under the hypothesis that the spatial increments along edges are independent (which is not the case here).

2.2. Tightness and regularity

Let $\mathbf{T}_n$ be a random binary tree taken under distribution $\mathbb{U}_n$. For $k \in \mathbb{Z}$, we set $M_k^n = M_k(\mathbf{T}_n)$, $\ell_n = \ell(\mathbf{T}_n)$, $r_n = r(\mathbf{T}_n)$, $\Delta^n = \Delta(\mathbf{T}_n)$, and let $S^n = S(\mathbf{T}_n)$ be the discrete derivative and integral of this process. We introduce the three-dimensional process $\zeta^n = (\zeta^n(t), t \in \mathbb{R})$ defined by

$$\zeta^n(t) := \left(\frac{1}{n^{1/2}}\Delta^n_{n^{1/4}t}, \frac{1}{n^{3/4}}M^n_{n^{1/4}t}, \frac{1}{n}S^n_{n^{1/4}t}\right)_{t\in\mathbb{R}},$$

where we recall that our notation uses implicitly the linear interpolation between integer values of $n^{1/4}t$. The choice of normalization for the three coordinates will become clear below.

Theorem 2.3. *The sequence of processes (ζ^n) is tight in $\mathcal{C}_0(\mathbb{R})^2 \times C_{0,1}(\mathbb{R})$, where $C_0(\mathbb{R})$ is the space of continuous functions with limit 0 at $\pm\infty$ and $C_{0,1}$ the space of continuous function with limit 0 in $-\infty$ and 1 in $+\infty$ (equipped with the topology of uniform convergence). Moreover, we have the convergence*

$$\zeta^n \to \zeta,$$

where

$$\zeta(t) := \left(\gamma^2 f'_{\mathrm{ISE}}(\gamma t), \gamma f_{\mathrm{ISE}}(\gamma t), \int_{-\infty}^{t} \gamma f_{\mathrm{ISE}}(\gamma s)\, ds\right),$$

where f'_{ISE} is a continuous function, which is the derivative of f_{ISE}, (the convergence holds in distribution, for the topology of uniform convergence, and $\gamma = 2^{-1/4}$).

For any $\varepsilon > 0$, the function f'_{ISE} is a.s. $(\frac{1}{2} - \varepsilon)$-Hölder continuous. However, for any $\varepsilon > 0$, the function f'_{ISE} is a.s. $(\frac{1}{2} + \varepsilon)$-Hölder continuous almost nowhere inside the support of f_{ISE}. In particular, f'_{ISE} is a.s. differentiable almost nowhere inside the support of f_{ISE}.

The fact that f_{ISE} has a.s. a continuous derivative, but no second derivative, was conjectured in [5, Conjecture 2.3]. Note also that the theorem shows that f'_{ISE} vanishes at the boundary of the support of f_{ISE} (since it is continuous).

Remark 2.4 (About the topology of convergence). We know from Proposition 2.2 that the sequence $(\ell(\mathbf{T}_n), r(\mathbf{T}_n))n^{-1/4}$ is tight. On the other hand, the random process ζ^n is constant (equal to $(0,0,0)$) on the interval $(-\infty, n^{-1/4}\ell(\mathbf{T}_n)]$ and this process is also constant (equal to $(0,0,1)$) on the interval $[n^{-1/4}(r(\mathbf{T}_n)+2), +\infty)$. Thanks to these simple observations, the tightness of the sequence ζ^n in $C_0(R)^2 \times C_{0,1}(R)$ will easily follow if we can verify that, for every $K > 0$, the sequence $(\zeta^n(t))_{-K\le t\le K}$ is tight in $C([-K,K], R^3)$. To get the latter tightness property, we will rely on a moment criterion (cf. Lemma 3.1 below).

2.3. A translated version of the integrated super-Brownian excursion on $\mathbb{R}^+$

The fact that the CRT is invariant under uniform rerooting (see [1]) has various consequences. The rerooting of a CRT with contour process $c = 2\mathsf{e}$ at position $r \in [0, 1]$, is defined thanks to its contour process

$$\begin{aligned} c^{(r)}(x) = c(x + r \bmod 1) + c(r) \\ - 2 \min\{c(u) : u \in [r \wedge (x + r \bmod 1), (r \vee (x + r \bmod 1)]\}, \end{aligned}$$

which gives the distance of the node encoded by $x + r$ to the node encoded by r in the tree with contour process c. The rerooted snake is obtained by rerooting the underlying tree at the new root, by setting 0 as its new spatial position, and by keeping the spatial variation along branches. The head of the obtained Brownian snake satisfies

$$(H^{(r)}(x), x \in [0, 1]) = (H(x + r \bmod 1) - H(r), x \in [0, 1]),$$

and

$$H^{(r)} \overset{(d)}{=} H.$$

Since for a uniform variable $u \in [0, 1]$, independent from H, the distribution of $H(u)$ is given by the occupation measure of H (which is μ_{ISE}), one has

$$\mu_{\mathrm{ISE}} \overset{(d)}{=} \mu_{\mathrm{ISE}}(. - X),$$

where X has distribution μ_{ISE} (to be clear, the translation value X is taken under the random measure μ_{ISE} that it translates). These considerations allow one to understand that if one takes $L = \min \mathsf{Support}(\mu_{\mathrm{ISE}})$, then

$$\mu^+ := \mu_{\mathrm{ISE}}(. - L)$$

is a random measure on $[0, +\infty)$ which can be used to describe μ_{ISE}:

$$\mu_{\mathrm{ISE}} \overset{(d)}{=} \mu^+_{\mathrm{ISE}}(. - Y),$$

where, again, Y is taken under the random measure μ^+_{ISE}. Let us call μ^+_{ISE} the translated version of μ_{ISE}, and denote by

$$\zeta^+(t) = \left(\gamma^2 f^{+\prime}_{\mathrm{ISE}}(\gamma t), \gamma f^+_{\mathrm{ISE}}(\gamma t), \int_{-\infty}^t \gamma f^+_{\mathrm{ISE}}(\gamma s)\, ds\right)$$

the corresponding encoding processes. The process ζ^+ is, in nature, a bit simpler than ζ since "it starts" at a deterministic abscissa, when ζ has a bilateral random support.

2.4. A companion process and a discrete diffusion

Bousquet-Mélou and the first author [4] have given a complete description of the law of the vertical profile $M(\mathbf{T}_n)$ under $\mathbb{U}_n$.

Proposition 2.5 ([4, Theorem 1]). *Let $\ell \in \mathbb{Z}^-$, $r \in \mathbb{Z}^+$, $m_i \in \{1,2,\dots\}$ for any $i \in [\![\ell, r]\!]$ and*

$$\sum_{i=\ell}^{r} m_i = n \quad \text{and} \quad m_{\ell-1} = m_{r+1} = 0.$$

We have

$$\begin{aligned}\#\{T \in \mathcal{B}_n : M(T) = (m_\ell,\dots,m_{-1};m_0,\dots,m_r)\}\\ = \frac{m_0\binom{m_{-1}+m_1}{m_0-1}}{m_\ell m_r} \prod_{\substack{\ell\le i\le r\\ i\neq 0}} \binom{m_{i-1}+m_{i+1}-1}{m_i-1},\end{aligned} \tag{2.3}$$

where $\binom{a}{b} = 0$ if $b > a$.

Of course, for $\mathbf{T}_n$ taken under $\mathbb{U}_n$, $\mathbb{P}(M(\mathbf{T}_n) = (m_\ell,\dots,m_{-1};m_0,\dots,m_r))$ is proportional to the right-hand side of (2.3). Rewrite this right-hand side a bit differently using that $\sum m_i = n$. For a normalizing sequence $(\alpha_n, n \ge 0)$, we have

$$\begin{aligned}\mathbb{P}(M(\mathbf{T}_n) = (m_\ell,\dots,m_{-1};m_0,\dots,m_r))\\ = \frac{m_0\frac{m_1+m_{-1}}{m_1+m_{-1}-m_0+1}}{\alpha_n\, m_\ell m_r 2^{m_\ell+m_r}} \prod_{\ell\le i\le r} \frac{\binom{m_{i-1}+m_{i+1}-1}{m_i-1}}{2^{m_{i-1}+m_{i+1}}}.\end{aligned} \tag{2.4}$$

We now arrive at the main idea at the origin of this note: the product in the formula of the law of $M(\mathbf{T}_n)$ will lead us to observe that the process $M(T)$ can be (roughly) represented with the help of a Markov chain $M^\star$.

However, because the i-th factor of the product depends on the numbers m_{i+1} and m_{i-1}, to obtain a Markov chain representation we need to consider a three-dimensional process: this is the reason for the introduction (in (2.1)) of the process $Z(T)$. Moreover, this Markovian representation will hold only if we condition on the values of $Z(T)$ at the two boundaries of an interval. We prove below that the companion process $M^\star$ (or rather $Z^\star$) possesses a diffusive limit: this will give the intuition that it should be also the case for f_{ISE} (but this will not prove it, because of the difficulty of obtaining the same statement under a double conditioning).

In order to parse (2.4), recall that for any fixed positive k, the distribution defined by

$$p_k(n+k) = \binom{k-1+n}{k-1} 2^{-n-k}, \quad n \ge 0,$$

is the negative binomial distribution $\mathsf{BNEG}(k)$; this distribution is that of the sum of k i.i.d. geometric $1/2$ random variables $g^{(j)}$ (with support $\{1, 2, \dots, \}$, and then, mean 2):

$$p_k(n+k) = \mathsf{P}\left[\sum_{j=1}^{k} g^{(j)} = n+k\right] = \mathsf{P}\left[\sum_{j=1}^{k} (g^{(j)} - 2) = n-k\right].$$

Each factor in the product in the right-hand side of (2.4) can thus be reinterpreted:

$$\begin{aligned}\binom{(m_{i-1} + m_{i+1} - m_i) + m_i - 1}{m_i - 1} & 2^{-m_{i-1} - m_{i+1}} \\ &= \mathsf{P}\left(\sum_{j=1}^{m_i} (g^{(j)} - 2) = \delta_{i+1} - \delta_i\right),\end{aligned} \tag{2.5}$$

where $\delta_i = m_i - m_{i-1}$. Hence, if the prefactors in (2.4) were not there, then conditionally on $(M_k, k \le i)$, the increment $\Delta_{i+1} = M_{i+1} - M_i$ would have the same distribution as $\Delta_i + \sum_{k=1}^{M_i} (g^{(k)} - 2)$, and the process (Z_j) would be a simple Markov chain (and (S_j) would be a Markov chain of order 3).

To shed more light, we introduce the *companion process*, a time homogeneous Markov chain $(Z^\star_k, k \ge 0) = ((\Delta^\star_k, M^\star_k, S^\star_k), k \ge 0)$ taking its values in $\mathbb{Z}^3$ as follows: conditionally on $(\Delta^\star_i, M^\star_i, S^\star_i) = (\delta_i, m_i, s_i)$,

$$\begin{cases} \Delta^\star_{i+1} = \delta_i + \displaystyle\sum_{k=1}^{|m_i|} (g^{(k)} - 2), \\ M^\star_{i+1} = m_i + \Delta^\star_{i+1}, \\ S^\star_{i+1} = s_i + M^\star_{i+1}. \end{cases} \tag{2.6}$$

Moreover, if $M^*_k \le 0$, then the process $Z^\star$ is stopped at time k (meaning that $Z^\star_{k+t} = Z^\star_k$ for all $t \ge 0$). Of course, to fully specify the process, we should specify a starting time and value – and we will when needed.

The companion process $Z^\star$ and the tree-related process $Z(\mathbf{T}_n)$ are related as follows.

Proposition 2.6. *The distribution of the companion process $Z^\star$ coincides with $Z(\mathbf{T}_n)$ on intervals which do not straddle* 0, *when one fixes boundary conditions at the two extremities of the interval: Formally, take $\mathbf{T}_n$ under $\mathbb{U}_n$. Fix integers $0 < k_1 < k_2$ (or $k_1 < k_2 < 0$) and (δ_1, m_1, s_1), (δ_2, m_2, s_2) in $\mathbb{Z} \times \mathbb{Z}_{>0} \times \mathbb{Z}$. Then the laws of the vectors $(Z^\star_j, k_1 \le j \le k_2)$ and $(Z_j(\mathbf{T}_n), k_1 \le j \le k_2)$ conditioned to take the value (δ_i, m_i, s_i) at k_i for $i = 1, 2$, are equal.*

Observe that since $m_2 > 0$, this condition implies that under the conditional distribution, the second component of $Z^\star$ stays positive on $[k_1, k_2]$.

Proof. Take an element $[z'_j = (d'_j, m'_j, s'_j), k_1 \le j \le k_2]$ such that for all $k_1 < j < k_2$, $m'_j > 0$, $m'_j = m'_{j-1} + d'_j$, $s'_j = s'_{j-1} + m'_j$ (so that (s'_j) is increasing), and moreover at the boundary $z'_{k_j} = (\delta_j, m_j, s_j)$ for $j \in \{1,2\}$. Using (2.5), it is immediate to check that $\mathsf{P}(Z^\star_j = z'_j, k_1 \le j \le k_2 \mid Z^\star_{k_1} = z'_{k_1}, Z^\star_{k_2} = z'_{k_2})$ and $\mathsf{P}(Z_j(\mathbf{T}_n) = z'_j, k_1 \le j \le k_2 \mid Z_{k_1}(\mathbf{T}_n) = z'_{k_1}, Z_{k_2}(\mathbf{T}_n) = z'_{k_2})$ are proportional, and then, are equal. ■

Notice that the prefactor in (2.4) says something about the root position (the position of "0" in the interval $[\ell, r]$), as well as a kind of cost of the extremal values (M_ℓ, M_r). For simplicity, we have chosen in Proposition 2.6 to consider only intervals avoiding zero. This allows one to work with nicer formulas. Section 2.3, in which the translated version ζ^+ is introduced, suggests that the root position is not important, and can be thought to be close to the left support.

Note also that the rerooting invariance of the continuum model (Section 2.3 again) is not exactly present in discrete binary trees.

Finally, note that the prefactor in (2.4) and the global condition of positivity of $M(\mathbf{T}_n)$ on $[\![\ell, r]\!]$ make the global study of (2.4) quite difficult; with intervals avoiding zero, we avoid (part of) these difficulties.

2.5. Convergence to a diffusion for the companion process attached at the left boundary

In view of (2.6), and since the law of $\sum_{k=1}^{|m_i|}(g^{(k)} - 2)$ of centred i.i.d. variables should be well approximated by the centred normal distribution with variance $|2m_i|$. It can be expected that $Z^\star$ (started at time 0) will converge in distribution, after an appropriate rescaling, to a process $\zeta^*_t = (\delta^*_t, m^*_t, s^*_t)$ that is solution to the stochastic differential equation (SDE)

$$\delta^\star_t = \delta_0 + \int_0^t \sqrt{2m^\star_x}\, dW(x), \quad m^\star_t = m_0 + \int_0^t \delta^\star_x\, dx, \quad s^\star_t = s_0 + \int_0^t m^\star_x dx, \tag{2.7}$$

where $W(t)$ is a standard Brownian motion. Note that the dynamics of the process can be encapsulated in the unique SDE

$$d((m^*_t)') = \sqrt{2m^*_t}\, dW_t, \tag{2.8}$$

which is some "order-2" diffusion (note that the s-coordinate plays no direct role in the dynamics).

Since $x \mapsto \sqrt{x}$ is not Lipschitz (at 0) and $M^\star$ is stopped when it becomes negative, however, some precautions will be needed.

A stopped version of the companion process. The process $Z^\star$ is well defined for any initial distribution with support in $\mathbb{Z}^3$, however, we are only interested in its behaviour when $M^\star \ge 0$ (we will condition on that event). Moreover, the convergence result we are about to state needs for the diffusion coefficient to be positive: we choose to stop $Z^\star$ when a certain level $K > 0$ is hit by $M^\star$. For $K > 0$ fixed, denote

$$T_K = \inf\{t > 0 : M_t^\star < K\}.$$

We define $\mathsf{P}_{\geq K}$ the distribution of the Markov chain $\bar{Z}^{[K]}$ obtained from $Z^\star$ as follows:

$$\bar{Z}_t^{[K]} = Z_{t\wedge T_K}^\star.$$

The process $Z^\star = (Z_k^\star, k \geq 0)$ is time homogeneous. To describe its limit, let us fix a constant $T > 0$, and set for all $k \geq 0$, $n \geq 1$,

$$t_k^n = kn^{-1/4}, \quad N_n = \min\{k : t_k^n > T\} = \lfloor Tn^{1/4} \rfloor + 1.$$

Set $\xi^{n,\star} = (\delta^{n,\star}, m^{n,\star}, s^{n,\star})$ as the càdlàg process, constant on $[t_k^n, t_{k+1}^n)$, and satisfying

$$\xi^{n,\star}(t_k^n) = \begin{bmatrix} n^{-1/2}\Delta_k^\star \\ n^{-3/4}M_k^\star \\ n^{-1}S_k^\star \end{bmatrix}.$$

The process $(\xi^{n,\star}(t_k^n), k \geq 0)$ is a Markov chain.

As usual, denote by $D([0,T],\mathbb{R}^3)$ the set of càdlàg functions defined on $[0,T]$ taking their values in $\mathbb{R}^3$, equipped with the Skorokhod topology. We write $\overline{\xi}^n$ for the stopped version.

Theorem 2.7. *Let $(\delta_0, m_0, s_0) \in \mathbb{R} \times (0,+\infty) \times \mathbb{R}$, $T > 0$ fixed, and $(\delta_0^n, m_0^n, s_0^n)$ be a sequence in $\mathbb{Z}^3$ such that $m_0^n > 0$. If*

$$z_0^n := \left(\frac{\delta_0^n}{n^{1/2}}, \frac{m_0^n}{n^{3/4}}, \frac{s_0^n}{n}\right) \to z_0 := (\delta_0, m_0, s_0),$$

then for any $0 < \varepsilon < m_0$, there are a Brownian motion $(W(t), t \geq 0)$ and a random process $\xi^\star(.)$ non-anticipative with respect to $W(.)$ so that, under $\mathsf{P}_{\geq \varepsilon n^{3/4}}$, $(\overline{\xi}^n)$ starting at the initial position z_0^n, converges in distribution in $D([0,T],\mathbb{R}^3)$ to $\xi^\star$, the unique solution of the stochastic differential equation

$$\xi^\star(t) = \xi(0) + \int_0^t f(\xi^\star(s), s)\, ds + \int_0^t \sigma(\xi^\star(s), s)\, dW(s), \tag{2.9}$$

where $\xi^\star(0) = z_0$, stopped when its second entry hits ε and

$$f\left(\begin{bmatrix} d \\ m \\ s \end{bmatrix}, t\right) = \begin{bmatrix} 0 \\ d \\ m \end{bmatrix}, \quad \sigma\left(\begin{bmatrix} d \\ m \\ s \end{bmatrix}, t\right) = \begin{bmatrix} \sqrt{2|m|} \\ 0 \\ 0 \end{bmatrix}.$$

This means that, before being stopped when $m_t^\star = \varepsilon$, $\zeta^\star(t) = (\delta_t^\star, m_t^\star, s_t^\star)$ satisfies (2.7).

Notice that (2.9) is the integral form of (2.8).

Unfortunately, we are not able to prove a "local-limit" statement that would be the analogue of Theorem 2.7 when the processes are conditioned by their value on the two boundaries of an interval of the form $[t_1, t_2]$. Moreover, since the process f_{ISE} vanishes at the boundary of its support (since it is a.s. continuous on $\mathbb{R}$), the approximation given by Theorem 2.7 is not sufficient to describe entirely the process ζ^+ (since $\varepsilon > 0$).

If we could overcome these difficulties, we would obtain by Proposition 2.6 that the process ζ (defined in Theorem 2.3) behaves as a diffusion, at least on any compact sub-interval of its support.

We hope that experts of diffusion approximations could be able to bridge these gaps. In the next section, we ask explicit questions in this direction.

2.6. Some questions and conjectures

2.6.1. Questions on the integrated super-Brownian excursion. Maybe the most direct question that follows from the previous discussion is to know whether one can add a dt term to (2.8) to obtain an SDE that would completely describe the process ζ. It is more natural to formulate it for the translated process ζ^+ (defined in Section 2.3) to avoid dealing with the bias at zero.

Conjecture 2.8. *There is a continuous function g such that the following stochastic differential equation holds, for $t > 0$:*

$$d\left(f_{\text{ISE}}^{+\prime}(t)\right) = 2\sqrt{f_{\text{ISE}}^{+}(t)}dB_t + g\left(f_{\text{ISE}}^{+\prime}(t), f_{\text{ISE}}^{+}(t), \int_{-\infty}^{t} f_{\text{ISE}}^{+}(s)\, ds\right) dt. \quad (2.10)$$

The conjecture implies a similar equation for the unshifted process ζ on $(0, +\infty)$. The law on $(-\infty, 0)$ should be more complex since each trajectory has to be biased by the value $f_{\text{ISE}}(0)$ (see the rerooting property in Section 2.3). It is natural to expect a similar SDE in which the function g depends on a fourth parameter t. Notice that the factor 2 in (2.10) is different from the constant $\sqrt{2}$ in (2.9) and in (2.8), since there is a change of time $t \mapsto \gamma t$ between them (see, e.g., Jacod [13], for more information on the change of time techniques in SDE).

Proving that conjecture would be very interesting, especially if the function g can be expressed explicitly. A possible approach to this question would be to try to re-sum the product formulas of [4] to obtain, at the discrete level, the explicit multivariate generating functions encoding the conditional transition probabilities for the process ζ^n. While we believe that it may be approachable while staying in the realm of algebraic functions (recall that a function is algebraic if it is a solution of a non-zero polynomial equation whose variables are the function itself and its parameters), the subsequent analytic combinatorics in several variables required might lead to considerable technical difficulties. We hope that a direct approach from the continuum could lead to better solutions. In any case, this suggests that the function g in Conjecture 2.8 could be algebraic and even quite explicit.

Another natural goal would be to try to prove the missing "local limit" version of Theorem 2.7.

Conjecture 2.9 (Two-sided version of Theorem 2.7). *For $i = 0, 1$, let $(\delta_i, m_i, s_i) \in \mathbb{R} \times (0, +\infty) \times [0, 1]$, and let $(\delta_i^n, m_i^n, s_i^n)$ be a sequence in $\mathbb{Z}^3$ such that $m_i^n > 0$. Assume*

$$z_i^n := \left(\frac{\delta_i^n}{n^{1/2}}, \frac{m_i^n}{n^{3/4}}, \frac{s_i^n}{n}\right) \to z_i := (\delta_i, m_i, s_i) \quad \text{for } i \in \{0, 1\}.$$

Let $0 < \varepsilon < s_1$, and consider the process $\xi^\star(t)$ as in Theorem 2.7, stopped when $m^\star$ hits ε, and started at z_0 for $t = 0$. Then, the discrete process $\mathsf{P}_{\geq \varepsilon n^{3/4}}$, $(\overline{\xi}^n)$ started at time 0 at position z_0^n and conditioned to take value z_1^n at time t_1, converges in distribution in $D([0, t_1], \mathbb{R}^3)$ to $\xi^\star$, started at position (δ_0, m_0, s_0) conditioned by $\xi^\star(t_1) = z_1$.

In the last sentence, notice that we can condition the process $\xi^\star$ by its terminal value $\xi^\star_{t_1}$ on $[0, t_1]$, on the support of this variable – and this characterizes (by disintegration), up to a negligible set, the conditional distribution given the ending point. It can be shown (personal communication of Nicolas Fournier), that the law of $m^\star_{t_1}$ is absolutely continuous with respect to the Lebesgue measure on any compact subset of $(0, +\infty)$.

The last conjecture seems to be related to "continuity" properties of the law of the process $\zeta^{n,*}$ on an interval $[k_1, k_2]$ and conditioned to its right boundary, according to the value on that right boundary. Although it seems difficult to obtain such a result in the general framework of approximating discrete Markov processes by diffusions, it might be doable for this particular case.

From the viewpoint of the convergence of ζ^n, and given Proposition 2.6, it would be even better to establish the following.

Conjecture 2.10. *Conjecture 2.9 also holds with $\varepsilon = 0$.*

Note that formula (2.3) is reminiscent of the closed formula for the distribution of "horizontal profile" that one may find for rooted plane trees, or rooted Cayley trees, taken under the uniform distribution on the corresponding sets of trees with n vertices. The horizontal profile, in general, just called "profile" in the literature, is the sequence $(z_i, i \geq 0)$ of the number of vertices at successive levels in the tree. If h is a positive integer, the number of rooted plane tree having $z_i > 0$ vertices at level i, for $1 \leq i \leq h$ is given by $\prod_{i=0}^{h-1} \binom{z_i + z_{i+1} - 1}{z_i - 1}$, where $z_0 = 1$: indeed, $(c_1, \ldots, c_{z_i})$ the number of children of the z_i individuals at level i forms a composition of z_{i+1} (this is well known, see, e.g., [4] again). The horizontal profile in these trees converges after space and time normalization to the local time of the Brownian excursion, which is the solution of a stochastic differential equation, as proven by Pitman [23] (see also Drmota and Gittenberger [10]). As a starting point to approach the questions above, one might try to reprove this differential equation directly from the discrete product formula, via a diffusion approximation and conditioning of the natural companion Markov process.

2.6.2. Family of distributions subject to boundary conditions. Finally, this discussion raises several questions about the characterization of a random continuum process by its law on proper compact subintervals of the support. Assume for the sake of the discussion that a one-dimensional process X has a continuous version on $[0,1]$. Consider the *family of distributions subject to boundary conditions* (FDBC) of X, which is the data of all the laws

$$\mathcal{L}([X(t), t \in [t_1, t_2]] \mid X(t_1) = x_1, X(t_2) = x_2)$$

for all $0 < t_1 < t_2 < 1$ (observe that 0 and 1 are excluded) and all (x_1, x_2) in the support of $(X(t_1), X(t_2))$. We observe that, in general, the FDBC is *not* sufficient to characterize the distribution of X. For example, the Brownian motion and the standard Brownian bridge have the same FDBC. Even the knowledge of the FDBC and the knowledge of the boundary distribution (for example, $X(0) = X(1) = 0$) is not sufficient to characterize X. For example, if

$$X^{(p)} = -\mathsf{Ber}(p)\mathsf{e} + (1 - \mathsf{Ber}(p))\mathsf{e},$$

where e is a normalized Brownian excursion, and $\mathsf{Ber}(p)$ is an independent Bernoulli random variable with parameter p, then $X_0^{(p)} = X_1^{(p)} = 0$ for all p, the $X^{(p)}$ have the same FDBC for all $p \in (0,1)$, but the law of $X^{(p)}$ clearly depends on p.

What can be shown to be sufficient to determine the distribution of X is, in addition to $X(0) = X(1) = 0$ and of the knowledge of the FDBC, the continuity[1] of the map

$$(t_1, t_2, x_1, x_2) \mapsto \mathcal{L}([X(t), t \in [t_1, t_2]] \mid X(t_1) = x_1,\ X(t_2) = x_2)$$

in $(0,1,0,0)$.

The weakness of the FDBC to characterize the distribution of processes implies that the convergence of the FDBC is also a much too coarse tool to entail convergence in distribution. That being said, of course, such a convergence still carries an important amount of information, and proving it in the case studied in this paper (Conjecture 2.9 or Conjecture 2.10) would be very interesting.

[1] A notion of continuity which is sufficient here, is the convergence of finite-dimensional distributions under $\mathcal{L}([X(t), t \in [t_1, t_2]] \mid X(t_1) = x_1, X(t_2) = x_2)$ when $(t_1, t_2, x_1, x_2) \to (0,1,0,0)$.

3. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. First note that by the results in [5] (see Proposition 2.2), the tightness of the sequence of processes $(n^{-3/4} M^n_{n^{1/4}t})_{t\in\mathbb{R}}$ in $C_0(\mathbb{R})$ equipped with the topology of uniform convergence and its convergence to $\gamma f_{\mathrm{ISE}}(\gamma.)$ are known. So to prove tightness of (ζ^n), it suffices to prove the tightness of its first component, namely, the sequence of processes $(n^{-1/2}\Delta^n_{n^{1/4}t})_{t\in\mathbb{R}}$: more precisely, it suffices to prove the tightness of its restriction to $C([-K, K])$ for any $K > 0$, as explained in Remark 2.4. Moreover, assuming this is done, pick any sub-sequential limit g of that process (in distribution) and a probability space on which this convergence is realized almost surely, then it follows from the discrete identity $M^n_k = \sum_{j\le k}\Delta^n_j$ that, almost surely, we have $\int_{-\infty}^t g(s)\,ds = \gamma f_{\mathrm{ISE}}(\gamma t)$. Therefore, $g(t) = (\gamma f_{\mathrm{ISE}}(\gamma t))'$ which shows that the convergence holds without taking subsequences – and that the limit is, as claimed, the derivative of $t \mapsto \gamma f_{\mathrm{ISE}}(\gamma t)$.

Therefore, we only have to prove the tightness of $(n^{-1/2}\Delta^n_{n^{1/4}.})$ on compact intervals. For this, we will rest on the well-known moment criterion of Kolmogorov (see [15, Theorem 2.23]), whose main point is to prove the following lemma. The value $p = 4$ would be sufficient to obtain tightness as a continuous process but some results of the paper, including the control of the Hölder regularity of the limit, require arbitrary values of p.

Lemma 3.1. *Let $p \ge 1$ be an integer. Let $a < b \in \mathbb{Z}$ such that $|\frac{b-a}{n^{1/4}}| \le 1$. Assume that $an^{-1/4}$ and $bn^{-1/4}$ belong to $[-K, K]$ for some $K > 0$. Then, for any integer $p \ge 1$, one has*

$$\left|\mathbf{E}\Big[\Big(\frac{\Delta^n_b - \Delta^n_a}{\sqrt{n}}\Big)^p\Big]\right| \le C_p\Big(\frac{b-a}{n^{1/4}}\Big)^{\lceil p/2\rceil} \tag{3.1}$$

for some constant C_p depending only on p and K.

The constant 1 in the upper-bound $\left|n^{-1/4}(b-a)\right| \le 1$ plays no special role in the proof of the lemma nor in its application (any positive constant would work).

3.1. Proof of Lemma 3.1

Let us first sketch the method, which is relatively straightforward even if it takes space to write: we will expand the p-th power, interpret the quantities obtained as counting (with signs) trees with p marked vertices, and proceed with generating functions. We will compute these generating functions combining a skeleton decomposition (whose origin in the context of labelled trees might be traced back to [6]) with analytic combinatorics via Hankel contours. The main point of the proof is that, after the proper analysis is done and sign cancellations analysed carefully, we can identify the dominating configurations as the ones in which the p marked vertices are grouped together "in $\lfloor \frac{p}{2} \rfloor$ pairs", which *in fine* is the explanation for the ratio between exponents p and $\frac{p}{2}$ in (3.1). This appearance of pairs in the analysis is in some sense the combinatorial incarnation of the Gaussian nature of the underlying continuum process, see also Section 3.3. Let us now proceed with the proof.

For the rest of Section 3.1, we fix integers $a, b \in \mathbb{Z}$ with $a < b$, and $n, p \geq 1$. We set $q := b - a$ and write $q = \mu n^{1/4}$. We assume that $|\frac{b-a}{n^{1/4}}| \leq 1$, so that

$$n^{-1/4} \leq |\mu| \leq 1.$$

We fix an arbitrary constant $K > 0$ and assume that $an^{-1/4}$, $bn^{-1/4}$ belong to the interval $[-K, K]$. Throughout Section 3, the notation C_p will denote a positive constant that may vary from line to line but depends only on p and K, and not on a, b, n, q, μ.

First, it will be convenient in the proof to avoid the case when the marked vertices are equal to the root. To do this, we define a modified version of the process in which the root (of label 0) does not contribute to the profile. Namely, using the Kronecker symbol δ, we define

$$\widetilde{M}_i^n := M_i^n - \delta_{i,0}, \quad \widetilde{\Delta}_i^n := \widetilde{M}_i^n - \widetilde{M}_{i-1}^n.$$

We first observe that to prove Lemma 3.1, it is enough to prove it for the modified process, i.e., to prove that one has[2]

$$\left|\mathbf{E}\Big[\Big(\frac{\widetilde{\Delta}_b^n - \widetilde{\Delta}_a^n}{\sqrt{n}}\Big)^p\Big]\right| \leq C_p\Big(\frac{b-a}{n^{1/4}}\Big)^{\lceil p/2 \rceil}. \tag{3.2}$$

Indeed, (3.1) clearly holds if one replaces the process (Δ_i^n) by $(\delta_{i,0} - \delta_{i-1,0})$, so for even p, and assuming that (3.1) holds for $(\widetilde{\Delta}_i^n)$, it holds for the process (Δ_i^n) which is the sum of both by the Minkowski inequality. For odd p, one can write, expanding the p-th power,

$$\left|\mathbf{E}\Big[\Big(\frac{\Delta_b^n - \Delta_a^n}{\sqrt{n}}\Big)^p\Big]\right| \leq \sum_{j=0}^{p} \binom{p}{j} \left|\mathbf{E}\Big[\Big(\frac{\widetilde{\Delta}_b^n - \widetilde{\Delta}_a^n}{\sqrt{n}}\Big)^j\Big]\right| \Big(\frac{4}{\sqrt{n}}\Big)^{p-j},$$

which assuming (3.2) for all $j \leq p$ implies (3.1). Thus, in order to prove Lemma 3.1, we will only need to prove (3.2).

[2] Since we use it here for the first time, recall our convention that the constant C_p may vary from line to line, but depends only on p and on the interval $[-K, K]$ to which $an^{-1/4}$ and $bn^{1-/4}$ are constrained.

Since the Catalan numbers satisfy $\mathrm{Cat}(n) = \Theta(4^n n^{-3/2})$, we need to show that

$$\left|\mathrm{Cat}(n) \times \mathbf{E}\Big[\Big(\widetilde{\Delta}^n_{b+1} - \widetilde{\Delta}^n_{a+1}\Big)^p\Big]\right| \leq C_p 4^n n^{p/2-3/2} \mu^{\lceil p/2 \rceil} \tag{3.3}$$

for some constant C_p. Note that compared to (3.2), we have incremented a and b by one, which will be convenient for notation.

3.1.1. Marked trees, skeletons. Since $\widetilde{\Delta}^n_{b+1} - \widetilde{\Delta}^n_{a+1} = \widetilde{M}^n_{b+1} - \widetilde{M}^n_b - \widetilde{M}^n_{a+1} + \widetilde{M}^n_a$, we can write

$$\mathbf{E}[(\widetilde{\Delta}^n_{b+1} - \widetilde{\Delta}^n_{a+1})^p] = \sum_{\substack{\varepsilon_1,\dots,\varepsilon_p \in \{0,1\} \\ \varepsilon'_1,\dots,\varepsilon'_p \in \{0,1\}}} (-1)^{\sum \varepsilon_i + \sum \varepsilon'_i} E\Big[\prod_{i=1}^{p} \widetilde{M}^n_{a+q\varepsilon_i+\varepsilon'_i}\Big].$$

Therefore,

$$\begin{aligned} &\mathrm{Cat}(n) \times \mathbf{E}[(\widetilde{\Delta}^n_{b+1} - \widetilde{\Delta}^n_{a+1})^p] \\ &\quad = \sum_{\substack{\varepsilon_1,\dots,\varepsilon_p \in \{0,1\} \\ \varepsilon'_1,\dots,\varepsilon'_p \in \{0,1\}}} (-1)^{\sum \varepsilon_i + \sum \varepsilon'_i} T_n(a + q\varepsilon_i + \varepsilon'_i,\ i = 1, \dots, p), \end{aligned} \tag{3.4}$$

where $T_n(i_1, \dots, i_p)$ denotes the number of binary trees of size n having p (numbered, possibly repeated) distinguished non-root[3] vertices, of respective abscissa $i_1, \dots, i_p$.

We will evaluate sum (3.4) by grouping trees with p marked vertices according to their skeleton and their scheme, which we now define.

Definition 3.2 (Skeleton, scheme, Figure 3). Let T be a rooted binary tree with root ρ, with p marked vertices $v_1, \dots, v_p$ distinct from ρ, numbered and possibly repeated.

Let V be the set formed by the vertices $v_1, \dots, v_p, \rho$ together with all their pairwise highest common ancestors. The *skeleton* S of T is the rooted binary tree on the vertex set V obtained from T by iteratively removing all vertices which are leaves but are not in V, until no such leaf remains, and then replacing each path of vertices of degree 2 joining two points of V (but containing no other vertex of V) by a single edge. We preserve the left/right order of edges outgoing from vertices of V, so S has a structure of binary tree.

Let $f_1, \dots, f_k$ be the leaves (vertices without children) of S, and let $g_1, \dots, g_\ell$ be the non-root vertices of S which are either unary or binary (1 or 2 children), where both lists are without repetition, and $k \geq 1$, $\ell \geq 0$. In both cases, vertices are numbered in the natural depth-first order of the tree. We have (without repetition)

$$V = \{f_1, \dots, f_k, g_1, \dots, g_\ell, \rho\}.$$

[3]Note that we are using the modified processes $\widetilde{M}$, $\widetilde{\Delta}$, which is why we require the marked vertices not to be the root.

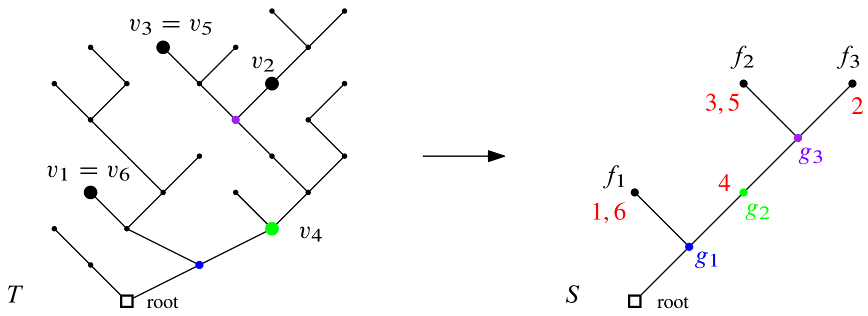

Figure 3. Left: A binary tree T with six marked vertices. Right: Its skeleton S. To help visualize which vertices of T are present in S, we represented them with colours. The decorations $1, 2, \ldots, 6$ indicating the position of the vertices v_i are in red.

Finally, the skeleton carries one additional information: for each vertex v of S, we record the subset of $\{1, \ldots, p\}$ formed by the values i such that $v_i = v$. One can think of this data as a function $[1, p] \to V \setminus \{\rho\}$, or as the fact that vertices of S carry p numbered decorations, where a vertex can be decorated zero or several times (decorations are represented in red in Figure 3). Note that a vertex f_i necessarily carries a decoration and the same is true for a unary vertex g_i, but a binary vertex g_i can be decorated or not. Note also that the root cannot be decorated.

The *labelled skeleton* of the marked tree T is the pair $\widehat{S} = (S, x)$, where S is the skeleton and $x = (x_1, \ldots, x_\ell)$ is the sequence of abscissas $x_1, \ldots, x_\ell$ of $g_1, \ldots, g_\ell$ in T.

The *scheme* of T is the pair $\widetilde{S} = (S, \lambda)$, where S is the skeleton of T and $\lambda = (\lambda(1), \ldots, \lambda(\ell)) \in \{0, 1, 2, 3, 4, 5\}^\ell$, where for each $i \in [\ell]$, we have $x_i \in I_{\lambda(i)}$, where $I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$ is the following partition of $\mathbb{Z}$:

$$\begin{aligned} \mathbb{Z} = {} & \underbrace{(\infty, a-1] \cup [b+2, \infty)}_{I_0} \cup \underbrace{[a+2, b-1]}_{I_1} \cup \underbrace{\{a\}}_{I_2} \\ & \cup \underbrace{\{a+1\}}_{I_3} \cup \underbrace{\{b\}}_{I_4} \cup \underbrace{\{b+1\}}_{I_5}. \end{aligned} \tag{3.5}$$

The number of non-root internal vertices whose abscissa belongs to I_0, I_1, or $\{a, a+1, b, b+1\}$ will play a key role in what follows. Let

$$n_0 = |\lambda^{-1}(\{0\})|, \quad n_1 = |\lambda^{-1}(\{1\})|, \quad n_2 = |\lambda^{-1}(\{2, 3, 4, 5\})|.$$

Note that the scheme $\widetilde{S}$ can be inferred from the labelled skeleton $\widehat{S}$. We say that $\widehat{S}$ is *compatible with* $\widetilde{S}$ if trees of the labelled skeleton $\widehat{S}$ have scheme $\widetilde{S}$.

Remark 3.3. There are finitely many binary trees having at most p leaves and unary vertices. Moreover, there is a finite number of ways to distribute the decorations $1, \dots, p$ among the vertices of S. So there is a finite number of skeletons – and a finite number of schemes.

In sum (3.4), we will group the configurations having the same labelled skeleton, and we will evaluate (upper bound) their contribution separately.

3.1.2. Branches and their generating function. Fix a labelled skeleton $\widehat{S} = (S, x)$, and use the notation of Definition 3.2. We will explain how to construct all trees with labelled skeleton $\widehat{S}$ by substituting edges with branches. A *branch* is just a rooted binary tree with a marked leaf, considered up to additive translation of all abscissas. The *increment* of the branch is the abscissa of its marked leaf (when the root is set to zero). The abscissas of vertices appearing along the path from the root to the mark leaf form a walk on $\mathbb{Z}$ with steps ± 1.

Now, for each edge e of S, let $\alpha_e \in V$ be the mother vertex of e in S, and let β_e be the other endpoint of e. Let $x(\alpha_e)$, $x(\beta_e)$ be their abscissa in the labelled skeleton (for the moment, the second one is defined only if β_e is not a leaf, since the labelled skeleton, by definition, does not record the abscissas of its leaves). Let r_e be equal to $+1, -1$ if e is leaning right or left from α_e, respectively.

All the marked trees contributing to (3.4) have p marked vertices (possibly repeated) whose abscissa is among $\{a, a+1, b, b+1\}$. Among these trees, the trees T having a labelled skeleton $\widehat{S}$ can be obtained in a unique way by the following procedure (Figure 4):

- For each leaf f of S, choose its abscissa $x(f)$ among a, $a+1$, b, $b+1$. Note that after this step the abscissas of all vertices of T are now defined.
- For each edge e of S, let $x_e^- := x(\alpha_e) + r_e$ and $x_e^+ := x(\beta_e)$. Replace the edge e by a branch T_e of increment $x_e^+ - x_e^-$, whose root and marked leaf are linked to α_e and identified with β_e, respectively.
- Attach a rooted binary tree G_e to each β_e which is a leaf of S.
- Use the decorations in $\{1, \dots, p\}$ of the vertices of S to recover the p marked vertices of T.

In order to evaluate expression (3.4), we consider its generating function, in a new variable t,

$$\sum_{n \geq 1} \sum_{\substack{\varepsilon_1, \dots, \varepsilon_p \in \{0,1\} \\ \varepsilon'_1, \dots, \varepsilon'_p \in \{0,1\}}} (-1)^{\sum \varepsilon_i + \sum \varepsilon'_i} T_n(a + q\varepsilon_i + \varepsilon'_i,\, i = 1, \dots, p) t^n. \tag{3.6}$$

Note that the coefficient of t^n in (3.6) is precisely (3.4). Let $F_{\widehat{S}}(t)$ be the contribution of the set of marked trees T having a given labelled skeleton $\widehat{S}$ to the generating

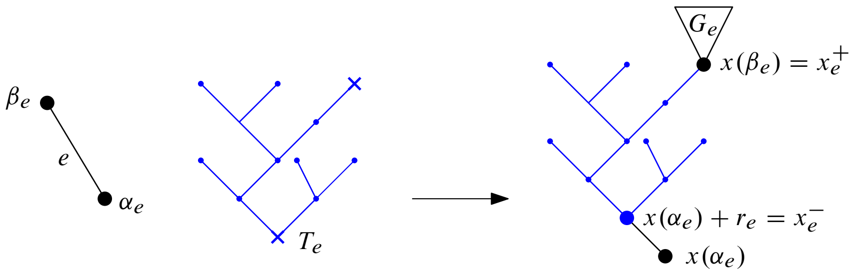

Figure 4. Replacing an edge e of the labelled skeleton by a branch T_e. On this example, e is a left-leaning edge ($r_e = -1$), and the increment of the branch to be substituted is $x(\beta_e) - (x(\alpha_e) - 1) = 2$.

function (3.6). This generating function can be evaluated by computing separately the contribution of each edge of $\widehat{S}$, as we now show. In all generating functions below, the variable t will mark the number of vertices of the underlying trees.

To start with, using classical last-passage decompositions, the generating function of branches of increment i is found[4] to be

$$H_i(t) = BU^{|i|},$$

where the series $U \equiv U(t)$ is defined by the system of equations

$$\begin{cases} T = t(1+T)^2, \\ U = y(1+U^2), \\ y = t(1+T), \end{cases}$$

and

$$B = (1-4y)^{-1/2} = \frac{1+U^2}{1-U^2}.$$

[4]Proof: The series $T(t)$ is clearly the generating function of binary trees by the number of vertices. Similarly, the series $U(y)$ is the generating function of Dyck paths extended by an extra up step, and the series $B(y) = \sum_{n \geq 0} \binom{2n}{n} y^{2n}$ is the generating function of Dyck bridges (± 1 path going from 0 to 0). By decomposing a walk ending at $i \geq 0$ at the last passage at each integer $j \in [0, i]$, we decompose it into a bridge followed by i translated Dyck paths separated by single up steps, which shows that the series BU^i is the generating function of ± 1 walks ending at i (note that the walk can be empty, which is not a problem for us).

This walk can be interpreted as a left/right path going from the root to a marked leaf of abscissa i. Finally, the relation $y = t(1+T)$ means that a (non-empty) binary tree is to be substituted at each step of that path.

The critical point of this system is readily found to be equal to $t = t_c := \frac{1}{4}$, corresponding to $U = U_c := 1$. More precisely, the algebraic series $B(t)$ and $U(t)$ have a unique singularity of minimal modulus at t_c, close to which they admit the Puiseux expansions

$$\begin{aligned} U &= 1 - c(1-4t)^{1/4} + O((1-4t)^{1/2}), \\ B &= c'(1-4t)^{-1/4} + O(1), \end{aligned} \tag{3.7}$$

with constants $c, c' > 0$ that we do not need to make explicit. In particular, there exists a neighbourhood V of $\frac{1}{4}$ in $\mathbb{C} \setminus (\frac{1}{4}, \infty)$ on which $|U(t)|$ has a unique maximum (equal to 1) at $t = \frac{1}{4}$.

We can now compute the generating function $F_{\widehat{S}}(t)$. We have

$$F_{\widehat{S}}(t) = \varepsilon_{\widehat{S}} \cdot t^{\ell} T^k \cdot \prod_{e \in E(S)} W_e(t), \tag{3.8}$$

where $E(S)$ is the set of edges of the skeleton, and $W_e(t)$ is the generating function of the branches that can be substituted at edge e, discussed below. Here the factors T^k and t^{ℓ} account respectively for the trees G_e to be inserted at each leaf β_e, and for the internal vertices of S. The sign $\varepsilon_{\widehat{S}} \in \{\pm 1\}$ is the contribution to the sign in (3.4) of the decorated vertices which are one of the internal vertices $g_1, \dots, g_{\ell}$ of the skeleton.[5]

The function W_e depends on the case considered

- "Internal" edge: β_e is not a leaf of S. Then the only constraint on the branch is its increment, which has to equal $x_e^+ - x_e^-$, therefore

$$W_e = H_{x_e^+ - x_e^-} = BU^{|x_e^+ - x_e^-|}. \tag{3.9}$$

- "External" edge: β_e is a leaf of S. We distinguish the following two cases:
  - β_e carries a unique decoration. If this is the case, in trees of scheme S, β_e corresponds to a unique marked vertex v_i, and it has to be counted, in (3.4), with a sign that depends on the value of $x(\beta_e)$ in $\{a, a+1, b, b+1\}$. Summing over these four possible values, we get the contribution

$$W_e = \sum_{\varepsilon, \varepsilon' \in \{0,1\}} (-1)^{\varepsilon + \varepsilon'} BU^{|a + q\varepsilon + \varepsilon' - x_e^-|}. \tag{3.10}$$

[5] Namely, for each i from 1 to p, if the i-th decoration in S is carried by the vertex g_j of the label x_j, then necessarily $x_j \in \{a, a+1, b, b+1\}$, and the multiplicative contribution of this index j to $\varepsilon_{\widehat{S}}$ is $+1$ if $x_j \in \{a, b+1\}$ and -1 otherwise. This sign will play no role in what follows, as we will only estimate the modulus of $F_{\widehat{S}}(t)$.

To evaluate this expression, we need to get rid of absolute values. This requires knowing the relative position of $x := x_e^-$ with numbers a, $a+1$, b, $b+1$. Sum (3.10) is immediately computed in each case:

$$\begin{aligned} &x \le a, && W_e = B(1-U)U^{a-x}(1-U^q),\\ &x \in [a+1,b], && W_e = B(U-1)(U^{b-x}+U^{x-a-1}),\\ &x \ge b+1, && W_e = B(1-U)U^{x-(b+1)}(1-U^q). \end{aligned} \tag{3.11}$$

Note that the information to determine cases in (3.11) is present in the labelled skeleton (and, in fact, it is present in the scheme, via the coordinate λ).

- β_e carries h decorations with $h \ge 2$. We call the edge e *frozen*. If this is the case, in trees of scheme S, β_e corresponds to h coinciding marked vertices v_i, all contributing to the same sign. We thus get

  $$W_e = \sum_{\varepsilon,\varepsilon' \in \{0,1\}} ((-1)^{\varepsilon+\varepsilon'})^h B U^{|a+q\varepsilon+\varepsilon'-x_e^-|}.$$

  In this case, we will only need to use the modulus upper bound

  $$|W_e| \le \sum_{x \in \{a,a+1,b,b+1\}} |B||U|^{|x-x_e^-|}. \tag{3.12}$$

  Note that we could be more precise when h is odd, but we will not need this.

3.1.3. Hankel contours. For a scheme $\widetilde{S} = (S,\lambda)$, we form the generating function

$$F_{\widetilde{S}} = \sum_{\widehat{S}=(S,x)} F_{\widehat{S}},$$

where the sum is taken over all the labelled skeletons $\widehat{S} = (S,x)$ which are compatible with $\widetilde{S}$.

The contribution of trees of scheme $\widetilde{S}$ to (3.4) is given by the coefficient

$$[t^n]F_{\widetilde{S}}(t) = \frac{1}{2\pi i}\oint_{\mathcal{I}_n} \frac{dt}{t^{n+1}} F_{\widetilde{S}}(t), \tag{3.13}$$

where $\mathcal{I}_n$ is the contour displayed in Figure 5, left, which follows a circle or radius r except close to the positive real axis, where it follows a small detour to the left of the singularity $\frac{1}{4}$ at distance $\frac{1}{n}$. Here $r > \frac{1}{4}$ is chosen so that $U(t)$ and $B(t)$ have no other singularity than $\frac{1}{4}$ inside the circle of radius r. We split the contour $\mathcal{I}_n$ into the "circle part" $\mathcal{C}_n$ which is a portion of the circle of radius r, and the "Hankel part" $\mathcal{H}_n$. We choose r close enough to $\frac{1}{4}$, so that $\mathcal{H}_n$ is entirely contained in the neighbourhood V previously chosen, and such that $U(t) < 1$ for all $t \in \mathcal{I}_n$ (this is

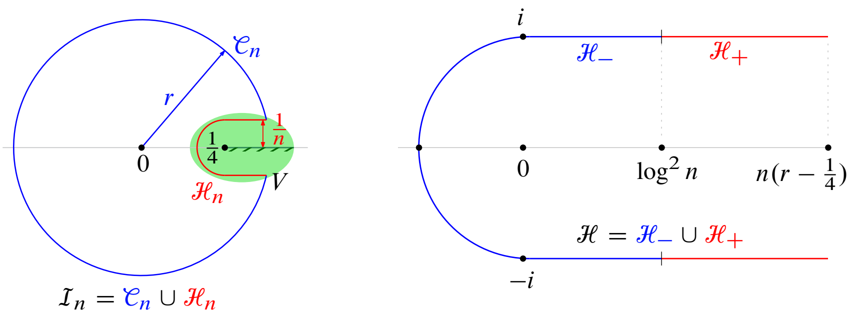

Figure 5. Contours of integration. Left: t-plane. Right: τ-plane.

possible since by Pringsheim's theorem, we have $|U(t)| < 1$ on the circle of radius $\frac{1}{4}$ from which V is removed. By compactness and continuity, we can thus increase the radius a little bit and keep this inequality).

We note that $|B(1-U)|$ is bounded independently of n on $\mathcal{C}_n$, and since $|U| \le 1$, we have $|U^k| \le 1$ for any $k = k(n) \ge 0$. Hence, putting together (3.8), (3.9), (3.11), and (3.12), we obtain that for $t \in \mathcal{C}_n$,

$$|F_{\widetilde{S}}(t)| \le C_p \cdot |B|^{m_{\mathrm{int}}+m_1} \cdot |U|^{\sum_{e\in F_{\mathrm{int}}\cup F_0\cup F_1} |x_e^+ - x_e^-|} \cdot |1-U^q|^{m_0}, \tag{3.14}$$

where

- F_{int} is the set of internal edges e of S, F_0 is the set of non-frozen external edges such that $x_e^- \notin [a+1, b]$, and F_1 is the set of frozen external edges;
- $m_{\mathrm{int}} = |F_{\mathrm{int}}|$ and $m_0 = |F_0|$, $m_1 = |F_1|$;
- for $e \in F_0$, we define x_e^+ to be equal to a or to $b+1$ if $x_e^- < a$ or $x_e^- > b+1$, respectively;
- for $e \in F_1$, we fix arbitrarily $x_e^+ \in \{a, a+1, b, b+1\}$ that minimizes $|x_e^+ - x_e^-|$.

It is important to note that all these notions are well defined from the scheme $\widetilde{S}$ (in particular, F_0 is well defined as we know the position of internal vertices in partition (3.5) from the component λ of the scheme). Note also that for $e \in F_0 \cup F_1$, we have *defined* the value x_e^+, which was undefined in $\widetilde{S}$, so x_e^-, x_e^+ are now defined for all edges e. We insist that, since we have already performed signed summations, we have lost the precise interpretation as combinatorial objects of the quantities we consider, and x_e^+ is not necessarily the value of the abscissa of the corresponding vertex in the underlying trees we were originally counting. Instead, the definition of x_e^+ should be understood as a convention made to be compatible[6] with the exponent of the series U in (3.11) and (3.12).

[6]In particular, we want the quantities U^{a-x} and $U^{x-(b+1)}$ in the first and last line of (3.11) to match the corresponding factor $U^{|x_e^+ - x_e^-|}$ in (3.14).

We will now sum estimate (3.14) over[7] all choices of abscissas $x_1, \dots, x_\ell$ in $\mathbb{Z}$ of the internal vertices of S which are compatible with the scheme $\widetilde{S}$.

In order to construct such a sequence $x_1, \dots, x_\ell$, we first choose a non-decreasing sequence

$$y_1 \leq \cdots \leq y_{\ell+1},$$

which *in fine* will be the reordering of the sequence $0, x_1, \dots, x_\ell$.

To construct a valid sequence y_i, we first choose which of the numbers y_i are equal to zero (at least one is), and we choose which remaining indices will belong to the sets $I_0, \dots, I_5$. There are at most C_p such choices. We then choose the values of each of the n_1 numbers inside $I_1 = (a+2, b-1)$: we have at most q^{n_1} choices.

Finally, we choose the n_0 remaining values, assuming $n_0 \geq 1$ (if $n_0 = 0$, there is no choice to make at this step).

We call $y_{i_1} \leq \cdots \leq y_{i_{n_0+1}}$ these n_0 numbers together with 0, and we set $N := y_{i_{n_0+1}} - y_{i_1}$. For N fixed, there are at most $C_p(N+1)^{n_0-1}$ choices for these numbers. Moreover, applying the triangle inequality for edge increments along the path between internal vertices of labels $y_{i_{n_0+1}}$ and y_{i_1} in S, we observe that

$$\sum_{e \in F_{\text{int}}} |x_e^+ - x_e^-| \geq N.$$

It remains to shuffle the sequence $y_1, \dots, \widehat{0}, \dots, y_{\ell+1}$ (one zero removed) to get the sequence $x_1 \dots, x_\ell$. The number of shuffles is at most C_p.

From (3.14), we thus deduce (recall that $|U| \leq 1$ for $t \in \mathcal{C}_n$) that, for $t \in \mathcal{C}_n$,

$$\begin{aligned} |F_{\widetilde{S}}(t)| &\leq C_p \cdot |B|^{m_{\text{int}}+m_1} \cdot |1-U^q|^{m_0} \cdot q^{n_1} \sum_{N \geq 0} (N+1)^{n_0-1} |U|^N \\ &\leq C_p |B|^{m_{\text{int}}+m_1} \cdot |1-U^q|^{m_0} \cdot \mu^{n_1} n^{n_1/4} \frac{1}{(1-|U|)^{n_0}} \\ &\leq C_p |B|^{m_{\text{int}}+m_1+n_0} \cdot |1-U^q|^{m_0} \cdot \mu^{n_1} n^{n_1/4} \end{aligned} \tag{3.15}$$

if $n_0 \geq 1$, and in fact the same bound holds if $n_0 = 0$ since the sum over N in the first line is not present in that case. Note that we have used that $|B(1-U)|$ is bounded away from 0 uniformly on the contour.

With this estimate in hand, we can now estimate integral (3.13). We start with the contribution of the Hankel part $\mathcal{H}_n$. We perform the change of variable

$$t = \frac{1}{4}\Big(1 + \frac{\tau}{n}\Big),$$

[7] Since we only work up to a constant multiplicative factor here, we will evaluate this sum approximately. An exact computation can be done following the lines of [6] but requires much heavier notation – and this is ultimately not needed here.

where τ now lives on the classical Hankel contour $\mathcal{H}$ represented in Figure 5, right. We furthermore split $\mathcal{H}$ into $\mathcal{H}_-$ and $\mathcal{H}_+$, consisting of the parts for which $\operatorname{Re}(\tau) \le \log^2 n$ or $\operatorname{Re}(\tau) > \log^2 n$, respectively.[8]

We first look at the contribution of $\mathcal{H}_-$. We have from (3.7), expressed in the τ variable,

$$\begin{aligned} 1-U &= c\Big(\frac{-\tau}{n}\Big)^{1/4} + O\Big(\sqrt{\frac{\log^2 n}{n}}\Big), \\ B &= c'\Big(\frac{-\tau}{n}\Big)^{-1/4} + O(1), \\ \frac{1}{t^{n+1}} &= 4^{n+1} e^{-\tau(1+O(\log^2 n/n))} \end{aligned} \tag{3.16}$$

and, recalling that $q = \mu n^{1/4}$ with $\mu \le 1$,

$$U^q = \exp\Big(-c\mu\Big((-\tau)^{1/4} + O\Big(\frac{\log n}{n^{1/4}}\Big)\Big)\Big), \tag{3.17}$$

where all big O are uniform in all parameters. Moreover, since $\operatorname{Re}((-\tau)^{1/4}) \ge 0$ and $c, \mu > 0$, we have[9]

$$\begin{aligned} |1-U^q| &= |e^{c\mu((-\tau)^{1/4} + O((\log n)/n^{1/4}))} - 1| \\ &\le c\mu|\tau|^{1/4}\Big(1 + O\Big(\frac{\log n}{n^{1/4}}\Big)\Big) = O(\mu|\tau|^{1/4}). \end{aligned} \tag{3.18}$$

We thus get from (3.15) that for $\tau \in \mathcal{H}_-$ we have

$$|F_{\tilde{S}}| \le C_p n^{(m_{\text{int}}+m_1+n_0+n_1)/4} \mu^{n_1+m_0} |\tau|^{(m_0-m_{\text{int}}-n_0-m_1)/4}.$$

It follows that the contribution of $\mathcal{H}_-$ to (3.13) is bounded, in modulus, by

$$\begin{aligned} C_p n^{(m_{\text{int}}+m_1+n_0+n_1)/4} \mu^{n_1+m_0} \oint_{\mathcal{H}_-} (n^{-1}|d\tau|) 4^n e^{-\operatorname{Re}(\tau)} |\tau|^{C_p} \\ \le C_p 4^n n^{(m_{\text{int}}+m_1+n_0+n_1-4)/4} \mu^{n_1+m_0}, \end{aligned} \tag{3.19}$$

since the remaining function of τ is integrable thanks to the exponential factor.

We now consider the contribution of the remaining contours. Note that for $\tau \in \mathcal{H}_+$ or for $t \in \mathcal{C}_n$, because of the factor t^{-n-1}, the integrand of the corresponding contour integral is dominated by 4^n times a superpolynomial factor $(\exp(-\Omega(\log^2 n))$ in the

[8] The contours we use and the way to split them are classical, see, e.g., the proof of the transfer theorems in [11].

[9] If $\operatorname{Re}(x) \le 0$, then $|e^x - 1| = |e^x - e^0| = |\int_{[0,x]} e^t\, dt| \le |x|$.

first case and $(4r)^{-n}$ in the second. Therefore, the contribution of these contours is more than polynomially smaller than the one of $\mathcal{H}_-$, and then estimate (3.19) is valid for the whole contribution. We thus obtain

$$|[t^n]F_{\widetilde{S}}| \leq C_p 4^n n^{(m_{\mathrm{int}}+m_1+n_0+n_1-4)/4} \mu^{n_1+m_0}. \tag{3.20}$$

An additional work is needed in the case $(p, m_0) = (1, 0)$, since this bound is not sufficient in the case for our purpose. In this case, the scheme is made by a unique leaf attached to the root, and moreover we have $0 \in [a, b+1]$ since the unique external edge is non-frozen and does not contribute to m_0. By (3.11), if $0 \in [a+1, b]$, the generating function is equal to

$$\begin{aligned} F_{\widetilde{S}}(t) &= B(U-1)(U^b + U^{-a-1}) \\ &= (1 + O(|\tau^{1/4}|))(2 + O(\mu|\tau|^{1/4})) = 2 + O(|\tau^{1/4}|), \end{aligned} \tag{3.21}$$

where we used (3.18) and the fact that $|a|, |b| = O(q)$ in this case. Therefore, we have in this case

$$|[t^n]F_{\widetilde{S}}| \leq C_p 4^n n^{-5/4} \quad \text{when } (p, m_0) = (1, 0). \tag{3.22}$$

Note that (3.20) would only give an upper bound of $O(4^n n^{-1})$ in this case. The reason for the improvement is that the leading term "2" in (3.21) does not contribute to the asymptotics of coefficients. Bound (3.22) is also true if $0 \in \{a, b+1\}$ as is easily checked going back to (3.11) once more.

This ends the complex-analytic part of the proof.

3.1.4. Exponent counting and dominant configurations. Recall that at this stage we are still working with a fixed scheme $\widetilde{S} = (S, \lambda)$, carrying p decorations.

Definition 3.4. In the case when p is even, we say that the scheme $\widetilde{S}$ is *dominant* if it is a binary tree with no unary vertex except from the root, having $k = p$ leaves (i.e., p external edges) in which the external edges are attached by pairs to $\frac{p}{2}$ internal vertices, such that these $\frac{p}{2}$ vertices have abscissa in $I_1 = [a+2, b-1]$.

See Figure 6 for a pictorial view of a dominant scheme. Note the that information that the abscissa of an internal vertex is or is not in I_1 can be inferred from the component λ of the scheme, so this definition makes sense.

To bound the right-hand side of (3.20), we will prove the following lemma. Note that we do not assume that p is even in the first part. Recall moreover that we have assumed that $n^{-1/4} \leq \mu \leq 1$. Note also that we exclude the case $(p, m_0) = (1, 0)$, which we already addressed separately in (3.22).

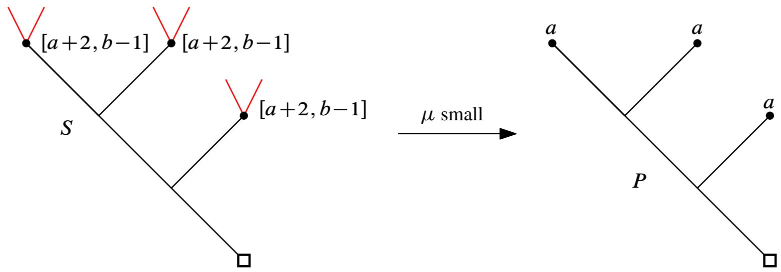

Figure 6. Left: A scheme with $p = 6$ leaves which dominates at first order the calculation of the moment $\mathbb{E}[(\Delta_a - \Delta_b)^p]$, after Lemma 3.1 (the external edges, in red, are grouped in pairs and attached to vertices with abscissa in $[a+2, b-1]$). Right: Calculations show that when $\mu = (b-a)n^{-1/4}$ goes to 0^+, the first-order contribution tends, up to identifiable factors, to the contribution that the "internal" scheme P (in fat black) would give to the computation of the moment $\mathbb{E}[(M_a)^{p/2}]$, at the first order. This observation leads to Proposition 3.7.

Lemma 3.5. *For any scheme $\widetilde{S}$, in notation above, if $(p, m_0) \neq (1, 0)$, we have*

$$n^{(m_{\text{int}}+m_1+n_0+n_1-4)/4}\mu^{n_1+m_0} \leq n^{p/2-3/2}\mu^{\lceil p/2\rceil}. \tag{3.23}$$

Moreover, in the case of p even, if $\widetilde{S}$ is not dominant, we have

$$n^{(m_{\text{int}}+m_1+n_0+n_1-4)/4}\mu^{n_1+m_0} \leq n^{p/2-3/2}\mu^{\lceil p/2\rceil} \times \max(n^{-1/8}\mu^{-1/2}, \mu). \tag{3.24}$$

Proof. Since $n \geq 1$ and $\mu \leq 1$, to maximize the wanted quantity we have, roughly speaking, to look for the largest possible value of $m_{\text{int}} + m_1 + n_0 + n_1$ and the smallest possible value of $n_1 + m_0$. But since these two quantities are not independent, we cannot optimize them individually, and we have to consider both factors in the left-hand side of (3.23) simultaneously. Moreover, in order to do this, we need to introduce other parameters of interest which will have a negative or positive influence on each quantity.

We write $\ell = \ell_{\text{un}} + \ell_{\text{bin}}$, where ℓ_{un} and ℓ_{bin} are the numbers of non-root internal vertices of S with 1 and 2 children, respectively. Since S has k leaves, we have $k = 1 + \ell_{\text{bin}} + \xi$, where ξ is equal to one if the root is binary in the scheme, and zero otherwise. Let $\delta = p - k$, which is non-negative since all leaves are decorated at least once.

We have, noting that $\ell = m_{\text{int}} = n_0 + n_1 + n_2$ and that $\ell = \ell_{\text{un}} + k - 1 - \xi = p + \ell_{\text{un}} - \xi - \delta - 1$,

$$\begin{aligned} m_{\text{int}}+m_1 + n_0 + n_1 - 4 &= 2\ell + m_1 - n_2 - 4 \\ &= 2p - 6 + 2\ell_{\text{un}} + m_1 - 2\xi - 2\delta - n_2 \\ &\leq 2p - 6 + \ell_{\text{un}} - \delta - 2\xi - n_2, \end{aligned}$$

where for the inequality we used that $\ell_{\mathrm{un}} - \delta + m_1 \leq 0$ (or equivalently, that $\ell_{\mathrm{un}} + k + m_1 \leq p$, which holds since by construction leaves and non-root unary vertices are necessarily decorated at least once, but the m_1 leaves incident to frozen edges are decorated at least once more). Hence

$$n^{(m_{\mathrm{int}}+m_1+n_0+n_1-4)/4} \leq n^{p/2-3/2} n^{(\ell_{\mathrm{un}}-n_2)/4} n^{-\xi/2} n^{-\delta/4}. \tag{3.25}$$

Now, write $n_2 = n_2^{\mathrm{e}} + n_2^{\mathrm{i}}$, where n_2^{e} and n_2^{i} are respectively, among the vertices of the scheme contributing to n_2, the vertices which are attached to at least one external edge, and the ones which are not attached to an external edge. Write furthermore $n_2^e = n_2^{\mathrm{e,un}} + n_2^{\mathrm{e,bin}}$, separating the contribution to n_2^e of unary and binary vertices of S.

The number of external edges of S is equal to the number of leaves k, so there are $k - m_0 - m_1$ external edges which are not frozen and are such that $x_e^- \in [a+1, b]$. For these edges, one necessarily has $x(\alpha_e) \in [a, b+1]$, and therefore their attachment vertex contributes to the quantity $n_1 + n_2$. Moreover, each internal vertex can be attached to at most two external edges, which implies that[10]

$$k - m_0 - m_1 \leq 2n_1 + 2n_2^{\mathrm{e,bin}} + n_2^{\mathrm{e,un}} + 2\xi, \tag{3.26}$$

hence

$$k - m_1 - 2n_2^{\mathrm{e,bin}} - n_2^{\mathrm{e,un}} - 2\xi \leq m_0 + 2n_1 \leq 2(m_0 + n_1), \tag{3.27}$$

and

$$n_1 + m_0 \geq \frac{k}{2} - \frac{m_1}{2} - n_2^{\mathrm{e,bin}} - \frac{n_2^{\mathrm{e,un}}}{2} + \frac{\chi}{2} - \xi, \tag{3.28}$$

where χ is equal to 1 if $k + n_2^{\mathrm{e,un}} + m_1$ is odd and to zero otherwise. Therefore,

$$\begin{aligned}\mu^{n_1+m_0} &\leq \mu^{k/2-m_1/2-n_2^{\mathrm{e,bin}}-n_2^{\mathrm{e,un}}/2+\chi/2-\xi} = \mu^{p/2-\delta/2-m_1/2-n_2^{\mathrm{e,bin}}-n_2^{\mathrm{e,un}}/2+\chi/2-\xi}\\ &\leq \mu^{p/2+\chi/2} n^{(\delta+m_1)/8+(n_2^{\mathrm{e,bin}}+n_2^{\mathrm{e,un}}/2)/4+\xi/4} (n^{-1/4}\mu^{-1})^{\Lambda},\end{aligned} \tag{3.29}$$

where $\Lambda := \frac{\delta+m_1}{2} + n_2^{\mathrm{e,bin}} + \frac{n_2^{\mathrm{e,un}}}{2} + \xi$. From (3.25) and (3.29), we get

$$\begin{aligned}&n^{(m_{\mathrm{int}}+m_1+n_0+n_1-4)/4} \mu^{n_1+m_0}\\ &\quad\leq n^{p/2-3/2} \mu^{p/2+\chi/2} n^{(m_1-\delta)/8} n^{((\ell_{\mathrm{un}}-n_2+n_2^{\mathrm{e,bin}}+n_2^{\mathrm{e,un}})/2)/4} (n^{-1/4}\mu^{-1})^{\Lambda}.\end{aligned} \tag{3.30}$$

[10] One has to be careful with (3.26) if the scheme is formed by a unique leaf attached to the root. If $p > 1$ (which implies $m_1 = 1$) or if $p = 1$ and $m_0 = 1$, the left-hand side is 0, so the inequality is true. Only the case $(p, m_0) = (1, 0)$ remains, but this case is excluded by hypothesis.

(Note that we used $n^{-\xi/2}n^{\xi/4} = n^{-\xi/4} \leq 1$). Now we have $\ell_{\mathrm{un}} \leq n_2^{\mathrm{i}} + n_2^{\mathrm{e,un}}$. Indeed, a non-root unary vertex of the scheme is necessarily decorated, so it only appears with a label in $\{a, a+1, b, b+1\}$ so it contributes to n_2.

It follows that

$$\ell_{\mathrm{un}} - n_2 + n_2^{\mathrm{e,bin}} + \frac{n_2^{\mathrm{e,un}}}{2} \leq n_2^{\mathrm{i}} + n_2^{\mathrm{e,un}} - n_2 + n_2^{\mathrm{e,bin}} + \frac{n_2^{\mathrm{e,un}}}{2} = \frac{n_2^{\mathrm{e,un}}}{2}.$$

Therefore, (3.30) and the fact that $n^{-1/4}\mu^{-1} \leq 1$ implies

$$n^{(m_{\mathrm{int}}+m_1+n_0+n_1-4)/4}\mu^{n_1+m_0} \leq n^{p/2-3/2}\mu^{p/2}\mu^{\chi/2}n^{(n_2^{\mathrm{e,un}}+m_1-\delta)/8}. \tag{3.31}$$

Note that $\mu^{\chi/2} \leq 1$ and that $n_2^{\mathrm{e,un}} + m_1 \leq \delta$, since vertices counted by $n_2^{\mathrm{e,un}}$ are unary, hence decorated.

Therefore, if p is even, we have the wanted inequality. If p is odd, then either $k + n_2^{\mathrm{e,un}} + m_1$ is odd, in which case $\frac{p+\chi}{2} = \lceil \frac{p}{2} \rceil$, and we also have the wanted inequality, or $k + n_2^{\mathrm{e,un}} + m_1$ is even. But this implies $k + n_2^{\mathrm{e,un}} + m_1 \neq p$, and since $k + n_2^{\mathrm{e,un}} + m_1 \leq p$ (by counting decorated vertices), this implies $k + n_2^{\mathrm{e,un}} + m_1 < p$, hence $\delta > n_2^{\mathrm{e,un}} + m_1$. Therefore, we have

$$n^{(n_2^{\mathrm{e,un}}+m_1-\delta)/8} \leq n^{-1/8} \leq \mu^{1/2}$$

(using again $\mu^{-1} \leq n^{1/4}$), and the wanted bound (3.23) holds in all cases.

The statement about non-dominant schemes for even p follows by studying the equality case in the inequalities used along the proof. More precisely, assume p even. To obtain (3.31), we have neglected a factor of $(n^{-1/4}\mu^{-1})^{\Lambda}$, so if $\Lambda \geq \frac{1}{2}$, we can strengthen the right-hand side of (3.31) by a factor of $n^{-1/8}\mu^{-1/2}$, therefore at the end of the analysis, (3.24) will hold (from the first coordinate of the maximum). So we can assume $\Lambda < \frac{1}{2}$, or equivalently $\Lambda = 0$. This implies that $\delta = m_1 = \xi = n_2^{\mathrm{e,bin}} = n_2^{\mathrm{e,un}} = 0$. Since $\delta = 0$, we have $k = p$ and all decorated vertices are leaves. Now, if $m_0 > 0$, the second equality in (3.27) is not tight; therefore, we have $k \leq 2(m_0 + n_1) - 1$ and (3.28) becomes $m_0 + n_1 \geq \frac{k}{2} + 1$ which is an improvement of 1 over (3.28) (note that here we use that $k = p$ is even and that $n_2^{\mathrm{e,un}} = m_1 = \chi = 0$). Thus in this case, we can improve the right-hand side of (3.29) by a factor of μ, and following the rest of the proof, (3.24) will hold (from the second coordinate of the maximum). So we can assume that $m_0 = 0$. For the same reason, we can assume that (3.26) is tight, for otherwise the same improvement over (3.28) will lead to an extra factor μ in the end. But (3.26) being tight precisely says that all the $k = p = 2n_1$ leaves are attached in pairs to internal vertices contributing to n_1, hence the scheme $\widetilde{S}$ is dominant. ∎

We get from the first inequality of the last lemma and from (3.20) that $|[t^n]F_{\widetilde{S}}| \leq C_p 4^n n^{p/2-3/2}\mu^{\lceil p/2 \rceil}$. Note that by (3.22), this bound is also true in the case $(p, m_0) =$

$(1, 0)$, since $n^{-5/4} \leq n^{-1}\mu$. Since quantity (3.4) is the sum of $[t^n]F_{\widetilde{S}}$ over the *finite* set all schemes $\widetilde{S}$, we have finally obtained (3.3). This ends the proof of Lemma 3.1.

3.2. Tightness and Hölder continuity

To end the proof of tightness, by the Kolmogorov continuity theorem [15, Theorem 2.23], it is enough to apply Lemma 3.1 with $p = 4$ and to prove that the sequence $(n^{-1/2}\Delta_0^n)_n$ is tight.

The tightness of $(n^{-1/2}\Delta_0^n)_n$ can be proved by estimating the second moment using a simpler variant of the scheme-based techniques we just developed, however we give a quicker proof, communicated to us by a referee, that deduces it from Lemma 3.1.

Lemma 3.6. *The sequence of real random variables $(n^{-1/2}\Delta_0^n)_n$ is tight.*

Proof. For $a = 0$ and $1 \leq b \leq n^{1/4}$, the case $p = 2$ of Lemma 3.1 together with the Cauchy–Schwarz inequality implies that

$$\mathbf{E}(|\Delta_b^n - \Delta_0^n|) \leq \sqrt{C_2}\sqrt{n}.$$

Using the telescopic sum

$$\Delta_{\lfloor n^{1/4}\rfloor}^n + \cdots + \Delta_1^n = M_{\lfloor n^{1/4}\rfloor}^n - M_0^n,$$

we obtain from the triangle inequality that

$$\mathbf{E}(|M_{\lfloor n^{1/4}\rfloor}^n - M_0^n - n^{1/4}\Delta_0^n|) \leq \sqrt{C_2}n^{3/4},$$

so the sequence $n^{-3/4}(M_{\lfloor n^{1/4}\rfloor}^n - M_0^n) - n^{-1/2}\Delta_0^n$ is tight. Since it is known [3] that $n^{-3/4}M_{\lfloor n^{1/4}\rfloor}^n$ and $n^{-3/4}M_0^n$ are tight, we are done. ∎

This concludes the proof of tightness (and convergence) in Theorem 2.3. The fact that the limiting process is $(\frac{1}{2} - \varepsilon)$-Hölder for any $\varepsilon > 0$, follows from Lemma 3.1 at all even values of $p \geq 2$, together with Kolmogorov continuity theorem (see [15, Theorem 2.23]).

Therefore, the only thing remaining to prove in Theorem 2.3 is the lack of $(\frac{1}{2} + \varepsilon)$-Hölder continuity.

3.3. Non-Hölder continuity and other consequences of the proof

Now that our estimate of (3.4) is complete, we can go back and estimate precisely the first order asymptotic contribution. This leads to the following proposition, whose second part completes the proof of Theorem 2.3.

Proposition 3.7. *Fix $\alpha \in \mathbb{R}$. We have the convergence in distribution*

$$\frac{f'_{\mathrm{ISE}}(\alpha+\mu) - f'_{\mathrm{ISE}}(\alpha)}{\sqrt{\mu}} \xrightarrow[\mu \overset{>}{\to} 0]{(d)} 2\sqrt{f_{\mathrm{ISE}}(\alpha)}\mathcal{N}, \tag{3.32}$$

where $\mathcal{N}$ is a standard centred and reduced Gaussian random variable independent of f_{ISE}. Consequently, the function f'_{ISE} is, almost surely, differentiable almost nowhere inside its support – and in fact, it is not $(\frac{1}{2}+\varepsilon)$-Hölder-continuous, for any $\varepsilon > 0$.

Note that (3.32) can be seen as a (weak) discrete version of the heuristic

$$df_{\mathrm{ISE}}(t)' \approx 2\sqrt{f_{\mathrm{ISE}}(t)}dB_t$$

(with, informally speaking, $dt \approx \mu$ and $dB_t \approx \mu^{1/2}\mathcal{N}$).

Proof of Proposition 3.7. We will prove convergence in law by proving convergence of moments, and we will first work at the discrete level. We fix some $K > |\alpha|$ and let as before $a, b \in \mathbb{Z}$ with $an^{-1/4}, bn^{-1/4} \in [-K, K]$, $q = (b-a)$ and $\mu = qn^{-1/4}$. As before, we assume $\mu \leq 1$, and we furthermore assume that $b \geq a+3$, i.e., $q \geq 3$. We will study the limit of (3.4).

First consider the case of p even, say $p = 2r$. We start by studying the contribution of dominant schemes. Let $\widetilde{S}$ be a dominant scheme, obtained as follows: start with a binary tree P with r leaves and attach a pair of dangling external edges to each leaf. Moreover, the leaves of P (internal vertices of S to which external edges are attached) need to have abscissa in $[a+2, b-1]$ (Figure 6).

Let $\widehat{S} = (S, x)$ be a labelled skeleton compatible with $\widetilde{S}$. Call $z_1, \ldots, z_{2r-1} \in \mathbb{Z}$ the labels of the vertices of P (which are precisely the internal vertices of S), with $z_r, \ldots, z_{2r-1} \in [a+2, b-1]$ being the labels of the leaves of P.

By (3.8), (3.9) and (3.11), the generating function corresponding to the labelled skeleton $\widehat{S}$ in the computation of (3.4) is given by

$$\begin{aligned} F_{\widehat{S}}(t) = {} & t^{2r-1} T^{2r} B^{|E(P)|} U^{\sum_{e \in E(P)} |x_e^- - x_e^+|} \\ & \times \prod_{i=r}^{2r-1} B^2 (1-U)^2 \prod_{\varepsilon \in \{\pm 1\}} (U^{b-(z_i+\varepsilon)} + U^{(z_i+\varepsilon)-a-1}), \end{aligned}$$

where we separated the contribution of internal and external edges ($E(P)$ denotes the set of edges of P, i.e., internal edges of S), and where the product over ε accounts for the two left/right leaning edges emanating for each leaf of P. Note that for $\tau \in \mathcal{H}_-$, we have $B(1-U) = 1 + o(1)$ and[11] if $z_i \in [a+2, b-1]$, then

$$U^{b-(z_i+\varepsilon)} + U^{(z_i+\varepsilon)-a-1} = 2 + O(\mu|\tau|^{1/4})$$

[11] The term $o(1)$ is relative to n going to infinity.

as in (3.17), with $\mu = (a-b)n^{-1/4}$ as before. These estimates hold uniformly in $\tau \in \mathcal{H}_-$, in the z_i, in μ, and in K. Moreover, let e be an external edge of P, we have $x_e^+ \in [a+2, b-1]$ and

$$U^{|x_e^- - x_e^+|} = U^{|x_e^- - a|}(1 + O(\mu|\tau|^{1/4})),$$

again uniformly in all parameters. We deduce that, again uniformly,

$$F_{\widehat{S}}(t) = 4^r(1 + O(\mu|\tau|^{1/4}) + o(1))t^{2r-1}T^{2r}B^{|E(P)|}U^{\sum_{e\in E(P)}|z_e^- - z_e^+|},$$

where the integers z_e^-, z_e^+ are defined as x_e^-, x_e^+, but starting from the labelling $z_1, \dots, z_{r-1}, a, a, \dots, a$ (in other words, we fix the leaves of P to abscissa a, but keep the abscissas of internal vertices). Using that t and T go respectively to $\frac{1}{4}$ and 1 when $\tau \in \mathcal{H}_-$, we can also write

$$F_{\widehat{S}}(t) = (1 + o(1) + O(\mu|\tau|^{1/4}))t^{r-1}T^r B^{|E(P)|}U^{\sum_{e\in E(P)}|z_e^- - z_e^+|}. \tag{3.33}$$

Since all big O and little o are uniform, and since as before contributions of the contour integrals outside of $\tau \in \mathcal{H}_-$ can be neglected in (3.13), we conclude that

$$[t^n]F_{\widehat{S}} = [t^n]t^{r-1}T^r B^{|E(P)|}U^{\sum_{e\in E(P)}|z_e^- - z_e^+|} + R, \tag{3.34}$$

where R is the contribution coming from the terms $o(1)$ and $O(\mu|\tau|^{1/4})$ in (3.33), to be addressed later. Now, the quantity $t^{r-1}T^r B^{|E(P)|}U^{\sum_{e\in E(P)}|z_e^- - z_e^+|}$ is easily recognized, by (3.9) and reasoning similarly as in Section 3.1.2, as the generating function of binary trees with r marked vertices with skeleton P, where marked vertices have all abscissas a, and such that the internal vertices of P have labels $z_1, \dots, z_{r-1}$.

Since all estimates are uniform, we can now sum the first term of (3.34) over $z_1, \dots, z_{2r-1}$ and over all binary trees P. We deduce that the total contribution of dominant schemes to this term is equal to

$$(2r-1)!!2^r(1+o(1))(q-2)^r T_n^*(\underbrace{a, a, \dots, a}_{r \text{ times}}), \tag{3.35}$$

where $T_n^*(i_1, \dots, i_r)$ is the contribution to the number $T_n(i_1, \dots, i_r)$ of marked trees whose r marked vertices are in generic position (i.e., their skeleton is a binary tree with r leaves). To obtain this expression, we have summed first over $z_r, \dots, z_{2r-1} \in [a+2, b-1]$ giving the factor $(q-2)^r$ (note that these variables do not appear in (3.34)), while the sum over P and over its internal labels $z_1, \dots, z_{r-1}$ accounts for the possible relative positions of the r marked vertices of label a and of the abscissas of the internal vertices of their skeleton, in a configuration counted by $T_n^*(a, a, \dots, a)$. Note also the combinatorial factor $(2r-1)!!2^r = \frac{(2r)!}{r!}$ which is the ratio between the possible numberings of the $2r$ leaves of S and the r leaves of P. Now, it is easily seen using again the same scheme techniques that we have (uniformly in K) $T_n^*(a,a,\dots,a) \sim T_n(a,a,\dots,a)$, therefore quantity (3.35) is equal to

$$(2r-1)!!2^r(1+o(1))(q-2)^r T_n(\underbrace{a,a,\dots,a}_{r \text{ times}}). \tag{3.36}$$

Moreover, one has (using again the same techniques or just observing that from [3] $\mathbf{E}M_a^r n^{-3r/4} > 0$) that $T_n(a,a,\dots,a) = \Omega(4^n n^{3r/4-3/2})$, therefore, recalling that $an^{-1/4}$, $bn^{-1/4}$ are in $[-K,K]$, expression (3.36) is equal to $\Omega(4^n n^{r-3/2}\overline{\mu}^r)$, where $\overline{\mu} = \mu(1-\frac{2}{q})$. Since we assume $q \geq 3$, we have $\overline{\mu} = \Theta(\mu)$ uniformly, so the last quantity is also $\Omega(4^n n^{r-3/2}\mu^r)$.

Equipped with this lower bound, we can now take into account the contribution of the remaining terms and schemes, and show that they are subdominant. First, the contribution of the term R in (3.34), summed over the abscissas of vertices of P, can be estimated in the same way we estimated (3.19). It follows that the contribution of the term R in (3.34) to $[t^n]F_{\widehat{S}}$ is $(o(1)+O(\mu))$ times (3.36). Moreover, by Lemma 3.5, the contribution of any non-dominant scheme to (3.4) is at most

$$4^n n^{r-3/2}\mu^r \cdot O(\mu + n^{-1/8}\mu^{-1/2}),$$

therefore it is at most $O(\mu + n^{-1/8}\mu^{-1/2})$ times (3.36). Putting everything together, we can now take into account the contribution of all schemes (dominant or not) to (3.4), and we finally obtain that

$$\begin{aligned}\mathrm{Cat}(n)\mathbb{E}[(\widetilde{\Delta}_{a+1}^n - \widetilde{\Delta}_{b+1}^n)^{2r}] = (2r-1)!!2^r(1 &+ O(\mu + n^{-1/8}\mu^{-1/2})\\ &+ o(1))n^{r/4}\overline{\mu}^r T_n(a,\dots,a).\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathbb{E}\Big[\Big(\frac{\widetilde{\Delta}_{a+1}^n - \widetilde{\Delta}_{b+1}^n}{\sqrt{n}}\Big)^{2r}\Big] &= (2r-1)!!\overline{\mu}^r\mathbb{E}\Big[\Big(\frac{2M_a^n}{n^{3/4}}\Big)^r\Big]\\ &\quad \times (1+O(\mu+n^{-1/8}\mu^{-1/2})+o(1)).\end{aligned} \tag{3.37}$$

Taking the limit $n \to \infty$ on both sides with $a = \lfloor \alpha n^{1/4} \rfloor$ and $b = \lfloor (\alpha+\mu)n^{1/4} \rfloor$ ($\mu > 0$ fixed), we obtain (noting that $\overline{\mu} \to \mu$)

$$\begin{aligned}\gamma^{4r}\mathbb{E}[(f'_{\mathrm{ISE}}(\gamma(\alpha+\mu)) - f'_{\mathrm{ISE}}(\gamma\alpha))^{2r}] &= (2r-1)!!\mu^r\mathbb{E}[\gamma^r(2f_{\mathrm{ISE}}(\gamma\alpha))^r](1+O(\mu))\\ &= \mu^r\gamma^r\mathbb{E}[(2f_{\mathrm{ISE}}(\gamma\alpha))^r\mathcal{N}^{2r}](1+O(\mu));\end{aligned}$$

the change of variables $\gamma\alpha = \widetilde{\alpha}$, $\gamma\mu = \widetilde{\mu}$ provides (taking into account $\gamma^{4r} = 2^{-r}$)

$$\begin{aligned}\mathbb{E}[(f'_{\mathrm{ISE}}(\widetilde{\alpha}+\widetilde{\mu}) - f'_{\mathrm{ISE}}(\widetilde{\alpha}))^{2r}] &= (2r-1)!!\widetilde{\mu}^r\mathbb{E}[(4f_{\mathrm{ISE}}(\widetilde{\alpha}))^r](1+O(\mu))\\ &= \widetilde{\mu}^r\mathbb{E}[(4f_{\mathrm{ISE}}(\widetilde{\alpha}))^r\mathcal{N}^{2r}](1+O(\mu)),\end{aligned}$$

where $\mathcal{N}$ is a standard Gaussian random variable independent of f_{ISE}. To deduce the first equality, we have used the convergence in law of Theorem 2.3, together with the fact that, by Lemma 3.1, the moments appearing on both sides of (3.37) are bounded independently of n for any $r > 0$ – these two facts imply the convergence of moments. This further implies

$$\mathbb{E}\Big[\Big(\frac{f'_{\mathrm{ISE}}(\alpha+\mu) - f'_{\mathrm{ISE}}(\alpha)}{\sqrt{\mu}}\Big)^p\Big] \xrightarrow{\mu\to 0} \mathbb{E}[(4 f_{\mathrm{ISE}}(\alpha))^{p/2} \mathcal{N}^p]. \tag{3.38}$$

Now consider the case of p odd. By the proof of Lemma 3.1 (equation (3.2)), we directly have

$$\Big|\mathbb{E}\Big[\Big(\frac{\widetilde{\Delta}^n_{a+1} - \widetilde{\Delta}^n_{b+1}}{\sqrt{n}}\Big)^p\Big]\Big| \leq C_p \mu^{p/2+1/2}.$$

Using the convergence of moments justified above, we deduce by taking the limit $n \to \infty$,

$$\Big|\mathbb{E}\Big[\Big(\frac{f'_{\mathrm{ISE}}(\alpha+\mu) - f'_{\mathrm{ISE}}(\alpha)}{\sqrt{\mu}}\Big)^p\Big]\Big| \leq C_p \sqrt{\mu},$$

and we deduce by taking the limit $\mu \to 0$ that (3.38) also holds for odd p (in that case, the right-hand side is null, since the Gaussian variable has null odd moments). This implies the convergence in distribution (3.32) (note that moments of $f_{\mathrm{ISE}}(\alpha)$ do not grow too fast, see, e.g., [3]).

Now take $\alpha = 0$, since $f_{\mathrm{ISE}}(0) > 0$ almost surely, this implies that f'_{ISE} is, almost surely, not differentiable at 0. By rerooting invariance, this implies that f'_{ISE} is, almost surely, differentiable almost nowhere inside of its support. The statement about non-hölderianity is obtained in the same way. ∎

Remark 3.8. It is possible, at the cost of heavier notation but with the same tools and without new significant difficulty, to enrich the counting techniques developed throughout Section 3 to estimate joint moments of the form

$$\mathbb{E}\Big[\prod_{i=1}^{A} (\Delta^n_{a_i} - \Delta^n_{b_i})^{r_i} (\Delta^n_{a_i})^{s_i} (M^n_{a_i})^{t_i}\Big] \tag{3.39}$$

for integer numbers a_i, b_i and $r_i, s_i, t_i \geq 0$. To do this, one only has to consider more general schemes in which the marked vertices can be of three types (corresponding to the three types of factors in (3.39)). The generating function W_e corresponding to each type of edges can be computed as before. At the asymptotic level, the same phenomenon will appear, and the dominating contributions are the ones in which for each $i \in [A]$, the r_i vertices of the first type share their attachment points in pairs. When furthermore $a_i - b_i = \mu_i n^{1/4}$ with μ_i small, each attachment vertex plays the same role as a vertex of label a_i up to easily identified factors. In this way, one can prove for even r_i,

$$\mathbb{E}\Big[\prod_{i=1}^{A}\Big(\frac{\Delta^n_{a_i}-\Delta^n_{b_i}}{\sqrt{\mu_i n}}\Big)^{r_i}\Big(\frac{\Delta^n_{a_i}}{n^{3/8}}\Big)^{s_i}\Big(\frac{M^n_{a_i}}{n^{3/4}}\Big)^{t_i}\Big]$$
$$= \mathbb{E}\Big[\prod_{i=1}^{A}\Big(\sqrt{\frac{2M^n_{a_i}}{n^{3/4}}}\mathcal{N}_i\Big)^{r_i}\Big(\frac{\Delta^n_{a_i}}{n^{3/8}}\Big)^{s_i}\Big(\frac{M^n_{a_i}}{n^{3/4}}\Big)^{t_i}\Big]$$
$$\times (1+o(1)+O(\max(\mu_i^{-1})n^{-1/4}+\max(\mu_i))),$$

when $a_i, b_i = O(n^{1/4})$ with $b_i - a_i = \mu_i n^{1/4} \geq 3$, and where the $\mathcal{N}_i$ are standard independent Gaussian random variables (centered, and having variance 1), independent of everything else. If one r_i is odd, the left-hand side is an $O(\max(\mu_i^{1/2}))$.

It follows, by taking the limit when n goes to infinity, that the convergence of Proposition 3.7 can be strengthened as follows. For any $\alpha_1, \ldots, \alpha_A$, we have the convergence in law of the vectors

$$\Big[\Big(f'_{\mathrm{ISE}}(\alpha_i), f_{\mathrm{ISE}}(\alpha_i), \frac{f'_{\mathrm{ISE}}(\alpha_i+\mu_i)-f'_{\mathrm{ISE}}(\alpha_i)}{\sqrt{\mu_i}}\Big), 1 \leq i \leq A\Big]$$
$$\xrightarrow[\mu_i\to 0^+,\forall i]{(d)} [(f'_{\mathrm{ISE}}(\alpha_i), f_{\mathrm{ISE}}(\alpha_i), 2\sqrt{f_{\mathrm{ISE}}(\alpha_i)}\mathcal{N}_i, 1 \leq i \leq A)],$$

where in the right-hand side, the $\mathcal{N}_i$ are i.i.d. $\mathcal{N}(0,1)$ Gaussian random variables, independent of $[(f'_{\mathrm{ISE}}(\alpha_i), f_{\mathrm{ISE}}(\alpha_i), 1 \leq i \leq A)]$. Setting up all the notation for a formal proof would go beyond the intent of this note, and we hope that more direct diffusion approximation techniques might give another approach to such results.

4. Proof of the diffusion approximation

In this section, we prove Theorem 2.7. We rely on the main theorem in Kushner [16], and we keep the notation of this reference: we warn the reader that without that paper in hand, this section should be difficult to understand, and the notation should seem a bit strange, since they are designed to treat more general processes.

We mainly work with $\xi^{n,\star} = (\delta^{n,\star}, m^{n,\star}, s^{n,\star})$ and use the stopped version (when $m^{n,\star} = \varepsilon$) only when needed. We denote by $\mathcal{B}^n_k$ the σ-algebra generated by the $(\xi^{n,\star}_i, i \leq k)$ (which includes $(\Delta^\star_0, M^\star_0, S^\star_0)$). We denote by $\mathsf{E}_{\mathcal{B}^n_k}$ the conditional expectation with respect to $\mathcal{B}^n_k$.

For all i, set $dt^n_i = n^{-1/4}$ (this is the homogeneous time increment). We have

$$\begin{aligned}
\mathsf{E}_{\mathcal{B}^n_k}(\xi^{n,\star}_{k+1}-\xi^{n,\star}_k) &= f_n(\xi^{n,\star}_k, t^n_k)\, dt^n_k, \\
\mathsf{Cov}_{\mathcal{B}^n_k}(\xi^{n,\star}_{k+1}-\xi^{n,\star}_k) &= \sigma^n_k(\xi^{n,\star}_k, t^n_k)\, \sigma^{n\mathsf{T}}_k(\xi^{n,\star}_k, t^n_k)\, dt^n_k,
\end{aligned} \tag{4.1}$$

where T denotes the transpose, with

$$f_n\left(\begin{bmatrix} d\\ m\\ s\end{bmatrix},t\right)=\begin{bmatrix}0\\ d\\ m\end{bmatrix},\quad \sigma_n\left(\begin{bmatrix} d\\ m\\ s\end{bmatrix},t\right)=\sqrt{2|m|}\begin{bmatrix}1\\ n^{-1/4}\\ n^{-1/2}\end{bmatrix}$$

(they are homogeneous, so that f_n as well as σ_n do not depend on t).

The functions f and σ have been defined in Theorem 2.7; for any n, set

$$nf_n=f\quad\text{and}\quad(\sigma_n-\sigma)\left(\begin{bmatrix} d\\ m\\ s\end{bmatrix},t\right)=\sqrt{2|m|}\begin{bmatrix}0\\ n^{-1/4}\\ n^{-1/2}\end{bmatrix}. \tag{4.2}$$

Formula (4.1) comes from the following simple facts:

$$\begin{aligned}
\mathsf{Var}_{\mathcal{B}_k^n}(\Delta^\star_{k+1}-\Delta^\star_k)&=2|M^\star_k|,\\
\mathsf{Var}_{\mathcal{B}_k^n}(M^\star_{k+1}-M^\star_k)&=\mathsf{Var}_{\mathcal{B}_k^n}(\Delta^\star_{k+1})=\mathsf{Var}_{\mathcal{B}_k^n}(\Delta^\star_{k+1}-\Delta^\star_k)=2|M^\star_k|,\\
\mathsf{Var}_{\mathcal{B}_k^n}(S^\star_{k+1}-S^\star_k)&=\mathsf{Var}_{\mathcal{B}_k^n}(M^\star_{k+1})=\mathsf{Var}_{\mathcal{B}_k^n}(M^\star_{k+1}-M^\star_k)=2|M^\star_k|,\\
\mathsf{Cov}_{\mathcal{B}_k^n}(M^\star_{k+1}-M^\star_k,\Delta^\star_{k+1}-\Delta^\star_k)&=\mathsf{Cov}_{\mathcal{B}_k^n}(\Delta^\star_{k+1},\Delta^\star_{k+1})=2|M^\star_k|,\\
\mathsf{Cov}_{\mathcal{B}_k^n}(S^\star_{k+1}-S^\star_k,\Delta^\star_{k+1}-\Delta^\star_k)&=\mathsf{Cov}_{\mathcal{B}_k^n}(M^\star_{k+1},\Delta^\star_{k+1}-\Delta^\star_k)=2|M^\star_k|,\\
\mathsf{Cov}_{\mathcal{B}_k^n}(S^\star_{k+1}-S^\star_k,M^\star_{k+1}-M^\star_k)&=\mathsf{Cov}_{\mathcal{B}_k^n}(M^\star_{k+1},M^\star_{k+1}-M^\star_k)=2|M^\star_k|;
\end{aligned}$$

indeed, the variance of the geometric random variables involved is 2; we use also that if Y is $\mathcal{F}$-measurable, $\mathsf{Cov}_{\mathcal{F}}(X,Z)=\mathsf{Cov}_{\mathcal{F}}(X-Y,Z)=\mathsf{Cov}_{\mathcal{F}}(X-Y,Z-Y)$. The (non-rescaled) covariance matrix of $Z^\star_{k+1}-Z^\star_k$ is then

$$2|M^\star_k|\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=2|M^\star_k|\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix},$$

so that if one writes $\xi_k^{n,\star}(i)$ for the i-th entry of $\xi_k^{n,\star}$, the covariance matrix is

$$A_{\mathcal{B}_k^n}=2|M^\star_k|\left[\frac{1}{n^{(i+j+2)/4}}\right]_{1\le i,j\le 3}=\frac{2|M^\star_k|}{n^{3/4}}\left[\frac{1}{n^{(i+j-2)/4}}\right]_{1\le i,j\le 3}dt_k^n.$$

We can now prove the diffusion approximation.

Proof of Theorem 2.7. In Kushner [16], there are 7 conditions to check, called there (1) and (A1) to (A6).

Let us introduce the good set

$$\mathrm{GS}(\varepsilon,C)=[-C,C]\times\left[\frac{\varepsilon}{2},C\right]\times[0,C]$$

for some $C > m_0 > \varepsilon$. The drift and diffusion functions f and σ are bounded and Lipschitz on $\mathrm{GS}(\varepsilon, C)$, and this will give a sufficient condition for the existence and uniqueness of a solution of SDE (2.9) (see condition (A6) below).

It suffices to check that the processes $\xi^{n,\star}$ and $\xi^\star$ satisfy the 7 Kushner constraints (on $\mathrm{GS}(\varepsilon, C)$, and that they do so, for all $C > 0$).

Condition (1). First, we have

$$\sum_{k=0}^{N_n-1} |f_n(\xi_k^{n,\star}, t_k^n) - f(\xi_k^{n,\star}, t_k^n)|^2 dt_k^n = 0,$$

(by (4.2)) when $\xi^{(n),\star}$ stays in $\mathrm{GS}(\varepsilon, C)$. Second,

$$\sigma_n(\xi_k^{n,\star}, t_k^n) - \sigma(\xi_k^{n,\star}, t_k^n) = 2\sqrt{|\xi_k^{n,\star}(2)|}\begin{bmatrix} 0 \\ n^{-1/4} \\ n^{-1/2} \end{bmatrix}.$$

We then need to prove that $\mathsf{E}(\sum_{k=0}^{N_n-1} \|\sigma_n(\xi_k^{n,\star}, t_k^n) - \sigma(\xi_k^{n,\star}, t_k^n)\|^2) dt_k^n \to 0$ which is equivalent to

$$\left(\sum_{k=0}^{N_n-1} \mathsf{E}(2|M_k^\star|n^{-3/4}) n^{-1/4}\right) n^{-1/4} \to 0.$$

Now, on $\mathrm{GS}(\varepsilon, C)$, $|M_k^\star| \le C n^{3/4}$, so the result follows immediately.

Condition (A1). $\max_{0\le k\le N_n-1} dt_k^n = n^{-1/4} \to 0$ when $n \to +\infty$.

Condition (A2). f and σ are indeed continuous and bounded on $\mathrm{GS}(\varepsilon, C)$, and f_n and σ_n are uniformly bounded on $\mathrm{GS}(\varepsilon, C)$.

Condition (A3). This condition concerns the convergence of the initial distribution of $\xi^{n,\star}(0)$ (which is one of the hypotheses of Theorem 2.7).

Condition (A4). Set

$$B_n := \mathsf{E}\left(\sum_{k=0}^{N_n-1} \|\xi_{k+1}^{n,\star} - \xi_k^{n,\star} - f_n(\xi_k^{n,\star}, t_k^n) dt_k^n\|^{2+\alpha}\right).$$

In fact, only the first entry of the vector $\xi_{k+1}^{n,\star} - \xi_k^{n,\star} - f_n(\xi_k^{n,\star}, t_k^n) dt_k^n$ is not zero: One has, since $|M_k^\star| \le C n^{3/4}$,

$$\begin{aligned} B_n &= \sum_{k=0}^{N_n-1} \mathsf{E}(|W_{|M_k^\star|} n^{-1/2}|^{1+\alpha/2}) \le n^{1/4} \max_{0\le m\le Cn^{3/4}} \mathsf{E}(|W_{|m|} n^{-1/2}|^{1+\alpha/2}) \\ &= n^{1/4} \max_{0\le m\le Cn^{3/4}} \mathsf{E}(|W_{|m|}|^{1+\alpha/2}) n^{-1/2-\alpha/4}. \end{aligned}$$

We need to prove that for a well-chosen $\alpha > 0$, $B_n \to 0$. Now, since the increments of W have all finite moments, by Marcinkiewicz–Zygmund inequality (see, e.g., [12])

$$\mathsf{E}(|W_{|n|}|^{1+\alpha/2}) \leq B(2Cn^{3/4})^{(1/2+\alpha/4)},$$

where the 2 comes from the variance of $g^{(k)} - 1$, and B is a positive function of α. It suffices then to take α such that

$$\frac{1}{4} + \Big(\frac{3}{4}\Big)\Big(\frac{1}{2} + \frac{\alpha}{4}\Big) - \frac{1}{2} - \frac{\alpha}{4} = \frac{1}{8} - \frac{\alpha}{16} < 0$$

and any $\alpha > 2$ does the job.

Condition (A5). Since $\frac{dt^n_{k+1}}{dt^n_k} = 1$, this condition is satisfied.

Condition (A6). Here, $f(\cdot,\cdot)$ is Lipschitz as well as σ on $\mathrm{GS}(\varepsilon, C)$, except at its boundary. These conditions are sufficient to entail the existence and uniqueness of the solution of SDE (2.9), started at z_0 (see [22, Theorem 5.2.1]), stopped when $m^\star = \varepsilon$. Notice that the value $\frac{\varepsilon}{2}$ in the second component in $\mathrm{GS}(\varepsilon, C)$ is taken smaller than ε, which is the point at which we stop $m^\star_n$ and $m^\star$. The value ε being at the interior of $[\frac{\varepsilon}{2}, C]$, one sees that the domain on which one can extend the existence of the solution of the SDE is sufficient to entail the convergence of the stopped version of $\zeta^{n,\star}$ to the stopped version of $\zeta^\star$. ■

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