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Blobbed topological recursion of the quartic Kontsevich model II: Genus = 0

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(with an appendix by Maciej Dołęga)

Abstract. We prove that the genus-0 sector of the quartic analogue of the Kontsevich model is completely governed by an involution identity which expresses the meromorphic differential $\omega_{0,n}$ at a reflected point ιz in terms of all $\omega_{0,m}$ with $m \leq n$ at the original point z. We prove that the solution of the involution identity obeys blobbed topological recursion, which confirms a previous conjecture about the quartic Kontsevich model.

1. Introduction and main result

1.1. Overview

This paper completes the solution of the genus-0 sector of the quartic analogue of the Kontsevich model. This is a model for $N \times N$ Hermitian matrices with the same covariance as the Kontsevich model [29] but with quartic instead of cubic potential. The non-linear Dyson–Schwinger equation [23] for the planar 2-point function of the quartic Kontsevich model was solved in a special case in [31] and then in full generality in [22, 33]. Building on this foundation, we identified in [12] three families of correlation functions and established interwoven loop equations between them. One family consists of meromorphic differential forms $\omega_{g,n}$ labelled by genus g and number n of marked points of a complex curve. By a lengthy evaluation of residues, the solution was found for $\omega_{0,2}$, $\omega_{0,3}$, $\omega_{0,4}$, and $\omega_{1,1}$. It strongly suggested that the family $\omega_{g,n}$ obeys blobbed topological recursion [7], a systematic extension of topological recursion [19] by additional terms holomorphic at ramification points of a covering $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

It is worth mentioning that our quartic Kontsevich model does not fit into the class of generalised Kontsevich models (see [11] for more details). The class of generalised

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Kontsevich models [4] is known to be governed by topological recursion with higher-order ramification [8]. However, the quartic Kontsevich model studied in this article is expected to satisfy blobbed topological recursion.

Recall that (blobbed) topological recursion (see, e.g., [7, 19] and references therein) starts from a spectral curve $(x : \Sigma \to \Sigma_0, \omega_{0,1} = y\,dx, \omega_{0,2})$. Here, $y : \Sigma \to \hat{\mathbb{C}}$ is also a covering such that $\omega_{0,1} = y\,dx$ is regular at the ramification points of x and $\omega_{0,2}$ is a symmetric bidifferential which extends (or is equal to) the Bergman kernel. We noticed in [12] that the two coverings x, y in the quartic Kontsevich model are related by $y(z) = -x(-z)$ (already visible in [22, 33]) and that

$$\omega_{0,2}(u, z) = -\omega_{0,2}(u, -z).$$

We show in this paper that this observation is far more than a coincidence: the properties of $x, y, \omega_{0,2}$ under reflection

$$z \mapsto \iota z := -z$$

completely characterise the genus-0 sector of the quartic Kontsevich model. There is a single global equation (1.4) which describes the behaviour of the $\omega_{0,n}$ under reflection. This equation can be solved, without connecting it to the matrix model, in the case of a general holomorphic involution $\iota z = \frac{az+b}{cz-a}$ of the Riemann sphere, with $a, b, c \in \mathbb{C}$ such that $a^2 + bc \neq 0$. The solution obeys blobbed topological recursion [7] (restricted to genus $g = 0$) and is for $b = c = 0$ identical to the solution of the complicated system of loop equations in [12]. We observe that the reflection formula (1.4) is related to the functional relations studied in the context of x-y symmetry in topological recursion (see, for instance, [20, Proposition 3.1], [27, Proposition 4.7], and [5]).

It is currently not known to us how to extend these results to genus $g > 0$. The loop equations of [12] bring in another structure which leads to poles of $\omega_{g>0,n}$ at the fixed point of the involution ι. In [28], we develop a different strategy for this situation. Nevertheless, we speculate that a global involution $z \mapsto \iota z$ is compatible with blobbed topological recursion and describes a geometric structure of the moduli space of stable complex curves.

1.2. Statement of the result

Let $x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a ramified covering of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with simple ramification points $\beta_1, \ldots, \beta_r$ (which solve $dx(\beta_i) = 0$). Let $\iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic global involution, $\iota^2(z) = z$, which

- does not fix or permute any ramification point(s)
- and such that $\iota\beta_i$ is not a pole of x for $i = 1, \ldots, r$.

Automorphisms of the Riemann sphere are Möbius transformations, and the involutions of them (different from the identity) are of the form

$$\iota z = \frac{az+b}{cz-a}, \qquad a,b,c \in \mathbb{C} \text{ with } a^2+bc \neq 0. \tag{1.1}$$

Another ramified covering of the Riemann sphere is introduced by[1]

$$y = -x \circ \iota : \hat{\mathbb{C}} \to \hat{\mathbb{C}}. \tag{1.2}$$

The assumptions imply that y is holomorphic at the ramification points β_i of x. The data are completed by a unique (up to a global constant factor) bidifferential $\omega_{0,2}$ on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ which is symmetric, odd under the involution of one variable and has a double pole on the diagonal without residue. These conditions give

$$\begin{aligned}\omega_{0,2}(w,z) &= \frac{1}{2}\frac{dw\,dz}{(w-z)^2} + \frac{1}{2}\frac{d(\iota w)\,d(\iota z)}{(\iota w-\iota z)^2} - \frac{1}{2}\frac{dw\,d(\iota z)}{(w-\iota z)^2} - \frac{1}{2}\frac{d(\iota w)\,dz}{(\iota w-z)^2}\\ &= \frac{dw\,dz}{(w-z)^2} + \frac{(a^2+bc)dw\,dz}{\big(cwz-a(w+z)-b\big)^2}.\end{aligned} \tag{1.3}$$

These data are extended by the following definition (see the first paragraph of Section 2.1 for the notation).

Definition 1.1 (Involution identity). A family $\{\omega_{0,m+1}\}_{m\geq 1}$ of meromorphic differentials on $\hat{\mathbb{C}}^{m+1}$ is introduced by (1.3) for $m=1$ and for $m \geq 2$ by

$$\begin{aligned}&\omega_{0,|I|+1}(I,q) + \omega_{0,|I|+1}(I,\iota q)\\ &= \sum_{s=2}^{|I|} \sum_{I_1 \uplus \cdots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{z\to q} \bigg(\frac{dy(q)dx(z)}{(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \bigg),\end{aligned} \tag{1.4}$$

where $I = \{u_1,\dots,u_m\}$.

We show that, under mild assumptions, the identity (1.4) completely determines $\omega_{0,|I|+1}(I,q)$, and that the required symmetry of the rhs of (1.4) under $q \mapsto \iota q$ is automatic.

Theorem 1.2. *For $\omega_{0,|I|+1}(I,z)$ with $I = \{u_1,\dots,u_m\}$ of length $|I| := m$, the following conventions are given:*

(a) *$\omega_{0,2}$ is given by (1.3);*

(b) *the meromorphic form $z \mapsto \omega_{0,|I|+1}(I,z)$ has for $m \geq 2$ poles at most in points z, where the rhs of (1.4) has poles;*

[1] This could be generalised to $y(z) = d - x(\iota z)$ for any $d \in \mathbb{C}$.

(c) $z \mapsto \omega_{0,|I|+1}(I,z)$ *is for* $m \geq 2$ *holomorphic at any* $z = u_k$*;*

(d) $z \mapsto \omega_{0,|I|+1}(I,\iota z)$ *is holomorphic at any ramification point* β_i *of* x*.*

Then, (1.4) *is for* $I = \{u_1,\dots,u_m\}$ *with* $m \geq 2$ *uniquely solved by*

$$\begin{aligned}
&\omega_{0,|I|+1}(I,z) \\
&= \sum_{i=1}^{r} \operatorname*{Res}_{q\to\beta_i} K_i(z,q) \sum_{I_1\uplus I_2=I} \omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q)) \\
&\quad - \sum_{k=1}^{m} d_{u_k}\Big[\operatorname*{Res}_{q\to\iota u_k} \sum_{I_1\uplus I_2=I} \tilde{K}(z,q,u_k) d_{u_k}^{-1}\big(\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,q)\big)\Big].
\end{aligned} \tag{1.5}$$

Here, $\beta_1,\dots,\beta_r$ *are the ramification points of* x *and* $\sigma_i \neq \mathrm{id}$ *denotes the local Galois involution in the vicinity of* β_i*, i.e.,* $x(\sigma_i(z)) = x(z)$, $\lim_{z\to\beta_i}\sigma_i(z) = \beta_i$*. By* d_{u_k} *we denote the exterior differential in* u_k*, which on 1-forms has a right inverse given by the primitive* $d_u^{-1}\omega(u) = \int_{u'=\infty}^{u'=u}\omega(u')$*. The recursion kernels are given by*

$$\begin{aligned}
K_i(z,q) &:= \frac{\frac{1}{2}\big(\frac{dz}{z-q} - \frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))}, \\
\tilde{K}(z,q,u) &:= \frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q} - \frac{d(\iota z)}{\iota z-u}\big)}{dx(q)(y(q)-y(\iota u))}.
\end{aligned} \tag{1.6}$$

The solution (1.5)+(1.6) *implies symmetry of* (1.4) *under* $q \mapsto \iota q$*.*

The proof is lengthy and will be divided into many steps. We rely on combinatorial identities proved in an appendix by Maciej Dołęga. We start to prove uniqueness: *if* a consistent solution of (1.4) exists, it must be of the form (1.5)+(1.6). Then, we prove that (1.5)+(1.6) implies consistency of (1.4).

In a second part, we show that the loop equations [12] of the quartic analogue of the Kontsevich model lead for the choice $\iota z = -z$ (i.e., $b = c = 0$) and $x(z) = R(z) = z - \lambda\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k+z}$ found in [33] to exactly the same solution (1.5)+(1.6). Thereby, we prove for genus 0 the main conjecture of [12] that the quartic Kontsevich model obeys blobbed topological recursion [7].

2. Proof of Theorem 1.2

2.1. Tools and conventions

Throughout this paper, we denote by $q,u,u_k,w,z,z_k \in \hat{\mathbb{C}}$ complex numbers and by $a,a',i,j,k,l,m,n,n_0,\dots,n_s,p,r,s,s'$ non-negative integers. By $I = \{u_1,\dots,u_m\}$

we understand a (multi-)set of length $|I| = m$ of complex numbers, which are allowed to coincide. By $\sum_{I_1 \uplus \cdots \uplus I_s = I}$ we denote the sum over all partitions of the multiset I into *disjoint non-empty* subsets $I_1, \ldots, I_s$ of any order. If we insist on a sum over ordered subsets, we write $\sum_{\substack{I_1 \uplus \cdots \uplus I_s = I \\ I_1 < \cdots < I_s}}$. We define $\{u_{k_1}, \ldots, u_{k_n}\} < \{u_{l_1}, \ldots, u_{l_m}\}$ iff $\min(k_i) < \min(l_j)$. This order is well defined because the subsets $\{u_{k_1}, \ldots, u_{k_n}\}$, $\{u_{l_1}, \ldots, u_{l_m}\}$ are disjoint. We will often write $I \uplus u_k$ or (I, u_k) for $I \uplus \{u_k\}$ and $I \setminus u_k$ for $I \setminus \{u_k\}$. In the second part, $\hat{q}^j$ for $j \geq 1$ denotes another preimage of q under x, i.e., $x(\hat{q}^j) = x(q)$.

Example 2.1. The set $I = \{u_1, u_2, u_3\}$ has 13 (Fubini number[2]) different partitions

$$\begin{aligned}
\{u_1, u_2, u_3\}, \quad & \{u_1\} \uplus \{u_2, u_3\}, \quad \{u_2\} \uplus \{u_3, u_1\}, \quad \{u_3\} \uplus \{u_1, u_2\}, \\
& \{u_1, u_2\} \uplus \{u_3\}, \quad \{u_2, u_3\} \uplus \{u_1\}, \quad \{u_3, u_1\} \uplus \{u_2\}, \\
& \{u_1\} \uplus \{u_2\} \uplus \{u_3\}, \quad \{u_2\} \uplus \{u_3\} \uplus \{u_1\}, \quad \{u_3\} \uplus \{u_1\} \uplus \{u_2\}, \\
& \{u_1\} \uplus \{u_3\} \uplus \{u_2\}, \quad \{u_2\} \uplus \{u_1\} \uplus \{u_3\}, \quad \{u_3\} \uplus \{u_2\} \uplus \{u_1\}
\end{aligned}$$

and $B_3 = 5$ (Bell number[3]) ordered partitions

$$\begin{aligned}
& \{u_1, u_2, u_3\}, \quad \{u_1\} \uplus \{u_2, u_3\}, \quad \{u_1, u_2\} \uplus \{u_3\}, \quad \{u_1, u_3\} \uplus \{u_2\}, \\
& \{u_1\} \uplus \{u_2\} \uplus \{u_3\}.
\end{aligned}$$

We will often need the projection of a meromorphic 1-form ω to the principal part $P^w \omega$ of its Laurent series about $w \in \mathbb{C}$. This projection is obtained by the residue

$$P^w \omega(z) = \operatorname*{Res}_{q \to w} \frac{\omega(q) dz}{z - q}.$$

In case of $w = \beta_i$ (a ramification point of x), we abbreviate

$$\mathcal{P}_z^i \omega(z) := P^{\beta_i} \omega(z).$$

An important tool will be the commutation rule of two iterated residues of a 1-form $\omega(q, z)$ in both complex variables q, z:

$$\operatorname*{Res}_{q \to w} \operatorname*{Res}_{z \to q} \omega(q, z) + \operatorname*{Res}_{q \to w} \operatorname*{Res}_{z \to w} \omega(q, z) = \operatorname*{Res}_{z \to w} \operatorname*{Res}_{q \to w} \omega(q, z). \tag{2.1}$$

It is an immediate consequence of contour integrations and holds under the assumption that $q = z, q = w, z = w$ are the only poles in a sufficiently small neighbourhood of w. We will encounter a situation where this assumption does not hold. In the

[2] OEIS A000670, visited on 11 July 2024.
[3] OEIS A000110, visited on 11 July 2024.

vicinity of a ramification point β_i, the contour integral must also enclose the local Galois conjugate $\sigma_i(q)$:

$$\operatorname*{Res}_{q\to\beta_i}\operatorname*{Res}_{z\to q}\omega(q,z)+\operatorname*{Res}_{q\to\beta_i}\operatorname*{Res}_{z\to\beta_i}\omega(q,z)+\operatorname*{Res}_{q\to\beta_i}\operatorname*{Res}_{z\to\sigma_i(q)}\omega(q,z)=\operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i}\omega(q,z). \tag{2.2}$$

The residue commutes with partial or exterior derivatives:

$$\begin{aligned}\operatorname*{Res}_{z\to u}\partial_u f(u,z)dz &= \partial_u\big(\operatorname*{Res}_{z\to u} f(u,z)dz\big),\\ \operatorname*{Res}_{z\to u} d_u f(u,z)dz &= d_u\big(\operatorname*{Res}_{z\to u} f(u,z)dz\big).\end{aligned} \tag{2.3}$$

To see this, let $\gamma_\epsilon(w)$ be the loop with centre w and radius ϵ. Then, for $\epsilon>2\delta>0$,

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}\delta}\bigg(\int_{\gamma_\epsilon(u+\delta)} f(u+\delta,z)dz-\int_{\gamma_\epsilon(u)} f(u,z)dz\bigg)\\ &=\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\epsilon(u)}\frac{f(u+\delta,z)-f(u,z)}{\delta}dz.\end{aligned}$$

The limit $\delta\to 0$ together with independence of all integrals from ϵ gives (2.3).

The residue does not change under the local Galois involution, that is,

$$\operatorname*{Res}_{q\to\beta_i}\omega(q)=\operatorname*{Res}_{q\to\beta_i}\omega(\sigma_i(q)). \tag{2.4}$$

Invariance of the term $\frac{c_{-1}dq}{q-\beta_i}$ of the Laurent expansion follows from $\sigma_i(q)-\beta_i=-(q-\beta_i)+\mathcal{O}((q-\beta_i)^2)$. For poles of order $n>1$, the term

$$\frac{c_{-n}d\sigma_i(q)}{(\sigma_i(q)-\beta_i)^n}=-\frac{1}{n}d\frac{c_{-n}}{(\sigma_i(q)-\beta_i)^{n-1}}$$

does not have a residue.

Of particular importance will be the following residue:

$$\begin{aligned}\nabla^n\omega_{0,|I|+1}(I,q) &:= \operatorname*{Res}_{z\to q}\frac{\omega_{0,|I|+1}(I,z)}{(y(z)-y(q))(x(q)-x(z))^n}\\ &\equiv\frac{(-1)^n}{n!}\lim_{z\to q}\frac{\partial^n}{\partial(x(z))^n}\bigg(\frac{x(z)-x(q)}{y(z)-y(q)}\frac{\omega_{0,|I|+1}(I,z)}{dx(z)}\bigg),\end{aligned} \tag{2.5}$$

which is a function of q and a 1-form in every variable in I. In particular,

$$\nabla^0\omega_{0,|I|+1}(I,q)=\frac{\omega_{0,|I|+1}(I,q)}{dy(q)}.$$

These functions arise in the Taylor expansion

$$\frac{x(z)-x(q)}{y(z)-y(q)}\frac{\omega_{0,|I|+1}(I,z)}{dx(z)}=\sum_{n=0}^{\infty}(x(q)-x(z))^n\nabla^n\omega_{0,|I|+1}(I,q). \tag{2.6}$$

Lemma 2.2. *The involution identity* (1.4) *can be expressed as*

$$\begin{aligned}
&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\
&\quad=-dy(q)\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{1}{s}\sum_{n_1+\cdots+n_s=s-1}\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q),
\end{aligned}$$

where $\sum_{n_1+\cdots+n_s=s-1}$ *is the sum over all partitions of* $s-1$ *into integers* $n_i\geq 0$.

Proof. We evaluate the residue (1.4) as limit of a partial derivative:

$$\begin{aligned}
&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\
&=\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{(-1)^s}{s}\operatorname*{Res}_{z\to q}\left(\frac{dy(q)dx(z)}{(x(z)-x(q))^s}\prod_{j=1}^{s}\frac{x(z)-x(q)}{y(z)-y(q)}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\right)\\
&=\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{(-1)^s}{s!}dy(q)\lim_{z\to q}\frac{\partial^{s-1}}{\partial(x(z))^{s-1}}\left(\prod_{j=1}^{s}\frac{x(z)-x(q)}{y(z)-y(q)}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\right).
\end{aligned}$$

Leibniz's rule for a higher derivative of a product together with (2.5) gives the assertion. ∎

Iterating Lemma 2.2, we obtain a variant with only a single term $\nabla\omega$.

Lemma 2.3. *The involution identity* (1.4) *can also be expressed as*

$$\begin{aligned}
&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\
&\quad=-dy(q)\sum_{s=1}^{|I|-1}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I}\nabla^{s}\omega_{0,|I_0|+1}(I_0,q)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota q)}{-dy(q)}.
\end{aligned}\tag{2.7}$$

Proof. By induction on $|I|$, starting from the true statement for $|I|=1$. For a partition $I=I_1\uplus\cdots\uplus I_s$ of $I=\{u_1,\ldots,u_m\}$ into $s\geq 2$ subsets together with a given partition $n_1+\cdots+n_s=s-1$, we let $u_\mu=\min(\biguplus_{\substack{k=1\\ n_k>0}}^{s}I_k)$ be the smallest element within those I_k with $n_k>0$. Moving the subset which contains u_μ to the first place allows us to get rid of the $\frac{1}{s}$-factor in Lemma 2.2:

$$\begin{aligned}
&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\
&\quad=-dy(q)\sum_{s=1}^{|I|-1}\sum_{n_0+\cdots+n_s=s}\sum_{\substack{I_0\uplus I_1\uplus\cdots\uplus I_s=I\\ u_\mu\in I_0}}\nabla^{n_0}\omega_{0,|I_0|+1}(I_0,q)\\
&\qquad\times\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q).
\end{aligned}\tag{2.8}$$

By construction, we have $n_0 > 0$. Take on the right-hand side of (2.8) a term of the form $X\nabla^0\omega_{0,|I_p|+1}(I_p,q)$ for any product X which contains I_0. Observe that (2.8) then contains for $|I_p|\geq 2$ also every term of the sum

$$X\sum_{k=2}^{|I_p|}\sum_{I_1'\uplus\cdots\uplus I_k'=I_p}\sum_{m_1+\cdots+m_k=k-1}\prod_{\ell=1}^{k}\nabla^{m_\ell}\omega_{0,|I_\ell'|+1}(I_\ell',q).$$

By induction hypothesis and with

$$\nabla^0\omega_{0,|I_p|+1}(I_p,q)=\frac{\omega_{0,|I_p|+1}(I_p,q)}{dy(q)},$$

we have

$$\begin{aligned}X\nabla^0\omega_{0,|I_p|+1}(I_p,q)+X\sum_{k=2}^{|I_p|}\sum_{I_1'\uplus\cdots\uplus I_k'=I_p}\sum_{m_1+\cdots+m_k=k-1}\prod_{\ell=1}^{k}\nabla^{m_\ell}\omega_{0,|I_\ell'|+1}(I_\ell',q)\\=X\frac{\omega_{0,|I_p|+1}(I_p,\iota q)}{-dy(q)},\end{aligned}$$

which is also true in case

$$|I_p|=1.$$

Repeat this procedure for the next $X\nabla^0\omega_{0,|I_p|+1}(I_p;q)$ for which the product X does not yet contain a factor $\frac{\omega_{0,|I_j|+1}(I_j,\iota q)}{-dy(q)}$. At the end of this procedure, the rhs of (2.8) is reduced to a sum of terms, each containing a factor $\frac{\omega_{0,|I_j|+1}(I_j,\iota q)}{-dy(q)}$.

Now, iterate the procedure for every $X\nabla^0\omega_{|I_p|+1}(I_p,q)$, where the product X contains $\nabla^{n_0}\omega_{|I_0|+1}(I_0;q)$ and precisely one factor $\frac{\omega_{0,|I_j|+1}(I_j,\iota q)}{-dy(q)}$. At the end of this step, we have reduced the second line of (2.8) to terms of the form

$$\nabla^1\omega_{|I_0|+1}(I_0;q)\frac{\omega_{0,|I_1|+1}(I_1,\iota q)}{-dy(q)}$$

or with a double factor $\prod_{j=1}^{2}\frac{\omega_{0,|I_j'|+1}(I_j',\iota q)}{-dy(q)}$. Iterate again until all ∇^0 in the second line of (2.8) are converted. This is the assertion. Since I_0 is anyway distinguished, we can omit the condition $u_\mu\in I_0$. ∎

2.2. Poles of $\omega_{0,m+1}(u_1,\dots,u_m,z)$ at $z=\iota u_k$

Lemma 2.4. *The involution identity* (1.4) *together with convention* (c) *in Theorem* 1.2 *that $\omega_{0,m+1}(u_1,\dots,u_m,z)$ is for $m\geq 2$ holomorphic at $z=u_k$ implies that*

$$z\mapsto\omega_{0,m+1}(u_1,\dots,u_m,\iota z)$$

has a pole at every $z = u_k$*. The principal part of the corresponding Laurent series is given by*

$$\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I, \iota q)dz}{z-q} = -d_{u_k}\Bigg[\sum_{s=1}^{|I|-1} \sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_k} \frac{1}{s!}\frac{\partial^s\big(\frac{1}{z-u_k}\big)}{\partial(y(u_k))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}\Bigg]dz. \tag{2.9}$$

Equivalently,

$$\omega_{0,|I|+1}(I,z) = -d_{u_k}\Bigg[\sum_{s=1}^{|I|-1} \sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_k} \frac{d(\iota z)}{s!}\frac{\partial^s\big(\frac{1}{\iota z-u_k}\big)}{\partial(y(u_k))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}\Bigg]$$
$$+ \textit{ terms which are holomorphic at } z = \iota u_k.$$

In particular, the poles of $\omega_{0,|I|+1}(I,z)$ *at* $z = \iota u_k$ *do not have a residue.*

Proof. We divide the involution identity (1.4) by $w - q$ and take the residue at $q = u_k$. By convention (c) in Theorem 1.2, the term $\omega_{0,|I|+1}(I, q)$ in the first line of (1.4) does not contribute to the residue. In the second line, we commute the two residues via (2.1). Since the inner integrand is holomorphic at $q = u_k$, we have

$$\begin{aligned}
&\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I, \iota q)}{w-q}\\
&= -\operatorname*{Res}_{q\to u_k}\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{1}{s}\operatorname*{Res}_{z\to u_k}\bigg(\frac{dy(q)dx(z)}{(w-q)(y(q)-y(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\bigg)\\
&= -\operatorname*{Res}_{q\to u_k}\sum_{s=1}^{|I|-1}\sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_k}\operatorname*{Res}_{z\to u_k}\bigg(\frac{dy(q)\omega_{0,2}(u_k,z)}{(w-q)(y(q)-y(z))^{s+1}}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\bigg).
\end{aligned}$$

We implemented the convention that there is only a pole at $z = u_k$ if a unique factor $\omega_{0,2}(u_k, z)$ is present. It can occur at all s places of the partition of I into s subsets so that $\frac{1}{s}$ cancels. We shifted $s - 1 \mapsto s$. We write $\omega_{0,2}(u_k, z)$ according to the second line of (1.3) and commute the differential d_{u_k} according to (2.3) in front of the residues:

$$\begin{aligned}
&\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I, \iota q)dw}{w-q}\\
&= -d_{u_k}\Bigg[\operatorname*{Res}_{q\to u_k}\sum_{s=1}^{|I|-1}\sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_k}\Bigg(\frac{dy(q)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}}{(w-q)(y(q)-y(u_k))^{s+1}}\Bigg)\Bigg]dw\\
&= -d_{u_k}\Bigg[\sum_{s=1}^{|I|-1}\sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_k}\frac{1}{s!}\frac{\partial^s\big(\frac{1}{w-q}\big)}{\partial(y(q))^s}\bigg|_{q=u_k}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}\Bigg]dw. \qquad (2.10)
\end{aligned}$$

This is the assertion (when renaming $w \mapsto z$). ■

We will derive an alternative formula as follows.

Proposition 2.5. *The poles of $z \mapsto \omega_{0,|I|+1}(I, \iota z)$ at $z = u_k$ can also be evaluated by*

$$\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I,\iota q)dz}{z-q} = -d_{u_k}\Bigg[\operatorname*{Res}_{q\to u_k} \sum_{I_0\uplus I_1=I} \frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-u_k}\big)}{dx(\iota q)(y(\iota q)-y(\iota u_k))} \times d_{u_k}^{-1}\big(\omega_{0,|I_0|+1}(I_0,\iota q)\omega_{0,|I_1|+1}(I_1,\iota q)\big)\Bigg].$$

Equivalently,

$$\omega_{0,|I|+1}(I,z) = -d_{u_k}\Bigg[\operatorname*{Res}_{q\to\iota u_k} \sum_{I_0\uplus I_1=I} \frac{\frac{1}{2}\big(\frac{d(\iota z)}{\iota z-\iota q}-\frac{d(\iota z)}{\iota z-u_k}\big)}{dx(q)(y(q)-y(\iota u_k))} d_{u_k}^{-1}\big(\omega_{0,|I_0|+1}(I_0,q)\omega_{0,|I_1|+1}(I_1,q)\big)\Bigg]$$
$$+ \textit{ terms which are holomorphic at } z = \iota u_k.$$

Proof. We shift $s \mapsto s-1$ in (2.9) and represent the term with $j = 0$ via Lemma 2.3 for $q \mapsto \iota u_k$:

$$\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I,\iota q)dw}{w-q}$$
$$= -d_{u_k}\Bigg[\sum_{s=0}^{|I|-2} \sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I\setminus u_k} \frac{1}{(s+1)!}\frac{\partial^{s+1}\big(\frac{1}{w-q}\big)}{\partial(y(q))^{s+1}}\Big|_{q=u_k} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}$$
$$\times \frac{(-dy(\iota u_k))}{dx(u_k)} \sum_{n=0}^{|I_0|-1} \sum_{I'_0\uplus\cdots\uplus I'_n=I_0} \nabla^n\omega_{0,|I'_0|+1}(I'_0,\iota u_k)\prod_{\ell=1}^{n}\frac{\omega_{0,|I'_\ell|+1}(I'_\ell,u_k)}{-dy(\iota u_k)}\Bigg]dw.$$

We have included the term $\omega_{|I_0|+1}(I_0,\iota u_k) = dy(\iota u_k)\nabla^0\omega_{|I_0|+1}(I_0,\iota u_k)$ as $n = 0$. Implementing (1.2), i.e., $-dy(\iota u_k) = dx(u_k)$ suggests changing summation variables to $s+n \mapsto s \in [0\ldots|I|-2]$. Then, we express the derivative with respect to $y(q)$ as a residue:

$$\operatorname*{Res}_{q\to u_k} \frac{\omega_{0,|I|+1}(I,\iota q)dw}{w-q}$$
$$= -d_{u_k}\Bigg[\sum_{s=0}^{|I|-2}\sum_{n=0}^{s} \sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I\setminus u_k} \frac{1}{(s+1-n)!}\frac{\partial^{s+1-n}\big(\frac{1}{w-q}\big)}{\partial(y(q))^{s+1-n}}\Big|_{q=u_k}$$
$$\times \nabla^n\omega_{0,|I_0|+1}(I_0,\iota u_k)\prod_{j=1}^{s}\frac{\omega_{0,|I_i|+1}(I_j,u_k)}{dx(u_k)}\Bigg]dw$$

$$
\begin{aligned}
&= -d_{u_k}\Big[\operatorname*{Res}_{q\to u_k}\sum_{s=0}^{|I|-2}\sum_{n=0}^{s}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I\backslash u_k}\frac{dy(q)}{(w-q)(y(q)-y(u_k))^{s-n+2}}\\
&\qquad\qquad\times\nabla^n\omega_{0,|I_0|+1}(I_0,\iota u_k)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}\Big]dw\\
&= -d_{u_k}\Big[\operatorname*{Res}_{q\to u_k}\sum_{s=0}^{|I|-2}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I\backslash u_k}\Big(\frac{dy(q)}{w-q}-\frac{dy(q)}{w-u_k}\Big)\frac{1}{(y(q)-y(u_k))^{s+2}}\\
&\qquad\times\prod_{j=1}^{s}\frac{\omega_{0,|I_i|+1}(I_j,u_k)}{dx(u_k)}\sum_{n=0}^{s}(x(\iota u_k)-x(\iota q))^n\nabla^n\omega_{0,|I_0|+1}(I_0,\iota u_k)\Big]dw.
\end{aligned}
$$

The term $\frac{dy(q)}{w-u_k}$ added in the last step has vanishing residue (obvious before setting $y(q)=-x(\iota q)$). It is added in order to extend the n-summation to any $n\geq 0$, giving with (2.6) for $q\mapsto\iota u_k$, $z\mapsto\iota q$ and again (1.2)

$$
\begin{aligned}
&\operatorname*{Res}_{q\to u_k}\frac{\omega_{0,|I|+1}(I,\iota q)dw}{w-q}\\
&\quad= -d_{u_k}\Big[\operatorname*{Res}_{q\to u_k}\sum_{s=0}^{|I|-2}\sum_{I_0\uplus\cdots\uplus I_s=I\backslash u_k}\frac{\frac{dy(q)}{w-q}-\frac{dy(q)}{w-u_k}}{(x(q)-x(u_k))(y(q)-y(u_k))^{s+1}}\\
&\qquad\qquad\times\frac{\omega_{0,|I_0|+1}(I_0,\iota q)}{(-dy(q))}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}\Big]dw.
\end{aligned}
\tag{2.11}
$$

The first two lines of (2.10) tell us that

$$
\begin{aligned}
&\sum_{s=1}^{|I'|}\sum_{I_1\uplus\cdots\uplus I_s=I'}\frac{dy(q)}{(y(q)-y(u_k))^{s+1}}\prod_{i=1}^{s}\frac{\omega_{0,|I_i|+1}(I_i,u_k)}{dx(u_k)}\\
&\quad= -d_{u_k}^{-1}(\omega_{0,|I'|+2}(I',u_k,\iota q))\\
&\qquad+\text{ terms which are regular at } q=u_k.
\end{aligned}
$$

Here, the inverse d_u^{-1} of the exterior differential of a 1-form ω is its primitive,

$$
d_u^{-1}\omega(u)=\int_{u'=\infty}^{u'=u}\omega(u').
$$

Inserting into (2.11), we confirm with

$$
I'\cup u_k=I_1
$$

and symmetrisation in I_1, I_0 the assertion. ∎

2.3. Symmetry of the involution identity I: $q \to \iota u_k$ and $q \to u_k$

We consider the ι-reflection of (1.4),

$$\begin{aligned}&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\&=\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{1}{s}\operatorname*{Res}_{z\to q}\bigg(\frac{dx(q)dy(z)}{(x(q)-x(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}\bigg),\end{aligned}\tag{2.12}$$

where (1.2) is used. We show that the rhs has the same pole at $q=u_k$ as the original equation (1.4), i.e., that the solution in Proposition 2.5 satisfies

$$\begin{aligned}&\operatorname*{Res}_{q\to u_k}\frac{\omega_{0,|I|+1}(I,\iota q)dw}{w-q}\\&=\operatorname*{Res}_{q\to u_k}\frac{dx(q)dw}{w-q}\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{1}{s}\operatorname*{Res}_{z\to q}\bigg(\frac{dy(z)}{(x(q)-x(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}\bigg).\end{aligned}\tag{2.13}$$

This is the same as

$$\begin{aligned}0&=-\operatorname*{Res}_{q\to u_k}\frac{dx(q)dw}{w-q}\sum_{s=1}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{1}{s}\operatorname*{Res}_{z\to q}\bigg(\frac{dy(z)}{(x(q)-x(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}\bigg)\\&=\operatorname*{Res}_{q\to u_k}\frac{dx(q)dw}{w-q}\sum_{s=1}^{|I|}\sum_{\substack{I_1\uplus\cdots\uplus I_s=I\\u_k\in I_1}}\operatorname*{Res}_{z\to u_k}\bigg(\frac{dy(z)}{(x(q)-x(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}\bigg),\end{aligned}$$

where (2.1) has been used. Fixing $u_k\in I_1$ gives a factor s. We write

$$\omega_{0,|I_1|+1}(I_1,\iota z)=d_{u_k}(d_{u_k}^{-1}\omega_{0,|I_1|+1}(I_1,\iota z)),$$

move d_{u_k} in front of the residues, and ignore it below. Then, we expand the denominator about $x(z)=x(u_k)$:

$$\begin{aligned}0=\sum_{p=1}^{\infty}&\operatorname*{Res}_{q\to u_k}\frac{dx(q)dw}{(w-q)(x(q)-x(u_k))^p}\\&\times\operatorname*{Res}_{z\to u_k}\Bigg(\sum_{s=1}^{\min(|I|,p)}\binom{p-1}{p-s}\sum_{\substack{I_1\uplus\cdots\uplus I_s=I\\u_k\in I_1}}dy(z)(x(z)-x(u_k))^{p-s}\\&\qquad\times d_{u_k}^{-1}\bigg[\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}\bigg]\Bigg).\end{aligned}\tag{2.14}$$

We will show that already the last two lines vanish for every $p \geq 1$. For $p = 1$, the equation to prove reduces to $\operatorname{Res}_{z\to u_k} d_{u_k}^{-1}\omega_{0,|I|+1}(I, \iota z) = 0$, which is true by Lemma 2.4 (only higher-order poles at $z = \iota u_k$). Next, for $p = 2$, we need to show that

$$0 = \operatorname*{Res}_{z\to u_k} (x(z)-x(u_k))\Big\{d_{u_k}^{-1}\omega_{0,|I|+1}(I, \iota z) + \sum_{\substack{I_1\uplus I_2=I\\ u_k\in I_1}} \frac{d_{u_k}^{-1}\omega_{0,|I_1|+1}(I_1, \iota z)\omega_{0,|I_2|+1}(I_2, \iota z)}{dx(\iota z)(y(\iota z)-y(\iota u))}\Big\}. \tag{2.15}$$

Indeed, by Proposition 2.5, the term in braces { } has at most a first-order pole at $z = u_k$, which is removed by a prefactor $(x(z) - x(u_k))^n$ for any $n \geq 1$. Hence, (2.15) is true. Next, for $p = 3$, we have to show that

$$\begin{aligned}0 = \operatorname*{Res}_{z\to u_k} \Big(&(x(z)-x(u_k))^2 d_{u_k}^{-1}\omega_{0,|I|+1}(I, \iota z)\\
&+ 2\sum_{I_1\uplus I_2=I,\ u_1\in I_1} \frac{(x(z)-x(u_k))}{dx(\iota z)} d_{u_k}^{-1}\omega_{0,|I_1|+1}(I_1, \iota z)\omega_{0,|I_2|+1}(I_2, \iota z)\\
&+ \sum_{I_1\uplus I_2\uplus I_3=I,\ u_1\in I_1} \frac{1}{(dx(\iota z))^2} d_{u_k}^{-1}\\
&\times \omega_{0,|I_1|+1}(I_1, \iota z)\omega_{0,|I_2|+1}(I_2, \iota z)\omega_{0,|I_3|+1}(I_3, \iota z)\Big).\end{aligned}$$

By the argument employed to prove (2.15), this reduces to

$$\begin{aligned}0 = \operatorname*{Res}_{z\to u_k} \frac{(x(z)-x(u_k))}{dx(\iota z)} \Big(&\sum_{I_1'\uplus I_2'=I,\ u_1\in I_1'} d_{u_k}^{-1}\omega_{0,|I_1'|+1}(I_1', \iota z)\omega_{0,|I_2'|+1}(I_2', \iota z)\\
&+ \sum_{I_1\uplus I_2\uplus I_3=I,\ u_1\in I_1} \frac{d_{u_k}^{-1}\omega_{0,|I_1|+1}(I_1, \iota z)\omega_{0,|I_2|+1}(I_2, \iota z)\omega_{0,|I_3|+1}(I_3, \iota z)}{dx(\iota z)(y(\iota z)-y(\iota u))}\Big).\end{aligned} \tag{2.16}$$

The sum in the first line will include the cases $I_2' = I_3$ and $I_2' = I_2$. Using again the argument based on Proposition 2.5, in the first case,

$$d_{u_k}^{-1}\omega_{0,|I_1'|+1}(I_1', \iota z) + \sum_{I_1\uplus I_2=I_1', u_k\in I_1} \frac{d_{u_k}^{-1}\omega_{0,|I_1|+1}(I_1, \iota z)\omega_{0,|I_2|+1}(I_2, \iota z)}{dx(\iota z)(y(\iota z)-y(\iota u_k))}$$

has at most a first-order pole at $z = u_k$. Multiplying this sum by

$$(x(z)-x(u_k))\frac{\omega_{0,|I_3|+1}(I_3, \iota z)}{dx(\iota z)}$$

gives a regular term without residue. The same is true for $I_2 \leftrightarrow I_3$. This proves (2.16). The same argument together with Pascal's triangle structure eventually shows that the second line of (2.14) vanishes identically for any $p \geq 1$. In conclusion, (2.13) is proved, which means that the rhs of (1.4), minus its reflection $q \mapsto \iota q$, is holomorphic at every $q = \iota u_k$ (and then also at $q = u_k$).

2.4. Linear loop equation

Let σ_i be the local Galois involution defined in a neighbourhood of the ramification point β_i. It satisfies $x(z) = x(\sigma_i(z))$, $\sigma_i(z) \neq z$ for $z \neq \beta_i$ and $\lim_{z\to\beta_i} \sigma_i(z) = \beta_i$.

Proposition 2.6. *The meromorphic differentials $\omega_{0,m+1}$ satisfy the linear loop equation* [6]*; i.e.,*

$$q \mapsto \omega_{0,|I|+1}(I,q) + \omega_{0,|I|+1}(I,\sigma_i(q))$$

is holomorphic at $q = \beta_i$.

Proof. We start from the involution identity (2.12), which arises by $q \mapsto \iota q$ from the original equation (1.4), and consider

$$\begin{aligned}
&\operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,|I|+1}(I,q)dw}{w-q} \\
&= \sum_{s=1}^{|I|} \frac{dw}{s} \sum_{I_1\uplus\cdots\uplus I_s=I} \operatorname*{Res}_{q\to\beta_i} \operatorname*{Res}_{z\to q} \frac{dx(q)dy(z)}{(w-q)(x(q)-x(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)},
\end{aligned}$$

where $\omega_{0,|I|+1}(I,\iota q)$ is included as $s=1$ on the rhs. Condition (d) in Theorem 1.2, i.e., holomorphicity of $\omega_{0,|I|+1}(I,\iota q)$ at $q = \beta_i$, implies that the integrand is regular at $z = \beta_i$ but has a pole at $z = \sigma_i(q)$. We thus have with commutation rule (2.2)

$$\begin{aligned}
&\operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,|I|+1}(I,q)dw}{w-q} \\
&= -\sum_{s=1}^{|I|} \frac{dw}{s} \sum_{I_1\uplus\cdots\uplus I_s=I} \operatorname*{Res}_{q\to\beta_i} \operatorname*{Res}_{z\to\sigma_i(q)} \frac{dx(q)dy(z)}{(w-q)(x(q)-x(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dy(z)}.
\end{aligned}$$

With

$$x(q) = x(\sigma_i(q)) \quad \text{and} \quad dx(q) = dx(\sigma_i(q)),$$

the inner integral evaluates to $\omega_{0,|I|+1}(I,\sigma_i(q))$, and we end up having

$$\operatorname*{Res}_{q\to\beta} \frac{\omega_{0,|I|+1}(I,q) + \omega_{0,|I|+1}(I,\sigma_i(q))}{w-q} dw = 0. \qquad \blacksquare$$

Remark 2.7. From (2.4) and the expansion $\sigma_i(q)-\beta_i=-(\beta_i-q)+\mathcal{O}((q-\beta_i)^2)$, we conclude that

$$\begin{aligned}&\operatorname*{Res}_{q\to\beta_i}\frac{\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\sigma_i(q))}{q-\beta_i}\\&\quad=\operatorname*{Res}_{q\to\beta_i}\frac{\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\sigma_i(q))}{\sigma_i(q)-\beta_i}\\&\quad=-\operatorname*{Res}_{q\to\beta_i}\frac{\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\sigma_i(q))}{q-\beta_i}.\end{aligned}$$

Hence, $\frac{\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\sigma_i(q))}{q-\beta_i}$ is regular at $q=\beta_i$, which means that

$$\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\sigma_i(q))$$

has at least a first-order zero at $q=\beta_i$.

2.5. The recursion kernel

We start from Lemma 2.3 for $q\mapsto \iota q$, where (1.2) is taken into account:

$$\begin{aligned}&\omega_{0,|I|+1}(I,q)+\omega_{0,|I|+1}(I,\iota q)\\&=dx(q)\sum_{s=1}^{|I|-1}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I}\nabla^s\omega_{0,|I_0|+1}(I_0,\iota q)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,q)}{dx(q)}\\&=dx(q)\sum_{s=1}^{|I|-1}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I}\operatorname*{Res}_{z\to q}\frac{\omega_{0,|I_0|+1}(I_0,\iota z)}{(x(q)-x(z))(y(z)-y(q))^s}\\&\quad\times\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,q)}{dx(q)}.\end{aligned}\tag{2.17}$$

We introduce

$$\begin{aligned}\mathfrak{W}_{a,s,s'}(I;q)&:=\sum_{\substack{I_0\uplus I_1\uplus\cdots\uplus I_s\uplus I'_1\uplus\cdots\uplus I'_{s'}\\ I_0=\emptyset\text{ for }a=0}}\big(\delta_{a,0}+(1-\delta_{a,0})\nabla^a\omega_{0,|I_0|+1}(I_0;\iota q)\big)\\&\qquad\times\prod_{k=1}^{s}\frac{\omega_{0,|I_k|+1}(I_k,q)}{dx(q)}\prod_{j=1}^{s'}\frac{\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{dx(\sigma_i(q))},\\ \mathfrak{B}_{a,a',s,s'}(I;q,z)&:=\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s\uplus I'_1\uplus\cdots\uplus I'_{s'}=I}\frac{\omega_{0,|I_0|+1}(I_0,\iota z)}{(x(q)-x(z))}\\&\qquad\times\frac{\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{(dx(q))^s(dx(\sigma_i(q)))^{s'}(y(z)-y(q))^a(y(z)-y(\sigma_i(q)))^{a'}}.\end{aligned}\tag{2.18}$$

These are functions of q and 1-forms in every variable in I, and $\mathfrak{B}_{a,a',s,s'}(I;q,z)$ is also a 1-form in z. A lengthy calculation gives the following important lemma.

Lemma 2.8. *Residues of* $\mathfrak{W}_{a,s,s'}$ *satisfy for* $0 < a \le s$

$$\begin{aligned}
&\operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\mathfrak{W}_{a,s,s'}(I;q)\\
&= \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\Bigg(\mathfrak{B}_{a,0,s,s'}(I;q,z) + \sum_{a'=1}^{|I|-s-s'-1} \frac{\mathfrak{B}_{0,a',s,s'+a'}(I;q,z)}{(y(\sigma_i(q))-y(q))^a}\Bigg)\\
&\quad + \operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\Bigg(-\frac{\mathfrak{W}_{0,s,s'+1}(I;q)}{(y(\sigma_i(q))-y(q))^a}\\
&\quad - \sum_{a'=1}^{|I|-s-s'-1} \frac{(-1)^{a'}\mathfrak{W}_{0,s+1,s'+a'}(I;q)}{(y(\sigma_i(q))-y(q))^{a+a'}}\\
&\quad + \sum_{a'=1}^{|I|-s-s'-2}\sum_{a''=1}^{|I|-s-s'-a'-1} \frac{(-1)^{a'}\mathfrak{W}_{a'',s+a'',s'+a'}(I;q)}{(y(\sigma_i(q))-y(q))^{a+a'}}\Bigg)
\end{aligned}\tag{2.19}$$

for any function $f_{a,s,s'}$ *meromorphic in a neighbourhood of* β_i.

Proof. We consider for a fixed partition $I_0 \uplus I_1 \uplus \cdots \uplus I_s \uplus I'_1 \uplus \cdots \uplus I'_{s'} = I$ and some $0 < a \le s$ the residue

$$\begin{aligned}
&\operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\nabla^a\omega_{0,|I_0|+1}(I_0,\iota q)\prod_{k=1}^{s}\frac{\omega_{0,|I_k|+1}(I_k,q)}{dx(q)}\\
&\quad\times\prod_{j=1}^{s'}\frac{\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{dx(\sigma_i(q))}\\
&\equiv \operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\\
&\quad\times\operatorname*{Res}_{z\to q}\frac{\omega_{0,|I_0|+1}(I_0,\iota z)\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{(x(q)-x(z))(y(z)-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}}\\
&= \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\\
&\quad\times\frac{\omega_{0,|I_0|+1}(I_0,\iota z)\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{(x(q)-x(z))(y(z)-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}}\\
&\quad-\operatorname*{Res}_{q\to\beta_i} \frac{f_{a,s,s'}(q)dx(q)}{w-q}\\
&\quad\times\operatorname*{Res}_{z\to\sigma_i(q)}\frac{\omega_{0,|I_0|+1}(I_0,\iota z)\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{(x(q)-x(z))(y(z)-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}}.
\end{aligned}$$

We have used (2.5) and (2.2) and the fact that the integrand is regular at $z=\beta_i$. The residue at $z=\sigma_i(q)$ in the last line can be evaluated immediately and gives rise to the function $-\frac{\omega_{0,|I_0|+1}(I_0,\iota\sigma_i(q))}{dx(\sigma_i(q))}$ for which we insert (2.17) at $q\mapsto\sigma_i(q)$:

$$\begin{aligned}
&\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\nabla^a\omega_{0,|I_0|+1}(I_0,\iota q)\prod_{k=1}^{s}\frac{\omega_{0,|I_k|+1}(I_k,q)}{dx(q)}\\
&\quad\times\prod_{j=1}^{s'}\frac{\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{dx(\sigma_i(q))}\\
&=\operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\\
&\quad\times\frac{\omega_{0,|I_0|+1}(I_0,\iota z)\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{(x(q)-x(z))(y(z)-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}}\\
&\quad-\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\\
&\quad\times\frac{\omega_{0,|I_0|+1}(I_0,\sigma_i(q))\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))}{dx(\sigma_i(q))(y(\sigma_i(q))-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}}\\
&\quad+\sum_{a'=1}^{|I_0|-1}\sum_{I''_0\uplus I''_1\uplus\cdots\uplus I''_{a'}=I_0}\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\operatorname*{Res}_{z\to\sigma_i(q)}\Bigg(\frac{\omega_{0,|I_0|+1}(I''_0,\iota z)}{(x(\sigma_i(q))-x(z))}\qquad(*)\\
&\quad\times\frac{\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))\prod_{j=1}^{a'}\omega_{0,|I''_j|+1}(I''_j,\sigma_i(q))}{(y(\sigma_i(q))-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}(y(z)-y(\sigma_i(q)))^{a'}}\Bigg).\\
&\qquad(*)
\end{aligned}$$

We process the last two lines (*) in the same manner, i.e., commute the two residues according to (2.2). There is again no contribution of a residue at $z=\beta_i$, but now an additional residue at $z=q$ arises. The resulting term $\frac{\omega_{0,|I_0|+1}(I''_0,\iota q)}{dx(q)}$ is expressed via (2.17):

$$\begin{aligned}
(*)&=\sum_{a'=1}^{|I_0|-1}\sum_{I''_0\uplus I''_1\uplus\cdots\uplus I''_{a'}=I_0}\operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\Bigg(\frac{\omega_{0,|I_0|+1}(I''_0,\iota z)}{(x(\sigma_i(q))-x(z))}\\
&\times\frac{\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))\prod_{j=1}^{a'}\omega_{0,|I''_j|+1}(I''_j,\sigma_i(q))}{(y(\sigma_i(q))-y(q))^a(dx(q))^s(dx(\sigma_i(q)))^{s'}(y(z)-y(\sigma_i(q)))^{a'}}\Bigg)\\
&-\sum_{a'=1}^{|I_0|-1}\sum_{I''_0\uplus I''_1\uplus\cdots\uplus I''_{a'}=I_0}\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}\Bigg(\frac{\omega_{0,|I_0|+1}(I''_0,q)}{dx(q)}\\
&\times\frac{(-1)^{a'}\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I'_j|+1}(I'_j,\sigma_i(q))\prod_{j=1}^{a'}\omega_{0,|I''_j|+1}(I''_j,\sigma_i(q))}{(y(\sigma_i(q))-y(q))^{a+a'}(dx(q))^s(dx(\sigma_i(q)))^{s'}}\Bigg)
\end{aligned}$$

$$+\sum_{a'=1}^{|I_0|-1}\sum_{I_0''\uplus I_1''\uplus\cdots\uplus I_{a'}''=I_0}\sum_{a''=1}^{|I_0''|-1}\sum_{I_0'''\uplus I_1'''\uplus\cdots\uplus I_{a''}'''=I_0''}\operatorname*{Res}_{q\to\beta_i}\frac{f_{a,s,s'}(q)dx(q)}{w-q}$$
$$\times\Bigg(\nabla^{a''}\omega_{0,|I_0'''|+1}(I_0''',\iota q)\frac{(-1)^{a'}\prod_{k=1}^{a''}\omega_{0,|I_k'''|+1}(I_k''',q)\prod_{j=1}^{a'}\omega_{0,|I_j''|+1}(I_j'',\sigma_i(q))}{(y(\sigma_i(q))-y(q))^{a+a'}}$$
$$\times\frac{\prod_{k=1}^{s}\omega_{0,|I_k|+1}(I_k,q)\prod_{j=1}^{s'}\omega_{0,|I_j'|+1}(I_j',\sigma_i(q))}{(dx(q))^s(dx(\sigma_i(q)))^{s'}}\Bigg).$$

This is inserted back into the equation we started with. We sum over all partitions

$$I_0\uplus I_1\uplus\cdots\uplus I_s\uplus I_1'\uplus\cdots\uplus I_{s'}'=I$$

for fixed s, s', a and express the result in terms of $\mathfrak{W}, \mathfrak{B}$ introduced in (2.18). The result is (2.19). ∎

Lemma 2.8 is our main tool to evaluate the polar part of (2.17) at $q=\beta_i$. Taking condition (d) of Theorem 1.2 into account, we need to evaluate

$$\operatorname*{Res}_{q\to\beta_i}\frac{\omega_{0,|I|+1}(I,q)dw}{w-q}=\sum_{s=1}^{|I|-1}\operatorname*{Res}_{q\to\beta_i}\frac{\mathfrak{W}_{s,s,0}(I;q)dx(q)dw}{w-q}.$$

In a first (also very lengthy) step, we show the following lemma.

Lemma 2.9. *We have*

$$\begin{aligned}0=&\operatorname*{Res}_{q\to\beta_i}\frac{dx(q)dw}{w-q}\Bigg(\sum_{s=1}^{|I|-1}\mathfrak{W}_{s,s,0}(I;q)+\frac{\mathfrak{W}_{0,1,1}(I;q)}{y(\sigma_i(q))-y(q)}\Bigg)\\&-\operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i}\frac{dx(q)dw}{w-q}\Bigg(\sum_{s=1}^{|I|-1}\sum_{s'=0}^{|I|-s-1}\mathfrak{B}_{s,s',s,s'}(I;q,z)\\&+\sum_{s=1}^{|I|-2}\sum_{s'=1}^{|I|-s-1}\frac{\mathfrak{B}_{s,s'-1,s,s'}(I;q,z)}{y(\sigma_i(q))-y(q)}\Bigg).\end{aligned}\tag{2.20}$$

Proof. We express $\sum_{s=1}^{|I|-1}\operatorname{Res}_{q\to\beta_i}\frac{dx(q)dw}{w-q}\mathfrak{W}_{s,s,0}(I;q)$ via (2.19) at $s'=0$, $a=s$, and $f_{a,s,s'}\equiv 1$. In the third line of (2.19), the case $a'=1$ of the second term cancels, when summing over s, every first term except for the single term with $s=1$ and $s'=0$. This surviving term with

$$s=a=1$$

is the last term in the first line of (2.20). When subtracting the second line of (2.20), the term with $s'=0$ in (2.20) cancels directly, and then the term with $s=1$ (and any

$s' \geq 1$) cancels after reordering partial fractions. After renaming the parameters, we arrive at

$$\begin{aligned}
&(2.20)_{\mathrm{rhs}} \\
&= \operatorname*{Res}_{q\to\beta_i} \frac{dx(q)dw}{w-q}\Bigg(-\sum_{s=2}^{|I|-2}\sum_{s'=2}^{|I|-s}(-1)^{s'}\frac{\mathfrak{W}_{0,s,s'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'-1}} \\
&\quad+\sum_{s=2}^{|I|-2}\sum_{s'=1}^{|I|-s-1}\sum_{a=1}^{s-1}(-1)^{s'}\frac{\mathfrak{W}_{a,s,s'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'-a}}\Bigg) \\
&\quad+\operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i}\frac{dx(q)dw}{w-q}\sum_{s=2}^{|I|-2}\sum_{s'=1}^{|I|-s-1}\bigg(\frac{\mathfrak{B}_{0,s',s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s}}-\frac{\mathfrak{B}_{s-1,s',s,s'}(I;q,z)}{y(\sigma_i(q))-y(q)}\bigg).
\end{aligned} \tag{2.21}$$

In the second term of the last line, we apply repeatedly the identity

$$\frac{\mathfrak{B}_{a,a',s,s'}(I;z,q)}{(y(\sigma_i(q))-y(q))^{s+s'-a-a'}}=\frac{\mathfrak{B}_{a-1,a',s,s'}(I;q,z)-\mathfrak{B}_{a,a'-1,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a-a'+1}}$$

to express $\frac{\mathfrak{B}_{s-1,s',s,s'}(I;q)}{(y(\sigma_i(q))-y(q))}$ as linear combination of functions $\mathfrak{B}_{a,0,s,s'}(I;q,z)$ and $\mathfrak{B}_{0,a',s,s'}(I;q,z)$. The coefficient of $\mathfrak{B}_{a,0,s,s'}(I;q,z)$ in this expansion is the number of paths made of steps up or right from $(a,0)$ to $(s-1,s')$ with a first step right. This is the same as the number $\binom{s+s'-a-2}{s'-1}$ of words of $s'-1$ letters R and $s-1-a$ letters U. Similarly, the coefficient of $\mathfrak{B}_{0,a',s,s'}(I;q,z)$ in this expansion is the number of up-right paths from $(0,a')$ to $(s-1,s')$ with a first step up. This is the same as the number $\binom{s+s'-a'-2}{s-2}$ of words of $s-2$ letters U and $s'-a'$ letters R. A right step comes with a factor (-1). We thus get

$$\begin{aligned}
\frac{\mathfrak{B}_{s-1,s',s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))}&=\sum_{a=1}^{s-1}\binom{s+s'-a-2}{s'-1}\frac{(-1)^{s'}\mathfrak{B}_{a,0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a}} \\
&\quad+\sum_{a'=1}^{s'}\binom{s+s'-a'-2}{s-2}\frac{(-1)^{s'-a'}\mathfrak{B}_{0,a',s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a'}}.
\end{aligned}$$

The term with $a'=s'$ cancels the first term of the last line of (2.21) so that we end up in the following equation in which $k\equiv 0$:

$$\begin{aligned}
&(2.20)_{\mathrm{rhs}} \\
&= \operatorname*{Res}_{q\to\beta_i}\frac{dx(q)}{w-q}\Bigg\{-\sum_{s=2+k}^{|I|-2-k}\sum_{s'=2+k}^{|I|-s}\binom{s-2}{k}\binom{s'-2}{k}\frac{(-1)^{s'}\mathfrak{W}_{0,s,s'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'-1}} && (\dagger) \\
&\quad+\sum_{s=2+k}^{|I|-2-k}\sum_{s'=1+k}^{|I|-s-1}\sum_{a=1}^{s-1-k}\binom{s-a-1}{k}\binom{s'-1}{k}\frac{(-1)^{s'}\mathfrak{W}_{a,s,s'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'-a}}\Bigg\}dw && (\ddagger)
\end{aligned}$$

$$
\begin{aligned}
&- \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{dx(q)}{w-q} \sum_{s=2}^{|I|-2}\sum_{s'=1}^{|I|-s-1} \Big\{ \sum_{a=1}^{s-1} \binom{s+s'-a-2}{s'-1} \frac{(-1)^{s'}\mathfrak{B}_{a,0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a}} \\
&+ \sum_{a'=1}^{s'-1} \binom{s+s'-a'-2}{s-2} \frac{(-1)^{s'-a'}\mathfrak{B}_{0,a',0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a'}} \Big\} dw. \qquad (2.22)
\end{aligned}
$$

Next, we process the line (‡) of (2.22) via (2.19). With the exception of one term, the 'hockey-stick identity' $\sum_{a=1}^{s-k-1}\binom{s-a-1}{k} = \binom{s-1}{k+1}$ occurs:

$$
\begin{aligned}
&(2.22)_{\ddagger} \\
&= \operatorname*{Res}_{q\to\beta_i} \frac{dx(q)dw}{w-q} \sum_{s=2+k}^{|I|-2-k}\sum_{s'=1+k}^{|I|-s-1} \Big\{ -\binom{s-1}{k+1}\binom{s'-1}{k} \frac{(-1)^{s'}\mathfrak{W}_{0,s,s'+1}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'}} \\
&- \binom{s-1}{k+1}\binom{s'-1}{k} \sum_{a'=1}^{|I|-s-s'-1} \frac{(-1)^{s'+a'}\mathfrak{W}_{0,s+1,s'+a'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'+a'}} \\
&+ \binom{s-1}{k+1}\binom{s'-1}{k} \sum_{a'=1}^{|I|-s-s'-2}\sum_{a''=1}^{|I|-s-s'-a'-1} \frac{(-1)^{s'+a'}\mathfrak{W}_{a'',s+a'',s'+a'}(I;q)}{(y(\sigma_i(q))-y(q))^{s+s'+a'}} \Big\} \\
&+ \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{dx(q)dw}{w-q} \sum_{s=2+k}^{|I|-2-k}\sum_{s'=1+k}^{|I|-s-1} \Big\{ \sum_{a=1}^{s-1-k} \binom{s-a-1}{k}\binom{s'-1}{k} \\
&\times \frac{(-1)^{s'}\mathfrak{B}_{a,0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a}} \\
&+ \binom{s-1}{k+1}\binom{s'-1}{k} \sum_{a'=1}^{|I|-s-s'-1} \frac{(-1)^{s'}\mathfrak{B}_{0,a',s,s'+a'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'}} \Big\}. \qquad (2.23)
\end{aligned}
$$

The following steps are performed:

- In the first line, we shift $s'+1 \mapsto s' \in [2+k \dots |I|-s]$.
- In the second line, we shift $s+1 \mapsto s \in [3+k \dots |I|-k-1]$. Then, we rename $s'+a' \mapsto s' \in [2+k \dots |I|-s]$ and sum over $a' \in [1 \dots s'-k-1]$. Recall that

$$\sum_{a'=1}^{s'-k-1}\binom{s'-a'-1}{k} = \binom{s'-1}{k+1}.$$

  The new ranges restrict $s \in [3+k \dots |I|-k-2]$.
- In the third line, we rename $s'+a' \mapsto s' \in [2+k \dots |I|-s]$ and sum over $a' \in [1 \dots s'-k-1]$. This gives $\sum_{a'=1}^{s'-k-1}\binom{s'-a'-1}{k} = \binom{s'-1}{k+1}$. We also rename $s+a'' \mapsto s \in [3+k \dots |I|-k-2]$ and keep the sum over $a'' \mapsto a \in [1 \dots s-2-k]$.

- In the final line, we rename $s'+a' \mapsto s' \in [2+k \dots I-s-1]$ and sum over $a' \in [1 \dots s'-k-1]$.

With the Pascal triangle identity $\binom{s-1}{k+1} - \binom{s-2}{k} = \binom{s-2}{k+1}$ and the corresponding adjustments of the ranges for s, s', we find that the first two lines $(\dagger, \ddagger)$ of (2.22), where $k=0$, equal the same two lines $(2.22)_{\dagger+\ddagger}$ with $k=1$, plus the iterated residue in the last three lines of (2.23), first for $k=0$. Iterating this procedure until $s \geq 2+k$ and $s' \geq 1+k$ becomes incompatible with the size $|I|$ gives for the first two lines of (2.22) the identity

$$\begin{aligned}(2.22)_{\dagger,\ddagger} = \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{dx(q)dw}{w-q} \sum_{k=0}^{[|I|/2]-2}\sum_{s=2+k}^{|I|-2-k}\sum_{s'=1+k}^{|I|-s-1}\Bigg\{\sum_{a=1}^{s-1-k}\binom{s-a-1}{k}\binom{s'-1}{k}\\ \times \frac{(-1)^{s'}\mathfrak{B}_{a,0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a}}\\ + \sum_{a'=1}^{s'-1-k}\binom{s-1}{k+1}\binom{s'-a'-1}{k}\frac{(-1)^{s'-a'}\mathfrak{B}_{0,a',s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a'}}\Bigg\}.\end{aligned}$$

Now, we change the summation order and sum first over k. With

$$\sum_{k=0}^{\min(n,s-1)}\binom{n}{k}\binom{s-1}{k} = \binom{n+s-1}{s-1},$$
$$\sum_{k=0}^{\min(n-1,s-1)}\binom{n}{k+1}\binom{s-1}{k} = \binom{n+s-1}{n-1}$$

(e.g., [21, Volume 4, (6.69)+(6.70)]), we conclude that

$$\begin{aligned}&(2.22)_{\dagger,\ddagger}\\ &= \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{dx(q)dw}{w-q}\sum_{s=2}^{|I|-2}\sum_{s'=1}^{|I|-s-1}\Bigg\{\sum_{a=1}^{s-1}\binom{s+s'-a-2}{s'-1}\frac{(-1)^{s'}\mathfrak{B}_{a,0,s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a}}\\ &\qquad + \sum_{a'=1}^{s'-1}\binom{s+s'-a'-2}{s-2}\frac{(-1)^{s'-a'}\mathfrak{B}_{0,a',s,s'}(I;q,z)}{(y(\sigma_i(q))-y(q))^{s+s'-a'}}\Bigg\}.\end{aligned}$$

Therefore, (2.22) and hence the rhs of (2.20) are identically zero. ∎

We will prove by induction that the second line of (2.20) vanishes identically. This requires a rearrangement of the forms $\mathfrak{B}$. To simplify notation, we introduce the split operator

$$\mathsf{S}\omega_{0,|I|+1}(I,q) := \sum_{I_1\uplus I_2=I}\frac{\omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q))}{dx(\sigma_i(q))(y(\sigma_i(q))-y(q))} \tag{2.24}$$

with $\mathsf{S}\omega_{0,2}(u,q)=0$. Then, (2.20) can be written with (2.17) as

$$
\begin{aligned}
0 = \operatorname*{Res}_{q\to\beta_i} & \frac{\omega_{0,|I|+1}(I,q)+\mathsf{S}\omega_{0,|I|+1}(I,q)}{w-q}dw \\
& - \sum_{\substack{I_0\uplus I'\uplus I''=I \\ \text{possibly } I''=\emptyset}} \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,|I_0|+1}(I_0,\iota z)}{x(q)-x(z)}\bigg\{\frac{\omega_{0,|I'|+1}(I',q)+\mathsf{S}\omega_{0,|I'|+1}(I',q)}{(w-q)(y(z)-y(q))} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \tilde{\mathfrak{B}}(I'';q,z)\bigg\}dw, \qquad (2.25)
\end{aligned}
$$

where $\tilde{\mathfrak{B}}(\emptyset;q,z)=1$ and for $I''\neq\emptyset$

$$
\begin{aligned}
&\tilde{\mathfrak{B}}(I'';q,z) \\
&= \sum_{s=1}^{|I''|}\sum_{s_0=0}^{s}\sum_{I_1\uplus\cdots\uplus I_s=I''}\prod_{j=1}^{s_0}\frac{\omega_{0,|I_j|+1}(I_j,q)}{dx(q)(y(z)-y(q))}\prod_{j=s_0+1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\sigma_i(q))}{dx(\sigma_i(q))(y(z)-y(\sigma_i(q)))}.
\end{aligned}
\tag{2.26}
$$

We claim that this expression can be reordered into

$$
\begin{aligned}
\tilde{\mathfrak{B}}(I'';q,z) = \sum_{p=1}^{|I''|}\sum_{I_1\uplus\cdots\uplus I_p=I''}\prod_{j=1}^{p}&\bigg\{\frac{\omega_{0,|I_j|+1}(I_j,q)+\omega_{0,|I_j|+1}(I_j,\sigma_i(q))}{dx(\sigma_i(q))(y(z)-y(\sigma_i(q)))} \\
& - \frac{y(\sigma_i(q))-y(q)}{dx(q)}\frac{\omega_{0,|I_j|+1}(I_j,q)+\mathsf{S}\omega_{0,|I_j|+1}(I_j,q)}{(y(z)-y(q))(y(z)-y(\sigma_i(q)))}\bigg\}.
\end{aligned}
\tag{2.27}
$$

Indeed, the term in braces expands with $dx(\sigma_i(q))=dx(q)$ into

$$
\begin{aligned}
\{\}_j = &\frac{\omega_{0,|I_j|+1}(I_j,q)}{dx(q)(y(z)-y(q))}+\frac{\omega_{0,|I_j|+1}(I_j,\sigma_i(q))}{dx(\sigma_i(q))(y(z)-y(\sigma_i(q)))} \\
&- \sum_{I_j'\uplus I_j''=I_j}\frac{\omega_{0,|I_j'|+1}(I_j',q)}{dx(q)(y(z)-y(q))}\frac{\omega_{0,|I_j''|+1}(I_j'',\sigma_i(q))}{dx(\sigma_i(q))(y(z)-y(\sigma_i(q)))}.
\end{aligned}
$$

A p-fold product is then of the form

$$
\begin{aligned}
\sum_{I_1\uplus\cdots\uplus I_p=I''}\prod_{j=1}^{p}\{\}_j = &\sum_{n+n_2+\tilde{n}=p}\frac{(-1)^{n_2}(n+n_2+\tilde{n})!}{n!n_2!\tilde{n}!}\sum_{I_1\uplus\cdots\uplus I_{n+n_2}\uplus I_1'\uplus\cdots\uplus I'_{\tilde{n}+n_2}=I''} \\
&\times\prod_{j=1}^{n+n_2}\frac{\omega_{0,|I_j|+1}(I_j,q)}{dx(q)(y(z)-y(q))}\prod_{k=1}^{\tilde{n}+n_2}\frac{\omega_{0,|I_k'|+1}(I_k',\sigma_i(q))}{dx(\sigma_i(q))(y(z)-y(\sigma_i(q)))}.
\end{aligned}
$$

We change the summation variables to $n+n_2=s_0$, $\tilde{n}+n_2=s-s_0$ and first sum over $n_2 \in [0 \dots \min(s_0, s - s_0)]$ and then over s, s_0. Because of

$$\sum_{n_2=0}^{\min(s_0,s-s_0)} \frac{(-1)^{n_2}(s-n_2)!}{(s_0-n_2)!n_2!(s-s_0-n_2)!} = \sum_{n_2=0}^{\min(s_0,s-s_0)} (-1)^{n_2}\binom{s-n_2}{s_0}\binom{s_0}{n_2} = 1$$

(see, e.g., [21, Volume 4, (10.13)]), we obtain the same expression as (2.26), which proves (2.27).

With these preparations, we complete the final step.

Proposition 2.10. *For all $|I| \geq 2$, one has*

$$\operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,|I|+1}(I,q) + \mathsf{S}\omega_{0,|I|+1}(I,q)}{w-q} dw = 0.$$

Equivalently, the meromorphic differentials $\omega_{0,m+1}$ satisfy the topological recursion

$$\begin{aligned}&\mathcal{P}^i_z \omega_{0,|I|+1}(I,z) \\ &= \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\big(\frac{dz}{z-q} - \frac{dz}{z-\sigma_i(q)}\big)}{dx(\sigma_i(q))(y(q)-y(\sigma_i(q)))} \sum_{I_1 \uplus I_2 = I} \omega_{0,|I_1|+1}(I_1,q)\omega_{0,|I_2|+1}(I_2,\sigma_i(q)).\end{aligned} \tag{2.28}$$

Proof. By induction on $|I| \geq 2$ using (2.25) together with (2.27), for $|I| = 2$, we necessarily have $|I_0| = |I'| = 1$ and $I'' = \emptyset$. This implies $\widetilde{\mathfrak{B}}(I''; q, z) = 1$ and $\mathsf{S}\omega_{0,2}(I_1, q) = 0$. Since $\omega_{0,2}(I_1, q)$ is regular at $q = \beta_i$, the integrand of the rhs of (2.25) has vanishing residue. Assume that the proposition is true for $|I| \leq \ell$. Because of $|I_0| \geq 1$, any $I', I'', I_1, \dots, I_p$ on the rhs of (2.25) and in (2.27) is of length strictly $< \ell$. Then, by induction hypothesis, the linear loop equation Proposition 2.6, and the regularity of $\frac{y(\sigma_i(q))-y(q)}{dx(q)}$ at $q \to \beta_i$, the whole integrand on the rhs of (2.25) is regular at $q = \beta_i$; i.e., its residue equals 0.

This shows that $\mathcal{P}^i_z \omega_{0,|I|+1}(I,z) = -\operatorname{Res}_{q\to\beta_i} \frac{\mathsf{S}\omega_{0,|I|+1}(I,q)}{z-q} dz$. With Remark 2.7, we have

$$\operatorname*{Res}_{q\to\beta_i} \frac{\mathsf{S}\omega_{0,|I|+1}(I,q)}{z-q} dz = \operatorname*{Res}_{q\to\beta_i} \left(\frac{1}{2}\frac{\mathsf{S}\omega_{0,|I|+1}(I,q)}{z-q} + \frac{1}{2}\frac{\mathsf{S}\omega_{0,|I|+1}(I,\sigma_i(q))}{z-\sigma_i(q)}\right) dz.$$

Now, (2.28) follows from $\mathsf{S}\omega_{0,|I|+1}(I,\sigma_i(q)) = -\mathsf{S}\omega_{0,|I|+1}(I,q)$. ∎

2.6. Symmetry of the involution identity II: $q \to \beta_i$ and $q \to \iota\beta_i$

Recall that the investigation of $\omega_{0,|I|+1}(I,q)$ for q near a ramification point β_i of x in Sections 2.4 and 2.5 started from the ι-reflection (2.12) of the involution identity (1.4). It thus remains to prove that the obtained solution is consistent with the original equation (1.4). This means we have to show that

$$\begin{aligned} &\operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,|I|+1}(I,q)dw}{w-q} \\ &= \operatorname*{Res}_{q\to\beta_i} \frac{dy(q)dw}{w-q} \sum_{s=2}^{|I|} \sum_{I_1\uplus\dots\uplus I_s=I} \frac{1}{s} \operatorname*{Res}_{z\to q} \left(\frac{dx(z)}{(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right). \end{aligned} \tag{2.29}$$

This is the same as

$$\begin{aligned} 0 &= -\operatorname*{Res}_{q\to\beta_i} \frac{dy(q)dw}{w-q} \sum_{s=1}^{|I|} \sum_{I_1\uplus\dots\uplus I_s=I} \frac{1}{s} \operatorname*{Res}_{z\to q} \left(\frac{dx(z)}{(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right) \\ &= \operatorname*{Res}_{q\to\beta_i} \frac{dy(q)dw}{w-q} \sum_{s=1}^{|I|} \sum_{I_1\uplus\dots\uplus I_s=I} \frac{1}{s} \operatorname*{Res}_{z\to\beta_i} \left(\frac{dx(z)}{(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right), \end{aligned}$$

where (2.1) has been used. We expand $\frac{1}{(y(q)-y(z))^s}$ about $y(z)=y(\beta_i)$ and then order into powers of $y(\beta_i)$. Hence, (2.29) holds iff

$$\begin{aligned} 0 &= \operatorname*{Res}_{q\to\beta_i} \frac{dy(q)dw}{w-q} \sum_{p=1}^{\infty} \frac{1}{p(y(q)-y(\beta_i))^p} \\ &\times \sum_{s=1}^{\min(|I|,p)} \sum_{I_1\uplus\dots\uplus I_s=I} \binom{p}{s} \operatorname*{Res}_{z\to\beta_i} \left(dx(z)(y(z)-y(\beta_i))^{p-s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right) \\ &= \operatorname*{Res}_{q\to\beta_i} \frac{dy(q)dw}{w-q} \sum_{p=1}^{\infty} \sum_{k=0}^{p} \frac{1}{p(y(q)-y(\beta_i))^p} \binom{p}{k} (-1)^k (y(\beta_i))^k \\ &\times \sum_{s=1}^{\min(|I|,p-k)} \sum_{I_1\uplus\dots\uplus I_s=I} \binom{p-k}{s} \operatorname*{Res}_{z\to\beta_i} \left(dx(z)(y(z))^{p-k-s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right). \end{aligned}$$

We prove that the last line vanishes identically for any $n=p-k$.

Proposition 2.11. *For any family $\omega_{0,|I|+1}(I,z)$ of 1-forms in z which satisfy the linear and quadratic loop equations*[4] [6, 7]*, one has, for any $n\geq 1$,*

$$\sum_{s=1}^{|I|} \sum_{I_1\uplus\dots\uplus I_s=I} \operatorname*{Res}_{z\to\beta_i} \left[\binom{n}{s} y(z)^{n-s} dx(z) \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right] = 0. \tag{2.30}$$

In particular, (2.29) holds under these assumptions.

[4] In our situation, the linear loop equations are proved in Proposition 2.6 and the quadratic loop equations are equivalent to Proposition 2.10.

Proof. In Remark 2.12 below, we indicate that the assertion would be an immediate consequence of existence of a loop insertion operator. Because we did not prove that such an operator exists in our case, we give a direct combinatorial proof based on a technical Lemma B.1.

We associate to the complex numbers $y, \bar{y}, w, \bar{w}$ in Lemma B.1 the functions (and forms in u_k) $y \mapsto y(z)$, $\bar{y} \mapsto y(\sigma_i(z))$, $w \mapsto \sum_{\emptyset \neq I' \subset I} t_{I'} \frac{\omega_{0,|I'|+1}(I',z)}{dx(z)}$,

$$\bar{w} \mapsto \sum_{\emptyset \neq I' \subset I} t_{I'} \frac{\omega_{0,|I'|+1}(I', \sigma_i(z))}{dx(z)}.$$

We keep only those terms which give rise to an admissible partition of I. (Restrict to admissible products of t_{I_k}; then set $t_{I_k} \mapsto 1$.) Lemma B.1 together with

$$e_2 := y\bar{w} + \bar{y}w + w\bar{w} = y(w + \bar{w}) + (\bar{y} - y)\Big(w + \frac{w\bar{w}}{\bar{y} - y}\Big)$$

gives

$$\begin{aligned}
&\sum_{k=0}^{n-1} \sum_{I_1 \uplus \cdots \uplus I_{n-k} = I} \binom{n}{k} \Big[y(z)^k \prod_{j=1}^{n-k} \frac{\omega_{0,|I_j|+1}(I_j, z)}{dx(z)} \\
&\qquad\qquad + y(\sigma_i(z))^k \prod_{j=1}^{n-k} \frac{\omega_{0,|I_j|+1}(I_j, \sigma_i(z))}{dx(z)} \Big] dx(z) \\
&= \sum_{(n_1,n_2,n_3,n_4) \in \mathfrak{D}_n} (-1)^{n_4} n \frac{\prod_{k=1}^{n_3+n_4-1}(n_1+k)(n_2+k)}{n_3! n_4! (n_3+n_4-1)!} y(z)^{n_1} y(\sigma_i(z))^{n_2} dx(z) \\
&\times \sum_{I_1 \uplus I_2 \uplus \cdots \uplus I_{n_3+n_4} = I} \prod_{j=1}^{n_3} \frac{(\omega_{0,|I_j|+1}(I_j, z) + \omega_{0,|I_j|+1}(I_j, \sigma_i(z)))}{dx(z)} \\
&\times \prod_{j=n_3+1}^{n_3+n_4} \Big[y(z) \frac{\omega_{0,|I_j|+1}(I_j, \sigma_i(z)) + \omega_{0,|I_j|+1}(I_j, z)}{dx(z)} \\
&\qquad + \frac{y(\sigma_i(z)) - y(z)}{dx(z)} \big(\omega_{0,|I_j|+1}(I_j, z) + \mathsf{S}\omega_{0,|I_j|+1}(I_j, z)\big) \Big],
\end{aligned} \tag{2.31}$$

where $\mathsf{S}\omega$ was defined in (2.24) and $\mathfrak{D}_n$ is a set of tuples specified in (B.3). The linear loop equation Proposition 2.6 and Remark 2.7 imply that

$$\frac{\omega_{0,|I_j|+1}(I_j, \sigma_i(z)) + \omega_{0,|I_j|+1}(I_j, z)}{dx(z)}$$

is holomorphic at $z = \beta_i$. Holomorphicity of the last line at $z = \beta_i$ follows from Proposition 2.10. Thus, (2.31) is regular at $z = \beta_i$. The projection to admissible partitions of I guarantees that contributions to (2.31) with $n - k > |I|$ are automatically zero.

We finish the proof of the proposition with the fact (2.4) that for any meromorphic 1-form $\omega(q)$ the residue does not change under the Galois involution,

$$\operatorname*{Res}_{q\to\beta_i}(\omega(q)+\omega(\sigma_i(q)))=2\operatorname*{Res}_{q\to\beta_i}\omega(q).$$

Since the residue of (2.31) vanishes[5], this implies the assertion (2.30). ∎

Remark 2.12. In topological recursion, the loop insertion operator D_w is a subtle object. A family of spectral curves needs to be considered with infinitely many parameters. The existence of a loop insertion operator is proved in [19] via deformation theory. It is unclear whether a similar construction holds in our case or in blobbed topological recursion in general. In our subsequent article [28], more information is provided about the existence and assumptions of a loop insertion operator in the quartic Kontsevich model. However, assuming that a loop insertion operator D_w exists[6] for any blobbed topological recursion, we could prove (2.30) by induction in $|I|\mapsto|I|+1$ with the following consideration:

$$\begin{aligned}
0&=D_w\sum_{s=1}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\operatorname*{Res}_{z\to\beta_i}\Bigg[\binom{n}{s}y(z)^{n-s}dx(z)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Bigg]dx(w)\\
&=\sum_{s=1}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\operatorname*{Res}_{z\to\beta_i}\Bigg[\binom{n}{s}(n-s)y(z)^{n-s-1}dx(z)\\
&\qquad\qquad\times\frac{\omega_{0,2}(w,z)}{dx(z)}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Bigg]\\
&\quad+\sum_{s=1}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\operatorname*{Res}_{z\to\beta_i}\Bigg[\binom{n}{s}y(z)^{n-s}dx(z)\\
&\qquad\qquad\times\sum_{\ell=1}^{s}\frac{\omega_{0,|I_\ell|+2}(I_\ell,w,z)}{dx(z)}\prod_{j=1,j\neq\ell}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Bigg]\\
&=\sum_{s=1}^{|I|+1}\sum_{I_1\uplus\cdots\uplus I_s=I\uplus w}\operatorname*{Res}_{z\to\beta_i}\Bigg[\binom{n}{s}y(z)^{n-s}dx(z)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Bigg].
\end{aligned}$$

We have used the fact that the sum $\sum_{I_1\uplus\cdots\uplus I_s=I\uplus w}$ should be symmetric such that all terms with the form $\omega_{0,2}(w,z)$ coming from the second line get a symmetry factor $\frac{1}{s+1}$ so that $\binom{n}{s}\frac{n-s}{s+1}=\binom{n}{s+1}$.

[5]Note that (2.4) only states equality of the residue. The integrand of (2.30) has, in general, higher-order poles but no residue.

[6]D_w acts as a derivation and satisfies $D_w(dx)=0$, $D_w(y(z)dx(z))dx(w)=\omega_{0,2}(w,z)$ and $D_w(\omega_{0,|I|+1}(I,z))dx(w)=\omega_{0,|I|+2}(I,w,z)$.

2.7. Finishing the proof of Theorem 1.2

We can now assemble the pieces into a proof of Theorem 1.2. In a first step, we assume that the rhs of (1.4) and of (2.12) are the same. By induction, these rhs have poles in the points $q \in \{\beta_i, \iota\beta_i, u_k, \iota u_k\}$. Then, conditions (b), (c), (d) imply that $\omega_{0,|I|+1}(I, q)$ is meromorphic on $\hat{\mathbb{C}}$ with poles only in $q \in \{\beta_i, \iota u_k\}$. Therefore,

$$\omega_{0,m+1}(u_1, \ldots, u_m, z) - \sum_{i=1}^{r} \operatorname*{Res}_{q\to\beta_i} \frac{\omega_{0,m+1}(u_1, \ldots, u_m, q)dz}{z-q} - \sum_{k=1}^{m} \operatorname*{Res}_{q\to\iota u_k} \frac{\omega_{0,m+1}(u_1, \ldots, u_m, q)dz}{z-q}$$

is a holomorphic 1-form on the Riemann sphere $\hat{\mathbb{C}} \ni z$, hence identically zero. Inserting the residues from Proposition 2.5 and Proposition 2.10 represents $\omega_{0,|I|+1}(I, z)$ as (1.5)+(1.6).

It remains to prove that the difference between the rhs of (1.4) and (2.12) is a holomorphic form on $\hat{\mathbb{C}} \ni q$, hence zero. By induction, it can have poles at most in $q \in \{\beta_i, \iota\beta_i, u_k, \iota u_k\}$. In Section 2.3, we have shown that the difference is holomorphic at every $q = u_k$ and $q = \iota u_k$, and in Section 2.6, it is shown that the difference is holomorphic at every $q = \beta_i$ and $q = \iota\beta_i$. This completes the proof of Theorem 1.2. ∎

Example 2.13. It is instructive to work out the pair-of-pants case $\omega_{0,3}$. For that, it is convenient to define $\omega_{0,2}(w, z) =: d_w Q(w; z)dz$ with

$$Q(w; z) := \frac{1}{z-w} - \frac{(a^2+bc)}{(cz-a)^2(w-\iota z)}; \tag{2.32}$$

see (1.3). One has $Q(w; \iota z) = \frac{a^2+bc}{(c\iota z-a)^2} Q(w; z)$. The involution identity (1.4) reads

$$\begin{aligned}&\omega_{0,3}(u_1, u_2, q) + \omega_{0,3}(u_1, u_2, \iota q)\\ &= \operatorname*{Res}_{z\to q} \Big(\frac{\omega_{0,2}(u_1, z)\omega_{0,2}(u_2, z)dy(q)}{dx(z)(y(q)-y(z))^2}\Big) = d_q d_{u_1} d_{u_2} \Big(\frac{Q(u_1; q)Q(u_2; q)}{x'(q)y'(q)}\Big).\end{aligned} \tag{2.33}$$

One has $y'(q) = \frac{a^2+bc}{(cq-a)^2} x'(\iota q)$. We take the poles in q apart. With the partial fraction expansion

$$\frac{1}{x'(q)} = c_1 + \sum_{i=1}^{r} \frac{1}{x''(\beta_i)(q-\beta_i)},$$
$$\frac{1}{y'(q)} = c_2 + \sum_{i=1}^{r} \frac{1}{y''(\iota\beta_i)(q-\iota\beta_i)}$$

and (2.32), one finds for the part of (2.33) with poles at $q \in \{\beta_i, \iota u_1, \iota u_2\}$

$$\begin{aligned}
&\omega_{0,3}(u_1,u_2,q)\\
&= d_q d_{u_1} d_{u_2}\Bigg(\sum_{i=1}^{r} \frac{Q(u_1;\beta_i)Q(u_2;\beta_i)}{x''(\beta_i)y'(\beta_i)(q-\beta_i)}\\
&\quad - \frac{Q(u_2;\iota u_1)}{x'(\iota u_1)y'(\iota u_1)\big(q-\frac{au_1+b}{cu_1-a}\big)} - \frac{Q(u_1;\iota u_2)}{x'(\iota u_2)y'(\iota u_2)\big(q-\frac{au_2+b}{cu_2-a}\big)}\Bigg)\\
&= -\sum_{i=1}^{r} \frac{\omega_{0,2}(u_1,\beta_i)\omega_{0,2}(u_2,\beta_i)\omega_{0,2}(q,\beta_i)}{x''(\beta_i)y'(\beta_i)(d\beta_i)^3} + d_q d_{u_1} d_{u_2}\Psi_{0,3}(u_1,u_2,q), \qquad (2.34)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{0,3}(u_1,u_2,q) := &\sum_{i=1}^{r} \frac{Q(u_1;\beta_i)Q(u_2;\beta_i)\iota'(\beta_i)}{x''(\beta_i)y'(\beta_i)(q-\iota\beta_i)}\\
&- \frac{Q(u_2;\iota u_1)}{x'(\iota u_1)y'(\iota u_1)(q-\iota u_1)} - \frac{Q(u_1;\iota u_2)}{x'(\iota u_2)y'(\iota u_2)(q-\iota u_2)}.
\end{aligned}$$

The function $\Psi_{0,3}$ is symmetric in all arguments, which can be seen by a lengthy but straightforward evaluation of the residues of $\Psi_{0,3}(u_1,u_2,q)du_1$ at

$$u_1 \in \{u_2, \iota u_2, \iota q, \beta_i, \iota\beta_i\}.$$

The first term in the last line of (2.34), which can be written as

$$-\sum_{i=1}^{r} \operatorname*{Res}_{z\to\beta_i} \frac{\omega_{0,2}(u_1,z)\omega_{0,2}(u_2,z)\omega_{0,2}(q,z)}{dx(z)dy(z)},$$

is the expression expected from topological recursion [19, Theorem 4.1], whereas $d_q d_{u_1} d_{u_2}\Psi_{0,3}(u_1,u_2,q)$ is the $(0,3)$-blob.

2.8. The sum over all preimages

Let $\omega^{\mathrm{TR}}_{g,n+1}$ be the differential forms generated by topological recursion only [19]. It is well known that for any $\omega^{\mathrm{TR}}_{g,n+1}$, except $(g,n)=(0,0)$, the following identity holds.

Theorem 2.14 ([19]). *Let $I=\{u_1,\dots,u_n\}$ and $\omega^{TR}_{g,n+1}$ be the differential forms generated by topological recursion, where $\omega^{TR}_{0,2}$ is the Bergman kernel and $\omega^{TR}_{0,1} = y\,dx$. Let further $\hat{z}^k$, $k=1,\dots,d$ be the preimages with $x(z)=x(\hat{z}^k)$ such that $z\neq\hat{z}^k$ and $\hat{z}^0\equiv z$. Then, the sum of $\omega^{TR}_{g,n+1}$ over all preimages vanishes, except for the Bergman kernel:*

$$\sum_{k=0}^{d} \frac{\omega^{TR}_{g,n+1}(I,\hat{z}^k)}{dx(z^k)} = \frac{\delta_{g,0}\delta_{n,1}dx(u_1)}{(x(z)-x(u_1))^2}.$$

For $\omega^{\mathrm{TR}}_{0,2}$, the theorem can be proved directly, and for any other $\omega^{\mathrm{TR}}_{g,n+1}$, it follows from the structure of the recursive kernel

$$K_i(z,q) = \frac{\frac{1}{2}\left(\frac{dz}{z-q} - \frac{dz}{z-\sigma_i(q)}\right)}{dx(\sigma_i(q))(y(q) - y(\sigma_i(q)))},$$

since

$$\sum_{k=0}^{d} \frac{\frac{1}{\hat{z}^k - q} - \frac{1}{\hat{z}^k - \sigma_i(q)}}{x'(\hat{z}^k)} = \frac{1}{x(z) - x(q)} - \frac{1}{x(z) - x(\sigma_i(q))} = 0.$$

Consequently, it is natural to ask whether a similar identity holds for the preimage sum of $\omega_{0,n+1}$ defined by (1.4) together with (1.3). Applying Theorem 1.2, we get the following proposition.

Proposition 2.15. *Let $I = \{u_1, \ldots, u_n\}$. For $n > 0$, the sum over all preimages is*

$$\begin{aligned}
&\sum_{k=0}^{d} \frac{\omega_{0,n+1}(I, \hat{z}^k)}{dx(z^k)} \\
&= \frac{\delta_{n,1} dx(u_1)}{(x(z) - x(u_1))^2} + \frac{\delta_{n,1} dy(u_1)}{(x(z) + y(u_1))^2} \\
&\quad + \sum_{j=1}^{n} d_{u_j} \left[\sum_{s=1}^{|I|-1} (-1)^{s+1} \sum_{I_1 \uplus \cdots \uplus I_s = I \setminus u_j} \frac{\prod_{i=1}^{s} \frac{\omega_{0,|I_1|+1}(I_i, u_j)}{(x(z)+y(u_j))dx(u_j)}}{x(z) + y(u_j)} \right],
\end{aligned}$$

where $\hat{z}^0 \equiv z$.

Proof. First, look at $\omega_{0,2}$ from the second line of (1.3). Dividing it by $dx(\hat{z}^k)$ and summing over k yields

$$\begin{aligned}
\sum_{k=0}^{d} \frac{\omega_{0,2}(u, \hat{z}^k)}{dx(z^k)} = -d_u \sum_{k=0}^{d} \bigg(&\frac{1}{2} \frac{1}{x'(\hat{z}^k)(u - \hat{z}^k)} + \frac{1}{2} \frac{\iota'(\hat{z}^k)}{x'(\hat{z}^k)(\iota u - \iota \hat{z}^k)} \\
&- \frac{1}{2} \frac{\iota'(\hat{z}^k)}{x'(\hat{z}^k)(u - \iota \hat{z}^k)} - \frac{1}{2} \frac{1}{x'(\hat{z}^k)(\iota u - \hat{z}^k)} \bigg).
\end{aligned} \tag{2.35}$$

Now, use the fact that $\chi_k = \iota \hat{z}^k$ are preimages of $\chi = \iota z$ under the map y, i.e.,

$$y(\chi) = y(\chi_k).$$

Furthermore, if a point z does not coincide with one of its preimages $\hat{z}^k$, it will generically also not coincide under the global involution, $\chi \neq \chi_k$. Together with

$$\frac{\iota'(z)}{x'(z)} = -\frac{1}{y'(\iota z)},$$

(2.35) breaks down to

$$\sum_{k=0}^{d}\frac{\omega_{0,2}(u,\hat{z}^k)}{dx(z^k)} = \frac{1}{2}d_u\Big(\frac{1}{x(z)-x(u)}-\frac{1}{y(\iota z)-y(\iota z)}+\frac{1}{y(\iota z)-y(u)}-\frac{1}{x(z)-x(\iota u)}\Big)$$
$$= d_u\Big(\frac{1}{x(z)-x(u)}-\frac{1}{x(z)+y(u)}\Big).$$

For $\omega_{0,n}$ with $n>2$, Theorem 1.2 proves by the same consideration as for topological recursion in Theorem 2.14 that the poles at the ramification points β_i do not contribute. For the remaining part, we use the equivalence given by Proposition 2.5 to Lemma 2.4. Interchanging the integral and the sum over all preimages in Lemma 2.4 gives

$$\sum_{k=0}^{d}\frac{\omega_{0,|I|+1}(I,\hat{z}^k)}{dx(z^k)}$$
$$= -\sum_{u_j\in I} d_{u_j}\Bigg[\sum_{s=1}^{|I|-1}\sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_j}\sum_{k=0}^{d}\frac{1}{s!}\frac{\partial^s\Big(\frac{\iota'(\hat{z}^k)}{x'(\hat{z}^k)(\iota\hat{z}^k-u_j)}\Big)}{\partial(y(u_j))^s}\prod_{i=1}^{s}\frac{\omega_{0,|I_i|+1}(I_i,u_j)}{dx(u_j)}\Bigg]$$
$$= -\sum_{u_j\in I} d_{u_j}\Bigg[\sum_{s=1}^{|I|-1}\sum_{I_1\uplus\cdots\uplus I_s=I\backslash u_j}\frac{1}{s!}\frac{\partial^s\Big(\frac{1}{x(z)+y(u_j)}\Big)}{\partial(y(u_j))^s}\prod_{i=1}^{s}\frac{\omega_{0,|I_i|+1}(I_i,u_j)}{dx(u_j)}\Bigg].$$

Carrying out the derivative with respect to $y(u_j)$ yields the assertion. ∎

2.9. A particular symmetry under the involution ι

In the second part, we prove that the planar sector (genus 0) of the quartic Kontsevich model is completely governed by the involution identity (1.4). In a decisive step of the proof, we will need an intriguing symmetry resulting from (1.4) alone:

$$0 = \operatorname*{Res}_{z\to q}\Bigg[\sum_{s=1}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\Bigg(\frac{dx(z)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}}{(x(z)-x(q))(y(q)-y(z))^s}\Bigg)\Bigg]$$
$$+\operatorname*{Res}_{z\to q}\Bigg[\sum_{s=1}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\Bigg(\frac{dx(\iota z)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dx(\iota z)}}{(x(\iota z)-x(\iota q))(y(\iota q)-y(\iota z))^s}\Bigg)\Bigg]. \tag{2.36}$$

The residues in (2.36) can be expressed as limits of partial derivatives of

$$\Big(\frac{x(z)-x(q)}{y(z)-y(q)}\Big)^s\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}$$

and $(\frac{x(\iota z)-x(\iota q)}{y(\iota z)-y(\iota q)})^s \prod_{j=1}^s \frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{dx(z)}$ with respect to $x(z)$ and $x(\iota z)$. Using (2.5), we thus bring (2.36) into an equation that we need:

$$0=\sum_{s=1}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\sum_{n_1+\cdots+n_s=s}\Big(\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q)\\+\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,\iota q)\Big). \tag{2.37}$$

The main combinatorial tool to verify (2.36) is Corollary A.8. Using Corollary A.8, we prove that the integrand in (2.36) is an exact 1-form in z.

Proposition 2.16. *We have*

$$\sum_{s=1}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\left\{\frac{dx(z)\prod_{j=1}^s\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}}{(x(z)-x(q))(y(q)-y(z))^s}-\frac{dy(z)\prod_{j=1}^s\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{(-dy(z))}}{(y(q)-y(z))(x(z)-x(q))^s}\right\}\\
=\sum_{s=2}^{|I|}\frac{1}{s!}\sum_{I_1\uplus\cdots\uplus I_s=I}d_z\left[\sum_{r=0}^{s-2}\left(\frac{1}{dy(z)}d_z\right)^{s-2-r}\left[\frac{1}{y(q)-y(z)}\right]\right.\\
\left.\times\left(-\frac{1}{dy(z)}d_z\right)^{r}\left[\frac{1}{(x(z)-x(q))}\frac{dx(z)}{dy(z)}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\right]\right]. \tag{2.38}$$

In particular, the residue (2.36) *at $z=q$ is zero.*

Proof. In the first line of (2.38), restricted to $s\geq 2$, we write $\frac{1}{(y(q)-y(z))^s}$ as a multiple differential and integrate by parts:

$$\sum_{s=2}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{dx(z)}{(x(z)-x(q))(y(q)-y(z))^s}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\\
=\sum_{s=2}^{|I|}\frac{1}{s!}\sum_{I_1\uplus\cdots\uplus I_s=I}d_z\left\{\sum_{r=0}^{s-2}\left(\frac{1}{dy(z)}d_z\right)^{s-2-r}\left[\frac{1}{(y(q)-y(z))}\right]\right.\\
\left.\times\left(-\frac{1}{dy(z)}d_z\right)^{r}\left[\frac{1}{(x(z)-x(q))}\frac{dx(z)}{dy(z)}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\right]\right\}\\
+\sum_{s=2}^{|I|}\frac{1}{s!}\sum_{I_1\uplus\cdots\uplus I_s=I}\frac{dy(z)}{(y(q)-y(z))}\\
\times\left(-\frac{1}{dy(z)}d_z\right)^{s-1}\left[\frac{1}{(x(z)-x(q))}\frac{dx(z)}{dy(z)}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\right].$$

Hence, the assertion is true if the 1-form in z

$$
\begin{aligned}
&f(I;z,q)\\
&:= \frac{\omega_{0,|I|+1}(I,z)+\omega_{0,|I|+1}(I,\iota z)}{(x(z)-x(q))}\\
&\quad+\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\Bigg\{-\frac{dy(z)}{s}\frac{\prod_{j=1}^{s}\omega_{0,|I_j|+1}(I_j,\iota z)}{(x(z)-x(q))^s(-dy(z))^s}\\
&\quad+\frac{dy(z)}{s!}\Big(-\frac{1}{dy(z)}d_z\Big)^{s-1}\Big[\frac{1}{(x(z)-x(q))}\Big(\frac{dx(z)}{dy(z)}\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Big)\Big]\Bigg\}
\end{aligned}
$$

is identically zero. The last line of $f(I;z,q)$ is of the form of Corollary A.8 with

$$
\begin{aligned}
\frac{1}{dy(z)}d_z &= b(y)\partial_x+\partial_y, &\quad a(x) &= \frac{1}{x-x_q},\\
b(y) &= \frac{dx(z)}{dy(z)}, &\quad c_j(y) &= \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)},
\end{aligned}
$$

and a constant $x_q=x(q)$. We thus get

$$
\begin{aligned}
&f(I;z,q)\\
&:= \frac{\omega_{0,|I|+1}(I,z)+\omega_{0,|I|+1}(I,\iota z)}{(x(z)-x(q))}\\
&\quad+\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\Bigg\{-\frac{dy(z)}{s}\frac{\prod_{j=1}^{s}\omega_{0,|I_j|+1}(I_j,\iota z)}{(x(z)-x(q))^s(-dy(z))^s}\\
&\quad+\frac{dy(z)}{s!}\sum_{r=1}^{s}\frac{(r-1)!}{(x(z)-x(q))^r}\\
&\quad\times\sum_{\substack{J_1\uplus\cdots\uplus J_r=\{1,2,\dots,s\}\\ J_1<\cdots<J_r}}\prod_{k=1}^{r}\Big(-\frac{1}{dy(z)}d_z\Big)^{|J_k|-1}\Big[\frac{dx(z)}{dy(z)}\prod_{j\in J_k}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Big]\Bigg\}.
\end{aligned}
\tag{2.39}
$$

The derivatives in the last line are expressed as a residue:

$$
\begin{aligned}
&\Big(-\frac{1}{dy(z)}d_z\Big)^{|J_k|-1}\Big[\frac{dx(z)}{dy(z)}\prod_{j\in J_k}\frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}\Big]\\
&\qquad= -(|J_k|-1)!\,\operatorname*{Res}_{w\to z}\Big(\frac{dx(w)}{(y(z)-y(w))^{|J_k|}}\prod_{j\in J_k}\frac{\omega_{0,|I_j|+1}(I_j,w)}{dx(w)}\Big).
\end{aligned}
$$

The case $r = 1$ combines to the involution identity (1.4) and is thus identified as the negative of the first line of (2.39). In the remainder, we order the partitions of I:

$$\begin{aligned} f(I;z,q) = \sum_{s=2}^{|I|} \sum_{\substack{I_1 \uplus \cdots \uplus I_s = I \\ I_1 < \cdots < I_s}} \Bigg\{ - \frac{(s-1)! dy(z) \prod_{j=1}^{s} \omega_{0,|I_j|+1}(I_j, \iota z)}{(x(z)-x(q))^s (-dy(z))^s} \\ + dy(z) \sum_{r=1}^{s} \frac{(-1)^r (r-1)!}{(x(z)-x(q))^r (dy(z))^r} \\ \times \sum_{\substack{J_1 \uplus \cdots \uplus J_r = \{1,2,\dots,s\} \\ J_1 < \cdots < J_r}} \prod_{k=1}^{r} \operatorname*{Res}_{w \to z} \Bigg(\frac{(|J_k|-1)! dy(z) dx(w)}{(y(z)-y(w))^{|J_k|}} \prod_{j \in J_k} \frac{\omega_{0,|I_j|+1}(I_j, w)}{dx(w)} \Bigg) \Bigg\}. \end{aligned} \tag{2.40}$$

We change the order of the summations. The outer summation is a sum over ordered partitions $I'_1 \uplus \cdots \uplus I'_r$ given by $I'_k = \bigcup_{j \in J_k} I_j$, which is combined with an inner summation over ordered partitions of the individual I'_k. Renaming in the first line of (2.40) $s \mapsto r$ and $I_j \mapsto I'_j$, we arrive at

$$\begin{aligned} & f(I;z,q) \\ & = \sum_{r=2}^{|I|} \sum_{\substack{I'_1 \uplus \cdots \uplus I'_r = I \\ I'_1 < \cdots < I'_r}} \frac{(r-1)! dy(z)}{(x(z)-x(q))^r (-dy(z))^r} \Bigg\{ - \prod_{j=1}^{r} \omega_{0,|I'_j|+1}(I'_j, \iota z) \\ & \quad + \prod_{k=1}^{r} \operatorname*{Res}_{w \to z} \Bigg(\sum_{s=1}^{|I'_k|} \sum_{\substack{I_{k1} \uplus \cdots \uplus I_{ks} = I'_k \\ I_{k1} < \cdots < I_{ks}}} \frac{(s-1)! dy(z) dx(w)}{(y(z)-y(w))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_{kj}|+1}(I_{kj}, w)}{dx(w)} \Bigg) \Bigg\}. \end{aligned}$$

The outcome is zero thanks to (1.4). ■

3. The quartic Kontsevich model

3.1. Summary of previous results

Let H_N be the real vector space of self-adjoint $N \times N$-matrices, H'_N its dual, and (e_{kl}) the standard matrix basis in the complexification of H_N. We define a measure $d\mu_{E,\lambda}$ on H'_N by

$$\begin{aligned} d\mu_{E,\lambda}(\Phi) &= \frac{1}{\mathcal{Z}} \exp\Big(- \frac{\lambda N}{4} \operatorname{Tr}(\Phi^4) \Big) d\mu_{E,0}(\Phi), \\ \mathcal{Z} &:= \int_{H'_N} \exp\Big(- \frac{\lambda N}{4} \operatorname{Tr}(\Phi^4) \Big) d\mu_{E,0}(\Phi), \end{aligned} \tag{3.1}$$

where $d\mu_{E,0}(\Phi)$ is a Gaußian measure with covariance

$$\left[\int_{H'_N} d\mu_{E,0}(\Phi)\ \Phi(e_{jk})\Phi(e_{lm})\right]_c = \frac{\delta_{jm}\delta_{kl}}{N(E_j+E_l)} \tag{3.2}$$

for some $0 < E_1 < \cdots < E_N$. The trace is understood as

$$\mathrm{Tr}(\Phi^4) = \sum_{k,l,m,n=1}^{N} \Phi(e_{kl})\Phi(e_{lm})\Phi(e_{mn})\Phi(e_{nk}).$$

Moments or cumulants of $d\mu_{E,\lambda}$ are viewed as general or connected correlation functions in a finite-dimensional approximation of a Euclidean quantum field theory.

We call the objects resulting from (3.1)+(3.2) the *quartic Kontsevich model* because of its formal analogy with the Kontsevich model [29] in which $\mathrm{Tr}(\Phi^4)$ in (3.1) is replaced with $\mathrm{Tr}(\Phi^3)$. The Gaußian measure $d\mu_{E,0}(\Phi)$ is the same as (3.2). Kontsevich proved in [29] that (3.1) with $\mathrm{Tr}(\Phi^3)$-term, viewed as function of the E_k, is the generating function for intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.

Derivatives of the Fourier transform

$$\mathcal{Z}(M) := \int_{H'_N} d\mu_{E,\lambda}(\Phi)\ e^{\mathrm{i}\Phi(M)}$$

with respect to matrix entries M_{kl} and parameters E_k of the free theory give rise to *Dyson–Schwinger equations* between the cumulants

$$\langle e_{k_1l_1}\cdots e_{k_nl_n}\rangle_c = \frac{1}{\mathrm{i}^n}\frac{\partial^n \log\mathcal{Z}(M)}{\partial M_{k_1l_1}\cdots\partial M_{k_nl_n}}\bigg|_{M=0}. \tag{3.3}$$

After $1/N$-expansion, one obtains a closed non-linear equation [23] for the $1/N$-leading part $G^{(0)}_{|kl|}$ of the 2-point function

$$N\langle e_{kl}e_{lk}\rangle_c = \sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|kl|}$$

and a hierarchy of affine equations [24, 25] for all other functions. The non-linear equation for $G^{(0)}_{|kl|}$ was solved in a special case in [31] and then in [22] in full generality. The solution introduces a ramified covering $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ given by (see [33])

$$R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d}\frac{\varrho_k}{\varepsilon_k + z}. \tag{3.4}$$

Here, $(\varepsilon_k, \varrho_k)$ are implicitly defined as solution of the system $R(\varepsilon_l) = e_l, \varrho_l R'(\varepsilon_l) = r_l$ when assuming that $(E_1, \ldots, E_N)$ consists of d pairwise different values $e_1, \ldots, e_d$ which arise with multiplicities[7] $r_1, \ldots, r_d$. The planar 2-point function is then given by $G^{(0)}_{|kl|} = \mathcal{G}^{(0)}(\varepsilon_k, \varepsilon_l)$, where $\mathcal{G}^{(0)}$ is the rational function

$$\mathcal{G}^{(0)}(z, w) = \frac{1 - \frac{\lambda}{N} \sum_{k=1}^{d} \frac{r_k \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon_k}^j)}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w) - R(-z)} \tag{3.5}$$

with poles located at $z + w = 0$ and $z, w \in \{\widehat{\varepsilon_k}^j\}$ for $k, j \in \{1, \ldots, d\}$. Here, $v \in \{z, \hat{z}^1, \ldots, \hat{z}^d\}$ is the set of solutions of $R(v) = R(z)$. One has

$$\mathcal{G}^{(0)}(z, w) = \mathcal{G}^{(0)}(w, z).$$

In [12], we identified an algorithm which constructs recursively, starting from (3.5), any cumulant (3.3) of the measure (3.1)+(3.2). Its core is a coupled system of loop equations [12, Propositions E.2, E.4, and Corollary 4.7] for three families of functions $\Omega^{(g)}_m(u_1, \ldots, u_m)$, $\mathcal{T}^{(g)}(u_1, \ldots, u_m \| z, w|)$ and $\mathcal{T}^{(g)}(u_1, \ldots, u_m \| z|w|)$ with $\mathcal{T}^{(0)}(\emptyset \| z, w|) = \mathcal{G}^{(0)}(z, w)$ and $\mathcal{T}^{(0)}(\emptyset \| z|w|)$ determined in [33]. Of particular importance are the functions $\Omega^{(g)}_m(u_1, \ldots, u_m)$ which arise from complexification of derivatives $\Omega^{(g)}_{a_1,\ldots,a_n}$ of the partially summed two-point function:

$$\sum_{g=0}^{\infty} N^{1-2g-n} \Omega^{(g)}_{a_1,\ldots,a_n} := \frac{\delta_{n,2}}{N(E_{a_1} - E_{a_2})^2} + \frac{\partial^{n-1}}{\partial E_{a_2} \cdots \partial E_{a_n}} \sum_{k=1}^{N} \langle e_{a_1 k} e_{k a_1} \rangle_c.$$

In [11], it is shown that the $\Omega^{(g)}_{a_1,\ldots,a_n}$ are distinguished polynomials of the cumulants (3.3).

The system of equations established in [12] permits to determine $\Omega^{(g)}_m(u_1, \ldots, u_m)$ without prior knowledge of $\langle e_{a_1 k} e_{k a_1} \rangle_c$. The solution of this system for $\Omega^{(0)}_2$, $\Omega^{(0)}_3$, $\Omega^{(0)}_4$, and $\Omega^{(1)}_1$ in [12] gave strong support for the conjecture that the meromorphic forms

$$\omega_{g,n}(z_1, \ldots, z_n) := \Omega^{(g)}_n(z_1, \ldots, z_n) dR(z_1) \cdots dR(z_n)$$

[7] Working with the assumption that the eigenvalues of the external matrix E can be split into a finite set of eigenvalues with certain multiplicities is the core assumption of the replica method used by Brézin and Hikami [13] for another type of generalised Kontsevich models. However, the type of generalised Kontsevich models studied in [13] is known to be governed by topological recursion, whereas the model under consideration in this article is not, as we are showing. This might also be a reason why the replica method was never successfully applied to the quartic Kontsevich model.

obey blobbed topological recursion [7] for the spectral curve $(x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \omega_{0,1} = y\,dx, \omega_{0,2})$ with

$$x(z) = R(z), \quad y(z) = -R(-z), \quad \omega_{0,2}(u,z) = \frac{du\,dz}{(u-z)^2} + \frac{du\,dz}{(u+z)^2}.$$

In the remainder of this paper, we prove this conjecture for $\omega_{0,n}$. More precisely, we prove that the solution of the system of equations given in [12] is identical to the solution of the involution identity (1.4) given in Theorem 1.2 for $\iota z = -z$ (i.e., $b = c = 0$ in (1.1)) and $x = R$ as in (3.4). In particular, the part of $\omega_{0,n}$ with poles at ramification points of $x = R$ obeys exactly the universal formula of topological recursion [19], and the other part with poles along opposite diagonals $z_i + z_j = 0$ is described by a residue formula of very similar type.

3.2. Loop equations

The loop equations derived in [12] imply that $\omega_{0,m+1}(u_1,\dots,u_m,z)$ is an exact 1-form in every variable $u_1,\dots,u_k$. We set

$$\omega_{0,m+1}(u_1,\dots,u_m,z) = d_{u_1}\cdots d_{u_m}\varpi_{0,m+1}(u_1,\dots,u_m;z).$$

The $\varpi_{0,m+1}(u_1,\dots,u_m;z)$ are 1-forms in z; they relate via

$$\varpi_{0,m+1}(u_1,\dots,u_m;z) = \lambda^{1-m}\mathcal{W}^{(0)}_{m+1}(u_1,\dots,u_m,z)dR(z)$$

to functions introduced in [12]. The loop equations derived in [12, Appendix G] translate as follows into equations between $\varpi_{0,m+1}$ and two classes of auxiliary functions.

Proposition 3.1. *The loop equations of the quartic Kontsevich model have in lowest degree the solution $\varpi_{0,2}(u;z) = -\frac{dz}{(u-z)} - \frac{dz}{(u+z)}$ and can be turned for $I = \{u_1,\dots,u_m\}$ with $m \geq 2$ into* [12, Proposition G.1, (G.4)+(G.5)]

$$\begin{aligned}\varpi_{0,|I|+1}(I;z) &= \operatorname*{Res}_{\substack{q\to -u_{1,\dots,m}\\ q\to\beta_{1,\dots,2d}}} \frac{dz}{z-q}\Big[\sum_{I_1\uplus I_2=I}\varpi_{0,|I_1|+1}(I_1;q)\mathfrak{v}_{0,|I_2|}(I_2\|q)\Big]\\ &\quad+\sum_{k=1}^{m}\frac{\mathfrak{v}_{0,|I|-1}(I\backslash u_k\|u_k)dz}{z+u_k},\end{aligned} \tag{3.6}$$

$$\begin{aligned}\textit{where}\quad \mathfrak{v}_{0,|I|}(I\|q) &= \sum_{\substack{I_1\uplus I_2=I\\ \textit{possibly } I_2=\emptyset}}\sum_{j=1}^{d}\frac{\varpi_{0,|I_1|+1}(I_1;-\hat{q}^j)\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|-\hat{q}^j,q|)}{dR(\hat{q}^j)(R(-q)-R(-\hat{q}^j))}\\ &\quad-\sum_{k=1}^{m}\frac{\tilde{\mathfrak{t}}_{0,|I|-1}(I\setminus u_k\|u_k,q|)}{(R(u_k)-R(-q))(R(-u_k)-R(q))}\end{aligned} \tag{3.7}$$

and $\tilde{\mathfrak{t}}_{0,0}(\emptyset\|z,q|) = 1$ *and for* $|I| \geq 1$

$$
\begin{aligned}
\tilde{\mathfrak{t}}_{0,|I|}(I\|z,q|) = &- \sum_{\substack{I_1 \uplus I_2 = I \\ \text{possibly } I_2=\emptyset}} \sum_{l=1}^{d} \frac{\varpi_{0,|I_1|+1}(I_1; -\hat{q}^l)\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|-\hat{q}^l, q|)}{dR(\hat{q}^l)(R(z) - R(-\hat{q}^l))} \\
&+ \sum_{k=1}^{m} \frac{\tilde{\mathfrak{t}}_{0,|I|-1}(I \setminus u_k\|u_k, q|)}{(R(z) - R(u_k))(R(q) - R(-u_k))} \\
&- \sum_{\substack{I_1 \uplus I_2 = I \\ \text{possibly } I_2=\emptyset}} \frac{\varpi_{0,|I_1|+1}(I_1; z)\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|z, q|)}{dR(z)(R(q) - R(-z))}.
\end{aligned} \tag{3.8}
$$

In (3.6), $\beta_1, \ldots, \beta_{2d}$ *are the ramification points of the ramified cover* R *given in (3.4). By* $\hat{q}^1, \ldots, \hat{q}^d$ *we denote the other preimages of* q *under* R*, i.e.,* $R(\hat{q}^j) = R(q)$*. Generically, they are pairwise different and different from* $q \equiv \hat{q}^0$*.*

Note that conditions (a), (c), (d) of Theorem 1.2 are automatically satisfied by (3.6). Compared with [12], we have set

$$
\tilde{\mathfrak{t}}_{0,|I|}(I\|z,w|) = \frac{\mathcal{U}^{(0)}(I\|z,w|)}{\lambda^{|I|}\mathcal{G}^{(0)}(z,w)}
$$

and

$$
\mathfrak{v}_{0,|I|}(I\|z) = -\lambda^{1-|I|}\mathfrak{U}^{(0)}(I\|z).
$$

The function $\tilde{\mathfrak{t}}_{0,|I|}(I\|z,q|)$ is regular at every $z = -\hat{q}^j$. To see this, we write in the last line of (3.8) the denominator as

$$
R(q) - R(-z) = R(-(-\hat{q}^j)) - R(-z)
$$

and insert the Taylor expansion (2.6) (for ϖ) and the usual Taylor expansion of $\tilde{\mathfrak{t}}_{0,|I|}(I_2\|z,q|)$:

$$
\frac{\varpi_{0,|I_1|+1}(I_1; z)\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|z, q|)}{dR(z)(R(q) - R(-z))} =
$$

$$
\sum_{n,p=0}^{\infty} \frac{(-1)^n}{p!}(R(z)-R(-\hat{q}^j))^{n+p-1}\nabla^n\varpi_{0,|I_1|+1}(I_1;-\hat{q}^j)\frac{\partial^p\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|z,q|)}{\partial(R(z))^p}\bigg|_{z=-\hat{q}^j}.
$$

Inserted back into (3.8), the case $p = n = 0$ cancels the term $l = j$ of the first line of (3.8) when taking

$$
\nabla^0\varpi_{0,|I_1|+1}(I_1; -\hat{q}^j) = \frac{\varpi_{0,|I_1|+1}(I_1; -\hat{q}^j)}{-dR(\hat{q}^j)}
$$

into account. Hence, all partial derivatives of $\tilde{\mathfrak{t}}_{0,|I|}(I\|z,q|)$ are regular at $z=-\hat{q}^j$:

$$
\begin{aligned}
&\frac{(-1)^n}{n!}\frac{\partial^n\tilde{\mathfrak{t}}_{0,|I|}(I\|z,q|)}{\partial(R(z))^n}\Big|_{z=-\hat{q}^j}\\
&=-\sum_{\substack{I_1\uplus I_2=I\\ \text{possibly } I_2=\emptyset}}\sum_{\substack{l=1\\ l\neq j}}^{d}\frac{\varpi_{0,|I_1|+1}(I_1;-\hat{q}^l)\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|-\hat{q}^l,q|)}{dR(\hat{q}^l)(R(-\hat{q}^j)-R(-\hat{q}^l))^{n+1}}\\
&\quad+\sum_{k=1}^{m}\frac{\tilde{\mathfrak{t}}_{0,|I|-1}(I\setminus u_k\|u_k,q|)}{(R(-\hat{q}^j)-R(u_k))^{n+1}(R(q)-R(-u_k))}\\
&\quad+\sum_{\substack{I_1\uplus I_2=I\\ \text{possibly } I_2=\emptyset}}\sum_{p=0}^{n+1}\nabla^{n-p+1}\varpi_{0,|I_1|+1}(I_1;-\hat{q}^j)\frac{(-1)^p}{p!}\frac{\partial^p\tilde{\mathfrak{t}}_{0,|I_2|}(I_2\|z,q|)}{\partial(R(z))^p}\Big|_{z=-\hat{q}^j}.
\end{aligned}
\tag{3.9}
$$

Formula (3.8) for $z\mapsto u_k$ and (3.9) provide a system of equations whose resolution provides $\tilde{\mathfrak{t}}_{0,|I|}(I\|-\hat{q}^l,q|)$ and $\tilde{\mathfrak{t}}_{0,|I|-1}(I\setminus u_k\|u_k,q|)$ as polynomials in $\nabla\varpi$ with coefficients in rational functions of R. Inserting into (3.6), we recursively express $\varpi_{0,|I|+1}(I;z)$ in terms of $\nabla^n\varpi_{0,|I'|+1}(I';z)$ for $|I'|<|I|$. We find it convenient to develop a graphical description for this resolution. With these tools, we can establish the following theorem.

Theorem 3.2. *Starting from $\varpi_{0,2}(u;z)=-\frac{dz}{(u-z)}-\frac{dz}{(u+z)}$, the system of equations (3.6), (3.7), (3.8) for $z\mapsto u_k$, and (3.9) has the solution*

$$
\begin{aligned}
\varpi_{0,|I|+1}(I;z)=&\sum_{i=1}^{r}\operatorname*{Res}_{q\to\beta_i}K_i(z,q)\sum_{I_1\uplus I_2=I}\varpi_{0,|I_1|+1}(I_1;q)\varpi_{0,|I_2|+1}(I_2;\sigma_i(q))\\
&-\sum_{k=1}^{m}\operatorname*{Res}_{q\to-u_k}\sum_{I_1\uplus I_2=I}\tilde{K}(z,q,u_k)\varpi_{0,|I_1|+1}(I_1;q)\varpi_{0,|I_2|+1}(I_2;q),
\end{aligned}
$$

where

$$
\begin{aligned}
K_i(z,q)&:=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z-\sigma_i(q)}\big)}{dR(\sigma_i(q))(R(-\sigma_i(q))-R(-q))},\\
\tilde{K}(z,q,u)&:=\frac{\frac{1}{2}\big(\frac{dz}{z-q}-\frac{dz}{z+u}\big)}{dR(q)(R(u)-R(-q))}.
\end{aligned}
$$

Hence,

$$
\omega_{0,m+1}(u_1,\dots,u_m,z)=d_{u_1}\cdots d_{u_m}\varpi_{0,m+1}(u_1,\dots,u_m;z)
$$

coincides with the solution of equation (1.4) for $x(z)=R(z)$ and $\iota(z)=-z$ given in Theorem 1.2.

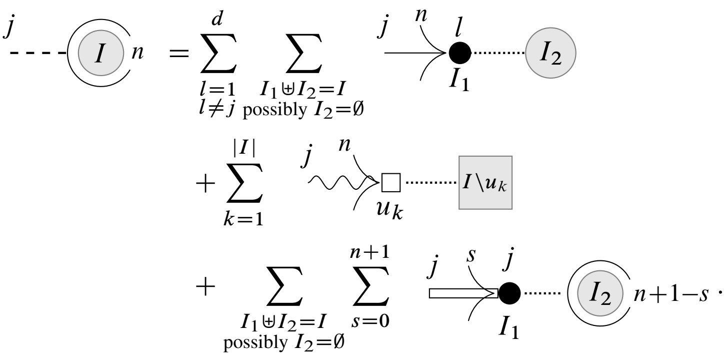
3.3. Graphical description

We introduce in Table 1 weighted functions, vertices, and edges. These are connected to chains which provide a graphical description for the terms

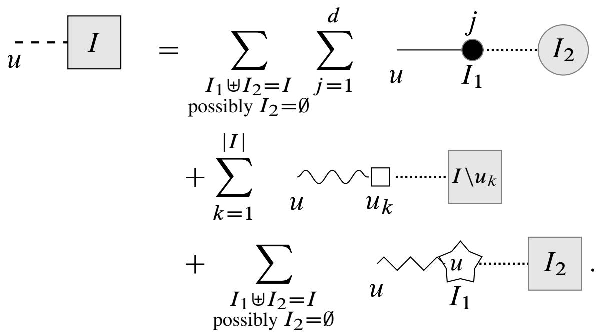
$$\frac{(-1)^p \partial^p}{\partial(R(z))^p} \tilde{\mathfrak{t}}_{0,|I|}(I \| z, q|)|_{z=-\hat{q}^j}$$

and $\tilde{\mathfrak{t}}_{0,|I|}(I \| u_k, q|)$ and its constituents. We agree that arrow tips with label $p = 0$ are not shown. Also, the surrounding circle segment indicating the n-th derivative with respect to $R(z)$ is not shown for $n = 0$.

Equation (3.9) has for $|I| \geq 1$ the following graphical description (we keep the order of the last three lines of (3.9)):



Similarly, equation (3.8) is for $|I| \geq 1$ represented as follows (we keep the order of lines):



The integrand of the first line of (3.6) is now iteratively obtained by distinguishing in $\tilde{\mathfrak{t}}_{0,|I|}(I \| z, q|)$ the cases $I = \emptyset$ from $I \neq \emptyset$. We describe this iteration graphically. The integrand of the first line of (3.6) is the sum of weights of chains made of initial vertex v0, subsequent vertices v1, v2, v3, and edges in between. A vertex v3 can follow v2 or another v3, whereas v1, v2 can be placed anywhere. The edge to choose is governed by the type of vertices at both ends. One multiplies the weights

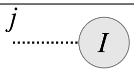
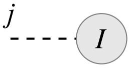
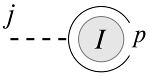
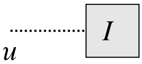
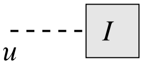
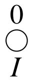
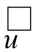
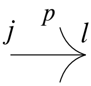
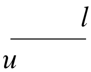
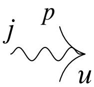
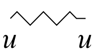
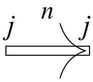
#	function	weight	remark
f1		$\tilde{\mathfrak{t}}_{0,\|I\|}(I\\|-\hat{q}^j,q)$	$\|I\| \geq 0$ equals 1 for $I=\emptyset$
		$\tilde{\mathfrak{t}}_{0,\|I\|}(I\\|-\hat{q}^j,q)$	$I\neq\emptyset$ case $p=0$ of f2
f2^p		$\frac{(-1)^p\partial^p}{\partial(R(z))^p}\tilde{\mathfrak{t}}_{0,\|I\|}(I\\|z,q)\big\|_{z=-\hat{q}^j}$	$I\neq\emptyset$
f3		$\tilde{\mathfrak{t}}_{0,\|I\|}(I\\|u,q)$	$\|I\|\geq 0$ equals 1 for $I=\emptyset$
		$\tilde{\mathfrak{t}}_{0,\|I\|}(I\\|u,q)$	$I\neq\emptyset$
#	**vertex**	**weight**	**remark**
v0		$-\varpi_{0,\|I\|+1}(I;q)$	initial vertex
v1		$\varpi_{0,\|I\|+1}(I;-\hat{q}^j)$	follows edges e1^p,e2,e6
v2		$\frac{1}{R(q)-R(-u)}$	follows edges e3^p,e4
v3		$\frac{\varpi_{0,\|I\|+1}(I;u)}{dR(u)(R(-u)-R(q))}$	follows edges e5
#	**edge**	**weight**	**remark**
e1^p		$\frac{1}{(R(-\hat{q}^j)-R(-\hat{q}^l))^{p+1}(-dR(\hat{q}^l))}$	follows vertices v0,v1 requires $l\neq j$ $\hat{q}^0\equiv q$, no tip for $p=0$
e2		$\frac{1}{(R(u)-R(-\hat{q}^l))(-dR(\hat{q}^l))}$	follows vertices v2,v3
e3^p		$\frac{1}{(R(-\hat{q}^j)-R(u))^{p+1}}$	follows vertices v0,v1 no tip for $p=0$
e4		$\frac{1}{R(v)-R(u)}$	follows vertices v2,v3 requires $u\neq v$
e5		1	follows vertices v2,v3
e6^n		∇^n	follows vertices v1 applies to next vertex

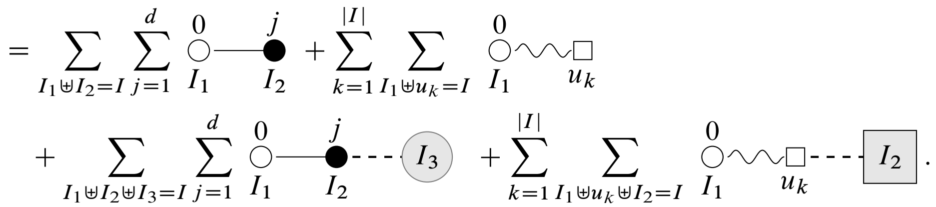
Table 1. Graphical rules for building blocks of chains.

given in Table 1 and sums for each order of vertices over partitions of I into subsets $I_1, \dots, I_s, u_{k_1}, \dots, u_{k_r}$ at the vertices, over the v1-labels $j, l, \dots$ (from 1 to d, but excluding the preceding label), and over the possible exponents n, p of the edges $\mathrm{e1}^p$, $\mathrm{e3}^p$, and $\mathrm{e6}^n$. These exponents are not arbitrary; we discuss later their pattern.

3.4. Examples

We write the first iteration in detail:

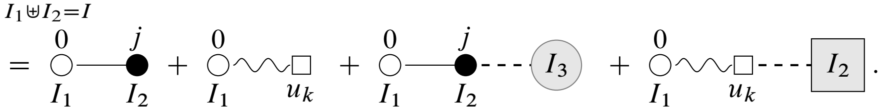
$$\sum_{I_1 \uplus I_2 = I} \varpi_{0,|I_1|+1}(I_1; q)\mathfrak{v}_{0,|I_2|}(I_2 \| q)$$



(3.10)

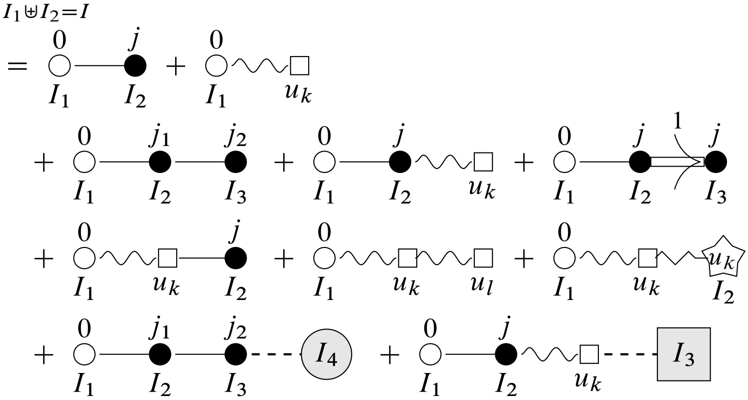
The necessary sum over partitions of I and over ranges of labels j are obvious from the vertex labels. We therefore employ from now on a simplified notation where these summations are omitted. This means that instead of (3.10) we simply write

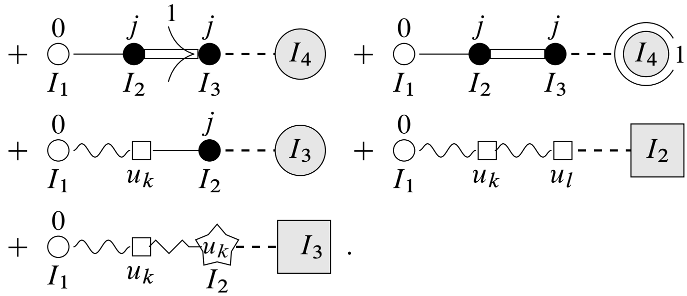
$$\sum_{I_1 \uplus I_2 = I} \varpi_{0,|I_1|+1}(I_1; q)\mathfrak{v}_{0,|I_2|}(I_2 \| q)$$



For $|I| = 2$, only the first two chains contribute. The next iteration reads in simplified notation

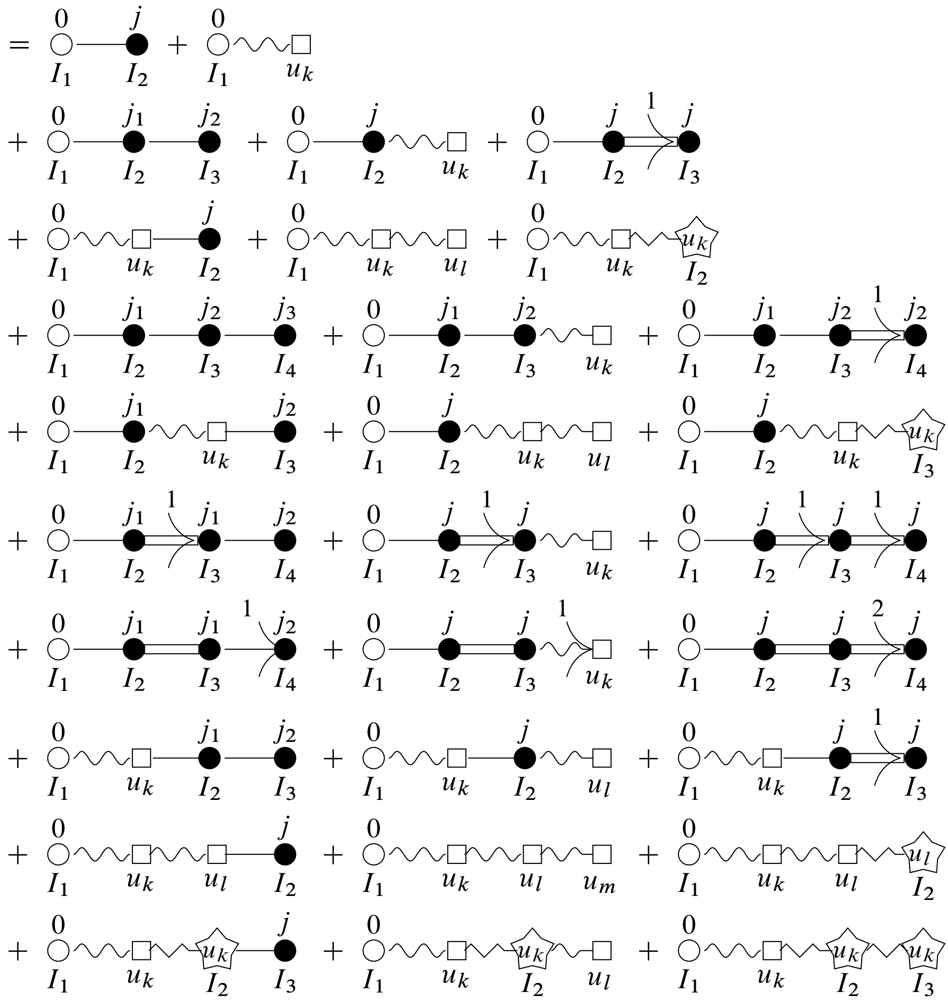
$$\sum_{I_1 \uplus I_2 = I} \varpi_{0,|I_1|+1}(I_1; q)\mathfrak{v}_{0,|I_2|}(I_2 \| q)$$





For $|I| = 3$, only the first three lines of the rhs are relevant. We give another iteration but stop it at $|I| = 4$:

$$\sum_{I_1 \uplus I_2 = I} \varpi_{0,|I_1|+1}(I_1; q)\mathfrak{v}_{0,|I_2|}(I_2 \| q)$$



$+$ chains with f1, f2^p, f3.

3.5. Cancellations between chains

The following tuples will occur in the subsequent combinatorial description.

Definition 3.3 ([16]). A Catalan tuple $\underline{\tilde{n}} = (n_0, \ldots, n_k)$ of length $k \in \mathbb{N}$ is a tuple of integers $n_j \geq 0$ for $j = 0, \ldots, k$ such that

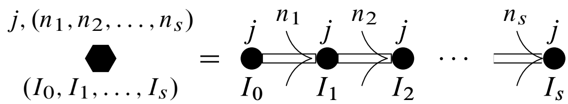
$$\sum_{j=0}^{k} n_j = k \quad \text{and} \quad \sum_{j=0}^{l} n_j > l \quad \text{for } l = 0, \ldots, k-1.$$

The set of Catalan tuples of length $|\underline{\tilde{n}}| := k$ is denoted by $\mathcal{C}_k$.

The cardinality of $\mathcal{C}_k$ is the k-th Catalan number[8].

Now, it will be convenient to collect subchains of consecutive vertices v1 with the same upper label j.

Definition 3.4. A v1-*block* is a subchain



of vertices v1 of the same label j, connected by edges $\mathrm{e6}^{n_i}$. We call j the *label*, $\underline{n} = (n_1, \ldots, n_s)$ the *degree*, and $\underline{I} = (I_0, I_1, \ldots, I_s)$ the *partition distribution* of the block. Moreover, we let $s = s(\underline{n})$ be the *size*, $|\underline{n}| = n_1 + \cdots + n_s$ the *length*, and $s(\underline{n}) - |\underline{n}|$ the *deficit* of the v1-block. We also regard vertices v1 as v1-blocks of size or length 0.

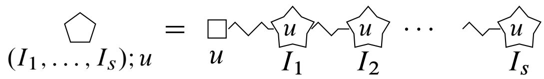
A v1-block can terminate a chain iff the deficit is 0. A v1-block can be followed by an edge $\mathrm{e1}^p$ or $\mathrm{e3}^p$; the label p of such an edge is then given by the deficit

$$p = s(\underline{n}) - |\underline{n}|$$

of the v1-block before it. Since a v1-block is formed by repeatedly attaching a function $\mathrm{f2}^p$ labelled $p \geq 0$, the condition on the deficit must hold at all intermediate steps. This amounts to a condition $\sum_{i=1}^{r} n_i \leq r$ on any partial sum. For blocks of total deficit 0 (those which terminate a chain or are followed by edges $\mathrm{e1}^0$ or $\mathrm{e3}^0$), this is equivalent to the opposite condition $\sum_{i=0}^{r} n_{s-i} > r$ for $0 \leq r \leq s-1$ and $\sum_{i=0}^{s} n_{s-i} = s$ when prepending $n_0 = 0$. This means that the reversely ordered tuple $\underline{\tilde{n}} := (n_s, n_{s-1}, \ldots, n_1, 0)$ is a *Catalan tuple*. We consider the subset of chains which differ only in the degrees $\underline{n}$ of a v1-block of size s but otherwise have identically labelled vertices. In this subset, any degree $\underline{n}$ of the v1-block compatible with the deficit condition is produced and precisely once.

[8] OEIS A000108, visited on 11 July 2024.

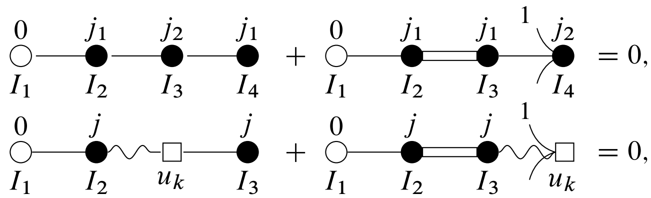
Definition 3.5. A v2-*block* is a subchain



starting with a vertex v2 of lower label u and several consecutive vertices v3 with the same inner label u, connected by edges e5. We let u be the *label*, s the *size*, and $(I_1,\dots,I_s)$ the *partition distribution* of a v2-block. A v2-block of size 0 is identified with a vertex v2 with lower label u.

If several v2-blocks arise in a chain, then its labels $u_k, u_l,\dots$ are necessarily different.

We will prove that, after taking cancellations into account, also the labels $j_1, j_2,\dots$ of v1-blocks in the surviving chains are pairwise different. These cancellations start with chains of 4 vertices:



which follows from the weights in Table 1 and with

$$\nabla^0 \varpi_{0,|I_3|+1}(I_3;-\hat{q}^j) = \frac{\varpi_{0,|I_3|+1}(I_3;-\hat{q}^j)}{(-dR(\hat{q}^j))}.$$

These identities reduce the set of graphs to a much simpler subset.

Lemma 3.6. *Let M be the set of chains generated by the loop equations for*

$$\sum_{I_1\uplus I_2=I} \varpi_{0,|I_1|+1}(I_1;q)\mathfrak{v}_{0,|I_2|}(I_2\|q).$$

Then, cancellations between weights remove all chains with edges e1^p *and* e3^p *having a tip labelled $p\geq 1$ and all chains with two or more identically labelled* v1*-blocks. The subset of surviving graphs is given by the set of chains made of* v2*-blocks and of* v1*-blocks which have deficit 0 and pairwise different labels, connected by appropriate edges without tip.*

Proof. Consider a v1-block of label j, partition distribution $\underline{I}$, and degree

$$\underline{n} = (n_1, n_2, \dots, n_s)$$

with deficit $p = s - n_1 - \dots - n_s \geq 1$. Its reverse degree $(n_s, n_{s-1},\dots,n_1,0)$ cannot be a Catalan tuple for $p\geq 1$. This means that either $n_s = 0$, or there is a unique

$2 \le t \le s$ such that $(n_s, n_{s-1}, \dots, n_t, 0)$ is a Catalan tuple but $(n_s, n_{s-1}, \dots, n_t, n_{t-1}, 0)$ is not. This necessarily means that $n_{t-1} = 0$. We define a unique splitting into two v1-blocks of degrees $\underline{n}^-$ and $\underline{n}^+$:

$$n_s = 0: \quad \text{set } \underline{n}^- = (n_1, \dots, n_{s-1}),\ \underline{I}^- = (I_0, \dots, I_{s-1}),\ \underline{n}^+ = \emptyset,\ \underline{I}^+ = (I_s),$$

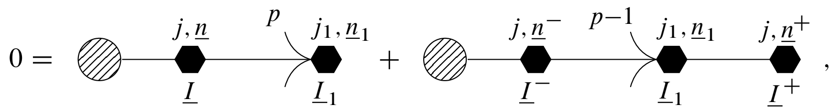
$$n_s \neq 0: \quad \text{set } \underline{n}^- = (n_1, \dots, n_{t-2}),\ \underline{I}^- = (I_0, \dots, I_{t-2}),$$
$$\underline{n}^+ = (n_t, \dots, n_s),\ \underline{I}^+ = (I_{t-1}, I_t, \dots, I_s).$$

By construction, $\underline{n}^+$ has deficit 0 so that it can terminate a chain or is followed by edges $e1^0$ or $e3^0$. The other label $\underline{n}^-$ has deficit $p-1$ and is followed by edges $e1^{p-1}$ or $e3^{p-1}$. Conversely, two degrees $\underline{n}^-$ of deficit $p-1 \ge 0$ and $\underline{n}^+$ of deficit 0 can be joined to a unique degree $\underline{n}$ of deficit p.

The weights given in Table 1 together with

$$\nabla^{n_{t-1}} \varpi_{0,|I_{t-1}|+1}(I_{t-1}; -\hat{q}^j) = \frac{\varpi_{0,|I_{t-1}|+1}(I_{t-1}; -\hat{q}^j)}{(-dR(\hat{q}^j))}$$

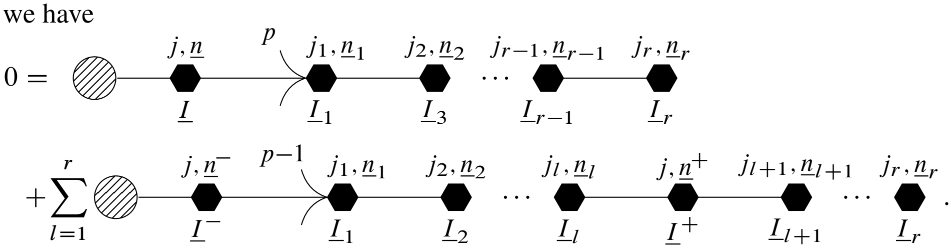
confirm the following identity:



(3.11)

where the shaded circle stands for any identical subchain in both chains. The same cancellation arises if the v1-block labelled j_1 is replaced by a v2-block and $e1^p$ by $e3^p$.

Next, for chains which extend by further blocks to the right, all with labels $\neq j$, we have



(3.12)

Again, the shaded circle stands for any identical subchain. The same cancellation arises if any subset of v1-blocks (other than the one labelled j) is replaced by corresponding v2-blocks.

After these preparations, we prove that (3.11) and (3.12) provide the claimed reduction in the set of chains describing $\sum_{I_1 \uplus I_2 = I} \varpi_{0,|I_1|+1}(I_1; q) \mathfrak{v}_{0,|I_2|}(I_2 \| q)$.

(1) We start with the type of chains indicated by the left graph in (3.11), with $p \ge 1$. Since the splitting of $\underline{n}$ into $\underline{n}^-$, $\underline{n}^+$ is unique, it cancels against a

unique chain indicated on the right of (3.11). Conversely, for any chain K terminating in a triple consisting of two v1-blocks of the same label j and any other block in between, there is a unique chain indicated on the left of (3.11) against which K cancels. As a result, we remove all chains with a single block after the last e1^p or e3^p edge (with $p \geq 1$) and all those chains which terminate in a triple of blocks in which two v1-blocks are equally labelled.

(2) We pass to (3.12) for $r = 2$. The chain in the first line is only present for $j_2 \neq j$ because the case $j_2 = j$ was removed in step (1). According to (3.12), the chain K indicated in the first line cancels against two uniquely determined chains terminating in a quadruple of blocks two of which are labelled j, and vice versa. After all, we remove all chains with two blocks after the last e1^p or e3^p edge (with $p \geq 1$) and all those chains terminating in a v1-block labelled j which is followed by three more blocks; one of them is also labelled j.

(r) Continuing in this manner removes all chains with an e1^p or e3^p edge with $p \geq 1$ and all chains with two or more identically labelled v1-blocks.

We are left with chains in which all blocks have different labels and are connected by edges e1^0, e3^0, i.e., without tip. ■

All surviving v1-blocks have degrees of deficit 0, i.e., are reversals of Catalan tuples. In the next step, we collect all v1-blocks which have the same union of their partition distribution (and deficit 0) to a v1-*group*:

$$\begin{aligned}
\underset{I}{\overset{j}{\hexagon}} &:= \sum_{s=0}^{|I|-1} \sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I} \sum_{(n_s,\dots,n_1,0)\in\mathcal{C}_s} \underset{(I_0,I_1,\dots,I_s)}{\overset{j,(n_1,n_2,\dots,n_s)}{\hexagonblack}}, \\
\mathrm{weight}\Big(\underset{I}{\overset{j}{\hexagon}}\Big) \\
&= (-dR(\hat{q}^j)) \sum_{s=0}^{|I|-1} \sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I} \sum_{(n_s,\dots,n_1,0)\in\mathcal{C}_s} \prod_{i=0}^{s} \nabla^{n_i} \varpi_{0,|I_i|+1}(I_i;-\hat{q}^j).
\end{aligned} \tag{3.13}$$

We have used the fact that the leftmost vertex of every v1-block has weight

$$\varpi_{0,|I_0|+1}(I_0;-\hat{q}^j) = (-dR(\hat{q}^j))\nabla^0 \varpi_{0,|I_0|+1}(I_0;-\hat{q}^j).$$

Similarly, we collect v2-blocks with the same union of their partition distribution to a v2-*group*:

$$\underset{I;u}{\Diamond} := \underset{u}{\square}\,\delta_{\|I\|,0} + \sum_{s=1}^{|I|} \sum_{I_1\uplus\cdots\uplus I_s=I} \underset{(I_1,\dots,I_s);u}{\pentagon},$$

$$\mathrm{weight}\Big(\underset{I;u}{\Diamond}\Big) = -\sum_{s=0}^{|I|}\ \sum_{I_1\uplus\cdots\uplus I_s=I}^{(I\neq\emptyset)} \frac{1}{(R(-u)-R(q))^{s+1}} \prod_{i=1}^{s} \frac{\varpi_{|I_i|+1}(I_i;u)}{dR(u)}. \tag{3.14}$$

The summation $\sum_{I_1\uplus\cdots\uplus I_s=I}^{(I\neq\emptyset)}$ is left out for $|I|=\emptyset$. For $I\neq\emptyset$, there is no contribution from $s=0$. We summarise the previous simplifications and collections.

Corollary 3.7. *The integrand* $\sum_{I_1\uplus I_2=I}\varpi_{0,|I_1|+1}(I_1;q)\mathfrak{v}_{0,|I_2|}(I_2\|q)$ *in the first line of* (3.6) *is the sum of weights of all different chains which meet the following criteria.*

- *The leftmost vertex is* v0 *with weight* $-\varpi_{0,|I_1|+1}(I_1;q)$.
- *Any other vertex is a* v1*-group with weight* (3.13) *or a* v2*-group with weight* (3.14). *The labels* j_i *of the* v1*-groups are pairwise different.*
- *The union of all subsets* I_i *at the initial vertex, the* v1*-groups, and the* v2*-groups, together with the labels* u_k *of the* v2*-groups, is* $I=\{u_1,\dots,u_m\}$.
- *The edges between the groups (and initial vertex) are given by* e1^0, e2, e3^0, e4 *depending on the groups they connect. Their weights are given in Table* 1.

3.6. Weight of a v1-group

Next, we prove a simpler formula for the weight (3.13) of a v1-group. Its main step is Corollary (A.3), a variant of Corollary A.2 given in the appendix. In the second line of (3.13), we write the sum over all partitions as sum over ordered partitions (introduced in the beginning of Section 2.1) together with a sum over permutations ς. Inserting the definition (2.5) of $\nabla\varpi$, we thus have

$$\begin{aligned}\mathrm{weight}\Big(\overset{j}{\underset{I}{\hexagon}}\Big) &= (-dR(\hat{q}^j))\sum_{s=0}^{|I|-1}\ \sum_{\substack{I_0\uplus I_1\uplus\cdots\uplus I_s=I\\ I_0<I_1<\cdots<I_s}}\ \sum_{\varsigma\in\mathcal{S}_{s+1}}\ \sum_{(n_s,\dots,n_1,0)\in\mathcal{C}_s}\ \lim_{z\to-\hat{q}^j}\Bigg\{\\ &\prod_{i=0}^{s}\frac{(-1)^{n_i}}{n_i!}\frac{\partial}{\partial(R(z))^{n_i}}\Bigg(\frac{R(z)-R(-\hat{q}^j)}{R(q)-R(-z)}\frac{\varpi_{0,|I_{\varsigma(i)}|+1}(I_{\varsigma(i)};z)}{dR(z)}\Bigg)\Bigg\}.\end{aligned}$$

With Corollary A.3 and the bijection between rooted plane trees and Catalan tuples, we can replace $\sum_{\varsigma\in\mathcal{S}_{s+1}}\sum_{(n_s,\dots,n_1,0)\in\mathcal{C}_s}\mapsto s!\sum_{n_0+\cdots+n_s=s}$. We reexpress the result in terms of $\nabla\varpi$ and admit again any order of partitions of I into $s+1$ subsets:

$$\begin{aligned}&\mathrm{weight}\Big(\overset{j}{\underset{I}{\hexagon}}\Big)\\ &= (-dR(\hat{q}^j))\sum_{s=0}^{|I|-1}\ \sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I}\frac{1}{s+1}\sum_{n_0+\cdots+n_s=s}\ \prod_{i=0}^{s}\nabla^{n_i}\varpi_{0,|I_i|+1}(I_i;-\hat{q}^j).\end{aligned} \tag{3.15}$$

Our aim is to prove Theorem 3.2, namely, that the solution $\varpi_{0,|I|+1}(I;q)$ of the system (3.6), (3.7), (3.8) (for $z \mapsto u_k$), and (3.9) is, after applying the exterior differentials d_{u_k} to pass from $\varpi_{0,|I|+1}$ to $\omega_{0,|I|+1}$, the same as the solution of (1.4) for $x(z) = R(z)$ and $\iota(z) = -z$. We prove this theorem by induction. The v1-group is always a genuine subchain because at least the initial vertex v0 is excluded. Therefore, Theorem 3.2 is the induction hypothesis for the v1-group, which gives the following proposition.

Proposition 3.8. *The* v1-*group has* $\mathrm{weight}\Big(\overset{j}{\underset{I}{\hexagon}}\Big) = -\varpi_{0,|I|+1}(I;\hat{q}^j)$.

Proof. This follows from Lemma 2.2 for $q \mapsto -\hat{q}^j$ when moving the first term

$$\omega_{0,|I|+1}(I,-\hat{q}^j) = dy(-\hat{q}^j)\nabla^0\omega_{0,|I|+1}(I,-\hat{q}^j)$$

to the rhs. Then, $d_{u_1}\cdots d_{u_m}$ applied to (3.15) equals $-\omega_{0,|I|+1}(I,\hat{q}^j)$ when taking Theorem 3.2 as induction hypothesis for $I = \{u_1,\dots,u_m\}$. Inverting the differentials d_{u_k} gives the assertion. ∎

3.7. Poles of $\varpi_{0,|I|+1}(I;z)$ at $z = \beta_i$

We let $\mathcal{P}^i_z\omega(z) = \mathrm{Res}_{q\to\beta_i}\frac{\omega(q)dz}{z-q}$ be the projection of a 1-form ω to its poles at $z = \beta_i$. Proposition 3.1 gives

$$\mathcal{P}^i_z\varpi_{0,|I|+1}(I;z) = \mathcal{P}^i_z\Big(\sum_{I_1\uplus I_2=I}\varpi_{0,|I_1|+1}(I_1;z)\mathfrak{v}_{0,|I_2|}(I_2\|z)\Big).$$

Proposition 3.9. *Let $\hat{q}^{j_i} = \sigma_i(q)$ be the preimage of q which corresponds to the local Galois involution near β_i. Then, for all $I = \{u_1,\dots,u_m\}$ with $m \geq 2$, one has*

$$\mathcal{P}^i_z\varpi_{0,|I|+1}(I;z) = -\mathcal{P}^i_z\bigg(\sum_{I_1\uplus I_2=I}\frac{\varpi_{0,|I_1|+1}(I_1;z)\varpi_{0,|I_1|+1}(I_1;\hat{z}^{j_i})}{dR(\hat{z}^{j_i})(R(-z)-R(-\hat{z}^{j_i}))}\bigg).$$

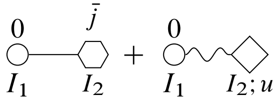
The application of $d_{u_1}\cdots d_{u_k}$ agrees with the restriction of (1.5) to poles at $z = \beta_i$.

Proof. In the graphical representation of Corollary 3.7, the assertion amounts to

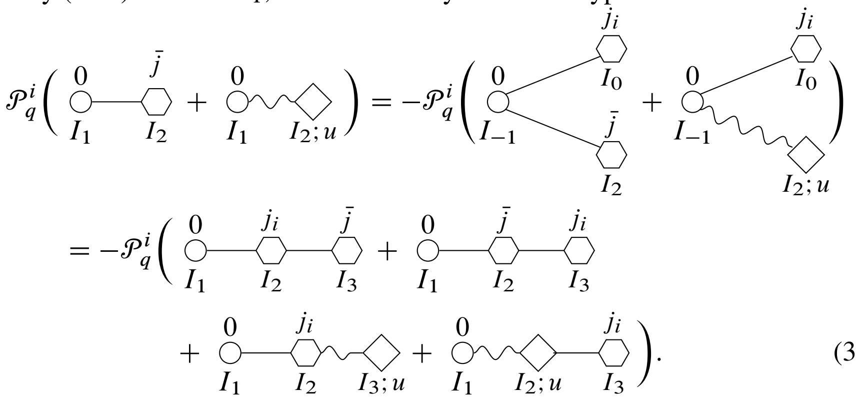
$$\mathcal{P}^i_q\Big(-\overset{0}{\underset{I}{\circ}}\Big) = \mathcal{P}^i_q\Big(\overset{0}{\underset{I_1}{\circ}}\!-\!\!\!-\!\!\!-\!\overset{j_i}{\underset{I_2}{\hexagon}}\Big). \tag{3.16}$$

The rhs is one of the chains contributing to the residue at $q = \beta_i$ in the first line of (3.6). We have to prove that all other chains described in Corollary 3.7 sum up to expressions regular at $q = \beta_i$.

We prove this regularity by induction on the length of chains (with v1/v2-groups as vertices). By $\bar{j}$ we denote a label different from j_i. There are two remaining chains of length 2, namely,



(in the first chain summation over $\bar{j} \neq j_i$ and over partitions $I_1 \uplus I_2 = I$, in the second chain summation over partitions $I_1 \uplus u \uplus I_2 = I$). Edges, v1-groups with label $\bar{j}$, and v2-groups are regular at $q = \beta_i$. The initial vertex v0 is regular for $|I_1| = 1$ so that these chains only contribute to $\mathcal{P}^i_q$ for $|I_1| \geq 2$. In that case, we can, up to terms holomorphic at β_i, replace the initial vertex by (3.16) for $I \mapsto I_1$, which is true by induction hypothesis. We thus have



(3.17)

This identity removes all chains of length 3 with a v1-group labelled j_i at any position. There remain only the chains of length 3 without v1-group labelled j_i. For $|I_1| = 1$, these are holomorphic at $q = \beta_i$ and can be discarded in the projection $\mathcal{P}^i_q$. The only poles come from initial v0-vertices with $|I_1| \geq 2$ multiplied by regular expressions. We can thus use (3.16) for $I \mapsto I_1$ again and express by the same mechanism as (3.17) the survived length-3 chains as $-\mathcal{P}^i_q$ of all length-4 chains which have a v1-group labelled j_i at any position. These cancel in the graphical representation. Since the v1-group labelled j_i can occur only once by Lemma 3.6, only the length-4 chains without any v1-group labelled j_i survive the cancellation.

We repeat this procedure up to chains of length $|I|$. The surviving ones have an initial v0-vertex and otherwise v1/v2-groups with other labels than j_i. Now, because the initial v0-vertex necessarily has

$$|I_1| = 1,$$

it is also regular at $q = \beta_i$. Therefore, all chains survived up to this point project with $\mathcal{P}^i_q$ to 0. ∎

3.8. Poles of $\varpi_{0,|I|+1}(I; z)$ at $z = -u_k$

We prove in Section 4 the following assumption.

Assumption 3.10. *Let* $I=\{u_1,\dots,u_m\}$ *with* $m\geq 2$. *Then, for every* $k=1,\dots,m$, *one has*

$$\operatorname*{Res}_{q\to -u_k} \varpi_{0,|I|+1}(I;q)=0.$$

We can thus focus on poles of second or higher-order captured by the projection

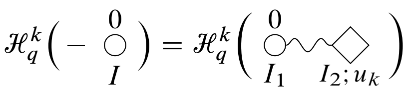
$$\mathcal{H}^k_z\omega(u_k,z):=\operatorname*{Res}_{q\to -u_k}\Big[\Big(\frac{dz}{z-q}-\frac{dz}{z+u_k}\Big)\omega(u_k,q)\Big]$$

for some 1-form $\omega(u_k,z)$ in z (which may depend on further variables). We prove the following proposition.

Proposition 3.11. *Let*

$$I=\{u_1,\dots,u_m\}$$

with $m\geq 2$. *The projection* $\mathcal{H}^k_q$ *of* $\varpi_{0,|I|+1}(I;q)$ *is recursively given by*



(3.18)

in the graphical description or explicitly by

$$\begin{aligned}\mathcal{H}^k_q&\varpi_{0,|I|+1}(I;q)\\ &=\sum_{s=0}^{|I|-2}\sum_{I_0\uplus\dots\uplus I_s=I\setminus u_k}\mathcal{H}^k_q\Bigg(\frac{\varpi_{0,|I_0|+1}(I_0;q)}{(R(-q)-R(u_k))(R(-u_k)-R(q))^{s+1}}\\ &\qquad\times\prod_{i=1}^{s}\frac{\varpi_{0,|I_i|+1}(I_i;u_k)}{dR(u_k)}\Bigg).\end{aligned} \tag{3.19}$$

Remark. Under Assumption 3.10, the expression (3.19) is equal to

$$\operatorname*{Res}_{q\to -u_k}\frac{dz}{z-q}\varpi_{0,|I|+1}(I;q).$$

Application of $d_{u_1}\cdots d_{u_m}$ thus coincides with (2.11) at $q\mapsto\iota q=-q$ and $w\mapsto -z$. This was shown to be equivalent to (2.9) and to (2.7), both for $z\mapsto \iota z=-z$ and

$$q\mapsto\iota z=-q.$$

Together with Proposition 3.9, it follows that

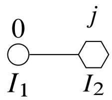
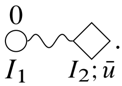
$$\omega_{0,m+1}(u_1,\dots,u_m,z):=d_{u_1}\cdots d_{u_m}\varpi_{0,m+1}(u_1,\dots,u_m;z)$$

agrees with (1.5)+(1.6). Hence, Theorem 3.2 is true if Assumption 3.10 holds.

Proof. Since the second line of (3.6) only has a first-order pole at $z = -u_k$, the projection of (3.6) to poles of higher-order reads

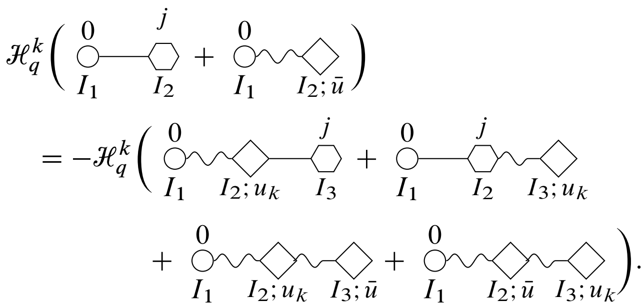
$$\mathcal{H}_q^k \varpi_{0,|I|+1}(I;q) = \mathcal{H}_q^k \Big[\sum_{I_1 \uplus I_2 = I} \varpi_{|I_1|+1}(I_1;q) \mathfrak{v}_{0,|I_2|}(I_2\|q) \Big]. \tag{3.20}$$

The rhs of (3.18) is one of the chains contributing to the rhs of (3.20). We prove by induction on the chain length (with v1/v2-groups as vertices) that all other chains sum to expressions which at $q = -u_k$ are holomorphic or have at most a first-order pole. By $\bar{u}$ we denote any $u_l \neq u_k$.

At length 2, we have in addition to the rhs of (3.18) the chains  + . In the case $u_k \in I_2$, these chains are holomorphic at $q = -u_k$ and can be discarded under $\mathcal{H}_q^k$. It remains the case $u_k \in I_1$. If $I_1 = \{u_k\}$, then the initial vertex v0 has weight

$$-\varpi_{0,2}(u_k;q) = \frac{dq}{u_k - q} + \frac{dq}{u_k + q}$$

and thus only a first-order pole at $q = -u_k$ which does not contribute to $\mathcal{H}_q^k$. The only contributions are thus from $u_k \in I_1$ with $|I_1| \geq 2$. Here, we can use the induction hypothesis (3.18) for $I \mapsto I_1 \mapsto I_2 \mapsto I_3$ so that


(3.21)

This identity removes all chains of length 3 with a v2-group labelled u_k at any position. The remaining length-3 chains have edges and v1/v2-groups which are holomorphic at $= -u_k$. Poles arise only if $u_k \in I_1$ in the initial vertex, and poles of second and higher-order require $|I_1| \geq 2$. Here, the induction hypothesis is available so that the same mechanism removes all chains of length 4 with a v2-group labelled u_k. We repeat this construction until the initial vertex necessarily has $|I_1| = 1$ and also projects to 0 under $\mathcal{H}_q^k$. This finishes the proof of (3.19). ∎

As discussed in the remark directly after Proposition 3.11, the proof of Theorem 3.2 is complete provided that Assumption 3.10 holds.

4. Proof of Assumption 3.10

4.1. The residue

The recursion formula (3.6) generates, a priori, also a first-order pole at $z = -u_k$ with residue

$$\begin{aligned}&\operatorname*{Res}_{q\to -u_k} \varpi_{0,|I|+1}(I;q)\\ &= \operatorname*{Res}_{q\to -u_k}\Big[\sum_{I_1\uplus I_2=I} \varpi_{0,|I_1|+1}(I_1;q)\mathfrak{v}_{0,|I_2|}(I_2\|q)\Big] + \mathfrak{v}_{0,|I|-1}(I\backslash u_k\|u_k). \qquad (4.1)\end{aligned}$$

Our goal is to prove Assumption 3.10, i.e., that the residue (4.1) vanishes for $|I| \geq 2$. Of particular importance will be the functions

$$\begin{aligned}\Delta\omega_{0,|I|+1}(I,z) &= \sum_{s=0}^{I-1}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I} \nabla^{s+1}\omega_{0,|I_0|+1}(I_0,z)\prod_{j=1}^{s}\frac{\omega_{0,|I_j|+1}(I_j,\iota z)}{-dy(z)},\\ \Delta\varpi_{0,|I|+1}(I;z) &= \sum_{s=0}^{I-1}\sum_{I_0\uplus I_1\uplus\cdots\uplus I_s=I} \nabla^{s+1}\varpi_{0,|I_0|+1}(I_0;z)\prod_{j=1}^{s}\frac{\varpi_{0,|I_j|+1}(I_j;-z)}{dR(-z)}.\end{aligned} \qquad (4.2)$$

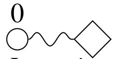
These arise as follows.

Lemma 4.1. *Let $I = \{u_1,\dots,u_m\}$ for $m \geq 2$. Suppose that Assumption 3.10 holds for $\varpi_{0,|I'|+1}$ with $u_k \in I'$ and $2 \leq |I'| < |I|$ (there is no condition for $m = 2$). Then,*

$$\begin{aligned}\operatorname*{Res}_{z\to -u_k}\varpi_{0|I|+1}(I;z) &= \mathfrak{v}_{0,|I|-1}(I\backslash u_k\|u_k) - \mathfrak{v}_{0,|I|-1}(I\backslash u_k\|-u_k)\\ &\quad - \Delta\varpi_{0,|I|}(I\backslash u_k;-u_k)\\ &\quad + \sum_{I_1\uplus I_2=I\backslash u_k}\mathfrak{v}_{0,|I_1|}(I_1\|-u_k)\Delta\varpi_{0,|I_2|+1}(I_2;-u_k). \qquad (4.3)\end{aligned}$$

Proof. We evaluate the residue on the rhs of (4.1). In the graphical representation, we have a contribution from the chain (in which $I' = \emptyset$ is allowed; the weight of the v2-group is given in (3.14))

$$\operatorname*{Res}_{q\to -u_k}\Big(\;0\quad I_0\quad I';u_k\Big)_{I_0\uplus I'=I\backslash u_k}$$

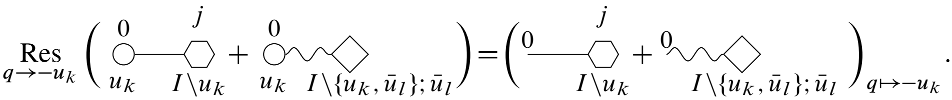


$$\begin{aligned}&= \sum_{s=0}^{|I|-2}\sum_{I_0\uplus\cdots\uplus I_s=I\backslash u_k}\operatorname*{Res}_{q\to -u_k}\left(\frac{\varpi_{0,|I_0|+1}(I_0;q)}{(R(-q)-R(u_k))(R(-u_k)-R(q))^{s+1}}\right)\\ &\quad\times\prod_{i=1}^{s}\frac{\varpi_{0,|I_i|+1}(I_i;u_k)}{dR(u_k)}\end{aligned}$$

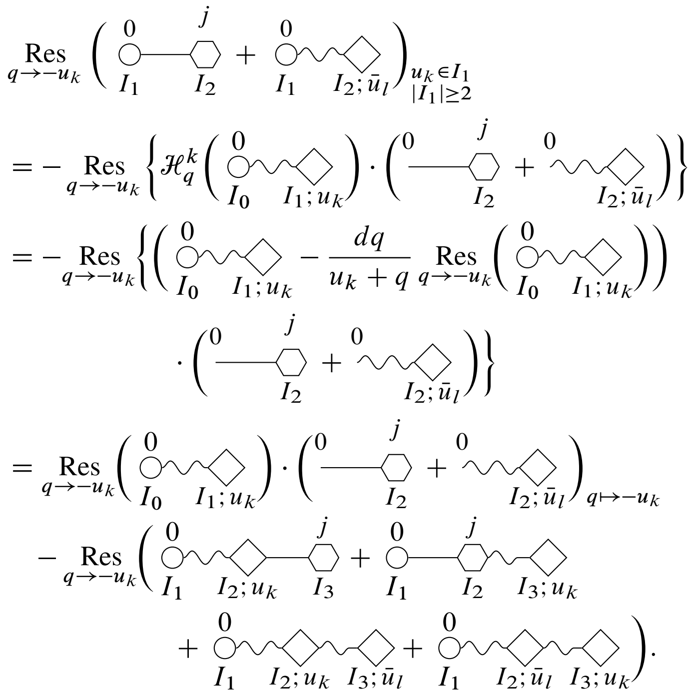
$$
\begin{aligned}
&= -\sum_{s=0}^{|I|-2} \sum_{I_0 \uplus \cdots \uplus I_s = I \backslash u_k} \nabla^{s+1} \varpi_{0,|I_0|+1}(I_0; -u_k) \prod_{i=1}^{s} \frac{\varpi_{0,|I_i|+1}(I_i; u_k)}{dR(u_k)} \\
&\equiv -\Delta \varpi_{0,|I|}(I \backslash u_k; -u_k),
\end{aligned}
\tag{4.4}
$$

where the definition (2.5) for $\omega \mapsto \varpi$ at $q \mapsto -u_k$ and $z \mapsto q$ has been used. This provides the third term on the rhs of (4.3). The first term is copied from (4.1).

We investigate the residues at $q = -u_k$ of all other chains. The remaining chains of length 2 (with v1/v2-groups as vertices) with residue at $q = -u_k$ are the same as in the first line of (3.21) with $u_k \in I_1$. Two cases are to be distinguish. For $I_1 = u_k$, we have a purely first-order pole and



Amputation of the initial v0-vertex gives the chains contributing to $-\mho_{0,|I'|}(I' \| q)$. Hence, the rhs of the above equation is the restriction of $-\mho_{0,|I|-1}(I \backslash u_k \| -u_k)$ to chains of length 1. The other case is $u_k \in I_1$ but $|I_1| \geq 2$. Since $|I_1| < |I|$, Assumption 3.10 holds for the initial vertex $-\varpi_{|I_1|+1}(I_1; q)$ whose poles at $q = -u_k$ are thus of purely higher-order. They are thus given by $-\mathcal{H}_q^k \varpi_{0,|I_1|+1}(I_1; q)$, which can be expressed by (3.18):



In the step from the second to third line, we have used the fact that only the whole projection $\mathcal{H}_q^k + \frac{dq}{q+u_k}\operatorname{Res}_{q\to -u_k}$ to the principal part of a Laurent series, but not $\mathcal{H}_q^k$ alone, is the identity operator under the residue. According to (4.4), the second line from below equals $-\Delta\varpi_{0,|I_0|+|I_1|+1}(I_0 \cup I_1; -u_k)$ times the restriction of $-\mathfrak{v}_{0,|I_3|}(I_3\|-u_k)$ to chains of length 1, here with $I_0 \uplus I_1 \uplus I_3 = I\backslash u_k$. The last line removes from the residue all chains of length 3 with a v2-group labelled u_k.

Thus, only those length-3 chains for which $u_k \in I_1$ is located at the initial vertex v0 contribute to the remaining residue. Again, the case $I_1 = u_k$ produces the restriction of $-\mathfrak{v}_{0,|I|-1}(I\backslash u_k\|-u_k)$ to chains of length 2. For $|I_1| \geq 2$, we use Assumption 3.10 that $-\varpi_{0,|I_1|+1}(I_1; q)$ has at $q = -u_k$ a pole of second or higher-order given by (3.18). The same argument as before produces on the one hand (4.4) times the restriction of $-\mathfrak{v}_{0,|I_3|}(I_3; -u_k)$ to chains of length 2 and on the other hand removes from the residue all chains of length 4 with a v2-group labelled u_k. Continuing this strategy until $I_1 = u_k$ is the only choice shows that the residue of all chains other than (4.4) evaluates to $-\mathfrak{v}_{0,|I|-1}(I\backslash u_k\|-u_k)$ plus (4.4) times $-\mathfrak{v}_{0,|I'|}(I'\|-u_k)$, summed over partitions of $I\backslash u_k$. ∎

Assumption 3.10 for $\varpi_{0,|I|+1}$, that the rhs of (4.3) evaluates to 0, is thus a condition on $\varpi_{0,|I'|+1}$ or $\omega_{0,|I'|+1}$ for $|I'| < I$. Here, Theorem 3.2 is the induction hypothesis. Its proof is complete (following the previous considerations) if Theorem 3.2 implies

$$
\begin{aligned}
0 = \mathfrak{v}_{0,|I|}(I\|-q) - \mathfrak{v}_{0,|I|}(I\|q) - \Delta\varpi_{0,|I|+1}(I; q) \\
+ \sum_{I_1 \uplus I_2 = I} \mathfrak{v}_{0,|I_1|}(I_1\|q)\Delta\varpi_{0,|I_2|+1}(I_2; q).
\end{aligned}
\tag{4.5}
$$

We are going to prove that the rhs of (4.5) is an entire holomorphic function on $\hat{\mathbb{C}}$, i.e., a constant, equal to its value 0 for $q \to \infty$. This implies (4.5).

We start to discuss the absence of poles at $q = \pm\beta_i$. Recall that (4.5) equals $\operatorname{Res}_{z\to q} \varpi_{0,|I|+2}(I, -q; z)$. The projection of (4.5) to poles at $q = \beta_i$ is thus given by

$$
\begin{aligned}
\operatorname*{Res}_{q\to\beta_i} \frac{(4.5)dq}{w-q} &= \operatorname*{Res}_{q\to\beta_i}\operatorname*{Res}_{z\to q} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w-q} \\
&= -\operatorname*{Res}_{q\to\beta_i}\operatorname*{Res}_{z\to\beta_i} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w-q} \\
&\quad + \operatorname*{Res}_{z\to\beta_i}\operatorname*{Res}_{q\to\beta_i} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w-q}
\end{aligned}
$$

when taking (2.1) into account. The final term gives zero because none of the chains contributing to $\varpi_{0,|I|+2}(I, -q; z)$ has a pole at $-q = -\beta_i$. The other term is also zero because $\varpi_{0,|I|+2}(I, -q; z)$ has due to the kernel $K_i(z, q)$ in Theorem 3.2 at

$z = \beta_i$ poles of purely higher-order without residue. We have established this fact in Proposition 3.9 without relying on Assumption 3.10. In summary, (4.5) is regular at $q = \beta_i$. We will show in Subsection 4.2 that (4.5) is antisymmetric under $q \mapsto -q$. This means that (4.5) is also regular at $q = -\beta_i$.

The same simple argument cannot be used to prove that (4.5) is regular at $q = \pm u_k$ because this would need Assumption 3.10. We therefore give in Subsection 4.3 a direct proof which uses the antisymmetry of (4.5).

In principle, the functions $\mathfrak{v}_{0,|I|}(I \| \pm q)$ may (and do) have poles at the other preimages $q = v$, where

$$v \in \{\pm \widehat{u_k}^j, \pm \widehat{(-u_k)}^j\}.$$

Recalling that (4.5) equals $\operatorname{Res}_{z \to q} \varpi_{0,|I|+2}(I, -q; z)$, the projection of (4.5) to a pole at such $q = v$ is

$$\begin{aligned}
\operatorname*{Res}_{q \to v} \frac{(4.5)dq}{w - q} &= \operatorname*{Res}_{q \to v} \operatorname*{Res}_{z \to q} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w - q} \\
&= -\operatorname*{Res}_{q \to v} \operatorname*{Res}_{z \to v} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w - q} + \operatorname*{Res}_{z \to v} \operatorname*{Res}_{q \to v} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w - q}.
\end{aligned}$$

The first term in the last line is trivially zero, but the second term can indeed have a pole at

$$-q = \widehat{u_k}^j$$

coming from the edge e4 in Table 1. An edge with these labels can only occur once in a chain so that it is a first-order pole. Its residue $\operatorname{Res}_{q \to v} \frac{\varpi_{0,|I|+2}(I, -q; z)dq}{w - q}$ is a 1-form in z from which we take the residue at the same $z = -\widehat{u_k}^j$. But there are no such poles so that (4.5) is regular at any $q \in \{\pm \widehat{u_k}^j, \pm \widehat{(-u_k)}^j\}$.

4.2. A necessary condition

Adding (4.5) and its copy for $q \to -q$ shows that necessary for (4.5) to be true is the identity (we apply $d_{u_1} \cdots d_{u_m}$ to pass to ω)

$$\begin{aligned}
0 = {} & \Delta\omega_{0,|I|+1}(I, q) + \Delta\omega_{0,|I|+1}(I, -q) \\
& - \sum_{I_1 \uplus I_2 = I} \Delta\omega_{0,|I_1|+1}(I_1, q) \Delta\omega_{0,|I_2|+1}(I_2, -q).
\end{aligned} \tag{4.6}$$

This is true for $I = \{u\}$. Equations of such type can be disentangled by repeated insertion into itself, which is the same operation as a treatment of formal power series (which here are in fact polynomials). This shows that (4.6) is equivalent to

$$-\Big[\log\big(1 - \Delta\omega_{0,|.|+1}(\cdot, q)\big)\Big](I) - \Big[\log\big(1 - \Delta\omega_{0,|.|+1}(\cdot, -q)\big)\Big](I) = 0, \tag{4.7}$$

where

$$-\big[\log\big(1-\Delta\omega_{0,|.|+1}(\cdot,q)\big)\big](I)\equiv\sum_{s=1}^{I}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\prod_{j=1}^{s}\Delta\omega_{0,|I_j|+1}(I_j,q).$$

This logarithm can also be represented as follows.

Proposition 4.2. *We have*

$$\begin{aligned}&\sum_{r=1}^{|I|}\frac{1}{r}\sum_{I_1\uplus\cdots\uplus I_r=I}\prod_{j=1}^{r}\Delta\omega_{0,|I_j|+1}(I_j,q)\\&=\sum_{s=1}^{|I|}\frac{1}{s}\sum_{I_1\uplus\cdots\uplus I_s=I}\sum_{n_1+\cdots+n_s=s}\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q).\end{aligned}\tag{4.8}$$

Proof. We write Lemma 2.2 as

$$\frac{\omega_{0,|I|+1}(I,\iota q)}{-dy(q)}=\sum_{s=1}^{|I|}\sum_{\substack{I_1\uplus\cdots\uplus I_s=I\\I_1<\cdots<I_s}}(s-1)!\sum_{n_1+\cdots+n_s=s-1}\prod_{i=1}^{s}\nabla^{n_i}\omega_{0,|I_i|+1}(I_i,q)\tag{4.9}$$

and insert it into the product in (4.2). This shows that products of $\Delta\omega_{0,|I_j|+1}(I_j,q)$ expand into products of $\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q)$ with the given condition on the sum of n_j. The fact that the prefactor reduces to $\frac{1}{s}$ is, however, by no means obvious. The first step of the proof is Lemma 4.3 below, which relies on Corollary A.5 in the Appendix. Then, a discussion given after the proof of Lemma 4.3 completes the proof. It relies on the same Corollary A.5. ∎

Lemma 4.3. *We have*

$$\begin{aligned}\Delta\omega_{0,|I|+1}(I,q)&=\nabla^1\omega_{0,|I|+1}(I,q)\\&\quad+\sum_{s=2}^{|I|}\sum_{I_1\uplus\cdots\uplus I_s=I}\sum_{n_1+\cdots+n_s=s}\frac{\#(n_j=0)}{s(s-1)}\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_j,q).\end{aligned}$$

Proof. As discussed before, we have a representation

$$\Delta\omega_{0,|I|+1}(I,q)=\sum_{s=1}^{|I|}\sum_{\substack{I_1\uplus\cdots\uplus I_s=I\\I_1<\cdots<I_s}}\sum_{\substack{n_1+\cdots+n_s=s\\\max(n_j)\geq 2\text{ if }s>1}}C^1_{n_1\cdots n_s}\prod_{j=1}^{s}\nabla^{n_j}\omega_{0,|I_j|+1}(I_i,q)\tag{4.10}$$

in which $C^1_{n_1\cdots n_s}$ is symmetric in all its arguments. To determine $C^1_{n_1\cdots n_s}$, we can consider a subsector of the n_j-summations where $n_1,\ldots,n_p>0$ for some p and

$n_{p+1} = \cdots = n_s = 0$. Other sectors are then obtained by symmetry. We will count the contributions from (4.2) which contribute to $C^1_{n_1 \cdots n_p 0 \cdots 0}$ for given positive integers $n_1, \ldots, n_p$ (which are followed by $n_1 + \cdots + n_p - p$ zeros). In a first step, we show that the number of these contributions is $C^1_{s0\cdots 0} = (s-1)!$ (which is clear) and for $p \geq 2$ given by

$$C^1_{n_1 \cdots n_p 0 \cdots 0} = \sum_{\ell=1}^{p-1} \sum_{j=1}^{p} (n_j - 1)! \sum_{\substack{J_1 \uplus \cdots \uplus J_\ell = \{1,\ldots,p\} \setminus \{j\} \\ J_1 < \cdots < J_\ell}} \frac{(n_1 + \cdots + n_p - p)!}{(n_j - \ell - 1)!} \times \prod_{i=1}^{\ell} \frac{(|\underline{n}|_{J_i})!}{(|\underline{n}|_{J_i} - |J_i| + 1)!}. \tag{4.11}$$

The number $C^1_{n_1 \cdots n_p 0 \cdots 0}$ is the sum over all j with $n_j \geq 2$ of specially ordered contributions from

$$\sum_{\substack{\tilde{I}_1 \uplus \cdots \uplus \tilde{I}_{n_j} = I \\ I_j = \tilde{I}_j,\ \tilde{I}_1 < \cdots < \tilde{I}_{n_j}}} \nabla^{n_j} \omega_{0,|I_j|+1}(I_j, q)(n_j - 1)! \prod_{\substack{i=1 \\ i \neq j}}^{n_j} \frac{\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i, \iota q)}{-dy(q)}. \tag{4.12}$$

The factors $\frac{\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i, \iota q)}{-dy(q)}$ are expressed via (4.9), but only contributions compatible with $n_{p+1} = \cdots = n_s = 0$ are retained. The positive $n_1, \ldots, n_p$, excluding n_j, arise from the part of (4.9) in which all factors $\nabla^0 \omega$ have a larger order than any $\nabla^r \omega$ with $r > 0$. In particular, contributions $\nabla^r \omega$ with $r > 0$ only arise from every of the first ℓ factors $\frac{\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i, \iota q)}{-dy(q)}$ in (4.12) for some ℓ with $1 \leq \ell < p - 1$ to sum over. From the last $n_j - 1 - \ell$ factors in (4.12), we only take the special term $\nabla^0 \omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i, q)$.

An expansion (4.9) used for the first ℓ factors (4.12) contributes to the specially ordered $C^1_{n_1 \cdots n_p 0 \cdots 0}$ whenever $\{1, \ldots, p\} \setminus \{j\}$ is partitioned into $J_1 \uplus \cdots \uplus J_\ell$ with $\min J_k < \min J_l$ for every pair $k < l$, where $\min J_k$ is the smallest integer in the set J_k. Then, the subset $\tilde{I}_i$ in (4.9)

- contains $\bigcup_{k \in J_i} I_k$ if $i < j$;
- contains $\bigcup_{k \in J_{i-1}} I_k$ if $j < i \leq p$.

We let $|\underline{n}|_{J_i} = \sum_{k \in J_i} n_k$. To be an admissible contribution to (4.9), the factors $\nabla^{n_k} \omega$ in the i-th block must be supplemented by $|\underline{n}|_{J_i} - |J_i| + 1$ factors $\nabla^0 \omega$.

Hence, the number $C^1_{n_1 \cdots n_p 0 \cdots 0}$ is given by the sum over j and ℓ of

- a sum over ordered partitions $\{1, \ldots, p\} \setminus \{j\} = J_1 \uplus \cdots \uplus J_\ell$,
- of a factor $(n_j - 1)!$ from (4.12),
- times a factor $(|\underline{n}|_{J_i})!$ for every $1 \leq i \leq \ell$ which is the factor $(s-1)!$ in (4.9),

- times the number of distributions of the $n_1+\dots+n_p-p$ factors $\nabla^0\omega$ into $\ell+1$ blocks, namely,
  - (a) a block of $(n_j-1-\ell)$ factors where from $\frac{\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i,\iota q)}{-dy(q)}$ only the special term $\nabla^0\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i,q)$ is retained;
  - (b) ℓ blocks of $|\underline{n}|_{J_i}-|J_i|+1$ factors which supplement the $\prod_{k\in J_i}\nabla^{n_k}\omega$ in a non-trivially expanded $\frac{\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i,\iota q)}{-dy(q)}$.

  There are $\frac{(n_1+\dots+n_p-p)!}{(n_j-1-\ell)!\prod_{i=1}^{\ell}(|\underline{n}|_{J_i}-|J_i|+1)!}$ such distributions, which is a valid multinomial coefficient due to

$$\sum_{i=1}^{\ell}|\underline{n}|_{J_i}=n_1+\dots+n_p-n_j \quad\text{and}\quad \sum_{i=1}^{\ell}|J_i|=p-1.$$

This number is (4.11). We remark that the restriction to $n_j\geq 2$ is automatic because $\frac{1}{(n_j-\ell-1)!}$ gives zero for $n_j=1$.

We write (4.11) in terms of falling factorials (see Corollary A.6), insert (A.7), and shift $\ell-1\mapsto r$:

$$\begin{aligned}
&C^1_{n_1\cdots n_p0\cdots0}\\
&=(n_1+\dots+n_p-p)!\sum_{\ell=1}^{p-1}\sum_{j=1}^{p}(n_j-1)(n_j-2)^{\underline{\ell-1}}\\
&\quad\times\sum_{\substack{J_1\uplus\dots\uplus J_\ell=\{1,\dots,p\}\setminus\{j\}\\ J_1<\dots<J_\ell}}\prod_{i=1}^{\ell}(|\underline{n}|_{J_i})^{\underline{|J_i|-1}}\\
&=(n_1+\dots+n_p-p)!\\
&\quad\times\sum_{r=0}^{p-2}\sum_{j=1}^{p}(n_j-1)(n_j-2)^{\underline{r}}\binom{p-2}{r}(n_1+\dots+n_p-n_j)^{\underline{p-2-r}}\\
&=(n_1+\dots+n_p-p)!\sum_{j=1}^{p}(n_j-1)(n_1+\dots+n_p-2)^{\underline{p-2}}\\
&=(n_1+\dots+n_p-2)!(n_1+\dots+n_p-p)\equiv(s-2)!(s-p).
\end{aligned}$$

In the fourth line, we have used the binomial theorem for the falling factorial. The final line is obvious.

For a general order of the n_i, we thus have $C^1_{n_1\cdots n_s}=(s-2)!\#(n_j=0)$, where $\#(n_j=0)$ is the number of n_j which equals zero. Relaxing the condition that the I_j are ordered amounts to an additional factor $\frac{1}{s!}$. This is the assertion. ∎

Lemma 4.3 is the starting point to evaluate the sum over r in the first line of (4.8). It is clear that this sum has a similar expansion as (4.10):

$$-\big[\log(1-\Delta\omega_{0,|.|+1}(\cdot,q))\big](I) = \sum_{s=1}^{|I|} \sum_{\substack{I_1\uplus\cdots\uplus I_s=I\\ I_1<\cdots<I_s}} \sum_{n_1+\cdots+n_s=s} C_{n_1\cdots n_s} \prod_{j=1}^{s} \nabla^{n_j}\omega_{0,|I_j|+1}(I_i,q),$$

where $C_{n_1\cdots n_s}$ is symmetric. We first show that for an order $n_1,\dots,n_p\geq 1$ and $n_{p+1}=\cdots=n_s=0$ it is given by

$$C_{n_1\cdots n_p0\cdots0} = \sum_{r=1}^{p}(r-1)! \sum_{\substack{J_1\uplus\cdots\uplus J_r=\{1,\dots,p\}\\ J_1<\cdots<J_r}} (n_1+\cdots+n_p-p)! \prod_{i=1}^{r} \frac{(|\underline{n}|_{J_i}-2)!(|\underline{n}|_{J_i}-|J_i|)}{(|\underline{n}|_{J_i}-|J_i|)!}. \tag{4.13}$$

The factor $(r-1)!$ combines the step from any partitions $I_1\uplus\cdots\uplus I_r=I$ into $r!$ ordered ones with the prefactor $\frac{1}{r}$ in (4.8). The subset J_i corresponds to $\Delta\omega_{0,|\tilde{I}_i|+1}(\tilde{I}_i,q)$ with

$$\tilde{I}_i = \bigcup_{j\in J_i} I_j \cup \bigcup_{k=1}^{|\underline{n}|_{J_i}-|J_i|} I'_k,$$

where the I'_k are taken from $I_{p+1},\dots,I_{n_1+\cdots+n_p}$. There are $\frac{(n_1+\cdots+n_p-p)!}{\prod_{i=1}^r(|\underline{n}|_{J_i}-|J_i|)!}$ different distributions of these I'_k, which explains the corresponding factor above. The numerator $(|\underline{n}|_{J_i}-2)!(|\underline{n}|_{J_i}-|J_i|)$ is the weight of $C^1_{\underline{n}_{J_i}0\cdots0}$ found in Lemma 4.3.

We write (4.13) in terms of rising factorials and insert (A.6), where $x_j\mapsto n_j-1$ and $\ell\mapsto r$:

$$\begin{aligned}
C_{n_1\cdots n_p0\cdots0} &= \sum_{r=1}^{p}(r-1)!(n_1+\cdots+n_p-p)! \prod_{i=1}^{r} \sum_{\substack{J_1\uplus\cdots\uplus J_r=\{1,\dots,p\}\\ J_1<\cdots<J_r}} (|\underline{n}|_{J_i}-|J_i|)^{\overline{|J_i|-1}} \\
&= \sum_{r=1}^{p}(r-1)!(n_1+\cdots+n_p-p)!\binom{p-1}{r-1}(n_1+\cdots+n_p-p)^{\overline{p-r}} \\
&= (n_1+\cdots+n_p-p)!(n_1+\cdots+n_p-p+1)^{\overline{p-2}} \\
&= (n_1+\cdots+n_p-1)! \equiv (s-1)!.
\end{aligned}$$

We have used $(r-1)!=1^{\overline{r-1}}$ and then applied the binomial theorem.

By symmetry, we thus have $C_{n_1\cdots n_s} = (s-1)!$ for any partition $n_1+\cdots+n_s = s$. Relaxing the condition that the I_j are ordered has to be corrected with an additional factor $\frac{1}{s!}$. This completes the proof of Proposition 4.2.

Proposition 4.2 together with (4.7) gives as necessary condition for (4.5) to be true the equality (2.37) which we have proved in Section 2.9.

4.3. Absence of poles of (4.5) at $q = -u_k$

The function $\mathfrak{v}_{0,|I|}(I\|q)$ is holomorphic at $u_k = q$ and has poles at $u_k = -q$ which exclusively come from v2-groups with label u_k. There are two possibilities. Either this v2-group is the single vertex of a length-1 chain $-\Big(\overset{0}{\sim\!\!\Diamond}_{I';u_k}\Big)_{I=I'\uplus u_k}$, or it is part of a chain of larger length. In the second case, it can be collectively taken out of all other chains; the remnant is just another copy of $-\mathfrak{v}_{0,|I|}(I'\|q)$ of smaller length with $I' \notin u_k$:

$$
\begin{aligned}
&\mathfrak{v}_{0,|I|}(I\|q) + \mathcal{O}((q+u_k)^0)\\
&= \sum_{s=0}^{|I|-1} \sum_{I_1\uplus\cdots\uplus I_s = I\setminus u_k}^{(I\setminus u_k\neq\emptyset)} \frac{1}{(R(-q)-R(u_k))(R(-u_k)-R(q))^{s+1}} \prod_{j=1}^{s} \frac{\varpi_{0,|I_j|+1}(I_j;u_k)}{dR(u_k)}\\
&\quad - \sum_{s=0}^{|I|-2} \sum_{I_{-1}\uplus I_1\uplus\cdots\uplus I_s = I\setminus u_k} \frac{\mathfrak{v}_{0,|I_{-1}|}(I_{-1}\|q)}{(R(-q)-R(u_k))(R(-u_k)-R(q))^{s+1}}\\
&\quad \times \prod_{j=1}^{s} \frac{\varpi_{0,|I_j|+1}(I_j;u_k)}{dR(u_k)}.
\end{aligned} \tag{4.14}
$$

We recall that $\sum^{(I'\neq\emptyset)}$ indicates that for $|I'| = 0$ the sum is omitted, whereas for $|I'| > 0$ the case $s = 0$ is left out. The following lemma (which we formulate for $q \mapsto -q$) characterises the polar part of the second line at $q = -u_k$.

Lemma 4.4. *We have*

$$
\begin{aligned}
&\sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1\uplus\cdots\uplus I_s = I} \sum_{n_1+\cdots+n_s = s} \prod_{j=1}^{s} \nabla^{n_i} \omega_{0,|I_j|+1}(I_j, q)\\
&= d_{u_k}\Bigg[\sum_{s=0}^{|I|-1} \sum_{I_1\uplus\cdots\uplus I_s = I\setminus u_k}^{(I\setminus u_k\neq\emptyset)} \frac{\prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}}{(x(q)-x(u_k))(y(q)-y(u_k))^{s+1}} \Bigg]\\
&\quad + \mathcal{O}((q-u_k)^0).
\end{aligned} \tag{4.15}
$$

Proof. The lhs of (4.15) can be rewritten with (2.5) as

$$
\begin{aligned}
&(4.15)_{\text{lhs}} \\
&= \sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1 \uplus \cdots \uplus I_s = I} \frac{(-1)^s}{s!} \lim_{z \to q} \frac{\partial^s}{\partial (x(z))^s} \left(\left(\frac{x(z)-x(q)}{y(z)-y(q)} \right)^s \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j, z)}{dx(z)} \right) \\
&= \operatorname*{Res}_{z \to q} \left[\sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1 \uplus \cdots \uplus I_s = I} \left(\frac{dx(z)}{(x(z)-x(q))(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j, z)}{dx(z)} \right) \right].
\end{aligned}
$$

Up to $\mathcal{O}((q-u_k)^0)$-contributions, it coincides with its projection to the polar part in which we change with (2.1) the order of residues:

$$
\begin{aligned}
&(4.15)_{\text{rhs}} + \mathcal{O}((q-u_k)^0) \\
&= \operatorname*{Res}_{w \to u_k} \frac{dw}{q-w} \operatorname*{Res}_{z \to w} \left[\sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1 \uplus \cdots \uplus I_s = I} \left(\frac{dx(z) \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}}{(x(z)-x(w))(y(w)-y(z))^s} \right) \right] \\
&= - \operatorname*{Res}_{w \to u_k} \frac{dw}{q-w} \operatorname*{Res}_{z \to u_k} \left[\sum_{s=0}^{|I|-1} \sum_{I_0 \uplus \cdots \uplus I_s = I \setminus u_k} \left(\frac{\omega_{0,2}(u_k, z) \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)}}{(x(z)-x(w))(y(w)-y(z))^{s+1}} \right) \right] \\
&= d_{u_k} \left[\operatorname*{Res}_{w \to u_k} \frac{dw}{q-w} \sum_{s=1}^{|I|-1} \sum_{I_0 \uplus \cdots \uplus I_s = I \setminus u_k} \left(\frac{\prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}}{(x(u_k)-x(w))(y(w)-y(u_k))^{s+1}} \right) \right] \\
&= d_{u_k} \left[\sum_{s=1}^{|I|-1} \sum_{I_0 \uplus \cdots \uplus I_s = I \setminus u_k} \left(\frac{\prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,u_k)}{dx(u_k)}}{(x(u_k)-x(q))(y(q)-y(u_k))^{s+1}} \right) + \mathcal{O}((q-u_k)^0) \right].
\end{aligned}
$$

We have used the fact that only $\omega_{0,2}$ has a pole at $z = u_k$ which is given by (1.3). ∎

With these preparations, we control the polar part of (4.5) at $q = \pm u_k$.

Proposition 4.5. *Holomorphicity of* (4.5) *at* $q = -u_k$ *is a consequence of* (2.37) *proved in Proposition* 2.16.

Proof. The first term $\mathfrak{v}_{0,|I|}(I\|-q)$ in (4.5) is holomorphic at $q = -u_k$. In the sum over

$$I_1 \uplus I_2 = I,$$

we distinguish $u_k \in I_1$ from $u_k \in I_2$. The function $\mathfrak{v}_{0,|I|}(I\|q)$ is for $u_k \in I$ written as (4.14) with Lemma 4.4 used for the rhs. We thus find

$$(4.5) = -A(I;q) + \sum_{\substack{I_1 \uplus I_2 = I \\ u_k \in I_1}} A(I_1;q)\mathfrak{v}_{0,|I_2|}(I_2\|q) + \mathcal{O}((q+u_k)^0),$$

where

$$\begin{aligned} A(I;q) := {} & \Delta\varpi_{0,|I|+1}(I;q) \\ & + \sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1\uplus\cdots\uplus I_s=I} \sum_{n_1+\cdots+n_s=s} \prod_{j=1}^{s} \nabla^{n_i} \varpi_{0,|I_j|+1}(I_j;-q) \\ & - \sum_{\substack{I'\uplus I''=I \\ u_k\in I''}} \Delta\varpi_{0,|I'|+1}(I';q) \\ & \times \sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1\uplus\cdots\uplus I_s=I''} \sum_{n_1+\cdots+n_s=s} \prod_{j=1}^{s} \nabla^{n_i} \varpi_{0,|I_j|+1}(I_j;-q). \end{aligned}$$

Consider the equation

$$0 = \Delta\varpi_{0,|I|+1}(I;q) + B(I;q) - \sum_{\substack{I'\uplus I''=I \\ u_k\in I''}} \Delta\varpi_{0,|I'|+1}(I';q) B(I'';q).$$

Its iterative solution is

$$\begin{aligned} B(I;q) &= -\Delta\varpi_{0,|I|+1}(I;q) \\ & \quad - \sum_{s=2}^{|I|} \sum_{\substack{I_1\uplus\cdots\uplus I_s=I \\ u_k\in I_1}} \Delta\varpi_{0,|I_1|+1}(I_1;q) \prod_{j=2}^{s} \Delta\varpi_{0,|I_j|+1}(I_j;q) \\ &= -\sum_{s=1}^{|I|} \frac{1}{s} \sum_{I_1\uplus\cdots\uplus I_s=I} \prod_{j=1}^{s} \Delta\varpi_{0,|I_j|+1}(I_j;q). \end{aligned}$$

The factor $\frac{1}{s}$ arises by symmetrisation when dropping the condition $u_k \in I_1$. The consistency condition (2.37) of (4.5) together with Proposition 4.2 thus implies that $A(I;q) \equiv 0$, which gives the assertion. ■

As a result, we have proved that (4.5) does not have any poles on $\hat{\mathbb{C}}$; it is thus a constant equal to its value 0 at $q = \infty$. This means that Assumption 3.10 is true and the proof of Theorem 3.2 is complete.

5. Conclusion and outlook

We have proved for genus $g = 0$ the main conjecture of [12] that meromorphic forms $\omega_{g,n}$ which naturally appear in the quartic analogue of the Kontsevich model follow blobbed topological recursion [7]. This makes the quartic Kontsevich model part of the growing family of structures in mathematics and physics governed by topological recursion [18, 19]. Other examples include the combinatorics of the Kontsevich model [29], the one- and two-matrix models [15], Hurwitz theory [10], Gromov–Witten theory [9], Weil–Petersson volumes of moduli spaces of hyperbolic Riemann surfaces [30], and many more.

We consider as most important result of this paper the discovery that the quartic Kontsevich model is completely characterised by the behaviour of its objects $\omega_{g,n}$ under the global involution $\iota z = -z$. We showed how a single equation (1.4),

$$\begin{aligned}\omega_{0,|I|+1}(I,q) &+ \omega_{0,|I|+1}(I,\iota q)\\ &= \sum_{s=2}^{|I|} \sum_{I_1 \uplus \cdots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{z \to q} \left(\frac{dy(q)dx(z)}{(y(q)-y(z))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,z)}{dx(z)} \right),\end{aligned} \tag{1.4}$$

governs the genus-0 case. This equation admits a naïve solution

$$\begin{aligned}&\omega_{0,|I|+1}(I,z)\\ &= \sum_{i=1}^{r} \sum_{s=2}^{|I|} \sum_{I_1 \uplus \cdots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{q \to \beta_i} \frac{dz}{z-q} \operatorname*{Res}_{w \to q} \left(\frac{dy(q)dx(w)}{(y(q)-y(w))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,w)}{dx(w)} \right)\\ &+ \sum_{k=1}^{|I|} \sum_{s=2}^{|I|} \sum_{I_1 \uplus \cdots \uplus I_s = I} \frac{1}{s} \operatorname*{Res}_{q \to \iota u_k} \frac{dz}{z-q} \operatorname*{Res}_{w \to q} \left(\frac{dy(q)dx(w)}{(y(q)-y(w))^s} \prod_{j=1}^{s} \frac{\omega_{0,|I_j|+1}(I_j,w)}{dx(w)} \right),\end{aligned}$$

which however leaves each of the following points obscure:

(a) Is (1.4) meaningful, i.e., is its rhs symmetric under $q \mapsto \iota q$?

(b) Has (1.4) anything to do with topological recursion?

(c) Is there any connection between (1.4) and the quartic Kontsevich model?

To answer the first two of these critical questions, we had to prove that the naïve solution is equivalent to the solution (1.5)+(1.6) given in Theorem 1.2. The generated material also allowed to affirm question (c), where a difficulty was to show that all poles are of purely higher-order. This property is a consequence of a hidden symmetry (2.36) resulting from (1.4) alone. As a result, we have established for genus $g = 0$ a precise connection between (b) and (c), which was conjectured in [12].

But the statement is more general: in [12] it is also shown that the poles of $\omega_{g\geq 1,n}(z_1, \ldots, z_n)$ are located, besides ramification points of x and diagonals $z_k = \iota z_l$, at the fixed points $z_k = \iota z_k$ of the involution. The next step in our programme will be to extend the involution identity (1.4) to higher genus. For that, one should take inspiration from functional relations in the context of the x-y duality in topological recursion [2, 5, 27]. It has to be seen to what extent these relations generalise to specific extensions of topological recursion in the spirit of blobbed topological recursion.

For instance, in [3] based on an observation in [26], a specific extension of topological recursion called *logarithmic topological recursion* for meromorphic dx, dy (including specifically logarithmic singularities of x, y) was defined and proved to be consistent with the x-y duality.

For the quartic Kontsevich model, we developed in [28] a different approach to prove blobbed topological recursion via extended loop equations. Explicit recursion formulae for $\omega_{1,n}$ were established. They take into account contributions from the so-called $(1+1)$-point function [33] and its generalisations. It seems that their governing equations [12] are compatible only with $\iota z = -z$, but one should investigate whether every

$$\iota z = \frac{az+b}{cz-a}$$

comes with its specific form of these equations. This should help to answer the exciting question whether the intersection numbers [7] encoded in the quartic Kontsevich model capture geometric information about a moduli space of curves equipped with an involution. It would also be interesting to investigate whether these structures relate to other extensions of topological recursion. We mention the work [4] (which contains a beautiful introduction to topological recursion and its ramifications) on r-spin intersection numbers to which the quartic Kontsevich model could be related.

A. Combinatorial identities involving labelled trees (by Maciej Dołęga)

A *set partition* of S is a (non-ordered) family of non-empty disjoint subsets of S (called parts of the partition), whose union is S. In the following, we always assume that S is finite. Denote by $\mathcal{P}(S)$ the set of set partitions of S and for any $\pi \in \mathcal{P}(S)$ and for any $B \in \pi$ denote by $|\pi|$ the number of parts of π and by $|B|$ the number of elements in the part B.

A graph $G = (V, E)$ is a *forest* if it has no cycles. If a forest is additionally connected, it is called a *tree*. Denote by $\mathcal{T}_V^*$ the set of *plane trees*, that is, trees embedded in a plane, with the set of vertices V. Denote by $\mathcal{T}_V$ the set of *labelled trees* with the set of vertices V, that is, the set of trees, whose vertices are labelled by distinct numbers $\{1, \dots, |V|\}$ (or, by isomorphism, any linearly ordered set of the cardinality $|V|$). Finally, a tree is *rooted* if it has a distinguished vertex $v_\bullet \in V$ called *the root*, and we denote by $\mathcal{T}_V^{*,\bullet}$ and $\mathcal{T}_V^\bullet$ the set of plane rooted trees and of labelled rooted trees, respectively, with the vertex set V. The degree of a vertex v in a tree T is the number of adjacent vertices to v.

The following classical theorem is a multivariate version of the celebrated Cayley's formula for the number of labelled trees.

Theorem A.1. *For any positive integer n and family of indeterminates $x_1, \dots, x_n$, x_{n+1}, the following formulas hold true:*

$$(x_1 + \cdots + x_{n+1})^{n-1} = \sum_{T \in \mathcal{T}_{[n+1]}} \prod_{v \in V} x_v^{\deg(v)-1} \tag{A.1}$$

and

$$(x_1 + \cdots + x_{n+1})^{n} = \sum_{T \in \mathcal{T}^{\bullet}_{[n+1]}} x_{v_\bullet}^{\deg(v_\bullet)} \prod_{v \in V \setminus \{v_\bullet\}} x_v^{\deg(v)-1}, \tag{A.2}$$

Cayley proved his formula by computing a certain determinant [14], and the first bijective proof was given by Prüfer [32]. Since then, many different proofs have been proposed and we would like to mention a relatively general method for counting trees by the use of the matrix-tree theorem; see, for instance, [1] for generalisations and applications.

Corollary A.2. *For any positive integer n and family of indeterminates $x_1, \dots, x_n$, x_{n+1}, the following formula holds true:*

$$(x_1 + \cdots + x_{n+1})^{n} = \sum_{T \in \mathcal{T}^{*,\bullet}_{[n+1]}} \sum_{\varsigma \in \mathcal{S}_{n+1}} \frac{x_{\varsigma(v_\bullet)}^{\deg(v_\bullet)}}{\deg(v_\bullet)!} \prod_{v \in V \setminus \{v_\bullet\}} \frac{x_{\varsigma(v)}^{\deg(v)-1}}{(\deg(v)-1)!}. \tag{A.3}$$

Proof. Note that the symmetric group $\mathcal{S}_{n+1}$ acts on the set $\mathcal{T}^{\bullet}_{[n+1]}$ of labelled rooted trees by permuting the labels. Moreover, each labelled rooted tree is uniquely constructed by choosing a plane tree $T \in \mathcal{T}^{*,\bullet}_{[n+1]}$, a label for its root, and for each $v \in V$ a subset $L_v \subset [n+1]$ of labels of its children. There are

$$\frac{(n+1)!}{\deg(v_\bullet)! \prod_{v \in V \setminus \{v_\bullet\}} (\deg(v)-1)!}$$

choices for such labellings. Therefore, acting by the permutation group on the labels and comparing it with the formula (A.2), we have got

$$(n+1)!(x_1+\cdots+x_{n+1})^{n} = \sum_{T \in \mathcal{T}^{*,\bullet}_{[n+1]}} \sum_{\varsigma \in \mathcal{S}_{n+1}} (n+1)! \frac{x_{\varsigma(v_\bullet)}^{\deg(v_\bullet)}}{\deg(v_\bullet)!} \prod_{v \in V \setminus \{v_\bullet\}} \frac{x_{\varsigma(v)}^{\deg(v)-1}}{(\deg(v)-1)!},$$

which finishes the proof. ∎

Corollary A.3. *For any $(n+1)$-tuple $f_0, \dots, f_n$ of differentiable functions, one has*

$$\begin{aligned} n! \sum_{k_0+\cdots+k_n=n} \prod_{i=0}^{n} \frac{f_i^{(k_i)}(x)}{k_i!} &\equiv (f_0 f_1 \cdots f_n)^{(n)}(x) \\ &= \sum_{T \in \mathcal{T}^{*,\bullet}_{[n+1]}} \sum_{\varsigma \in \mathcal{S}_{n+1}} \frac{f_{\varsigma(v_\bullet)}^{(\deg(v_\bullet))}(x)}{\deg(v_\bullet)} \prod_{v \in V \setminus \{v_\bullet\}} \frac{f_{\varsigma(v)}^{(\deg(v)-1)}(x)}{(\deg(v)-1)!}. \end{aligned}$$

Proof. Set $x_i \mapsto \partial_{x_i}$ in (A.3), apply it to $f_0(x_0)f_1(x_1)\cdots f_n(x_n)$, and substitute $x_0 = x_1 = \cdots = x_n = x$. ■

Here is a corollary from Theorem A.1 which gives an identity expressed in terms of set-partitions.

Corollary A.4. *For any positive integer n and family of indeterminates $x_1, \ldots, x_n$, the following formula holds true:*

$$\Big(1 + \sum_{i=1}^{n} x_i\Big)^{n-1} = \sum_{\pi \in \mathcal{P}([n])} \prod_{B \in \pi} \Big(\sum_{b \in B} x_b\Big)^{|B|-1}. \tag{A.4}$$

Proof. For any $T \in \mathcal{T}_{[n+1]}$, removing the vertex $n+1$ from it yields the decomposition into a collection of disjoint rooted labelled trees on the set $[n]$. This decomposition establishes a bijection between rooted, labelled forests F on the vertex set $[n]$ and labelled trees T on $n+1$ vertices. Moreover, the degrees of the non-root vertices of F coincide with their degrees in T, and the degrees of the root vertices of F are equal to their degrees in T minus one. This decomposition gives the following identity by plugging $x_{n+1} = 1$ in (A.1):

$$(x_1 + \cdots + x_n + 1)^{n-1} = \sum_{F} \prod_{v \in V_\bullet} x_v^{\deg(v)} \prod_{v \in V \setminus V_\bullet} x_v^{\deg(v)-1}.$$

Note that for any set-partition $\pi \in \mathcal{P}([n])$ and for any collection of rooted, labelled trees $\{T_B \in \mathcal{T}_B^\bullet : B \in \pi\}$ there exists a rooted, labelled forest F on $[n]$, which is the disjoint union of $\{T_B \in \mathcal{T}_B^\bullet : B \in \pi\}$ and every rooted, labelled forest F on $[n]$ is obtained in this way. Therefore,

$$\begin{aligned}
&(x_1 + \cdots + x_n + 1)^{n-1} \\
&\quad = \sum_{\pi \in \mathcal{P}([n])} \prod_{B \in \pi} \Big(\sum_{T_B \in \mathcal{T}_B^\bullet} x_{v_\bullet(T_B)}^{\deg(v_\bullet(T_B))} \prod_{v \in V(T_B) \setminus \{v_\bullet(T_B)\}} x_v^{\deg(v)-1} \Big) \\
&\quad = \sum_{\pi \in \mathcal{P}([n])} \prod_{B \in \pi} \Big(\sum_{b \in B} x_b\Big)^{|B|-1}
\end{aligned}$$

by (A.2), which finishes the proof. ■

Corollary A.5. *For any positive integers $\ell < n$ and family of indeterminates $x_1, \ldots, x_n$, the following formula holds true:*

$$\binom{n-1}{\ell-1}(x_1 + \cdots + x_n)^{n-\ell} = \sum_{\substack{\pi \in \mathcal{P}([n]): \\ |\pi| = \ell}} \prod_{B \in \pi} \Big(\sum_{b \in B} x_b\Big)^{|B|-1}. \tag{A.5}$$

Proof. It is enough to apply the binomial formula for the left-hand side of (A.4) and compare the homogenous parts of degree $n-\ell$. ∎

Let $x^{\bar{n}} := \prod_{i=0}^{n-1}(x+i)$ and $x^{\underline{n}} := \prod_{i=0}^{n-1}(x-i)$ denote the raising and the falling factorials.

Corollary A.6. *The following identities hold true:*

$$\sum_{\substack{\pi\in\mathcal{P}([n])\\|\pi|=\ell}}\prod_{B\in\pi}\Big(\sum_{b\in B}x_b\Big)^{\overline{|B|-1}} = \binom{n-1}{\ell-1}(x_1+\cdots+x_n)^{\overline{n-\ell}}, \tag{A.6}$$

$$\sum_{\substack{\pi\in\mathcal{P}([n])\\|\pi|=\ell}}\prod_{B\in\pi}\Big(\sum_{b\in B}x_b\Big)^{\underline{|B|-1}} = \binom{n-1}{\ell-1}(x_1+\cdots+x_n)^{\underline{n-\ell}}. \tag{A.7}$$

Proof. It is enough to realise that

$$\Big(\sum_{b\in B}x_b\Big)^{\bar{k}} = \Big(-\sum_{b\in B}\partial_{t_b}\Big)^k\prod_{j=1}^n t_j^{-x_j}\Big|_{t_j=1},$$

$$\Big(\sum_{b\in B}x_b\Big)^{\underline{k}} = \Big(\sum_{b\in B}\partial_{t_b}\Big)^k\prod_{j=1}^n t_j^{x_j}\Big|_{t_j=1}.$$

Substitute $x_j \mapsto \mp\partial_{t_j}$ in (A.5), apply it to $\prod_{j\in J}t_j^{\mp x_j}$, and set all $t_j \equiv 1$. ∎

Let $a(x), b(y), c_1(y), \dots, c_n(y) \in C^\infty(\mathbb{R})$ be smooth functions. (In fact, they might be formal elements of a ring equipped with the formal derivations ∂_x, ∂_y; see [17].) In the following, we are going to prove an explicit combinatorial formula for the expression

$$\big(b(y)\partial_x + \partial_y\big)^{n-1}\big(a(x)\cdot b(y)\cdot c_1(y)\cdots c_n(y)\big)$$

in terms of special labelled trees, where we allow repetitions.

Consider the set of rooted, labelled trees T such that

- the root vertex $v_\bullet(T)$ has label -1,
- the set of vertices adjacent to the root is denoted by $V_0(T)$ and for any $v\in V_0(T)$ one has $\deg(v) > 1$ and v is labelled by 0,
- the set $V_{[n]}(T)$ of the remaining vertices has cardinality n and its elements are labelled by distinct numbers $1,\dots,n$.

We denote the set of these trees by $\mathcal{T}_{[n]}^{\bullet;0}$.

Theorem A.7. *For any $(n+2)$-tuple of functions $a(x), b(y), c_1(y), \ldots, c_n(y)$, the following identity holds true:*

$$\begin{aligned}&\big(b(y)\partial_x + \partial_y\big)^{n-1}\big(a(x)\cdot b(y)\cdot c_1(y)\cdots c_n(y)\big)\\ &= \sum_{T\in\mathcal{T}^{\bullet;0}_{[n]}} a(x)^{(\deg(v_\bullet(T))-1)} \prod_{v\in V_0(T)} b(y)^{(\deg(v)-2)} \prod_{v\in V_{[n]}(T)} c_{\mathrm{label}(v)}(y)^{(\deg(v)-1)},\end{aligned} \tag{A.8}$$

where $f^{(n)}(z) = \partial_z^n f(z)$ with the convention $f^{(0)}(z) = f(z)$.

Proof. We can decompose a tree $T \in \mathcal{T}^{\bullet;0}_{[n]}$ as follows. Suppose that the degree of the root of T is equal to r. Let $T' \in \mathcal{T}^{\bullet}_{[n+1]}$ be a tree obtained from T by identifying all the vertices labelled by 0 with the root vertex $v_\bullet$. Note that the degree of the root of T' is equal to $r \le \deg(v_\bullet(T')) \le n$. In particular, T is uniquely determined by T' and by a set-partition $\pi \in \mathcal{P}(N(v_\bullet(T')))$, where $N(v_\bullet(T'))$ is the set of vertices in T' adjacent to the root $v_\bullet(T')$. Each block of $B \in \pi$ corresponds to a vertex of T labelled by 0. This decomposition gives us the following equality:

$$\begin{aligned}&\sum_{T\in\mathcal{T}^{\bullet;0}_{[n]}} a(x)^{(\deg(v_\bullet(T))-1)} \prod_{v\in V_0(T)} b(y)^{(\deg(v)-2)} \prod_{v\in V_{[n]}(T)} c_{\mathrm{label}(v)}(y)^{(\deg(v)-1)}\\ &= \sum_{r=1}^{n} \sum_{\substack{T'\in\mathcal{T}^{\bullet}_{[n+1]},\\ \deg(v_\bullet)=r}} \prod_{v\in V\setminus\{v_\bullet\}} c_{\mathrm{label}(v)}(y)^{(\deg(v)-1)} \sum_{\pi\in\mathcal{P}([r])} a(x)^{(|\pi|-1)} \prod_{B\in\pi} b(y)^{(|B|-1)}.\end{aligned}$$

Define a transformation $f : \mathbb{C}[y, x_1, \ldots, x_n] \to \mathbb{C}(y)$ by declaring its action on monomials

$$f(y^k \cdot x_1^{\alpha_1}\cdots x_n^{\alpha_n}) := b^{(k)}(y)\prod_{i=1}^{n} c_i^{(\alpha_i)}(y). \tag{A.9}$$

Using Leibniz rule, we can compute

$$\partial_y^{|B|-k}\Big(\prod_{b\in B} c_b(y)\Big) = f\Big(\Big(\sum_{b\in B} x_b\Big)^{|B|-k}\Big),$$

and using the proof of Corollary A.4, we rewrite the rhs of (A.8) as

$$\sum_{r=1}^{n}\binom{n-1}{r-1}\Big(\prod_{i=1}^{n} c_i(y)\Big)^{(n-r)} \sum_{\pi\in\mathcal{P}([r])} a(x)^{(r-1)} \prod_{B\in\pi} b(y)^{(|B|-1)}.$$

It is enough to notice that

$$\sum_{\pi\in\mathcal{P}([r])} a(x)^{(|\pi|-1)} \prod_{B\in\pi} b(y)^{(|B|-1)} = (b(y)\partial_x + \partial_y)^{r-1} a(x) b(y),$$

which is easy to prove by induction on r (every set-partition $\pi \in \mathcal{P}([r+1])$ is either constructed from a set partition $\pi' \in \mathcal{P}([r+1])$ by adding a new block $\{r+1\}$, which corresponds to the action of $\partial_x b(y)$ on $(b(y)\partial_x + \partial_y)^{r-1}a(x)b(y)$ or it is constructed from $\pi' \in \mathcal{P}([r+1])$ by adding $r+1$ to one of its blocks, which corresponds to the action of ∂_y on $(b(y)\partial_x + \partial_y)^{r-1}a(x)b(y))$. Summing up, we have that

$$\begin{aligned}
&\sum_{r=1}^{n}\binom{n-1}{r-1}\left(\prod_{i=1}^{n} c_i(y)\right)^{(n-r)} \sum_{\pi\in\mathcal{P}([r])} a(x)^{(|\pi|-1)} \prod_{B\in\pi} b(y)^{(|B|-1)} \\
&= \sum_{r=1}^{n}\binom{n-1}{r-1}\left(\prod_{i=1}^{n} c_i(y)\right)^{(n-r)} (b(y)\partial_x + \partial_y)^{r-1} a(x)b(y) \\
&= (b(y)\partial_x + \partial_y + \partial_z)^{n-1}\left(a(x)b(y)\prod_{i=1}^{n} c_i(z)\right)\Bigg|_{z=y} \\
&= (b(y)\partial_x + \partial_y)^{n-1}\left(a(x)b(y)\prod_{i=1}^{n} c_i(y)\right),
\end{aligned}$$

which finishes the proof of (A.8). ■

Corollary A.8. *For any $(n+2)$-tuple of functions $a(x), b(y), c_1(y), \dots, c_n(y)$, the following identity holds true:*

$$\begin{aligned}
&\big(b(y)\partial_x + \partial_y\big)^{n-1}\big(a(x)\cdot b(y)\cdot c_1(y)\cdots c_n(y)\big) \\
&= \sum_{\pi\in\mathcal{P}([n])} \partial_x^{|\pi|-1} a(x) \prod_{B\in\pi}\Big(\partial_y^{|B|-1}\big(b(y)\prod_{b\in B} c_b(y)\big)\Big).
\end{aligned} \tag{A.10}$$

Proof. Note that any $T \in \mathcal{T}_{[n]}^{\bullet;0}$ is uniquely determined by the following data: pick a set-partition $\pi \in \mathcal{P}([n])$. For each part $B \in \pi$, pick a labelled tree $T_B \in \mathcal{T}_{B\cup\{0\}}$. Take the disjoint union of $(T_B)_{B\in\pi}$ and connect all the vertices labelled by 0 to a new vertex labelled by -1. In this way, we obtain a tree $T \in \mathcal{T}_{[n]}^{\bullet;0}$, and conversely, every $T \in \mathcal{T}_{[n]}^{\bullet;0}$ decomposes into a collection of labelled trees $\big(T_B \in \mathcal{T}_{B\cup\{0\}}\big)_{B\in\pi}$. This decomposition yields the following identity:

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_{[n]}^{\bullet;0}} a(x)^{(\deg(v_\bullet(T))-1)} \prod_{v\in V_0(T)} b(y)^{(\deg(v)-2)} \prod_{v\in V_{[n]}(T)} c_{\mathrm{label}(v)}(y)^{(\deg(v)-1)} \\
&= \sum_{\pi\in\mathcal{P}([n])} a^{(|\pi|-1)}(x) \prod_{B\in\pi} \sum_{T_B\in\mathcal{T}_{B\cup\{0\}}} \Big(b(y)^{(\deg(v_0(T_B))-1)} \prod_{b\in B} c_b^{(\deg(v_b(T_B))-1)}(y)\Big).
\end{aligned} \tag{A.11}$$

Similarly, as before, we can compute

$$\partial_y^{|B|-1}\Big(b(y)\prod_{b\in B} c_b(y)\Big) = f\Big(\Big(y + \sum_{b\in B} x_b\Big)^{|B|-1}\Big),$$

where f is a transformation given by (A.9). Using (A.1) and the definition of f, we can further transform it into

$$\partial_y^{|B|-1}\Big(b(y)\prod_{b\in B}c_b(y)\Big) = \sum_{T_B\in\mathcal{T}_{B\cup\{0\}}}\Big(b(y)^{(\deg(v_0(T_B))-1)}\prod_{b\in B}c_b^{(\deg(v_b(T_B))-1)}(y)\Big).$$

Plugging it into the rhs of (A.11) and using (A.8), we end up precisely with (A.10), which finishes the proof. ∎

B. An identity used in Section 2.6

We recall two well-known identities [21, Volume 4, (10.18) and Volume 5, (1.18)]:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{r+k} = (-1)^n\binom{x}{r+n}, \tag{B.1}$$

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{x-i}{k}\frac{1}{y+i} = \frac{\binom{x+y}{k}}{y\binom{y+k}{k}}, \tag{B.2}$$

which hold for $r\in\mathbb{N}$ and $x,y\in\mathbb{C}$.

Let $\mathcal{D}_n$ be the set of tuples (n_1,n_2,n_3,n_4) of non-negative integers with $n_1+n_2+n_3+2n_4=n$ and $n_3+n_4\neq 0$, that is,

$$\mathcal{D}_n := \{(n_1,n_2,n_3,n_4)|n_i\in\mathbb{N}, n_3+n_4\neq 0, n_1+n_2+n_3+2n_4=n\}. \tag{B.3}$$

Then, the following decomposition holds.

Lemma B.1. *Let $y,\bar{y},w,\bar{w}\in\mathbb{C}$ and $e_1:=w+\bar{w}$ and $e_2:=y\bar{w}+\bar{y}w+w\bar{w}$. Then, we have for any n*

$$\begin{aligned}&\sum_{k=0}^{n-1}\binom{n}{k}(y^k w^{n-k}+\bar{y}^k\bar{w}^{n-k})\\ &= \sum_{(n_1,n_2,n_3,n_4)\in\mathcal{D}_n}(-1)^{n_4}n\frac{\prod_{k=1}^{n_3+n_4-1}(n_1+k)(n_2+k)}{n_3!n_4!(n_3+n_4-1)!}y^{n_1}\bar{y}^{n_2}e_1^{n_3}e_2^{n_4}.\end{aligned} \tag{B.4}$$

Proof. We expand $y^{n_1}\bar{y}^{n_2}e_1^{n_3}e_2^{n_4}$ into a linear combination of $y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}$. For given n_1, n_2, n_3, n_4, k, $\bar{k}$, t, $\bar{t}$, at most one term of the multinomial expansion of e_1,e_2 contributes. The coefficient of $y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}$ in such a contribution is

$$\begin{aligned}&[y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}](y^{n_1}\bar{y}^{n_2}e_1^{n_3}e_2^{n_4})\\ &= \frac{n_4!}{(k-n_1)!(\bar{k}-n_2)!(n_4+n_1+n_2-k-\bar{k})!}\frac{n_3!}{(t+k-n_4-n_1)!(\bar{t}+\bar{k}-n_4-n_2)!},\end{aligned}$$

where $k+\bar{k}+t+\bar{t}=n=n_1+n_2+n_3+2n_4$. It is only non-zero if $n_1\in[k-\bar{t}\dots k]$, $n_2\in[\bar{k}-t\dots\bar{k}]$, and $n_3\in[k+\bar{k}-n_1-n_2\dots\min(k-n_1+t,\bar{k}-n_3+\bar{t})]$.

We thus need to evaluate the sum

$$[y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}](\text{B.4})=\sum_{n_1=\max(0,k-\bar{t})}^{k}\;\sum_{n_2=\max(0,\bar{k}-t)}^{\bar{k}}\;\sum_{n_4=k+\bar{k}-a-b}^{\min(k-n_1-t,\bar{k}-n_2+\bar{t})}T_{n_1,n_2,n_4},\tag{B.5}$$

where

$$\begin{aligned}&T_{n_1,n_2,n_4}\\&=n(-1)^{n_4}\frac{(n_1+n_3+n_4-1)!(n_2+n_3+n_4-1)!}{n_1!n_2!n_3!n_4!(n_3+n_4-1)!}[y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}](y^{n_1}\bar{y}^{n_2}e_1^{n_3}e_2^{n_4})\end{aligned}\tag{B.6}$$

with $n_3=n-n_1-n_2-2n_4$. The aim is to prove that (B.5)+(B.6) breaks down to

$$\binom{n}{k}\delta_{t,n-k}\delta_{\bar{k},0}\delta_{\bar{t},0}+\binom{n}{\bar{k}}\delta_{k,0}\delta_{t,0}\delta_{\bar{t},n-\bar{k}}.$$

Shifting summation indices to $n_1=a+k-\bar{t}$, $n_2=b+\bar{k}-t$, $n_4=c+k+\bar{k}-n_1-n_2=c+t+\bar{t}-a-b$ leads to

$$\begin{aligned}[y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}](\text{B.4})&=\sum_{a=0}^{\bar{t}}\sum_{b=0}^{t}\sum_{c=0}^{\min(a,b)}\frac{(t+k+a-c-1)!(\bar{t}+\bar{k}+b-c-1)!}{(a+k-\bar{t})!(b+\bar{k}-t)!(t+\bar{t}-c-1)!}\\&\qquad\times\frac{n(-1)^{c+a+\bar{t}+b+t}}{(\bar{t}-a)!(t-b)!c!(b-c)!(a-c)!}.\end{aligned}$$

Next, change the order of the sums by

$$\sum_{a=0}^{\bar{t}}\sum_{b=0}^{t}\sum_{c=0}^{\min(a,b)}f_{a,b,c}=\sum_{c=0}^{\min(t,\bar{t})}\sum_{a=c}^{\bar{t}}\sum_{b=c}^{t}f_{a,b,c}=\sum_{c=0}^{\min(t,\bar{t})}\sum_{a=0}^{\bar{t}-c}\sum_{b=0}^{t-c}f_{a+c,b+c,c}$$

to derive

$$\begin{aligned}&[y^k\bar{y}^{\bar{k}}w^t\bar{w}^{\bar{t}}](\text{B.4})\\&=\sum_{c=0}^{\min(t,\bar{t})}\sum_{a=0}^{\bar{t}-c}\sum_{b=0}^{t-c}\frac{n(-1)^{c+a+\bar{t}+b+t}(t+k+a-1)!(\bar{t}+\bar{k}+b-1)!}{(a+c+k-\bar{t})!(b+c+\bar{k}-t)!(t+\bar{t}-c-1)!(\bar{t}-c-a)!(t-c-b)!c!b!a!}\\&=\sum_{c=0}^{\min(t,\bar{t})}\frac{n(-1)^{c+\bar{t}+t}(t+\bar{t}-1-c)!}{(\bar{t}-c)!(t-c)!c!}\\&\quad\times\sum_{a=0}^{\bar{t}-c}(-1)^a\binom{\bar{t}-c}{a}\binom{t+k-1+a}{c+k-\bar{t}+a}\sum_{b=0}^{t-c}(-1)^b\binom{t-c}{b}\binom{\bar{t}+\bar{k}-1+b}{c+\bar{k}-t+b}.\end{aligned}$$

The sums over a, b can be evaluated separately with the identity (B.1). Consequently, one concludes with identity (B.2)

$$\begin{aligned}
[y^k \bar{y}^{\bar{k}} w^t \bar{w}^{\bar{t}}](\text{B.4}) &= n\binom{t+k-1}{k}\binom{\bar{t}+\bar{k}-1}{\bar{k}} \sum_{c=0}^{\min(t,\bar{t})} \frac{(-1)^c (t+\bar{t}-1-c)!}{(\bar{t}-c)!(t-c)!c!} \\
&= n\binom{t+k-1}{k}\binom{\bar{t}+\bar{k}-1}{\bar{k}}\Big(\frac{\delta_{t,0}}{\bar{t}} + \frac{\delta_{\bar{t},0}}{t}\Big) \\
&= n\bigg(\delta_{t,0}\underbrace{\binom{k-1}{k}}_{=\delta_{k,0}}\binom{\bar{t}+\bar{k}-1}{\bar{k}}\frac{1}{\bar{t}} + \delta_{\bar{t},0}\binom{t+k-1}{k}\underbrace{\binom{\bar{k}-1}{\bar{k}}}_{=\delta_{\bar{k},0}}\frac{1}{t}\bigg) \\
&= n\bigg(\delta_{t,0}\delta_{k,0}\binom{n-1}{\bar{k}}\frac{1}{n-\bar{k}} + \delta_{\bar{t},0}\delta_{\bar{k},0}\binom{n-1}{k}\frac{1}{n-k}\bigg) \\
&= \delta_{t,0}\delta_{k,0}\binom{n}{\bar{k}} + \delta_{\bar{t},0}\delta_{\bar{k},0}\binom{n}{k}. \qquad \blacksquare
\end{aligned}$$

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