

Chaotic properties of smooth dynamical systems

Adam Kanigowski

1 Introduction

The fields of dynamical systems and ergodic theory are concerned with evolution of a given system under time. Given an initial condition (or a set of initial conditions) one is interested in its long term behavior. One of the founders of modern dynamical systems is Henri Poincaré. The famous *Poincaré recurrence theorem* [47] states that a measure-preserving dynamical system will return (infinitely often) arbitrarily close to its initial state. Subsequent fundamental achievements came with *ergodic theorems*, proven by John von Neumann in the L^2 setting [41] and by George Birkhoff in a stronger almost everywhere situation [3]. They imply that, under the natural ergodicity assumption on the system, the time average is equal to the space average. In particular, they yield quantitative information on the set of return times in the Poincaré recurrence theorem. These theorems gave first general statements on random behavior in dynamics. This naturally sparked questions on what finer chaotic properties one can (typically) observe in naturally appearing dynamical systems, which became one of the central directions of study in the realm of (smooth) dynamical systems in the second half of the last century. More precisely, people were interested in random behavior of deterministic systems, i.e., in how sensitive can the evolution of a deterministic system be (depending on the initial condition). One of the main motivations was coming from dynamical systems modeling real-life phenomena, such as evolution of weather or motion of planets in the solar system. Most notably, in the 1960s Edward Lorenz [38], a meteorologist and mathematician, was conducting an experiment on predicting the weather in a given place on earth. He came up with a deterministic dynamical system modeling the evolution of weather and noticed that even a tiny perturbation of the initial condition would typically lead to a completely different outcome after a relatively short time. This phenomenon became known as the *butterfly effect* and this led to the discovery of the so-called (chaotic) *Lorenz attractor* (see Figure 1¹).

At around the same time some fundamental mathematical tools were developed which allowed to rigorously *measure* ran-

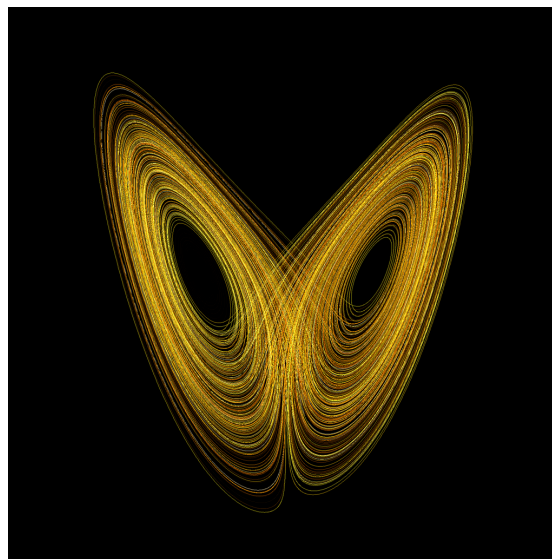


Figure 1. The Lorenz attractor. Nearby initial conditions diverge at exponential speed.

domness of deterministic dynamical systems. The entropy of an abstract measure-preserving system was defined by Kolmogorov and Sinai in the late 1950s [35, 56]. Systems with positive entropy would then exhibit strong randomness, at least in ‘parts’ of the system. Other fundamental objects defined for smooth systems on manifolds are the *Lyapunov exponents*. Positivity of Lyapunov exponents is another manifestation of strongly chaotic properties. There are also other qualitative and quantitative properties that yield a hierarchy of possible chaotic behaviors. By qualitative properties we mean properties that can be defined for an abstract dynamical system. Some main examples are: ergodicity, weak mixing, mixing, absolutely continuous spectrum, positive entropy, K -property and Bernoulli property. Quantitative properties use some additional structure of the system (in most cases smooth or piecewise-smooth structure). Some main examples are: rates of deviation of ergodic averages, rates of weak mixing and mixing (decay of correlations), central limit theorem, large deviations. The last seventy years have seen a spectacular development in studying chaotic properties of

¹ Taken from https://commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png.

(smooth) dynamical systems. In the sections below I plan to give a selective account of some important classical results in the theory, highlight some of the more recent developments, and discuss some open problems and future directions.

2 Randomness in deterministic systems

One of the fundamental discoveries in smooth dynamical systems in the last seventy years is the fact that deterministic (smooth) dynamical systems can behave very randomly. In particular, in many aspects such systems can behave like a system generated by a sequence of independent coin tosses. The first class of smooth systems for which such phenomena were observed is that of the *uniformly hyperbolic* systems, or *Anosov* systems [1], i.e., systems for which the tangent space at every point splits into contracting and expanding directions for the derivative of the map f and moreover these directions are invariant under f . This in particular implies that all *Lyapunov exponents* are non-zero.

2.1 Uniformly hyperbolic systems

The main class of uniformly hyperbolic examples is given by *hyperbolic matrices* with integer entries. Rather than giving a general definition of a uniformly hyperbolic system, we will focus on these examples which should provide some intuition for the general situation.

Example 2.1. Let $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Note that $C: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and since C has integer entries, it descends to a map $\tilde{C}: \mathbb{T}^2 \rightarrow \mathbb{T}^2$. Moreover, \tilde{C} preserves the Lebesgue measure since $\det C = 1$. The matrix C has eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda_2 = \lambda_1^{-1} < 1$, with corresponding eigendirections E_1 and E_2 given by the vectors $v_1 = (\frac{1+\sqrt{5}}{2}, 1)$ and $v_2 = (\frac{1-\sqrt{5}}{2}, 1)$, respectively. This implies that for the derivative of \tilde{C} (which is C itself) the tangent space splits into the contracting E_2 and expanding E_1 directions. This hyperbolic matrix is called the *Arnold's cat map*.²

More generally, one can take a hyperbolic matrix $B \in SL(n, \mathbb{Z})$ (i.e., with no eigenvalues on the unit circle) and consider the action of the associated map \tilde{B} on \mathbb{T}^n .

Readers not familiar with dynamical systems may restrict their attention to hyperbolic matrices in what follows. The main mechanism that allows to study chaotic properties of uniformly hyperbolic systems is the existence of so-called *Markov partitions*, constructed in [5, 57]. They allow showing that a uniformly hyperbolic dynamical system is 'almost' Hölder conjugate to a *subshift of finite type* (SFT). SFTs are a class of symbolic systems which essentially exhibit the same statistical and chaotic properties as systems generated by

sequences of independent coin tosses (Bernoulli shifts). Rather than defining Markov partitions in full generality, we will just present one simple example of a Markov partition for a non-invertible map. This should give the readers a general idea on usefulness of such partitions.

Example 2.2. Let $f(x) = 2x \bmod 1$ on $\mathbb{T} = [0, 1)$ with addition mod 1. Consider a partition of \mathbb{T} given by $P_0 = [0, 1/2)$ and $P_1 = [1/2, 1)$. Note that we can code the orbit of any x by a sequence $\omega \in \{0, 1\}^{\mathbb{N}}$ in the following way: x corresponds to ω if for any $i \in \mathbb{N}$, $f^i(x) \in P_{\omega_i}$. This coding is uniquely defined (and invertible) for points x whose orbit $f^i(x)$ avoids the discontinuities at 0 and 1/2. So for all but countably many points the system f is Hölder conjugate to the full shift σ on $\{0, 1\}^{\mathbb{N}}$. The countably many exceptions account for the word 'almost' in the paragraph above. Then $\{P_0, P_1\}$ is a Markov partition for f . One can then study statistical properties of f by studying σ , which is much easier.

Unfortunately, the powerful tool of Markov partitions is usually not available for systems that are not hyperbolic, i.e., for partially hyperbolic systems, or, in particular, systems with some Lyapunov exponents equal to 0. This is a much wider class of systems for which there might be some directions in the tangent space which do not contract or expand (as fast) when iterated by the derivative. The simplest examples would be an integer matrix like in Example 2.1, but for which some of the eigenvalues have absolute value one. Another class of straightforward examples would be time-1 maps of *Anosov flows* (think about the geodesic flow on the unit tangent bundle of a hyperbolic surface). The flow direction is then an isometry. Statistical and chaotic properties of systems for which not all Lyapunov exponents are non-zero are much less understood. There are two main classes for which one usually applies completely different methods: (a) systems with positive entropy, or equivalently, with some of the Lyapunov exponents different from zero (i.e., there are some directions in which we see contraction or expansion); and (b) systems of zero entropy (with all Lyapunov exponents equal to zero). We will discuss these two classes in separate subsections.

2.2 Systems with positive entropy

As mentioned above, the general theory of systems with some Lyapunov exponents equal to zero is much less understood and there is no general framework to study chaotic properties in this class. In the last century, significant partial progress was made on understanding qualitative (ergodic) properties of such systems that we will now describe.

Qualitative properties

Qualitative properties can be defined for abstract measure-preserving systems. The strongest such property is being *Bernoulli*, i.e.,

² See <https://galileo-unbound.blog/2019/06/16/vladimir-arnolds-cat-map/>.

being measure-theoretically isomorphic to a Bernoulli shift (a system generated by a sequence of independent coin tosses). A weaker property is called the *K-property* (*K* for Kolmogorov) which is equivalent to saying that every factor of the system has positive entropy [54]. Some weaker properties are mixing, weak mixing and ergodicity, but for simplicity we will focus here on the *K* and Bernoulli properties only. Anatole Katok [30], and then Brin, Feldman and Katok [6], have shown that every smooth manifold of dimension at least 2 supports a Bernoulli diffeomorphism, and so there are no topological restrictions for Bernoullicity in the smooth setting. The main development for establishing *K* and Bernoulli properties was to use some geometric structures derived from the existence of non-zero exponents (stable and unstable manifolds). Sinai [56] has shown that the *K*-property follows from ergodicity of the so-called *unstable foliation*. This is still the main mechanism for showing *K*-property for smooth dynamical systems, used widely in literature, see for example [8]. The theory of Bernoulli shifts was developed in the second half of the last century, culminating in *Ornstein's theory* which allowed for their full classification in terms of entropy [42]. In particular, the *very weak Bernoulli* property introduced in [42] turned out to be a very useful tool in establishing Bernoullicity of smooth systems with (some) non-zero exponents. This came with the work of Ornstein and Weiss [44], where they used a geometric method to establish the very weak Bernoulli property for hyperbolic toral automorphisms, as well as for geodesic flows. Even though the systems they considered were algebraic, the introduced method proved to be applicable in a much more general setting. Specifically, the method is based on finding, for a given N , a matching between two pieces of typical unstable manifolds, i.e., a measure-preserving map Θ_N between these unstable manifolds such that the points x and $\Theta_N(x)$ are close for most of the iterates under the dynamics when the iterates are taken from the set $[0, N]$. This mechanism is to this day the main technique to study the Bernoulli property for smooth systems and has been used in various settings, e.g., [7, 10, 11, 36, 46, 48, 49]. Let us mention that despite considerable progress in the study of qualitative properties of smooth systems there are a number of open questions regarding whether a given dynamical system is Bernoulli or not. In particular, in [16] the authors showed that a smooth diffeomorphism which is exponentially mixing, is in fact Bernoulli. The methods used in [16] seem to suggest that there is a 'competition' between mixing rates of the system and growth on the center space which would determine if a system is Bernoulli or not. It is also not known if exponential mixing (for Hölder functions) implies Bernoulli in the non-smooth (e.g., symbolic) setting. Finally, let us mention that Kolmogorov initially conjectured that the *K* and Bernoulli properties are equivalent. This was shown not to be the case by Ornstein [43] in the abstract setting and independently by Katok [31] and Rudolph [55] in the smooth setting. In [29] the authors showed that *K* does not imply Bernoulli in dimension 4. It is known that it does in dimension 2 [45]. The problem in dimension 3 is one of the big open questions in the field [32].

Quantitative properties

Quantitative properties are those which use the fact that the system is a smooth diffeomorphism acting on a smooth manifold. Main quantitative properties are quantitative mixing (or decay of correlations), central limit theorem (CLT), large deviations, quantitative ergodicity. Quantitative properties are much less understood than qualitative ones. There are still open questions regarding interplay of quantitative and qualitative properties. In what follows we will mostly focus on quantitative mixing and central limit theorem. In fact, if the system has fast decay of correlations of all orders, then a CLT follows, see, e.g., [4]. The problem of establishing quantitative mixing rates for a given smooth system is difficult and widely open in general, even though a lot of progress was made for algebraic systems where one uses representation theory, see, e.g., [25, 34, 40]. The most successful methods of establishing (exponential) rates of mixing is by using *transfer operators* and their *spectral gaps*, initiated in the fundamental work of Dmitry Dolgopyat [13]. In fact, it seems that this method (or variants thereof) is the only approach for establishing rates of mixing in the smooth setting. It has been used by many authors in various situations, see, e.g., [37, 60] and references therein. In fact, this method also allows one to get other statistical properties, such as large deviations and CLT. It is based on translating the problem on (exponential) mixing into a problem concerning the spectral nature of a naturally associated operator (the transfer operator). One can then use tools from complex and harmonic analysis to study the spectrum of the system on so-called *anisotropic Banach spaces*, see, e.g., the survey [12] and references therein. As mentioned, existence of a spectral gap usually implies all statistical properties, and so if one wants systems with exotic behavior (enjoying some, but not all of the properties) one needs to introduce different methods. In particular, it is not known if there exists a smooth dynamical system which is exponentially mixing, but not exponentially mixing of order 3. In [14] the authors have produced exotic examples of smooth dynamical systems (with finite smoothness) which have zero entropy and still satisfy the CLT. A C^∞ flow of zero entropy and with CLT was constructed in [15]. Existence of a C^∞ map with zero entropy and with a CLT is still an open question. As mentioned in the previous section, it is known that every manifold of dimension at least 2 supports a Bernoulli diffeomorphism. It is not known if every manifold of dimension at least 2 supports a smooth diffeomorphism which satisfies a CLT as well as a smooth diffeomorphism which is exponentially mixing.

2.3 Systems with zero entropy

The world of zero-entropy systems in general is too wide to have any meaningful general structures. According to, e.g. [33], there are two rather different main classes of zero-entropy systems, *elliptic* systems and *parabolic* systems. The main model for elliptic systems is provided by irrational rotations on compact Abelian

groups (irrational rotations of the circle being the main example), but there are also less rigid examples of elliptic dynamics [20]. Elliptic dynamics is usually characterized by a rather non-chaotic, tame behavior where *mixing* is often precluded. We will focus on the class of parabolic systems and briefly highlight typical chaotic features in this class. Unlike the hyperbolic world, there is no strict definition of a parabolic system. It is rather a set of common characteristics that is an indicator of parabolic behavior. The prime class of parabolic systems are *horocycle flows*. In algebraic language, the horocycle flow (h_t) is given by the action of the matrix $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ acting on finite-volume quotients of the group $SL(2, \mathbb{R})$. Ergodic and statistical features of horocycle flows are the model behavior in the class of parabolic systems. More generally, unipotent flows on quotients of semisimple Lie group constitute a wider class of (algebraic) parabolic systems. Yet another class of (non-algebraic) systems that can be classified as parabolic (even though they also exhibit some elliptic-like behavior) is the class of smooth flows on surfaces, known as locally Hamiltonian flows (see Figure 2).

This is the class in which recently many tools from homogeneous dynamics were adapted to the non-algebraic setting. There are also systems which share both elliptic and parabolic characteristics: interval exchange transformations and translation flows, nilrotations on nilmanifolds, reparametrizations (or perturbations) of all the above-mentioned examples. One of the typical features of parabolic systems is that the orbit growth (or orbit divergence) happens typically at polynomial (or logarithmic) rate. This is the difference with hyperbolic systems, where the growth is exponential, and also with elliptic systems, where the growth is often sub-logarithmic. Moreover, typically the divergence of all nearby conditions occurs along well-understood, structured directions. Some characteristic features that one usually observes for parabolic systems are: small set of invariant measures (often a unique such measure), polynomial (or logarithmic) rates of mixing, which are usually obtained by the *mixing via shearing mechanism*, polynomial deviation of ergodic averages, absolutely continuous or Lebesgue spectrum, variants of Ratner's property (shearing in the direction of the dynamics), structured sets of *joinings*, 'nice' orbit closures. These properties for horocycle flows (or more generally unipotent flows) were established by a number of authors: Marina Ratner in her seminal work [50, 52] studied orbit closures and measure

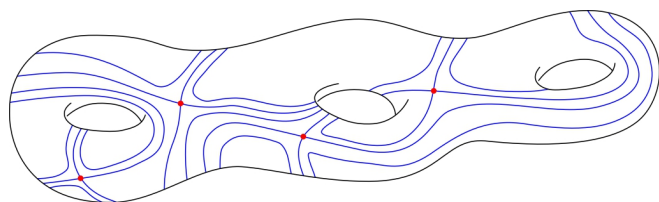


Figure 2. A locally Hamiltonian flow. The red points are fixed points of the flow.

rigidity for unipotent flows, in [51] Ratner also studied rates of mixing for horocycle flows. Higher-order mixing was established by Marcus [39]. Deviation of ergodic averages was studied in [21] and [59].

Much less was known until recently for parabolic systems that are not algebraic. This has changed significantly in the last 20 years, in which major progress in the study of spectral, mixing and rigidity properties of these systems was made. Due to limited space we will again only provide a list of references on the topic: deviation of ergodic averages was studied in [22–24]. Mixing properties were established in [17, 53, 58, 61, 62]. Higher-order mixing was studied in [19, 26, 28]. Joinings and rigidity were studied in [2, 27]. Spectral properties were studied in [9, 18].

Even though there has been significant progress on general parabolic systems, there are still many directions which are quite open: spectral features (the spectral type and the multiplicity function) of non-algebraic parabolic systems are still not known in many cases. Optimal rates of mixing and higher-order mixing is another open problem in the area. And despite some partial results, the theory of joinings is another problem that is rather widely open.

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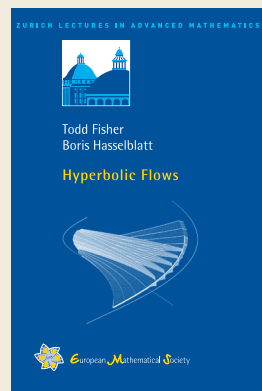
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