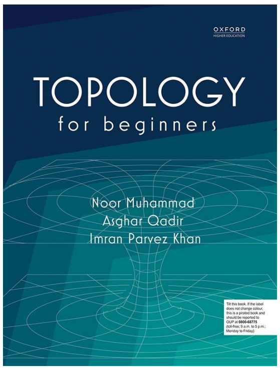
Book reviews

Topology for Beginners by Noor Muhammad, Asghar Qadir and Imran Parvez Khan

Reviewed by M. Ali Khan



There are wonderful books in topology and considerable material on hand on the internet, and so this particular reviewer may be forgiven for approaching this volume with some skepticism even if, or rather especially if, note is taken of the word "beginners" in its title. The question as to what the proverbial beginner would/could get from this book initially loomed large when texts such as those of Dugundji, Engelking, Kelley, Munkres, Schubert, Simmons, Ward, Willard are all easily available. There is also Wilder's classic piece on "Topology: its nature and significance," in *Mathematical Teaching*, **55(6)**, 462–475 (1962). The authors cite Steen–Seebach on examples – why does it not suffice as a text for a term? Why the need for another book? Can none of the texts listed above adequately serve the beginner? Do they require more commitment than a beginner can give? What is then unique and singular to what this book does offer?

The authors furnish a five-fold answer to the one question that underlies this medley already in their preface. They point to their use of examples (and counterexamples) to illustrate definitions, and to their explications of theorems by illustrating their use on well-chosen applications, excursions into applying the theory. Their third reason nods to history in the assertion that the "*raison d'être* for any definition really comes out by exploring changes in it." Yes, every theorem has its past. Surely, none of these reasons are controversial, let alone original, but the final two reasons concerning style and signature stand out for this reviewer: Wittgenstein's pictorial theory of meaning, and to his emphasis on the language of the everyday. However, there is a note of mild defensiveness when the authors note these.

> Early intuition is based in geometrical concepts which are best explained by "pictures," or diagrams, which may often be quite misleading. Students need to be gradually "weaned" away from using them rather than plunged into a bewildering world of great abstraction.

And as they move on to style and language, the authors rely on anthropomorphizing the book:

> [The] style of presentation is extremely *informal*, not to say *downright chatty*. We, as serious mathematicians, would not have it so, but that is the way the book insisted on being written, and we had to go along with it.

One imagines that they have the style of more demanding classical texts like that of Hardy–Littlewood–Pólya's on inequalities, or Fremlin's treatise on measure theory, in their minds as a counterpoint. However, speaking only for myself, I am glad that their book asserted itself. It knew its own strength.

The introductory chapter is most interesting. As the authors write, "it is as brief as the first chapter is long, and as simple and heuristic as that chapter." With the aid of 19 pictures/diagrams, the chapter is sectioned into twelve parts, and the individual titles succinctly convey what all is in the chapter. If permitted a little gloss on the titles, they may be listed as follows: (i) topology in the broad context of mathematics, (ii) the development of mathematics, (iii) the advent of topology, (iv) the mathematical background for it, (v) a psychology aside, (vi) topological constructions, (vii) the need for a language of sets, (viii) a diversion into logic, (ix) more on the language of sets, (x) cardinal numbers and counting, (xi) transfinite numbers and uncountability, (xii) more general topological transformations. This reviewer found some sections especially fascinating and provocative, and crying out for serious engagement for all interested in mathematics as language and its unreasonable effectiveness, not only in the natural sciences but also in the social sciences and the humanities. I shall resist this temptation in this short review, other than to send the readers of this newsletter to the standard references. I do so in response to the authors saying that "mathematical argument is there for all to see and judge without any ambiguity and without inducing any prejudices." If only the sociology of knowledge was as simple as that.

The material in Chapters 2 to 7 is standard. The concluding Chapter 8 is to whet the beginner's appetite by talking in the language the beginner has so far been talked to. In the concluding paragraph of the first section on further topological directions, while asserting that topology is "a *must* for theoretical physics and can be expected to rapidly extend its domain of influence," the authors write:

> Topology is required in Economics and is becoming important in Game Theory and Decision Making. Part of the reason lies in its use of the Bolzano–Weierstrass theorem, which is used for finding optimal solutions, or proving that there is no optimal solution available.

Relevant to optimization and economics is also the earlier statement:

> Since all Dynamics derives especially from the minimization of a function called the Lagrangian, the study of Dynamics and Dynamical Systems is based on the study of the connectedness of the space of permissible functions.

They might also want to confront the algebraic and the topological approaches to additive representations, as is done by Peter Wakker in the *Journal of Mathematical Psychology* **32**, 421–435 (1988).

I have only one minor quibble: it is that *compactness* and *upper semicontinuity* of the associated topology on the choice set, and the objective function respectively for the existence of an optimum, and the need for a *convex* structure for its uniqueness, may be as important as that for *connectedness*. Pushing a bit further, the applications of topology in the social sciences have by now gone considerably beyond the Bolzano-Weierstrass theorem which, strictly speaking, belongs more to analysis than to topology. This then leads to a plea to the authors, with their admirable expository style, to follow up this volume, if not with another written on topology for beginners in social sciences, at least to include another chapter in a second edition. The authors might enjoy Paul Samuelson's 1944 *Foundations of Economic Analysis* and his 1960 tribute to Harold Hotelling on the "structure of a minimum equilibrium system" in Ralph Pfouts (ed.) *Essays in Economics and Econometrics*, North Carolina Press. Samuelson was an avid reader of classical theoretical physics. They might also want to look at the reviewer's work with Metin Uyanik on a "deconstruction and integration of the continuity postulate" and on "On an extension of a theorem of Eilenberg and a characterization of topological connectedness": the first in the *Journal of Mathematical Economics* and the second in *Topology and its Applications*.

Let me conclude my strong recommendation of this book to interested beginners by drawing attention to its thirty-three references: fourteen are on physics, ten on topology, nine to articles of general interest, five of which concern mathematical issues. This reviewer, coming as he does from a background in the human sciences, found invaluable the references to Cabrera on superconductivity, to Glashow–Weinberg on unified theory, to Guth–Linde on an inflationary universe, to Misner–Thorne–Wheeler on gravitation, to Penrose on laws, and to Qadir himself on Einstein's relativity. What more could a beginner want in 160 pages of symbols, pictures and prose?

Acknowledgements. The author thanks Eduardo Fernández, Ema Nakayama and Asghar Qadir for their careful reading.

Noor Muhammad, Asghar Qadir and Imran Parvez Khan, *Topology for Beginners.* Oxford University Press Pakistan, 2022, 210 pages; Softcover ISBN 978-9697-34182-5.

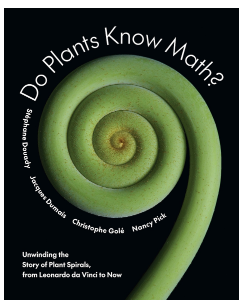
M. Ali Khan is the Abram Hutzler professor of political economy at the Johns Hopkins University, and adjunct professor at the New School of Social Research. He obtained his MPhil as well as his PhD from Yale University in June 1972 and June 1973, respectively. He works in mathematical economics with methods of non-standard analysis (Loeb spaces), non-smooth analysis and optimization (Mordukhovich–Ioffe cones), chaotic dynamics, and the law of large numbers with a continuum of random variables.

akhan@jhu.edu

DOI 10.4171/MAG/240

Do Plants Know Math? Unwinding the Story of Plant Spirals, from Leonardo Da Vinci to Now by Stéphane Douady, Jacques Dumais, Christophe Golé and Nancy Pick

Reviewed by Adhemar Bultheel



Since Nancy Pick, an American science writer, heard Stéphane Douady, a French physicist, discuss the Fibonacci numbers and the golden ratio during his lecture, they joined forces to start this book project. Later, they were able to convince Jacques Dumais, a Canadian biologist, and Christophe Golé, an Algerian-born mathematician, to join. All of them are fascinated by the fact that Fibonacci numbers continuously appear in phyllotaxis, and more generally, in nature. This book reports their search for an answer to the intriguing question: why is nature so 'obsessed' with these numbers?

The book starts by explaining the terminology. There are, of course, the Fibonacci numbers and the golden ratio, appearing in the spiral—or the helix—that can be connected to how plants grow leaves along their stem, but also to the spirals observed in the seeds of sunflowers, pine cones, or pineapples. These are called *parastichies* in phyllotaxis and are characterized by two integers, n and m: new leaves appear after a turn of n/m of a complete circle (the divergence angle) around the stem. These numbers turn out to be Fibonacci-like, approaching a golden angle of about 137.5° in the limit. They also describe the presence of n left-turning spirals and m right-turning ones. Later in the book, concepts such as dynamical systems, fractals, grids, and circle packing will be introduced. There are also biological terms, such as *meristem,* which is the tip where the plant grows. It contains the stem cells that differentiate and grow into primordia, which will develop into the plant's organs, like leaves, petals, etc. From the beginning, the authors take the reader along on their journey, encouraging them to observe patterns in nature, explore the many marvelous illustrations in the book, and even complete homework assignments to create crafts that verify the claims presented.

The authors tell the historical development of the subject from antiquity up to their own involvement. Each chapter starts with a 'Fibonacci poem' having seven lines, where the number of syllables in each line follows a Fibonacci sequence. Between the introduction and the conclusion, the main body of the book is arranged into five parts, each consisting of several chapters: (1) the early recognition of Fibonacci numbers and the golden ratio in mathematics and nature, (2) how science became interested in the topic and what additional information was revealed by the introduction of the (3) microscope, and later the (4) computer, and (5) how biologists investigated new pathways at a cellular level, eventually attempting to answer the why-question.

So, we start with the number sequence recognized long before Fibonacci in ancient Egypt and in Sanskrit poetry, move on to Leonardo da Vinci (who had a notebook classifying different kinds of leaves growing on plants), and of course Fibonacci (with his legendary example of population growth in rabbits), to Kepler (whose mother was a herbal healer, accused of being a witch), who tried to fit the planetary system into the mathematical rules of Platonic solids and who, in his treatise on the six-pointed snowflake, observed that the ratio of Fibonacci numbers approached the golden ratio, a concept traditionally attributed to Luca Pacioli.

Charles Bonnet (1720–1793) explained the placement of leaves as being optimal because he thought that plants grew by absorbing the dew coming from below. The word *phyllotaxis* seems to have been coined by Karl Friedrich Schimper in 1830, who observed the spirals in the placement of the leaves on a stem and in scales on a pine cone. But it was Alexander Braun who introduced this concept in his book, marvelously illustrated by drawings of his sister Cécile. In 1837, the Bravais brothers (one of whom was trained in crystallography) linked these spirals to plane grids (representing the surface of the cylindrical stem) defined by the n/m ratio. On this basis, Bonnet developed a continued fraction that converged to the irrational number $(3-\sqrt{5})/2$, and this is the portion of the circle giving the golden angle.

Wilhelm Hofmeister (1824–1877) rejected the prefixed spirals and looked at cell growth with the microscope. Swiss biologist Simon Schwendener (1829–1919) took a mechanical approach, considering cells on the stem surface to be circles. The stacking of these circles, that grow as they age, determines that a leaf starts growing where there is the most space available. The Dutch botanist Gerrit van Iterson Jr. (1878–1972) described possible solutions using a bifurcation diagram (1907), which has become a standard concept in today's phyllotaxis. One may notice these different branches as irregular transitions occurring, for example, at where the scales of the pine cone have different sizes. The Fibonacci branches are chosen by minimizing the energy.

With Alan Turing (1912–1954) we arrive in the computer age. He applied a theory of diffusion to explain biological patterns like the stripes of a zebra or the spots of a leopard. Much later, this evolved into chaos theory and (nonlinear) dynamical systems. Near the end of his life, Turing worked on morphogenesis and phyllotaxis, but, unfortunately, he did not live to see this work published. His notes, including the hypothesis of geometrical phyllotaxis, were only released 40 years after his death, when Douady published related results at approximately the same time.

Meanwhile, Aristid Lindenmayer (1925–1989) and Arthur Veen studied spirals in sunflower seeds. Lindenmayer modeled plant growth using what became known as *L-systems,* which established a formal computer language with an alphabet representing different elements and rules for their interaction. Their simulations of the diffusion of growth inhibitors produced very realistic images of sunflowers.

Douady shows with lab experiments on repulsing magnetic droplets and numerical simulations that, because the Schwendener circles grow as they move along, there are actual gaps at the bifurcation points of the Iterson diagram. This implies that always, the Fibonacci branch is taken, as earlier explained by Iterson on the basis of an energetic argument. Primordia in plants appear not only where, but also when the occasion is favorable.

That did not work for corn, which Douady considered a monster generated by extensive breeding, but here the work of Dumais and Golé comes in to explain the zigzagging growing front. Near the center of the meristem, the cells have different sizes, which explains an irregular (i.e., not hexagonal but rhombic) grid in the circle packing and hence the zigzag front. The teeth of the zigzag line are formed by the m left and n right spirals intersecting the growing front at different angles. Therefore, the rhombic pattern will, at some point in the growing process, produce a degeneration in the form of a triangle or pentagon, explaining the choice of Fibonacci branches.

What follows is a relatively short *intermezzo* on fractals and kirigami (the art of folding paper and cutting it to achieve particular effects when unfolded). Fractals describe accurately the shape of broccoli, ferns, leaves, etc., and kirigami is applied to show how leaves are packed in buds before they unfold.

But then they return to spirals from a biologist's point of view. As in physics, there are the empiricists who observe and do experiments, and there are the ones who try to explain everything through mathematics. Here, cell division is studied using a mathematical soap bubble model, something observed by the Belgian botanist Leo Errera (1858–1905), who was inspired by the work of Joseph Plateau (1801–1883) and popularized by D'Arcy Wentworth Thompson in his famous book *On Growth and Form* (1917). Cells divide in such a way that they form minimal surfaces, but there are small deviations. Depending on their shape, divisions of cells can happen at local minima rather than at the global one. Then Douady used dynamical systems again to show that cell division converges to an attractor and that generations of offspring cells will arrange again in spirals, bringing us back, in a fractal-like way, to the same story all over.

The authors add a chapter on animal analogs of the same story, like the spiral of the nautilus shell, the spiral patterns of the scales of fish or snakes, the tail of a peacock, or the multi-faceted eye of a fly. However, the Fibonacci sequence is not so frequent here, which might be explained by the mirror symmetry in animal bodies.

So, what is the conclusion? Do plants know math? The authors' answer is that these mathematical patterns are just the result of morphogenesis, and that there is no mathematical god that imposes them on nature. Each plant cell is just following the basic laws of science, and as the plant grows, the patterns spontaneously arise.

To finish with a celebration dinner, the last chapter presents several recipes to cook the plants with all their fantastic patterns, to make use of what they are really good at—i.e., to feed us.

This is a whirling journey through history and through different, seemingly unrelated, scientific topics. It is brought to the reader in a most entertaining and readable way.

Stéphane Douady, Jacques Dumais, Christophe Golé and Nancy Pick, *Do Plants Know Math? Unwinding the Story of Plant Spirals, from Leonardo Da Vinci to Now.* Princeton University Press, 2024, xiii+352 pages, Hardcover ISBN 978-0691-15865-5, eBook ISBN 978-0691-26108-9.

Adhemar Bultheel is emeritus professor at the Department of Computer Science of the KU Leuven (Belgium). He has been teaching mainly undergraduate courses in analysis, algebra, and numerical mathematics.

adhemar.bultheel@kuleuven.be

DOI 10.4171/MAG/234