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Short geodesics and small eigenvalues on random hyperbolic punctured spheres

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Abstract. We study the number of short geodesics and small eigenvalues on Weil–Petersson random genus zero hyperbolic surfaces with n cusps in the regime $n \to \infty$. Inspired by work of Mirzakhani and Petri (2019), we show that the random multiset of lengths of closed geodesics converges, after a suitable rescaling, to a Poisson point process with explicit intensity. As a consequence, we show that the Weil–Petersson probability that a hyperbolic punctured sphere with n cusps has at least $k = o(n)$ arbitrarily small eigenvalues tends to 1 as $n \to \infty$.

1. Introduction

1.1. Overview of main results

For hyperbolic surfaces, understanding the lengths of closed geodesics offers a deep insight into both the geometry and spectral theory of the surface. In this paper, we consider the genus zero setting and study the distribution of short closed geodesics on random surfaces sampled from the moduli space $\mathcal{M}_{0,n}$ of hyperbolic punctured spheres with respect to the Weil–Petersson probability measure $\mathbb{P}_n$, as the number of cusps tends to infinity (see Section 2 for further details on this model). In particular, we show that the number of short closed geodesics exhibit Poissonian statistics. Using similar ideas, we gain an understanding about the number of small Laplacian eigenvalues that exist on typical such surfaces.

To state these results more precisely, we introduce the following notation. Let $X \in \mathcal{M}_{0,n}$ and let $0 \leq a < b$ be real numbers. Denote by

$$N_{n,[a,b]}(X) \colon (\mathcal{M}_{0,n}, \mathbb{P}_n) \to \mathbb{N}$$

the random variable that counts the number of primitive closed geodesics on X with lengths in the interval $[a/\sqrt{n}, b/\sqrt{n}]$. We remark that when n is sufficiently large, $b/\sqrt{n} < 2\operatorname{arcsinh}(1)$, and so in this case, the geodesics that are counted by the random

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variable will be simple by the Collar theorem [3, Theorem 4.4.6]. In Section 3 we prove the following, inspired by the result of Mirzakhani and Petri [19] for closed surfaces of large genus. Recall that on a probability space $(X, \mathbb{P})$, a random variable $X\colon \Omega \to \mathbb{N}$ is called Poisson distributed with mean $\lambda \in [0, \infty)$ if

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

for all $k \in \mathbb{N}$.

Theorem 1.1. *Let $\ell \in \mathbb{N}$ and suppose that $0 \leq a_i < b_i$ are real numbers for $i = 1, \ldots, \ell$ such that the intervals $[a_i, b_i]$ are pairwise disjoint. Then, the sequence of random vectors*

$$\left(N_{n,[a_1,b_1]}(X), \ldots, N_{n,[a_\ell,b_\ell]}(X)\right)_{n\geq 3}$$

converges in distribution as $n \to \infty$ to a vector of independent Poisson distributed random variables with means

$$\lambda_{[a_i,b_i]} = \frac{b_i^2 - a_i^2}{2} \frac{(j_0\pi)^2}{4}\left(1 - \frac{J_3(j_0)}{J_1(j_0)}\right),$$

where J_α are Bessel functions of the first kind and j_0 is the first positive zero of J_0.

Remark 1.2. One can compute that

$$\frac{(j_0\pi)^2}{4}\left(1 - \frac{J_3(j_0)}{J_1(j_0)}\right) \approx 8.7997.$$

It is equal to the constant

$$\sum_{i=2}^{\infty} \frac{V_{0,i+1}}{i!}\left(\frac{x_0}{2\pi^2}\right)^{i-1},$$

where $x_0 = -\frac{1}{2} j_0 J_0'(j_0)$ and $V_{0,m}$ is the volume of the moduli space $\mathcal{M}_{0,m}$, and this constant arises from using volume asymptotics of Manin and Zograf [16] which we reproduce in Theorem 2.2.

Remark 1.3. For $X \in \mathcal{M}_{0,n}$, let $\xi_X^{(n)}$ be the point measure associated with the multiset $\{\sqrt{n}\ell_\gamma(X)\}_\gamma$ where the indexing runs over homotopy classes of closed curves on an n punctured sphere, and let $\xi^{(n)}$ denote the corresponding point process on $[0, \infty)$ with respect to $(\mathcal{M}_{0,n}, \mathbb{P}_n)$. Theorem 1.1 is equivalent to saying that $\xi^{(n)}$ converges in distribution to a Poisson point process on $[0, \infty)$ with intensity

$$\frac{(j_0\pi)^2}{4}\left(1 - \frac{J_3(j_0)}{J_1(j_0)}\right) x\, \mathrm{d}x.$$

This result is proven using the method of factorial moments (see Proposition 3.1), which requires the computation of the expectation of the random variables

$$N_{n,[a,b]}(X)\big(N_{n,[a,b]}(X)-1\big)\cdots\big(N_{n,[a,b]}(X)-k\big)$$

for each $k = 0, 1, \ldots$ These random variables have the useful interpretation of being the number of ordered lists of length $k+1$ consisting of distinct primitive closed geodesics with lengths in $[a/\sqrt{n}, b/\sqrt{n}]$. The expected number of such lists is then ripe for computation using the integration machinery developed by Mirzakhani [17], which we recall in Section 2. A key insight of our proof is classifying the topological types of multicurves that make the dominant contribution to these expectations (see Definition 3.3).

An interesting consequence of Theorem 1.1 is that we can gain an understanding of the systole of a typical surface. For example, we see that for $x > 0$,

$$\lim_{n\to\infty} \mathbb{P}_n\Big(\mathrm{sys}(X) < \frac{x}{\sqrt{n}}\Big) = 1-\exp\Big(-\frac{x^2}{2}\frac{(j_0\pi)^2}{4}\Big(1-\frac{J_3(j_0)}{J_1(j_0)}\Big)\Big),$$

where $\mathrm{sys}(X)$ is the random variable measuring the length of the systole of the surface X. We can compare this to the deterministic bounds obtained by Schmutz [26, Theorem 14] which show that for any $X \in \mathcal{M}_{0,n}$,

$$\mathrm{sys}(X) \le 4\,\mathrm{arcosh}\Big(\frac{3n-6}{n}\Big), \tag{1.1}$$

and this bound is sharp for $n = 4, 6$ and 12. See also Lakeland and Young [11] for sharper bounds on the systole in the case of arithmetic hyperbolic punctured spheres. We show that asymptotically almost surely (that is, with probability tending to 1 as $n \to \infty$) the systole of a surface is much smaller than this in the Weil–Petersson model. In fact, we also have the complementary result.

Proposition 1.4. *There exists a constant $B > 0$ such that for any constants $0 < \varepsilon < \frac{1}{2}$, $A > 0$, any $c_n < An^{\varepsilon}$ and n sufficiently large,*

$$\mathbb{P}_n\Big(\mathrm{sys}(X) > \frac{c_n}{\sqrt{n}}\Big) \le \max\Big\{Bc_n^{-2}, \frac{B}{\sqrt{n}}\Big\}.$$

Remark 1.5. Proposition 1.4 also remains true on $\mathcal{M}_{g,n}$ when the genus g is fixed and non-zero.

Remark 1.6. By similar methods to [19, Theorem 5.1] it is possible to write

$$\sqrt{n}\mathbb{E}_n\big(\mathrm{sys}(X)\big) = \int_0^\infty \mathbb{P}_n\Big(\mathrm{sys}(X) > \frac{x}{\sqrt{n}}\Big)\,\mathrm{d}x = \int_0^\infty \mathbb{P}_n\big(N_{n,[0,x]}(X)=0\big)\,\mathrm{d}x.$$

By taking the $n \to \infty$ limit, interchanging the limit and integral and using the convergence in distribution of $N_{n,[0,x]}(X)$ from Theorem 1.1, one would obtain

$$\lim_{n\to\infty} \sqrt{n}\mathbb{E}_n\big(\mathrm{sys}(X)\big) = \frac{\sqrt{2}}{j_0\sqrt{\pi}\sqrt{1 - J_3(j_0)/J_1(j_0)}} \approx 0.4225. \tag{1.2}$$

However, to justify this interchange of limit and integral we require better bounds on $\mathbb{P}_n(\mathrm{sys}(X) > x/\sqrt{n})$ than those obtained in Proposition 1.4 to apply the dominated convergence theorem. By (1.1) it would be sufficient to obtain an improvement of the bound in Proposition (1.4) to just Bc_n^{-2}. The extra term $B/\sqrt{n}$ in Proposition 1.4 essentially arises from using the trivial bounds in Lemma 2.4. In a forthcoming work of the authors, we prove a large-n asymptotic for $V_{g,n}(x_1, \dots, x_k)$ from which (1.2) can be deduced. However, the methods of the current paper fall short of this so we do not claim it here.

Our methods also allow us to consider vectors consisting of random variables counting different topological types of short geodesics. For an integer $c \geqslant 2$ and real numbers $a, b \geqslant 0$, we let $N_{n,c,[a,b]}(X)$: $(\mathcal{M}_{0,n}, \mathbb{P}_n) \to \mathbb{N}$ denote the random variable which counts the number of primitive closed geodesics with lengths in $[a/\sqrt{n}, b/\sqrt{n}]$ which bound c cusps. We prove the following.

Theorem 1.7. *Let $\ell \in \mathbb{N}$, $c_1, \dots, c_\ell \geq 2$ be distinct integers and $0 \leq a_i < b_i$ be real numbers for $i = 1, \dots, \ell$. Then, the sequence of random vectors*

$$\big(N_{n,c_1,[a_1,b_1]}(X), \dots, N_{n,c_\ell,[a_\ell,b_\ell]}(X)\big)_{n\geq 3}$$

converges in distribution as $n \to \infty$ to a vector of independent Poisson distributed random variables with means

$$\lambda_{c_i,[a_i,b_i]} = \frac{b_i^2 - a_i^2}{2} \frac{V_{0,c_i+1}}{c_i!} \left(\frac{x_0}{2\pi^2}\right)^{c_i-1},$$

where $V_{0,m}$ is the volume of the moduli space $\mathcal{M}_{0,m}$ and $x_0 = -\frac{1}{2} j_0 J_0'(j_0)$ with J_0 the Bessel function of the first kind and j_0 its first positive zero.

As a consequence of Theorem 1.7, we gain an understanding of the topological nature of the systole in the large n limit. This can be seen from the following corollary.

Corollary 1.8. *Let $k \geq 2$ be an integer, $n \in \mathbb{N}$ and $x > 0$ be real. Suppose that $\mathcal{A}_{k,x,n} \subseteq \mathcal{M}_{0,n}$ is the collection of surfaces X that satisfy the following conditions:*

(1) *for each $2 \leq i < k$, no systolic curve of X separates off exactly i cusps;*

(2) *there exists a systolic curve on X that separates off at least k cusps, with length less than $x/\sqrt{n}$.*

Then,

$$\lim_{n\to\infty} \mathbb{P}_n(\mathcal{A}_{k,x,n}) \geq e^{-\frac{x^2}{2}\sum_{i=2}^{k-1}\frac{V_{0,i+1}}{i!}\left(\frac{x_0}{2\pi^2}\right)^{i-1}}\left(1-e^{-\frac{x^2}{2}\frac{V_{0,k+1}}{k!}\left(\frac{x_0}{2\pi^2}\right)^{k-1}}\right) > 0.$$

Remark 1.9. It is also not too difficult to show that Corollary 1.8 (2) can be changed to *exactly* k cusps. Indeed, this follows from an analogue of Theorem 1.7 for the random vector

$$\left(N_{n,c_1,[a_1,b_1]}(X),\ldots,N_{n,c_\ell,[a_\ell,b_\ell]}(X),N_{n,\geq \max(c_i)+1,[a_{\ell+1},b_{\ell+1}]}(X)\right)_{n\geq 3},$$

where $N_{n,\geq c,[a,b]}(X)$ counts the number of primitive geodesics with lengths in the interval $[a/\sqrt{n}, b/\sqrt{n}]$ that separate off at least c cusps.

Understanding the distribution of closed geodesics on a surface also offers insight into its spectral properties. In Section 5 we demonstrate the existence of many short closed geodesics on typical hyperbolic punctured spheres that each separate off two distinct cusps from the surface. The existence of these curves, when combined with the Mini-max Lemma 5.1 for the Laplacian, and an argument similar to Buser [3, Theorem 8.1.3], can be used to deduce the existence of $o(n)$ arbitrarily small eigenvalues on a typical hyperbolic punctured sphere. Recall, that the spectrum of a hyperbolic punctured sphere consists of absolutely continuous spectrum in the range $[1/4,\infty)$, a simple eigenvalue at 0, possibly finitely many eigenvalues in the range $(0, 1/4)$ and potentially embedded eigenvalues above $1/4$. Then, let $\lambda_k(X)$ denote the $(k+1)$th smallest eigenvalue of the Laplacian on $X \in \mathcal{M}_{0,n}$ *if it exists*. A Theorem of Zograf (see [33]) says that there is a constant $C > 0$ such that for any $X \in \mathcal{M}_{g,n}$,

$$\lambda_1(X) \leqslant C\frac{g+1}{n}.$$

In particular, if $g = o(n)$ then any surface in $\mathcal{M}_{g,n}$ has a small eigenvalue. Our next result complements this by showing a random surface has many small eigenvalues. We prove the following.

Theorem 1.10. *There is a constant $C > 0$ such that for any function $k\colon \mathbb{N} \to \mathbb{N}$ with $k = o(n)$ and $k \to \infty$ as $n \to \infty$, then*

$$\mathbb{P}_n\left(\lambda_k(X) < C\sqrt{\frac{k}{n}}\right) \to 1,$$

as $n \to \infty$. In particular, for any $\varepsilon > 0$, $\mathbb{P}_n[\lambda_k(X) < \varepsilon] \to 1$ as $n \to \infty$.

By work of Ballmann, Mathiesen and Mondal [2], we have that $\lambda_{n-1}(X) > 1/4$, in particular Theorem 1.10 says that a random surface with many cusps is not far from saturating this bound to leading order.

Remark 1.11. Theorem 1.10 can be easily extended to surfaces with fixed genus $g > 0$, cf. Remark 5.5.

1.2. Relations to existing work

It is worthwhile to compare the results obtained here with existing literature in the large genus and mixed large genus and large cusp regimes. For closed hyperbolic surfaces of genus g, Mirzakhani and Petri [19] have also obtained Poissonian statistics for the distributions of closed geodesics in the regime $g \to \infty$, and their work is the main inspiration for our investigation here. More precisely, they consider the random variables

$$N_{g,[a,b]}(X) \colon (\mathcal{M}_g, \mathbb{P}_g) \to \mathbb{N},$$

where $\mathbb{P}_g$ is the associated Weil–Petersson probability measure, which count the number of primitive closed geodesics on the surface X with lengths in the interval $[a, b]$. In the $g \to \infty$ limit, they show that these random variables converge in distribution to a Poisson distributed random variable with mean

$$\lambda'_{[a,b]} = \int_a^b \frac{e^t + e^{-t} - 2}{2t} \, \mathrm{d}t.$$

Remark 1.12. We note that

$$\frac{e^t + e^{-t} - 2}{2t} = \frac{2\sinh^2(t/2)}{t} = \frac{t}{2} + O(t^3),$$

and so for small a and b the integral gives a leading order of $(b^2 - a^2)/4$ akin to the constant arising in Theorem 1.1.

They use this to show that the limit of the expected systole length is $\approx 1.61498\ldots$ This is in contrast to the hyperbolic punctured sphere setting that we study here, where the systole is on scales of order $n^{-1/2}$. In fact, in the large genus limit, Mirzakhani [18, Theorem 4.2] showed that for $\varepsilon > 0$ sufficiently small, the Weil–Petersson probability of a closed hyperbolic surface having systole smaller than ε is proportional to ε^2. Moreover, if one considers the length of the separating systole, that is, the shortest closed geodesic that separates the surface, its expected size is $2\log(g)$ in the large genus regime by the work of Parlier, Wu and Xue [23]. Note, in the setting we consider here, the systole is always separating.

In addition to this, Nie, Wu and Xue showed in [22] that with probability tending to 1 as $g \to \infty$, the separating systole on $X \in \mathcal{M}_g$ separates X into $\Sigma_{1,1} \cup \Sigma_{g-1,1}$. In contrast, Corollary 1.8 demonstrates that there is not an analogue of this result for the hyperbolic punctured sphere setting. Indeed, with positive probability as $n \to \infty$, a hyperbolic surface has only separating systoles that cut off at least k cusps for any $k \geq 2$.

For the spectral consequences, there is a stark contrast between the cusp and genus limits. Indeed, in the closed hyperbolic surface setting, it is conjectured that typical surfaces (with respect to a reasonable probability model such as the Weil–Petersson model) should have no small, non-trivial eigenvalues. More precisely, Wright makes the conjecture [30, Problem 10.4] that for any $\varepsilon > 0$,

$$\lim_{g\to\infty} \mathbb{P}_g\Big(X : \lambda_1(X) \geq \frac{1}{4} - \varepsilon\Big) = 1, \tag{1.3}$$

where $\lambda_1(X)$ is the first non-zero eigenvalue of the Laplacian on X. The current state of the art for this is $1/4$ replaced by $2/9$ in (1.3) above obtained by Anantharaman and Monk [1]. See also the work of Wu and Xue [31] and Lipnowski and Wright [14] where $1/4$ is replaced by $3/16$. These results proceed the earlier work of Magee, Naud and Puder [15] who obtain a relative spectral gap of size $3/16$ in a random covering probability model. More precisely, given a compact hyperbolic surface, one can consider a degree d Riemannian covering uniformly at random. Their result then states that for any $\varepsilon > 0$, a covering will have no new eigenvalues from those on the base surface in the interval $[0, 3/16 - \varepsilon]$ with probability tending to 1 as $d \to \infty$.

In the non-compact setting, Magee and the first named author [9] recently showed that in the same random covering model, this relative spectral gap can be improved to the interval $[0, 1/4 - \varepsilon]$. In particular, via a compactification procedure, they demonstrated the existence of a sequence of closed hyperbolic surfaces with genus tending to ∞ with $\lambda_1 \to 1/4$.

In the mixed regime of genus $g \to \infty$ and number of cusps $n = O(g^\alpha)$ for $0 \leqslant \alpha < 1/2$, the first named author [8] demonstrated in the Weil–Petersson model that with probability tending to 1 as $g \to \infty$, a surface has an explicit spectral gap with size dependent upon α. Moving past the $n = o(g^{1/2})$ threshold, Shen and Wu [27] showed that for $g^{1/2+\delta} \ll n \ll g$ and any $\varepsilon > 0$, the Weil–Petersson probability of λ_1 being less than ε tends to one as $g \to \infty$. For the scale $n \asymp g^{1/2}$ as $g \to \infty$, Shen and Wu [27, Theorem 2] determine that for any $\varepsilon > 0$, $\lambda_1 < \varepsilon$ with positive probability as $g \to \infty$. Theorem 1.10 complements these results by providing information about $o(n)$ eigenvalues.

Other related work in the Weil–Petersson model includes the study of Laplacian eigenfunctions [4, 28], quantum ergodicity [12, 13], local Weyl law [20], Gaussian Orthogonal Ensemble energy statistics [24] (see [21] in the case of random covers), and prime geodesic theorem error estimates [32]. See also [7] and the references therein for results concerning the lengths of boundaries of pants decompositions for random surfaces.

2. Background

In this section we introduce the necessary background on moduli space, the Weil–Petersson metric and Mirzakhani's integration formula. One can see [30] for a nice exposition of these topics.

2.1. Moduli space

Let $\Sigma_{g,c,d}$ denote a topological surface with genus g, c labeled punctures and d labeled boundary components, where $2g+n+d \geqslant 3$. A marked surface of signature (g,c,d) is a pair (X,φ) where X is a hyperbolic surface and $\varphi\colon \Sigma_{g,c,d} \to X$ is a homeomorphism. Given $(l_1,\dots,l_d) \in \mathbb{R}^d_{>0}$, we define the Teichmüller space $\mathcal{T}_{g,c+d}(l_1,\dots,l_d)$ by

$$\begin{aligned}\mathcal{T}_{g,c,d}(l_1,\dots,l_d) \overset{\text{def}}{=} \big\{&\text{Marked surfaces } (X,\varphi) \text{ of signature } (g,c,d)\\ &\text{with labeled totally geodesic boundary components}\\ &(\beta_1,\dots,\beta_d) \text{ with lengths } (l_1,\dots,l_d)\big\}/\sim,\end{aligned}$$

where $(X_1,\varphi_1) \sim (X_2,\varphi_2)$ if and only if there exists an isometry $m\colon X_1 \to X_2$ such that φ_2 and $m \circ \varphi_1$ are isotopic. Let $\mathrm{Homeo}^+(\Sigma_{g,c,d})$ denote the group of orientation preserving homeomorphisms of $\Sigma_{g,c,d}$ which leave every boundary component setwise fixed and do not permute the punctures. Let $\mathrm{Homeo}_0^+(\Sigma_{g,c,d})$ denote the subgroup of homeomorphisms isotopic to the identity. The mapping class group is defined as

$$\mathrm{MCG}_{g,c,d} \overset{\text{def}}{=} \mathrm{Homeo}^+(\Sigma_{g,c,d})/\,\mathrm{Homeo}_0^+(\Sigma_{g,c,d}).$$

$\mathrm{Homeo}^+(\Sigma_{g,c,d})$ acts on $\mathcal{T}_{g,c,d}(l_1,\dots,l_d)$ by pre-composition of the marking, and $\mathrm{Homeo}_0^+(\Sigma_{g,c,d})$ acts trivially, hence $\mathrm{MCG}_{g,c,d}$ acts on $\mathcal{T}_{g,c,d}(l_1,\dots,l_d)$ and we define the moduli space $\mathcal{M}_{g,c,d}(l_1,\dots,l_d)$ by

$$\mathcal{M}_{g,c,d}(l_1,\dots,l_d) \overset{\text{def}}{=} \mathcal{T}_{g,c,d}(l_1,\dots,l_d)/\,\mathrm{MCG}_{g,c,d}\,.$$

By convention, a geodesic of length 0 is a cusp and we suppress the distinction between punctures and boundary components in our notation by allowing $l_i \geqslant 0$. In particular,

$$\mathcal{M}_{g,c+d} = \mathcal{M}_{g,c,d}(0,\dots,0).$$

Throughout the sequel we shall restrict our study to the case that $g=0$.

2.2. Weil–Petersson metric

For $\boldsymbol{l} = (l_1,\dots,l_n)$ with $l_i \geqslant 0$ for $1 \leqslant i \leqslant n$, the space $\mathcal{T}_{g,n}(\boldsymbol{l})$ carries a natural symplectic structure known as the Weil–Petersson symplectic form and is denoted by ω_{WP} [5]. It is invariant under the action of the mapping class group and descends to a symplectic form on $\mathcal{M}_{g,n}(\boldsymbol{l})$. The form ω_{WP} induces the volume form

$$\mathrm{dVol}_{\mathrm{WP}} \stackrel{\mathrm{def}}{=} \frac{1}{(3g-3+n)!} \bigwedge_{i=1}^{3g-3+n} \omega_{\mathrm{WP}},$$

which is also invariant under the action of the mapping class group and descends to a volume form on $\mathcal{M}_{g,n}(\boldsymbol{l})$. By a theorem of Wolpert [29], this volume form can be made explicit in terms of Fenchel–Nielsen coordinates. We write $\mathrm{d}X$ as shorthand for $\mathrm{dVol}_{\mathrm{WP}}$. We let $V_{g,n}(\boldsymbol{l})$ denote $\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}(\boldsymbol{l}))$, the total volume of $\mathcal{M}_{g,n}(\boldsymbol{l})$, which is finite. We write $V_{g,n}$ to denote $V_{g,n}(\mathbf{0})$ and since we shall only consider the case $g = 0$ in this article, we shall often write $V_n \stackrel{\mathrm{def}}{=} V_{0,n}$.

We will define the Weil–Petersson probability measure on $\mathcal{M}_{0,n}$ by normalizing $\mathrm{dVol}_{\mathrm{WP}}$. Indeed, for any Borel subset $\mathcal{B} \subseteq \mathcal{M}_{0,n}$,

$$\mathbb{P}_n[\mathcal{B}] \stackrel{\mathrm{def}}{=} \frac{1}{V_n} \int_{\mathcal{M}_{0,n}} \mathbb{1}_{\mathcal{B}}\, \mathrm{d}X.$$

We write $\mathbb{E}_n$ to denote expectation with respect to $\mathbb{P}_n$.

2.3. Mirzakhani's integration formula

We define a k-multicurve to be an ordered k-tuple $\Gamma = (\gamma_1, \dots, \gamma_k)$ of disjoint non-homotopic non-peripheral simple closed curves on $\Sigma_{0,n}$ and we write

$$[\Gamma] = [\gamma_1, \dots, \gamma_k]$$

to denote its homotopy class. The mapping class group $\mathrm{MCG}_{0,n}$ acts on homotopy classes of multicurves and we denote the orbit containing $[\Gamma]$ by

$$\mathcal{O}_\Gamma = \{(g \cdot \gamma_1, \dots, g \cdot \gamma_k) \mid g \in \mathrm{MCG}_{0,n}\}.$$

Given a simple, non-peripheral closed curve γ on $\Sigma_{0,n}$, for $(X, \varphi) \in \mathcal{T}_{0,n}$ we define $\ell_\gamma(X)$ to be the length of the unique geodesic in the free homotopy class of $\varphi(\gamma)$. Then given a function $f : \mathbb{R}^k_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, for $X \in \mathcal{M}_{0,n}$ we define

$$f^\Gamma(X) \stackrel{\mathrm{def}}{=} \sum_{(\alpha_1,\dots,\alpha_k)\in\mathcal{O}_\Gamma} f\big(\ell_{\alpha_1}(X), \dots, \ell_{\alpha_k}(X)\big),$$

which is well defined on $\mathcal{M}_{0,n}$ since we sum over the orbit $\mathcal{O}_\Gamma$. Let $\Sigma_n(\Gamma)$ denote the result of cutting the surface $\Sigma_{0,n}$ along $(\gamma_1, \dots, \gamma_k)$, then

$$\Sigma_n(\Gamma) = \bigsqcup_{i=1}^{s} \Sigma_{0,c_i,d_i}$$

for some $\{(c_i, d_i)\}_{i=1}^{s}$. Each γ_i gives rise to two boundary components γ_i^1 and γ_i^2 of $\Sigma_n(\Gamma)$. Given $\boldsymbol{x} = (x_1, \ldots, x_k)$, let $\boldsymbol{x}^{(i)}$ denote the tuple of coordinates x_j of $\boldsymbol{x}$ such that γ_j is a boundary component of Σ_{c_i,d_i}. We define

$$V_n(\Gamma, \underline{x}) \stackrel{\text{def}}{=} \prod_{i=1}^{s} V_{c_i+d_i}(\boldsymbol{x}^{(i)}).$$

We can now state Mirzakhani's integration formula.

Theorem 2.1 (Mirzakhani's integration formula [17, Theorem 7.1]). *Given a k-multicurve $\Gamma = (\gamma_1, \ldots, \gamma_k)$,*

$$\int_{\mathcal{M}_{0,n}} f^{\Gamma}(X)\,\mathrm{d}X = \int_{\mathbb{R}^k_{\geqslant 0}} f(x_1, \ldots, x_k) V_n(\Gamma, \underline{x}) x_1 \cdots x_k \,\mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

2.4. Volume bounds

Finally, we state some results on Weil–Petersson volumes of moduli space which we will need later on. We make essential use of the following asymptotic for $V_{0,n}$ due to Manin and Zograf [16, Theorem 6.1].

Theorem 2.2. *There exists a constant B_0 such that as $n \to \infty$,*

$$V_n = (2\pi^2)^{n-3} n!(n+1)^{-7/2} x_0^{-n}\Big(B_0 + O\Big(\frac{1}{n}\Big)\Big),$$

where $x_0 = -1/2 j_0 J_0'(j_0)$ and J_0 is the Bessel function and j_0 is the first positive zero of J_0.

The following bound of Mirzakhani [18, Lemma 3.2 (3)] is needed for comparing moduli space volumes with differing cusps.

Lemma 2.3. *There exist constants $\gamma_1, \gamma_2 > 0$ such that for any $n \geq 3$,*

$$\gamma_1 \frac{V_{n+1}}{n-2} \leq V_n \leq \gamma_2 \frac{V_{n+1}}{n-2}.$$

The following well-known bounds will be sufficient for our purposes, e.g. [19, Proposition 3.1].

Lemma 2.4. *Suppose that $n \geq 3$, then for any $b_1, \ldots, b_n \geq 0$,*

$$1 \leqslant \frac{V_{0,n}(2b_1, \ldots, 2b_n)}{V_{0,n}} \leqslant \prod_{i=1}^{n} \frac{\sinh(b_i)}{b_i}.$$

3. Poisson statistics

In this section, we will prove Theorems 1.1 and 1.7. We begin by proving Theorem 1.1 for $\ell = 1$, that is, for a sequence of random variables; the extension to random vectors is straightforward, and we outline the necessary changes in Section 3.3 so that we can drastically simplify indexing and notation here. Furthermore, many of the same ideas can be used for the proof of Theorem 1.7 which we will prove in Section 3.4.

To this end, for $0 \leq a < b$ and $X \in \mathcal{M}_{0,n}$, recall that we defined $N_{n,[a,b]}(X)$ to be the number of primitive closed geodesics on X whose length are in the interval $[a/\sqrt{n}, b/\sqrt{n}]$. For fixed a, b, we will show that in the $n \to \infty$ regime, the random variables $N_{n,[a,b]}(X)$ converge in distribution to a Poisson distributed random variable with mean

$$\lambda = \frac{b^2 - a^2}{2} \sum_{i=2}^{\infty} \frac{V_{i+1}}{i!} x_0^{i-1}.$$

We then manipulate this series to obtain the stated constant in terms of Bessel functions. To demonstrate convergence, we use the method of factorial moments.

Proposition 3.1 (Method of factorial moments). *Let $\ell \in \mathbb{N}$ and suppose $(\Omega_n, \mathbb{P}_n)_{n \geq 1}$ are a sequence of probability spaces, and $X_{i,n} \colon \Omega_n \to \mathbb{N}$ are a sequence of random variables for $i = 1, \dots, \ell$. For $k \in \mathbb{N}$, we denote by*

$$(X_{i,n})_k := X_{i,n}(X_{i,n} - 1) \cdots (X_{i,n} - k + 1).$$

Suppose there exists $\lambda_i \in (0, \infty)$ for $i = 1, \dots, \ell$ such that for every $k_1, \dots, k_\ell \in \mathbb{N}$,

$$\lim_{n \to \infty} \mathbb{E}_n\big[(X_{1,n})_{k_1} \cdots (X_{\ell,n})_{k_\ell}\big] = \lambda_1^{k_1} \cdots \lambda_\ell^{k_\ell}.$$

Then, the random vector $(X_{1,n}, \dots, X_{\ell,n})$ converges in distribution to a random vector of independent Poisson distributed random variable with parameter λ_i.

We will first apply Proposition 3.1 in the case that $\ell = 1$. Since we are interested in understanding the expectation of the random variables as $n \to \infty$ for fixed $0 \leq a < b$, we can assume that n is sufficiently large so that

$$\frac{b}{\sqrt{n}} < 2 \operatorname{arcsinh}(1).$$

Then $(N_{n,[a,b]}(X))_k$ is precisely the number of k-tuples consisting of distinct, disjoint, primitive simple closed geodesics on X, whose lengths are in $[a/\sqrt{n}, b/\sqrt{n}]$. Disjointness of the curves follows from the fact that any two geodesics of length less than $2 \operatorname{arcsinh}(1)$ do not intersect [3, Theorem 4.1.6].

Remark 3.2. Consider the mapping class group orbit of an ordered multicurve. Since the mapping class group fixes the punctures of the surface, any two disjoint, non-peripheral, simple closed curves on $\Sigma_{0,n}$ will be in the same mapping class group orbit if and only if they separate the same punctures from the surface. Moreover, two ordered multicurves are in the same mapping class group orbit if and only if their corresponding curve components each separate the surface with the same topological decomposition and the punctures of $\Sigma_{0,n}$ that are on the subsurfaces are the same.

We will separate out the ordered multicurves that we consider into two types: nested multicurves and unnested multicurves.

Definition 3.3. A *nested multicurve* is an ordered multicurve $\Gamma = (\gamma_1, \ldots, \gamma_k)$ on $\Sigma_{0,n}$ consisting of distinct, disjoint, non-peripheral simple closed curves such that for some i, the two subsurfaces in the disconnected surface $\Sigma_{0,n} \setminus \gamma_i$ each contain at least one of the remaining multicurve components γ_j in their interior. An *unnested multicurve* is an ordered multicurve that is not nested. In other words, it is an ordered multicurve $\Gamma = (\gamma_1, \ldots, \gamma_k)$ on $\Sigma_{0,n}$ consisting of distinct, disjoint, non-peripheral simple closed curves such that for every i, one of the subsurfaces in the disconnected surface $\Sigma_{0,n} \setminus \gamma_i$ contains all of the other multicurve components in its interior.

We will write

$$\big(N_{n,[a,b]}(X)\big)_k = N^{N,k}_{n,[a,b]}(X) + N^{U,k}_{n,[a,b]}(X), \tag{3.1}$$

where $N^{N,k}_{n,[a,b]}(X)$ counts the nested k-multicurves and $N^{U,k}_{n,[a,b]}(X)$ counts the unnested k-multicurves. We will see that the main contribution to the expectation arises from the unnested multicurves.

3.1. Contribution of nested multicurves

Recall from Section 2 that Σ_n denotes a topological surface with n punctures labeled with the alphabet $\{1, \ldots, n\}$. More generally, $\Sigma_{g,c,d}$ denotes a topological surface with genus g, c labeled punctures and d labeled boundaries.

Given a nested ordered k-multicurve Γ on Σ_n, we associate an unordered collection of $k+1$ triples $\{(c_i, d_i, \{a^i_1, \ldots, a^i_{c_i}\})\}_{i=1}^{k+1}$, where

(1) $\sum_{i=1}^{k+1} c_i = n$, $c_i \geq 0$;

(2) $\sum_{i=1}^{k+1} d_i = 2k$, $1 \leq d_i < k$;

(3) $c_i + d_i \geq 3$;

(4) $\{a^i_1, \ldots, a^i_{c_i}\} \subseteq \{1, \ldots, n\}$ are pairwise disjoint and their union over all i is $\{1, \ldots, n\}$;

(5) Cutting Σ_n along the multicurve Γ gives rise to

$$\Sigma_n \setminus \Gamma = \bigsqcup_{i=1}^{k+1} \Sigma_{0,c_i,d_i},$$

where Σ_{0,c_i,d_i} has the labels $\{a_1^i, \ldots, a_{c_i}^i\}$ on its punctures inherited from the puncture labels on Σ_n. Indeed, the decomposition is obtained by recording the puncture and boundary numbers on the subsurfaces in the fifth condition along with the corresponding labels of the punctures.

Under such a cutting for the resulting tuples, condition (1) is immediate since any subsurface has at least zero punctures and the sum of the number of punctures of each subsurface is precisely n since the subsurfaces glue back together to give Σ_n.

Each component of Γ gives rise to exactly two boundary components in the decomposition so that the sum of the boundaries is $2k$. Moreover, each subsurface has at least one boundary obtained from the multicurve component that separates it from the surface. The number of boundary components on each subsurface is strictly less than k precisely because the multicurve is nested. Indeed, if one of the subsurfaces has k boundaries, then by $\sum_{i=1}^{k+1} d_i = 2k$, each of the other k subsurfaces has precisely one boundary corresponding to the k components in Γ. This means that cutting Σ_n by a curve component of Γ gives two subsurfaces, one of which corresponds to a subsurface in the decomposition with one boundary component. This subsurface hence does not contain any other multicurve components and thus the multicurve is unnested. Put together, these observations mean that condition (2) holds for the collection of tuples arising from Γ.

Condition (3) holds since otherwise in the cut surface there would be a component with exactly one boundary and one puncture or with two boundary components and no punctures. In the first case, the curve component in the multicurve that corresponds to this boundary component homotopes down to the puncture which is a contradiction because the curves are non-peripheral. In the second case, two curve components must bound an annulus which contradicts the assumption that the curve components are non-homotopic.

Denote the collection of unordered triples satisfying the conditions (1)–(4) by $\widetilde{\mathcal{A}}_k$.

Lemma 3.4. *A collection $\{(c_i, d_i, \{a_1^i, \ldots, a_{c_i}^i\})\}_{i=1}^{k+1} \in \widetilde{\mathcal{A}}_k$ corresponds to exactly $f(d_1, \ldots, d_{k+1})$ mapping class group orbits of nested multicurves for some function*

$$f(d_1, \ldots, d_{k+1}) \le (2k-1)!!k!.$$

Proof. Define the mapping from the collection of mapping class group orbits of nested multicurves to $\widetilde{\mathcal{A}}_k$ by $\mathcal{O}(\Gamma) \mapsto \{(c_i, d_i, \{a_1^i, \ldots, a_{c_i}^i\})\}_{i=1}^{k+1}$ with the collection obtained by cutting along a representative of the orbit. The mapping is well defined by Remark 3.2. It is obvious that from this map, that every collection in $\widetilde{\mathcal{A}}_k$ has at least one multicurve orbit associated to it. Moreover, applying a permutation to the components of the multicurves in an orbit provides a distinct orbit giving rise to the same collection. Any other mapping class group orbit associated to a collection is obtained by some gluing of the subsurfaces Σ_{0,c_i,d_i} with the cusp labels $\{a_1^i,\dots,a_{c_i}^i\}$ to obtain $\Sigma_{0,n}$. The number of such gluings depends only on the number of boundaries d_i on each subsurface component and is clearly bounded by the number of ways to pair the $2k$ boundaries of the subsurfaces. Note that this is an over-count since some gluings can give rise to the same multicurve orbit and some pairings are not permissible as they do not give rise to $\Sigma_{0,n}$. The number of such pairing is $(2k-1)!!$ and so when accounting also for the permutations of the multicurves, we obtain an upper bound of $(2k-1)!!k!$. ∎

Now for each collection $\{(c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\})\}_{i=1}^{k+1}\in\widetilde{\mathcal{A}}_k$ we will fix an arbitrary ordering on the pairs in the collection. We put this ordering on so that we can refer to specific indexed elements of a given collection. We will denote by $\widetilde{\mathcal{A}}_k^o$ the collection $\widetilde{\mathcal{A}}_k$ with a fixed ordering on the pairs in each of its elements. We then have

$$N_{n,[a,b]}^{N,k}(X)=\sum_{\{(c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\})\}_{i=1}^{k+1}\in\widetilde{\mathcal{A}}_k^o}\ \sum_{\mathcal{O}(\Gamma)}\ \sum_{\Gamma=(\gamma_1,\dots,\gamma_n)\in\mathcal{O}(\Gamma)}\mathbb{1}\Big(\frac{a}{\sqrt{n}}\le\ell_X(\gamma_1),\dots,\ell_X(\gamma_k)\le\frac{b}{\sqrt{n}}\Big),$$

where the middle summation is over all mapping class group orbits of nested multicurves that are associated with the ordered collection $\{(c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\})\}_{i=1}^{k+1}$ which from Lemma 3.4, there are at $(2k-1)!!k!$ such orbits.

By Mirzakhani's integration formula (Theorem 2.1) the expectation of

$$\sum_{\Gamma=(\gamma_1,\dots,\gamma_k)\in\mathcal{O}(\Gamma)}\mathbb{1}\Big(\frac{a}{\sqrt{n}}\le\ell_X(\gamma_1),\dots,\ell_X(\gamma_k)\le\frac{b}{\sqrt{n}}\Big),$$

depends only upon the topological decomposition of the cut surface $X\setminus\Gamma$, the information of which is contained in the associated collection

$$\big\{\big(c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\}\big)\big\}_{i=1}^{k+1}\in\widetilde{\mathcal{A}}_k^o.$$

In fact, it is independent of the puncture labels in each triple of a collection, and so we will consider the collection $\mathcal{A}_k^o$ which contains the ordered collections $\{(c_i,d_i)\}_{i=1}^{k+1}$ such that $\{(c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\})\}_{i=1}^{k+1}\in\widetilde{\mathcal{A}}_k^o$ for some partition of the labels $\{1,\dots,n\}$. Each element of $\mathcal{A}_k^o$ corresponds to precisely

$$\binom{n}{c_1,\dots,c_k}=\frac{n!}{c_1!\cdots c_k!(n-\sum_{i=1}^k c_i)!}$$

elements of $\widetilde{\mathcal{A}}^o_k$. Recall from Section 2 that since we will only deal with the moduli space of genus zero surfaces, we use the notations

$$V_m(x_1,\dots,x_m) := V_{0,m}(x_1,\dots,x_m) \quad \text{and} \quad V_m := V_{0,m}$$

for the moduli space volumes. We then obtain the following.

Lemma 3.5. *For any $k \geq 1$,*

$$\mathbb{E}_n\big(N^{N,k}_{n,[a,b]}(X)\big) \leq \sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}^o_k} (2k-1)!!k! \times \binom{n}{c_1,\dots,c_k} \frac{b^{2k}}{2^k} \frac{\prod_{i=1}^{k+1} V_{c_i+d_i}}{n^k V_n} \Big(1 + O\Big(\frac{kb^2}{n}\Big)\Big).$$

Proof. By Theorem 2.1, we have

$$\mathbb{E}_n\big(N^{N,k}_{n,[a,b]}(X)\big) = \sum_{\{(c_i,d_i,\{a^i_1,\dots,a^i_{c_i}\})\}_{i=1}^{k+1} \in \widetilde{\mathcal{A}}^o_k} \sum_{\mathcal{O}(\Gamma)} \frac{1}{V_n} \times \int_{a/\sqrt{n}}^{b/\sqrt{n}} \cdots \int_{a/\sqrt{n}}^{b/\sqrt{n}} x_1 \cdots x_k \prod_{i=1}^{k+1} V_{c_i+d_i}(\mathbf{0}_{c_i}, \mathbf{x}_i)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

where $\mathbf{0}_a := (0,\dots,0)$ is of length a and the $\mathbf{x}_i$ are length d_i vectors containing information about the lengths of the boundaries of the subsurfaces obtained when cutting along each multicurve whose lengths are assigned $x_1,\dots,x_k$; so in particular, each x_j appears exactly twice among all of the vectors $\mathbf{x}_i$. Using the upper bound of Lemma 2.4, the correspondence of Lemma 3.4 and the passage from $\widetilde{\mathcal{A}}^o_k$ to $\mathcal{A}^o_k$, we obtain

$$\begin{aligned}
\mathbb{E}_n\big(N^{N,k}_{n,[a,b]}(X)\big) &\leq \sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}^o_k} (2k-1)!!k! \times \binom{n}{c_1,\dots,c_k} \frac{\prod_{i=1}^{k+1} V_{c_i+d_i}}{V_n} \Big(\int_{a/\sqrt{n}}^{b/\sqrt{n}} 4\frac{\sinh^2(x/2)}{x}\,\mathrm{d}x\Big)^k \\
&\leq \sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}^o_k} (2k-1)!!k! \times \binom{n}{c_1,\dots,c_k} \frac{b^{2k}}{2^k} \frac{\prod_{i=1}^{k+1} V_{c_i+d_i}}{n^k V_n} \Big(1 + O\Big(\frac{kb^2}{n}\Big)\Big),
\end{aligned}$$

concluding the proof. ∎

We prove the following estimate for this volume summation.

Proposition 3.6. *For any $k \geq 1$,*

$$\mathbb{E}_n\big(N^{N,k}_{n,[a,b]}(X)\big) = O_k\Big(\frac{b^{2k}}{n}\Big).$$

Proof. We will show that

$$\sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}_k^o} \binom{n}{c_1,\ldots,c_k} \frac{\prod_{i=1}^{k+1} V_{c_i+d_i}}{n^k V_n} = O_k\Big(\frac{1}{n}\Big),$$

and then the result follows from Lemma 3.5. Note that

$$\begin{aligned}\sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}_k^o} &\binom{n}{c_1,\ldots,c_k} \frac{\prod_{i=1}^{k+1} V_{c_i+d_i}}{n^k V_n} \\ &\leq \sum_{t=1}^{k+1} \sum_{\substack{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}_k^o \\ c_t \geq n/(k+1)}} \frac{n! V_{c_t+d_t}}{c_t! n^k V_n} \prod_{\substack{i=1 \\ i \neq t}}^{k+1} \frac{V_{c_i+d_i}}{c_i!},\end{aligned} \tag{3.2}$$

since $\sum_{i=1}^{k+1} c_i = n$, so at least one c_i must be at least $n/(k+1)$. Now if

$$c_t \geq \frac{n}{k+1},$$

then for n sufficiently large, by Stirling's approximation for the factorial,

$$\begin{aligned}\frac{(c_t+d_t)!}{n^k c_t!} &\leq \frac{(c_t+d_t)^{c_t+d_t+1/2}}{n^k c_t^{c_t+1/2}} e^{-d_t} \\ &= e^{-d_t} \frac{c_t^{d_t}}{n^k}\Big(1+\frac{d_t}{c_t}\Big)^{c_t+d_t+1/2} \leq n^{d_t-k}\Big(1+\frac{k(k+1)}{n}\Big)^{k+1/2},\end{aligned}$$

using the fact that $n/(k+1) \leq c_t \leq n$ and $d_t \leq k$. Thus for n sufficiently large, Theorem 2.2 gives

$$\begin{aligned}\frac{n! V_{c_t+d_t}}{c_t! n^k V_n} &\leq C_1 \frac{n^{7/2}}{(c_t+d_t)^{7/2}} \frac{(c_t+d_t)!}{n^k c_t!}\Big(\frac{x_0}{2\pi^2}\Big)^{n-(c_t+d_t)} \\ &\leq C_2 n^{d_t-k}\Big(\frac{x_0}{2\pi^2}\Big)^{n-(c_t+d_t)},\end{aligned}$$

for some constants $C_1, C_2 > 0$ possibly dependent upon k. Next note that from Lemma 2.3, there exists a universal constant $\gamma > 0$ such that $V_{m+1} \leq \gamma(m-2)V_m$ for all $m \geq 3$. Thus, for $c_i \geq 4$ (since $d_i \geq 1$),

$$\frac{V_{c_i+d_i}}{c_i!} \leq \gamma^{d_i} \frac{(c_i+d_i-3)!}{c_i!(c_i-3)!} V_{c_i}.$$

But, by Stirling's approximation,

$$\begin{aligned}\frac{(c_i+d_i-3)!}{(c_i-3)!} &\leq e^{-d_i}\frac{(c_i+d_i-3)^{c_i+d_i-5/2}}{(c_i-3)^{c_i-5/2}}\\ &\leq e^{-d_i}c_i^{d_i}(1+d_i)^{d_i+1/2}\Big(1+\frac{d_i}{c_i-3}\Big)^{c_i-3}\leq c_i^{d_i}(1+k)^{d_i+1/2}.\end{aligned}$$

Thus,

$$\frac{V_{c_i+d_i}}{c_i!}\leq \gamma^{d_i}(1+k)^{d_i+1/2}\frac{c_i^{d_i}V_{c_i}}{c_i!}.$$

Moreover, if one sets $V_0=V_1=V_2=1$, then for $c_i\leq 3$, we also obtain

$$\begin{aligned}\frac{V_{c_i+d_i}}{c_i!} &\leq \gamma^{d_i}(c_i+d_i-3)!\frac{V_3}{c_i!}\\ &\leq \gamma^{d_i}d_i!\frac{V_{c_i}}{c_i!}\leq \gamma^{d_i}(1+k)^{d_i+1/2}\frac{\max\{1,c_i^{d_i}\}V_{c_i}}{c_i!}.\end{aligned}$$

Returning to the bound in equation (3.2), for some constant $C>0$ dependent only upon k, we have

$$\begin{aligned}&\sum_{t=1}^{k+1}\sum_{\substack{\{(c_i,d_i)\}_{i=1}^{k+1}\in\mathcal{A}_k^o\\ c_t\geq n/(k+1)}}\frac{n!V_{c_t+d_t}}{c_t!n^kV_n}\prod_{\substack{i=1\\ i\neq t}}^{k}\frac{V_{c_i+d_i}}{c_i!}\\ &\leq C\sum_{t=1}^{k+1}\sum_{\substack{\{(c_i,d_i)\}_{i=1}^{k+1}\in\mathcal{A}_k^o\\ c_t\geq n/(k+1)}} n^{d_t-k}\Big(\frac{x_0}{2\pi^2}\Big)^{n-(c_t+d_t)}\prod_{\substack{i=1\\ i\neq t}}^{k+1}\frac{\max\{1,c_i^{d_i}\}V_{c_i}}{c_i!}\\ &\leq C\sum_{t=1}^{k+1}\sum_{\substack{\{(c_i,d_i)\}_{i=1}^{k+1}\in\mathcal{A}_k^o\\ c_t\geq n/(k+1)}} n^{d_t-k+\sum_{j=1,\,j\neq t}^{k+1}(d_j-2)\mathbb{1}(d_j\geq 2)}\prod_{\substack{i=1\\ i\neq t}}^{k+1}\frac{\max\{1,c_i^2\}V_{c_i}x_0^{c_i}}{c_i!(2\pi^2)^{c_i}},\end{aligned}$$

where we use

$$\Big(\frac{x_0}{2\pi^2}\Big)^{n-c_t}=\Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1,\,i\neq t}^{k}c_i}$$

and absorb the constant $(x_0/2\pi^2)^{-d_t}$ into the constant C since d_t is bounded by k, and $c_i^{d_i}=c_i^2c_i^{d_i-2}\leq c_i^2n^{(d_i-2)\mathbb{1}(d_i\geq 2)}$. Note that because $d_j<k$ for each j, we have

$$d_t-k+\sum_{\substack{j=1\\ j\neq t}}^{k+1}(d_j-2)\mathbb{1}(d_j\geq 2)\leq -1.$$

Indeed, let $\mathcal{I} \subseteq \{j \in \{1, \ldots, k+1\} \setminus \{t\} : d_j = 1\}$. Then,

$$\begin{aligned} d_t - k + \sum_{\substack{j=1 \\ j \neq t}}^{k+1} (d_j - 2)\mathbb{1}(d_j \geq 2) &= -k + \sum_{j=1}^{k+1} d_j - \sum_{j \in \mathcal{I}} d_j - \sum_{\substack{j \notin \mathcal{I} \\ j \neq t}} 2 \\ &= |\mathcal{I}| - k, \end{aligned}$$

but $|\mathcal{I}| \leq k-1$ since if $d_j = 1$ for every $j \neq t$, then $d_t = k$, which is a contradiction to a collection being in $\mathcal{A}_k$ and hence the result follows. Next observe that in an ordered collection $\{(c_i, d_i)\}_{i=1}^{k+1}$, the tth pair is determined entirely by the other pairs since

$$c_t = n - \sum_{\substack{i=1 \\ i \neq t}}^{k+1} c_i \quad \text{and} \quad d_t = 2k - \sum_{\substack{i=1 \\ i \neq t}}^{k+1} d_i.$$

Thus, we see that $\mathcal{A}_k^o$ is contained in the set

$$\Bigg\{ \{(c_i, d_i)\}_{i=1}^{k+1} : c_i \in \mathbb{N},\ 1 \leqslant d_i \leqslant k, \\ c_t = \max\Bigg\{0, n - \sum_{i=1, i \neq t}^{k+1} c_i\Bigg\},\ d_t = \max\Bigg\{0, 2k - \sum_{i=1, i\neq t}^{k+1} d_i\Bigg\}\Bigg\}.$$

Thus,

$$\begin{aligned} &C \sum_{t=1}^{k+1} \sum_{\substack{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}_k^o \\ c_t \geq n/(k+1)}} n^{d_t - k + \sum_{j=1,\, j \neq t}^{k+1} (d_j - 2)\mathbb{1}(d_j \geq 2)} \prod_{\substack{i=1 \\ i \neq t}}^{k+1} \frac{\max\{1, c_i^2\} V_{c_i} x_0^{c_t}}{c_i!(2\pi^2)^{c_i}} \\ &\leq \frac{C}{n} \sum_{t=1}^{k+1} \sum_{\{(c_i,d_i)\}_{i=1}^{k+1} \in \mathcal{A}_k^o} \prod_{\substack{i=1 \\ i \neq t}}^{k+1} \frac{\max\{1, c_i^2\} V_{c_i} x_0^{c_i}}{c_i!(2\pi^2)^{c_i}} \\ &\leq \frac{C}{n} \sum_{t=1}^{k+1} \underbrace{\sum_{d_1=1}^{k} \cdots \sum_{d_{k+1}=1}^{k}}_{\text{no } d_t \text{ term}} \underbrace{\sum_{c_i=0}^{\infty} \cdots \sum_{c_{k+1}=0}^{\infty}}_{\text{no } c_t \text{ term}} \prod_{\substack{i=1 \\ i \neq t}}^{k+1} \frac{\max\{1, c_i^2\} V_{c_i} x_0^{c_i}}{c_i!(2\pi^2)^{c_i}} \\ &= \frac{B}{n} \left(1 + \sum_{c=1}^{\infty} \frac{c^2 V_c x_0^c}{c!(2\pi^2)^c}\right)^k, \end{aligned}$$

where $B > 0$ is some constant dependent only upon k. The latter summation then converges due to Theorem 2.2, and hence we obtain the desired result. ■

3.2. Contribution of unnested curves

We now compute the contribution of $N^{U,k}_{n,[a,b]}(X)$ to the expectation. We start with an enumeration system of the multicurves as was done for the nested multicurves. For $k \geq 1$, we let $\mathcal{B}_k$ be the collection of ordered tuples $((c_i, d_i, \{a^i_1, \dots, a^i_{c_i}\}))_{i=1}^{k+1}$ satisfying

(1) $\sum_{i=1}^{k+1} c_i = n$ and $c_i \geq 2$ for $1 \leq i \leq k$;

(2) $d_1 = \cdots = d_k = 1$ and $d_{k+1} = k$;

(3) $\{a^i_1, \dots, a^i_{c_i}\} \subseteq \{1, \dots, n\}$ are pairwise disjoint and their union over all i is $\{1, \dots, n\}$;

(4) $c_i + d_i \geq 3$.

Lemma 3.7. *For $k \geq 2$, there is a bijection between mapping class group orbits of unnested multicurves and $\mathcal{B}_k$.*

Proof. We define a bijective mapping in the following way. Cutting Σ_n along a representative $\Gamma = (\gamma_1, \dots, \gamma_k)$ of a mapping class group orbit of an unnested multicurve gives a decomposition of Σ_n as

$$\Sigma_n \setminus \Gamma = \left(\bigsqcup_{i=1}^{k} \Sigma_{c_i,1} \right) \sqcup \Sigma_{c_{k+1},k},$$

where there are labels $\{a^i_1, \dots, a^i_{c_i}\}$ on the c_i punctures of Σ_{c_i,d_i} inherited from the labeling of Σ_n, with the topological decomposition being independent of the orbit representative chosen by Remark 3.2. Indeed, cutting Σ_n along γ_i separates Σ_n into two subsurfaces $\Sigma_{c_i,1} \sqcup \Sigma_{n-c_i,1}$ with one of these subsurfaces containing no other component of Γ in its interior, which we take to be $\Sigma_{c_i,1}$. The cutting of the remainder of the surfaces along any other curve component leaves $\Sigma_{c_i,1}$unchanged, and so repeating this procedure recursively for each of the curve components gives the stated decomposition. So, for $1 \leq i \leq k$, c_i is the number of cusps on the subsurface of X obtained from cutting along γ_i that does not contain any other curve component of Γ. The multicurve is then identified with the tuple $((c_i, d_i, \{a^i_1, \dots, a^i_{c_i}\}))_{i=1}^{k+1}$.

Surjectivity of this identification is clear. For injectivity, we note that if $\mathcal{O}(\Gamma)$ and $\mathcal{O}(\Gamma')$ are two unnested multicurve mapping class orbits with the same decomposition, then taking representatives

$$\Gamma = (\gamma_1, \dots, \gamma_k) \in \mathcal{O}(\Gamma) \quad \text{and} \quad \Gamma' = (\gamma'_1, \dots, \gamma'_k) \in \mathcal{O}(\Gamma'),$$

we see that for each $1 \leqslant i \leqslant k$, $\Sigma_n \setminus \gamma_i$ and $\Sigma_n \setminus \gamma'_i$ are puncture label preserving homeomorphic by construction of the mapping. This means that they are in the same mapping class group orbit and hence the multicurves are also using Remark 3.2. ■

In the case of $k=1$, we instead have the following.

Lemma 3.8. *There is a* 1-2 *correspondence between mapping class group orbits of unnested multicurves and* $\mathcal{B}_1$.

Proof. We construct the association as in the proof of Lemma 3.7, the difference is that after cutting the labeled surface, there is no canonical choice for the ordered tuple that is obtained. We thus associate both choices with the single curve, namely

$$\big((c_1,1,\{a_1^1,\dots,a_{c_1}^1\}),(c_2,1,\{a_1^2,\dots,a_{c_2}^2\})\big)$$

and

$$\big((c_2,1,\{a_1^2,\dots,a_{c_2}^2\}),(c_1,1,\{a_1^1,\dots,a_{c_1}^1\})\big). \qquad \blacksquare$$

Using this combinatorial interpretation of the mapping class group orbits, we obtain the following estimate. As before, we shall use the notation $V_m(x_1,\dots,x_m):=V_{0,m}(x_1,\dots,x_m)$ and $V_m:=V_{0,m}$.

Lemma 3.9. *We have the following estimates:*

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,1}(X)\big)=\sum_{c=2}^{\lfloor n/2\rfloor}\binom{n}{c}\Big(\frac{b^2-a^2}{2}\Big)\frac{1}{n}\frac{V_{c+1}V_{n-c+1}}{V_n}\Big(1+O\Big(\frac{b^2}{n}\Big)\Big),$$

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,2}(X)\big)=\sum_{\substack{(c_1,c_2)\in\mathbb{N}^2\\ c_1+c_2\le n-1\\ c_i\ge 2}}\binom{n}{c_1,c_2}\Big(\frac{b^2-a^2}{2}\Big)^2 \times\frac{1}{n^2}\frac{V_{c_1+1}V_{c_2+1}V_{n-c_1-c_2+2}}{V_n}\Big(1+O\Big(\frac{b^2}{n}\Big)\Big),$$

and for $k\ge 3$,

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,k}(X)\big)=\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ \sum_{i=1}^k c_i\le n\\ c_i\ge 2}}\binom{n}{c_1,\dots,c_k}\Big(\frac{b^2-a^2}{2}\Big)^k \times\frac{1}{n^k}\frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{V_n}\Big(1+O_k\Big(\frac{b^2}{n}\Big)\Big).$$

Proof. Notice that by Lemmas 3.7 and 3.8, we have

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,k}(X)\big)=C_k\sum_{((c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\}))_{i=1}^{k+1}\in\mathcal{B}_k}\mathbb{E}_n \times\sum_{\Gamma=(\gamma_1,\dots,\gamma_k)\in\mathcal{O}(\Gamma)}\mathbb{1}\Big(\frac{a}{\sqrt{n}}\le\ell_X(\gamma_1),\dots,\ell_X(\gamma_k)\le\frac{b}{\sqrt{n}}\Big),$$

where the mapping class group orbit $\mathcal{O}(\Gamma)$ is the unique such orbit associated with a given ordered collection $((c_i, d_i, \{a_1^i, \dots, a_{c_i}^i\}))_{i=1}^{k+1}$, $C_1 = 1/2$ and $C_k = 1$ for $k \geq 2$. By Theorem 2.1, we have

$$\begin{aligned}
\mathbb{E}_n & \sum_{\Gamma=(\gamma_1,\dots,\gamma_k)\in\mathcal{O}(\Gamma)} \mathbb{1}\Big(\frac{a}{\sqrt{n}} \leq \ell_X(\gamma_1), \dots, \ell_X(\gamma_k) \leq \frac{b}{\sqrt{n}}\Big) \\
&= \frac{1}{V_n} \int_{a/\sqrt{n}}^{b/\sqrt{n}} \cdots \int_{a/\sqrt{n}}^{b/\sqrt{n}} V_{c_{k+1}+k}(\mathbf{0}_{c_{k+1}}, x_1, \dots, x_k) \\
&\qquad \times \prod_{i=1}^{k} x_i V_{c_i+1}(\mathbf{0}_{c_i}, x_i)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k.
\end{aligned}$$

By Lemma 2.4, we have

$$V_{c_i+1} \leqslant V_{c_i+1}(\mathbf{0}_{c_i}, x_i) \leqslant V_{c_i+1} \frac{\sinh(x_i/2)}{(x_i/2)},$$

$$V_{c_{k+1}+k} \leqslant V_{c_{k+1}+k}(\mathbf{0}_{c_{k+1}}, x_1, \dots, x_k) \leqslant V_{c_{k+1}+k} \prod_{i=1}^{k} \frac{\sinh(x_i/2)}{(x_i/2)}.$$

It then follows from the Taylor expansion

$$4\frac{\sinh^2(x_i/2)}{x_i} = x_i + O(x_i^3) = x_i + O\Big(\frac{b^3}{n^{3/2}}\Big),$$

that when $x_i \leqslant b/\sqrt{n}$, we have

$$V_{c_{k+1}+k}(\mathbf{0}_{c_{k+1}}, x_1, \dots, x_k) \prod_{i=1}^{k} x_i V_{c_i+1}(\mathbf{0}_{c_i}, x_i) = \Big(\prod_{i=1}^{k} x_i\Big)\Big(1 + O\Big(\frac{b^2}{n}\Big)\Big)^k.$$

We then see that

$$\begin{aligned}
\mathbb{E}_n & \sum_{\Gamma=(\gamma_1,\dots,\gamma_k)\in\mathcal{O}(\Gamma)} \mathbb{1}\Big(\frac{a}{\sqrt{n}} \leq \ell_X(\gamma_1), \dots, \ell_X(\gamma_k) \leq \frac{b}{\sqrt{n}}\Big) \\
&= \Big(\frac{b^2 - a^2}{2}\Big)^k \frac{1}{n^k} \frac{V_{c_1+1} \cdots V_{c_k+1} V_{c_{k+1}+k}}{V_n} \Big(1 + O_k\Big(\frac{b^2}{n}\Big)\Big).
\end{aligned}$$

By definition of the collection $\mathcal{B}_k$ we have $c_{k+1} = n - \sum_{i=1}^k c_i$, and so

$$\begin{aligned}
\mathbb{E}_n\big(N_{n,[a,b]}^{U,k}(X)\big) = C_k & \sum_{((c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\}))_{i=1}^{k+1}\in\mathcal{B}_k} \Big(\frac{b^2 - a^2}{2}\Big)^k \\
& \times \frac{1}{n^k} \frac{V_{c_1+1} \cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i+k}}{V_n} \Big(1 + O_k\Big(\frac{b^2}{n}\Big)\Big),
\end{aligned}$$

where the summand is independent of the puncture labeling. When the collection of tuples $((c_i,1))_{i=1}^k$ are fixed, there is a single choice of value for c_{k+1} and $\binom{n}{c_1,\dots,c_k}$ choices of assigning the cusp labels to the subsurfaces according to the definition of $\mathcal{B}_k$. In the case of $k \geq 3$ the summation is thus equal to

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,k}(X)\big) = \sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ \sum_{i=1}^k c_i \leq n\\ c_i\geq 2}} \binom{n}{c_1,\dots,c_k}\Big(\frac{b^2-a^2}{2}\Big)^k \times \frac{1}{n^k}\frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{V_n}\Big(1+O_k\Big(\frac{b^2}{n}\Big)\Big).$$

For $k=2$, the requirement on the indices that $c_{k+1}+2\geq 3$, means the indexing runs over the replaced condition $c_1+c_2\leq n-1$. Similarly for $k=1$, by Lemma 3.8,

$$\begin{aligned}\mathbb{E}_n\big(N_{n,[a,b]}^{U}(X)\big) &= \frac{1}{2}\sum_{((c_i,d_i,\{a_1^i,\dots,a_{c_i}^i\}))_{i=1}^2\in\mathcal{B}_1}\binom{n}{c_1}\Big(\frac{b^2-a^2}{2}\Big) \times \frac{1}{n}\frac{V_{c_1+1}V_{n-c_1+1}}{V_n}\Big(1+O\Big(\frac{b^2}{n}\Big)\Big)\\ &= \sum_{c=2}^{\lfloor n/2\rfloor}\binom{n}{c}\Big(\frac{b^2-a^2}{2}\Big)\frac{1}{n}\frac{V_{c+1}V_{n-c+1}}{V_n}\Big(1+O\Big(\frac{b^2}{n}\Big)\Big). \qquad \blacksquare\end{aligned}$$

We now determine the leading order of this summand.

Proposition 3.10. *Define*

$$\alpha = \sum_{i=2}^{\infty}\frac{V_{i+1}}{i!}\Big(\frac{x_0}{2\pi^2}\Big)^{i-1},$$

then

$$\mathbb{E}_n\big(N_{n,[a,b]}^{U,k}(X)\big) = \Big(\frac{b^2-a^2}{2}\Big)^k\alpha^k\Big(1+O_k\Big(\frac{1}{n^{1/2}}\Big)\Big).$$

Proof. We first consider the case when $k\geq 3$, the result is similar for $k=1$ and 2. By Lemma 3.9, it suffices to show that

$$\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ \sum_{i=1}^k c_i \leq n\\ c_i\geq 2}} \binom{n}{c_1,\dots,c_k}\frac{1}{n^k}\frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{V_n} = \alpha^k + O_k\Big(\frac{1}{\sqrt{n}}\Big).$$

Observe first that

$$
\begin{aligned}
\alpha^k &= \sum_{c_1=2}^{\infty} \cdots \sum_{c_k=2}^{\infty} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \\
&= \sum_{c_1=2}^{\lfloor n^{1/2} \rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2} \rfloor} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \\
&\qquad + \sum_{\substack{(c_1,\ldots,c_k)\in\mathbb{N}^k \\ \exists i : c_i \geq \lceil n^{1/2} \rceil \\ c_j \geq 2}} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \\
&\leq \sum_{c_1=2}^{\lfloor n^{1/2} \rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2} \rfloor} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \\
&\qquad + k \sum_{c_1=\lceil n^{1/2} \rceil}^{\infty} \sum_{c_2=2}^{\infty} \cdots \sum_{c_k=2}^{\infty} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Bigg| \sum_{\substack{(c_1,\ldots,c_k)\in\mathbb{N}^k \\ \sum_{i=1}^k c_i \leq n \\ c_i \geq 2}} \binom{n}{c_1,\ldots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i + k}}{V_n} - \alpha^k \Bigg| \\
&\quad \leqslant \Bigg| \sum_{\substack{(c_1,\ldots,c_k)\in\mathbb{N}^k \\ 2 \leq c_i \leq n^{1/2}}} \binom{n}{c_1,\ldots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i + k}}{V_n} - \alpha^k \Bigg| \\
&\qquad + \Bigg| \sum_{\substack{(c_1,\ldots,c_k)\in\mathbb{N}^k \\ \exists i : c_i \geq n^{1/2}, c_i \geq 2 \\ \sum_{i=1}^k c_i \leq n}} \binom{n}{c_1,\ldots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i + k}}{V_n} \Bigg| \\
&\quad \leqslant \underbrace{\Bigg| \sum_{\substack{(c_1,\ldots,c_k)\in\mathbb{N}^k \\ 2 \leq c_i \leq n^{1/2}}} \binom{n}{c_1,\ldots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i + k}}{V_n} - \sum_{c_1=2}^{\lfloor n^{1/2} \rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2} \rfloor} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \Bigg|}_{\text{(a)}}
\end{aligned}
$$

$$
\underbrace{+\,k \sum_{c_1=\lceil n^{1/2}\rceil}^{\infty} \sum_{c_2=2}^{\infty} \cdots \sum_{c_k=2}^{\infty} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!}}_{\text{(b)}}
$$

$$
\underbrace{+ \sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k \\ \exists i : c_i \geq n^{1/2}, c_i \geq 2 \\ \sum_{i=1}^{k} c_i \leq n}} \binom{n}{c_1,\dots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^{k} c_i + k}}{V_n}}_{\text{(c)}}.
$$

We first bound the contribution of (a). We calculate that

$$
\begin{aligned}
&\Bigg| \sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k \\ 2\leq c_i \leq n^{1/2}}} \binom{n}{c_1,\dots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^{k} c_i + k}}{V_n} \\
&\qquad - \sum_{c_1=2}^{\lfloor n^{1/2}\rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2}\rfloor} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \prod_{i=1}^{k} \frac{V_{c_1+1}}{c_i!} \Bigg| \\
&= \Bigg| \sum_{c_1=2}^{\lfloor n^{1/2}\rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2}\rfloor} \Bigg(\frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^{k} c_i + k}}{c_1!\cdots c_k!(n-\sum_{i=1}^{k} c_i)!} \frac{n!}{n^k V_n} \\
&\qquad - \frac{V_{c_1+1}\cdots V_{c_k+1}}{c_1!\cdots c_k!} \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \Bigg) \Bigg| \\
&\leqslant \sum_{c_1=2}^{\lfloor n^{1/2}\rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2}\rfloor} \frac{V_{c_1+1}\cdots V_{c_k+1}}{c_1!\cdots c_k!} \left| \frac{n! V_{n-\sum_{i=1}^{k} c_i + k}}{n^k V_n (n-\sum_{i=1}^{k} c_i)!} - \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \right|.
\end{aligned}
$$

Since each $c_i \leqslant n^{1/2}$, we can calculate using the volume asymptotics of Theorem 2.2 that

$$
\begin{aligned}
&\left| \frac{n! V_{n-\sum_{i=1}^{k} c_i + k}}{n^k V_n (n-\sum_{i=1}^{k} c_i)!} - \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \right| \\
&= \Bigg| \frac{n^{7/2}(n-\sum_{i=1}^{k} c_i + k)! x_0^n (2\pi^2)^{n-\sum_{i=1}^{k} c_i + k - 3}}{n^k (n-\sum_{i=1}^{k} c_i + k)^{7/2} (n-\sum_{i=1}^{k} c_i)! x_0^{n-\sum_{i=1}^{k} c_i + k} (2\pi^2)^{n-3}} \\
&\qquad \times \left(\frac{B_0 + O(1/(n-\sum_{i=1}^{k} c_i + k))}{B_0 + O(1/n)} \right) - \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \Bigg| \\
&= \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^{k}(c_i-1)} \left| \frac{n^{7/2}}{(n-\sum_{i=1}^{k} c_i + k)^{7/2}} \frac{(n-\sum_{i=1}^{k} c_i + k)!}{n^k (n-\sum_{i=1}^{k} c_i)!} \left(1 + O\left(\frac{1}{n}\right)\right) - 1 \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)} \bigg| \Big(1+\frac{\sum_{i=1}^k c_i - k}{n-\sum_{i=1}^k c_i + k}\Big)^{7/2} \Big(1-\frac{\sum_{i=1}^k c_i - k}{n}\Big) \cdots \\
&\qquad \cdots \Big(1-\frac{\sum_{i=1}^k c_i - 1}{n}\Big)\Big(1+O\Big(\frac{1}{n}\Big)\Big) - 1 \bigg| \\
&= \Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)} \bigg| \Big(1+O\Big(\frac{k^2}{n^{1/2}}\Big)\Big) - 1 \bigg| = O\Big(\frac{k^2}{n^{1/2}}\Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)}\Big).
\end{aligned}
$$

Thus we can control the contribution of (a) by

$$
\begin{aligned}
&\sum_{c_1=2}^{\lfloor n^{1/2}\rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2}\rfloor} \frac{V_{c_1+1}\cdots V_{c_k+1}}{c_1!\cdots c_k!} \bigg| \frac{n V_{n-\sum_{i=1}^k c_i + k}}{n^k V_n (n-\sum_{i=1}^k c_i)!} - \Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)} \bigg| \\
&\qquad = O\bigg(\frac{k^2}{n^{1/2}} \sum_{c_1=2}^{\lfloor n^{1/2}\rfloor} \cdots \sum_{c_k=2}^{\lfloor n^{1/2}\rfloor} \frac{V_{c_1+1}\cdots V_{c_k+1}}{c_1!\cdots c_k!} \Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)}\bigg) \\
&\qquad = O\Big(\frac{k^2\alpha^k}{\sqrt{n}}\Big).
\end{aligned}
$$

To bound the contribution of (b), we notice that

$$
\sum_{c_1=\lceil n^{1/2}\rceil}^{\infty} \sum_{c_2=2}^{\infty} \cdots \sum_{c_k=2}^{\infty} \Big(\frac{x_0}{2\pi^2}\Big)^{\sum_{i=1}^k (c_i-1)} \prod_{i=1}^{k} \frac{V_{c_i+1}}{c_i!} = \bigg(\sum_{c=\lceil n^{1/2}\rceil}^{\infty} \frac{V_{c+1} x_0^{c-1}}{c!(2\pi^2)^{c-1}}\bigg) \alpha^{k-1}. \tag{3.3}
$$

Applying volume estimates of Theorem 2.2, we can control the summation by

$$
\begin{aligned}
\sum_{c=\lceil n^{1/2}\rceil}^{\infty} \frac{V_{c+1} x_0^{c-1}}{c!(2\pi^2)^{c-1}} &= \frac{x_0^{-2}}{2\pi^2}\Big(B_0 + O\Big(\frac{1}{n^{1/2}}\Big)\Big) \sum_{c=\lceil n^{1/2}\rceil}^{\infty} \frac{c+1}{(c+2)^{7/2}} \\
&\le \frac{x_0^{-2}}{2\pi^2}\Big(B_0 + O\Big(\frac{1}{n^{1/2}}\Big)\Big) \frac{1}{n^{3/4}} \qquad (3.4) \\
&= O\Big(\frac{1}{n^{3/4}}\Big), \qquad (3.5)
\end{aligned}
$$

which together with (3.3) shows that (b) $= O(\alpha^{k-1}/n^{3/4})$. Finally, we control the contribution of (c). First we look at

$$
\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k \\ \exists i : c_j \ge n^{1/2} \\ \sum_{i=1}^k c_i \le n/2,\, c_i \ge 2}} \binom{n}{c_1,\dots,c_k} \frac{1}{n^k} \frac{V_{c_1+1}\cdots V_{c_k+1} V_{n-\sum_{i=1}^k c_i + k}}{V_n}.
$$

If $\sum_{i=1}^k c_i \leqslant n/2$, then by the volume estimates of Theorem 2.2, we have

$$\frac{V_{n-\sum_{i=1}^k c_i+k}}{(n-\sum_{i=1}^k c_i)!}\frac{n!}{n^k V_n} = \frac{(n-\sum_{i=1}^k c_i+k)!}{n^k(n-\sum_{i=1}^k c_i)!}\frac{(n+1)^{7/2}}{(n-\sum_{i=1}^k c_i+k+1)^{7/2}} \times \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k(c_i-1)}\left(1+O\left(\frac{1}{n}\right)\right) \leq \left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=1}^k(c_i-1)}\left(1+O\left(\frac{1}{n}\right)\right).$$

Then

$$\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ \exists i:c_j\geq n^{1/2}\\ \sum_{i=1}^k c_i\leqslant n/2,\, c_i\geq 2}} \binom{n}{c_1,\dots,c_k}\frac{1}{n^k}\frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{V_n}$$
$$\leq \sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ \exists i:c_j\geq n^{1/2}\\ \sum_{i=1}^k c_i\leqslant n/2,\, c_i\geq 2}} \frac{V_{c_1+1}(x_0/2\pi^2)^{c_1-1}\cdots V_{c_k+1}(x_0/2\pi^2)^{c_k-1}}{c_1!\cdots c_k!}\left(1+O\left(\frac{1}{n}\right)\right)$$
$$\leq k\left(\sum_{c=\lceil n^{1/2}\rceil}^{\infty}\frac{V_{c_1+1}x_0^{c-1}}{c!(2\pi^2)^{c-1}}\right)\alpha^{k-1}\left(1+O\left(\frac{1}{n}\right)\right) = O\left(\frac{k\alpha^{k-1}}{n^{3/4}}\right),$$

where the last bound follows from (3.5). We now consider the contribution from when $\sum_{i=1}^k c_i \geq n/2$, that is, of the term

$$\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ n/2\leqslant\sum_{i=1}^k c_i\leqslant n,\, c_i\geq 2}} \binom{n}{c_1,\dots,c_k}\frac{1}{n^k}\frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{V_n}.$$

Since $\sum_{i=1}^k c_i \geqslant n/2$, there exists some ℓ such that $c_\ell \geqslant n/2k$, and thus it follows that

$$\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ n/2\leqslant\sum_{i=1}^k c_i\leqslant n,\, c_i\geq 2}} \frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{c_1!\cdots c_k!(n-\sum_{i=1}^k c_i)!}\frac{n!}{n^k V_n}$$
$$\leq k\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ n/2\leqslant\sum_{i=1}^k c_i\leqslant n\\ c_1\geq\lfloor n/2k\rfloor,\, c_i\geq 2}} \frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{c_1!\cdots c_k!(n-\sum_{i=1}^k c_i)!}\frac{n!}{n^k V_n}$$

$$= k \sum_{\substack{(c_1,\dots,c_k,c_{k+1})\in\mathbb{N}^{k+1}\\ \sum_{i=1}^{k+1} c_i=n,\, c_1\geq\lfloor n/2k\rfloor\\ 0\leq c_{k+1}\leq n/2,\, c_i\geq 2,\ i<k+1}} \frac{V_{c_1+1}\cdots V_{c_k+1}V_{c_{k+1}+k}}{c_1!\cdots c_k!c_{k+1}!}\frac{n!}{n^k V_n}.$$

By the volume estimate of Theorem 2.2, we then obtain

$$\begin{aligned}\frac{V_{c_1+1}}{c_1!}\frac{n!}{n^k V_n} &= \frac{(n+1)^{7/2}}{(c_1+2)^{5/2}}\frac{1}{n^k}\left(\frac{x_0}{2\pi^2}\right)^{n-c_1-1}\left(1+O\left(\frac{k}{n}\right)\right)\\ &= \frac{1}{n^{k-1}}\left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=2}^{k+1} c_i-1}\left(1+O\left(\frac{k}{n}\right)\right).\end{aligned}$$

Moreover, by Lemma 2.3, there exists a constant $\gamma > 0$ such that

$$V_{c_{k+1}+k} \leqslant \gamma^{k-3}(c_{k+1}+k)^{k-3}V_{c_{k+1}+3}.$$

Hence,

$$\begin{aligned}&k \sum_{\substack{(c_1,\dots,c_k,c_{k+1})\in\mathbb{N}^{k+1}\\ \sum_{i=1}^{k+1} c_i=n,\, c_1\geq\lfloor n/2k\rfloor,\\ 0\leq c_{k+1}\leq n/2,\, c_i\geq 2,\ i<k+1}} \frac{V_{c_1+1}\cdots V_{c_k+1}V_{c_{k+1}+k}}{c_1!\cdots c_k!c_{k+1}!}\frac{n!}{n^k V_n}\\
&\leq k \sum_{\substack{(c_2,\dots,c_k,c_{k+1})\in\mathbb{N}^{k}\\ \sum_{i=2}^{k+1} c_i\leq n-\lfloor n/2k\rfloor,\\ 0\leq c_{k+1}\leq n/2,\, c_i\geq 2,\, i<k+1}} \frac{V_{c_2+1}\cdots V_{c_{k+1}+k}}{c_2!\cdots c_{k+1}!}\left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=2}^{k+1} c_i-1}\frac{1}{n^{k-1}}\left(1+O\left(\frac{k}{n}\right)\right)\\
&\leq \gamma^{k-3}k \sum_{\substack{(c_2,\dots,c_k,c_{k+1})\in\mathbb{N}^{k}\\ \sum_{i=2}^{k+1} c_i\leq n-\lfloor n/2k\rfloor\\ 0\leq c_{k+1}\leq n/2,\, c_i\geq 2,\ i<k+1}} \frac{V_{c_2+1}\cdots V_{c_{k+1}+3}}{c_2!\cdots (c_{k+1}+1)!}\left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=2}^{k+1} c_i-1}\\
&\qquad\qquad\times\frac{(c_{k+1}+k)^{k-2}}{n^{k-1}}\left(1+O\left(\frac{k}{n}\right)\right).\end{aligned}$$

Since $c_{k+1} \leq n/2$, for n sufficiently large (i.e. $k \leq n/2$), we have

$$\frac{(c_{k+1}+k)^{k-2}}{n^{k-1}} \leq \frac{1}{n},$$

and thus

$$\sum_{\substack{(c_1,\dots,c_k)\in\mathbb{N}^k\\ n/2\leqslant\sum_{i=1}^k c_i\leqslant n,\, c_i\geq 2}} \frac{V_{c_1+1}\cdots V_{c_k+1}V_{n-\sum_{i=1}^k c_i+k}}{c_1!\cdots c_k!(n-\sum_{i=1}^k c_i)!}\frac{n!}{n^k V_n}$$

$$\leq \frac{\gamma^{k-3}k}{n} \sum_{\substack{(c_2,\ldots,c_k,c_{k+1})\in\mathbb{N}^k \\ \sum_{i=2}^{k+1} c_i \leq n-\lfloor n/2k \rfloor \\ 0\leq c_{k+1}\leq n/2,\, c_i\geq 2,\, i<k+1}} \frac{V_{c_2+1}\cdots V_{c_{k+1}+3}}{c_2!\cdots(c_{k+1}+1)!}\left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=2}^{k+1} c_i - 1}\left(1+O\left(\frac{k}{n}\right)\right)$$

$$\leq \frac{\gamma^{k-3}k}{n}\left(\sum_{c_{k+1}=0}^{\lfloor n/2\rfloor} \frac{V_{c_{k+1}+3}}{(c_{k+1}+1)!}\left(\frac{x_0}{2\pi^2}\right)^{c_{k+1}-1}\right)$$

$$\times\left(\sum_{c_2=2}^{\infty}\cdots\sum_{c_k=2}^{\infty}\left(\frac{x_0}{2\pi^2}\right)^{\sum_{i=2}^{k} c_i}\prod_{i=2}^{k}\frac{V_{c_i+1}}{c_i!}\right)\left(1+O\left(\frac{k}{n}\right)\right)$$

$$\leq \frac{\gamma^{k-3}k\alpha^{k-1}}{n}\left(\frac{x_0}{2\pi^2}\right)^{k-1}\left(\sum_{c_{k+1}=0}^{\infty}\frac{V_{c_{k+1}+3}}{(c_{k+1}+1)!}\left(\frac{x_0}{2\pi^2}\right)^{c_{k+1}-1}\right)\left(1+O\left(\frac{k}{n}\right)\right).$$

This latter summation converges since for sufficiently large c_{k+1}, we have the asymptotic from Theorem 2.2 asserting that

$$\frac{V_{c_{k+1}+3}}{(c_{k+1}+1)!}\left(\frac{x_0}{2\pi^2}\right)^{c_{k+1}-1} = 2\pi^2 x_0^{-4}\frac{1}{c_{k+1}^{3/2}}\left(1+O\left(\frac{1}{c_{k+1}}\right)\right).$$

In summary, we have that

$$\text{(a)} = O\left(\frac{k^2\alpha^k}{\sqrt{n}}\right),$$

$$\text{(b)} = O\left(\frac{\alpha^{k-1}}{n^{3/4}}\right),$$

$$\text{(c)} = O\left(\frac{\gamma^{k-3}k\alpha^{k-1}}{n} + \frac{k\alpha^{k-1}}{n^{3/4}}\right),$$

and hence the result follows for $k \geq 3$.

For $k = 1$, by Lemma 3.9 it is sufficient to show

$$\sum_{c=2}^{\lfloor n/2\rfloor}\binom{n}{c}\frac{1}{n}\frac{V_{c+1}V_{n-c+1}}{V_n} = \alpha + O\left(\frac{1}{\sqrt{n}}\right).$$

As before we observe that

$$\left|\sum_{c=2}^{\lfloor n/2\rfloor}\binom{n}{c}\frac{1}{n}\frac{V_{c+1}V_{n-c+1}}{V_n} - \alpha\right| \leq \underbrace{\left|\sum_{c=2}^{\lfloor\sqrt{n}\rfloor}\binom{n}{c}\frac{1}{n}\frac{V_{c+1}V_{n-c+1}}{V_n} - \frac{V_{c+1}}{c!}\left(\frac{x_0}{2\pi^2}\right)^{c-1}\right|}_{\text{(a)}}$$

$$+ \underbrace{\sum_{c=\lfloor\sqrt{n}\rfloor+1}^{\infty}\frac{V_{c+1}}{c!}\left(\frac{x_0}{2\pi^2}\right)^{c-1}}_{\text{(b)}} + \underbrace{\sum_{c=\lfloor\sqrt{n}\rfloor+1}^{\lfloor n/2\rfloor}\frac{1}{n}\frac{V_{c+1}V_{n-c+1}n!}{V_n(n-c)!c!}}_{\text{(c)}}.$$

Identically to the analysis of the terms (a) and (b) for $k \geq 3$, we obtain

$$(\mathrm{a}) = O\Big(\frac{1}{\sqrt{n}}\Big),$$
$$(\mathrm{b}) = O\Big(\frac{1}{n^{3/4}}\Big).$$

The bound for (c) is simpler than before since $c \leq n/2$ it follows that $n-c+1 \geq n/2$, and so

$$\frac{n!V_{n-c+1}}{(n-c)!nV_n} = \Big(\frac{x_0}{2\pi^2}\Big)^{c-1}\Big(1 + O\Big(\frac{1}{n}\Big)\Big).$$

Thus by Theorem 2.2,

$$\sum_{c=\lfloor\sqrt{n}\rfloor+1}^{\lfloor n/2\rfloor} \frac{1}{n}\frac{V_{c+1}V_{n-c+1}n!}{V_n(n-c)!c!} = \sum_{c=\lfloor\sqrt{n}\rfloor+1}^{\lfloor n/2\rfloor} \frac{V_{c+1}}{c!}\Big(\frac{x_0}{2\pi^2}\Big)^{c-1}\Big(1 + O\Big(\frac{1}{n}\Big)\Big) = O\Big(\frac{1}{n^{3/4}}\Big).$$

This leaves the case for $k = 2$, where again by Lemma 3.9, it is sufficient to show

$$\sum_{\substack{(c_1,c_2)\in\mathbb{N}^2\\ c_1+c_2\leq n-1,\, c_i\geq 2}} \binom{n}{c_1, c_2}\frac{1}{n^2}\frac{V_{c_1+1}V_{c_2+1}V_{n-c_1-c_2+2}}{V_n} = \alpha^2 + O\Big(\frac{1}{\sqrt{n}}\Big).$$

We observe that

$$\Bigg|\sum_{\substack{(c_1,c_2)\in\mathbb{N}^2\\ c_1+c_2\leq n-1,\, c_i\geq 2}} \binom{n}{c_1, c_2}\frac{1}{n^2}\frac{V_{c_1+1}V_{c_2+1}V_{n-c_1-c_2+2}}{V_n} - \alpha^2\Bigg|$$
$$\leq \underbrace{\Bigg|\sum_{c_1=2}^{\lfloor\sqrt{n}\rfloor}\sum_{c_2=2}^{\lfloor\sqrt{n}\rfloor}\binom{n}{c_1, c_2}\frac{1}{n^2}\frac{V_{c_1+1}V_{c_2+1}V_{n-c_1-c_2+2}}{V_n} - \Big(\frac{x_0}{2\pi^2}\Big)^{c_1-1+c_2-1}\frac{V_{c_1+1}V_{c_2+1}}{c_1!c_2!}\Bigg|}_{(\mathrm{a})}$$
$$+ \underbrace{2\sum_{c_1=\lfloor\sqrt{n}\rfloor}^{\infty}\sum_{c_2=2}^{\infty}\Big(\frac{x_0}{2\pi^2}\Big)^{c_1-1+c_2-1}\frac{V_{c_1+1}V_{c_2+1}}{c_1!c_2!}}_{(\mathrm{b})}$$
$$+ \underbrace{\sum_{\substack{(c_1,c_2)\in\mathbb{N}^2\\ c_1+c_2\leq n-1,\, c_i\geq 2\\ \exists i: c_i\geq\sqrt{n}}} \binom{n}{c_1, c_2}\frac{1}{n^2}\frac{V_{c_1+1}V_{c_2+1}V_{n-c_1-c_2+2}}{V_n}}_{(\mathrm{c})}.$$

The bounds on each of these terms follow very similarly to the case when $k \geq 3$. The only difference is for (c), where now the analogue of c_3 takes values between 1 and $\lfloor n/2 \rfloor$, and there is no need to use the Mirzakhani volume bounds from Lemma 2.3 on the V_{c_3+2} term. ∎

We now conclude the proof of Theorem 1.1 for $\ell = 1$.

Proof of Theorem 1.1 for $\ell = 1$. Combining (3.1) and Propositions 3.6 and 3.10, we see that for fixed k,

$$\mathbb{E}_n\left(\left(N_{n,[a,b]}(X)\right)_k\right) = \left(\frac{b^2 - a^2}{2}\right)^k \alpha^k \left(1 + O_k\left(\frac{1}{n^{1/2}}\right)\right) + O_k\left(\frac{b^{2k}}{n}\right)$$
$$\to \left(\frac{b^2 - a^2}{2}\right)^k \alpha^k$$

as $n \to \infty$, and so Theorem 1.1 follows from Proposition 3.1 with the mean

$$\frac{b^2 - a^2}{2}\alpha.$$

To see that

$$\alpha = \sum_{i=2}^{\infty} \frac{V_{i+1}}{i!}\left(\frac{x_0}{2\pi^2}\right)^{i-1} = \frac{(j_0\pi)^2}{4}\left(1 - \frac{J_3(j_0)}{J_1(j_0)}\right),$$

consider the function

$$\varphi_0(x) = \sum_{i=3}^{\infty} \frac{V_i}{i!} \frac{x^i}{(2\pi^2)^{i-3}},$$

which by Manin and Zograf [16, Proof of Theorem 6.1] has radius of convergence given by x_0. By differentiating, we see that

$$\alpha = \frac{2\pi^2 \varphi_0'(x_0)}{x_0}.$$

Define $y(x) = \varphi_0''(x)$, then from [16, Proof of Theorem 6.1] and the introduction of [10], $y(x)$ can be obtained by inverting $x(y) = -\sqrt{y}J_0'(2\sqrt{y})$.

By direct computation and using the recurrence relations satisfied by Bessel functions, we see that $x'(y) = J_0(2\sqrt{y})$. Thus on the domain $[0, j_0^2/4]$, this derivative is monotonically decreasing from 1 to 0, since j_0 is the first positive zero of J_0. It follows that $x(y)$ is monotonically increasing on the same domain from $x(0) = 0$ to $x(j_0^2/4) = x_0$. We wish to compute $\varphi_0'(x_0)$ which, since $\varphi_0'(0) = 0$, is given by

$$\varphi_0'(x_0) = \int_0^{x_0} y(x)\, \mathrm{d}x.$$

Using the definition of x and its properties demonstrated above, the area under the curve $y(x)$ in the domain $[0, x_0] = [x(0), x(j_0^2/4)]$ is precisely $x_0(j_0^2/4)$ minus the area under the curve $x(y)$ in the domain $[0, j_0^2/4]$. Thus,

$$\begin{aligned}\varphi_0'(x_0) &= x_0 \frac{j_0^2}{4} - \int_0^{j_0^2/4} x(y)\,\mathrm{d}y \\ &= \int_0^{j_0^2/4} y J_0(2\sqrt{y})\,\mathrm{d}y = \frac{1}{8}\int_0^{j_0} y^3 J_0(y)\,\mathrm{d}y.\end{aligned}$$

To evaluate this latter integral, we recall the identity (see [6, Equation 5.52])

$$\int x^{p+1} J_p(x)\,\mathrm{d}x = x^{p+1} J_{p+1}(x),$$

and so integration by parts gives

$$\begin{aligned}\varphi_0'(x_0) &= \frac{1}{8}\left(j_0^3 J_1(j_0) - 2\int_0^{j_0} y^2 J_1(y)\,\mathrm{d}y \right) \\ &= \frac{1}{8}\big(j_0^3 J_1(j_0) - 2j_0^2 J_2(j_0)\big) = \frac{j_0^3}{16}\big(J_1(j_0) - J_3(j_0)\big),\end{aligned} \tag{3.6}$$

where in the last step we use the Bessel function identity $\frac{4}{x}J_2(x) = J_1(x) + J_3(x)$. Since $J_0'(x) = -J_1(x)$, we also have $x_0 = -\frac{j_0}{2}J_0'(j_0) = \frac{j_0}{2}J_1(j_0)$, and so from (3.6) we see that

$$\alpha = \frac{2\pi^2\varphi_0'(x_0)}{x_0} = \frac{(j_0\pi)^2}{4}\left(1 - \frac{J_3(j_0)}{J_1(j_0)}\right). \qquad \blacksquare$$

3.3. Proof of Theorem 1.1

We now prove Theorem 1.1 using the results on nested and unnested multicurves from before. Let $\ell \in \mathbb{N}$ and suppose that $0 \le a_i < b_i$ are real numbers for $i = 1, \ldots, \ell$ such that the intervals $[a_i, b_i]$ are pairwise disjoint. For sufficiently large n, we have $b_i/\sqrt{n} < 2\operatorname{arcsinh}(1)$ for each $i = 1, \ldots, \ell$ and so given $k_1, \ldots, k_\ell \in \mathbb{N}$ and $X \in \mathcal{M}_{0,n}$, the product

$$\big(N_{n,[a_1,b_1]}(X)\big)_{k_1} \cdots \big(N_{n,[a_\ell,b_\ell]}(X)\big)_{k_\ell}$$

counts the number of ordered ℓ tuples whose ith entry is an ordered k_i tuple consisting of distinct, disjoint, primitive simple closed geodesics on X, whose lengths are in the interval $[a_i/\sqrt{n}, b_i/\sqrt{n}]$. Since the intervals $[a_i, b_i]$ are pairwise disjoint, the geodesics in the ith and jth tuples are distinct. This means, the product of the factorials counts the number of ordered $\sum_{i=1}^{\ell} k_i$ multigeodesics on X such that the ith block (of length k_i) of curves have lengths in $[a_i/\sqrt{n}, b_i/\sqrt{n}]$.

As before, we can separate the different mapping class group orbits of these multicurves into nested and unnested curves. Moreover, we can use identical topological descriptions of these multicurve types as in Sections 3.1 and 3.2 to see that with an application of Mirzakhani's integration formula Theorem 2.1, the contribution of the nested multicurves to the expectation is then bounded by

$$\sum_{\{(c_j,d_j)\}_{j=1}^{j=1+\tilde{k}} \in \mathcal{A}_{\tilde{k}}^{\mathcal{O}}} \tilde{k}! \binom{n}{c_1, \dots, c_{\tilde{k}}} \frac{\tilde{b}^{2\tilde{k}}}{2^{\sum_{i=1}^{\ell} k_i}} \frac{\prod_{j=1}^{1+\tilde{k}} V_{c_j+d_j}}{n^{\tilde{k}} V_n} \Big(1 + O\Big(\frac{\tilde{k}\tilde{b}^2}{n}\Big)\Big), \tag{3.7}$$

where $\tilde{k} = \sum_{i=1}^{\ell} k_i$ and $\tilde{b} = \max_{j=1,\dots,\ell} b_i$. Applying Proposition 3.6 with $\tilde{k}$ and $\tilde{b}$, we see that this is $O(1/n)$. In a similar vein, we can identically consider the contribution of the unnested multicurves to the expectation, and see it is equal to

$$\sum_{\substack{(c_1,\dots,c_{\tilde{k}}) \in \mathbb{N}^{\tilde{k}} \\ \sum_{i=1}^{\tilde{k}} c_i \le n,\, c_i \ge 2}} \binom{n}{c_1, \dots, c_{\tilde{k}}} \frac{1}{n^{\tilde{k}}} \frac{V_{c_1+1} \cdots V_{\tilde{k}+1} V_{n-\sum_{i=1}^{\tilde{k}} c_i + \tilde{k}}}{V_n}$$
$$\times \prod_{j=1}^{\ell} \Big(\frac{b_j^2 - a_j^2}{2}\Big)^{k_j} \Big(1 + O\Big(\frac{\tilde{k}\tilde{b}^2}{n}\Big)\Big),$$

again, where $\tilde{k} = \sum_{i=1}^{\ell} k_i$. By Proposition 3.10, this is equal to

$$\Big(\prod_{i=1}^{\ell} \Big(\alpha \frac{b_i^2 - a_i^2}{2}\Big)^{k_i}\Big) \Big(1 + O_{\tilde{k}}\Big(\frac{1}{\sqrt{n}}\Big)\Big),$$

where

$$\alpha = \sum_{i=2}^{\infty} \frac{V_{i+1}}{i!} \Big(\frac{x_0}{2\pi^2}\Big)^{i-1} = \frac{(j_0\pi)^2}{4} \Big(1 - \frac{J_3(j_0)}{J_1(j_0)}\Big).$$

Combining these contributions to evaluate $\mathbb{E}_n((N_{n,[a_1,b_1]}(X))_{k_1} \cdots (N_{n,[a_\ell,b_\ell]}(X))_{k_\ell})$ and taking $n \to \infty$ we obtain Theorem 1.1 by an application of Proposition 3.1.

3.4. Proof of Theorem 1.7

The proof of Theorem 1.7 is similar to Theorem 1.1 except that we are only required to examine a single topological type of multicurves. Recall that for any integer $c \ge 2$, any $n \in \mathbb{N}$ and any real numbers $0 \le a < b$, we defined $N_{n,c,[a,b]}(X)$ to be the number of primitive closed geodesics on X that separate off exactly c cusps from X with length in the interval $[a/\sqrt{n}, b/\sqrt{n}]$.

We will use the method of factorial moments to prove Theorem 1.7, and so for $k_1, \dots, k_\ell \in \mathbb{N}$, we are required to compute the expected value of

$$\big(N_{n,c_1,[a_1,b_1]}(X)\big)_{k_1} \cdots \big(N_{n,c_\ell,[a_\ell,b_\ell]}(X)\big)_{k_\ell}, \tag{3.8}$$

where $c_1, \ldots, c_\ell \geq 2$ are distinct integers and $0 \leq a_i < b_i$ are real numbers for $i = 1, \ldots, \ell$.

Notice that this product counts the number of ordered ℓ tuples whose ith entry is an ordered k_i tuple of distinct primitive closed geodesics on X that separate off exactly c_i cusps from X, and whose lengths are in the interval $[a_i/\sqrt{n}, b_i/\sqrt{n}]$. Since the integers c_i are distinct, the geodesics in the ith entry of this tuple are distinct from those in the jth tuple for any $i \neq j$. Moreover, for n sufficiently large, $b_i/\sqrt{n} < 2\operatorname{arcsinh}(1)$ for each $i = 1, \ldots, \ell$, and so the geodesics considered are simple and disjoint from one another. This means that the product counts the number of ordered multigeodesics of length $\sum_{i=1}^{k} k_i$ on X such that geodesics in the ith block separate off exactly c_i cusps, and have lengths in $[a_i/\sqrt{n}, b_i/\sqrt{n}]$.

Now, we split this count into counts of nested and unnested multicurves of this topological type. The number of nested multicurves is bounded by the number of nested multicurves considered in the proof of Theorem 1.1 since we are considering only a subset of the possible topological types that can occur due to the limitations on how many cusps each multicurve component separates off. It follows that the expected number of nested multicurves included in the count here is bounded by (3.7) which we saw to be $O_{\widetilde{k}}(\widetilde{b}^{2\widetilde{k}}/n)$, where $\widetilde{k} = \sum_{i=1}^{\ell} k_i$ and $\widetilde{b} = \max_{i=1,\ldots,\ell} b_i$.

Once again, the dominant contribution to the expectation will arise from the unnested multicurves. Since we know precisely how many cusps are separated off by each multicurve component, there is precisely one topological type of mapping class group orbit of unnested multicurves up to the labeling of the cusps that is considered. Indeed, a mapping class group orbit of an unnested multicurve that is counted by (3.8) corresponds to the following decomposition of a puncture labeled topological surface $\Sigma_{0,n,0}$:

$$\Sigma_{0,n-\sum_{i=1}^{\ell} k_i c_i, \sum_{i=1}^{\ell} k_i} \sqcup \bigsqcup_{i=1}^{\ell} \bigsqcup_{j=1}^{k_i} \Sigma_{0,c_i,1},$$

with labels on the punctures of each subsurface inherited from labels on $\Sigma_{0,n,0}$. The number of ways that the labels can be inherited on the subsurfaces is given precisely by the multinomial coefficient

$$C \stackrel{\text{def}}{=} \binom{n}{\underbrace{c_1, \ldots, c_1}_{k_1 \text{ times}}, \ldots, \underbrace{c_\ell, \ldots, c_\ell}_{k_\ell \text{ times}}}.$$

Note that if $\sum_{i=1}^{\ell} k_i c_i > n$ then there are no such unnested multigeodesics since the cusps separated by each curve component are distinct, and so we assume that this summation is bounded by n from now on.

Using Mirzakhani's integration formula Theorem 2.1, we are led to compute

$$\frac{C}{V_n}\int_{[a_1/\sqrt{n},b_1/\sqrt{n}]^{k_1}}\cdots\int_{[a_\ell/\sqrt{n},b_\ell/\sqrt{n}]^{k_\ell}}\left(\prod_{i=1}^{\ell}\prod_{j=1}^{k_i} t_{i,j}V_{c_i+1}(\mathbf{0}_{c_i},t_{i,j})\right)$$
$$\times V_{n-\sum_{i=1}^{\ell}k_i(c_i-1)}(\mathbf{0}_{n-\sum_{i=1}^{\ell}k_ic_i},\mathbf{t})\bigwedge_{i=1}^{\ell}\bigwedge_{j=1}^{k_i}\mathrm{d}t_{i,j},$$

where $\mathbf{t}=(t_{1,1},\dots,t_{1,k_1},\dots,t_{\ell,1},\dots,t_{\ell,k_\ell})$. We have the following estimate.

Proposition 3.11. *For distinct integers $c_1,\dots,c_\ell\geq 2$ and $k_1\dots,k_\ell\in\mathbb{N}$ satisfying $\sum_{i=1}^{\ell}k_ic_i\leq n$, we have that as $n\to\infty$,*

$$\frac{C}{V_n}\int_{[a_1/\sqrt{n},b_1/\sqrt{n}]^{k_1}}\cdots\int_{[a_\ell/\sqrt{n},b_\ell/\sqrt{n}]^{k_\ell}}\left(\prod_{i=1}^{\ell}\prod_{j=1}^{k_i} t_{i,j}V_{c_i+1}(\mathbf{0}_{c_i},t_{i,j})\right)$$
$$\times V_{n-\sum_{i=1}^{\ell}k_i(c_i-1)}(\mathbf{0}_{n-\sum_{i=1}^{\ell}k_ic_i},\mathbf{t})\bigwedge_{i=1}^{\ell}\bigwedge_{j=1}^{k_i}\mathrm{d}t_{i,j}$$
$$=\left(\prod_{i=1}^{\ell}\left(\frac{b_i^2-a_i^2}{2}\frac{V_{c_i+1}}{c_i!}\left(\frac{x_0}{2\pi^2}\right)^{c_i-1}\right)^{k_i}\right)\left(1+O_{\widetilde{k},\widetilde{b},\widetilde{c},\ell}\left(\frac{1}{n}\right)\right),$$

where $\widetilde{k}=\sum_{i=1}^{\ell}k_i$, $\widetilde{c}=\max(c_i)$ and $\widetilde{b}=\max(b_i)$.

Proof. The proof is similar but simpler than the proof of Proposition 3.10. For each of the volumes in the integrand, we use Lemma 2.4 to pass to a sinh approximation with error term and then consider a Taylor expansion of the resulting integrand to obtain

$$\frac{C}{V_n}\int_{[a_1/\sqrt{n},b_1/\sqrt{n}]^{k_1}}\cdots\int_{[a_\ell/\sqrt{n},b_\ell/\sqrt{n}]^{k_\ell}}\left(\prod_{i=1}^{\ell}\prod_{j=1}^{k_i} t_{i,j}V_{c_i+1}(\mathbf{0}_{c_i},t_{i,j})\right)$$
$$\times V_{n-\sum_{i=1}^{\ell}k_i(c_i-1)}(\mathbf{0}_{n-\sum_{i=1}^{\ell}k_ic_i},\mathbf{t})\bigwedge_{i=1}^{\ell}\bigwedge_{j=1}^{k_i}\mathrm{d}t_{i,j}$$
$$=\frac{C(\prod_{i=1}^{\ell}V_{c_i+1}^{k_i})V_{n-\sum_{i=1}^{\ell}k_i(c_i-1)}}{V_n}$$
$$\times\int_{[a_1/\sqrt{n},b_1/\sqrt{n}]^{k_1}}\cdots\int_{[a_\ell/\sqrt{n},b_\ell/\sqrt{n}]^{k_\ell}}\left(\prod_{i=1}^{\ell}\prod_{j=1}^{r_i}t_{i,j}\right)\bigwedge_{i=1}^{\ell}\bigwedge_{j=1}^{k_i}\mathrm{d}t_{i,j}\left(1+O\left(\frac{\widetilde{k}\widetilde{b}^2}{n}\right)\right)$$
$$=\frac{C(\prod_{i=1}^{\ell}V_{c_i+1}^{k_i})V_{n-\sum_{i=1}^{\ell}k_i(c_i-1)}}{n^{\sum_{i=1}^{\ell}k_i}V_n}\prod_{i=1}^{\ell}\left(\frac{b_i^2-a_i^2}{2}\right)^{k_i}\left(1+O\left(\frac{\widetilde{k}\widetilde{b}^2}{n}\right)\right).$$

By Theorem 2.2, since the c_i and k_i are fixed, we have that as $n \to \infty$,

$$\frac{C V_{n-\sum_{i=1}^{\ell} k_i(c_i-1)}}{n^{\sum_{i=1}^{\ell} k_i} V_n} = \frac{(n-\sum_{i=1}^{\ell} k_i(c_i-1))!}{n^{\sum_{i=1}^{\ell} k_i}(n-\sum_{i=1}^{\ell} k_i c_i)!}\Big(\frac{n+1}{n-\sum_{i=1}^{\ell} k_i(c_i-1)+1}\Big)^{7/2} \times \frac{x_0^{\sum_{i=1}^{\ell} k_i(c_i-1)}}{(c_1!)^{k_1}\cdots(c_k!)^{k_\ell}(2\pi^2)^{\sum_{i=1}^{\ell} k_i(c_i-1)}}\Big(1+O\Big(\frac{1}{n}\Big)\Big).$$

Then we note that

$$\frac{(n-\sum_{i=1}^{\ell} k_i(c_i-1))!}{n^{\sum_{i=1}^{\ell} k_i}(n-\sum_{i=1}^{\ell} k_i c_i)!} = \Big(1-\frac{\sum_{i=1}^{\ell} k_i c_i - \sum_{i=1}^{\ell} c_i}{n}\Big)\cdots\Big(1-\frac{\sum_{i=1}^{\ell} k_i c_i - 1}{n}\Big) = 1+O\Big(\frac{\ell\tilde{k}\tilde{c}^2}{n}\Big),$$

and

$$\Big(\frac{n+1}{n-\sum_{i=1}^{\ell} k_i(c_i-1)+1}\Big)^{7/2} = \Big(1+\frac{\sum_{i=1}^{\ell} k_i(c_i-1)}{n+1-\sum_{i=1}^{\ell} k_i(c_i-1)}\Big)^{7/2} = 1+O\Big(\frac{\ell\tilde{k}\tilde{c}}{n}\Big).$$

Put together, we obtain

$$\frac{C(\prod_{i=1}^{\ell} V_{c_i+1}^{k_i})V_{n-\sum_{i=1}^{\ell} k_i(c_i-1)}}{n^{\sum_{i=1}^{\ell} k_i} V_n}\prod_{i=1}^{\ell}\Big(\frac{b_i^2-a_i^2}{2}\Big)^{k_i}\Big(1+O_{\tilde{k},\tilde{b},\tilde{c},\ell}\Big(\frac{1}{n}\Big)\Big) = \Big(\prod_{i=1}^{\ell}\Big(\frac{b_i^2-a_i^2}{2}\frac{V_{c_i+1}}{c_i!}\Big(\frac{x_0}{2\pi^2}\Big)^{c_i-1}\Big)^{k_i}\Big)\Big(1+O_{\tilde{k},\tilde{b},\tilde{c},\ell}\Big(\frac{1}{n}\Big)\Big),$$

as required. ■

Proof of Theorem 1.7. The result now follows from Proposition 3.11 and the method of factorial moments Proposition 3.1. ■

Proof of Corollary 1.8. Notice by definition, that a surface satisfying

$$\big(N_{n,2,[0,x]}(X),\dots,N_{n,k-1,[0,x]}(X),N_{n,k,[0,x]}(X)\big) = (0,\dots,0,m)$$

for some $m \geq 1$ is contained in $\mathcal{A}_{k,x,n}$. By Theorem 1.7, this sequence of random vectors converges in distribution as $n \to \infty$ to a vector of independent Poisson random variables with means

$$\lambda_{i,[0,x]} = \frac{x^2}{2}\frac{V_{i+1}}{i!}\Big(\frac{x_0}{2\pi^2}\Big)^{i-1}$$

for $i = 2, \dots, k$, respectively. It thus follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}_n(\mathcal{A}_{k,x,n}) \\
&\quad \geqslant \lim_{n\to\infty} \mathbb{P}_n\big(X : N_{n,2,[0,x]}(X) = 0, \dots, N_{n,k-1,[0,x]}(X) = 0, N_{n,k,[0,x]}(X) \geq 1\big) \\
&\quad = e^{-\sum_{i=2}^{k-1} \lambda_{i,[0,x]}}(1 - e^{-\lambda_{k,[0,x]}}),
\end{aligned}$$

as required. ■

4. Systole

In this section, we prove Proposition 1.4, providing information about the systole of hyperbolic punctured spheres. Recall that for an integer $c \geqslant 2$ and real numbers $a, b \geqslant 0$, the random variable

$$N_{n,c,[a,b]}(X) \colon (\mathcal{M}_{0,n}, \mathbb{P}_n) \to \mathbb{N}$$

counts the number of closed, primitive geodesics with lengths in $[a/\sqrt{n}, b/\sqrt{n}]$ which separate off c cusps.

Proposition 4.1 (Proposition 1.4). *There exists a constant $B > 0$ such that for any constants $0 < \varepsilon < 1/2$, $A > 0$, any $c_n < An^{\varepsilon}$ and n sufficiently large,*

$$\mathbb{P}_n\Big(\mathrm{sys}(X) > \frac{c_n}{\sqrt{n}}\Big) \leq \max\Big\{Bc_n^{-2}, \frac{B}{\sqrt{n}}\Big\}.$$

Proof. The systole of a surface is always a simple closed geodesic, and so if the systole has length greater than $c_n/\sqrt{n}$, then $N_{n,2,[0,c_n]}(X) = 0$, thus we have

$$\mathbb{P}_n\Big(\mathrm{sys}(X) > \frac{c_n}{\sqrt{n}}\Big) \leq \mathbb{P}_n\big(N_{n,2,[0,c_n]}(X) = 0\big).$$

By the second moment method

$$\begin{aligned}
&\mathbb{P}_n\big(N_{n,2,[0,c_n]}(X) = 0\big) \\
&\qquad \leq \frac{\mathbb{E}_n((N_{n,2,[0,c_n]}(X))^2) - \mathbb{E}_n(N_{n,2,[0,c_n]}(X))^2}{\mathbb{E}_n((N_{n,2,[0,c_n]}(X))^2)} \\
&\qquad = \frac{\mathbb{E}_n((N_{n,2,[0,c_n]}(X))_2) + \mathbb{E}_n(N_{n,2,[0,c_n]}(X)) - \mathbb{E}_n(N_{n,2,[0,c_n]}(X))^2}{\mathbb{E}_n((N_{n,2,[0,c_n]}(X))_2) + \mathbb{E}_n(N_{n,2,[0,c_n]}(X))}.
\end{aligned}$$

Given any $A > 0$ and $0 < \varepsilon < 1/2$, for n sufficiently large, $An^{\varepsilon-1/2} < 2\,\mathrm{arcsinh}(1)$ and so the second factorial moment of $N_{n,2,[0,c_n]}(X)$ has the natural interpretation of counting the number of ordered multicurves of length two consisting of distinct, non-intersecting simple closed geodesics with lengths in $[0, c_n/\sqrt{n}]$ that separate off 2 cusps. Using Theorem 2.1 we compute for any $t = o_{n\to\infty}(n^{1/2})$,

$$\begin{aligned}\mathbb{E}_n\big(N_{n,2,[0,t]}(X)\big) &= \frac{1}{V_{0,n}}\binom{n}{2}\int_0^{t/\sqrt{n}} x V_{0,3}(0,0,x)V_{0,n-1}(\mathbf{0}_{n-2},x)\,\mathrm{d}x \\ &= \frac{V_{0,n-1}}{V_{0,n}}\binom{n}{2}\int_0^{t/\sqrt{n}} 4\frac{\sinh^2(x/2)}{x}\big(1+O(x^2)\big)\,\mathrm{d}x \\ &= \frac{V_{0,n-1}}{V_{0,n}}\binom{n}{2}\frac{t^2}{2n}\Big(1+O\Big(\frac{t^2}{n}\Big)\Big).\end{aligned}$$

By Theorem 2.2, we obtain

$$\begin{aligned}\frac{V_{0,n-1}}{V_{0,n}}\binom{n}{2} &= \frac{1}{2}\frac{x_0}{2\pi^2}\frac{(n-1)!(n+1)^{7/2}}{n^{7/2}(n-2)!}\Big(1+O\Big(\frac{1}{n}\Big)\Big) \\ &= \frac{1}{2}\frac{x_0}{2\pi^2}n\Big(1+O\Big(\frac{1}{n}\Big)\Big).\end{aligned}$$

Thus,

$$\mathbb{E}_n\big(N_{n,2,[0,t]}(X)\big) = \frac{t^2x_0}{8\pi^2}\Big(1+O\Big(\frac{t^2}{n}\Big)\Big).$$

Similarly, we compute

$$\begin{aligned}\mathbb{E}_n\big(\big(N_{n,2,[0,t]}(X)\big)_2\big) &= \frac{1}{V_{0,n}}\binom{n}{2,2}\int_0^{t/\sqrt{n}}\int_0^{t/\sqrt{n}} xyV_{0,3}(0,0,x)V_{0,3}(0,0,y) \\ &\qquad\times V_{0,n-2}(\mathbf{0}_{n-4},x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{V_{0,n-2}}{V_{0,n}}\binom{n}{2,2}\Big(\int_0^{t/\sqrt{n}} 4\frac{\sinh^2(x/2)}{x}\,\mathrm{d}x\Big)^2\Big(1+O\Big(\frac{t^2}{n}\Big)\Big) \\ &= \frac{V_{0,n-2}}{V_{0,n}}\binom{n}{2,2}\frac{t^4}{4n^2}\Big(1+O\Big(\frac{t^2}{n}\Big)\Big).\end{aligned}$$

Again using Theorem 2.2, we can compute

$$\mathbb{E}_n\big(\big(N_{n,2,[0,t]}(X)\big)_2\big) = \frac{t^4x_0^2}{64\pi^4}\Big(1+O\Big(\frac{t^2}{n}\Big)\Big).$$

Using $t = c_n$ in these estimates, we find that there is a uniform constant $B > 0$ such that

$$\begin{aligned}\mathbb{P}_n\big(N_{n,2,[0,c_n]}(X) = 0\big) &\le \frac{1 + (c_n^2x_0/8\pi^2)O(c_n^2/n) + O(c_n^2/n)}{(c_n^2x_0/8\pi^2+1)(1+O(c_n^2/n))} \\ &\le \frac{B}{c_n^2} + B\frac{c_n^2}{n}.\end{aligned} \tag{4.1}$$

If $c_n \leqslant \sqrt{n}$, then (4.1) gives the claim. Otherwise if $c_n > \sqrt{n}$, we use that

$$\mathbb{P}_n\big(N_{n,2,[0,c_n]}(X) = 0\big) \leqslant \mathbb{P}_n\big(N_{n,2,[0,\sqrt{n}]}(X) = 0\big)$$

and (4.1) to finish the proof. ∎

Remark 4.2. The previous theorem also remains true for the moduli space $\mathcal{M}_{g,n}$ where g is a fixed constant. This follows from large cusp volume asymptotics of $\mathcal{M}_{g,n}$ from Manin and Zograf [16, Theorem 6.1] when g is fixed and non-zero.

5. Small eigenvalues

In this section we prove Theorem 1.10. First we want to establish a geometric criterion for the existence of multiple small eigenvalues. We start with the following mini-max principle which can for example be found in [25].

Lemma 5.1 (Mini-max). *Let A be a non-negative self-adjoint operator on a Hilbert space H with domain $\mathcal{D}(A)$. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_k$ denote the eigenvalues of A below the essential spectrum $\sigma_{\text{ess}}(A)$. Then for $1 \leqslant j \leqslant k$,*

$$\lambda_j = \min_{\psi_1,\dots,\psi_j} \max\left\{\frac{\langle\psi, A\psi\rangle}{\|\psi\|^2} \mid \psi \in \operatorname{span}(\psi_1,\dots,\psi_j)\right\},$$

where the minimum is taken over linearly independent $\psi_1,\dots,\psi_j \in \mathcal{D}(A)$. If A only has $l \geqslant 0$ eigenvalues below the essential spectrum then for any integer $s \geqslant 1$,

$$\inf \sigma_{\text{ess}}(A) = \inf_{\psi_1,\dots,\psi_{l+s}} \sup\left\{\frac{\langle\psi, A\psi\rangle}{\|\psi\|^2} \mid \psi \in \operatorname{span}(\psi_1,\dots,\psi_{l+s})\right\},$$

where again the infimum is taken over linearly independent $\psi_1,\dots,\psi_{\ell+s} \in \mathcal{D}(A)$.

Using Lemma 5.1 we prove the following result which gives us a criterion for the existence of small eigenvalues.

Lemma 5.2. *Let $X \in \mathcal{M}_{g,n}$ and assume that there exists $f_0,\dots,f_k \in C_c^\infty(X)$ with $\|f_i\|_{L^2} = 1$ for $0 \leqslant i \leqslant k$ and $\operatorname{Supp}(f_i) \cap \operatorname{Supp}(f_j) = \emptyset$ for $i \neq j$, such that*

$$\max_{0\leqslant j\leqslant k} \int_X \|\operatorname{grad} f_j(z)\|^2 \,\mathrm{d}\mu(z) < \kappa < \frac{1}{4}.$$

Then $0 < \lambda_1(X) \leqslant \cdots \leqslant \lambda_k(X)$ exist and satisfy $\lambda_k(X) < \kappa$.

Proof. Let $f_0, \dots, f_k$ satisfy the assumptions of the lemma. Since $C_c^\infty(X) \subset \mathcal{D}(\Delta)$, we see that

$$\min_{\psi_1,\dots,\psi_{k+1}} \max\left\{\frac{\langle\psi, \Delta\psi\rangle}{\|\psi\|^2} \mid \psi \in \operatorname{span}(\psi_1, \dots, \psi_{k+1})\right\}$$
$$\leqslant \max\left\{\langle\psi, \Delta\psi\rangle \mid \psi \in \operatorname{span}(f_0, \dots, f_k), \|\psi\|_{L^2} = 1\right\}.$$

Let $\psi = \sum_{i=0}^k a_i f_i$ with $a_i \in \mathbb{C}$, then $\|\psi\|_{L^2} = 1$ implies that

$$1 = \left\langle \sum_{i=0}^k a_i f_i, \sum_{i=0}^k a_i f_i \right\rangle = \sum_{i=0}^k |a_i|^2 \langle f_i, f_i\rangle = \sum_{i=0}^k |a_i|^2,$$

and

$$\langle\psi, \Delta\psi\rangle = \left\langle \sum_{i=0}^k a_i f_i, \Delta \sum_{i=0}^k a_i f_i \right\rangle = \sum_{i=0}^k |a_i|^2 \langle f_i, \Delta f_i\rangle < \kappa < \frac{1}{4},$$

where we used that $\operatorname{Supp}(f_i) \cap \operatorname{Supp}(f_j) = \emptyset$ for $i \neq j$. We deduce that

$$\max\left\{\langle\psi, A\psi\rangle \mid \psi \in \operatorname{span}(f_0, \dots, f_k), \|\psi\|_{L^2} = 1\right\} < \kappa < \frac{1}{4}.$$

Assume that X does not have $k+1$ eigenvalues below $1/4$, then $\inf \sigma_{\mathrm{ess}}(X) < \kappa < 1/4$ by Lemma 5.1, giving a contradiction. Then $\lambda_1(X) \leqslant \dots \leqslant \lambda_k(X)$ exist and

$$\lambda_k(X) < \kappa,$$

by Lemma 5.1. ∎

Lemma 5.3. *Let $\varepsilon < 1/4$ and assume $X \in \mathcal{M}_{g,n}$ has $k+1$ geodesics of length $\leqslant \varepsilon/6$, each of which separates off two cusps. Then $\lambda_k(X)$ exists and satisfies*

$$\lambda_k(X) \leqslant \varepsilon.$$

Proof. We adapt the proof of [3, Theorem 8.1.3]. First label the geodesics from $\gamma_0, \dots, \gamma_k$. Let Y_j be the subsurface with two cusps with geodesic boundary γ_j. Denote the two cusps bounded by γ_j by $C_{j,1}, C_{j,2}$ and their unique length $\varepsilon/6$ horocycles by $\beta_{j,1}, \beta_{j,2}$. Let $\widetilde{Y}_j \subset Y_j$ denote the compact region of Y_j bounded by $\beta_{j,1}, \beta_{j,2}$. We then define the function

$$g_k(z) \stackrel{\text{def}}{=} \begin{cases} \operatorname{dist}(z, \gamma_k) & \text{if } z \in \widetilde{Y}_k \text{ and } \operatorname{dist}(z, \gamma_k) \leqslant 1, \\ \operatorname{dist}(z, \beta_{k,1}) & \text{if } z \in \widetilde{Y}_k \text{ and } \operatorname{dist}(z, \beta_{k,1}) \leqslant 1, \\ \operatorname{dist}(z, \beta_{k,2}) & \text{if } z \in \widetilde{Y}_k \text{ and } \operatorname{dist}(z, \beta_{k,2}) \leqslant 1, \\ 0 & \text{if } z \notin \widetilde{Y}_k, \\ 1 & \text{otherwise,} \end{cases}$$

which is well defined by the choice of ε. Then writing

$$Z_k \stackrel{\text{def}}{=} \{z \in \widetilde{Y}_k \mid \operatorname{dist}(z, \gamma_k) \leqslant 1\} \cup \{z \in \widetilde{Y}_k \mid \operatorname{dist}(z, \beta_{k,1}) \leqslant 1\} \\ \cup \{z \in \widetilde{Y}_k \mid \operatorname{dist}(z, \beta_{k,1}) \leqslant 1\},$$

we have that $\| \operatorname{grad} g_k(z)\|^2 = 1$ for $z \in Z_k$, $\| \operatorname{grad} g_k(z)\|^2 = 0$ elsewhere and one can calculate that

$$\operatorname{Vol}(Z_k) \leq \frac{5}{6}\varepsilon \sinh 1.$$

Note that each g_k is compactly supported by the definition of $\widetilde{Y}_k$. By L^2 normalizing and smoothly approximating, we can find functions $f_0, \dots, f_k \in C_c^\infty(X)$ with $\|f_i\|_{L^2} = 1$ for $0 \leqslant i \leqslant k$ and $\operatorname{Supp}(f_i) \cap \operatorname{Supp}(f_j) = \emptyset$ for $i \neq j$, such that

$$\max_{0 \leqslant j \leqslant k} \int_X \| \operatorname{grad} f_j(z)\|^2 \, \mathrm{d}\mu(z) < \varepsilon < \frac{1}{4},$$

and the conclusion follows from Lemma 5.2. ■

We conclude with the proof of Theorem 1.10.

Theorem 5.4 (Theorem 1.10). *There is a constant $C > 0$ such that for any function $k\colon \mathbb{N} \to \mathbb{N}$ with $k = o(n)$ and $k \to \infty$ as $n \to \infty$,*

$$\mathbb{P}_n\Big(\lambda_k(X) < C\sqrt{\frac{k}{n}}\Big) \to 1,$$

as $n \to \infty$. In particular, for any $\varepsilon > 0$, $\mathbb{P}_n[\lambda_k(X) < \varepsilon] \to 1$ as $n \to \infty$.

Proof. For $X \in \mathcal{M}_n$, let $\widetilde{N}_{n,2,[0,l]}(X)$ be the function that counts the number of primitive closed geodesics on X which separate off 2 cusps with lengths in the window $[0, l]$, note that here there is no rescaling of the window. We claim that there exists a constant $C' > 0$ such that for any function $L\colon \mathbb{N} \to (0, \infty)$ with $L(n) \to \infty$ as $n \to \infty$ and $L = o(\sqrt{n})$, the probability that $X \in \mathcal{M}_{0,n}$ satisfies

$$\widetilde{N}_{n,2,[0,1/6L]}(X) \geqslant C' \frac{n}{L^2}, \tag{5.1}$$

tends to 1 as $n \to \infty$. We now show that Theorem 5.4 follows from this claim. Let $k = o(n)$ and pick L with $L \to \infty$ and $L \leqslant \sqrt{C'n/k + 1}$. It follows from the above claim that with probability tending to 1, there are at least $k + 1$ curves with lengths $\leqslant 1/6L$. Then by Lemma 5.3, $\lambda_k < 1/L$.The remainder of the proof is dedicated to showing (5.1).

Consider the sequence of random variables

$$Y_n \stackrel{\text{def}}{=} \frac{L^2 \widetilde{N}_{n,2,[0,1/L]}(X)}{n}.$$

First we want to show that $\mathrm{Var}(Y_n) \to 0$ as $n \to \infty$. By nearly identical calculations to the proof of Proposition 4.1, we see that

$$\mathbb{E}_n[Y_n] = \binom{n}{2} \frac{L^2}{nV_n} \int_0^{1/L} l\, V_{n-1}(l)\, \mathrm{d}l = \frac{x_0}{8\pi^2} + o(1). \tag{5.2}$$

Calculating the second moment,

$$\mathbb{E}\Bigg[\bigg(\frac{L^2 \widetilde{N}_{n,2,[0,1/L]}(X)}{n}\bigg)^2\Bigg] = \mathbb{E}\Bigg[\bigg(\frac{L^2}{n} \sum_{\substack{\gamma \in \mathcal{P}(X) \\ \gamma \text{ separates off exactly 2 cusps}}} \mathbb{1}_{1/L}\big(l_\gamma(X)\big)\bigg)^2\Bigg]$$

$$= \frac{L^4}{n^2} \mathbb{E}\Bigg[\sum_{\substack{\gamma \in \mathcal{P}(X) \\ \gamma \text{ separates off exactly 2 cusps}}} \mathbb{1}_{1/L}\big(l_\gamma(X)\big)\Bigg]$$

$$+ \frac{L^4}{n^2} \mathbb{E}\Bigg[\sum_{\substack{(\gamma_1,\gamma_2) \in \mathcal{P}(X) \times \mathcal{P}(X),\, \gamma_1 \neq \gamma_2 \\ \gamma_1, \gamma_2 \text{ separate off exactly 2 cusps}}} \mathbb{1}_{1/L}\big(l_{\gamma_1}(X), l_{\gamma_2}(X)\big)\Bigg].$$

Then

$$\frac{L^4}{n^2} \mathbb{E}\Bigg[\sum_{\substack{\gamma \in \mathcal{P}(X) \\ \gamma \text{ separates off exactly 2 cusps}}} \mathbb{1}_{1/L}\big(l_\gamma(X)\big)'\Bigg] = \frac{L^2}{n} \mathbb{E}[Y_n] = o(1).$$

Again by similar calculations to the proof of Proposition 4.1 and the fact that the pairs of curves γ_1 and γ_2 are disjoint when they have length at most $1/L \to 0$,

$$\frac{L^4}{n^2} \mathbb{E}\Bigg[\sum_{\substack{(\gamma_1,\gamma_2) \in \mathcal{P}(X) \times \mathcal{P}(X),\, \gamma_1 \neq \gamma_2 \\ \gamma_1, \gamma_2 \text{ separate off exactly 2 cusps}}} \mathbb{1}_{1/L}\big(l_{\gamma_1}(X), l_{\gamma_2}(X)\big)\Bigg]$$

$$= \binom{n}{2,2} \frac{L^4}{n^2 V_n} \int_0^{1/L} \int_0^{1/L} l\, V_{n-2}(l_1, l_2)\, \mathrm{d}l_1\, \mathrm{d}l_2 = \left(\frac{x_0}{8\pi^2}\right)^2 + o(1).$$

We conclude that as $n \to \infty$.

$$\mathrm{Var}[Y_n] = \mathbb{E}[Y_n^2] - \mathbb{E}_n[Y_n]^2 = o(1). \tag{5.3}$$

By Chebyshev's inequality, we have

$$\mathbb{P}\left[|Y_n - \mathbb{E}[Y_n]| \geqslant \frac{x_0}{16\pi^2}\right] \leqslant \frac{256\pi^4}{x_0^2} \mathrm{Var}[Y_n] \to 0,$$

as $n \to \infty$. In particular, with probability at least $1 - (256\pi^4/x_0^2)\,\mathrm{Var}[Y_n]$, one has

$$Y_n > \mathbb{E}[Y_n] - \frac{x_0}{16\pi^2}.$$

Moreover, by (5.2), there exists $A > 0$ such that for all $n \geq A$, one has

$$\mathbb{E}[Y_n] > \frac{3}{2} \frac{x_0}{16\pi^2}.$$

Put together, this shows that $Y_n \geq x_0/32\pi^2$ for any $n \geq A$, with probability at least $1 - (256\pi^4/x_0^2)\operatorname{Var}[Y_n]$. It follows that

$$\widetilde{N}_{n,2,[0,1/6L]}(X) \geqslant \frac{x_0}{32\pi^2} \frac{n}{(6L)^2},$$

with probability tending to 1 as $n \to \infty$. ∎

Remark 5.5. The proof of Theorem 1.10 also works for surfaces with *fixed* genus $g > 0$ with $n \to \infty$, i.e. there is a constant $C > 0$ such that for any function $k\colon \mathbb{N} \to \mathbb{N}$ with $k = o(n)$ and $k \to \infty$ as $n \to \infty$ and any fixed g,

$$\mathbb{P}_{g,n}\Big(\lambda_k(X) < C\sqrt{\frac{k}{n}}\Big) \to 1$$

as $n \to \infty$. For this, one simply uses [16, Theorem 6.1] with $g \geqslant 0$ fixed in the calculations leading to (5.3).

Remark 5.6. The proof Theorem 1.10 shows that for $L = L(n)$ with $L \to \infty$ as $n \to \infty$ and $L = o(\sqrt{n})$, the random variable $L^2 \widetilde{N}_{n,2,[0,1/L]}(X)/n$ converges in distribution to the constant distribution $x_0/8\pi^2$.

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