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Analysis and Differential Geometry. – *Primitives of volume forms in Carnot groups*, by Annalisa Baldi, Bruno Franchi and Pierre Pansu, communicated on 14 March 2025.

Abstract. – In the Euclidean space, it is known that a function $f \in L^2$ of a ball, with vanishing average, is the divergence of a vector field $F \in L^2$ with $\|F\|_{L^2(B)} \le C\|f\|_{L^2(B)}$. In this note, we prove a similar result in any Carnot group $\mathbb{G}$ for a vanishing average $f \in L^p$, $1 \le p < Q$, where Q is the so-called homogeneous dimension of $\mathbb{G}$.

Keywords. – Carnot groups, differential forms, Sobolev inequalities.

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1. Introduction

This note is motivated by a question raised by Michael Cowling [6]: in $\mathbb{R}^n$, it is known that a function $f \in L^2(B_{\mathrm{Euc}}(0,1))$ with vanishing average can be expressed as the divergence of a vector field $F \in L^2(B_{\mathrm{Euc}}(0,1))^n$, satisfying

$$\|F\|_{L^2(B_{\mathrm{Euc}}(0,1))^n} \le C\|f\|_{L^2(B_{\mathrm{Euc}}(0,1))}.$$

The question is whether a similar result holds in Heisenberg groups, which can be identified with $\mathbb{R}^{2n+1}$ or, more generally, in the so-called Carnot groups (of which Heisenberg groups are a special case), provided the usual divergence is replaced by a suitable "intrinsic" divergence (see (5) below).

This problem can be rephrased in terms of Sobolev inequalities for differential forms in the Rumin complex $(E_0^\bullet, d_c)$ (see Section 3.1 for precise definitions). Specifically, given a compactly supported volume form $\omega = f\,dV$ with vanishing average, does there exist an $(n-1)$-compactly supported primitive ϕ whose L^2-norm is controlled by the L^2-norm of ω?

Sobolev inequalities for the Rumin complex in Heisenberg groups have been studied in [2], but unfortunately, the results in [2] do not cover the case of volume forms. The aim of this paper is to fill this gap by providing a positive answer to Cowling's question. Furthermore, as already mentioned, the results of this note are formulated in the more general setting of Carnot groups, and the L^2-norms are replaced with any suitable L^p-norms.

The main result is presented in Theorem 3.1 in Section 3 (see also Theorem 3.17 for an equivalent formulation). Section 2 provides some preliminary definitions, while Section 3.1 gives a brief introduction to Rumin's complex (for more details, see [3, 11, 15]). Finally, Section 4 is an appendix which collects various results on convolution kernels in Carnot groups, some of which are well known.

2. Preliminary results and notations

A *Carnot group* $\mathbb{G}$ of step κ and dimension n is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ has dimension n and admits a *step κ stratification*. This means there exist linear subspaces $V_1, \dots, V_\kappa$ such that

$$(1)\quad \mathfrak{g} = V_1 \oplus \cdots \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \quad \text{for } i > \kappa,$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. Let $m_i = \dim(V_i)$ for $i = 1, \dots, \kappa$, and define $h_i = m_1 + \cdots + m_i$, with $h_0 = 0$ and, clearly, $h_\kappa = n$.

Choose a basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$, adapted to the stratification, i.e., such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \text{ is a basis of } V_j \quad \text{for each } j = 1, \dots, \kappa.$$

This basis $\{e_1, \dots, e_n\}$ will be fixed throughout this note.

Let $X = \{X_1, \dots, X_n\}$ be the family of left-invariant vector fields such that $X_i(0) = e_i$. Given (1), the subset $X_1, \dots, X_{m_1}$ generates, by commutations, all the other vector fields. We will refer to $X_1, \dots, X_{m_1}$ as the *generating vector fields* of the group.

The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$, making $\{X_1, \dots, X_n\}$ an orthonormal basis. The group $\mathbb{G}$ can be identified with its Lie algebra $\mathfrak{g}$, endowed with the product defined by the Campbell–Hausdorff–Dynkin formula. In particular, we can identify $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell–Hausdorff–Dynkin formula.

A point $p \in \mathbb{G}$ can be written as $p = (p_1, \dots, p_n)$ or as $p = p^{(1)} + \cdots + p^{(\kappa)}$, where $p^{(i)} \in V_i$ for $i = 1, \dots, \kappa$. The Haar measure on $\mathbb{G}$ can be taken to be equal to the Lebesgue measure on $\mathfrak{g} \equiv \mathbb{R}^n$.

Two important families of maps from $\mathbb{G}$ to $\mathbb{G}$ are the group translations and dilations. For any $x \in \mathbb{G}$, the *(left) translation* $\tau_x : \mathbb{G} \to \mathbb{G}$ is defined as

$$z \mapsto \tau_x z := x \cdot z.$$

For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ is defined as

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n),$$

where $d_i \in \mathbb{N}$ is the *homogeneity* of the variable x_i in $\mathbb{G}$ (see [9, Chapter 1]) and is given by

$$d_j = i \quad \text{whenever } h_{i-1} + 1 \le j \le h_i.$$

Hence, $1 = d_1 = \cdots = d_{m_1} < d_{m_1+1} = 2 \le \cdots \le d_n = \kappa$.

If f is a real function defined on $\mathbb{G}$, we denote by $\check{f}$ the function defined by $\check{f}(p) := f(p^{-1})$.

Following [9], we adopt the following multi-index notation for higher-order derivatives. If $I = (i_1, \dots, i_n)$ is a multi-index, we set

$$X^I = X_1^{i_1} \cdots X_n^{i_n}.$$

By the Poincaré–Birkhoff–Witt theorem (see, e.g., [5, Chapter 1, Section 2.7]), the differential operators X^I form a basis for the algebra of left-invariant differential operators on $\mathbb{G}$. Moreover, we define the order of the differential operator X^I as $|I| := i_1 + \cdots + i_n$, and its degree of homogeneity with respect to dilations as $d(I) := d_1 i_1 + \cdots + d_n i_n$.

Again, following [9], we define the group convolution in $\mathbb{G}$. If $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G})$, we set

$$f * g(p) := \int f(q) g(q^{-1} p)\, dq \quad \text{for } p \in \mathbb{G}.$$

It is important to note that if g is a smooth function and L is a left-invariant differential operator, then

$$L(f * g) = f * Lg.$$

The convolution is also well defined when $f, g \in \mathcal{D}'(\mathbb{G})$, provided at least one of them has compact support. In this case, the following identities hold:

$$\langle f * g, \varphi \rangle = \langle g, \check{f} * \varphi \rangle \quad \text{and} \quad \langle f * g, \varphi \rangle = \langle f, \varphi * \check{g} \rangle \tag{2}$$

for any test function φ.

If $f \in \mathcal{E}'(\mathbb{G})$ and $g \in \mathcal{D}'(\mathbb{G})$, then for $\psi \in \mathcal{D}(\mathbb{G})$, we have

$$\begin{aligned} \big\langle (X^I f) * g, \psi \big\rangle &= \big\langle X^I f, \psi * \check{g} \big\rangle = (-1)^{|I|} \big\langle f, \psi * (X^I \check{g}) \big\rangle \\ &= (-1)^{|I|} \big\langle f * (X^I \check{g})\check{\ }, \psi \big\rangle. \end{aligned}$$

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$, and let $W^{m,p}_{\mathrm{Euc}}(U)$ denote the usual Sobolev space. We also recall the definition of the (integer order) Folland–Stein Sobolev space (see, e.g., [8, 9] for a general presentation).

Definition 2.1. If $U \subset \mathbb{G}$ is an open set, $1 \le p \le \infty$, and $m \in \mathbb{N}$, the space $W^{m,p}(U)$ consists of all $u \in L^p(U)$ such that

$$X^I u \in L^p(U) \quad \text{for all multi-indices } I \text{ with } d(I) \le m,$$

endowed with the norm

$$\|u\|_{W^{m,p}(U)} := \sum_{d(I)\le m} \|X^I u\|_{L^p(U)}.$$

When $p = 2$, we will simply write $H^m(U) = W^{m,2}(U)$.

THEOREM 2.2. *Let $U \subset \mathbb{G}$ be an open set, $1 \le p \le \infty$, and $m \in \mathbb{N}$. Then,*

(i) $W^{m,p}(U)$ *is a Banach space.*

In addition, if $p < \infty$, the following hold:

(ii) $W^{m,p}(U) \cap C^\infty(U)$ *is dense in* $W^{m,p}(U)$.

(iii) *If $U = \mathbb{G}$, then $\mathcal{D}(\mathbb{G})$ is dense in $W^{m,p}(U)$.*

(iv) *If $1 < p < \infty$, then $W^{m,p}(U)$ is reflexive.*

(v) $W^{m,p}_{\mathrm{Euc,loc}}(U) \subset W^{m,p}(U)$, *i.e., for any $V \subset\subset U$ and for any $u \in W^{m,p}_{\mathrm{Euc,loc}}(U)$,*

$$\|u\|_{W^{m,p}(V)} \le C_V \|u\|_{W^{m,p}_{\mathrm{Euc}}(V)}.$$

(vi) $W^{\kappa m,p}(U) \subset W^{m,p}_{\mathrm{Euc,loc}}(U)$, *i.e., for any $V \subset\subset U$ and for any $u \in W^{\kappa m,p}(U)$,*

$$\|u\|_{W^{m,p}_{\mathrm{Euc}}(V)} \le C_V \|u\|_{W^{\kappa m,p}(U)}.$$

DEFINITION 2.3. Let $\mathbb{G}$ be a Carnot group. A *homogeneous norm* $\|\cdot\|$ on $\mathbb{G}$ is a continuous function

$$\|\cdot\| : \mathbb{G} \to [0, +\infty)$$

such that

$$\begin{aligned} &\|p\| = 0 \iff p = 0;\\ &\|p^{-1}\| = \|p\|;\\ &\big\|\delta_\lambda(p)\big\| = \lambda\|p\|;\\ &\|p\cdot q\| \le \|p\| + \|q\|, \end{aligned} \tag{3}$$

for all $p, q \in \mathbb{G}$ and all $\lambda > 0$.

A homogeneous norm induces a homogeneous left-invariant distance d in $\mathbb{G}$ in a standard way. If $p \in \mathbb{G}$ and $r > 0$, we denote by $B_d = B_d(p,r)$ the open d-ball centered at p with radius r.

In a Carnot group $\mathbb{G}$, we shall consider in particular the homogeneous norm defined in the following theorem.

THEOREM 2.4 (see [10]). *Let $\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa$ be a Carnot group. Let $\|\cdot\|_{V_1}, \dots, \|\cdot\|_{V_\kappa}$ be fixed Euclidean norms on the layers.*

Then, there exist constants $\varepsilon_1, \dots, \varepsilon_\kappa$, with $\varepsilon_1 = 1$ and $\varepsilon_2, \dots, \varepsilon_\kappa \in (0, 1]$, depending only on the group $\mathbb{G}$ and the norms $\|\cdot\|_{V_1}, \dots, \|\cdot\|_{V_\kappa}$, such that the functions

$$\|x\|_\infty := \max_j \varepsilon_j \left(\|x^{(j)}\|_{V_j}\right)^{1/j} \tag{4}$$

are homogeneous norms on $\mathbb{G}$.

We denote by d_∞ the homogeneous left-invariant distance associated with $\|\cdot\|_\infty$ and by B_∞ the metric balls of d_∞.

We stress that the balls $B_\infty(e, r)$ are convex.

The vectors of V_1, also called horizontal vectors, define by left translations the *horizontal bundle*, which we also denote by V_1. A section of the horizontal bundle is called a horizontal vector field.

If $F = \sum_{i=1}^{m_1} F_i X_i$ is a horizontal vector field,

$$F \in L^1_{\mathrm{loc}}(\mathbb{G}, V_1),$$

we define

$$\operatorname{div}_{\mathbb{G}} F := \sum_j X_j F_j \tag{5}$$

in the sense of distributions.

3. Main result

The main result of this note is stated in the following theorem.

THEOREM 3.1. *Let d be a left-invariant distance on a Carnot group associated with a homogeneous norm. Suppose $1 \le p < Q$ and $\lambda > 1$. Set $B := B_d(e, r)$ and $B' := B_d(e, \lambda r)$. If $f \in L^p(B)$ is compactly supported and satisfies*

$$\int_B f(p)\, dp = 0,$$

then there exists a compactly supported horizontal vector field $F \in L^q(B', V_1)$, where

(i) $1 \le q \le \frac{pQ}{Q-p}$ *if $p > 1$, or*

(ii) $1 \le q < \frac{Q}{Q-1}$ *if $p = 1$,*

such that

$$f = \operatorname{div}_{\mathbb{G}} F \quad \text{in } B.$$

Additionally, there exists a constant $C = C(p, q, \lambda, B)$, independent of f, such that

$$\|F\|_{L^q(B', V_1)} \le C \|f\|_{L^p(B)}.$$

If $p > 1$ and $q = \frac{pQ}{Q-p}$, then the constant C does not depend on B.

Our proof of Theorem 3.1 involves several steps and relies on Sobolev inequalities for differential forms in Rumin's complex. In the next subsection, we recall the key features of the Rumin's complex.

3.1. Rumin's Complex

Let $\mathfrak{g}$ be the Lie algebra of the Carnot group $\mathbb{G}$. The dual space of $\mathfrak{g}$ is denoted by $\bigwedge^1 \mathfrak{g}$. The basis dual to $\{X_1, \dots, X_n\}$ is the family of covectors $\{\theta_1, \dots, \theta_n\}$.

Following Federer (see [7, Section 1.3]), the exterior algebras of $\mathfrak{g}$ and of $\bigwedge^1 \mathfrak{g}$ are the graded algebras indicated as

$$\bigwedge_\bullet \mathfrak{g} := \bigoplus_{k=0}^{n} \bigwedge_k \mathfrak{g} \quad \text{and} \quad \bigwedge^\bullet \mathfrak{g} := \bigoplus_{k=0}^{n} \bigwedge^k \mathfrak{g}$$

where $\bigwedge_0 \mathfrak{g} = \bigwedge^0 \mathfrak{g} = \mathbb{R}$ and, for $1 \le k \le n$,

$$\bigwedge_k \mathfrak{g} := \operatorname{span}\{X_{i_1} \wedge \dots \wedge X_{i_k} : 1 \le i_1 < \dots < i_k \le n\},$$
$$\bigwedge^k \mathfrak{g} := \operatorname{span}\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} : 1 \le i_1 < \dots < i_k \le n\}.$$

The elements of $\bigwedge_k \mathfrak{g}$ and $\bigwedge^k \mathfrak{g}$ are called *k-vectors* and *k-covectors*.

We denote by Θ^k the basis $\{\theta_{i_1} \wedge \dots \wedge \theta_{i_k} : 1 \le i_1 < \dots < i_k \le n\}$ of $\bigwedge^k \mathfrak{g}$.

We denote also by $dV := \theta_1 \wedge \dots \wedge \theta_n$ the volume form associated with our adapted basis of $\mathfrak{g}$, which can be though as the Lebesgue measure on $\mathbb{R}^n$ up to a suitable normalization constant. Obviously, $\bigwedge^n \mathfrak{g} := \operatorname{span}\{dV\}$.

The dual space $\bigwedge^1(\bigwedge_k \mathfrak{g})$ of $\bigwedge_k \mathfrak{g}$ can be naturally identified with $\bigwedge^k \mathfrak{g}$. The action of a k-covector φ on a k-vector v is denoted as $\langle \varphi | v \rangle$.

The inner product $\langle \cdot, \cdot \rangle$ extends canonically to $\bigwedge_k \mathfrak{g}$ and to $\bigwedge^k \mathfrak{g}$ making the bases $X_{i_1} \wedge \dots \wedge X_{i_k}$ and $\theta_{i_1} \wedge \dots \wedge \theta_{i_k}$ orthonormal.

Definition 3.2. For $1 \le k \le n$, we define linear isomorphisms (Hodge duality: see [7, Section 1.7.8])

$$\star : \bigwedge_k \mathfrak{g} \longleftrightarrow \bigwedge_{n-k} \mathfrak{g}$$

as follows.

If $I = \{i_1, \dots, i_k\}$, $1 \le i_1 < \dots < i_k \le n$, $I^\star = \{i_1^\star < \dots < i_{n-k}^\star\} = \{1, \dots, n\} \setminus I$, and

$$X_I = X_{i_1} \wedge \dots \wedge X_{i_k},$$

we write

$$\star X_I := (-1)^{\sigma(I)} X_{I^\star},$$

where $\sigma(I)$ is the number of couples $(i_h, i_\ell^\star)$ with $i_h > i_\ell^\star$. Hence, putting $v = \sum_I v_I X_I$ we set

$$\star v := \sum_I v_I(\star X_I).$$

Analogously, the Hodge operator

$$\star : \bigwedge^k \mathfrak{g} \longleftrightarrow \bigwedge^{n-k} \mathfrak{g}$$

can be defined as follows.

If $\theta_I = \theta_{i_1} \wedge \cdots \wedge \theta_{i_k}$, we write

$$\star\theta_I := (-1)^{\sigma(I)}\theta_{I^\star}.$$

For $\varphi = \sum_I \varphi_I \theta_I$, we put

$$\star\varphi := \sum_I \varphi_I(\star\theta_I).$$

Notice that if $v = v_1 \wedge \cdots \wedge v_k$ is a simple k-vector, then $\star v$ is a simple $(n-k)$-vector. If $v \in \bigwedge_k \mathfrak{g}$, we define $v^\natural \in \bigwedge^k \mathfrak{g}$ by the identity $\langle v^\natural | w\rangle := \langle v, w\rangle$, and analogously we define $\varphi^\natural \in \bigwedge_k \mathfrak{g}$ for $\varphi \in \bigwedge^k \mathfrak{g}$.

Definition 3.3. If $\alpha \in \bigwedge^1 \mathfrak{g}$, $\alpha \neq 0$, we say that α has *pure weight* k, and we write $w(\alpha) = k$ if $\alpha^\natural \in V_k$. More generally, if $\alpha \in \bigwedge^h \mathfrak{g}$, we say that α has pure weight k if α is a linear combination of covectors $\theta_{i_1} \wedge \cdots \wedge \theta_{i_h}$ with $w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = k$.

Obviously, if for example $\alpha \in \bigwedge^1 \mathfrak{g}$,

$$w(\alpha) = k \quad \text{if and only if } \alpha = \sum_{j=h_{k-1}+1}^{h_k} \alpha_j \theta_j,$$

with $\alpha_{h_{k-1}+1}, \dots, \alpha_{h_k} \in \mathbb{R}$.

Remark 3.4 (see [3, Remark 2.6]). If $\alpha, \beta \in \bigwedge^h \mathfrak{g}$ and $w(\alpha) \neq w(\beta)$, then $\langle \alpha, \beta\rangle = 0$.

We have

$$\bigwedge^h \mathfrak{g} = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \bigwedge^{h,p} \mathfrak{g}, \tag{6}$$

where $\bigwedge^{h,p} \mathfrak{g}$ is the linear span of the h–covectors of pure weight p and $M_h^{\min}$, $M_h^{\max}$ are respectively the smallest and the largest weight of h-covectors.

Since the elements of the basis Θ^h have pure weights, a basis of $\bigwedge^{h,p}\mathfrak{g}$ is given by $\Theta^{h,p} := \Theta^h \cap \bigwedge^{h,p}\mathfrak{g}$ (in Section 2, we called such a basis an adapted basis).

We denote by $\Omega^{h,p}$ the vector space of all smooth h–forms in $\mathbb{G}$ of pure weight p, i.e. the space of all smooth sections of $\bigwedge^{h,p}\mathfrak{g}$. We have

$$\Omega^h = \bigoplus_{p=M_h^{\min}}^{M_h^{\max}} \Omega^{h,p}. \tag{7}$$

Lemma 3.5. *We have $d(\bigwedge^{h,p}\mathfrak{g}) = \bigwedge^{h+1,p}\mathfrak{g}$; i.e., if $\alpha \in \bigwedge^{h,p}\mathfrak{g}$ is a left invariant h-form of weight p, then $w(d\alpha) = w(\alpha)$.*

Proof. See [15, Section 2.1]. ■

Let now $\alpha \in \Omega^{h,p}$ be a (say) smooth form of pure weight p. We can write

$$\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i\, \theta_i^h, \quad \text{with } \alpha_i \in \mathcal{E}(\mathbb{G}).$$

Then,

$$d\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{n} (X_j\alpha_i)\theta_j \wedge \theta_i^h + \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i\, d\theta_i^h.$$

Hence, we can write

$$d = d_0 + d_1 + \cdots + d_\kappa,$$

where

$$d_0\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \alpha_i\, d\theta_i^h$$

does not increase the weight,

$$d_1\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{j=1}^{m_1} (X_j\alpha_i)\theta_j \wedge \theta_i^h$$

increases the weight of 1, and, more generally,

$$d_k\alpha = \sum_{\theta_i^h \in \Theta^{h,p}} \sum_{w(\theta_j)=k} (X_j\alpha_i)\theta_j \wedge \theta_i^h, \quad k = 1, \dots, \kappa.$$

In particular, d_0 is an algebraic operator.

Definition 3.6. If $0 \le h \le n$ and we denote by d_0^* the L^2-adjoint of d_0, we set

$$E_0^h := \ker d_0 \cap \ker d_0^* = \ker d_0 \cap (\operatorname{Im} d_0)^\perp \subset \Omega^h.$$

Since the construction of E_0^h is left invariant, this space of forms can be viewed as the space of sections of a fiber bundle, generated by left translation and still denoted by E_0^h.

We denote by $N_h^{\min}$ and $N_h^{\max}$ the minimum and the maximum, respectively, of the weights of forms in E_0^h.

If we set $E_0^{h,p} := E_0^h \cap \Omega^{h,p}$, then

$$E_0^h = \bigoplus_{p=N_h^{\min}}^{N_h^{\max}} E_0^{h,p}.$$

We notice that also the space of forms $E_0^{h,p}$ can be viewed as the space of smooth sections of a suitable fiber bundle generated by left translations, which we still denote by $E_0^{h,p}$.

As customary, if $\Omega \subset \mathbb{G}$ is an open set, we denote by $\mathcal{E}(\Omega, E_0^h)$ the space of smooth sections of E_0^h.

The spaces $\mathcal{D}(\Omega, E_0^h)$ and $\mathcal{S}(\mathbb{G}, E_0^h)$ are defined analogously.

Since both $E_0^{h,p}$ and E_0^h are left invariant as $\bigwedge^h \mathfrak{g}$, they are subbundles of $\bigwedge^h \mathfrak{g}$ and inherit the scalar product on the fibers.

In particular, we can obtain a left invariant orthonormal basis $\Xi_0^h = \{\xi_j^h\}$ of E_0^h such that

$$\Xi_0^h = \bigcup_{p=N_h^{\min}}^{N_h^{\max}} \Xi_0^{h,p}, \tag{8}$$

where $\Xi_0^{h,p} := \Xi^h \cap \bigwedge^{h,p} \mathfrak{g}$ is a left invariant orthonormal basis of $E_0^{h,p}$. All the elements of $\Xi_0^{h,p}$ have pure weight p.

Once the basis Θ_0^h is chosen, the spaces $\mathcal{E}(\Omega, E_0^h)$, $\mathcal{D}(\Omega, E_0^h)$, $\mathcal{S}(\mathbb{G}, E_0^h)$ can be identified with $\mathcal{E}(\Omega)^{\dim E_0^h}$, $\mathcal{D}(\Omega)^{\dim E_0^h}$, $S(\mathbb{G})^{\dim E_0^h}$, respectively.

PROPOSITION 3.7 ([15]). *If $0 \le h \le n$ and $*$ denote the Hodge duality (see Definition 3.2), then*

$$\star E_0^h = E_0^{n-h}.$$

By a simple linear algebra argument, we can prove the following lemma.

LEMMA 3.8. *If $\beta \in \Omega^{h+1}$, then there exists a unique $\alpha \in \Omega^h \cap (\ker d_0)^\perp$ such that*

$$d_0^* d_0 \alpha = d_0^* \beta.$$

With the notation of the lemma, we set $\alpha := d_0^{-1}\beta$.

REMARK 3.9. We stress that d_0^{-1} is an algebraic operator, like d_0 and its adjoint d_0^*.

LEMMA 3.10 ([15]). *The map* $d_0^{-1}d$ *induces an isomorphism from* $\mathrm{Im}(d_0^{-1})$ *to itself. In addition, there exists a differential operator*

$$P = \sum_{k=1}^{N}(-1)^k D^k, \quad N \in \mathbb{N} \textit{ suitable},$$

such that

$$P d_0^{-1} d = d_0^{-1} d P = \mathrm{Id}_{\mathrm{Im}(d_0^{-1})}.$$

We set $Q := P d_0^{-1}$.

REMARK 3.11. If α has pure weight k, then $P\alpha$ is a sum of forms of pure weight greater than or equal to k.

We state now the following key results.

THEOREM 3.12 ([15]). *The de Rham complex* $(\Omega^\bullet, d)$ *splits as the direct sum of two sub-complexes* $(E^\bullet, d)$ *and* $(F^\bullet, d)$*, with*

$$E^h := \ker d_0^{-1} \cap \ker(d_0^{-1} d) \quad \textit{and} \quad F^h := \mathrm{Im}(d_0^{-1}) + \mathrm{Im}(d d_0^{-1}),$$

for $h = 0, \dots, n$*, such that we have the following:*

(i) *The projection* Π_E *on* E *along* F *is given by* $\Pi_E = \mathrm{Id} - Qd - dQ$*. In particular,* Π_E *is a differential operator of order* $s \geq 0$ *in the horizontal derivatives, where* s *depends on* $\mathbb{G}$ *and on the degree of the forms it acts on.*

(ii) *If* Π_{E_0} *is the orthogonal projection from* Ω^h *on* E_0^h*, then* $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$ *and* $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$.

THEOREM 3.13 ([15]). *If we set*

$$d_c := \Pi_{E_0}\, d\, \Pi_E,$$

then $d_c : E_0^h \to E_0^{h+1}$ *satisfies*

(i) $d_c^2 = 0$*;*

(ii) *the complex* $(E_0^\bullet, d_c)$ *is exact.*

In particular, if $h = 0$ and $f \in E_0^0 = \mathcal{E}(\mathbb{G})$, then

$$d_c f = \sum_{i=1}^{m}(X_i f)\theta_i^1$$

is the *horizontal differential of* f.

In addition, by Proposition 3.7, $E_0^n = \{f\, dV,\ f \in \mathcal{E}(\mathbb{G})\}$.

REMARK 3.14 (see [3, Remark 2.17]). We have

$$d\,\Pi_E = \Pi_E d. \tag{9}$$

It follows from [3, Proposition 2.18] that if $\alpha \in E_0^h$ has weight p, then

$$\Pi_E\alpha = \alpha + \text{terms of weight greater than } p.$$

REMARK 3.15. In particular, if $\alpha \in E_0^n$ (and therefore has weight Q), then $\Pi_E\alpha = \alpha$ since there are no forms of weight $> Q$.

DEFINITION 3.16. We denote by d_c^* the L^2-(formal) adjoint of d_c.

We recall that on E_0^h,

$$d_c^* = (-1)^{n(h+1)+1} \star d_c \star .$$

3.2. *Equivalent formulation and proof of Theorem 3.1*

Let us start by noticing that d_c^* on 1-forms can be identified with the horizontal divergence. Indeed, if $F = \sum_{i=1}^{m_1} F_i X_i \in L^1_{\mathrm{loc}}(\mathbb{G}, V_1)$, we denote by $F^\sharp$ the differential 1-form defined by

$$\langle F^\natural | V\rangle = \langle F, V\rangle = \sum_j \int_{\mathbb{G}} F_j V_j \, dp$$

for any $V = \sum_{i=1}^{m_1} V_i X_i \in \mathcal{D}(\mathbb{G}, V_1)$, i.e.

$$F^\natural = \sum_i F_i\, \theta_i.$$

If now $\phi \in \mathcal{D}(\mathbb{G})$, then (keeping in mind that $d_c^* F^\natural$ is a 0-form)

$$\int_{\mathbb{G}} \phi\, d_c^* F^\natural\, dp = \int_{\mathbb{G}} \langle F^\natural, d_c\phi\rangle\, dp = \sum_j \int_{\mathbb{G}} F_j\, X_j\phi = -\int_{\mathbb{G}} \phi\, \mathrm{div}_{\mathbb{G}} F, \tag{10}$$

where the above identities are meant in the sense of distributions. Hence, $f = \mathrm{div}_{\mathbb{G}} F$ if and only if $d_c^* F^\natural = f$, i.e.

$$-\star d_c \star F^\natural = f.$$

Applying Hodge operator to identity, and keeping in mind that $d_c \star F^\natural$ is an n-form and hence $\star\star = \mathrm{Id}$, we obtain

$$-\star \star d_c \star F^\natural = \star f,$$

i.e.

$$d_c(-\star F^{\natural}) = f\, dV.$$

If we set $\phi := -\star F^{\natural}$ and $\omega := f\, dV$, an equivalent formulation of Theorem 3.1 becomes as follows.

THEOREM 3.17. *Let d be a left invariant distance on a Carnot group associated with a homogeneous norm. Let $1 \le p < Q$ and $\lambda > 1$, and set $B := B_d(e, r)$ and $B' := B_d(e, \lambda r)$. If $\omega \in L^p(B, E_0^n)$ is compactly supported and satisfies*

$$\int_B \omega = 0,$$

then there exists a compactly supported differential form $\phi \in L^q(B', E_0^{n-1})$ with

(i) $1 \le q \le pQ/(Q-p)$ *if* $p > 1$

or

(ii) $1 \le q < Q/(Q-1)$ *if* $p = 1$,

so that

$$d_c\phi = \omega \quad \text{in } B.$$

In addition, there exists $C = C(p, q, \lambda, B)$ independent of ω such that

$$\|\phi\|_{L^q(B', E_0^{n-1})} \le C\|\omega\|_{L^p(B, E_0^n)}.$$

If $p > 1$ and $q = pQ/(Q-p)$, the constant does not depend on B.

Since different homogeneous norms are equivalent (see, e.g., [12, Section 1.2]), without loss of generality, from now on we may assume that $d = d_\infty$ and for the sake of simplicity, we shall write $B(e, r)$ for $B_\infty(e, r)$. From now on, for the sake of simplicity, by a rescaling argument and since d_c is homogeneous with respect to the group dilations, we take $r = 1$ in Theorems 3.1 and 3.17, that is, $B = B(e, 1)$.

The first step in order to prove Theorem 3.17 will be to define an operator acting on n-forms which inverts Rumin's differential d_c (albeit with a loss of regularity). Inspired by the work of [13], Mitrea, Mitrea, and Monniaux, in [14], define a compact homotopy operator $J_{\mathrm{Euc},h}$ in Lipschitz star-shaped domains in Euclidean space $\mathbb{R}^n$, providing an explicit representation formula for $J_{\mathrm{Euc},h}$, together with continuity properties among Sobolev spaces. Since in this note we are interested in forms of top degree n, we recall what [14, Theorem 4.1] states only in this particular case. Theorem 4.1 of [14] says that if $D \subset \mathbb{R}^N$ is a star-shaped Lipschitz domain, then there exists

$$J_{\mathrm{Euc},h} : L^p\Big(D, \bigwedge\nolimits^n\Big) \to W^{1,p}_{\mathrm{Euc}}\Big(D, \bigwedge\nolimits^{n-1}\Big) \hookrightarrow W^{\kappa,p}(D, E_0^{n-1})$$

such that

$$\omega = dJ_{\mathrm{Euc},n}\omega + \Big(\int_D \omega\Big)\theta\, dV \quad \text{for all } \omega \in \mathcal{D}\Big(D, \bigwedge^n\Big), \tag{11}$$

where $\theta \in \mathcal{D}(\mathbb{G})$ satisfies

$$\int_{\mathbb{G}} \theta\, dp = 1.$$

Furthermore, $J_{\mathrm{Euc},n}$ maps smooth compactly supported forms to smooth compactly supported forms.

For the sake of simplicity, from now on we drop the index n – the degree of the form – writing, e.g., J_{Euc} instead of $J_{\mathrm{Euc},n}$.

To our aim, take now $D = B$. If $\omega \in \mathcal{D}(B, E_0^n)$, with vanishing average, we set

$$J = \Pi_{E_0} \circ \Pi_E \circ J_{\mathrm{Euc}} \circ \Pi_E. \tag{12}$$

Since $\Pi_E \omega = \omega$ on E_0^n, we can also write

$$J\omega = \Pi_{E_0} \circ \Pi_E \circ J_{\mathrm{Euc}}\omega. \tag{13}$$

Then, J inverts Rumin's differential d_c on forms of degree n in the sense of the following result.

Lemma 3.18. *If $\alpha \in E_0^n$ is a compactly supported smooth form in a ball $\widetilde{B}$ with*

$$\int_{\widetilde{B}} \alpha = 0,$$

then

$$\alpha = d_c J\alpha. \tag{14}$$

In addition, $J\alpha$ is compactly supported in $\widetilde{B}$.

Proof. By (11),

$$\alpha = dJ_{\mathrm{Euc}}\alpha. \tag{15}$$

We recall now that $\Pi_E\Pi_{E_0}\Pi_E = \Pi_E$ and $\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0}$. In addition, on forms of degree $n-1$, $d\Pi_E = \Pi_E d$. Thus, by (15),

$$\begin{aligned} d_c J\alpha &= \Pi_{E_0} d\Pi_E\Pi_{E_0}\Pi_E J_{\mathrm{Euc}}\alpha = \Pi_{E_0} d\Pi_E J_{\mathrm{Euc}}\alpha \\ &= \Pi_{E_0}\Pi_E dJ_{\mathrm{Euc}}\alpha = \Pi_{E_0}\Pi_E\alpha = \Pi_{E_0}\alpha = \alpha \end{aligned}$$

since $\alpha \in E_0^n$. Finally, if $\operatorname{supp}\alpha \subset \widetilde{B}$, then $\operatorname{supp} J\alpha \subset \widetilde{B}$ since both Π_E and Π_{E_0} preserve the support. ■

Unfortunately, the operator J contains the differential operator Π_E that yields a loss of regularity. We can get rid of this inconvenient combining J with a smoothing operator coming from an approximated homotopy formula. The approximated homotopy formula is based on a global homotopy identity relying on the inverse of Rumin's Laplacian.

Indeed, if $\omega = f dV \in \mathcal{D}(\mathbb{G}, E_0^n)$, we can define its sub-Laplacian as

$$\Delta_{\mathbb{G},n}\omega := d_c d_c^* \omega.$$

Since $\star\star = \mathrm{Id}$ on n-forms,

$$\Delta_{\mathbb{G},n}\omega = \star\Delta_{\mathbb{G},0} \star \omega,$$

and the fundamental solution $\Delta_{\mathbb{G},n}^{-1}$ of $\Delta_{\mathbb{G},n}$ is given by

$$\Delta_{\mathbb{G},n}^{-1} = \star\Delta_{\mathbb{G},0}^{-1}\star$$

that is associated with a kernel of type 2 (see [8]).

We are now able to prove the equivalent formulation of Theorem 3.1 arguing as in [2, Theorem 5.12].

PROOF OF THEOREM 3.17. Suppose first that $\omega \in \mathcal{D}(B, E_0^n)$. If ω is continued by zero on all of $\mathbb{G}$, we notice preliminarily that

$$\omega = \Delta_{\mathbb{G},n}\Delta_{\mathbb{G},n}^{-1}\omega = d_c(d_c^*\Delta_{\mathbb{G},n}^{-1})\omega,$$

where $d_c^*\Delta_{\mathbb{G},n}^{-1}$ is associated with a matrix-valued kernel k_1 of type 1 acting on f. Keeping in mind that, by Hodge duality, ω can be identified with the function f, without loss of generality, we can treat k_1 as it were a scalar kernel. We consider a cut-off function ψ_R supported in an R-neighborhood of the origin, such that $\psi_R \equiv 1$ near the origin. We can write

$$k_1 = \psi_R k_1 + (1-\psi_R)k_1. \tag{16}$$

Since the kernel of $\Delta_{\mathbb{G},n}^{-1}$ is of type 2, the kernel $\psi_R k_1$ belongs to $L^1(\mathbb{G})$. Let us denote by $K_{1,R}$ the convolution operator associated with $\psi_R k_1$ and by S the convolution operator associated with the kernel

$$K_S := d_c\big((1-\psi_R)k_1\big). \tag{17}$$

It follows from (16) that

$$\omega = d_c K_{1,R}\omega + S\omega \quad \text{if } \omega \in \mathcal{D}(B, E_0^n). \tag{18}$$

The kernel K_S is smooth. We stress also that supp $K_{1,R}\omega$ is contained in an R-neighborhood of B so that

$$\operatorname{supp} K_{1,R}\omega \subset B' \tag{19}$$

provided $R = R(\lambda) < d(B, \partial B')$. By (18), also

$$\operatorname{supp} S\omega \subset B'. \tag{20}$$

Finally, by (18), $S\omega \in E_0^n$.

The homotopy formula (18) still holds in the sense of distributions when $\omega \in L^p$. To prove that, we need the following lemma.

Lemma 3.19. *With S and $K_{1,R}$ defined as above, we have the following:*

(i) *S is regularizing from $\mathcal{E}'(\mathbb{G})$ to $\mathcal{E}(\mathbb{G})$. In addition, if $p, q \geq 1$ and $m \in \mathbb{N} \cup \{0\}$, then S can be continued as a bounded map from $L^p(B, E_0^n) \cap \mathcal{E}'(B, E_0^n)$ to $W^{m,q}(B', E_0^n)$*
$$S : L^p(B, E_0^n) \to W^{m,q}(B', E_0^n).$$
In particular, by Theorem 2.2 (vi), due to the arbitrariness of the choice of m, we also have
$$S : L^p(B, E_0^n) \to W^{m,q}_{\text{Euc}}(B', E_0^n);$$

(ii) *if $p \geq 1$, the map $K_{1,R}$ can be continued as a bounded map from $L^p(B, E_0^n) \cap \mathcal{E}'(B, E_0^n)$ to $L^p(B', E_0^n)$;*

(iii) *if $p > 1$, then the map $K_{1,R}$ can be continued as a bounded map from $L^p(B, E_0^n) \cap \mathcal{E}'(B, E_0^n)$ to $W^{1,p}(B', E_0^n)$ and the identity (16) still holds for*
$$\omega \in L^p(B, E_0^n) \cap \mathcal{E}'(B, E_0^n);$$

(iv) *the identity (16) still holds for $\omega \in L^1(B, E_0^n) \cap \mathcal{E}'(B, E_0^n)$ in the sense of distributions;*

(v) *if $p > 1$, then*
$$K_{1,R} : L^p(B, E_0^n) \cap \mathcal{E}'(B, E_0^n) \to L^q(B', E_0^{n-1}) \quad \text{for } p \leq q \leq Q/(Q-1);$$

(vi) $K_{1,R} : L^1(B, E_0^n) \cap \mathcal{E}'(B, E_0^n) \to L^q(B', E_0^{n-1})$ *for* $1 \leq q < Q/(Q-1)$.

Proof. Let us prove (i). Since the kernel K_S is smooth and the convolution maps $\mathcal{E}'(\mathbb{G}) \times \mathcal{E}(\mathbb{G})$ into $\mathcal{E}(\mathbb{G})$, the operator S is regularizing from $\mathcal{E}'(\mathbb{G})$ to $\mathcal{E}(\mathbb{G})$ (see [16, p. 167]). In addition, since B is bounded, then without loss of generality, we may assume that $p = 1$.

Remember $\omega = f dV$; hence, we can identify ω and the scalar function f. We have

$$
\begin{aligned}
\|S\omega\|_{W^{m,q}(B',E_0^n)} &= \|\omega * K_S\|_{W^{m,q}(B')} \\
&= \sum_{d(I)\le m} \|\omega * X^I K_S\|_{L^q(B')} \\
&= \sum_{d(I)\le m} \left(\int_{B'} \left(\int_B |\omega(y)| \left| X^I K_S(y^{-1}x) \right| dy \right)^q dx \right)^{1/q}.
\end{aligned}
$$

Notice now that if $x \in B'$ and $y \in B$, then $y^{-1}x \in B(e, 1+\lambda)$. Thus, if $\chi \in \mathcal{D}(\mathbb{G})$ is a cut-off function, $\chi \equiv 1$ on $B(e, 1+\lambda)$, then $\chi X K_S \in L^q(\mathbb{G})$, so that, by Young's inequality (see Theorem 4.3 (i) and [9, Proposition 1.18]),

$$
\begin{aligned}
\|S\omega\|_{W^{m,q}(B',E_0^n)} &\le \sum_{d(I)\le m} \left\| |\chi\omega(y)| * |X^I K_S| \right\|_{L^q(\mathbb{G})} \\
&\le C \left\|\omega(y)\right\|_{L^1(\mathbb{G})} = C \left\|\omega(y)\right\|_{L^1(B)}.
\end{aligned}
$$

Proof of (ii). By a similar argument,

$$
\begin{aligned}
\|K_{1,R}\omega\|_{L^p(B',E_0^n)} &\le \|\omega * \psi_R k_1\|_{L^p(B',E_0^n)} \\
&\le \left(\int_{B'} \left(\int_B |\omega(y)| \left| \psi_R k_1(y^{-1}x) \right| dy \right)^p dx \right)^{1/p} \\
&\le C \|\psi_R k_1\|_{L^1(B(1+\lambda))} \|\omega\|_{L^p(B,E_0^n)}.
\end{aligned}
$$

Proof of (iii). Let X be a horizontal derivative. Then, we have only to estimate the L^p-norm of

$$
X(\omega * \psi_R k_1) = \omega * (X\psi_R) k_1 + \omega * (\psi_R X k_1).
$$

By Lemma 4.4,

$$
\begin{aligned}
\left\|\omega * (X\psi_R)k_1\right\|_{L^p(B')} &\le C \left\|\omega * (X\psi_R)k_1\right\|_{L^{pQ/(Q-p)}(B')} \\
&\le C \left\|\omega * (X\psi_R)k_1\right\|_{L^{pQ/(Q-p)}(\mathbb{G})} \\
&\le C \|\omega\|_{L^p(\mathbb{G})} = C \|\omega\|_{L^p(B)};
\end{aligned}
$$

analogously, since Xk_1 is a kernel of type 0,

$$
\left\|\omega * (\psi_R X k_1)\right\|_{L^p(B')} \le C \|\omega\|_{L^p(\mathbb{G})} \le C \|\omega\|_{L^p(B)}.
$$

Finally, since ω is compactly supported in B, it can be approximated in $L^p(B)$ by a sequence $(\omega_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(B)$. Thus,

$$
d_c K_{1,R}\omega_k \to d_c K_{1,R}\omega \quad \text{in } L^p(\mathbb{G}) \text{ as } k \to \infty.
$$

In addition, by (i),

$$S\omega_k \to S\omega \quad \text{in } L^p(\mathbb{G}) \text{ as } k \to \infty,$$

and (iii) is proved

Proof of (iv). Take a sequence $(\omega_k)_{k\in\mathbb{N}}$ as in the proof of (iii). By (ii),

$$K_{1,R}\omega_k \to K_{1,R}\omega \quad \text{in } L^p(\mathbb{G}) \text{ as } k \to \infty.$$

In particular, $d_c K_{1,R}\omega_k \to d_c K_{1,R}\omega$ in the sense of distributions. Then, (iv) follows from (16).

Proof of (v). The statement follows by Lemma 4.4.

Proof of (vi). The statement follows by Remark 4.10. ∎

Let us resume the proof of Theorem 3.1. Since S is a smoothing operator, then $S\omega \in \mathcal{D}(B', E_0^n)$, keeping also in mind that $S\omega$ is supported in B' (see (20)).

We notice also that for any $p \geq 1$, $S\omega$ has vanishing average since ω has vanishing average. Indeed, take $\chi \in D(\mathbb{G})$, $\chi \equiv 1$ on B'. Again, identify $\omega = f\,dV$ with the scalar function f; we have, by Lemma 3.19 (iii) and (iv), that the homotopy formula (18) holds in the sense of distributions. Therefore,

$$\begin{aligned}\int_{B'} S\omega\, dV &= \int_{B'} \chi S\omega\, dV \\ &= \int_{\mathbb{G}} \chi\omega\, dV + \int_{\mathbb{G}} (d_c\chi) \wedge K_{1,R}\omega = 0\end{aligned}$$

since $d_c\chi = 0$ on $\operatorname{supp} K_{1,R}\omega$.

Since $S\omega$ has vanishing average, we can apply (14) to $\alpha := S\omega$ and we get $S\omega = d_c JS\omega$, where J is defined in (12). By Lemma 3.18, $JS\omega$ is supported in B'. Thus, if we set $\phi := (JS + K_{1,R})\omega$, then ϕ is supported in B'. Moreover, $d_c\phi = d_c JS\omega + d_c K_{1,R}\omega = S\omega + \omega - S\omega = \omega$.

Remember now that, by Theorem 3.12 (i), Π_E on forms of degree $(n-1)$ is a differential operator of order $s \geq 0$ in the horizontal derivatives. Thus, by Lemma 3.19,

$$\begin{aligned}\|\phi\|_{L^q(B',E_0^{n-1})} &\leq \|JS\omega\|_{L^q(B',E_0^{n-1})} + \|K_{1,R}\omega\|_{L^q(B',E_0^{n-1})} \\ &\leq \|JS\omega\|_{L^q(B',E_0^{n-1})} + C\|\omega\|_{L^p(B',E_0^n)} \\ &\leq \|S\omega\|_{W^{s-1,q}_{\mathrm{Euc}}(B',E_0^{n-1})} + C\|\omega\|_{L^p(B',E_0^n)} \\ &\leq \|S\omega\|_{W^{s,q}_{\mathrm{Euc}}(B',E_0^{n-1})} + C\|\omega\|_{L^p(B',E_0^n)} \\ &\leq C\big(\|S\omega\|_{W^{\kappa s,q}(B,E_0^h)} + \|\omega\|_{L^p(B',E_0^n)}\big) \\ &\leq C\|\omega\|_{L^p(B',E_0^n)}.\end{aligned} \tag{21}$$

This completes the proof of the theorem. ∎

4. Appendix: Kernels in Carnot groups

Following [8, 9], we now recall the notion of a *kernel of type* μ and some related properties, as outlined in Propositions 4.2 and 4.3 below. For these results, we refer to [1, Section 3.2].

Definition 4.1. A kernel of type μ is a homogeneous distribution of degree $\mu - Q$ (with respect to group dilations) that is smooth outside of the origin.

The convolution operator with a kernel of type μ is still called an operator of type μ.

Proposition 4.2. *Let $K \in \mathcal{D}'(\mathbb{G})$ be a kernel of type μ.*

(i) ${}^{\mathrm{v}}K$ *is again a kernel of type* μ*;*

(ii) *WK and KW are associated with kernels of type $\mu - 1$ for any horizontal derivative W;*

(iii) *if $\mu > 0$, then $K \in L^1_{\mathrm{loc}}(\mathbb{G})$.*

Theorem 4.3. *We have the following:*

(i) *Hausdorff–Young inequality holds; i.e., if $f \in L^p(\mathbb{G})$, $g \in L^q(\mathbb{G})$, $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, then $f * g \in L^r(\mathbb{G})$ (see* [9, Proposition 1.18]*).*

(ii) *If K is a kernel of type 0, $1 < p < \infty$, then the mapping $T : u \to u * K$ defined for $u \in \mathcal{D}(\mathbb{G})$ extends to a bounded operator on $L^p(\mathbb{G})$ (see* [8, Theorem 4.9]*).*

(iii) *Suppose $0 < \mu < Q$, $1 < p < Q/\mu$ and $\frac{1}{q} = \frac{1}{p} - \frac{\mu}{Q}$. Let K be a kernel of type μ. If $u \in L^p(\mathbb{G})$, the convolutions $u * K$ and $K * u$ exist a.e. and are in $L^q(\mathbb{G})$ and there is a constant $C_p > 0$ such that*

$$\|u * K\|_q \le C_p \|u\|_p \quad \textit{and} \quad \|K * u\|_q \le C_p \|u\|_p$$

(see [8, Proposition 1.11]*).*

Lemma 4.4 (see [2, Lemma 3.5]). *Suppose $0 < \mu < Q$. If K is a kernel of type μ and $\psi \in \mathcal{D}(\mathbb{G})$, $\psi \equiv 1$ in a neighborhood of the origin, then the statement* (iii) *of Theorem 4.3 still holds if we replace K by ψK or $(1 - \psi)K$.*

Analogously, if K is a kernel of type 0 and $\psi \in \mathcal{D}(\mathbb{G})$, then statement (ii) *of the same theorem still holds if we replace K by ψK or $(\psi - 1)K$.*

Definition 4.5. Let f be a measurable function on $\mathbb{G}$. If $t > 0$, we set

$$\lambda_f(t) = \big|\{|f| > t\}\big|.$$

If $1 \le r \le \infty$ and

$$\sup_{t>0} t^r \lambda_f(t) < \infty,$$

we say that $f \in L^{r,\infty}(\mathbb{G})$.

DEFINITION 4.6. Following [4, Definition A.1], if $1 < r < \infty$, we set

$$\|u\|_{M^r} := \inf \left\{ C \geq 0;\ \int_K |u|\, dx \leq C|K|^{1/r'} \text{ for all } L\text{-measurable set } K \subset \mathbb{G} \right\},$$

and $M^r = M^r(\mathbb{G})$ is the set of measurable functions u on $\mathbb{G}$ satisfying $\|u\|_{M^r} < \infty$.

Repeating verbatim the arguments of [4, Lemma A.2], we obtain the following.

LEMMA 4.7. *If $1 < r < \infty$, then*

$$\frac{(r-1)^r}{r^{r+1}} \|u\|^r_{M^r} \leq \sup_{t>0} \left\{ t^r \left| \{|u| > t\} \right| \right\} \leq \|u\|^r_{M^r}.$$

In particular, if $1 < r < \infty$, then $M^r = L^{r,\infty}(\mathbb{G})$.

COROLLARY 4.8. *If $1 \leq q < r$, then $M^r \subset L^q_{\mathrm{loc}}(\mathbb{G}) \subset L^1_{\mathrm{loc}}(\mathbb{G})$.*

PROOF. By Lemma 4.7, if $u \in M^r$, then $|u|^q \in M^{r/q}$, and we can conclude thanks to Definition 4.6. ∎

LEMMA 4.9. *Let K be a kernel of type $\mu \in (0, Q)$. Then, for all $f \in L^1(\mathbb{G})$, we have*

$$f * K \in M^{Q/(Q-\mu)}$$

and there exists $C > 0$ such that

$$\|f * K\|_{M^{Q/(Q-\mu)}} \leq C \|f\|_{L^1(\mathbb{G})}$$

for all $f \in L^1(\mathbb{G})$. In particular, by Corollary 4.8, if $1 \leq q < Q/(Q-\mu)$, then

$$f * K \in L^q_{\mathrm{loc}}(\mathbb{G}) \subset L^1_{\mathrm{loc}}(\mathbb{G}).$$

As in [1, Remark 3.10], we have the following remark.

REMARK 4.10. Suppose $0 < \mu < Q$. If K is a kernel of type μ and $\psi \in \mathcal{D}(\mathbb{G})$, $\psi \equiv 1$ in a neighborhood of the origin, then the statements of Lemma 4.9 still hold if we replace K by $(1-\psi)K$ or by ψK.

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