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**Mathematical Physics.** – *Towards a global classification of the generalized elastic materials*, by Franco Cardin and Adriano Montanaro, communicated on 11 April 2025.

*This paper is dedicated to the memory of our mentor, Aldo Bressan.*
*It is also dedicated to the memory of Adriano who passed away*
*during the final stages of this work, written jointly.*

Abstract. – The aim of this note is to clarify exhaustively the global equivalence of the generalized stored energy functions for the elastic materials with polarizations.

Keywords. – constitutive functions, generalized elasticity, Lagrangian submanifolds.

Mathematics Subject Classification 2020. – 74A20 (primary); 53D12 (secondary).

## 1. Introduction

In the foreword of the 1972 book [3] by Aldo Bressan, Nuel D. Belnap wrote:

> *The book in your hands, written by a professor of physics working in Padua in the very shadow of the chair from which Galileo first preached the marriage of mathematics and nature, is the most important contribution to date concerning the introduction of quantifiers into modal logic. It surpasses any article or book in the generality of its conceptions, the degree of their development, and the profundity of the attendant analysis. Perhaps one should credit the author's near total isolation from the logical community for allowing him to proceed with the elaboration of his fresh ideas unobstructed by premature criticism, and doubtless one must credit his uncompromising insistence on 'usability' to the fact that his enterprise arose from and has been continually nourished by the felt needs of a practicing physicist.*

This extraordinary admission of scientific greatness comes from far away, from the strong unstoppable will of Bressan to weave a rigorous logical tissue around the notion, properly of physical mathematical nature, of *physical possibility*. The road traveled by Bressan from the axiomatic foundations of classical mechanics, according to Mach and Painlevé, to the book mentioned was long: Mach and Painlevé were exactly first to systematically use this notion in a primitive way. That theoretical but also genuinely physical point of view (see [2]) was next exported by Bressan in the context of the thermo-mechanics of continuous media, see [5–7]. Among other things, he displayed his criticism around the fully quantification, or arbitrary possibility of choice, of **b** (body force) and $r$ (radiation) in the well-known standard way to obtain constitutive restrictions from Clausius–Duhem inequality following Coleman–Noll's line of thought [11, 19].

Bressan spent a lot of energy to clarify in depth the notion of *equivalence* in the description of materials in continuum mechanics. His mathematical logical apparatus was fundamental in this task. Everything is well testified in [4–6].

It is important to note that the search for Bressan and Montanaro (see [7, 16, 17]) headed towards the study of the classes of functions constitutive for the *same* material. The use of the physical possibility of being able to have arbitrary body forces and radiation had led, as is well known, to important constitutive restrictions for the materials; nevertheless, the problem of the determination of these constitutive functions, more exactly, of the class of functions representing of the same material, remained open. Once again, the guiding idea of the physical possibility led towards the introduction of ideal 'cuts' to be carried out in order to isolate and theoretically test the body in the study of its material characteristics.

In the last decades of the past century, inside a culturally and only apparently far mathematical environment, symplectic geometry, the possibility of identifying elastic *materials* with polarizations was showed through suitable *Lagrangian submanifolds* and their *generating functions* (see below definition) thought of as constitutive energy functions. This link will be proposed and clarified in the next section. This idea, this description of materials, gradually spread (see e.g. [1, 8–10, 13]) especially in dealing with elastic materials with singular and/or multivalued behavior.

The fundamental problem of *equivalence* remained open, exactly in the sense of Bressan: when do two generating functions define the same material, that is, the same Lagrangian submanifold? This question is nowadays exhaustively answered by a series of theorems that we report and conveniently repurposed here that manage to draw the whole class of generating functions defining exactly the same Lagrangian submanifold, that is, the same material.

### 1.1. *Elastic materials with polarizations: The Landau–Ericksen–Pitteri–Zanzotto experience*

There is some experimental evidence that the classical setting for hyperelasticity (e.g. elastostatics with Dirichlet boundary conditions) is sometimes *not adequate*: in fact, *multi-valued* stress-strain functions with possible singularities (see later, in particular examples 1, 2, and 3) may appear. In order to investigate these phenomena, it is possible to set up a generalized framework strictly analogous to Analytical Mechanics. Here, as usual, we consider the *gradient of deformation* matrices $F$ related to motions $x = \chi(t, X)$,

$$F = F_{iL} = \frac{\partial \chi_i}{\partial X_L}(t, X).$$

Set the cone $\mathrm{Lin}^+ := \{F \in \mathrm{Lin} : \det F > 0\}$, where Lin denotes the manifold of real $(3 \times 3)$-matrices. Then, for some $\sigma : \mathrm{Lin}^+ \to \mathbb{R}$, $F \mapsto \sigma(F)$, named *stored energy* function, the *Piola–Kirchhoff stress tensor* $S$ of a (standard) hyperelastic material is given by

$$S = \frac{\partial \sigma}{\partial F}(F). \tag{1.1}$$

*Objectivity* is satisfied when $\sigma$ depends on $F$ through $C = F^T F$.

Endow the cotangent bundle $T^*(\mathrm{Lin}^+)$ – to put it roughly, the space of pairs $(F, S)$ – with the natural symplectic structure defined by the 2-form given by the exterior derivative $d\Theta$ of $\Theta$, the 1-form 'mechanical work' on $T^*(\mathrm{Lin}^+)$:

$$\Theta = S_{iL} dF_{iL}, \quad i, L = 1, 2, 3. \tag{1.2}$$

A *Lagrangian submanifold* $\Lambda$ of the symplectic manifold $(T^*(\mathrm{Lin}^+), \Theta)$, with inclusion map $j : \Lambda \hookrightarrow T^*(\mathrm{Lin}^+)$, is a submanifold satisfying

(i) $\dim \Lambda = \dim \mathrm{Lin}^+$,

(ii) the restriction of the work form on $\Lambda$, $j^*\Theta = \Theta|_\Lambda$ is *closed*, $d(j^*\Theta) = 0$, where $j^*$ is the pull-back of $j$.

- Around the points $(F, S)$ where the Lagrangian submanifold $\Lambda$ is *transverse* to the fibers of the canonical projection $\pi_{\mathrm{Lin}^+} : T^*(\mathrm{Lin}^+) \to \mathrm{Lin}^+$, that is, where the composed map

$$\begin{array}{ccccc} \Lambda & \xhookrightarrow{j} & T^*(\mathrm{Lin}^+) & \xrightarrow{\pi_{\mathrm{Lin}^+}} & \mathrm{Lin}^+, \\ \zeta & \longmapsto & \big(F(\zeta), S(\zeta)\big) & \longmapsto & F(\zeta) \end{array} \tag{1.3}$$

has maximal rank,

$$\operatorname{rank}\big(D(\pi_{\mathrm{Lin}^+} \circ j)\big) = 9 = \max, \tag{1.4}$$

then $\Lambda$ is locally described by the pairs $(F, S) \in T^*(\mathrm{Lin}^+)$ such that the Piola–Kirchhoff stress tensor is given by (1.1), for some smooth function $\sigma(F)$. Such a Lagrangian submanifold $\Lambda$ of $T^*(\mathrm{Lin}^+)$ corresponds (at least locally) to a *hyper-elastic material*: $S = \partial\sigma/\partial F(F)$ and $\sigma(F)$ takes on the role of its generating function.

- Conversely, it is also clear that every graph of the differential of a function $\sigma(F)$, $\{(F, \partial\sigma/\partial F(F)),\ F \in \mathrm{Lin}^+\}$, is a Lagrangian submanifold.

By relaxing the transversality condition (1.4), we obtain a generalized definition of hyperelastic materials in a natural way; we are led to say that *a generalized hyperelastic material is characterized by a Lagrangian submanifold* $\Lambda$ *of* $T^*(\mathrm{Lin}^+)$, not necessarily everywhere transverse to the fibers of $T^*\mathrm{Lin}^+$ (see [8, 9, 13, 21]).

A celebrated theorem arisen inside the symplectic framework of asymptotic wave propagation, Maslov–Hörmander's theorem [9], characterizes locally any Lagrangian submanifold. It was first proven by Maslov [15] in 1965 and refined by Hörmander [12] in 1971.

It shows that, for any Lagrangian submanifold $\Lambda$, locally there exists always some real smooth function $\sigma$ (called *generating function or Morse family*)

$$\mathrm{Lin}^+ \times \mathbb{R}^k \ni (F, p) \longmapsto \sigma(F, p) \in \mathbb{R}, \tag{1.5}$$

such that $\Lambda$ is described by the pairs $(F, S) \in T^*(\mathrm{Lin}^+)$ satisfying

$$S = \frac{\partial\sigma}{\partial F}(F, p), \quad 0 = \frac{\partial\sigma}{\partial p}(F, p), \tag{1.6}$$

for some *auxiliary parameters*[1] $p \in \mathbb{R}^k$; furthermore, the condition

$$\mathrm{rk}\left(\frac{\partial^2\sigma}{\partial p\partial F}\ \frac{\partial^2\sigma}{\partial p\partial p}\right)\Bigg|_{\frac{\partial\sigma}{\partial p}(F,p)=0} = k = \max \tag{1.7}$$

has to be fulfilled. Conversely, given a function $\sigma(F, p)$ as in (1.7), relations (1.6) define a Lagrangian submanifold in $T^*(\mathrm{Lin}^+)$.

The above theorem fits perfectly inside the framework of Ericksen, where the indefinite elastostatic equations become

$$\mathrm{div}\,\frac{\partial\sigma}{\partial F} + b = 0, \quad \frac{\partial\sigma}{\partial p} = 0.$$

The auxiliary parameters $p = (p_\alpha)_{\alpha=1,\ldots,k}$ can be removed if, at least locally, $\Lambda$ is transversal to the fibers of $\pi_{\mathrm{Lin}^+} : T^*(\mathrm{Lin}^+) \to \mathrm{Lin}^+$:

- *if* $\det \frac{\partial^2\sigma}{\partial p\partial p}(F, p)|_{\frac{\partial\sigma}{\partial p}=0} \neq 0$*, then, by the implicit function theorem, from* $\frac{\partial\sigma}{\partial p}(F, p) = 0$*, we can define a unique local* $p = \tilde{p}(F)$*, so that, eventually, the new function* $\overline{\sigma}(F) := \sigma(F, \tilde{p}(F))$ *is an equivalent generating function (without polarizations* $p$*) for the*

(1) called *polarizations* in the Landau–Ericksen–Pitteri–Zanzotto theories.

*same material* $\Lambda$. *In detail,*

$$\Lambda = \left\{ (F, S) : S = \frac{\partial \sigma}{\partial F}(F, p) \text{ and } \frac{\partial \sigma}{\partial p}(F, p) = 0,\ F \in \mathrm{Lin}^+,\ \text{for some } p \in \mathbb{R}^k \right\},$$

*or, equivalently,*

$$\Lambda = \left\{ (F, S) : S = \frac{\partial \overline{\sigma}}{\partial F}(F),\ F \in \mathrm{Lin}^+ \right\}.$$

- *But the auxiliary parameters are essential whenever*

$$\text{rank}\left( D(\pi_{\mathrm{Lin}^+} \circ j) \right) < \max \tag{1.8}$$

*and this happens precisely at the* caustics $(F^*, S^*)$ *of the material surface* $\Lambda$*:*

$$\det \frac{\partial^2 \sigma}{\partial p \partial p}(F^*, p)\Big|_{\frac{\partial \sigma}{\partial p}=0} = 0.$$

*We briefly recall some crucial examples.*

(1) The above definition leads to an intrinsic geometrical understanding of the 2-lattices of Ericksen, and the $\nu$+1-lattices of Pitteri (see [20] and the book [21]), in the sense that the structure (1.6) is the *most general* local representation of a generalized hyperelastic material. In those theories the role of the parameters $p$ is played by the so-called *polarizations* or *shifts*. Phenomenologically, these theories can be thought of as continuum mechanical versions of underlying discrete lattice theories.

(2) A concrete example of generalized hyperelastic material is given by $\alpha$-*quartz* and $\beta$-*quartz*: a generalized density with non-trivial auxiliary parameters is worked out, thereby exhibiting a structure with *multi-branches* in (an analogue of) the above $(F, S)$-space (see [21]). The *phase transition* from $\alpha$-*quartz* to $\beta$-*quartz* is occurring precisely when the material is locally going through the caustic region

$$\det \frac{\partial^2 \sigma}{\partial p \partial p}\Big|_{\frac{\partial \sigma}{\partial p}=0} = 0.$$

(3) Another example is provided by the theory of *Phase Transitions* and *Pseudo-elastic hysteresis* by Ingo Müller, see [18]. In that theory, Müller introduces a new parameter $z \in [0, 1]$ (the *phase fraction*) and a free energy function $f(d; z)$, depending on the *deformation* $d$ and $z$. Furthermore, he points out that the admissible *deformation-load* states $(d, P)$ at phase equilibria are given by the pairs $(d, P)$ such that, for some $z$,

$$P = \frac{\partial f}{\partial d}(d; z), \quad 0 = \frac{\partial f}{\partial z}(d; z). \tag{1.9}$$

### *1.2. The complete classification of the equivalent generalized stored energy density*

In the above section, we have seen that the notion of Lagrangian submanifold $\Lambda$ into the space of the strain-stress pairs $(F, S)$, the cotangent bundle $T^*\mathrm{Lin}^+$, is resuming in an intrinsic unified way the concept of hyper-elastic and generalized hyper-elastic material, possibly involving further auxiliary parameters, said *polarizations*, $p$. In this new framework, we are able to draw the full class of all the generating functions defining the same material $\Lambda$.

First, we point out three operations giving us equivalent generalized stored energy:

(1) Given $\sigma_1(F, p)$, then $\sigma_2(F, p) := \sigma_1(F, p) + \mathrm{const.}$ is (obviously) equivalent to $\sigma_1$ since both ones draw the same set of strain-stress pairs $(F, S)$, i.e. the same Lagrangian submanifold.

(2) Given $\sigma_1(F, p)$, and given a *non-degenerate quadratic form* $v^T A v$, $p \in \mathbb{R}^{k_1}$, $v \in \mathbb{R}^{k_2}$, then $\sigma_2(F, p, v) := \sigma_1(F, p) + v^T A v$ is equivalent to $\sigma_1$; again, they draw the same Lagrangian submanifold.

(3) Given a *fibered diffeomorphism*

$$\begin{aligned} \mathrm{Lin}^+ \times \mathbb{R}^k &\longrightarrow \mathrm{Lin}^+ \times \mathbb{R}^k \\ (F, \bar{p}) &\longmapsto \big(F, p(F, \bar{p})\big) \end{aligned}$$

then, for any $\sigma_1(F, p)$, we have that $\sigma_2(F, \bar{p}) := \sigma_1(F, p(F, \bar{p}))$ is equivalent in the above sense.

The very exciting fact, concluding this note, is the following: after a theorem by Weinstein [22], clearly well exposed also in [14, Appendix 7, Section 1.17, Theorem 3, p. 472], we finally can say that two generating functions $\sigma_1 : \mathrm{Lin}^+ \times \mathbb{R}^h$ and $\sigma_2 : \mathrm{Lin}^+ \times \mathbb{R}^k$, each satisfying the conditions (1.6) and (1.7), are defining the *same* set of strain-stress pairs $(F, S)$, i.e. the same Lagrangian submanifold $\Lambda$, *if and only* you get one from the other by means of a suitable sequence of the above three operations.

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