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Partial Differential Equations. – *Dirichlet problems with unbalanced growth and double resonance*, by Yunru Bai, Nikolaos S. Papageorgiou and Shengda Zeng, communicated on 9 May 2025.

Abstract. – We consider a nonlinear, nonhomogeneous Dirichlet problem driven by a nonautonomous $(p,2)$-Laplacian with unbalanced growth. The reaction is resonant both at $\pm\infty$ and at zero (double resonance). Using variational tools together with truncation and comparison techniques and critical groups, we show that the problem has at least three nontrivial bounded solutions, all with sign information (positive, negative and nodal), which are ordered.

Keywords. – generalized Orlicz spaces, double resonance, extremal constant sign solutions, nodal solution, critical groups.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear and nonhomogeneous Dirichlet problem:

$$(1) \qquad \begin{cases} -\Delta_p^\alpha u(z) - \Delta u(z) = f\big(z, u(z)\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $2 < p < N$. In this problem, Δ_p^α denotes the weighted p-Laplacian with weight $\alpha(\cdot) \in C^{0,1}(\overline{\Omega}) \setminus \{0\}$, $\alpha(z) \geq 0$ for a.a $z \in \Omega$, defined by

$$\Delta_p^\alpha u = \operatorname{div}\big(\alpha(z)|Du|^{p-2}Du\big).$$

We observe that if $\min_{\overline{\Omega}} \alpha > 0$, then problem (1) is a nonautonomous version of a $(p,2)$-equation. Such equations were studied extensively recently and in the literature we can find many existence and multiplicity results. We mention the works of Barile–Figueiredo [2], Benouhiba–Belyacine [4], Bobkov–Tanaka [5], Marano–Mosconi–Papageorgiou [27], Papageorgiou–Qin–Rădulescu [32], Papageorgiou–Rădulescu [33], Papageorgiou–Winkert [37], Pei–Zhang [39], Tanaka [41] and for anisotropic problems Vetro–Vetro [42]. Such equations arise in the physical models. We refer to the works of Benci–D'Avenia–Fortunato–Pisani [3] (quantum physics) and Cherfils–Il'yasov [7] (reaction-diffusion systems). The pleasant feature of these problems is that there is a global (up to the boundary) regularity theory for the solutions, see Lieberman [22]. This means that the pool of available analytical tools is very rich and makes the analysis of the problem easier. The operator of (1) is related to the density function

$$\theta(z,t) = \alpha(z)t^p + t^2 \quad \text{for all } z \in \Omega,\ \text{all } t \ge 0$$

with $2 < p$. When $\min_{\overline{\Omega}} \alpha > 0$, then $\theta(z,\cdot)$ exhibits balanced growth; namely, we have

$$c_0 t^p \le \theta(z,t) \le c_1(1 + t^p),$$

for all $z \in \Omega$, all $t \ge 0$, some $c_0, c_1 > 0$. We see that $\theta(z,\cdot)$ is trapped between two same powers of $t \ge 0$. This feature permits the use of the regularity theory of Lieberman [22]. If $\min_{\overline{\Omega}} \alpha = 0$, then the situation changes drastically. Now, $\theta(z,\cdot)$ exhibits unbalanced growth; namely, it holds that

$$t^2 \le \theta(z,t) \le c_2(1 + t^p),$$

for all $z \in \Omega$, all $t \ge 0$, some $c_2 > 0$. So, now $\theta(z,\cdot)$ is trapped between two different powers of $t \ge 0$. This changes completely the framework of the problem. In this new setting, the standard Lebesgue and Sobolev spaces are not adequate to study such equations, and instead we need to use generalized Orlicz spaces. In addition, for these equations there is no global regularity theory, only local regularity results, see the survey papers of Marcellini [28], Mingione–Rădulescu [29]. The absence of a global regularity removes from consideration the powerful tools mentioned earlier and makes unbalanced growth problems more difficult and require new tools and techniques. Recently, there have been same existence and multiplicity results for such equations. We mention the works of Deręgowska–Gasiński–Papageorgiou [10], Gasiński–Papageorgiou [13], Ho–Winkert [18], Gasiński–Winkert [14], Liu–Dai [23], Crespo-Blanco–Gasiński–Harjulehto–Winkert [8], Liu–Papageorgiou [24, 26], Crespo-Blanco–Gasiński–Winkert [9] and Papageorgiou–Vetro–Vetro [36]. In all the aforementioned works, the reaction (the right-hand side of the equation) is $(p-1)$-superlinear and the method of proof is based on the Nehari manifold technique. This approach works well when the reaction is of power type. Otherwise, it requires restrictive monotonicity conditions on the quotient function $0 \ne x \mapsto \frac{f(z,x)}{|x|^{p-2}x}$. Recently, Liu–Papageorgiou [25] proposed an alternative approach based on critical groups (Morse theory). In [25], the authors considered a "concave-convex" problem. Here instead, we assume that the reaction $f(z,x)$ is $(p-1)$-linear in x, and it can be resonant with respect to the principal eigenvalue of $(-\Delta_p^\alpha, W_0^{1,\theta_0}(\Omega))$, with $W_0^{1,\theta_0}(\Omega)$ being the relevant generalized Orlicz space. We use the spectral analysis of Papageorgiou–Pudelko–Rădulescu [31]. The reaction is also resonant at zero with respect to higher eigenvalues of $(-\Delta, H_0^1(\Omega))$.

We prove the existence of three nontrivial bounded solutions with sign information for all of them (positive, negative and nodal (sign changing)). In addition, the three solutions are ordered.

2. Mathematical background and hypotheses

Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue-null set. Let $\alpha \in C^{0,1}(\overline{\Omega})\setminus\{0\}$ with $\alpha(z) \geq 0$ for all $z \in \Omega$ and consider the function

$$\theta(z,t) = \alpha(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

The Lebesgue–Orlicz space $L^\theta(\Omega)$ is defined by

$$L^\theta(\Omega) := \left\{u \in L^0(\Omega) : \rho_\theta(u) = \int_\Omega \theta\big(z, |u|\big)\, dz < \infty\right\}.$$

The function $\rho_\theta(\cdot)$ is known as the modular function corresponding to $\theta(\cdot,\cdot)$. We equip $L^\theta(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_\theta$ defined by

$$\|u\|_\theta = \inf\left\{\lambda > 0 : \rho_\theta\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then, $L^\theta(\Omega)$ becomes a Banach space which is separable and uniformly convex (thus reflexive). Using $L^\theta(\Omega)$, we can introduce the corresponding Sobolev–Orlicz space $W^{1,\theta}(\Omega)$

$$W^{1,\theta}(\Omega) := \big\{u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega)\big\}.$$

Here Du denotes the weak gradient of u. The norm $\|\cdot\|_{1,\theta}$ on $W^{1,\theta}(\Omega)$ is defined by

$$\|u\|_{1,\theta} = \|u\|_\theta + \|Du\|_\theta \quad \text{with } \|Du\|_\theta = \big\||Du|\big\|_\theta.$$

Also, to treat Dirichlet problems, we introduce

$$W_0^{1,\theta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\theta}}.$$

Both spaces $W^{1,\theta}(\Omega)$ and $W_0^{1,\theta}(\Omega)$ are Banach spaces which are separable and uniformly convex (thus, reflexive). Suppose that $\frac{p}{2} < 1 + \frac{1}{N}$. Then, on $W_0^{1,\theta}(\Omega)$, the Poincaré inequality holds; namely, we have

$$\|u\|_\theta \leq \hat{c}\|Du\|_\theta \quad \text{for some } \hat{c} > 0, \text{ all } u \in W_0^{1,\theta}(\Omega).$$

Therefore, on $W_0^{1,\theta}(\Omega)$, we can consider the equivalent norm

$$\|u\| = \|Du\|_\theta \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Since we study a Dirichlet equation, our main interest is on the space $W_0^{1,\theta}(\Omega)$. We have some useful embeddings for the spaces $L^\theta(\Omega)$ and $W_0^{1,\theta}(\Omega)$. In what follows, by $\hookrightarrow$ we denote continuous embedding.

PROPOSITION 1. *The following statements are valid:*

(a) $L^\theta(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *for all* $1 \le r \le 2$;

(b) $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ *for all* $1 \le r \le 2^* = \frac{2N}{N-2}$ *and the embedding is compact if* $1 \le r < 2^*$;

(c) $L^p(\Omega) \hookrightarrow L^\theta(\Omega)$.

The modular function $\rho_\theta(\cdot)$ and the norm $\|\cdot\|$ are closely related.

PROPOSITION 2. *Let* $u \in W_0^{1,\theta}(\Omega)$. *The following statements hold:*

(a) $\|u\| = \eta \Leftrightarrow \rho_\theta(\frac{Du}{\eta}) = 1$;

(b) $\|u\| < 1$ *(resp.* $= 1$, > 1*)* $\Leftrightarrow \rho_\theta(Du) < 1$ *(resp.* $= 1$, > 1*)*;

(c) $\|u\| < 1 \Rightarrow \|u\|^p \le \rho_\theta(Du) \le \|u\|^2$;

(d) $\|u\| > 1 \Rightarrow \|u\|^2 \le \rho_\theta(Du) \le \|u\|^p$;

(e) $\|u\| \to 0$ *(resp.* $\to +\infty$*)* $\Leftrightarrow \rho_\theta(Du) \to 0$ *(resp.* $\to +\infty$*)*.

For more information about generalized Orlicz spaces, we refer to the book of Harjulehto–Hästö [16].

Let $V\colon W_0^{1,\theta}(\Omega) \to W_0^{1,\theta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle := \int_\Omega \big[\alpha(z)|Du|^{p-2} + 1\big](Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,\theta}(\Omega).$$

This operator has the following properties (see Papageorgiou–Winkert [38, p. 683]).

PROPOSITION 3. $V(\cdot)$ *is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (maximal monotone too) and of type* $(S)_+$*; that is,*

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W_0^{1,\theta}(\Omega) \text{ and } \limsup_{n\to\infty} \langle V(u_n), u_n - u\rangle \le 0,$$
$$\text{then } u_n \to u \text{ in } W_0^{1,\theta}(\Omega)\text{”}.$$

In what follows, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. A function $\alpha \in L^1_{\mathrm{loc}}(\Omega)$ is said to be a “weight” if $\alpha(z) > 0$ for a.a $z \in \Omega$. We will consider a particular class of weights, the so-called “p-Muckenhoupt weights”. We denote this class by $\mathcal{A}_p$ and we have

$$\alpha \in \mathcal{A}_p \text{ implies } \alpha \in L^1_{\mathrm{loc}}(\Omega) \text{ and } \alpha^{\frac{1}{1-p}} \in L^1_{\mathrm{loc}}(\Omega).$$

Equivalently, we can say that $\alpha \in \mathcal{A}_p$ if and only if

$$\sup\left[\frac{1}{|B|_N}\int_B \alpha(z)\,dz\right]\left[\frac{1}{|B|_N}\int_B \alpha(z)^{\frac{1}{1-p}}\,dz\right]^{p-1} < \infty,$$

the supremum taken over all balls $B \subseteq \Omega$. The balls can be replaced by cubes Q with sides parallel to the coordinate axes (see Harjulehto–Hästö [16, p. 114]). The function $\alpha(z) = |z|^\eta$ with $-N < \eta < N(p-1)$ is in $\mathcal{A}_p$.

Suppose $\alpha \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p$ and let

$$\theta_0(z,t) := \alpha(z)t^p \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

We introduce the corresponding generalized Orlicz spaces $L^{\theta_0}(\Omega)$ and $W_0^{1,\theta_0}(\Omega)$ which coincide with the weighted spaces $L^p(\Omega,\mu)$ and $W_0^{1,p}(\Omega,\mu)$ with $\mu(C) = \int_C \alpha(z)\,dz$ for all $C \subseteq \Omega$ measurable (see Heinonen–Kilpeläinen–Martio [17]). From Lemma 2 of Papageorgiou–Rădulescu–Zhang [35], we have that

$$W_0^{1,\theta_0}(\Omega) \hookrightarrow L^{1,\theta_0}(\Omega) \text{ compactly.} \tag{2}$$

Using (2), Papageorgiou–Pudelko–Rădulescu [31] developed the spectral properties of $-\Delta_p^\alpha$. So, let $m \in L^\infty(\Omega) \setminus \{0\}$ be $m(z) \geq 0$ for a.a $z \in \Omega$, and consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p^\alpha u(z) = \widehat{\lambda} m(z)\alpha(z)\big|u(z)\big|^{p-2}u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1^\alpha(m) > 0$ which has the following variational characterization:

$$\widehat{\lambda}_1^\alpha(m) := \inf\left\{\frac{\rho_{\theta_0}(Du)}{\int_\Omega m(z)\alpha(z)|u|^p\,dz} : u \in W_0^{1,\theta_0}(\Omega) \text{ and } u \neq 0\right\}. \tag{4}$$

If $m \equiv 1$, then we write $\widehat{\lambda}_1^\alpha(1) =: \widehat{\lambda}_1^\alpha$ and the Rayleigh quotient becomes $\frac{\rho_{\theta_0}(Du)}{\rho_{\theta_0}(u)}$. This eigenvalue has the following properties:

- $\widehat{\lambda}_1^\alpha(m)$ is isolated; that is, if $\widehat{\sigma}_\alpha(p)$ denotes the spectrum of (3), then we can find $\varepsilon > 0$ such that
$$\big(\widehat{\lambda}_1^\alpha(m), \widehat{\lambda}_1^\alpha(m) + \varepsilon\big) \cap \widehat{\sigma}_\alpha(p) = \emptyset.$$
- $\widehat{\lambda}_1^\alpha(m)$ is simple; that is, if $\hat{u}, \hat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1^\alpha(m)$, then $\hat{u} = \tau\hat{v}$ for some $\tau \in \mathbb{R} \setminus \{0\}$. So, the eigenspace of $\widehat{\lambda}_1^\alpha(m)$ is one-dimensional.
- The infimum in (4) is realized on the one-dimensional eigenspace of $\widehat{\lambda}_1^\alpha(m)$, the elements of which have fixed sign.
- All eigenvalues $\widehat{\lambda} \neq \widehat{\lambda}_1^\alpha$ have nodal eigenfunctions.

- All eigenfunctions of (3) belong in $W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.
- If $m, \hat{m} \in L^\infty(\Omega) \setminus \{0\}$, $0 \le m(z) \le \hat{m}(z)$ for a.a $z \in \Omega$ and $m \neq \hat{m}$, then $\widehat{\lambda}_1^\alpha(\hat{m}) < \widehat{\lambda}_1^\alpha(m)$.

By $\hat{u}_1$ we denote the positive, normalized (that is, $\int_\Omega m(z)\alpha(z)\hat{u}_1^p\,dz = 1$) eigenfunction corresponding to $\widehat{\lambda}_1^\alpha(m) > 0$. We have

$$\begin{cases} \hat{u}_1 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \\ \text{for every compact set } K \subseteq \Omega,\ 0 < c_K \le \hat{u}_1(z) \text{ for a.a } z \in K. \end{cases} \tag{5}$$

If a function $u \in L^0(\Omega)$ satisfies the second part of (5), then we write

$$0 \prec u.$$

Evidently, $0 \prec u \Rightarrow 0 < u(z)$ for a.a $z \in \Omega$. Moreover, if $u \in C(\Omega)$, then $0 \prec u$. We write $v \prec 0$ if $0 \prec -v$.

We will also need the spectrum of the Dirichlet Laplacian. So, we consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) = \widehat{\lambda} u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Using the spectral theorem for compact self-adjoint operators, we have a complete description of the spectrum of (6), which consists of a sequence $\{\widehat{\lambda}_k := \widehat{\lambda}_k(2)\}_{k\in\mathbb{N}} \subseteq (0, +\infty)$ such that $\widehat{\lambda}_k \to +\infty$ as $k \to \infty$. By $E(\widehat{\lambda}_k)$ we denote the eigenspace for the eigenvalue $\widehat{\lambda}_k$, $k \in \mathbb{N}$. We know that $E(\widehat{\lambda}_k)$ is finite-dimensional and $E(\widehat{\lambda}_k) \subseteq C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ (classical linear regularity theory, see Gilbarg–Trudinger [15]). We have the following orthogonal direct sum decomposition:

$$H_0^1(\overline{\Omega}) = \overline{\bigoplus_{k\in\mathbb{N}} E(\widehat{\lambda}_k)}.$$

The eigenspaces have the "Unique Continuation Property" (UCP for short); that is, if $u \in E(\widehat{\lambda}_k)$ and $u(\cdot)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. Moreover, we have the following variational characterizations of the eigenvalues:

$$\widehat{\lambda}_1 = \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right\}, \tag{7}$$

$$\begin{aligned} \widehat{\lambda}_k &= \inf\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \overline{\bigoplus_{i\ge k} E(\widehat{\lambda}_i)},\ u \neq 0\right\} \\ &= \sup\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bigoplus_{i=1}^{k} E(\widehat{\lambda}_i),\ u \neq 0\right\} \quad \text{for } k \ge 2. \end{aligned} \tag{8}$$

The infimum and the supremum in (7) and (8) are actually attained on the corresponding eigenspaces. From (7), it is clear that $\widehat{\lambda}_1$ has eigenfunctions of constant sign, while every $\widehat{\lambda}_k$, $k \geq 2$ has nodal eigenfunctions.

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We denote

$$K_\varphi := \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi\text{)},$$
$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Let $Y_2 \subseteq Y_1 \subseteq X$ and $k \in \mathbb{N}_0$. By $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group for the pair (Y_1, Y_2), with real coefficients. Let $u_0 \in K_\varphi$ be isolated and let $c = \varphi(u_0)$. Then, the critical groups of $\varphi(\cdot)$ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k\big(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}\big) \quad \text{for all } k \in \mathbb{N}_0,$$

with U being a neighborhood of u_0 such that $\varphi^c \cap U \cap K_\varphi = \{u_0\}$. The excision property of singular homology implies that the above definition is independent of the isolating neighborhood U.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be locally Lipschitz if for every $K \subseteq \mathbb{R}$ compact, $\varphi|_K$ is Lipschitz continuous with Lipschitz constant $c_K > 0$. This definition is equivalent to saying that every $x \in \mathbb{R}$ has a neighborhood U such that $f|_U$ is Lipschitz continuous. We say that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^∞-locally Lipschitz function if for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a $z \in \Omega$, $x \mapsto f(z, x)$ is locally Lipschitz with Lipschitz constants $c_K \in L^\infty(\Omega)$.

For every $u \in L^0(\Omega)$, we define $u^+ = \max\{u, 0\}$, $u^- = \min\{-u, 0\}$ and we have $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W_0^{1,\theta}(\Omega)$, then $u^\pm \in W_0^{1,\theta}(\Omega)$.

Now we introduce the hypotheses on the data of (1).

$\underline{H_0}$: $\alpha \in C^{0,1}(\overline{\Omega}) \cap \mathcal{A}_p$, $1 < 2 < p < N$ and $\frac{p}{2} < 1 + \frac{1}{N}$.

REMARK 4. Note that the last inequality implies $p < 2^* = \frac{2N}{N-2}$ and so we can use the embeddings from Proposition 1.

$\underline{H_1}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^∞-locally Lipschitz function such that

(i) for every $\rho > 0$, there exists $\widehat{\alpha}_\rho \in L^\infty(\Omega)$ such that

$$\big|f(z, x)\big| \leq \widehat{\alpha}_\rho(z) \quad \text{for a.a } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exist $\beta_0 > 0$ and a function $\widehat{\beta}(z) \in L^1(\Omega)_+$ such that

$$-\beta_0 \leq \liminf_{x \to \pm\infty} \frac{f(z, x)}{\alpha(z)|x|^{p-2}x} \leq \limsup_{x \to \pm\infty} \frac{f(z, x)}{\alpha(z)|x|^{p-2}x} \leq \widehat{\lambda}_1^\alpha,$$

uniformly for a.a $z \in \Omega$, and if $F(z, x) = \int_0^x f(z, s)\, ds$, then

$$-\widehat{\beta}(z) \leq f(z, x)x - pF(z, x) \quad \text{for a.a } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iii) there exist $\delta > 0$, an integer $m \geq 2$ and $\eta \in L^\infty(\Omega)_+$ such that

$$\widehat{\lambda}_m(2) \leq \eta(z) \qquad \text{for a.a } z \in \Omega,\ \eta \not\equiv \widehat{\lambda}_m(2),$$
$$\eta(z)x^2 \leq f(z,x)x \leq \widehat{\lambda}_{m+1}(2)x^2 \quad \text{for a.a } z \in \Omega,\ \text{all } 0 < |x| \leq \delta,$$

and the second inequality is strict on a set of positive measure;

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a $z \in \Omega$, the function

$$x \mapsto f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x \text{ is nondecreasing on } [-\rho, \rho].$$

Remark 5. Hypothesis H_1 (ii) permits resonance with respect to $\widehat{\lambda}_1^\alpha$ as $x \to +\infty$. On the other hand, hypothesis H_1 (iii) implies that we can have resonance with respect to a higher eigenvalue $\widehat{\lambda}_{m+1}(2) > 0$ of the Dirichlet Laplacian since $\limsup_{x\to 0} \frac{f(z,x)}{x} \leq \widehat{\lambda}_{m+1}(2)$ uniformly for a.a $z \in \Omega$. Therefore, we deal with a double resonance situation.

We conclude this section with a lemma which is an outgrowth of the properties of the principal eigenvalue $\widehat{\lambda}_1^\alpha > 0$ and of the corresponding.

Lemma 6. *If hypotheses H_0 hold, $\eta \in L^\infty(\Omega)_+$, $\eta(z) \leq \widehat{\lambda}_1^\alpha$ for a.a $z \in \Omega$ and $\eta \not\equiv \widehat{\lambda}_1^\alpha$, then there exists $\hat{c} > 0$ such that*

$$\hat{c}\rho_{\theta_0}(Du) \leq \rho_{\theta_0}(Du) - \int_\Omega \eta(z)\alpha(z)|u|^p\,dz \quad \textit{for all } u \in W_0^{1,\theta_0}(\Omega).$$

Proof. Arguing by contradiction, suppose that the assertion of the lemma is not true. Then, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta_0}(\Omega)$ such that

$$\rho_{\theta_0}(Du_n) - \int_\Omega \eta(z)\alpha(z)|u_n|^p dz < \frac{1}{n}\rho_{\theta_0}(Du_n) \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

We set $y_n = \frac{u_n}{\|u_n\|_{1,\theta_0}}$ with $\|\cdot\|_{1,\theta_0}$ being the norm of the generalized Orlicz space $W_0^{1,\theta_0}(\Omega)$. Then,

$$\|y_n\|_{1,\theta_0} = 1 \quad \text{for all } n \in \mathbb{N}.$$

On account of the reflexivity of $W_0^{1,\theta_0}(\Omega)$ and using (2), we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\theta_0}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^{\theta_0}(\Omega). \tag{10}$$

From (9), we have

$$\left(1 - \frac{1}{n}\right)\rho_{\theta_0}(Dy_n) < \int_\Omega \eta(z)\alpha(z)|y_n|^p\,dz \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

We pass to the limit as $n \to \infty$ and use (10) and the fact that the modular function $\rho_{\theta_0}(\cdot)$ is continuous, convex; therefore, $\rho_{\theta_0}(\cdot)$ is sequentially weakly lower semicontinuous.

So, we obtain

$$
\begin{aligned}
(12)\qquad & \rho_{\theta_0}(Dy) \le \int_\Omega \eta(z)\alpha(z)|y|^p\,dz \le \widehat{\lambda}_1^\alpha \rho_{\theta_0}(y) \quad \text{(see (4))},\\
&\implies \rho_{\theta_0}(Dy) = \widehat{\lambda}_1^\alpha \rho_{\theta_0}(y),\\
&\implies y = 0 \quad \text{or} \quad y = \pm\hat{u}^{\pm}.
\end{aligned}
$$

If $y = 0$, then from (10) and (11), we see that

$$\rho_{\theta_0}(Dy_n) \to 0 \implies \|y_n\|_{1,\theta_0} \to 0 \quad \text{(see Proposition 2)}.$$

This contradicts the fact that $\|y_n\|_{1,\theta_0} = 1$ for all $n \in \mathbb{N}$. If $y = \pm\hat{u}_1$, then since $0 \prec \hat{u}_1$, we see that $|y(z)| > 0$ for a.a $z \in \Omega$. Then, from the first inequality in (12) and the hypothesis on $\eta(\cdot)$, we have

$$\rho_{\theta_0}(Dy) < \widehat{\lambda}_1^\alpha \rho_{\theta_0}(y)$$

contradicting (4). ∎

Similarly, as a consequence of the UCP property, we have that if $\eta \in L^\infty(\Omega)$ satisfies $\eta(z) \ge \widehat{\lambda}_m(z)$ with $m \in \mathbb{N}$ for a.a $z \in \Omega$ and the inequality is strict on a set of positive Lebesgue measure, then

$$\|D\bar{u}\|_2^2 - \int_\Omega \eta(z)\bar{u}^2\,dz \le -\hat{c}\|D\bar{u}\|_2^2 \quad \text{for some } \hat{c} > 0, \text{ all } \bar{u} \in \overline{H}_m = \bigoplus_{i=1}^m E(\widehat{\lambda}_i).$$

3. Solutions of constant sign

Let $\varphi\colon W_0^{1,\theta}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Evidently, $\varphi \in C^1(W_0^{1,\theta}(\Omega))$.

To produce constant sign solutions for problem (1), we will use the positive and negative truncations of $\varphi(\cdot)$, namely, the C^1-functionals $\varphi_\pm : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,\pm u^\pm)\,dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Proposition 7. *If hypotheses H_0 and H_1 hold, then the functionals $\varphi_\pm(\cdot)$ are coercive.*

PROOF. We do the proof for $\varphi_+(\cdot)$, the proof for $\varphi_-(\cdot)$ being similar. We argue by contradiction. So, suppose that $\varphi_+(\cdot)$ is not coercive. Then, we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$ and $M_0 > 0$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad \varphi_+(u_n) \le M_0 \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

On $\mathbb{R}_{++} = (0,\infty)$, we have

$$\begin{aligned}\frac{d}{dx}\left[\frac{F(z,x)}{x^p}\right] &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}}\\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}}\\ &\ge -\frac{\widehat{\beta}(z)}{x^{p+1}} \quad \text{(see hypothesis } H_1\text{ (ii))},\end{aligned}$$

$$\Longrightarrow \frac{F(z,x)}{x^p} - \frac{F(z,y)}{y^p} \ge \frac{\widehat{\beta}(z)}{p}\left[\frac{1}{x^p} - \frac{1}{y^p}\right] \text{for a.a } z \in \Omega, \text{ all } x \ge y > 0. \tag{14}$$

Hypothesis H_1 (ii) implies that

$$\limsup_{x\to+\infty} \frac{pF(z,x)}{\alpha(z)x^p} \le \widehat{\lambda}_1^{\alpha} \quad \text{uniformly for a.a } z \in \Omega. \tag{15}$$

If in (14), we let $x \to +\infty$ and use (15), then

$$\alpha(z)\widehat{\lambda}_1^{\alpha} y^p - pF(z,y) \ge -\widehat{\beta}(z) \quad \text{for a.a } z \in \Omega, \text{ all } y \ge 0. \tag{16}$$

Using (13) and (16), we have

$$\begin{aligned}&\frac{1}{p}\rho_{\theta_0}(Du_n^-) + \frac{1}{2}\|Du_n^-\|_2^2 + \frac{1}{p}\big[\rho_{\theta_0}(Du_n^+) - \widehat{\lambda}_1^{\alpha}\rho_{\theta_0}(u_n^+)\big] + \frac{1}{2}\|Du_n^+\|_2^2\\ &\qquad\qquad \le M_0 + \|\widehat{\beta}\|_1 \quad \text{for all } n \in \mathbb{N},\\ &\Longrightarrow \frac{1}{p}\rho_\theta(Du_n^-) \le M_0 + \|\widehat{\beta}\|_1 \quad \text{for all } n \in \mathbb{N} \text{ (recall } 2 < p \text{ and see (4))},\end{aligned}$$

$$\Longrightarrow \{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \text{ is bounded (see Proposition 2)}. \tag{17}$$

From (13) and (17), it follows that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \tag{18}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for $n \in \mathbb{N}$. Then, $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\theta}(\Omega), \quad y_n \to y \text{ in } L^r(\Omega) \text{ with } r \in (p, 2^*) \quad \text{(see Proposition 1)}. \tag{19}$$

From (13), we have

$$\frac{1}{p}\rho_{\theta_0}(Dy_n) + \frac{1}{2\|u_n^+\|^{p-2}}\|Dy_n\|_2^2 \leq \frac{M_0}{\|u_n^+\|^p} + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^p}\,dz \quad \text{for all } n \in \mathbb{N}. \tag{20}$$

Hypotheses H_1 (i) and (ii) imply that

$$\begin{aligned}&\left|F(z,x)\right| \leq c_3(1+x^p) \quad \text{for a.a } z \in \Omega,\ \text{some } c_3 > 0,\\ &\implies \left\{\frac{F\big(\cdot,u_n^+(\cdot)\big)}{\|u_n^+\|^p}\right\}_{n\in\mathbb{N}} \subseteq L^1(\Omega) \text{ is uniformly integrable.}\end{aligned}$$

By the Dunford–Pettis theorem (see Hu–Papageorgiou [19, p. 187]), we have

$$\frac{F\big(\cdot,u_n^+(\cdot)\big)}{\|u_n^+\|^p} \xrightarrow{w} \frac{1}{p}\eta(\cdot)\alpha(\cdot)y(\cdot)^p \quad \text{in } L^1(\Omega), \tag{21}$$

with $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\lambda}_1^\alpha$ for a.a $z \in \Omega$ (see [1, proof of Proposition 16]).

We return to (20), pass to the limit as $n \to \infty$ and use (18), (19), (21) and the fact that the modular function $\rho_{\theta_0}(\cdot)$ is sequentially weakly lower semicontinuous. We obtain

$$\rho_{\theta_0}(Dy) \leq \int_\Omega \eta(z)\alpha(z)y^p\,dz. \tag{22}$$

First we assume that

$$\eta \not\equiv \hat{\lambda}_1^\alpha \quad \text{(see (21))}.$$

Using Lemma 6 in (22), we have

$$\hat{c}\rho_{\theta_0}(Dy) \leq 0 \implies y = 0.$$

From (20), (2) and (18), we see that

$$\rho_{\theta_0}(Dy_n) \to 0 \quad \text{as } n \to \infty. \tag{23}$$

We know that $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Hence, from Proposition 2, we have

$$\begin{aligned}&1 = \rho_\theta(Dy_n) = \rho_{\theta_0}(Dy_n) + \|Dy_n\|_2^2 \quad \text{for all } n \in \mathbb{N},\\ &\implies \|Dy_n\|_2 \to 1 \quad \text{as } n \to \infty \quad \text{(see (23))},\\ &\implies \|Du_n^+\|_2 \to +\infty \quad \left(\text{recall } y_n = \frac{u_n^+}{\|u_n^+\|} \text{ and see (18)}\right).\end{aligned} \tag{24}$$

From (13), we have

$$\frac{1}{p}\rho_{\theta_0}(Du_n^+) - \int_\Omega F(z,u_n^+)\,dz + \frac{1}{2}\|Du_n^+\|_2^2 \le M_0,$$

$$\Longrightarrow \int_\Omega \left[\widehat{\lambda}_1^\alpha \alpha(z)(u_n^+)^p - pF(z,u_n^+)\right]dz + \frac{p}{2}\|Du_n^+\|_2^2 \le pM_0,$$

$$\Longrightarrow \|Du_n^+\|_2^2 \le M_1, \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N} \quad \text{(see (17))}. \tag{25}$$

Comparing (24) and (25), we have a contradiction.

Next, we assume that $\eta(z) = \widehat{\lambda}_1^\alpha$ for a.a $z \in \Omega$ (see (21)). From (4) and (22), we have

$$\rho_{\theta_0}(Dy) = \widehat{\lambda}_1^\alpha \rho_{\theta_0}(y) \implies y = 0 \text{ or } y = \hat{u}_1 \quad (\text{recall } y \ge 0).$$

If $y = 0$, then as above we reach a contradiction. If $y = \hat{u}_1$, then $0 \prec y$ and so we have

$$u_n^+(z) \to \infty \quad \text{for a.a } z \in \Omega. \tag{26}$$

From (25), we have

$$\widehat{\lambda}_1(2)\int_\Omega (u_n^+)^2\,dz \le M_1 \quad \text{for all } n \in \mathbb{N}. \tag{27}$$

Using (26) and Fatou's lemma, we obtain

$$\int_\Omega (u_n^+)^2\,dz \to \infty,$$

which contradicts (27). This proves the coercivity of $\varphi_+(\cdot)$. Similarly, we show the coercivity of $\varphi_-(\cdot)$. ∎

REMARK 8. The above proof reveals that the resonance with respect to $\widehat{\lambda}_1^\alpha > 0$ occurs from the left of the eigenvalue in the sense that for any $\tau > 0$, we have

$$0 \ge \limsup_{x\to\pm\infty} \frac{pF(z,x) - \widehat{\lambda}_1^\alpha \alpha(z)|x|^p}{|x|^\tau} \quad \text{uniformly for a.a } z \in \Omega.$$

Using Proposition 7, we can produce two constant sign solutions for problem (1).

PROPOSITION 9. *If hypotheses H_0 and H_1 hold, then problem* (1) *has at least two constant sign solutions*

$$u_0, v_0 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \textit{and} \quad v_0 \prec 0 \prec u_0.$$

PROOF. From Proposition 7, we know that $\varphi_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous (see Proposition 1). So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,\theta}(\Omega)$ such that

$$\varphi_+(u_0) = \inf\left\{\varphi_+(u) : u \in W_0^{1,\theta}(\Omega)\right\}. \tag{28}$$

Recall that the positive, normalized eigenfunction corresponding to $\widehat{\lambda}_1(2)$ satisfies $\hat{u}_1 \in C_0^1(\overline{\Omega})$ and $\hat{u}_1(z) > 0$ for all $z \in \Omega$. So, we can find $t \in (0,1)$ small such that

$$0 \le t\hat{u}_1(z) \le \delta \quad \text{for all } z \in \overline{\Omega}, \tag{29}$$

with $\delta > 0$ as postulated by hypothesis H_1 (iii). Then,

$$\begin{aligned} \varphi_+(t\hat{u}_1) &\le \frac{t^p}{p}\rho_{\theta_0}(D\hat{u}_1) \\ &\quad + \frac{t^2}{2}\int_\Omega \big[\widehat{\lambda}_1(2) - \eta(z)\big]\hat{u}_1^2\,dz \quad \text{(see (29) and hypothesis } H_1 \text{ (iii))}, \\ &= c_4 t^p - c_5 t^2 \end{aligned}$$

for some $c_4, c_5 > 0$ (since $\hat{u}_1(z) > 0$ for all $z \in \Omega$). Since $2 < p$, choosing $t \in (0,1)$ even smaller, we have

$$\varphi_+(t\hat{u}_1) < 0 \implies \varphi_+(u_0) < 0 = \varphi_+(0) \quad \text{(see (28))} \implies u_0 \ne 0.$$

From (28), we have

$$\begin{aligned} &\big\langle \varphi_+'(u_0), h\big\rangle = 0 \quad \text{for all } h \in W_0^{1,\theta}(\Omega), \\ &\implies \big\langle V(u_0), h\big\rangle = \int_\Omega f(z, u_0^+)h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega). \end{aligned} \tag{30}$$

In (30), we choose the test function $h = -u_0^- \in W_0^{1,\theta}(\Omega)$ and obtain

$$\rho_\theta(Du_0^-) = 0 \implies u_0 \ge 0 \text{ and } u_0 \ne 0 \quad \text{(see Proposition 2)}.$$

Let $k > 1$ and define the set

$$E_k := \big\{z \in \Omega : u_0(z) > k\big\}.$$

Let $k > 1$ be large so that

$$|E_k|_N \le 1 \quad \text{and} \quad \big\|(u_0 - k)^+\big\| \le 1. \tag{31}$$

On account of hypotheses H_1 (i), (ii), (iii), given $r \in (p, 2^*)$, we can find $c_6 > 0$ such that

$$f(z,x) \le c_6[x + x^{r-1}] \quad \text{for a.a } z \in \Omega, \text{ all } x \ge 0. \tag{32}$$

In (30), we choose the test function $h = (u_0 - k)^+ \in W_0^{1,\theta}(\Omega)$ to get

$$\begin{aligned} \big\|(u_0 - k)^+\big\|^p &\le \rho_\theta\big(D(u_0 - k)^+\big) \quad \text{(see (31) and Proposition 2)} \\ &= \int_\Omega f(z, u_0)(u_0 - k)^+\,dz \quad \text{(recall } u_0 \ge 0) \\ &\le c_6 \int_\Omega (u_0 + u_0^{r-1})(u_0 - k)^+\,dz \quad \text{(see (32))} \\ &\le c_7 |E_k|_N^{\frac{1}{r'}} \big\|(u_0 - k)^+\big\| \end{aligned}$$

by using Hölder's inequality, (31) and since $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$. This means that

$$\|(u_0-k)^+\|^{p-1} \le c_7 |E_k|_N^{\frac{1}{r'}}. \tag{33}$$

Let $m > k$. We have

$$\begin{aligned}(m-k)^p|E_m|_N^{\frac{p}{r}} &\le \left[\int_{E_m}(u_0-k)^r\,dz\right]^{\frac{p}{r}} \le \left[\int_{E_k}(v_0-k)^r\,dz\right]^{\frac{p}{r}} \text{ (since } E_m \subseteq E_k)\\ &\le c_8\|(u_n-k)^+\|^p \text{ for some } c_8>0 \ \big(\text{recall that } W_0^{1,\theta}(\Omega)\hookrightarrow L^r(\Omega)\big).\end{aligned}$$

Hence, we have

$$\begin{aligned}(m-k)^{p-1}|E_m|_N^{\frac{p-1}{r}} &\le c_9\|(u_0-k)^+\|^{p-1} \quad \text{for some } c_9>0\\ &\le c_{10}|E_k|_N^{\frac{1}{r'}} \quad \text{for some } c_{10}>0 \quad (\text{see } (33));\end{aligned}$$

namely,

$$|E_m|_N \le \frac{c_{11}}{(m-k)^r}|E_k|^{\frac{r}{r'}\cdot\frac{p'}{p}} \quad \text{for some } c_{11}>0.$$

Note that $\frac{r}{r'}\cdot\frac{p'}{p} = \frac{r-1}{p-1} > 1$ (recall $p<r$). So, using Lemma B.1 of Kinderlehrer–Stampacchia [21, p. 63], we see that there exists $M>1$ large such that

$$|E_M|_N = 0 \implies u_0 \in L^\infty(\Omega).$$

Let $\rho = \|u_0\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(iv). Then, from (30), we have

$$-\Delta_p^\alpha u_0 - \Delta u_0 + \widehat{\xi}_\rho u_0^{p-1} \ge 0 \text{ in } \Omega \implies 0 \prec u_0,$$

see Papageorgiou–Vetro–Vetro [36, Proposition 2.4].

Similarly, working this time with the functional $\varphi_-(\cdot)$, we produce a negative solution $v_0 \in W_0^{1,\theta}(\Omega)\cap L^\infty(\Omega)$ and $v_0 \prec 0$ (that is, $0 \prec -v_0$). ∎

In fact, we can produce extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution.

Given $r \in (p, 2^*)$, from hypotheses H_1 (i), (ii), (iii), we see that we can find $c_{12} > 0$ such that

$$f(z,x)x \ge \eta(z)x^2 - c_{12}|x|^r \quad \text{for a.a } z\in\Omega, \text{ all } x \in \mathbb{R}. \tag{34}$$

This unilateral growth condition on $f(z,\cdot)$ leads to the following auxiliary Dirichlet problem:

$$\begin{cases} -\Delta_p^\alpha u(z) - \Delta u(z) = \eta(z)u(z) - c_{12}|u(z)|^{r-2}u(z) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega\end{cases} \tag{35}$$

with $2 < p < r < 2^*$.

PROPOSITION 10. *If hypotheses H_0 hold and $\eta \in L^\infty(\Omega)$ satisfies*

$$\widehat{\lambda}_m(2) \leq \eta(z) \text{ for a.a } z \in \Omega \quad \text{and} \quad \eta \not\equiv \widehat{\lambda}_m(2) \text{ for } m \geq 2,$$

then problem (35) *has a unique positive solution*

$$\bar{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad 0 \prec \bar{u},$$

and since problem (35) *is odd, $\bar{v} = -\bar{u} \prec 0$ is the unique negative solution of* (35).

PROOF. To produce a positive solution for problem (35), we consider the C^1-functional $\psi_+ : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\psi_+(u) := \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 + \frac{c_{12}}{r}\|u^+\|_r^r \\ - \frac{1}{2}\int_\Omega \eta(z)(u^+)^2\,dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).\end{aligned}$$

Since $r > p > 2$, we see that $\psi_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W_0^{1,\theta}(\Omega)$ such that

$$\psi_+(\bar{u}) = \inf\left\{\psi_+(u) : u \in W_0^{1,\theta}(\Omega)\right\}. \tag{36}$$

Let $t \in (0,1)$ and $\hat{u}_1 \in C_0^1(\overline{\Omega})$ as before the positive, normalized eigenfunction for $\widehat{\lambda}_1(2)$. Hence,

$$\begin{aligned}\psi_+(t\hat{u}_1) &= \frac{t^p}{p}\rho_{\theta_0}(D\hat{u}_1) + \frac{t^2}{2}\int_\Omega \left[\widehat{\lambda}_1(2) - \eta(z)\right]\hat{u}_1^2\,dz + \frac{c_{12}t^r}{r}\|\hat{u}_1\|_r^r \\ &\leq c_{13}t^p - c_{14}t^2 \quad \text{for some } c_{13}, c_{14} > 0 \ \big(\text{recall that } p < r \text{ and } t \in (0,1)\big) \\ &< 0 \quad \text{choosing } t \in (0,1) \text{ small (since } 2 < p).\end{aligned}$$

This leads to

$$\psi_+(\bar{u}) < 0 = \psi_0(0) \quad (\text{see } (36)) \implies \bar{u} \neq 0.$$

From (36), we have

$$\begin{aligned}&\left\langle \psi_+'(\bar{u}), h\right\rangle = 0 \quad \text{for all } h \in W_0^{1,\theta}(\Omega) \\ &\implies \langle V(\bar{u}), h\rangle = \int_\Omega \left[\eta(z)(\bar{u}^+) - c_{12}(\bar{u}^+)^{r-1}\right]h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega).\end{aligned} \tag{37}$$

In (37), we choose the test function $h = -\bar{u}^- \in W_0^{1,\theta}(\Omega)$ to obtain

$$\rho_\theta(D\bar{u}^-) = 0 \implies \bar{u} \geq 0 \text{ and } \bar{u} \neq 0.$$

So, $\bar{u} \in W_0^{1,\theta}(\Omega)$ is a positive solution of (35). As in the proof of Proposition 9, we show that

$$\bar{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad 0 \prec \bar{u}.$$

Next we show the uniqueness of this positive solution. To this end, we introduce the integral functional $j\colon L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_{\theta_0}(Du^{1/2}) + \frac{1}{2}\|Du^{1/2}\|_2^2 & \text{if } u \geq 0,\ u^{1/2} \in W_0^{1,\theta}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). From Díaz–Saá [11] (see also Papageorgiou–Rădulescu [34]), we know that $j(\cdot)$ is convex. Suppose that $\bar{y} \in W_0^{1,\theta}(\Omega)$ is another positive solution of (35). Again we have

$$\bar{y} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad 0 \prec \bar{y}.$$

Let $\varepsilon > 0$ and set

$$\bar{u}_\varepsilon = \bar{u} + \varepsilon \quad \text{and} \quad \bar{y}_\varepsilon = \bar{y} + \varepsilon.$$

Recall that $L^\infty(\Omega)$ is an ordered Banach space with positive (order) cone $L^\infty(\Omega)_+ := \{u \in L^\infty(\Omega) : u(z) \geq 0 \text{ for a.a } z \in \Omega\}$. This cone has a nonempty interior given by

$$\operatorname{int} L^\infty(\Omega)_+ := \left\{u \in L^\infty(\Omega)_+ : \operatorname{ess\,inf}_\Omega u > 0\right\}.$$

Evidently, we have

$$\bar{u}_\varepsilon, \bar{y}_\varepsilon \in \operatorname{int} L^\infty(\Omega)_+. \tag{38}$$

Clearly, we have

$$\frac{\bar{u}_\varepsilon}{\bar{y}_\varepsilon} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\bar{y}_\varepsilon}{\bar{u}_\varepsilon} \in L^\infty(\Omega) \tag{39}$$

(see also Hu–Papageougiou [20, Proposition 2.86, p. 90] and (38)). Let $h = \bar{u}_\varepsilon^2 - \bar{y}_\varepsilon^2 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$. Using (38), we can check that for $t \in (0,1)$ small, we have

$$\bar{u}_\varepsilon^2 + th \in \operatorname{dom} j \quad \text{and} \quad \bar{y}_\varepsilon^2 + th \in \operatorname{dom} j.$$

Exploiting the convexity of $j(\cdot)$, we can compute the directional derivatives of $j(\cdot)$ at $\bar{u}_\varepsilon^2$ and at $\bar{v}_\varepsilon^2$ in the direction h. Using the nonlinear Green's identity (see Hu–Papageorgiou [19, p. 216]), we obtain

$$j'(\bar{u}_\varepsilon^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p^\alpha \bar{u} - \Delta \bar{u}}{\bar{u}_\varepsilon} h\,dz = \frac{1}{2}\int_\Omega \frac{\eta(z)\bar{u} - c_{12}\bar{u}^{r-1}}{\bar{u}_\varepsilon} h\,dz,$$
$$j'(\bar{y}_\varepsilon^2)(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p^\alpha \bar{y} - \Delta \bar{y}}{\bar{y}_\varepsilon} h\,dz = \frac{1}{2}\int_\Omega \frac{\eta(z)\bar{y} - c_{12}\bar{y}^{r-1}}{\bar{y}_\varepsilon} h\,dz.$$

On account of the convexity of $j(\cdot)$, we have

$$0 \leq \int_\Omega \left[\eta(z)\left(\frac{\bar{u}}{\bar{u}_\varepsilon} - \frac{\bar{y}}{\bar{y}_\varepsilon}\right) - c_{12}\left(\frac{\bar{u}^{r-1}}{\bar{u}_\varepsilon} - \frac{\bar{y}^{r-1}}{\bar{y}_\varepsilon}\right)\right] h \, dz.$$

We let $\varepsilon \to 0^+$ and use the Lebesgue dominated convergence theorem to get

$$0 \leq -c_{12} \int_\Omega (\bar{u}^{r-2} - \bar{y}^{r-2})(\bar{u}^2 - \bar{y}^2) \, dz \leq 0 \implies \bar{u} = \bar{y}.$$

This proves the uniqueness of the positive solution $\bar{u}$ of problem (35).

The problem is odd and so we infer that

$$\bar{v} = -\bar{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad \bar{v} \prec 0$$

is the unique negative solution of (35). ∎

Let S_+ be the set of positive solutions of (1) and S_- the set of negative solutions of (1). From Proposition 9, we have that

$$\emptyset \neq S_+ \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and } 0 \prec u \quad \text{for all } u \in S_+,$$
$$\emptyset \neq S_- \subseteq W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \quad \text{and } v \prec 0 \quad \text{for all } v \in S_-.$$

We show that the solutions $\bar{u}, \bar{v}$ of (35) (see Proposition 10) provide bounds for S_+ and S_-, respectively.

PROPOSITION 11. *If hypotheses H_0, H_1 hold, then $\bar{u} \leq u$ for all $u \in S_+$ and $v \leq \bar{v}$ for all $v \in S_-$.*

PROOF. Let $u \in S_-$ and introduce the Carathéodory function k_+ defined by

$$k_+(z,x) := \begin{cases} \eta(z)x^+ - c_{12}(x^+)^{r-1} & \text{if } x \leq u(z), \\ \eta(z)u(z) - c_{12}\big(u(z)\big)^{r-1} & \text{if } u(z) \leq x. \end{cases} \tag{40}$$

We set $K_+(z,x) := \int_0^x k_+(z,s)\,ds$ and consider the C^1-functional $\widehat{\psi}_+ : W_0^{1,\theta}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) := \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega K_+(z,u)\,dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

From (40), it is clear that $\widehat{\psi}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u}_* \in W_0^{1,\theta}(\Omega)$ such that

$$\widehat{\psi}_+(\bar{u}_*) = \inf\big\{\widehat{\psi}_+(u) : u \in W_0^{1,\theta}(\Omega)\big\}. \tag{41}$$

For $t \in (0,1)$ and $\hat{u}_1$ the positive normalized eigenfunction for $\widehat{\lambda}_1(2) > 0$, we have

$$\begin{aligned}
\widehat{\psi}_+(t\hat{u}_1) &\le \frac{t^p}{p}\rho_{\theta_0}(D\hat{u}_1) + c_{12}t^r\|\hat{u}_1\|_r^r + c_{12}\int_{\{u<t\hat{u}_1\}} u^{r-1}(t\hat{u}_1 - u)\,dz \\
&\quad - \frac{t^2}{2}\int_\Omega [\eta(z) - \widehat{\lambda}_1(2)]\hat{u}_1^2\,dz + \int_{\{u<t\hat{u}_1\}} \eta(z)u^2\,dz \quad \text{(see (40))} \\
&\le c_{13}t^p - c_{14}t^2 + t^2\|\eta\hat{u}_1\|_\infty \big|\{u < t\hat{u}_1\}\big|_N .
\end{aligned} \tag{42}$$

Since $|\{u < t\hat{u}_1\}|_N \to 0$ as $t \to 0^+$ (recall $0 \prec u$), from (42), we see that for $t \in (0,1)$ small, we have

$$\widehat{\psi}_+(t\hat{u}_1) < 0 \text{ (recall } 2 < p) \implies \widehat{\psi}_+(\bar{u}_*) < 0 = \widehat{\psi}_+(0) \text{ (see (41))} \implies \bar{u}_* \ne 0.$$

From (41), we have

$$\begin{aligned}
&\langle \widehat{\psi}_+'(\bar{u}_*), h\rangle = 0 \quad \text{for all } h \in W_0^{1,\theta}(\Omega) \\
&\implies \langle V(\bar{u}_*), h\rangle = \int_\Omega k_+(z,\bar{u}_*)h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega).
\end{aligned} \tag{43}$$

In (43), we choose the test function $h = -\bar{u}_*^- \in W_0^{1,\theta}(\Omega)$ to yield

$$\rho_\theta(D\bar{u}_*^-) = 0 \implies \bar{u}_* \ge 0 \text{ and } \bar{u}_* \ne 0 \quad \text{(see Proposition 2)}.$$

Next in (43), we choose the test function $h = (\bar{u}_* - u)^+ \in W_0^{1,\theta}(\Omega)$ to find

$$\begin{aligned}
\langle V(\bar{u}_*), (\bar{u}_* - u)^+\rangle &= \int_\Omega [\eta(z)u - c_{12}u^{r-1}](\bar{u}_* - u)^+\,dz \quad \text{(see (40))} \\
&\le \int_\Omega f(z,u)(\bar{u}_* - u)^+\,dz \quad \text{(see (34))} \\
&= \langle V(u), (\bar{u}_* - u)^+\rangle \quad (\text{since } u \in S_+) \\
&\Rightarrow \bar{u}_* \le u \quad \text{(see Proposition 3)}.
\end{aligned}$$

We have proved that

$$0 \le \bar{u}_* \le u \quad \text{and} \quad \bar{u}_* \ne 0. \tag{44}$$

From (40), (43), (44) and Proposition 10, it follows that

$$\bar{u}_* = \bar{u} \implies \bar{u} \le u \quad \text{for all } u \in S_+ \quad \text{(see (44))}.$$

Similarly, we show that

$$v \le \bar{v} \quad \text{for all } v \in S_-. \qquad \blacksquare$$

Now we are ready to produce the extremal constant sign solutions for problem (1).

PROPOSITION 12. *If hypotheses H_0 and H_1 hold, then we can find $u_* \in S_+$ and $v_* \in S_-$ such that $u_* \leq u$ for all $u \in S_+$ and $v \leq v_*$ for all $v \in S_-$.*

PROOF. From Filippakis–Papageorgiou [12], we know that S_+ is downward directed; that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leq u_1$ and $u \leq u_2$. Then, using Theorem 5.109 of Hu–Papageorgiou [19, p. 308], we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ decreasing such that

$$\inf S_+ = \inf_{u\in\mathbb{N}} u_n.$$

Also, we have

$$\langle V(u_n), h\rangle = \int_\Omega f(z, u_n)h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega), \text{ all } n \in \mathbb{N}, \tag{45}$$

and

$$\bar{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N} \quad \text{(see Proposition 11)}. \tag{46}$$

In (45), we choose the test function $h = u_n \in W_0^{1,\theta}(\Omega)$. Using (46) and hypothesis H_1 (i), we see that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\theta}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W_0^{1,\theta}(\Omega) \quad \text{and} \quad u_n \to u_* \text{ in } L^p(\Omega) \quad \text{(see Proposition 1)}. \tag{47}$$

In (45), we choose the test function $h = u_n - u_* \in W_0^{1,\theta}(\Omega)$ and then pass to the limit as $n \to \infty$ and use (47) to obtain

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_*\rangle = 0 \implies u_n \to u_* \text{ in } W_0^{1,\theta}(\Omega) \quad \text{(see Proposition 3)}. \tag{48}$$

In (45), we pass to the limit as $n \to \infty$ and use (48) to find

$$\langle V(u_*), h\rangle = \int_\Omega f(z, u_*)h\,dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega).$$

Moreover, from (46) and (48), we infer that

$$\bar{u} \leq u_* \implies u_* \in S_+,\ u_* = \inf S_+.$$

In a similar fashion, we produce $v_* \in S_-$ such that $v \leq v_*$ for all $v \in S_-$. We point out that S_- is upward directed; that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v$ and $v_2 \leq v$. ∎

4. Nodal solutions and multiplicity theorem

In this section, we produce a third nontrivial solution which is nodal (sign changing). To do this, we will use the extremal constant sign solutions produced in Section 3, truncation and comparison techniques and critical groups.

The idea is to use truncation to focus on the order interval

$$[v_*, u_*] := \{u \in W_0^{1,\theta}(\Omega) : v_*(z) \le u(z) \le u_*(z) \text{ for a.a } z \in \Omega\}.$$

On account of the extremality of v_* and u_*, any nontrivial solution of (1) distinct from v_* and u_*, will be nodal. Due to the lack of a global regularity theory for unbalanced growth problems, to show the nontriviality of the solution in $[v_*, u_*]$, we will use critical groups.

Proposition 13. *If hypotheses H_0 and H_1 hold, then*

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{R} \quad \textit{for all } k \in \mathbb{N}_0 \textit{ with } d_m = \dim \bigoplus_{i=1}^{m} E\big(\widehat{\lambda}_i(2)\big).$$

Proof. Let $\widehat{\eta} \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ and consider the C^2-functional $\sigma\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\sigma(u) := \frac{1}{2}\|Du\|_2^2 - \frac{\widehat{\eta}}{2}\|u\|_2^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Clearly, $u = 0$ is nondegenerate critical point of $\sigma(\cdot)$ with Morse index d_m. So, by Proposition 3.100 of Hu–Papageorgiou [20, p. 168], we have

$$C_k(\sigma, 0) = \delta_{k,d_m}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0. \tag{49}$$

Let $\varphi_0\colon H_0^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi_0(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

Consider the homotopy $H(t,u)$ defined by

$$h(t,u) = (1-t)\varphi_0(u) + t\sigma(u) \quad \text{for all } t \in [0,1], \text{ all } u \in H_0^1(\Omega).$$

For $0 < t \le 1$, let $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \le \delta$, with $\delta > 0$ as postulated by hypothesis H_1 (iii). By $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(H^{-1}(\Omega) := H_0^1(\Omega)^*$, $H_0^1(\Omega))$. Then, we have

$$\big\langle h'_u(t,u), h\big\rangle_0 = (1-t)\big\langle\varphi_0'(u), h\big\rangle_0 + t\big\langle\sigma'(u), h\big\rangle_0 \quad \text{for all } h \in H_0^1(\Omega). \tag{50}$$

We consider the following orthogonal direct sum decomposition:

$$H_0^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1}$$

where

$$\bar{H}_m = \bigoplus_{i=1}^{m} \left[E\big(\hat{\lambda}_i(2)\big)\right] \quad \text{and} \quad \hat{H}_{m+1} = \bar{H}_m^{\perp} = \overline{\bigoplus_{i\geq m+1} E\big(\hat{\lambda}_i(2)\big)}.$$

If $v \in H_0^1(\Omega)$, it admits the following unique sum decomposition:

$$v = \bar{v} + \hat{v} \quad \text{with } \bar{v} \in \bar{H}_m \text{ and } \hat{v} \in \hat{H}_{m+1}.$$

In (50), we choose the test function $h = \hat{u} - \bar{u} \in H_0^1(\Omega)$ to yield

$$\begin{aligned}\langle \varphi_0'(u), \hat{u} - \bar{u}\rangle_0 &= \int_\Omega (Du, D\hat{u} - D\bar{u})_{\mathbb{R}^N}\, dz - \int_\Omega f(z,u)(\hat{u}-\bar{u})\, dz \\ &= \|D\hat{u}\|_2^2 - \|D\bar{u}\|_2^2 - \int_\Omega f(z,u)(\hat{u}-\bar{u})\, dz.\end{aligned} \tag{51}$$

Hypothesis H_1 (iii) implies that

$$\eta(z) \leq \frac{f(z,x)}{x} \leq \hat{\lambda}_{m+1}(2) \quad \text{for a.a } z \in \Omega,\ \text{all } 0 < |x| \leq \delta. \tag{52}$$

Then, from the choice of u, we have

$$\begin{aligned} f(z,u)(\hat{u}-\bar{u}) &= f(z,u)h = \frac{f(z,u)}{u} uh \\ &\leq \begin{cases} \hat{\lambda}_{m+1}(2)(\hat{u}^2 - \bar{u}^2) & \text{if } uh > 0, \\ \eta(z)(\hat{u}^2 - \bar{u}^2) & \text{if } uh < 0 \end{cases} \quad \text{(see (52))} \\ &\leq \hat{\lambda}_{m+1}(z)\hat{u}^2 - \eta(z)\bar{u}^2 \end{aligned} \tag{53}$$

for a.a $z \in \Omega$. We return to (51) and use (53) to derive

$$\begin{aligned}\langle \varphi_0'(u), \hat{u} - \bar{u}\rangle_0 &\geq \|D\hat{u}\|_2^2 - \|D\bar{u}\|_2^2 - \hat{\lambda}_{m+1}(2)\|\hat{u}\|_2^2 + \int_\Omega \eta(z)\bar{u}^2\, dz \\ &= \left[\|D\hat{u}\|_2^2 - \hat{\lambda}_{m+1}(2)\|\hat{u}\|_2^2\right] - \left[\|D\bar{u}\|_2^2 - \int_\Omega \eta(z)\bar{u}^2\, dz\right] \geq 0.\end{aligned} \tag{54}$$

We also have

$$\begin{aligned}\langle \sigma'(u), \hat{u} - \bar{u}\rangle &= \int_\Omega (Du, D\hat{u} - D\bar{u})_{\mathbb{R}^N}\, dz - \hat{\eta} \int_\Omega u(\hat{u}-\bar{u})\, dz \\ &= \|D\hat{u}\|_2^2 - \|D\bar{u}\|_2^2 - \eta\|\hat{u}\|_2^2 + \eta\|\bar{u}\|_2^2 \\ &= \left[\|D\hat{u}\|_2^2 - \eta\|\hat{u}\|_2^2\right] - \left[\|D\bar{u}\|_2^2 - \hat{\eta}\|\bar{u}\|_2^2\right] \\ &\geq c_{15}\|u\|^2 \quad \text{for some } c_{15} > 0.\end{aligned} \tag{55}$$

Recall that $\widehat{\eta} \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$. We return to (50) and choose the test function $y = \hat{u} - \bar{u} \in H_0^1(\Omega)$ and use (54) and (55) to derive

$$\langle h'_u(t,u), \hat{u} - \bar{u}\rangle_0 \geq t c_{15}\|u\|^2 > 0 \quad (\text{recall } 0 < t \leq 1).$$

For $t = 0$, we have $h(0,\cdot) = \varphi_0(\cdot)$. We show that $u = 0 \in K_{\varphi_0}$ is isolated. We argue by contradiction. So, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq H_0^1(\Omega)$ such that

$$u_n \to 0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad \varphi_0'(u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{56}$$

From the equality in (56), we have

$$\begin{cases} -\Delta u_n(z) = f\big(z, u_n(z)\big) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{57}$$

for all $n \in \mathbb{N}$. Standard linear regularity theory (see Gilbarg–Trudinger [15, p. 186] and Struwe [40, p. 218]) implies that exist $\zeta \in (0,1)$ and $c_{16} > 0$ such that

$$u_n \in C_0^{1,\zeta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\zeta}(\overline{\Omega})} \leq c_{16} \quad \text{for all } n \in \mathbb{N}. \tag{58}$$

We know that $C_0^{1,\zeta}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ (see Arzela–Ascoli theorem). Therefore, from (56) and (58), it follows that

$$u_n \to 0 \text{ in } C_0^1(\overline{\Omega}) \implies \eta(z)u_n(z)^2 \leq f\big(z, u_n(z)\big)u_n(z) \leq \widehat{\lambda}_{m+1}(2)u_n(z)^2$$

for a.a $z \in \Omega$, all $n \geq n_0$ (see hypothesis H_1 (iii)). From earlier calculations we know that

$$f\big(z, u_n(z)\big)(\hat{u}_n - \bar{u}_n)(z) \leq \widehat{\lambda}_{m+1}(2)\hat{u}_n(z)^2 - \eta(z)\bar{u}_n(z) \quad \text{for a.a } z \in \Omega, \text{ all } n \geq n_0.$$

On (57), we act that with $h = \hat{u}_n - \bar{u}_n \in H_0^1(\Omega)$, we obtain

$$\begin{aligned} \|D\hat{u}_n\|_2^2 - \|D\bar{u}_n\|_2^2 &= \int_\Omega f(z, u_n)(\hat{u}_n - \bar{u}_n)\, dz \\ &\leq \widehat{\lambda}_{m+1}(2)\|\hat{u}_n\|_2^2 - \int_\Omega \eta(z)\bar{u}_n^2\, dz. \end{aligned}$$

Hence, it holds that

$$0 \leq \|D\hat{u}_n\|_2^2 - \widehat{\lambda}_{m+1}(2)\|\hat{u}_n\|_2^2 \leq \|D\bar{u}_n\|_2^2 - \int_\Omega \eta(z)\bar{u}_n^2\, dz \leq -c_{17}\|\bar{u}_n\|_2^2$$

for some $c_{17} > 0$ for all $n \geq n_0$ (see (8)); namely,

$$\bar{u}_n = 0 \text{ and } \|D\hat{u}_n\|_2^2 = \widehat{\lambda}_{m+1}(2)\|\hat{u}_n\|_2^2; \quad \text{hence, } \hat{u}_n \in E\big(\widehat{\lambda}_{m+1}(2)\big) \text{ for } n \geq n_0.$$

The eigenspace $E(\widehat{\lambda}_{m+1}(2))$ has the UCP and so $\hat{u}_n(z) \neq 0$ for a.a $z \in \Omega$, all $n \geq n_0$. Therefore,

$$\widehat{\lambda}_{m+1}(2)\|u_n\|_2^2 = \|Du_n\|_2^2 = \int_\Omega f\big(z, u_n(z)\big)u_n(z)\,dz < \widehat{\lambda}_{m+1}(2)\|u_n\|_2^2$$

(see hypothesis H_1 (iii)) generates a contradiction.

So, $u = 0$ is isolated. Then, the homotopy invariance property of critical groups (see Hu–Papageorgiou [20, Theorem 3.131, p. 179]) implies that

$$C_k(\varphi_0|_{C_0^1(\bar{\Omega})}, 0) = C_k(\sigma|_{C_0^1(\bar{\Omega})}, 0) \implies C_k(\varphi_0, 0) = C_k(\sigma, 0)$$

for all $k \in \mathbb{N}_0$ (see Hu–Papageorgiou [20, Theorem 3.128, p. 178]). Hence,

$$C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (49))}. \tag{59}$$

Let $\widehat{\varphi}_0 = \varphi_0|_{W_0^{1,\theta}(\Omega)}$. Since $W_0^{1,\theta}(\Omega) \hookrightarrow H_0^1(\Omega)$ densely (see Proposition 1), from Palais [30, Theorem 16], we have

$$C_k(\widehat{\varphi}_0, 0) = C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (59))}. \tag{60}$$

For every $u \in W_0^{1,\theta}(\Omega)$, we have

$$\big|\varphi(u) - \widehat{\varphi}_0(u)\big| = \frac{1}{p}\rho_{\theta_0}(Du), \tag{61}$$

and for all $h \in W_0^{1,\theta}(\Omega)$, we obtain

$$\begin{aligned}
&\big|\langle \varphi'(u) - \widehat{\varphi}_0'(u), h\rangle\big| \\
&\quad = \left|\int_\Omega \alpha(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\,dz\right| \\
&\quad \leq c_{18}\rho_{\theta_0}(Du)^{1/p'}\|h\| \quad \text{for some } c_{18} > 0 \quad \big(\text{since } W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,\theta_0}(\Omega)\big).
\end{aligned}$$

This means that

$$\big\|\varphi'(u) - \widehat{\varphi}_0'(u)\big\|_* \leq c_{18}\rho_{\theta_0}(Du)^{1/p'}. \tag{62}$$

From (61), (62), Proposition 2 and the C^1-continuity property of critical groups (see Hu–Papageorgiou [20, Theorem 3.129, p. 179]), we conclude that

$$C_k(\varphi, 0) = C_k(\widehat{\varphi}_0, 0) = \delta_{k,d_m}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (60))}. \qquad \blacksquare$$

Let u_*, v_* be the two extremal constant sign solutions produced in Proposition 12. We introduce the Carathéodory function $g(z, x)$ defined by

$$g(z,x) := \begin{cases} f\big(z, v_*(z)\big) & \text{if } x < v_*(z), \\ f(z,x) & \text{if } v_*(z) \leq x \leq u_*(z), \\ f\big(z, u_*(z)\big) & \text{if } u_*(z) < x. \end{cases} \tag{63}$$

In addition, we consider the positive and negative truncations of $g(z,\cdot)$; namely, the Carathéodory functions

$$g_{\pm}(z,x) = g(z,\pm x^{\pm}). \tag{64}$$

We set

$$G(z,x) = \int_0^x g(z,s)\,ds, \quad G_{\pm}(z,x) := \int_0^x g_{\pm}(z,s)\,ds$$

and consider the C^1-functionals $\psi, \psi_{\pm} : W_0^{1,\theta}(\Omega) \to \mathbb{R}$

$$\psi(u) := \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G(z,u)\,dz,$$
$$\psi_{\pm}(u) := \frac{1}{p}\rho_{\theta_0}(Du) + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_{\pm}(z,u)\,dz$$

for all $u \in W_0^{1,\theta}(\Omega)$. Using (63) and (64), we see that

$$K_\psi \subseteq [v_*, u_*] \quad \text{and} \quad K_{\psi_+} \subseteq [0, u_*] \quad \text{and} \quad K_{\psi_-} \subseteq [v_*, 0].$$

Taking into account the extremality of u_* and v_*, we conclude that

$$K_\psi \subseteq [v_*, u_*] \quad \text{and} \quad K_{\psi_+} = \{0, u_*\} \quad \text{and} \quad K_{\psi_-} = \{0, v_*\}. \tag{65}$$

PROPOSITION 14. *If hypotheses H_0 and H_1 hold, then u_* and v_* are local minimizers of $\psi(\cdot)$.*

PROOF. Clearly, $\psi_+(\cdot)$ is coercive (see (63) and (64)) and sequentially weakly continuous. So, we can find $\hat{u}_* \in W_0^{1,\theta}(\Omega)$ such that

$$\psi_+(\hat{u}_*) = \inf\left\{\psi_+(u) : u \in W_0^{1,\theta}(\Omega)\right\}. \tag{66}$$

As before (see the proof of Proposition 11), using (34), we show that

$$\psi_+(\hat{u}_*) < 0 = \psi_+(0) \implies \hat{u}_* \neq 0.$$

Since $\hat{u}_* \in K_{\psi_+}$ from (65), it follows that

$$\hat{u}_* = u_*.$$

So, it follows that

$$C_k(\psi_+, u_*) = \delta_{k,0}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0. \tag{67}$$

CLAIM. $C_k(\psi, u_*) = C_k(\psi_+, u_*)$ for all $k \in \mathbb{N}_0$.

Let $u \in W_0^{1,\theta}(\Omega)$. Then,

$$\begin{aligned}|\psi(u) - \psi_+(u)| &\leq \int_\Omega |G(z,u) - G_+(z,u)|\,dz \\ &\leq \int_\Omega |G(z,u) - G_+(z,u_*)|\,dz \\ &\quad + \int_\Omega |G_+(z,u_*) - G_+(z,u)|\,dz \quad \text{(since } u_* \geq 0\text{)}.\end{aligned} \tag{68}$$

We estimate the two integrals in the right-hand side of (68). Since $F(z,\cdot)$ is an L^∞-locally Lipschitz function, we have

$$\begin{aligned}&\int_\Omega |G(z,u) - G(z,u^*)|\,dz \\ &\quad\leq \int_{\{u<v_*\}} \big(|f(z,v_*)(v_* - u)| + |F(z,v_*) - F(z,u_*)|\big)\,dz \\ &\qquad + \int_{\{v_*\leq u\leq u_*\}} |F(z,u) - F(z,u_*)|\,dz \\ &\qquad + \int_{\{u_*<u\}} |f(z,u_*)(u - u_*)|\,dz \quad \text{(see (63))} \\ &\quad\leq c_{19}\|u - u_*\| \quad \text{for some } c_{19} > 0\end{aligned}$$

since $F(z,\cdot)$ is an L^∞-locally Lipschitz function.

Similarly, we show that

$$\int_\Omega |G_+(z,u_*) - G_+(z,u)|\,dz \leq c_{20}\|u - u_*\| \quad \text{for some } c_{20} > 0.$$

Therefore, we can say that

$$|\psi(u) - \psi_+(u)| \leq c_{21}\|u - u_*\| \quad \text{with } c_{21} = \max\{c_{20}, c_{21}\}. \tag{69}$$

Also, for all $u, h \in W_0^{1,\theta}(\Omega)$, we have

$$\begin{aligned}&|\langle \psi'(u) - \psi_+'(u), h\rangle| \\ &\quad\leq \int_\Omega |g(z,u) - g_+(z,u)||h|\,dz \\ &\quad\leq \int_\Omega |g(z,u) - g_+(z,u_*)||h|\,dz + \int_\Omega |g_+(z,u_*) - g_+(z,u)||h|\,dz \\ &\quad\leq c_{22}\|u - u_*\|\|h\| \quad \text{for some } c_{22} > 0\end{aligned}$$

because $g(z,\cdot)$, $g_+(z,\cdot)$ are Lispchitz continuous; recall $v_*, u_* \in L^\infty(\Omega)$. This leads to

$$\|\psi'(u) - \psi_+'(u)\|_* \leq c_{22}\|u - u_*\|. \tag{70}$$

From (69) and (70), we see that given $\varepsilon_0 > 0$, we can find $\delta_0 > 0$ such that

$$\|\psi - \psi_+\|_{C^1(\bar{B}_{\delta_0}(u_*))} \leq \varepsilon_0. \tag{71}$$

We assume that K_ψ is finite. Otherwise, on account of (65), we see that we already have an infinity of bounded nodal solutions and so, we are finished. Then, (71) and the C^1-continuity property of critical groups (see Hu–Papageorgiou [20, Theorem 3.129, p. 179]) imply that

$$C_k(\psi, u_*) = C_k(\psi_+, u_*) \quad \text{for all } k \in \mathbb{N}_0.$$

This proves the claim.

From (67) and the claim, we obtain

$$C_k(\psi, u_*) = \delta_{k,0}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0.$$

Invoking Theorem 4.6 of Chang [6, p. 43], we conclude that

$$u_* \text{ is a local minimizer of } \psi(\cdot).$$

Similarly, for v_* use this time $\psi_-(\cdot)$. ∎

Now, we are ready to produce a nodal solution.

Proposition 15. *If hypotheses H_0 and H_1 hold, then problem (1) has a nodal solution $y_0 \in W_0^{1,\theta}(\Omega)$ such that*

$$v_* \leq y_0 \leq u_*.$$

Proof. We assume that $\psi(v_*) \leq \psi(u_*)$ (the reasoning is the same if the opposite inequality holds). Also recall that we assume that K_ψ is finite (otherwise, on account of (65), we already have a sequence of distinct bounded nodal solutions and so we are done). Then, from Proposition 14 and Proposition 3.132 of Hu–Papageorgiou [20, p. 179] (see (73)), we can find $\rho \in (0, 1)$ small such that

$$\psi(v_*) \leq \psi(u_*) < \inf\left\{\psi(u) : \|v - u_*\| = \rho\right\} = m_0. \tag{72}$$

Clearly, $\psi(\cdot)$ is coercive (see (63)). Therefore, from Hu–Papageorgiou [20, p. 123], we have that

$$\psi(\cdot) \text{ satisfies the } C\text{-condition.} \tag{73}$$

Then, (72) and (73) permit the use of the Mountain pass theorem. Therefore, we can find $y_0 \in W_0^{1,\theta}(\Omega)$ such that

$$y_0 \in K_\psi \quad \text{and} \quad m_0 \leq \psi(y_0). \tag{74}$$

From (65), (72) and (74), we have

$$y_0 \in [v_*, u_*] \quad \text{and} \quad y_0 \in \{v_*, u_*\}. \tag{75}$$

Corollary 3.123 of Hu–Papageorgiou [20, p. 178] implies that

$$C_1(\psi, y_0) \neq 0. \tag{76}$$

Let $V = W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$. We have $V \hookrightarrow W_0^{1,\theta}(\Omega)$ densely. As above using the C^1-continuity property of critical groups, we show

$$C_k(\psi|_V, 0) = C_k(\varphi|_V, 0) \implies C_k(\psi, 0) = C_k(\varphi, 0) \quad \text{for all } k \in \mathbb{N}_0,$$

see Palais [30, Theorem 16]. This indicates that

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{R} \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{see Proposition 13}). \tag{77}$$

Since $d_m \geq 2$ (recall $m \geq 2$), from (76) and (77), we infer that $y_0 \neq 0$ and so $y_0 \in [v_*, u_*]$ is a nodal solution of (1). ∎

We can state the following multiplicity theorem for problem (1).

THEOREM 16. *If hypotheses H_0 and H_1 hold, then problem* (1) *has at least three nontrivial solutions:*

$$\begin{aligned}
&u_* \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \text{ with } 0 \prec u_*,\\
&v_* \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega) \text{ with } v_* \prec 0,\\
&y_0 \in W_0^{1,\theta}(\Omega) \text{ is nodal and } v_* \leq y_0 \leq u_*.
\end{aligned}$$

REMARK 17. In this multiplicity theorem, we provide sign information for all the solutions produced and these solutions are ordered.

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