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Differential Geometry. – *Nijenhuis operators on Banach homogeneous spaces*, by Tomasz Goliński, Gabriel Larotonda and Alice Barbara Tumpach, communicated on 13 June 2025.

Abstract. – For a Banach–Lie group G and an embedded Lie subgroup K, we consider the homogeneous Banach manifold $\mathcal{M} = G/K$. In this context, we establish the most general conditions for a bounded operator N acting on $\mathrm{Lie}(G)$ to define a homogeneous vector bundle map $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$. In particular, our considerations extend all previous settings in the matter and are well suited for the case where $\mathrm{Lie}(K)$ is not complemented in $\mathrm{Lie}(G)$. We show that the vanishing of the Nijenhuis torsion for a homogeneous vector bundle map $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$ (defined by an admissible bounded operator N on $\mathrm{Lie}(G)$) is equivalent to the Nijenhuis torsion of N having values in $\mathrm{Lie}(K)$. As an application, we consider the question of the integrability of an almost complex structure $\mathcal{J}$ on $\mathcal{M}$ induced by an admissible bounded operator J, and we give a simple characterization of the integrability in terms of certain subspaces of the complexification of $\mathrm{Lie}(G)$.

Keywords. – Nijenhuis operator, homogeneous space, almost complex manifold, Banach–Lie group.

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1. Introduction

For a smooth Banach vector bundle map $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$ (covering identity), its *Nijenhuis torsion* is defined as

$$\Omega_{\mathcal{N}}(X, Y) = \mathcal{N}[\mathcal{N}X, Y] + \mathcal{N}[X, \mathcal{N}Y] - [\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}^2[X, Y]$$

in terms of vector fields X, Y in $\mathcal{M}$. Here, $[\cdot,\cdot]$ is the usual Lie bracket of vector fields. Sometimes $\Omega_{\mathcal{N}}$ is called the *Nijenhuis tensor* of $\mathcal{N}$ in the literature. It was defined in [33] in order to describe the behavior of distributions spanned by eigenvectors of $\mathcal{N}$, see [28] for a review of the history around this subject. The Nijenhuis torsion is closely related to the problem of integrability of almost complex structures solved in the finite-dimensional real-analytic case by Eckmann and Frölicher in [15, 16] and for the smooth (or even less regular case) by Newlander and Nirenberg in [32].

The map $\mathcal{N}$ is a *Nijenhuis operator* if its Nijenhuis torsion vanishes. Nijenhuis operators are useful in the study of integrable systems, see e.g. [5,28,29,31,35] and the references therein. For example, the vanishing of the torsion $\Omega_{\mathcal{N}}$ is also equivalent to the Jacobi identity for a new *deformed* bracket on vector fields of $\mathcal{M}$ defined as follows:

$$[X,Y]_{\mathcal{N}} = [\mathcal{N}X,Y] + [X,\mathcal{N}Y] - \mathcal{N}[X,Y]$$

(see [27,29]), and in fact, $\mathcal{N}$ gives rise to a Lie algebra morphism from the new Lie algebra structure on the vector fields to the old one. It allows one also to deform Poisson brackets on the manifold via so-called Poisson–Nijenhuis structures [29]. They are also linked with Poisson–Lie groups and even Poisson groupoids and Lie bialgebroids [13].

There is a recent growing interest in Nijenhuis operators and their applications, as can be seen in the series of recent papers, for instance, [6,7] or [10] and references therein.

To the best of our knowledge, so far Nijenhuis operators were only studied in finite-dimensional context or formally. The aim of this paper is to generalize known results to the context of Banach manifolds taking into account (extending and correcting) results known for the specific case of complex structures from e.g. [2]. Our setting is as follows: we consider G a Banach–Lie group and K an embedded Banach–Lie subgroup of G, such that the quotient map $\pi : G \to G/K = \mathcal{M}$ is a smooth submersion. It is well known that in this setting, the existence of a closed linear complement for $\mathrm{Lie}(K)$ in $\mathrm{Lie}(G)$ is equivalent to the existence of smooth local cross-sections σ for the quotient map π. In our approach, we assume neither of those (see Remark 5.1 below) and extend some classical results to that setting.

On $\mathcal{M} = G/K$, a Banach vector bundle map $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$ is *homogeneous* if it is equivariant with respect to the natural action of the Lie group G on $\mathcal{M}$. In this paper, we are interested in homogeneous vector bundle maps that can be described by operators $N \in \mathcal{B}(\mathfrak{g})$ with certain properties (*admissible operators* of Definition 2.9), where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Banach–Lie algebra of G. The main purpose of this paper is to prove the following.

Theorem A. *Let $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$ be a homogeneous Banach vector bundle map induced by an admissible operator $N \in \mathcal{B}(\mathfrak{g})$ and the action of G on $\mathcal{M}$. Then, $\mathcal{N}$ is Nijenhuis if and only if for any $v, w \in \mathfrak{g}$,*

$$N[v,Nw] + N[Nv,w] - [Nv,Nw] - N^2[v,w] \in \mathfrak{k} = \mathrm{Lie}(K).$$

This is Theorem 3.6 below. In the process, we clarify certain aspects of known proofs of related results. As a corollary of Theorem 3.6, for homogeneous almost complex structures $\mathcal{J}$ defined by an admissible operator J, we give a linear characterization in the complexification of $\mathfrak{g}$ for $\mathcal{J}$ to be integrable, invoking the Banach version of the Newlander–Nirenberg theorem [2, Theorem 7] and our previous theorem. More precisely, let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and $J^{\mathbb{C}}$ the complexification of J. Let

$$Z_+ = \{v \in \mathfrak{g}^{\mathbb{C}} : J^{\mathbb{C}}v - iv \in \mathfrak{k}^{\mathbb{C}}\}.$$

By combining Theorem 4.6 and Corollary 4.7 below, we get the following.

THEOREM B. *Assuming that G/K has a real-analytic structure, the almost complex structure $\mathcal{J}$ is integrable if and only if Z_+ is a Banach–Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.*

Both theorems giving the characterization on homogeneous spaces are known in finite-dimensional case, but the existing proofs employ certain properties of vector fields and almost complex structures that do not always hold in Banach context. In particular, the proof by Frölicher in the finite-dimensional setting [16, Section 19, Satz 2] involves the existence of local cross-sections of the quotient map $\pi : G \to \mathcal{M} = G/K$ (see Remark 5.1), which may not exist in general in the Banach setting. On the other hand, in [2, Theorem 13], which is stated in the Banach setting, there is no mention of Nijenhuis operators and the discussion concerns only almost complex structures, which involves the additional constraint $\mathcal{J}^2 = -1$ (a particular case of our results). We also note that there is a problem with the proof of [2, Theorem 13] (see Remarks 2.7 and 2.14): we show on a very simple example that properties used in that proof do not hold, even in the finite-dimensional case (Section 6.1). Our approach avoids and clarifies the problem but at the same time extends the result to the case when $\mathfrak{k}$ is not complemented in $\mathfrak{g}$.

The results of this paper will also be applied in the study of almost Kähler structures on the coadjoint orbits of the unitary groups [19].

The paper is organized as follows: in Section 2, we introduce the necessary ideas and objects from the theory of Banach vector bundles and homogeneous spaces, and we discuss vector fields in the homogeneous space and homogeneous vector bundle maps. In particular, we comment on some possible pitfalls, which are later illustrated on an elementary example of the unit sphere of $\mathbb{R}^3$ in Section 6.1. In Section 3, we recall the notion of Nijenhuis torsion for a Banach vector bundle map, and using the exponential chart of the group G together with what we call projected vector fields in G/K, we prove the first main Theorem 3.6. In Section 4, we present the almost complex structures as special cases of the homogeneous maps discussed before, and we prove the second main result of the paper, Theorem 4.6. In Section 5, the classical approach to homogeneous operators is presented (when a splitting of the Lie algebra is at hand), illustrating how our approach is in fact more general and includes that one as a particular case. We finish the paper discussing some examples and applications of our main theorems in Section 6. This section also includes cases where our approach yields no Nijenhuis operators or complex structures, to point out the limitations. In the paper [18], more examples related to C^*-algebras are presented.

2. Homogeneous structures

In this section, main structures are introduced and the terminology and notations are fixed. This material does not pretend to be original; however, not all results were previously explicitly stated in the Banach context. For the sake of self-consistency of the paper, some proofs are also included.

2.1. *Notations and basic properties*

Notation 2.1. Let $\mathcal{M}_1, \mathcal{M}_2$ be smooth real manifolds and E a real Banach space.

- For $f : \mathcal{M}_1 \to \mathcal{M}_2$ a smooth map, we denote with $f_* : T\mathcal{M}_1 \to T\mathcal{M}_2$ its differential, which at each point $m \in \mathcal{M}_1$ will be denoted by $f_{*m} : T_m\mathcal{M}_1 \to T_{f(m)}\mathcal{M}_2$.
- We say that f is an *immersion* if, for all $m \in \mathcal{M}_1$, the map f_{*m} is an injection with closed range, and we say that f is a *submersion* if f_{*m} is a surjection for all $m \in \mathcal{M}_1$.
- If X is a vector field in $\mathcal{M}_1$ and $p \in \mathcal{M}_1$, we sometimes write X_p instead of $X(p)$ for convenience.
- If X_1, X_2 are vector fields in $\mathcal{M}_1, \mathcal{M}_2$, respectively, they are f-related if $X_2(f(m)) = f_{*m}(X_1(m))$ for all $m \in \mathcal{M}_1$. It is well known that in this case if also a vector field Y_2 is f-related with X_2, then

$$[X_2, Y_2]\big(f(m)\big) = f_{*m}\big([X_1, Y_1](m)\big), \tag{2.1}$$

  where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields.
- We use $\mathcal{B}(E)$ to denote the space of bounded linear operators acting on E and we denote with $\mathrm{GL}(E)$ the group of invertible bounded operators.

Notation 2.2. Let G be a real Banach–Lie group with Banach–Lie algebra $\mathfrak{g}$.

- The left and right multiplication by elements $g \in G$ will be denoted by $l_g(h) := gh$ and $r_g(h) := hg$, and the conjugation is $l_g r_{g^{-1}}$, i.e., $h \mapsto ghg^{-1}$.
- The differential of l_g at the unit element $h = 1$ will be denoted as L_g, i.e., $L_g = (l_g)_{*1}$ and likewise $R_g = (r_g)_{*1}$.
- The adjoint map on the Lie algebra (the differential of conjugation at the identity) is denoted as Ad_g, i.e., $\mathrm{Ad}_g = L_g R_{g^{-1}}$.
- The Lie bracket in $\mathfrak{g}$ will be denoted as $[v, w] = \mathrm{ad}_v w$, where $\mathrm{ad} = (\mathrm{Ad})_{*1}$ is the differential at $g = 1$ of the adjoint representation of the group $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

DEFINITION 2.3 (Homogeneous spaces). Let K be an immersed Banach–Lie subgroup of G with Banach–Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$. We say that G/K is a *homogeneous space* of G if the quotient space for the right action of K on G

$$G/K = \{gK,\ g \in G\}$$

has a Hausdorff Banach manifold structure such that the quotient map $\pi(g) = gK$ is a submersion.

This is guaranteed for instance when K is a split Banach–Lie subgroup (i.e., the Lie algebra $\mathfrak{k}$ is closed in $\mathfrak{g}$ and has a closed complement), but we do not require it. Since $K = \pi^{-1}(\pi(K))$ is a closed subgroup of G, and since we are imposing π to be a submersion, then in fact K must be embedded in G, see [1, Corollary 4.3] (however, it may not be split as mentioned before, see Section 6.2).

NOTATION 2.4. Let $\pi : G \to G/K$ be a homogeneous space.

- The base point in G/K will be denoted as $p_0 = \pi(K)$.
- The action of G will be denoted as $\alpha : G \times G/K \to G/K$, i.e., $\alpha(g, p) = \pi(gh)$ for $p = \pi(h) \in G/K$.
- For a fixed $g \in G$, we denote by $\alpha_g \in \mathrm{Aut}(G/K)$ the mapping $\alpha_g(p) = \alpha(g, p)$. Similarly, for a fixed $p \in G/K$, we denote by α^p the mapping $\alpha^p(g) = \alpha(g, p)$.
- The differential of α_g will be denoted by $(\alpha_g)_* : T(G/K) \to T(G/K)$. The point at which it is evaluated will be indicated as long as it is relevant or necessary. The same considerations will apply to $(\alpha^p)_* : TG \to T(G/K)$.

The following lemmas and remarks collect the trivial (but useful for our purposes) relations among the differentials of the various maps and vector fields on G/K.

LEMMA 2.5. *Let $g \in G$, $v \in \mathfrak{g}$. Then, the following hold:*

(1) $(\alpha_g)_*\pi_{*1} = \pi_{*g}L_g$ *or equivalently* $\pi_{*g} = (\alpha_g)_*\pi_{*1}L_g^{-1}$.

(2) *If* $p = \pi(g) \in G/K$, *then* $(\alpha^p)_{*1} = \pi_{*g}R_g = (\alpha_g)_*\pi_{*1}\mathrm{Ad}_g^{-1}$.

(3) *For any $h \in G$, the action property for derivative of α reads*

$$(\alpha_h)_*(\alpha_g)_* = (\alpha_{hg})_*. \tag{2.2}$$

It also implies that $(\alpha_g)_*$ is a diffeomorphism of $T(G/K)$.

(4) *For any $k \in K$, we have* $(\alpha_k)_*\pi_{*1} = \pi_{*1}\mathrm{Ad}_k$.

(5) *The kernel of the differential* $\pi_{*1} : \mathfrak{g} \to T_{p_0}(G/K) \cong \mathfrak{g}/\mathfrak{k}$ *is equal to* $\mathfrak{k}$.

(6) $\pi_{*g}L_g v = \pi_{*h}L_h w$ *iff there exists $k \in K$ such that $h = gk$ and* $v - \mathrm{Ad}_k w \in \mathfrak{k}$.

PROOF. Differentiating the equality $\alpha_g \circ \pi = \pi \circ l_g$ at the identity $1 \in G$, by chain rule we get the first claim. Similarly, for $p = \pi(g)$ differentiating $\alpha^p = \pi \circ r_g$ at the identity and using the previous identity, we get

$$\pi_{*g} R_g = (\alpha_g)_* \pi_{*1} L_g^{-1} R_g = (\alpha_g)_* \pi_{*1} \mathrm{Ad}_g^{-1},$$

which proves the second claim. The third claim follows from differentiating the property $\alpha_g \alpha_h = \alpha_{gh}$ and applying chain rule. For the fourth claim, we write $\pi(g) = \pi(gk) = \pi(r_k g)$ and we differentiate with respect to g at $g = 1$ to get $\pi_{*1} = \pi_{*k} R_k$. Thus, using the first identity, we get

$$(\alpha_k)_* \pi_{*1} = \pi_{*k} L_k = \pi_{*1} R_k^{-1} L_k = \pi_{*1} \mathrm{Ad}_k.$$

The fifth assertion follows from

$$\frac{d}{dt} \pi(e^{tv}) = (\alpha_{e^{tv}})_{*p_0} \pi_{*1} v.$$

Namely, if $v \in \ker \pi_{*1}$, then $\pi(e^{tv}) = \pi(1) = p_0$. Thus, $\{e^{tv}\}_{t \in \mathbb{R}} \subset K$ what implies $v \in \mathfrak{k}$. The sixth assertion is immediate from the fact that the base point must be the same (hence, $h = gk$) and then by the previous identities

$$\begin{aligned}(\alpha_g)_* \pi_{*1} v &= \pi_{*g} L_g v = \pi_{*h} L_h w = (\alpha_h)_* \pi_{*1} w \\ &= (\alpha_{gk})_* \pi_{*1} w = (\alpha_g)_* \pi_{*1} \mathrm{Ad}_k w.\end{aligned}$$

Since $(\alpha_g)_*$ is an isomorphism and $\ker \pi_{*1} = \mathfrak{k}$, the conclusion follows. ■

DEFINITION 2.6 (Projected vector fields in G/K). The right-invariant vector fields $X^v(g) = R_g v$ on the Lie group G can be pushed down to G/K as follows: for $p = \pi(g)$, consider

$$\widetilde{X^v}(p) = (\alpha^p)_{*1} v = \pi_{*g} R_g v = (\alpha_g)_* \pi_{*1} \mathrm{Ad}_g^{-1} v \in T_p G/K, \tag{2.3}$$

where the second and third equalities come from Lemma 2.5 (2). Since $p \mapsto (\alpha^p)_{*1} v = \alpha_{*(1,p)}(v, 0)$ depends smoothly on p, this defines a vector field in G/K, which is the vector field generated by the infinitesimal action of $v \in \mathfrak{g}$ on G/K, in other words an *infinitesimal generator* of the group action α.

The vector fields X^v and $\widetilde{X^v}$ are π-related, and it is easy to check that for $v, w \in \mathfrak{g}$ we have

$$[\widetilde{X^v}, \widetilde{X^w}] = -\widetilde{X^{[v,w]}},$$

where on the left we have the Lie bracket of vector fields on the manifold G/K, and on the right $[v, w] = \mathrm{ad}_v w$ is the Lie bracket in $\mathfrak{g}$.

REMARK 2.7 (A word of caution about invariance). If $p = \pi(h)$, then by Lemma 2.5 (3), we have

$$(\alpha_g)_{*p}\widetilde{X^v}(p) = (\alpha_{gh})_{*p_0}\pi_{*1}\mathrm{Ad}_h^{-1}v,$$

while

$$\widetilde{X^v}\big(\alpha_g(p)\big) = \widetilde{X^v}\big(\pi(gh)\big) = (\alpha_{gh})_{*p_0}\pi_{*1}\mathrm{Ad}_{gh}^{-1}v.$$

Since $(\alpha_{gh})_{*p_0}$ is an isomorphism, the equality $(\alpha_g)_{*p}\widetilde{X^v}(p) = \widetilde{X^v}(\alpha_g(p))$ can only happen if

$$\mathrm{Ad}_h^{-1}(\mathrm{Ad}_g^{-1}v - v) \in \ker \pi_{*1} = \mathfrak{k}.$$

So the vector fields $\widetilde{X^v}$ *are not* α_g*-related with themselves* in general, unlike left-invariant vector fields on a Lie group.

However, if we fix $h \in G$ and $v \in \mathfrak{g}$, then it is straightforward to check that $\widetilde{X^{\mathrm{Ad}_h v}}$ is the unique vector field in G/K which is α_h-related to the projected vector field $\widetilde{X^v}$. In other words,

$$(\alpha_h)_*\widetilde{X^v}\alpha_h^{-1} = \widetilde{X^{\mathrm{Ad}_h v}}. \tag{2.4}$$

2.2. *Homogeneous vector bundle maps and admissible operators*

In this section, we discuss Banach homogeneous vector bundle maps acting in the tangent bundle $T(G/K)$, and then we discuss the ones which come from a linear map defined "upstairs" in $\mathfrak{g}$. All vector bundle maps under consideration in this paper are understood to be covering identity, i.e., mapping each fiber to the same fiber.

DEFINITION 2.8. A smooth vector bundle map $\mathcal{N} : T(G/K) \to T(G/K)$ is called *homogeneous* if it is equivariant for the action by the automorphisms α_g:

$$(\alpha_g)_*\,\mathcal{N}_p = \mathcal{N}_{\alpha_g(p)}(\alpha_g)_* \quad \text{for any } p \in G/K \text{ and any } g \in G.$$

By homogeneity, any such map comes from some $\mathcal{N}_{p_0} \in \mathcal{B}(T_{p_0}(G/K))$ at the base point p_0.

We now look at the situation from the side of the Lie algebra $\mathfrak{g}$ of the group G.

DEFINITION 2.9. We will consider the following *admissible* linear bounded operators on $\mathfrak{g}$:

$$\mathcal{A}(G,K) = \big\{N \in \mathcal{B}(\mathfrak{g}) : N\mathfrak{k} \subset \mathfrak{k},\ \mathrm{Ran}(\mathrm{Ad}_k N - N\mathrm{Ad}_k) \subset \mathfrak{k}\ \forall k \in K\big\}. \tag{2.5}$$

The conditions imposed in this definition are necessary to make sure the operator will give rise to a homogeneous vector bundle map $\mathcal{N} : T(G/K) \to T(G/K)$, see Proposition 2.12 below. In particular, the condition $N\mathfrak{k} \subset \mathfrak{k}$ ensures that N descends to an operator on $\mathfrak{g}/\mathfrak{k}$. The other condition ensures that the resulting operator can be propagated by group action to $T(G/K)$.

Lemma 2.10. *Let $N \in \mathcal{B}(\mathfrak{g})$, and consider the assertions:*

(1) $\operatorname{Ran}(\operatorname{Ad}_k N - N \operatorname{Ad}_k) \subset \mathfrak{k}$ *for all* $k \in K$.

(2) $\operatorname{Ran}(N \circ \operatorname{ad}_z - \operatorname{ad}_z \circ N) \subset \mathfrak{k}$ *for all* $z \in \mathfrak{k}$.

Then, (1) *implies* (2) *and if K is connected, then* (1) *and* (2) *are equivalent.*

Proof. If (1) holds, then writing $k = e^{tz}$ with $z \in \mathfrak{k}$ and differentiating $e^{t \operatorname{ad}_z} N v - N e^{t \operatorname{ad}_z} v \in \mathfrak{k}$ we obtain (2). If K is connected, then it is generated by an exponential neighborhood of the unit element; hence, for any k, we can write $\operatorname{Ad}_k = \prod_i \operatorname{Ad}_{e^{z_i}} = \prod_i e^{\operatorname{ad}_{z_i}}$ for a finite number of $z_i \in \mathfrak{k}$, and this together with (2) readily implies (1). ■

Definition 2.11. The *homogeneous vector bundle map induced by $N \in \mathcal{A}(G, K)$* is the smooth vector bundle map $\mathcal{N} : T(G/K) \to T(G/K)$ given at each $p = \pi(g) \in G/K$ by

$$\mathcal{N}_p = (\alpha_g)_* \mathcal{N}_{p_0} (\alpha_g)_*^{-1}, \tag{2.6}$$

where $\mathcal{N}_{p_0} : T_{p_0} G/K \to T_{p_0} G/K$ with $p_0 = \pi(K)$ is defined as

$$\mathcal{N}_{p_0} \pi_{*1} v := \pi_{*1} N v, \quad v \in \mathfrak{g}. \tag{2.7}$$

Proposition 2.12. *For any $N \in \mathcal{A}(G, K)$, the map $\mathcal{N} : T(G/K) \to T(G/K)$ in the previous definition is a well-defined homogeneous vector bundle map in G/K.*

Proof. First let us show that $\mathcal{N}_{p_0} : T_{p_0} G/K \to T_{p_0} G/K$ is well defined by (2.7). Since π is a submersion, any tangent vector in $T_{p_0} G/K$ can be written as $\pi_{*1} X$, for some $X \in \mathfrak{g}$. We need to show that the value of $\mathcal{N}_{p_0} \pi_{*1} X \in T_{p_0} G/K$ does not depend on the representative $X \in \mathfrak{g}$. Consider $X_1, X_2 \in \mathfrak{g}$ such that $\pi_{*1}(X_1) = \pi_{*1}(X_2)$, i.e., $X_1 - X_2 \in \mathfrak{k}$. We have $\pi_{*1} N(X_1 - X_2) = 0$ since $N\mathfrak{k} \subset \mathfrak{k}$. By linearity, we therefore have

$$\pi_{*1} N(X_1) = \pi_{*1} N(X_2).$$

Since π_{*1} is a bounded surjection, the norm in $T_{p_0}(G/K) \simeq \mathfrak{g}/\mathfrak{k}$ must be equivalent to the quotient norm $\|\cdot\|_{\mathrm{quot}}$ by the open mapping theorem. To show that $\mathcal{N}_{p_0}$ is bounded for the quotient norm in $T_{p_0}(G/K)$, note that

$$\left\| \mathcal{N}_{p_0}[v] \right\|_{\mathrm{quot}} = \inf_{z \in \mathfrak{k}} \|Nv - z\| \le \inf_{z \in \mathfrak{k}} \|Nv - Nz\| \le \|N\| \inf_{z \in \mathfrak{k}} \|v - z\| = \|N\| \|v\|_{\mathrm{quot}}.$$

Now we take an arbitrary vector $X_p \in T_p(G/K)$ and we write it as $X_p = \pi_{*g} L_g v = \pi_{*h} L_h w$ for some $v, w \in \mathfrak{g}$. Using Lemma 2.5 (6), it follows that $h = gk$ and $v - \operatorname{Ad}_k w \in \mathfrak{k}$; hence,

$$\pi_{*1} v = \pi_{*1} \operatorname{Ad}_k w.$$

Comparing values of $\mathcal{N}$ on those two presentations of X_p, we get on one hand

$$\begin{aligned}(\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}X_p &= (\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}\pi_{*g}L_g v = (\alpha_g)_*\mathcal{N}_{p_0}\pi_{*1}v \\ &= (\alpha_g)_*\mathcal{N}_{p_0}\pi_{*1}\mathrm{Ad}_k w = (\alpha_g)_*\pi_{*1}N\,\mathrm{Ad}_k w \\ &= (\alpha_g)_*\pi_{*1}\mathrm{Ad}_k N w\end{aligned}$$

since N commutes with Ad_k modulo $\ker \pi_{*1} = \mathfrak{k}$. On the other hand,

$$\begin{aligned}(\alpha_h)_*\mathcal{N}_{p_0}(\alpha_h)_*^{-1}X_p &= (\alpha_h)_*\mathcal{N}_{p_0}(\alpha_h)_*^{-1}\pi_{*h}L_h w = (\alpha_h)_*\mathcal{N}_{p_0}\pi_{*1}w \\ &= (\alpha_h)_*\pi_{*1}N w = (\alpha_g)_*(\alpha_k)_*\pi_{*1}N w \\ &= (\alpha_g)_*\pi_{*1}\mathrm{Ad}_k N w\end{aligned}$$

by means of Lemma 2.5 (4). This proves that $\mathcal{N}$ is well defined by (2.6). By construction, $\mathcal{N}$ is a vector bundle map, and since it is given by the composition of smooth maps, it is smooth. The α_g-equivariance is a consequence of the definition $\mathcal{N}_p = (\alpha_g)_* N(\alpha_g)_*^{-1}$, the definition of $(\alpha_g)_*$, and the chain rule for α. ∎

Remark 2.13. In general, not every linear morphism of $T_{p_0}(G/K)$ comes from a linear operator on $\mathfrak{g}$. This is related to the so-called quotient lifting property of Banach spaces, see e.g. [23, 30]. However, when it happens, the linear operator necessarily belongs to $\mathcal{A}(G,K)$.

Remark 2.14 (Projected vector fields and the homogeneous vector bundle map). The Banach vector bundle maps can be seen also as maps on vector fields. Let us apply the map $\mathcal{N} : T(G/K) \to T(G/K)$ defined by means of $N \in \mathcal{A}(G,K)$ as in Definition 2.11 to the projected vector fields on G/K. If we compute $\mathcal{N}\widetilde{X^v}$, $v \in \mathfrak{g}$, we note that it differs from $\widetilde{X^{Nv}}$ in general. It can be seen as follows. For $p = \pi(g)$, by definition (2.3), we have

$$\widetilde{X^{Nv}}(p) = (\alpha_g)_*\pi_{*1}\mathrm{Ad}_g^{-1}Nv,$$

while

$$\begin{aligned}\mathcal{N}_p\widetilde{X^v}(p) &= (\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}\widetilde{X^v}(p) = (\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}(\alpha_g)_*\pi_{*1}\mathrm{Ad}_g^{-1}v \\ &= (\alpha_g)_*\mathcal{N}_{p_0}\pi_{*1}\mathrm{Ad}_g^{-1}v = (\alpha_g)_*\pi_{*1}N\,\mathrm{Ad}_g^{-1}v.\end{aligned}$$

Thus, for them to be equal, one must have

$$\mathrm{Ad}_g^{-1}(Nv) - N(\mathrm{Ad}_g^{-1}v) \in \mathfrak{k},$$

for all $g \in G$ (and not only $g \in K$), which is usually not the case. Compare with Remark 5.1 below.

3. The Nijenhuis torsion of a vector bundle map $\mathcal{N}$

Definition 3.1. Let $\mathcal{M}$ be any smooth manifold and let $\mathcal{N} : T\mathcal{M} \to T\mathcal{M}$ be a smooth Banach vector bundle map. The *Nijenhuis torsion* of $\mathcal{N}$ is defined as

$$\Omega_{\mathcal{N}}(X, Y) = \mathcal{N}[\mathcal{N}X, Y] + \mathcal{N}[X, \mathcal{N}Y] - [\mathcal{N}X, \mathcal{N}Y] - \mathcal{N}^2[X, Y]$$

for X, Y vector fields in $\mathcal{M}$. We say that $\mathcal{N}$ is a *Nijenhuis operator* in $\mathcal{M}$ if its torsion vanishes.

Note that $\Omega_{\mathcal{N}}$ is anti-symmetric. The following is well known for finite-dimensional manifolds. We omit the proof since it is an easy modification of [2, Lemma 2] for almost complex structures in the Banach setting (i.e., vector bundle maps such that $\mathcal{N}^2 = -1$). Let E denote the Banach space modeling the manifold $\mathcal{M}$.

Theorem 3.2. *The Nijenhuis torsion of $\mathcal{N}$ at $p \in \mathcal{M}$ depends only on the values of the vector fields at the point p; i.e., $\Omega_{\mathcal{N}}$ is a tensor. In any manifold chart (U, φ), using the local expression of $\mathcal{N} : \varphi(U) \subset E \to \mathcal{B}(E)$ and the local expressions of the vector fields $X, Y : \varphi(U) \to E$, one has*

$$\begin{aligned}\Omega_{\mathcal{N}}(X, Y)_p = {} & \mathcal{N}_p\big((\mathcal{N}_{*p}X_p)(Y_p) - (\mathcal{N}_{*p}Y_p)(X_p)\big) \\ & + \big(\mathcal{N}_{*p}(\mathcal{N}_p Y_p)\big)(X_p) - \big(\mathcal{N}_{*p}(\mathcal{N}_p X_p)\big)(Y_p).\end{aligned}$$

We now return to the homogeneous structures to give a local/global expression of the torsion of $\mathcal{N}$.

Remark 3.3 (Exponential map of G). Let $V \subset \mathfrak{g}$ be a 0-neighborhood such that $\exp|_V : V \to U = \exp(V)$ is a diffeomorphism. Recall the formula for the differential of the exponential map

$$\exp_{*z} x = L_{e^z} F(\mathrm{ad}_z)x = R_{e^z} f(\mathrm{ad}_z)x, \tag{3.1}$$

where $F(\lambda) = (1 - e^{-\lambda})/\lambda$ and $f(\lambda) = e^{\lambda} F(\lambda)$. The proof of these formulas for finite-dimensional Lie groups can be found in Helgason's book [24, Chapter IV, Theorem 4.1]; for a proof adapted to the Banach setting, see for instance [44, Appendix A].

From now on we denote by L_g the differential $(\ell_g)_{*h}$ at any $h \in G$, for short. We note that

$$F(\lambda) = 1 - \tfrac{1}{2}\lambda + O(\lambda^2) \quad \text{while } f(\lambda) = 1 + \tfrac{1}{2}\lambda + O(\lambda^2).$$

Lemma 3.4. *For an admissible operator $N \in \mathcal{A}(G, K)$ and $v \in \mathfrak{g}$, consider the right-invariant vector fields X^v (Definition 2.6) and the vector fields $X^{v,N}$ defined as*

$$X^{v,N}(g) = L_g N \mathrm{Ad}_g^{-1} v.$$

Then, one has

(1) $L_g^{-1}[X^{v,N}, X^w](g) = -N[v_0, w_0]$,

(2) $L_g^{-1}[X^{v,N}, X^{w,N}](g) = [Nv_0, Nw_0] - N[v_0, Nw_0] - N[Nv_0, w_0]$,

where $v, w \in \mathfrak{g}$, $v_0 = \mathrm{Ad}_g^{-1} v$, *and* $w_0 = \mathrm{Ad}_g^{-1} w$.

Proof. We use the exponential chart (gU, φ) around $g \in G$ to compute the Lie brackets, i.e., $\varphi : gU \to V \subset \mathfrak{g}$ and $\varphi^{-1}(z) = ge^z$ for $z \in V$. Consider the local expressions $\bar{X}^{v,N}$, $\bar{X}^v$ of $X^{v,N}$, X^v in this chart, i.e.,

$$\bar{X}^v(z) = \varphi_{*ge^z} X^v(ge^z), \quad \bar{X}^{v,N}(z) = \varphi_{*ge^z} X^{v,N}(ge^z)$$

for $z \in V$. We note that $L_{ge^z} = L_g L_{e^z}$ and $R_{ge^z} = R_{e^z} R_{e^g}$, and that L commutes with R. Differentiating the identity $\varphi(ge^z) = z$, we obtain $\varphi_{*ge^z}(l_g)_{*e^z} \exp_{*z} = \mathrm{id}_{\mathfrak{g}}$ or equivalently using the formula (3.1)

$$\varphi_{*ge^z} = F(\mathrm{ad}_z)^{-1} L_{e^z}^{-1} L_g^{-1} = f(\mathrm{ad}_z)^{-1} R_{e^z}^{-1} L_g^{-1} = f(\mathrm{ad}_z)^{-1} L_g^{-1} R_{e^z}^{-1}$$

by the previous remark. Then, plugging in the formula from Definition 2.6, we get

$$\bar{X}^v(z) = f(\mathrm{ad}_z)^{-1} L_g^{-1} R_g v = f(\mathrm{ad}_z)^{-1} \mathrm{Ad}_g^{-1} v$$

or equivalently $f(\mathrm{ad}_z)\bar{X}^v(z) = \mathrm{Ad}_g^{-1} v$. Replacing z with tz, we see that

$$\bar{X}^v(tz) + \tfrac{1}{2}\big[tz, \bar{X}^v(tz)\big] + O(t^2) = \mathrm{Ad}_g^{-1} v.$$

Then, by differentiating at $t = 0$, we get

$$\bar{X}^v_{*0}(z) = -\tfrac{1}{2}\big[z, \bar{X}^v(0)\big] = -\tfrac{1}{2}[z, \mathrm{Ad}_g^{-1} v].$$

Applying the same approach to $X^{v,N}$, we get

$$\bar{X}^{v,N}(z) = F(\mathrm{ad}_z)^{-1} N(\mathrm{Ad}_{e^z}^{-1} \mathrm{Ad}_g^{-1} v)$$

or equivalently $F(\mathrm{ad}_z)\bar{X}^{v,N}(z) = N(e^{-\mathrm{ad}_z} \mathrm{Ad}_g^{-1} v)$. Then, analogously to the previous case, we obtain

$$\bar{X}^{v,N}(tz) - \tfrac{1}{2}\big[tz, \bar{X}^{v,N}(tz)\big] + O(t^2) = N \mathrm{Ad}_g^{-1} v - N[tz, \mathrm{Ad}_g^{-1} v] + O(t^2).$$

Thus, again differentiating at $t = 0$, we arrive at

$$\bar{X}^{v,N}_{*0}(z) = \tfrac{1}{2}[z, \bar{X}^{v,N}(0)] - N[z, \mathrm{Ad}_g^{-1} v] = \tfrac{1}{2}[z, N \mathrm{Ad}_g^{-1} v] - N[z, \mathrm{Ad}_g^{-1} v].$$

Now we compute

$$[\bar{X}^{v,N}, \bar{X}^w](0) = \bar{X}^w_{*0}(\bar{X}^{v,N}_0) - \bar{X}^{v,N}_{*0}(\bar{X}^w_0)$$

which gives us

$$[\bar{X}^{v,N}, \bar{X}^{w}](0) = -\tfrac{1}{2}[N\mathrm{Ad}_g^{-1}v, \mathrm{Ad}_g^{-1}w] - \tfrac{1}{2}[\mathrm{Ad}_g^{-1}w, N\mathrm{Ad}_g^{-1}v] + N[\mathrm{Ad}_g^{-1}w, \mathrm{Ad}_g^{-1}v]$$
$$= N[\mathrm{Ad}_g^{-1}w, \mathrm{Ad}_g^{-1}v] = N[w_0, v_0].$$

Applying $\varphi_{*0}^{-1} = L_g$ gives the first formula of the lemma. The other Lie bracket

$$[\bar{X}^{v,N}, \bar{X}^{w,N}](0) = \bar{X}_{*0}^{w,N}(\bar{X}_0^{v,N}) - \bar{X}_{*0}^{v,N}(\bar{X}_0^{w,N})$$

can be computed in a similar fashion in order to arrive at the second formula of the lemma. ■

LEMMA 3.5. *Let $v, w \in \mathfrak{g}$ and consider the projected vector fields $\widetilde{X^v}, \widetilde{X^w}$ in G/K (Definition 2.6). Then, at $p = \pi(g)$, we have*

$$\mathcal{N}[\widetilde{X^v}, \mathcal{N}\widetilde{X^w}](p) + \mathcal{N}[\mathcal{N}\widetilde{X^v}, \widetilde{X^w}](p) = -2(\alpha_g)_*\pi_{*1}N^2[v_0, w_0]$$

and

$$[\mathcal{N}\widetilde{X^v}, \mathcal{N}\widetilde{X^w}](p) = (\alpha_g)_*\pi_{*1}\big([Nv_0, Nw_0] - N[v_0, Nw_0] - N[Nv_0, w_0]\big),$$

where $v_0 = \mathrm{Ad}_g^{-1}v$ and $w_0 = \mathrm{Ad}_g^{-1}w$.

PROOF. We first note that the vector fields $\mathcal{N}\widetilde{X^v}$ are π-related to the vector fields $X^{v,N}$ of the previous lemma, i.e., for any $g \in G$,

$$\pi_{*g}X^{v,N}(g) = \pi_{*g}L_g N\mathrm{Ad}_g^{-1}v = (\alpha_g)_*\mathcal{N}_{p_0}\pi_{*1}\mathrm{Ad}_g^{-1}v$$
$$= (\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}\widetilde{X^v}(p) = \mathcal{N}_p\widetilde{X^v}(p)$$

by Lemma 2.5 (1). We also recall that $\widetilde{X^v}$ are π-related with X^v. Thus, again by Lemma 2.5 (1), the previous lemma, and equality (2.1), we obtain

$$[\mathcal{N}\widetilde{X^v}, \widetilde{X^w}](p) = \pi_{*g}[X^{v,N}, X^w](g) = (\alpha_g)_*\pi_{*1}L_g^{-1}[X^{v,N}, X^w](g)$$
$$= -(\alpha_g)_*\pi_{*1}N[v_0, w_0].$$

By reversing the bracket and exchanging v, w, we also get

$$[\widetilde{X^v}, \mathcal{N}\widetilde{X^w}](p) = -(\alpha_g)_*\pi_{*1}N[v_0, w_0].$$

Thus, summing and applying $\mathcal{N}_p = (\alpha_g)_*\mathcal{N}_{p_0}(\alpha_g)_*^{-1}$, with $\mathcal{N}_{p_0}\pi_{*1}v := \pi_{*1}Nv$, we obtain the first formula of the lemma. To compute the second bracket, we use the previous lemma and the result is straightforward. ■

The following theorem, albeit in the setting of *complemented* Banach–Lie algebras (Section 5), can be found in [2, Theorem 13]. However, the method of proof in that paper has flaws (see Remarks 2.7, 2.14 and Example 6.1).

THEOREM 3.6. *Assume G/K is equipped with a homogeneous vector bundle map $\mathcal{N}$ induced by $N \in \mathcal{A}(G, K)$. Let X, Y be vector fields on G/K. Let $p = \pi(g)$ and take $v, w \in \mathfrak{g}$ such that $X(p) = (\alpha_g)_* \pi_{*1} v$ and $Y(p) = (\alpha_g)_* \pi_{*1} w$. Then, the Nijenhuis torsion of $\mathcal{N}$ can be expressed as*

$$\Omega_{\mathcal{N}}(X, Y)(p) = (\alpha_g)_* \pi_{*1}\big(N[v, Nw] + N[Nv, w] - [Nv, Nw] - N^2[v, w]\big).$$

In particular, $\mathcal{N}$ is a Nijenhuis operator in G/K, i.e., $\Omega_{\mathcal{N}} \equiv 0$, if and only if

$$N[v, Nw] + N[Nv, w] - [Nv, Nw] - N^2[v, w] \in \mathfrak{k} \tag{3.2}$$

for all $v, w \in \mathfrak{g}$.

PROOF. Since the value of the torsion tensor depends only on the values of the vector fields at the considered point (Theorem 3.2), we fix g and we replace X, Y with the projected vector fields with speeds $\mathrm{Ad}_g v$ and $\mathrm{Ad}_g w$, respectively (i.e., $\widetilde{X}(\pi(h)) = \widetilde{X^{\mathrm{Ad}_g v}}(\pi(h)) = (\alpha_h)_* \pi_{*1} \mathrm{Ad}_h^{-1} \mathrm{Ad}_g v$ and likewise $\widetilde{Y} = \widetilde{X^{\mathrm{Ad}_g w}}$, as in Definition 2.6). Then, we compute the torsion $\Omega_{\mathcal{N}}$ of these two vector fields, and almost all the computations were done in the previous lemmas. We only need to add that their Lie bracket is

$$[\widetilde{X}, \widetilde{Y}](p) = \pi_{*g} R_g [\mathrm{Ad}_g w, \mathrm{Ad}_g v] = -\pi_{*g} R_g \mathrm{Ad}_g [v, w] = -(\alpha_g)_* \pi_{*1} [v, w]$$

by means of Lemma 2.5(2). Therefore,

$$\mathcal{N}^2[\widetilde{X}, \widetilde{Y}](p) = -(\alpha_g)_* \pi_{*1} N^2[v, w],$$

which then cancels out one of the brackets in Lemma 3.5. ∎

REMARK 3.7. The expression in (3.2) is actually the value at the identity of the Nijenhuis torsion of the left-invariant bundle map $TG \to TG$ defined by N.

COROLLARY 3.8. *If either v or w belong to $\mathfrak{k}$, then (3.2) is automatically fulfilled for admissible N. If there exists a linear complement $\mathfrak{m}$ of $\mathfrak{k}$, it suffices to check (3.2) for $v, w \in \mathfrak{m}$.*

PROOF. By the anti-symmetry in v, w of (3.2), it suffices to verify the first claim for $v \in \mathfrak{k}$. By Lemma (2.10), N commutes with the adjoint action by elements in $\mathfrak{k}$ modulo $\mathfrak{k}$ and we have

$$N[v, Nw] = N^2[v, w] + k_1,$$

for some $k_1 \in \mathfrak{k}$; hence, the first term in equation (3.2) cancels with the fourth. Since $Nv \in \mathfrak{k}$ also, we have

$$N[Nv, w] = [Nv, Nw] + k_2$$

for some $k_2 \in \mathfrak{k}$, and the second term cancels with the third one. Now if $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$, by the bilinearity of the torsion $\Omega_{\mathcal{N}}$ and the previous claim, the second claim follows. ■

COROLLARY 3.9. *In the case when N is defined as ad_d for some $d \in \mathfrak{g}$, it is an admissible operator (see Definition 2.9) if and only if for all $v \in \mathfrak{g}$ and $k \in \mathfrak{k}$,*

$$[k, d] \in \mathfrak{k}$$

and

$$\big[v, [k, d]\big] \in \mathfrak{k}.$$

The condition (3.2) in this case simplifies to

$$\big[[d, v], [d, w]\big] \in \mathfrak{k}$$

for all $v, w \in \mathfrak{g}$.

PROOF. Simply expand (3.2) and apply the Jacobi identity. ■

4. ALMOST COMPLEX STRUCTURES

Let us recall that by almost complex structure $\mathcal{J}$ on a manifold $\mathcal{M}$, we mean a Banach vector bundle map $\mathcal{J} : TM \to TM$ such that $\mathcal{J}^2 = -1$. Its Nijenhuis torsion defined in Definition 3.1 is

$$\Omega_{\mathcal{J}}(X, Y) = \mathcal{J}\big([\mathcal{J}X, Y] + [X, \mathcal{J}Y]\big) - [\mathcal{J}X, \mathcal{J}Y] + [X, Y], \tag{4.1}$$

where X and Y are vector fields in $\mathcal{M}$, and the bracket $[\cdot,\cdot]$ denotes the bracket of vector fields.

REMARK 4.1. For finite-dimensional manifolds, the vanishing of this tensor is equivalent to the integrability of the almost complex structure by the Newlander–Nirenberg theorem [15,32]. In the infinite-dimensional setting, this is not always true. An example of an infinite-dimensional smooth almost complex Banach manifold with a vanishing Nijenhuis tensor, which is not integrable, was given by Patyi in [36]. However, as it was shown in [2] and in the appendix of [39], for real-analytic Banach manifolds endowed with real-analytic almost complex structures, the Newlander–Nirenberg theorem reduces to the Frobenius theorem for the eigenspaces of the complex linear extension $\mathcal{J}^{\mathbb{C}}$ of $\mathcal{J}$ to the complex analytic extension of the tangent bundle $T\mathcal{M}^{\mathbb{C}}$ by the same argument as employed in [15]. It is therefore true in this context.

The example in [36] shows that the construction of the complex analytic extension of the tangent bundle $T^{\mathbb{C}}\mathcal{M}$ may not be possible when the structure is only smooth and not real-analytic. The obstruction is the lack of certain properties of PDEs which hold in finite-dimensional vector spaces, but are not available in the Banach setting. For context and better explanation of these remarks, see the proof of Malgrange in Nirenberg's lecture notes [34, Theorem 4].

DEFINITION 4.2. A *homogeneous almost complex structure* is a homogeneous Banach vector bundle map $\mathcal{J}$ in G/K (Definition 2.8) with the additional requirement that $\mathcal{J}^2 = -1$.

In our homogeneous setting, we are interested in those $\mathcal{J}$ that are induced by admissible linear bounded operators via Definition 2.11.

DEFINITION 4.3. Consider the following subset of admissible operators on $\mathfrak{g}$:

$$\mathcal{AC}(G, K) = \{J \in \mathcal{A}(G, K) \mid \operatorname{Ran}(J^2 + 1) \subset \mathfrak{k}\}.$$

Note that if $\mathcal{J}$ is induced by $J \in \mathcal{AC}(G, K)$, one has $\mathcal{J}^2_{p_0} = -1$ in $T_{p_0}(G/K)$; therefore, $\mathcal{J}^2 = -1$ in the whole tangent bundle $T(G/K)$.

NOTATION 4.4 (Complexification). Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ be the complexification of the Banach–Lie algebra $\mathfrak{g}$, and denote by $\mathfrak{k}^{\mathbb{C}}$ the complexification of $\mathfrak{k}$. Relative to the splitting $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, the complex conjugation maps an element $x = a + ib \in \mathfrak{g}$ to its complex conjugate defined by $\bar{x} = a - ib$. For the complexified Lie-bracket, it is plain that

$$\overline{[x, y]} = [\bar{x}, \bar{y}]. \tag{4.2}$$

We will denote by $J^{\mathbb{C}}$ the complex linear extension of $J \in \mathcal{B}(\mathfrak{g})$ to $\mathfrak{g}^{\mathbb{C}}$, i.e., $J^{\mathbb{C}}(a + ib) = Ja + iJb$. Note that $J^{\mathbb{C}}$ is a bounded operator on $\mathfrak{g}^{\mathbb{C}}$, which satisfies the following:

- $J^{\mathbb{C}}(\bar{x}) = \overline{J^{\mathbb{C}}(x)}$ for any $x \in \mathfrak{g}^{\mathbb{C}}$;
- for $J \in \mathcal{A}(G, K)$, its complexification $J^{\mathbb{C}}$ preserves $\mathfrak{k}^{\mathbb{C}}$ and we have

  $$J^{\mathbb{C}}[k, v] = [k, J^{\mathbb{C}} v] \tag{4.3}$$

  for any $k \in \mathfrak{k}^{\mathbb{C}}$ and $v \in \mathfrak{g}^{\mathbb{C}}$;
- for $J \in \mathcal{AC}(G, K)$, $\operatorname{Ran}((J^{\mathbb{C}})^2 + 1) \subset \mathfrak{k}^{\mathbb{C}}$ holds.

DEFINITION 4.5. Define the following two subspaces:

$$Z_{\pm} = \{v \in \mathfrak{g}^{\mathbb{C}} : (J^{\mathbb{C}} \mp i)v \in \mathfrak{k}^{\mathbb{C}}\},$$

that is, $v \in Z_+$ if $Jv = iv + k$ for some $k \in \mathfrak{k}^{\mathbb{C}}$ and likewise with Z_-. Note that $\overline{Z_+} = Z_-$ and that Z_+ and Z_- are closed as preimages of the closed subalgebra $\mathfrak{k}^{\mathbb{C}}$ by a continuous map.

THEOREM 4.6. *A homogeneous almost complex structure* $\mathcal{J}$ *in* G/K *induced by* $J \in \mathcal{AC}(G, K)$ *is Nijenhuis (i.e., the condition* $\Omega_{\mathcal{J}} \equiv 0$ *holds) if and only if* $[v, w] \in Z_+$ *for all* $v, w \in Z_+$.

PROOF. We use the formula (3.2) from Theorem 3.6, with the addition that for $v, w \in \mathfrak{g}$,

$$J^2[v,w] + [v,w] \in \mathfrak{k}. \tag{4.4}$$

Let us define an anti-symmetric bilinear form β on $\mathfrak{g}$ by

$$\beta(v,w) := J[v,Jw] + J[Jv,w] - [Jv,Jw] - J^2[v,w]. \tag{4.5}$$

In this setting, the vanishing of the torsion is therefore equivalent to β taking values in $\mathfrak{k}$:

$$\Omega_{\mathcal{J}} \equiv 0 \iff \beta(v,w) \in \mathfrak{k}, \quad \forall v, w \in \mathfrak{g}.$$

- Suppose that $\Omega_{\mathcal{J}}$ vanishes. By complexifying the bilinear form β defined by equation (4.5) and using equation (4.4), we have for $v, w \in \mathfrak{g}^{\mathbb{C}}$

  $$J^{\mathbb{C}}[v, J^{\mathbb{C}}w] + J^{\mathbb{C}}[J^{\mathbb{C}}v, w] - [J^{\mathbb{C}}v, J^{\mathbb{C}}w] + [v,w] \in \mathfrak{k}^{\mathbb{C}}.$$

  Let us prove that if $v, w \in Z_+$, then the bracket $[v, w]$ belongs to Z_+ as well. For $v, w \in Z_+$, we have $J^{\mathbb{C}}v = iv + k_1$ and $J^{\mathbb{C}}w = iw + k_2$. Therefore,

  $$2iJ^{\mathbb{C}}[v,w] + J^{\mathbb{C}}[v,k_2] + J^{\mathbb{C}}[k_1,w] + 2[v,w] - i[v,k_2] - i[k_1,w] \in \mathfrak{k}^{\mathbb{C}}.$$

  By equation (4.3), we have

  $$J^{\mathbb{C}}[v,k_2] = -J^{\mathbb{C}}[k_2,v] = -[k_2, J^{\mathbb{C}}v] = -[k_2, iv + k_1] = i[v,k_2] + k_3$$

  and likewise $J^{\mathbb{C}}[k_1, w] = i[k_1, w] + k_4$; therefore,

  $$2iJ^{\mathbb{C}}[v,w] = -2[v,w] + k_5,$$

  which proves that $[v, w] \in Z_+$.
- Now we prove the implication in reverse direction. Suppose that for $v, w \in Z_+$, the bracket $[v, w]$ belongs to Z_+. Let us prove that β takes values in $\mathfrak{k}$.
  Let $v, w \in \mathfrak{g}$. Then,

  $$(J^{\mathbb{C}} - i)(J^{\mathbb{C}} + i)v = (J^2 + 1)v \in \mathfrak{k};$$

  therefore, $(J^{\mathbb{C}} + i)v \in Z_+$, and likewise $(J^{\mathbb{C}} + i)w \in Z_+$. Then, the hypothesis of the theorem tells us that

  $$J^{\mathbb{C}}\big[(J^{\mathbb{C}} + i)v, (J^{\mathbb{C}} + i)w\big] = i\big[(J^{\mathbb{C}} + i)v, (J^{\mathbb{C}} + i)w\big] + k$$

for some $k \in \mathfrak{k}^{\mathbb{C}}$. After expanding, we get that

$$J[Jv, Jw] + iJ[v, Jw] + iJ[Jv, w] - J[v, w] \tag{4.6}$$

equals

$$i[Jv, Jw] - [Jv, w] - [v, Jw] - i[v, w] + k.$$

Note that by using the hypothesis, conjugating, using equation (4.2) and the fact that $\overline{Z_+} = Z_-$, we also have $J^{\mathbb{C}}[x, y] + i[x, y] \in \mathfrak{k}^{\mathbb{C}}$ for all $x, y \in Z_-$. Since we also have $(J^{\mathbb{C}} - i)v \in Z_-$ and $(J^{\mathbb{C}} - i)w \in Z_-$, with a similar reasoning we obtain that

$$J[Jv, Jw] - iJ[v, Jw] - iJ[Jv, w] - J[v, w] \tag{4.7}$$

equals

$$-i[Jv, Jw] - [Jv, w] - [v, Jw] + i[v, w] + k'.$$

Adding equations (4.6) and (4.7) (and halving) and canceling out, we arrive at

$$[v, Jw] + [Jv, w] + J[Jv, Jw] - J[v, w] \in \mathfrak{k}.$$

If we apply J, we get

$$J[v, Jw] + J[Jv, w] + J^2[Jv, Jw] - J^2[v, w] \in \mathfrak{k}.$$

Finally, using equation (4.4),

$$J^2[Jv, Jw] = -[Jv, Jw] + k_2 \quad \text{for } k_2 \in \mathfrak{k};$$

hence, $\beta(v, w) \in \mathfrak{k}$. ■

Combining Definitions 2.3, 2.9, 4.5, Theorem 4.6 and the Newlander–Nirenberg theorem in real-analytic Banach context [2, Theorem 7], we obtain the following.

COROLLARY 4.7. *Let G/K be a real-analytic homogeneous space equipped with a real-analytic homogeneous almost complex structure $\mathcal{J}$ given by $J \in \mathcal{AC}(G, K)$. Then, $\mathcal{J}$ is integrable (i.e., G/K admits complex charts compatible with $\mathcal{J}$) if and only if it is Nijenhuis, i.e., if and only if*

$$Z_+ = \{v \in \mathfrak{g}^{\mathbb{C}} : (J^{\mathbb{C}} - i)v \in \mathfrak{k}^{\mathbb{C}}\}$$

is a complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

REMARK 4.8. By conjugation, Z_+ is a complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$ if and only if Z_- is.

5. A brief look at the split case

In this section, we discuss the situation when K is split in G, i.e., if $\mathfrak{k}$ has a closed complement $\mathfrak{m}$ in $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Under this condition one can consider the following subset of *admissible* linear bounded operators on $\mathfrak{g}$:

$$\mathcal{B}(G,K) = \big\{N \in \mathcal{B}(\mathfrak{g}) : \mathfrak{k} \subset \ker N;\ N\mathfrak{m} \subset \mathfrak{m};\ \mathrm{Ran}(\mathrm{Ad}_k N - N\mathrm{Ad}_k) \subset \mathfrak{k}\ \forall k \in K\big\}.$$

Similarly, one can consider the following subset of $\mathcal{AC}(G,K)$:

$$\mathcal{I}_{\mathfrak{m}}(G,K) = \big\{N \in \mathcal{AC}(G,K) : \mathfrak{k} \subset \ker N;\ N\mathfrak{m} \subset \mathfrak{m};\ N^2_{|\mathfrak{m}} = -1_{|\mathfrak{m}}\big\}.$$

Note that $\mathcal{B}(G,K)$ is strictly contained in $\mathcal{A}(G,K)$ and $\mathcal{I}_{\mathfrak{m}}(G,K)$ is strictly contained in $\mathcal{AC}(G,K)$. By [2, Proposition 12], any almost complex structure on $\mathcal{M} = G/K$ is induced by a linear map in $\mathcal{I}_{\mathfrak{m}}(G,K)$. A similar construction was used in [12] to define Riemannian metrics on the manifold of non-linear flags.

Remark 5.1. If K is split in G, then around each $p \in G/K$ there exists a smooth local cross-section $\sigma : U \subset G/K \to G$ for the quotient map (i.e., $\pi \circ \sigma = \mathrm{id}_U$, see [3, Theorem 4.19]). Then, the proof of Theorem 3.6 can be simplified (following Frölicher [16, Satz 2, Section 19] for almost complex structures) by considering the local vector fields on the homogeneous space

$$\hat{v}_p = \pi_{*\sigma(p)} L_{\sigma(p)} v, \quad p \in U,\ v \in \mathfrak{g}.$$

It is plain that $\hat{v}$ is π-related to the restriction of the left-invariant vector field generated by v to the submanifold $\sigma(U)$, and also that

$$(\mathcal{N}\,\hat{v})_p = \mathcal{N}_p \hat{v}_p = \pi_{*\sigma(p)} L_{\sigma(p)} N v = (\widehat{Nv})_p;$$

i.e., $\mathcal{N}$ exchanges the field induced by $v \in \mathfrak{g}$ with the one induced by $Nv \in \mathfrak{g}$.

Remark 5.2. One can consider $J \in \mathcal{I}_{\mathfrak{m}}(G,K)$. In this case, the spaces $Z_\pm$ defined in Definition 4.5 are given by

$$Z_\pm = \mathfrak{k}^{\mathbb{C}} \oplus \mathrm{Eig}_{\pm i}(J^{\mathbb{C}}_{|\mathfrak{m}^{\mathbb{C}}}),$$

where $\mathrm{Eig}_{\pm i}(J^{\mathbb{C}}_{|\mathfrak{m}^{\mathbb{C}}})$ is the eigenspace with eigenvalue $\pm i$ of the complex linear extension $J^{\mathbb{C}}$ restricted to $\mathfrak{m}^{\mathbb{C}}$. Note that in this case, we have

- $\mathfrak{g}^{\mathbb{C}} = Z_+ + Z_-$,
- $Z_+ \cap Z_- = \mathfrak{k}^{\mathbb{C}}$,
- $\mathrm{Ad}_{\mathfrak{k}} Z_\pm \subset Z_\pm$.

By [2, Theorem 15] (see also [16]), in this complemented case, any homogeneous complex structure on $M = G/K$ comes from this kind of decomposition of $\mathfrak{g}^{\mathbb{C}}$. In particular, in the complemented case, Corollary 4.7 reduces to [2, Theorem 13].

6. Examples

We end this paper with some examples that illustrate the definitions, applications, and possible pitfalls.

6.1. *Example: spheres*

Identify the sphere $\mathbb{S}^2$ with the homogeneous space $\mathrm{SO}(3)/\mathrm{SO}(2)$ in such a way that the base point corresponds to the north pole $p_0 = (0,0,1)^T$. The action α of $G = \mathrm{SO}(3)$ (and $\mathfrak{g} = \mathfrak{so}(3)$ as well) is given by left matrix multiplication. Consider the basis of $\mathfrak{so}(3)$:

$$k_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Note that k_0 spans the Lie algebra $\mathfrak{k} \cong \mathfrak{so}(2)$ and that $\{e_1 p_0, e_2 p_0\}$ spans the tangent space to the sphere $\mathbb{S}^2$ at p_0. The infinitesimal generator $v \in \mathfrak{so}(3)$ gives us

$$\widetilde{X^v}(p) = (\alpha^p)_{*1} v = vp, \quad \text{for } p \in \mathbb{S}^2.$$

The diffeomorphism $\alpha_g : \mathbb{S}^2 \to \mathbb{S}^2$ transforms $\widetilde{X^v}$ in the following manner (see equation (2.4)):

$$gvp = g\widetilde{X^v}(p) = (\alpha_g)_*(\widetilde{X^v})(p) = (\alpha_g)_*(\widetilde{X^v})\big((\alpha_g)^{-1}(\alpha_g p)\big) = \widetilde{X^{\mathrm{Ad}_g v}}(gp),$$

while $\widetilde{X^v}(gp) = vgp$; thus, in general, $g\widetilde{X^v}(p) \neq \widetilde{X^v}(gp)$ (Remark 2.7).

Now consider $N = \mathrm{ad}_{k_0}$ on $\mathfrak{so}(3)$. It preserves $\mathfrak{k} = \mathbb{R}k_0$. To show that it is admissible, we verify the claim $\mathrm{Ran}(\mathrm{Ad}_k N - N \mathrm{Ad}_k) \subset \mathfrak{k}$ for $k \in K$ (Definition 2.9): since K is a connected group, by Lemma 2.10, it is enough to verify it on the level of the Lie algebra, which is trivial. Therefore, N descends to a linear operator $\mathcal{N}_{p_0}$ on the tangent space to the sphere at p_0 of the following form:

$$\mathcal{N}_{p_0}(vp_0) = (Nv)(p_0), \quad v \in \mathfrak{so}(3). \tag{6.1}$$

With this we define the vector bundle map $\mathcal{N}$ on $\mathbb{S}^2$ using the homogeneous action of the group, following Definition 2.11:

$$\mathcal{N}_{gp_0} z := g\mathcal{N}_{p_0}(g^{-1}z), \quad z \in T_{gp_0}\mathbb{S}^2 \subset \mathbb{R}^3.$$

Regarding Remark 2.14, we have

$$(\mathcal{N}\widetilde{X^v})(gp_0) = \mathcal{N}_{gp_0}\widetilde{X^v}(gp_0) = g\mathcal{N}_{p_0}g^{-1}\widetilde{X^v}(gp_0) = g\mathcal{N}_{p_0}(g^{-1}vgp_0)$$
$$= gN(g^{-1}vg)p_0 = gN(\mathrm{Ad}_g^{-1}v)p_0,$$

and on the other hand, $\widetilde{X^{Nv}}(gp_0) = (Nv)gp_0$, which are far from being equal unless $g \in K$.

Note that from the definition of N, it follows directly that $\mathrm{Ran}(N^2 + 1) = \{0\} \subset \mathfrak{k}$. Therefore, N induces an almost complex structure $\mathcal{N}$ on $\mathbb{S}^2 \simeq \mathrm{SO}(3)/\mathrm{SO}(2)$. By Remark 5.2, the vanishing of its torsion is equivalent to $Z_+ = \{E_1 + iE_2\}$ being a Lie subalgebra of $\mathfrak{so}(3)$, which is trivial because it is (complex) one-dimensional. Therefore, this particular complex structure is integrable, giving the usual complex structure on the sphere $\mathbb{S}^2$. We remark here that it is known that the real sphere $\mathbb{S}^n$ admits an almost complex structure if and only if $n = 2$ or $n = 6$ (see [8] or the survey [26] for further details). The known almost complex structure on $\mathbb{S}^6$ is also homogeneous and can be constructed by considering $\mathbb{S}^6$ inside the subspace of purely imaginary octonions; however, this almost complex structure is not integrable (see [26]).

For the infinite-dimensional sphere $\mathcal{S}$ (the unit sphere of a real Hilbert space $\mathcal{H}$), it is known that $\mathcal{S}$ is real-analytic isomorphic to $\mathcal{H}$, see [14]; therefore, $\mathcal{S}$ admits an almost complex structure, being a complex manifold (an infinite-dimensional real Hilbert space $\mathcal{H}$ is also a complex Hilbert space, halving the basis).

6.2. *Example: non-complemented setting—left and right multiplication*

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and denote with $\mathcal{B}(\mathcal{H})$ the bounded linear operators acting in $\mathcal{H}$, with $\mathcal{K}(\mathcal{H})$ the ideal of compact operators. Note that $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is known as the Calkin algebra, see [18] for more details. Consider the group of invertible operators $G = \mathrm{GL}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. For the Lie subgroup K, consider the group of invertible operators which differ from the identity by a compact operator $K = \mathrm{GL}(\mathcal{H}) \cap (\mathbb{1} + \mathcal{K}(\mathcal{H}))$. Note that since $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$, then K is an immersed subgroup of G (moreover, it is embedded since the topology of K is the norm topology). But K is not split in G since the compact operators are not complemented in the bounded operators. Now, since compact operators are a closed ideal in the algebra of bounded operators, the group K is a normal subgroup of G; therefore, the quotient has a structure of Banach–Lie group, which makes of the quotient map $\pi : G \to G/K$ a smooth submersion (see [17, Theorem II.2]).

Consider an operator N given by right and left multiplication by bounded operators:

$$N(X) = AXB, \tag{6.2}$$

for $A, B, X \in \mathcal{B}(\mathcal{H})$. In this case, the condition $N\mathcal{K}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ is automatically satisfied since $\mathcal{K}(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(H)$.

The other condition for N to be admissible (see Lemma 2.10) is

$$[k, AXB] - A[k, X]B \in \mathcal{K}(\mathcal{H}),$$

for $k \in \mathcal{K}(\mathcal{H})$. It is also automatically satisfied for the same reason.

PROPOSITION 6.1. *The operator N defined by (6.2) descends to the operator $\mathcal{N}$ on the homogeneous space. If we choose A and B such that A^2 and B^2 are multiples of $\mathbb{1}$ and $A^2B^2 = -\mathbb{1}$, we get an almost complex structure on G/K.*

Let us verify using Theorem 3.6 and Corollary 4.7 if $\mathcal{N}$ is a Nijenhuis operator and integrable complex structure. Note that it is known that G/K has a real analytic manifold structure. The condition (3.2) does not hold in general; however, in a simpler case when either $A = \mathbb{1}$ or $B = \mathbb{1}$, it is always satisfied.

PROPOSITION 6.2. *If N is left (or right) multiplication by a bounded operator from $\mathcal{B}(\mathcal{H})$, then $\mathcal{N}$ is a Nijenhuis operator on G/K. Moreover, if the square of this operator is -1, we obtain an integrable complex structure on G/K.*

6.3. *Example: non-complemented setting—rank one case*

Consider G and K as in previous example. Let us look for another simple case of the operator N. First, consider the linear functional ℓ on $S = \mathbb{C}\mathbb{1} + \mathcal{K}(\mathcal{H})$ defined as $\ell(\mathcal{K}(\mathcal{H})) = 0$ and $\ell(\mathbb{1}) = 1$. Since

$$\|t\mathbb{1} + k\| = |t|\|\mathbb{1} + k'\| \geq |t| = \left|\ell(t\mathbb{1} + k)\right|,$$

it follows that ℓ is bounded in $S = \mathbb{C}\mathbb{1} + \mathcal{K}(\mathcal{H})$. By means of the Hahn–Banach theorem, one extends it to a bounded functional on the whole $\mathcal{B}(\mathcal{H})$, also denoted by ℓ. Now consider $N \in \mathcal{B}(\mathcal{B}(\mathcal{H}))$ given by

$$N(X) = \ell(X) \cdot \mathbb{1}. \tag{6.3}$$

PROPOSITION 6.3. *N defined by (6.3) gives rise to a homogeneous vector bundle map $\mathcal{N}$ on G/K. However, by Theorem 3.6, it is never a Nijenhuis operator.*

PROOF. Let us verify that indeed $N \in \mathcal{A}(G, K)$. By definition, it vanishes on $\mathcal{K}(\mathcal{H})$, so it preserves it in a trivial manner. The other condition is equivalent to

$$\mathrm{Ad}_{\mathbb{1}+k} N(X) - N(\mathrm{Ad}_{\mathbb{1}+k} X) \in \mathcal{K}(\mathcal{H}) \tag{6.4}$$

for all $k \in \mathcal{K}(\mathcal{H})$ such that $\mathbb{1} + k \in \mathrm{GL}(\mathcal{H})$, $X \in \mathcal{B}(\mathcal{H})$. Since $N(X)$ lies in the center of $\mathcal{B}(\mathcal{H})$, condition (6.4) can be written as

$$N(X) - N\big((\mathbb{1} + k)X(\mathbb{1} + \tilde{k})\big) \in \mathcal{K}(\mathcal{H}),$$

where $\mathbb{1} + \tilde{k} = (\mathbb{1} + k)^{-1}$ with $\tilde{k}$ compact. Since kX, $X\tilde{k}$, and $kX\tilde{k}$ are all compact and hence in the kernel of N, the identity holds.

For the second claim, note that in the condition (3.2), the first three terms vanish identically since the image of N lies inside the center of the Lie algebra $\mathcal{B}(\mathcal{H})$. Thus, the condition for $\mathcal{N}$ to be Nijenhuis is

$$N^2\big([v,w]\big) \in \mathcal{K}(\mathcal{H}) \quad \text{for all } v, w \in \mathcal{B}(\mathcal{H}).$$

Note that by definition, N is idempotent and never takes value in $\mathcal{K}(\mathcal{H}) \setminus \{0\}$. Thus, for $\mathcal{N}$ to be Nijenhuis, the following identity should be satisfied:

$$N\big([v,w]\big) = 0 \quad \text{for all } v, w \in \mathcal{B}(\mathcal{H}). \tag{6.5}$$

It was demonstrated in [22] that every operator in $\mathcal{B}(\mathcal{H})$ is the sum of four commutators; thus, the linear span of all commutators is equal to the whole $\mathcal{B}(\mathcal{H})$. Thus, the condition (6.5) never holds, as it would imply $N = 0$. ∎

Let us also mention that vector bundle maps N of the discussed form never give rise to an almost complex structure since they are idempotent $N^2 = N$.

By replacing identity operator $\mathbb{1}$ in the definition (6.3) of the map N by another operator, it is possible to obtain examples of Nijenhuis operators, see [18, Section 4.1.2].

6.4. *Example: the restricted Grassmannian*

Consider a separable infinite-dimensional complex Hilbert space $\mathcal{H}$ endowed with the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$$

onto two infinite-dimensional closed subspaces. Denote by $P_\pm$ an orthogonal projection onto $\mathcal{H}_\pm$ and by $d = i(P_+ - P_-)$.

Consider the Banach–Lie group $G = U_{\mathrm{res}}$ defined as follows (see e.g. [37]):

$$U_{\mathrm{res}} = \{U \in \mathcal{B}(\mathcal{H}) \mid U^*U = UU^* = \mathbb{1}, [U,d] \in L^2(\mathcal{H})\},$$

where $L^2(\mathcal{H})$ is the ideal of Hilbert–Schmidt operators. Its Banach–Lie algebra is

$$\mathfrak{g} = \mathfrak{u}_{\mathrm{res}} = \{u \in \mathcal{B}(\mathcal{H}) \mid u^* = -u, [u,d] \in L^2(\mathcal{H})\}.$$

One verifies readily that $d \in \mathfrak{u}_{\mathrm{res}}$.

The group U_{res} acts on the Hilbert space $\mathcal{H}$ in the natural way and in consequence it also acts on the Grassmannian of $\mathcal{H}$, i.e., the set of all closed subspaces of $\mathcal{H}$. The action on the Grassmannian is not transitive. The orbit of the closed infinite-dimensional subspace $\mathcal{H}_+$ is known as the *restricted Grassmannian* $\mathrm{Gr}_{\mathrm{res}}$ [37]. The stabilizer of $\mathcal{H}_+$ is a product of two unitary groups $K = \mathcal{U}(\mathcal{H}_+) \times \mathcal{U}(\mathcal{H}_-)$. The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ possesses a manifold structure and the quotient map is a submersion. It is thus a homogeneous space $G/K = U_{\mathrm{res}}/(\mathcal{U}(\mathcal{H}_+) \times \mathcal{U}(\mathcal{H}_-))$.

We can construct Nijenhuis operators on $\mathrm{Gr}_{\mathrm{res}}$ by considering bounded operators on the Banach–Lie algebra $\mathfrak{u}_{\mathrm{res}}$ of the form $N = \mathrm{ad}_{\tilde{d}}$, where $\tilde{d}$ belongs to the center of $\mathfrak{k} = \mathfrak{u}(\mathcal{H}_+) \times \mathfrak{u}(\mathcal{H}_-)$.

PROPOSITION 6.4. *The operator* $\mathrm{ad}_{\tilde{d}}$ *descends to a vector bundle map* $\mathcal{N}$ *on* $T\mathrm{Gr}_{\mathrm{res}}$. *Moreover,* $\mathcal{N}$ *is a Nijenhuis operator.*

PROOF. From Corollary 3.9, it follows that $\mathrm{ad}_{\tilde{d}}$ belongs to $\mathcal{A}(G, K)$, and thus by Proposition 2.12, it descends to a vector bundle map $\mathcal{N}$.

By Theorem 3.6 and again Corollary 3.9, $\mathcal{N}$ is a Nijenhuis operator if and only if

$$\big[[\tilde{d}, v], [\tilde{d}, w]\big] \in \mathfrak{u}(\mathcal{H}_+) \times \mathfrak{u}(\mathcal{H}_-) \quad \text{for all } v, w \in \mathfrak{u}_{\mathrm{res}}.$$

That this condition holds can be checked by direct computation, or else by noticing that it is equivalent to the fact that the restricted Grassmannian is a (locally) symmetric space [42]. ■

The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$ is a Kähler manifold, which means among others that it possesses a complex structure. It is induced by $N = \mathrm{ad}_{\tilde{d}}$ with $\tilde{d} = \frac{1}{2}d = \frac{1}{2}i(P_+ - P_-)$. Direct computation shows that $\mathrm{Ran}((\mathrm{ad}_{\tilde{d}})^2 + 1) \subset \mathfrak{u}(\mathcal{H}_+) \times \mathfrak{u}(\mathcal{H}_-)$; thus, we obtain an almost complex structure. Previous considerations prove that it is indeed integrable (since this homogeneous space has a real analytic manifold structure). Let us note that the restricted Grassmannian is also a symplectic leaf in a certain Banach Lie–Poisson space (central extension of the predual space of $\mathfrak{u}_{\mathrm{res}}$, see [4]) and a Poisson homogeneous space of a Banach Poisson–Lie group [43]. It is related to numerous hierarchies of integrable systems [20, 21], in particular to the Korteweg–de Vries hierarchy [38, 43]. As Hermitian-symmetric space, the restricted Grassmannian admits a hyperkähler extension which can be identified with its cotangent space or with the coadjoint orbit of the complexification of U_{res} [39–41].

REMARK 6.5. More examples of Nijenhuis operators and complex structures can be found in the paper [18], where the constructions mentioned above are applied to several classes of C^*-algebras.

REMARK 6.6. In finite dimensions, there is a well-known method of obtaining an almost complex structure on coadjoint orbits of Lie groups, see e.g. [11, Section 1.2, Theorem 2], [9, Part V, Section 12.2], and [45]. It goes by considering a polar decomposition of the ad_d operator. In the paper [19], the generalization of this approach will be applied to the study of unitary orbits of trace-class operators, in the spirit of Kirillov's orbit method [25]. The results of the present paper will be used to address the question of integrability of these structures.

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Tomasz Goliński
Faculty of Mathematics, University of Białystok
Ciołkowskiego 1M, 15-245 Białystok, Poland
tomaszg@math.uwb.edu.pl

Gabriel Larotonda (corresponding author)
Departmento de Mathematica, Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires
Ciudad Universitaria, Avenida Cantilo S/N, 1423 Buenos Aires
Consejo Nacional de Investigaciones Científicas y Técnicas
Buenos Aires, Argentina
glaroton@dm.uba.ar

Alice Barbara Tumpach
Wolfgang Pauli Institut
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
Laboratoire Paul Painlevé
59665 Villeneuve d’Ascq, France
alice-barbora.tumpach@univ-lille.fr