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Algebraic Geometry. – *Dimension-free Maz'ya–Shaposhnikova limiting formulas in Grushin spaces*, by Huaiqian Li and Ke Wang, communicated on 11 April 2025.

Abstract. – Motivated by the paper [Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 23 (2022), no. 2, 837–875], we provide a simplified and robust proof of the dimension-free Maz'ya–Shaposhnikova limiting formula for seminorms associated with the Baouendi–Grushin operator. This operator, which is generally non-hypoelliptic, arises frequently in the study of Carnot–Carathéodory spaces. Our approach not only streamlines the original argument but also extends a recent result from [arXiv:2401.03409].

Keywords. – Besov seminorm, fractional Sobolev space, Grushin space, heat kernel, Maz'ya–Shaposhnikova formula.

Mathematics Subject Classification 2020. – 46E30 (primary); 53C17, 46E35, 35R11 (secondary).

1. Introduction

Let $p \in [1, \infty)$ and $s \in (0, 1)$. Let $\mathbb{R}^d$ be the Euclidean space endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Consider the fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$:

$$W^{s,p}(\mathbb{R}^d) := \{ f \in L^p(\mathbb{R}^d) : \|f\|_{W^{s,p}} < \infty \},$$

where $\| \cdot \|_{W^{s,p}}$ stands for the Gagliardo seminorm given by

$$\|f\|_{W^{s,p}} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{\|x - y\|^{d+ps}} \, \mathrm{d}y \, \mathrm{d}x \right)^{1/p}.$$

For more information on fractional Sobolev spaces, see for instance [8].

On the one hand, in [2, 3], J. Bourgain et al. explored the limiting behavior of the Gagliardo seminorm $\| \cdot \|_{W^{s,p}}$ as $s \nearrow 1$, and in particular, they proved that

$$\lim_{s \nearrow 1} (1 - s) \|f\|^p_{W^{s,p}} = \mathrm{c}(d, p) \|\nabla f\|^p_{L^p}, \quad f \in W^{1,p}(\mathbb{R}^d), \tag{1.1}$$

where $\| \cdot \|_{L^p}$ denotes the L^p-norm on the standard L^p-space $L^p(\mathbb{R}^d)$, ∇f is understood in the distribution sense, and

$$\mathrm{c}(d, p) = 2\pi^{\frac{d-1}{2}} \Gamma\left(\frac{p+1}{2}\right) / \Gamma\left(\frac{p+d}{2}\right).$$

Here and below, Γ stands for the Gamma function. Indeed,

$$\mathrm{c}(d,p)=\frac{1}{p}\int_{\mathbb{S}^{d-1}}|\langle \mathrm{e},\theta\rangle|\,\mathscr{H}^{d-1}(\mathrm{d}\theta),$$

where $\mathrm{e}\in\mathbb{S}^{d-1}$ is arbitrary and $\mathscr{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. On the other hand, as a complement to (1.1), it is natural to consider the limiting behavior of $\|\cdot\|_{W^{s,p}}$ as $s\searrow 0$. V. Maz'ya et al. provided a response and proved in [23] that, for every $f\in\bigcup_{s\in(0,1)}W^{s,p}(\mathbb{R}^d)$,

$$\lim_{s\searrow 0}s\|f\|^p_{W^{s,p}}=\frac{2d\omega_d}{p}\|f\|^p_{L^p}, \tag{1.2}$$

where $\omega_d=\pi^{d/2}/\Gamma(d/2+1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. We refer to (1.2) as the Maz'ya–Shaposhnikova (MS) limiting formula. See for instance [1, 4–6, 9, 19, 21, 22, 24] for intensive studies on (1.2) in various settings. Additionally, we briefly mention that fractional Orlicz–Sobolev spaces, along with their magnetic counterparts, have been investigated in detail in recent works [10, 11].

It is well known that (1.2) can be formulated in a dimension-free version. Let $(P_t)_{t\geq 0}$ be the standard heat semigroup generated by the Laplacian Δ, i.e., for any bounded measurable function f on $\mathbb{R}^d$, $P_0f:=f$ and

$$P_tf(x):=\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}f(y)e^{-\frac{\|x-y\|^2}{4t}}\,\mathrm{d}y,\quad t>0,\ x\in\mathbb{R}^d.$$

Let $s\in(0,1)$ and $p\in[1,\infty)$. Consider the Besov seminorm associated with the heat semigroup $(P_t)_{t\geq 0}$:

$$\mathrm{N}_{s,p}(f):=\left(\int_0^\infty\int_{\mathbb{R}^d}P_t\big(|f-f(x)|^p\big)(x)\,\mathrm{d}x\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}\right)^{1/p}.$$

Then, it is easy to calculate that (see e.g. [4, (1.4)])

$$\mathrm{N}_{s,p}(f)^p=\frac{2^{ps}}{\pi^{d/2}}\Gamma\left(\frac{d+ps}{2}\right)\|f\|^p_{W^{s,p}}. \tag{1.3}$$

Combining (1.2) with (1.3), we can easily derive the following dimension-free MS limiting formula:

$$\lim_{s\searrow 0}s\mathrm{N}_{s,p}(f)^p=\frac{4}{p}\|f\|^p_{L^p},\quad f\in\bigcup_{s\in(0,1)}W^{s,p}(\mathbb{R}^d). \tag{1.4}$$

Recently, analogous formulas to (1.4) have been established in various settings. For instance, [4] established such results for Besov seminorms associated with the Kolmogorov operator on $\mathbb{R}^d$, while [16] extended them to the sub-Laplacian on Carnot groups. Similar findings were also obtained recently in [20] for Dunkl heat semigroup associated Besov seminorms. In addition, a partial formula for Besov seminorms related to the sub-Laplacian on Grushin spaces was recently proved in [29], which can be regarded as one of the motivations for our work.

In the present note, we further develop the ideas behind (1.4) by investigating the same phenomenon in the setting of Grushin spaces. Section 2 begins by introducing the necessary notation, key concepts, and known results about Grushin spaces, culminating in the statement of our main result (Theorem 2.3). The proof of this theorem is then detailed in Section 3.

2. Preliminaries and main results

Let $\alpha \geq 0$, and let k, m be positive integers such that $\mathfrak{n} = k + m$. Consider the following system of vector fields on $\mathbb{R}^{\mathfrak{n}} = \mathbb{R}^k \times \mathbb{R}^m$:

$$X_i = \partial_{x_i},\ i = 1, \ldots, k, \quad Y_j = \|x\|^\alpha \partial_{y_j},\ j = 1, \ldots, m,$$

where $\|x\|^2 = \sum_{i=1}^k x_i^2$ is the Euclidean norm of $x \in \mathbb{R}^k$. Let $\mathcal{H}$ denote the subbundle spanned by $\{X_i, Y_j : i = 1, \ldots, k,\ j = 1, \ldots, m\}$.

A curve $\gamma : [0, 1] \to \mathbb{R}^{\mathfrak{n}}$ is called horizontal with respect to (w.r.t.) $\mathcal{H}$ if it is absolutely continuous and satisfies $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ for a.e. $t \in [0, 1]$, where $\dot{\gamma}(t)$ stands for the time derivative of $\gamma(t)$. The length of γ is given by

$$L_\alpha(\gamma) = \int_0^1 \left[\sum_{i=1}^k \left|\dot{\gamma}_i(t)\right|^2 + \left(\sum_{i=1}^k \gamma_i(t)^2 \right)^{-\alpha} \sum_{j=1}^m \left|\dot{\gamma}_{k+j}(t)\right|^2 \right]^{1/2} \mathrm{d}t,$$

where we denoted $\gamma(t) = (\gamma_1(t), \ldots, \gamma_{\mathfrak{n}}(t))$. For every two points $g, g' \in \mathbb{R}^{\mathfrak{n}}$, let $\mathcal{C}(g, g')$ be the class of all horizontal curves $\gamma : [0, 1] \to \mathbb{R}^{\mathfrak{n}}$ w.r.t. $\mathcal{H}$ such that $\gamma(0) = g$ and $\gamma(1) = g'$. The Carnot–Carathéodory distance ρ_α induced by $\mathcal{H}$ is given by

$$\rho_\alpha(g, g') = \inf \left\{ L_\alpha(\gamma) : \gamma \in \mathcal{C}(g, g') \right\}, \quad g, g' \in \mathbb{R}^{\mathfrak{n}}.$$

The pair $(\mathbb{R}^k \times \mathbb{R}^m, \rho_\alpha)$ defines the Grushin space, denoted as $\mathbb{G}_\alpha^{\mathfrak{n}}$. It is well known that $\mathbb{G}_\alpha^{\mathfrak{n}}$ is a locally compact geodesic metric space. In the special case $k = m = 1$, $\mathbb{G}_\alpha^{\mathfrak{n}}$ reduces to the Grushin plane, which can be identified as the metric completion of the (open) Riemannian manifold $(\mathbb{M}, \mathfrak{g})$, where $\mathbb{M} = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ and $\mathfrak{g} = \mathrm{d}x^2 + \|x\|^{-2\alpha} \mathrm{d}y^2$.

The sub-Laplacian (or Baouendi–Grushin operator) on $\mathbb{G}_\alpha^{\mathfrak{n}}$ is defined as

$$\Delta_\alpha = \sum_{i=1}^k X_i^2 + \sum_{j=1}^m Y_j^2 = \Delta_x + \|x\|^{2\alpha} \Delta_y,$$

where Δ_x (resp. Δ_y) is the standard Laplace operator acting on the variable $x \in \mathbb{R}^k$ (resp. $y \in \mathbb{R}^m$). Indeed, if we introduce the α-gradient operator

$$\nabla_\alpha = (X_1, \ldots, X_k, Y_1, \ldots, Y_m) = \big(\nabla_x, \|x\|^\alpha \nabla_y\big),$$

and the α-divergence

$$\mathrm{div}_\alpha = \sum_{i=1}^{k} X_i + \sum_{j=1}^{m} Y_j,$$

then we may express Δ_α in divergence form as $\Delta_\alpha = \mathrm{div}_\alpha \circ \nabla_\alpha$, where ∇_x (resp. ∇_y) is the standard gradient operator acting on the variable $x \in \mathbb{R}^k$ (resp. $y \in \mathbb{R}^m$). It is important to note that, unlike the hypoelliptic operators studied in [4, 16], the operator Δ_α may fail to be hypoelliptic except that α is a positive integer. In this special case, Δ_α becomes a sum of squares of smooth vector fields satisfying Hörmander's finite rank condition [17, 18]. Two noteworthy special cases are when $\alpha = 0$, Δ_α boils down to the standard Laplacian on $\mathbb{R}^{\mathfrak{n}}$; when $\alpha = 1$ with $m = 1$ and k even, Δ_α is closely related to the Kohn–Laplacian on the Heisenberg group $\mathbb{H}^{\frac{k}{2}}$ [14, 15].

The family of anisotropic dilations associated with the Baouendi–Grushin operator Δ_α is given by

$$\delta_\lambda^\alpha(x, y) = (\lambda x, \lambda^{\alpha+1} y), \quad (x, y) \in \mathbb{G}_\alpha^{\mathfrak{n}},\ \lambda > 0.$$

It is well known that ρ_α is 1-homogeneous w.r.t. $(\delta_\lambda^\alpha)_{\lambda>0}$:

$$\rho_\alpha(\delta_\lambda^\alpha g, \delta_\lambda^\alpha g') = \lambda \rho_\alpha(g, g'), \quad g, g' \in \mathbb{G}_\alpha^{\mathfrak{n}},\ \lambda > 0.$$

The homogeneous dimension of $\mathbb{G}_\alpha^{\mathfrak{n}}$ w.r.t. $(\delta_\lambda^\alpha)_{\lambda>0}$ is denoted by

$$\mathrm{Q} = \mathfrak{n} + \alpha m.$$

Consider the $\mathfrak{n}$-dimensional Lebesgue measure $\mathrm{d}g = \mathrm{d}x\, \mathrm{d}y$ on $\mathbb{G}_\alpha^{\mathfrak{n}}$, where $\mathrm{d}x$ (resp. $\mathrm{d}y$) denotes the Lebesgue measure on $\mathbb{R}^k$ (resp. $\mathbb{R}^m$). It is well known that

$$\mathrm{d}\delta_\lambda^\alpha(x, y) = \lambda^{\mathrm{Q}} \mathrm{d}x\, \mathrm{d}y, \quad \lambda > 0.$$

For every measurable subset E of $\mathbb{G}_\alpha^{\mathfrak{n}}$, we write $|E|$ as the $\mathfrak{n}$-dimensional Lebesgue measure of E. For every $p \in [1, \infty]$, let $L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$ indicate the standard L^p-space over $\mathbb{G}_\alpha^{\mathfrak{n}}$ endowed with the L^p-norm $\|\cdot\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}$.

Let $B_\alpha(g, r) = \{g' \in \mathbb{G}_\alpha^{\mathfrak{n}} : \rho_\alpha(g, g') < r\}$ be the open ball with center $g \in \mathbb{G}_\alpha^{\mathfrak{n}}$ and radius $r > 0$ in the metric ρ_α. According to [13, Proposition 2.2] and [12, Theorem 2.3], there is a constant $b \in (0, 1)$ such that

$$\mathfrak{C}_\alpha(g, br) \subset B_\alpha(g, r) \subset \mathfrak{C}_\alpha(g, b^{-1} r), \quad g \in \mathbb{G}_\alpha^{\mathfrak{n}},\ r > 0,$$

where for any $g = (x, y) \in \mathbb{G}_\alpha^{\mathfrak{n}}$ and any $r > 0$, the set $\mathfrak{C}_\alpha(g, r)$ is the product of closed intervals:

$$\mathfrak{C}_\alpha(g,r) = \prod_{i=1}^{k}[x_i - r,\, x_i + r] \times \prod_{j=1}^{m}\big[y_j - r\big(\|x\| + r\big)^\alpha,\; y_j + r\big(\|x\| + r\big)^\alpha\big].$$

This geometric control implies the following volume asymptotics: there exists a constant $c > 1$ such that

$$c^{-1}r^{\mathfrak{n}}\big(\|x\|+r\big)^{m\alpha} \le \big|B_\alpha(g,r)\big| \le c\,r^{\mathfrak{n}}\big(\|x\|+r\big)^{m\alpha}, \quad g=(x,y)\in\mathbb{G}_\alpha^{\mathfrak{n}},\ r>0. \tag{2.1}$$

Consequently, the space $(\mathbb{G}_\alpha^{\mathfrak{n}}, \mathrm{d}g)$ satisfies the volume doubling property, i.e., there exists a constant $D > 0$ such that

$$\big|B_\alpha(g, 2r)\big| \le D\big|B_\alpha(g,r)\big|, \quad g \in \mathbb{G}_\alpha^{\mathfrak{n}},\ r > 0,$$

and the volume comparison property, i.e., there exists a constant $C > 0$ such that

$$C^{-1}\left(\frac{R}{r}\right)^{\mathfrak{n}} \le \frac{\big|B_\alpha(g,R)\big|}{\big|B_\alpha(g,r)\big|} \le C\left(\frac{R}{r}\right)^{\mathrm{Q}}, \quad g \in \mathbb{G}_\alpha^{\mathfrak{n}},\ 0 < r \le R < \infty. \tag{2.2}$$

These properties establish that $(\mathbb{G}_\alpha^{\mathfrak{n}}, \mathrm{d}g)$ is a space of homogeneous type in the sense of [7, p. 66]. However, we note that $(\mathbb{G}_\alpha^{\mathfrak{n}}, \mathrm{d}g)$ is not generally Ahlfors regular.

Let $(H_t)_{t\ge 0}$ be the Grushin heat semigroup generated by Δ_α. For every bounded measurable function f on $\mathbb{G}_\alpha^{\mathfrak{n}}$, the semigroup is defined by $H_0 f := f$ and

$$H_t f(g) := \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')f(g')\,\mathrm{d}g', \quad g \in \mathbb{G}_\alpha^{\mathfrak{n}},\ t > 0,$$

where h_t is the Grushin heat kernel associated with H_t for every $t > 0$. It is well known that H_t is essentially self-adjoint on $L^2(\mathbb{G}_\alpha^{\mathfrak{n}})$ for any $t > 0$, and $(H_t)_{t\ge 0}$ extends to a strongly continuous contraction semigroup on $L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$ for any $p \in [1, \infty)$, while acting as a contraction semigroup on $L^\infty(\mathbb{G}_\alpha^{\mathfrak{n}})$. Moreover, $(H_t)_{t\ge 0}$ is conservative (or stochastically complete), i.e., $H_t 1 = 1$ for every $t > 0$, and sub-Markovian, i.e., $0 \le H_t f \le 1$ for every $t > 0$ and every measurable function f on $\mathbb{G}_\alpha^{\mathfrak{n}}$ satisfying $0 \le f \le 1$. The kernel $(h_t)_{t>0}$ is symmetric, i.e., $h_t(g, g') = h_t(g', g)$ for every $g, g' \in \mathbb{G}_\alpha^{\mathfrak{n}}$ and every $t > 0$, and satisfies the following Gaussian upper bound:

$$0 < h_t(g,g') \le \frac{c_1}{\big|B_\alpha(g,\sqrt{t})\big|}\exp\left(-\frac{\rho_\alpha(g,g')^2}{c_2 t}\right), \quad g, g' \in \mathbb{G}_\alpha^{\mathfrak{n}},\ t > 0, \tag{2.3}$$

for some constants $c_1, c_2 > 0$. A comparable Gaussian lower bound also holds though we do not require it in our subsequent analysis. Refer to [25, 26] for details.

Equivalently, $(H_t)_{t\geq 0}$ corresponds to the diffusion process $(U_t, V_t)_{t\geq 0}$ solving the following stochastic differential equations:

$$\begin{cases} \mathrm{d}U_t = \sqrt{2}\,\mathrm{d}B_t^{(1)}, \\ \mathrm{d}V_t = \sqrt{2}\|U_t\|^{\alpha}\,\mathrm{d}B_t^{(2)}, \end{cases}$$

where $(B_t^{(1)})_{t\geq 0}$ and $(B_t^{(2)})_{t\geq 0}$ are independent standard Brownian motions in $\mathbb{R}^k$ and $\mathbb{R}^m$, respectively, defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. The semigroup admits the probabilistic representation: for any bounded measurable function f on $\mathbb{G}_\alpha^{\mathfrak{n}}$,

$$H_t f(x,y) = \mathbb{E}_{(x,y)}\big[f(U_t, V_t)\big], \quad (x,y) \in \mathbb{G}_\alpha^{\mathfrak{n}},\ t \geq 0,$$

where $\mathbb{E}_{(x,y)}$ stands for expectation for the process $(U_t, V_t)_{t\geq 0}$ starting at $(x,y)\in\mathbb{G}_\alpha^{\mathfrak{n}}$. For studies including the Bismut formula and the log-Harnack inequalities for generalized Grushin-type operators, we refer to [27, 28].

An important consequence of the Gaussian upper bound (2.3) combined with (2.1), (2.2) and the Riesz–Thorin interpolation theorem is that the semigroup $(H_t)_{t\geq 0}$ is ultra-contractive. We refer to [29, Propositions 2.4 and 2.5] for complete details on this derivation.

LEMMA 2.1. (1) *Let* $p \in [1, \infty]$. *There exists a constant* $c_1 > 0$ *such that*

$$\big|H_t f(g)\big| \leq c_1 t^{-\frac{\mathrm{Q}}{2p}} \|f\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad g \in \mathbb{G}_\alpha^{\mathfrak{n}},\ t > 0,\ f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}}).$$

(2) *Let* $1 \leq p < q \leq \infty$. *There exists a constant* $c_2 > 0$ *such that*

$$\|H_t f\|_{L^q(\mathbb{G}_\alpha^{\mathfrak{n}})} \leq c_2 t^{-\frac{q-p}{2pq}\mathrm{Q}} \|f\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad t > 0,\ f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}}).$$

Let $p \in [1, \infty)$ and $s \in (0, \infty)$. Define the Besov space associated with the Grushin heat semigroup $(H_t)_{t\geq 0}$ as

$$B_{s,p}(\mathbb{G}_\alpha^{\mathfrak{n}}) = \big\{f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}}) : \mathrm{N}_{s,p}^{\alpha}(f) < \infty\big\},$$

where $\mathrm{N}_{s,p}^{\alpha}(\cdot)$ is the Besov seminorm given by

$$\mathrm{N}_{s,p}^{\alpha}(f) = \left(\int_0^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f - f(g)|^p\big)(g)\,\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}}\right)^{1/p}.$$

REMARK 2.2. For every $(p, s) \in [1, \infty) \times (0, \infty)$ and every $f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$, it holds that $\mathrm{N}_{s,p}^{\alpha}(f) < \infty$ if and only if

$$\mathrm{N}(f) := \int_0^1 \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f - f(g)|^p\big)(g)\,\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}} < \infty.$$

Indeed, the necessity is clear. For the sufficiency, assuming that $\mathrm{N}(f) < \infty$, we have

$$\begin{aligned}
\mathrm{N}^{\alpha}_{s,p}(f)^p &= \mathrm{N}(f) + \int_1^{\infty} \int_{\mathbb{G}^{\mathfrak{n}}_{\alpha}} H_t\big(|f - f(g)|^p\big)(g)\,\mathrm{d}g \frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}} \\
&\le \mathrm{N}(f) + 2^{p-1} \int_1^{\infty} \int_{\mathbb{G}^{\mathfrak{n}}_{\alpha}} \big[H_t|f|^p(g) + \big|f(g)\big|^p\big]\,\mathrm{d}g \frac{\mathrm{d}t}{t^{1+\frac{sp}{2}}} \\
&\le \mathrm{N}(f) + 2^p \|f\|^p_{L^p(\mathbb{G}^{\mathfrak{n}}_{\alpha})} \int_1^{\infty} t^{-(1+\frac{sp}{2})}\,\mathrm{d}t \\
&= \mathrm{N}(f) + \frac{2^{p+1}}{sp} \|f\|^p_{L^p(\mathbb{G}^{\mathfrak{n}}_{\alpha})} < \infty.
\end{aligned}$$

The main result of this note is the following dimension-free MS limiting formula for the seminorm $\mathrm{N}^{\alpha}_{s,p}$.

THEOREM 2.3. *Let $p \in [1, \infty)$. Then, for every $f \in \bigcup_{s\in(0,1)} B_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha})$,*

$$\lim_{s \searrow 0} s\mathrm{N}^{\alpha}_{s,p}(f)^p = \frac{4}{p} \|f\|^p_{L^p(\mathbb{G}^{\mathfrak{n}}_{\alpha})}.$$

REMARK 2.4. (1) Let $p \in [1, \infty)$ and $s \in (0, 1)$. In their recent work [29], the authors introduced an alternative Besov space $\mathfrak{B}_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha})$, which is defined as the completion of $C^{\infty}_c(\mathbb{G}^{\mathfrak{n}}_{\alpha})$ (the space of compactly supported and infinitely differentiable functions on $\mathbb{G}^{\mathfrak{n}}_{\alpha}$) w.r.t. the norm $\|\cdot\|_{B_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha})}$ given by

$$\|f\|_{B_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha})} = \|f\|_{L^p(\mathbb{G}^{\mathfrak{n}}_{\alpha})} + \mathrm{N}^{\alpha}_{s,p}(f).$$

Following methods similar to those in the paper [4], they proved that (see [29, Theorem 3.21])

$$\lim_{s \searrow 0} s\mathrm{N}^{\alpha}_{s,p}(u)^p = \frac{4}{p} \|u\|^p_{L^p(\mathbb{G}^{\mathfrak{n}}_{\alpha})}, \quad u \in \bigcup_{s\in(0,1)} \mathfrak{B}_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha}).$$

While [29, Proposition 3.4] shows the inclusion

$$\mathfrak{B}_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha}) \subseteq B_{s,p}(\mathbb{G}^{\mathfrak{n}}_{\alpha}),$$

the converse inclusion remains an open question (see [29, Remark 3.5]). This situation differs notably from the settings in both [4, Proposition 3.2] and [16, Lemma 3.4], where such inclusions were established. This distinction necessitates a novel approach to prove Theorem 2.3, as existing methods from these references cannot be directly applied.

(2) The key point in the proof of Theorem 2.3 relies on approximating the L^p-function using simple functions, in contrast to the Besov norm approximation method

central to [4, 29]. It turns out that our approach is simple and robust, which can be employed to establish similar formulas in more general settings—such as conservative, symmetric Markov semigroups with the ultra-contractive property as demonstrated in Lemma 2.1 (1). A detailed treatment of this extension will be presented in a work in preparation.

3. Proofs

In this section, we present the complete proof for Theorem 2.3. The main contribution is Lemma 3.1. We emphasize that Lemma 3.1 holds for all $f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$ and all $p \in (1,\infty)$ and Lemma 3.2 holds for all $f \in L^1(\mathbb{G}_\alpha^{\mathfrak{n}})$, improving [29, Lemma 3.20] which is shown only for $f \in C_c^\infty(\mathbb{G}_\alpha^{\mathfrak{n}})$.

Let $\mathcal{S}(\mathbb{G}_\alpha^{\mathfrak{n}})$ be the class of all finitely simple functions defined on $\mathbb{G}_\alpha^{\mathfrak{n}}$. Recall that a simple function on $\mathbb{G}_\alpha^{\mathfrak{n}}$ is called finitely simple if it is supported in a set with finite $\mathfrak{n}$-dimensional Lebesgue measure. Recall also the well-known fact that $\mathcal{S}(\mathbb{G}_\alpha^{\mathfrak{n}})$ is dense in $L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$ for every $p \in (1,\infty)$.

Lemma 3.1. *Let $p \in (1,\infty)$. Then, for any $f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$,*

$$\lim_{s\searrow 0} s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|^p\big)(g)\,\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} = \frac{4}{p}\|f\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}.$$

Proof. Let $p \in (1,\infty)$. The proof is divided into four parts.

Part 1. Let $f \in \mathcal{S}(\mathbb{G}_\alpha^{\mathfrak{n}})$. By the elementary inequality

$$\big||a-b|^p-|a|^p-|b|^p\big| \le C_p\big(|a|^{p-1}|b|+|a||b|^{p-1}\big), \quad a,b\in\mathbb{R},$$

for some positive constant C_p depending only on p, we derive that

$$\begin{aligned}
&\left|\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t(|f-f(g)|^p)(g)\,\mathrm{d}g - \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big(|f(g')|^p+|f(g)|^p\big)\,\mathrm{d}g'\mathrm{d}g\right| \\
&\quad\le \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big||f(g')-f(g)|^p-|f(g')|^p-|f(g)|^p\big|\,\mathrm{d}g'\mathrm{d}g \\
&\quad\le C_p\int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big[|f(g)|^{p-1}|f(g')|+|f(g)||f(g')|^{p-1}\big]\,\mathrm{d}g'\mathrm{d}g \\
&\quad= 2C_p\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} |f(g)|^{p-1}H_t|f|(g)\,\mathrm{d}g, \quad t>0,
\end{aligned}$$

where we used the symmetry of $(h_t)_{t>0}$ in the equality. Then, by Hölder's inequality and the ulta-contractivity in Lemma 2.1 (2), we have

$$
\begin{aligned}
(3.1)\quad \mathrm{J}_s(f) :=& \Bigg| s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|^p\big)(g)\,\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} \\
&- s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big(\big|f(g')\big|^p + \big|f(g)\big|^p\big)\,\mathrm{d}g'\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}\Bigg| \\
&\le 2C_p s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \big(|f|^{p-1}H_t|f|\big)(g)\,\mathrm{d}g\,\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} \\
&\le 2C_p s\int_1^\infty \|f\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\big\|H_t|f|\big\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\,\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} \\
&\le C_p'\|f\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\, s\int_1^\infty t^{-[1+\frac{ps}{2}+\frac{Q}{2}(1-\frac{1}{p})]}\,\mathrm{d}t \\
&= C_p'\|f\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\,\frac{s}{\frac{ps}{2}+\frac{Q}{2}\big(1-\frac{1}{p}\big)},\quad s>0,
\end{aligned}
$$

for some constant $C_p' > 0$.

PART 2. Let $f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$. By the density of $\mathcal{S}(\mathbb{G}_\alpha^{\mathfrak{n}})$ in $L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$, we may choose a sequence $(f_k)_{k\ge 1} \subset \mathcal{S}(\mathbb{G}_\alpha^{\mathfrak{n}})$ such that $|f_k| \le |f|$ a.e. for every $k \ge 1$ and $f_k \to f$ a.e. as $k \nearrow \infty$. Then, by the symmetry and the conservativeness of h_t, we have

$$
\begin{aligned}
(3.2)\quad \mathrm{I}_1 :=& \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big(\big|f(g')\big|^p + \big|f(g)\big|^p\big)\,\mathrm{d}g'\,\mathrm{d}g \\
&- \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big(\big|f_k(g')\big|^p + \big|f_k(g)\big|^p\big)\,\mathrm{d}g'\,\mathrm{d}g \\
=& \int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big[\big(\big|f(g')\big|^p - \big|f_k(g')\big|^p\big) + \big(\big|f(g)\big|^p - \big|f_k(g)\big|^p\big)\big]\,\mathrm{d}g'\,\mathrm{d}g \\
=& 2\big(\|f\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})} - \|f_k\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\big),\quad k\ge 1,\ t>0.
\end{aligned}
$$

For a given function u on $\mathbb{G}_\alpha^{\mathfrak{n}}$, let

$$\Phi_t(u)(g,g') := h_t(g,g')^{1/p}\big[u(g)-u(g')\big],\quad t>0,\ g,g' \in \mathbb{G}_\alpha^{\mathfrak{n}}.$$

It is easy to see that, for any fixed elements $g, g' \in \mathbb{G}_\alpha^{\mathfrak{n}}$ and any fixed $t > 0$, the mapping $u \mapsto \Phi_t(u)(g, g')$ is a linear functional. Applying the elementary inequality

$$|a^p - b^p| \le p\max\{a^{p-1},\, b^{p-1}\}|a-b|,\quad a,b\ge 0,$$

together with the triangle inequality for the L^p-norm $\|\cdot\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}}\times\mathbb{G}_\alpha^{\mathfrak{n}})}$ on the L^p-space $L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})$ over the product space $\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}}$ endowed with the Lebesgue measure, we obtain

$$\begin{aligned}
\mathrm{I}_2 &:= \bigg| \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big| f(g') - f(g) \big|^p \, \mathrm{d}g' \mathrm{d}g \\
&\qquad - \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big| f_k(g') - f_k(g) \big|^p \, \mathrm{d}g' \mathrm{d}g \bigg| \\
&= \Big| \big\| \Phi_t(f) \big\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} - \big\| \Phi_t(f_k) \big\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} \Big| \\
&\leq p \max \Big\{ \big\| \Phi_t(f) \big\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})}, \big\| \Phi_t(f_k) \big\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} \Big\} \\
&\qquad \times \big\| \Phi_t(f) - \Phi_t(f_k) \big\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})}, \quad k \geq 1, \ t > 0.
\end{aligned}$$

By the symmetry and the conservativeness of h_t along with the triangle inequality, we have the following estimates:

$$\begin{aligned}
&\big\| \Phi_t(f) - \Phi_t(f_k) \big\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} \\
&\quad \leq \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big(\big| f(g') - f_k(g) \big| + \big| f(g) - f_k(g) \big| \big)^p \, \mathrm{d}g' \mathrm{d}g \\
&\quad \leq 2^{p-1} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big(\big| f(g') - f_k(g') \big|^p + \big| f(g) - f_k(g) \big|^p \big) \, \mathrm{d}g' \mathrm{d}g \\
&\quad = 2^p \| f - f_k \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad k \geq 1, \ t > 0,
\end{aligned}$$

and

$$\begin{aligned}
\big\| \Phi_t(f_k) \big\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} &\leq 2^{p-1} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big(\big| f_k(g') \big|^p + \big| f_k(g) \big|^p \big) \, \mathrm{d}g' \, \mathrm{d}g \\
&= 2^p \| f_k \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})} \leq 2^p \| f \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad k \geq 1, \ t > 0, \\
\big\| \Phi_t(f) \big\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}} \times \mathbb{G}_\alpha^{\mathfrak{n}})} &\leq 2^{p-1} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big(\big| f(g') \big|^p + \big| f(g) \big|^p \big) \, \mathrm{d}g' \, \mathrm{d}g \\
&\leq 2^p \| f \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad t > 0,
\end{aligned}$$

where we also used the elementary inequality $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ for any $a, b \geq 0$ and any $q \geq 1$. As a consequence, we arrive at

$$\mathrm{I}_2 \leq 2^p p \| f \|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})} \| f - f_k \|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad k \geq 1, \ t > 0. \tag{3.3}$$

Part 3. Let $f \in L^p(\mathbb{G}_\alpha^{\mathfrak{n}})$. By the conservativeness and the symmetry of h_t again, it is easy to see that

$$\begin{aligned}
&s \int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g') \big(\big| f(g') \big|^p + \big| f(g) \big|^p \big) \, \mathrm{d}g' \mathrm{d}g \frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} \\
&\qquad = 2s \| f \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})} \int_1^\infty t^{-(1+\frac{ps}{2})} \, \mathrm{d}t = \frac{4}{p} \| f \|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})} < \infty, \quad s > 0.
\end{aligned} \tag{3.4}$$

PART 4. Let f and $(f_k)_{k\geq 1}$ be the same as in Part 2. Combining (3.1), (3.2), (3.3), and (3.4) together, we have

$$\begin{aligned}(3.5)\quad &\left|s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}-\frac{4}{p}\|f\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\right|\\ &\leq s\int_1^\infty(\mathrm{I}_1+\mathrm{I}_2)\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}+\mathrm{J}_s(f_k)\\ &\leq 2^{p+1}\|f\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\|f-f_k\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}+\frac{4}{p}\Big|\|f\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}-\|f_k\|^p_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\Big|\\ &\quad+C_p'\|f_k\|^{p-1}_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\|f_k\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\frac{s}{\frac{ps}{2}+\frac{Q}{2}\big(1-\frac{1}{p}\big)},\quad k\geq 1,\ s>0.\end{aligned}$$

It is clear that $\|f_k-f\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}\to 0$ as $k\nearrow\infty$, by the dominated convergence theorem. Therefore, letting $s\searrow 0$ first and then sending $k\nearrow\infty$ in (3.5), we immediately obtain the desired result. ■

The proof for the next lemma, which we include the details here for the sake of completeness, follows the same argument as in [4, Lemma 4.4].

LEMMA 3.2. *For every* $f\in L^1(\mathbb{G}_\alpha^{\mathfrak{n}})$,

$$(3.6)\qquad \lim_{s\searrow 0}s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}=4\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}.$$

PROOF. (1) By the symmetry and the conservativeness of h_t, we see that

$$\begin{aligned}&s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}\\ &\leq s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big(|f(g')|+|f(g)|\big)\,\mathrm{d}g'\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}\\ &=2s\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\int_1^\infty t^{-(1+\frac{s}{2})}\,\mathrm{d}t\\ &=4\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})},\quad f\in L^1(\mathbb{G}_\alpha^{\mathfrak{n}}),\ s>0,\end{aligned}$$

which clearly implies that

$$(3.7)\quad \limsup_{s\searrow 0}s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}}\leq 4\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})},\quad f\in L^1(\mathbb{G}_\alpha^{\mathfrak{n}}).$$

(2) Let $f\in L^1(\mathbb{G}_\alpha^{\mathfrak{n}})$ and $\varepsilon>0$. Then, there exists a compact set $D_\varepsilon\subseteq\mathbb{G}_\alpha^{\mathfrak{n}}$ such that

$$(3.8)\qquad \|f\|_{L^1(D_\varepsilon)}\geq\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}-\varepsilon.$$

For convenience, we write $\complement D_\varepsilon$ instead of $\mathbb{G}_\alpha^{\mathfrak{n}} \setminus D_\varepsilon$. Then, for every $t > 0$, we deduce that

$$
\begin{aligned}
(3.9)\quad &\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f - f(g)|\big)(g)\,\mathrm{d}g \\
&= \int_{D_\varepsilon}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big|f(g') - f(g)\big|\,\mathrm{d}g'\mathrm{d}g \\
&\quad + \int_{\complement D_\varepsilon}\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} h_t(g,g')\big|f(g') - f(g)\big|\,\mathrm{d}g'\,\mathrm{d}g \\
&\geq \int_{D_\varepsilon}\int_{\complement D_\varepsilon} h_t(g,g')\big|f(g') - f(g)\big|\,\mathrm{d}g'\mathrm{d}g \\
&\quad + \int_{\complement D_\varepsilon}\int_{D_\varepsilon} h_t(g,g')\big|f(g') - f(g)\big|\,\mathrm{d}g'\mathrm{d}g \\
&\geq \int_{D_\varepsilon}\int_{\complement D_\varepsilon} h_t(g,g')\big(\big|f(g)\big| - \big|f(g')\big|\big)\,\mathrm{d}g'\mathrm{d}g \\
&\quad + \int_{\complement D_\varepsilon}\int_{D_\varepsilon} h_t(g,g')\big(\big|f(g')\big| - \big|f(g)\big|\big)\,\mathrm{d}g'\mathrm{d}g \\
&= 2\int_{D_\varepsilon}\big|f(g)\big|\int_{\complement D_\varepsilon} h_t(g,g')\,\mathrm{d}g'\,\mathrm{d}g - 2\int_{D_\varepsilon}\int_{\complement D_\varepsilon} h_t(g,g')\big|f(g')\big|\,\mathrm{d}g'\,\mathrm{d}g \\
&= 2\int_{D_\varepsilon}\big|f(g)\big|\,\mathrm{d}g - 2\int_{D_\varepsilon}\big|f(g)\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g'\,\mathrm{d}g \\
&\quad - 2\int_{\complement D_\varepsilon}\big|f(g')\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g\,\mathrm{d}g' \\
&\geq 2\big(\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})} - \varepsilon\big) - 2\int_{D_\varepsilon}\big|f(g)\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g'\,\mathrm{d}g \\
&\quad - 2\int_{\complement D_\varepsilon}\big|f(g')\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g\,\mathrm{d}g',
\end{aligned}
$$

where we applied the symmetry of h_t in the first equality, the conservativeness and Fubini's theorem in the second equality, and (3.8) in the last inequality.

By Fubini's theorem, (3.8), and the fact that $H_t 1_E \leq 1$ for any measurable set $E \subseteq \mathbb{G}_\alpha^{\mathfrak{n}}$ and any $t \geq 0$, we have

$$
(3.10)\qquad \int_{\complement D_\varepsilon}\big|f(g')\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g\mathrm{d}g' \leq \int_{\complement D_\varepsilon}\big|f(g)\big|\,\mathrm{d}g < \varepsilon, \quad t > 0,
$$

where 1_A denotes the indicator function of the set $A \subseteq \mathbb{G}_\alpha^{\mathfrak{n}}$. Applying Lemma 2.1 (1), we find some constant $c > 0$ such that

$$
\begin{aligned}
(3.11)\qquad \int_{D_\varepsilon}\big|f(g)\big|\int_{D_\varepsilon} h_t(g,g')\,\mathrm{d}g'\mathrm{d}g &= \int_{D_\varepsilon}\big|f(g)\big|H_t 1_{D_\varepsilon}(g)\,\mathrm{d}g \\
&\leq ct^{-\frac{Q}{2}}|D_\varepsilon|\int_{D_\varepsilon}\big|f(g)\big|\,\mathrm{d}g \leq ct^{-\frac{Q}{2}}|D_\varepsilon|\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}, \quad t > 0.
\end{aligned}
$$

Thus, combining (3.9), (3.10), and (3.11) together, we arrive at

$$\begin{aligned} s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}} &\geq 2s\int_1^\infty \big(\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}-2\varepsilon-ct^{-\frac{Q}{2}}|D_\varepsilon|\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\big)\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}} \\ &= 4\big(\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}-2\varepsilon\big)-4c|D_\varepsilon|\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}\frac{s}{s+Q} \\ &\to 4\big(\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}-2\varepsilon\big), \quad \text{as } s\searrow 0, \end{aligned}$$

where c is a positive constant. By the arbitrariness of $\varepsilon>0$, we obtain

$$\liminf_{s\searrow 0} s\int_1^\infty \int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{s}{2}}} \geq 4\|f\|_{L^1(\mathbb{G}_\alpha^{\mathfrak{n}})}. \tag{3.12}$$

(3) Therefore, putting (3.7) and (3.12) together, we immediately complete the proof of (3.6). ■

Now we are ready to prove our main result.

Proof of Theorem 2.3. Let $p\in[1,\infty)$. Take $f\in B_{\sigma,p}(\mathbb{G}_\alpha^{\mathfrak{n}})$ for some $\sigma\in(0,1)$. Then,

$$\begin{aligned} \int_0^1\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(|f-f(g)|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} &\leq \int_0^1\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{\sigma s}{2}}} \\ &\leq \mathrm{N}_{\sigma,p}^\alpha(f)^p<\infty, \quad s\in(0,\sigma]. \end{aligned}$$

Hence, multiplying s and taking the limit as $s\searrow 0$, we obtain

$$\lim_{s\searrow 0} s\int_0^1\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}=0. \tag{3.13}$$

Note that

$$\begin{aligned} &\Big|s\mathrm{N}_{s,p}^\alpha(f)^p-\frac{4}{p}\|f\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}^p\Big| \\ &\quad\leq s\int_0^1\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}} \\ &\qquad+\Big|s\int_1^\infty\int_{\mathbb{G}_\alpha^{\mathfrak{n}}} H_t\big(\big|f-f(g)\big|^p\big)(g)\,\mathrm{d}g\frac{\mathrm{d}t}{t^{1+\frac{ps}{2}}}-\frac{4}{p}\|f\|_{L^p(\mathbb{G}_\alpha^{\mathfrak{n}})}^p\Big|, \quad s\in(0,1). \end{aligned} \tag{3.14}$$

Therefore, combining (3.14) and (3.13) with Lemma 3.1 and Lemma 3.2, we have

$$\lim_{s\searrow 0}\left|s\mathrm{N}^{\alpha}_{s,p}(f)^p-\frac{4}{p}\|f\|^p_{L^p(\mathbb{G}^n_\alpha)}\right|=0,$$

which implies the desired result. ■

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