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Sharp regularization effect for the non-cutoff Boltzmann equation with hard potentials

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Abstract. For the Maxwellian molecules or hard potentials case, we verify the smoothing effect for the spatially inhomogeneous Boltzmann equation without angular cutoff. Given initial data with low regularity, we prove solutions at any positive time are analytic for strong angular singularity, and in the Gevrey class with optimal index for mild angular singularity. To overcome the degeneracy in the spatial variable, a family of well-chosen vector fields with time-dependent coefficients will play a crucial role, and the sharp regularization effect of weak solutions relies on a quantitative estimate on directional derivatives in these vector fields.

1. Introduction and main result

Due to the diffusion property, the regularization effect is well explored for parabolic-type equations. As a typical example, solutions to the Cauchy problem of the heat equation will become analytic at positive times for given initial data with low regularity. This kind of parabolic regularization effect has been observed in several classical equations which describe the motion of dilute gas and fluid dynamics in different physical scales. For instance, at macroscopic scales, the motion of fluid may be described by the classical Navier–Stokes equations, which indeed enjoy the analytic smoothing effect (cf., e.g., Foias–Temam [23]). Meanwhile, in mesoscopic kinetic theory, the Boltzmann equation plays a fundamental role, and the regularization properties of weak solutions were observed in Lions [37] and further verified by Desvillettes [18]. Since then there have been extensive works on the C^∞-smoothing effect for the non-cutoff Boltzmann equation and related models, most of which are concerned with the spatially homogeneous case; the breakthrough for the inhomogeneous counterpart was achieved in the very recent work of Imbert–Silvestre [32]. In this work, we aim to explore the analytic and sharp Gevrey class regularization effect for the spatially inhomogeneous Boltzmann equation without angular cutoff. Different from the heat or the Navier–Stokes equations, the spatially inhomogeneous Boltzmann equation is a degenerate parabolic equation. Although sometimes we may expect C^∞-smoothness for general degenerate equations, it is highly non-trivial

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to get the analytic regularity. In fact, for the inhomogeneous Boltzmann equations, so far very few analytic solutions are available.

To understand the transport properties of a dilute gas described by the Boltzmann equation, explicit solutions would be useful to capture the non-equilibrium phenomena. Due to the high non-linearity of the Boltzmann collision operator, it is usually not easy to find an explicit solution and in this case, it would be more convenient to solve the Boltzmann equation via analytic approximation with the help of numerical methods. In this paper we will theoretically verify analyticity at positive time of mild solutions to the spatially inhomogeneous Boltzmann equation with strong angular singularity. On the other hand, for mild angular singularity, the sharp regularization that we may expect will be in the Gevrey class rather than in analytic space. To investigate the sharp regularity, the main difficulty arises from the degeneracy in the spatial direction coupled with the highly non-linear feature in the Boltzmann collision operator. For the spatial homogeneous case, the regularity issue reduces to a parabolic problem, and motivated by the heat equation, analytic solutions to the Boltzmann equation and related models have been proven for rather weak initial data; cf. [9, 15, 38] for instance and also [6, 13, 19, 20, 24, 39, 40, 42] for regularity in other function spaces. However, analytic solutions are much less known for the spatially inhomogeneous counterpart, and well-posedness in analytic space was obtained by Ukai [45], where the author required analytic regularity for initial data so that the Cauchy–Kovalevskaya theorem may apply, and to the best of our knowledge, no analytic solution is known for non-analytic initial data. Motivated by diffusive models such as the hypoelliptic Fokker–Planck and Landau equations, it is natural to expect a smoothing effect for the spatially inhomogeneous Boltzmann equation in analytic space or sharp Gevrey class rather than in the C^∞ setting.

The spatially inhomogeneous Boltzmann equation in a torus reads

$$\partial_t F + v \cdot \partial_x F = Q(F, F), \quad F|_{t=0} = F_0, \tag{1.1}$$

where $F(t, x, v)$ stands for the probability density function at position $x \in \mathbb{T}^3$, time $t \geq 0$ with velocity $v \in \mathbb{R}^3$. If $F = F(t, v)$ is independent of x, then equation (1.1) reduces to the spatial homogeneous Boltzmann equation. The Boltzmann collision operator on the right-hand side of (1.1) is a bilinear operator defined by

$$Q(G, F)(t, x, v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(G'_* F' - G_* F)\, dv_*\, d\sigma, \tag{1.2}$$

where, as throughout the paper, we use the standard shorthand $F' = F(t, x, v')$, $F = F(t, x, v)$, $G'_* = G(t, x, v'_*)$ and $G_* = G(t, x, v_*)$, and the pairs (v, v_*) and (v', v'_*) are the velocities of particles after and before collisions, with the following momentum and energy conservation rules fulfilled:

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

From the above relations we have the so-called σ-representation, with $\sigma \in \mathbb{S}^2$,

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma, \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma. \end{cases}$$

The cross-section $B(v-v_*,\sigma)$ in (1.2) depends on the relative velocity $|v-v_*|$ and the deviation angle θ with

$$\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma.$$

Without loss of generality, we may assume that $B(v-v_*,\sigma)$ is supported on $0 \le \theta \le \frac{\pi}{2}$ such that $\cos\theta \ge 0$ and also assume that it takes the following specific form:

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta), \tag{1.3}$$

where $|v-v_*|^\gamma$ is called the kinetic part with $-3 < \gamma \le 1$, and $b(\cos\theta)$ is called the angular part satisfying

$$0 \le \sin\theta b(\cos\theta) \approx \theta^{-1-2s} \tag{1.4}$$

for $0 < s < 1$, where here and throughout the paper, $p \approx q$ means $C^{-1}q \le p \le Cq$ for some generic constant $C \ge 1$. So the angular part $b(\cos\theta)$ has a singularity near 0 in the sense that

$$\int_0^{\frac{\pi}{2}} \sin\theta b(\cos\theta)\,d\theta = +\infty.$$

In the following discussion, by strong angular singularity we mean that $\frac{1}{2} \le s < 1$, and mild angular singularity means that $0 < s < \frac{1}{2}$. Recall that $\gamma = 0$ is the Maxwellian molecules case, while the cases $-3 < \gamma < 0$ and $0 < \gamma$ correspond respectively to the soft potential and the hard potential. In this text, we will restrict our attention to the cases of Maxwellian molecules and hard potential, i.e., $\gamma \ge 0$.

We are concerned with the solution to the Boltzmann equation (1.1) around the normalized global Maxwellian $\mu = \mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$. Thus, let $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v)$ and similarly for the initial datum F_0. Then the reformulated unknown $f = f(t,x,v)$ satisfies

$$\partial_t f + v\cdot\nabla_x f + \mathcal{L} f = \Gamma(f,f), \quad f|_{t=0} = f_0, \tag{1.5}$$

with the linearized collision operator $\mathcal{L}$ and the non-linear collision operator $\Gamma(\cdot,\cdot)$ given respectively by

$$\mathcal{L} f = -\mu^{-\frac{1}{2}} Q(\mu, \sqrt{\mu} f) - \mu^{-\frac{1}{2}} Q(\sqrt{\mu} f, \mu), \tag{1.6}$$

and

$$\Gamma(g,h) = \mu^{-\frac{1}{2}} Q(\sqrt{\mu} g, \sqrt{\mu} h). \tag{1.7}$$

Initiated by [18, 37], so far it is well understood that an angular singularity will lead to fractional diffusion in velocity, so that it is a natural conjecture that the Boltzmann collision operator without cutoff should behave essentially as the fractional Laplacian:

$$-Q(g, f) \sim C_g(-\Delta_v)^s + \text{l.o.t.}, \tag{1.8}$$

where l.o.t. refers to lower-order terms that are easier to control. Note that(1.8) is rigorously verified by Alexandre–Desvillettes–Villani–Wennberg [1], where the velocity should vary in a bounded region. For the global counterpart of (1.8), an accurate characterization by fractional Laplacian $(-\Delta_v)^s$ and fractional Laplacian on a sphere $(v \wedge \partial_v)^{2s}$ is given by [2] with the help of pseudo-differential calculus. Moreover, fractional diffusion in the spatial variable x may also be archived due to the non-trivial interaction between the diffusion in velocity and the transport part. Thus, even though the spatially inhomogeneous Boltzmann equation is degenerate in the spatial direction, it admits an intrinsic hypoelliptic structure just like the diffusive variants such as the Fokker–Planck equation or the Landau equation. Inspired by the analytic regularization effect observed by [11,41] for these specific diffusive models, it is natural to require the same phenomena for the Boltzmann equation with strong angular singularity, and in this work, we will confirm it by virtue of a family of well-chosen vector fields. Moreover, for the remaining case of mild angular singularity, we verify the Gevrey smoothing effect with sharp index.

Before stating the main result, we first recall the extensive studies on the regularization properties of weak solutions to the spatially inhomogeneous Boltzmann equation. The mathematical verification of the regularization phenomena may go back to Desvillettes [18] for a one-dimensional model of the Boltzmann equation. Later on, the intrinsic diffusion structure in velocity was proven by Alexandre–Desvillettes–Villani–Wennberg [1]. Since then substantial developments have been achieved, and here we only mention the works [3, 4, 17, 25, 26, 28] for the C^∞ or Sobolev regularization effect. The smoothing effect in more regular Gevrey classes with Gevrey index $1 + \frac{1}{2s}$ was proven by [12, 21, 33, 35], based on the hypoelliptic structure explored in [2, 10, 14, 16, 27, 34, 36]. Another effective tool refers to De Giorgi–Nash–Moser theory, with the help of a strong averaging lemma that plays a crucial role in capturing the regularizing effect; this approach was recently applied to study conditional regularity for the spatially inhomogeneous Boltzmann equation with general initial data (cf. [29–32, 43, 44] for instance) and well-posedness for the close-to-equilibrium problem with polynomial tails (cf. [7, 8, 44]).

1.1. Notation and function spaces

Given two operators P_1 and P_2 we denote by $[P_1, P_2]$ the commutator between P_1 and P_2, that is, $[P_1, P_2] = P_1P_2 - P_2P_1$.

We denote by $\hat{f}$ or $\mathcal{F}_x f$ the partial Fourier transform of $f(t, x, v)$ with respect to the spatial variable $x \in \mathbb{T}^3$, that is,

$$\hat{f}(t, m, v) = \mathcal{F}_x f(t, m, v) = \int_{\mathbb{T}^3} e^{-im\cdot x} f(t, x, v)\, dx, \quad m \in \mathbb{Z}^3,$$

where here and below we use $m \in \mathbb{Z}^3$ to stand for the Fourier dual variable of $x \in \mathbb{T}^3$. Similarly, $\mathcal{F}_{x,v} f$ represents the full Fourier transform of $f(t,x,v)$ with respect to (x,v) and we will denote by (m,η) the Fourier dual variable of (x,v). For the sake of convenience, we will denote by $\widehat{\Gamma}(\hat{f},\hat{g})$ the partial Fourier transform of $\Gamma(f,g)$ defined in (1.7), meaning that

$$\begin{aligned}
&\widehat{\Gamma}(\hat{f},\hat{g})(t,m,v)\\
&\quad := \mathcal{F}_x(\Gamma(f,g))(t,m,v)\\
&\quad = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\mu^{\frac{1}{2}}(v_*)\big([\hat{f}(v_*')*\hat{g}(v')](m)-[\hat{f}(v_*)*\hat{g}(v)](m)\big)\,d\sigma\,dv_*,
\end{aligned}$$

where the convolutions are taken with respect to the Fourier variable $m \in \mathbb{Z}^3$:

$$[\hat{f}(u)*\hat{g}(v)](m) := \int_{\mathbb{Z}^3_\ell} \hat{f}(t,m-\ell,u)\hat{g}(t,\ell,v)\,d\Sigma(\ell), \tag{1.9}$$

for any velocities $u, v \in \mathbb{R}^3$. Here and below $d\Sigma(m)$ stands for the discrete measure on $\mathbb{Z}^3$, i.e.,

$$\int_{\mathbb{Z}^3} g(m)\,d\Sigma(m) := \sum_{m\in\mathbb{Z}^3} g(m)$$

for any summable function $g = g(m)$ on $\mathbb{Z}^3$. When applying Leibniz's formula, it will be convenient to work with the trilinear operator $\mathcal{T}$ defined by

$$\mathcal{T}(g,h,\omega) = \iint B(v-v_*,\sigma)\omega_*(g_*'h'-g_*h)\,dv_*\,d\sigma, \tag{1.10}$$

where B is given in (1.3), and ω is a function of the variable v only. The bilinear operator Γ in (1.7) and the above $\mathcal{T}$ are linked by

$$\Gamma(g,h) = \mathcal{T}(g,h,\mu^{\frac{1}{2}}). \tag{1.11}$$

Similarly to above we denote by $\widehat{\mathcal{T}}(\hat{g},\hat{h},\omega)$ the partial Fourier transform of $\mathcal{T}(g,h,\omega)$ with respect to x, that is, for any functions $\omega=\omega(v)$ of the variable v only,

$$\begin{aligned}
&\widehat{\mathcal{T}}(\hat{g},\hat{h},\omega)(m,v)\\
&\quad = \mathcal{F}_x(\mathcal{T}(g,h,\omega))(m,v)\\
&\quad = \iint B(v-v_*,\sigma)\omega(v_*)\big([\hat{g}(v_*')*\hat{h}(v')](m)-[\hat{g}(v_*)*\hat{h}(v)](m)\big)\,dv_*\,d\sigma,
\end{aligned} \tag{1.12}$$

where the conclusions are taken with respect to the Fourier variable $m \in \mathbb{Z}^3$, seeing definition (1.9).

Throughout the paper, without confusion we will use L^2_v to stand for the classical Lebesgue space L^2 consisting of functions of the specified variable v, and similarly for $L^2_{x,v}$. We denote by H^p_v the classical Sobolev space H^p in the variable v, and similarly for $H^p_{x,v}$. We recall the mixed Lebesgue spaces $L^p_m L^q_T L^r_v$ introduced in [22], which are defined by

$$L^p_m L^q_T L^r_v = \{g = g(t,x,v);\ \|g\|_{L^p_m L^q_T L^r_v} < +\infty\},$$

where

$$\|g\|_{L^p_m L^q_T L^r_v} := \begin{cases} \left(\int_{\mathbb{Z}^3} \left(\int_0^T \|\hat{g}(t,m,\cdot)\|^q_{L^r_v}\,dt\right)^{\frac{p}{q}} d\Sigma(m)\right)^{\frac{1}{p}}, & q<\infty,\\ \left(\int_{\mathbb{Z}^3} \left(\sup_{0<t<T} \|\hat{g}(t,m,\cdot)\|_{L^r_v}\right)^p d\Sigma(m)\right)^{\frac{1}{p}}, & q=\infty, \end{cases}$$

for $1 \le p, r < \infty$ and $1 \le q \le \infty$. In particular,

$$L^p_m L^r_v = \{g = g(x,v);\ \|g\|_{L^p_m L^r_v} := \left(\textstyle\int_{\mathbb{Z}^3} \|\hat{g}(m,\cdot)\|^p_{L^r_v}\,d\Sigma(m)\right)^{\frac{1}{p}} < +\infty\}$$

and

$$L^1_m = \{g = g(x);\ \|g\|_{L^1_m} := \textstyle\int_{\mathbb{Z}^3} |\hat{g}(m)|\,d\Sigma(m) < +\infty\}.$$

Finally, we recall the triple norm $|||\cdot|||$ introduced by Alexandre–Morimoto–Ukai–Xu–Yang [5], defined as

$$\begin{aligned} |||f|||^2 := &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\mu_*(f-f')^2\,d\sigma\,dv\,dv_* \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma) f_*^2(\sqrt{\mu'}-\sqrt{\mu})^2\,d\sigma\,dv\,dv_*. \end{aligned} \tag{1.13}$$

Note that the triple norm is indeed equivalent to the anisotropic norm $|\cdot|_{N^{\gamma,s}}$ introduced in Gressman–Strain [25]. Both norms can be characterized by an explicit norm $\|(a^{\frac{1}{2}})^w f\|_{L^2_v}$, with $(a^{\frac{1}{2}})^w$ standing for the Weyl quantization of symbol $a^{\frac{1}{2}}$ (cf. [2] for detail). In this text, we will use the above triple norm to avoid the need for pseudo-differential calculus.

1.2. Statement of the main result

Let $L^1_m L^2_v$ and $L^1_m L^\infty_T L^2_v$ be the spaces defined in the previous part. We first recall the existence and uniqueness of solutions to (1.5) established by Duan–Liu–Sakamoto–Strain [22] in the setting of $L^1_m L^2_v$. Assume that the cross-section satisfies (1.3) and (1.4) with $0 \le \gamma$ and $0 < s < 1$. It is proven in [22] that for a given initial datum $f_0 \in L^1_m L^2_v$ satisfying

$$\|f_0\|_{L^1_m L^2_v} \le \epsilon$$

for some sufficiently small constant $\epsilon > 0$, the non-linear Boltzmann equation (1.5) admits a unique global solution in $L^1_m L^\infty_T L^2_v$ for any $T > 0$. Moreover, the higher-order regularity of the mild solution f is obtained in [21], which says that f is in Gevrey class

$G^{1+\frac{1}{2s}}(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ for $t>0$. Recall that $f=f(x,v)\in G^r(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ with index $r>0$ if $f\in C^\infty(\mathbb{T}^3_x\times\mathbb{R}^3_v)$, and there exists a constant $C>0$ such that

$$\forall\alpha,\beta\in\mathbb{Z}^3_+,\quad \|\partial^\alpha_x\partial^\beta_v f\|_{L^2_{x,v}}\le C^{|\alpha|+|\beta|+1}[(|\alpha|+|\beta|)!]^r.$$

Here, r is called the Gevrey index. In particular, $G^1(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ is just the space of analytic functions, and that $G^r(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ with $0<r<1$ is the space of ultra-analytic functions. We have an equivalent expression of the Gevrey class $G^r(\mathbb{Z}^3_x\times\mathbb{R}^3_v)$ by virtue of the Fourier multiplier $e^{c(-\Delta_x-\Delta_v)^{\frac{1}{2r}}}$, with $c>0$ a constant, that is, we say $f\in G^r(\mathbb{Z}^3_x\times\mathbb{R}^3_v)$ if

$$e^{c(-\Delta_x-\Delta_v)^{\frac{1}{2r}}}f\in L^2_{x,v}. \tag{1.14}$$

Here, $e^{c(-\Delta_x-\Delta_v)^{\frac{1}{2r}}}f$ is defined by

$$\mathcal{F}_{x,v}(e^{c(-\Delta_x-\Delta_v)^{\frac{1}{2r}}}f)(m,\eta)=e^{c(|m|^2+|\eta|^2)^{\frac{1}{2r}}}\mathcal{F}_{x,v}f(m,\eta),$$

recalling that $\mathcal{F}_{x,v}$ represents the full Fourier transform with respect to (x,v), and that (m,η) are the Fourier dual variables of (x,v).

This work aims to prove the sharp Gevrey class smoothing effect, improving the previous Gevrey regularity index $1+\frac{1}{2s}$ in [21]. The main result can be stated as follows.

Theorem 1.1. *Let $G^r(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ be the Gevrey space defined above. Assume that the cross-section satisfies* (1.3) *and* (1.4) *with $\gamma\ge 0$ and $0<s<1$. There exists a sufficiently small constant $\epsilon>0$ such that if*

$$\|f_0\|_{L^1_m L^2_v}\le\epsilon, \tag{1.15}$$

then the Boltzmann equation (1.5) *admits a global-in-time solution f satisfying that $f\in G^\tau(\mathbb{T}^3_x\times\mathbb{R}^3_v)$ for all $t>0$, where*

$$\tau=\max\Big\{\frac{1}{2s},1\Big\}. \tag{1.16}$$

Moreover, for any $T\ge 1$ and any number λ satisfying $\lambda>1+\frac{1}{2s}$, there exists a constant $C>0$ depending on T and λ, such that

$$\forall\alpha,\beta\in\mathbb{Z}^3_+,\quad \sup_{0<t\le T}t^{(\lambda+1)|\alpha|+\lambda|\beta|}\|\partial^\alpha_x\partial^\beta_v f(t)\|_{L^2_{x,v}}\le C^{|\alpha|+|\beta|+1}[(|\alpha|+|\beta|)!]^\tau. \tag{1.17}$$

Remark 1.2. As will be seen below, our argument relies on the restriction that $\gamma\ge 0$. It is interesting to extend the result above to the case of soft potentials, which would require some new ideas. We hope the method in this text may give insights into the regularity of the soft potentials case and other related topics for more general spatially inhomogeneous Boltzmann equations.

1.3. Sharpness of the Gevrey index

In view of (1.16), we have an analytic regularization effect for the strong angular singularity case (i.e., $\frac{1}{2} \leq s < 1$). For the mild angular singularity case of $0 < s < \frac{1}{2}$, only Gevrey class regularization with index $\frac{1}{2s}$ can be expected. In this part, we will confirm the sharpness of the Gevrey index through the some toy models of the Boltzmann equation. To do so, we first consider the following fractional Fokker–Planck equation in $\mathbb{T}_x^3 \times \mathbb{R}_v^3$:

$$\begin{cases} \partial_t g + v \cdot \partial_x g + (-\Delta_v)^s g = 0, \quad 0 < s < 1, \\ g|_{t=0} = g_0 \in L^2_{x,v}, \end{cases} \tag{1.18}$$

which is a toy model of the Boltzmann equation with Maxwellian molecules (i.e., $\gamma = 0$ in (1.3)). By performing the full Fourier transform, we could reformulate (1.18) as the following transport equation:

$$\begin{cases} \partial_t \mathcal{F}_{x,v} g - m \cdot \partial_\eta \mathcal{F}_{x,v} g + |\eta|^{2s} \mathcal{F}_{x,v} g = 0, \\ \mathcal{F}_{x,v} g|_{t=0} = \mathcal{F}_{x,v} g_0, \end{cases}$$

recalling that (m, η) are the Fourier dual variables of (x, v). By solving the above transport equation we get an explicit solution g to (1.18) satisfying

$$(\mathcal{F}_{x,v} g)(t, m, \eta) = e^{-\int_0^t |\eta + \rho m|^{2s} \, d\rho} (\mathcal{F}_{x,v} g_0)(m, \eta + tm). \tag{1.19}$$

Moreover, observe that (cf. [41, Lemma 3.1] for instance)

$$-t(|\eta|^2 + t^2|m|^2)^s / c_s \leq -\int_0^t |\eta + \rho m|^{2s} \, d\rho \leq -c_s t(|\eta|^2 + t^2|m|^2)^s,$$

and thus, for any $t > 0$,

$$-(|m|^2 + |\eta|^2)^s / c_{s,t} \leq -\int_0^t |\eta + \rho m|^{2s} \, d\rho \leq -c_{s,t}(|m|^2 + |\eta|^2)^s, \tag{1.20}$$

where $c_s > 0$ is a small constant depending only on s, and $c_{s,t} > 0$ is a small constant depending only on s and t. Then combining (1.19) and (1.20) yields that, for any $t > 0$,

$$\begin{aligned} &\|e^{c_{s,t}(-\Delta_x - \Delta_v)^s} g(t)\|^2_{L^2_{x,v}} \\ &\quad = \int_{\mathbb{Z}^3 \times \mathbb{R}^3} e^{2c_{s,t}(|m|^2 + |\eta|^2)^s} e^{-2\int_0^t |\eta + \rho m|^{2s} \, d\rho} |(\mathcal{F}_{x,v} g_0)(m, \eta + tm)|^2 \, d\Sigma(m) \, d\eta \\ &\quad \leq \|g_0\|^2_{L^2_{x,v}}. \end{aligned}$$

Then, in view of the equivalent definition (1.14) of Gevrey space,

$$\forall t > 0, \quad g(t, \cdot, \cdot) \in G^{\frac{1}{2s}}(\mathbb{T}_x^3 \times \mathbb{R}_v^3).$$

Next we will show that the Gevrey index $\frac{1}{2s}$ is sharp. To do so, let r be any given number satisfying $0 < r < \frac{1}{2s}$, and we choose an initial datum g_0 in (1.18) such that

$$\forall \varepsilon > 0, \quad \|e^{\varepsilon(-\Delta_x - \Delta_v)^{\frac{1}{2r}}} g_0\|_{L^2_{x,v}} = +\infty, \tag{1.21}$$

which means $g_0 \notin G^r(\mathbb{T}^3_x \times \mathbb{R}^3_v)$. Moreover, for any constant $c_* > 0$, we can find a constant R depending only on c_* and the constant $c_{s,t}$ in (1.20), such that

$$(|m|^2 + |\eta|^2)^s / c_{s,t} \le \frac{c_*}{2}(|m|^2 + |\eta|^2)^{\frac{1}{2r}} + R,$$

due to the fact that $0 < r < \frac{1}{2s}$. Thus, with (1.20), it follows that

$$e^{2c_*(|m|^2+|\eta|^2)^{\frac{1}{2r}}} e^{-2\int_0^t |\eta+\rho m|^{2s} d\rho} \ge e^{-2R} e^{c_*(|m|^2+|\eta|^2)^{\frac{1}{2r}}}.$$

As a result, we use (1.19) to conclude that, for any given $t > 0$,

$$\begin{aligned}
&\|e^{c_*(-\Delta_x - \Delta_v)^{\frac{1}{2r}}} g(t)\|^2_{L^2_{x,v}} \\
&\quad = \int_{\mathbb{Z}^3 \times \mathbb{R}^3} e^{2c_*(|m|^2+|\eta|^2)^{\frac{1}{2r}}} e^{-2\int_0^t |\eta+\rho m|^{2s} d\rho} |(\mathcal{F}_{x,v} g_0)(m, \eta + tm)|^2 \, d\Sigma(m)\, d\eta \\
&\quad \ge e^{-2R} \int_{\mathbb{Z}^3 \times \mathbb{R}^3} e^{c_*(|m|^2+|\eta|^2)^{\frac{1}{2r}}} |(\mathcal{F}_{x,v} g_0)(m, \eta + tm)|^2 \, d\Sigma(m)\, d\eta,
\end{aligned}$$

which, with (1.21) and the fact that

$$|m|^2 + |\eta|^2 \ge \frac{|m|^2 + |\eta + tm|^2}{2(t^2+1)},$$

implies, for any given $t > 0$,

$$\forall c_* > 0, \quad \|e^{c_*(-\Delta_x - \Delta_v)^{\frac{1}{2r}}} g(t)\|_{L^2_{x,v}} = +\infty.$$

Thus $g(t) \notin G^r(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ for $t > 0$, and we have proven that $\frac{1}{2s}$ is the sharp Gevrey index we may expect when investigating the regularization effect for the toy model (1.18) of the Boltzmann equation.

(i) *Mild angular singularity case*. For $0 < s < \frac{1}{2}$, in Theorem 1.1 we get the regularization effect in the sharp Gevrey class $\frac{1}{2s}$, coinciding with the index for the toy model (1.18).

(ii) *Strong angular singularity and hard potentials*. For the Boltzmann equation with strong angular singularity and hard potentials, a more approximate model than (1.18) is

$$\begin{cases} \partial_t g + v \cdot \partial_x g + \langle v \rangle^\gamma (-\Delta_v)^s g = 0, \\ g|_{t=0} = g_0 \in L^2_{x,v}, \end{cases} \tag{1.22}$$

where $\langle v \rangle := (1 + v^2)^{\frac{1}{2}}$, and $0 < \gamma \le 1$, $\frac{1}{2} \le s < 1$. Note that the coefficient $\langle v \rangle^\gamma = (1 + |v|^2)^{\frac{\gamma}{2}}$ in (1.22) is only (locally) analytic but not ultra-analytic for $0 < \gamma \le 1$. Then heuristically it seems reasonable that ultra-analyticity is not achievable and analyticity is the best regularity setting we may expect for the toy model (1.22), and so too for the original Boltzmann equation. In Theorem 1.1, the analytic smoothing effect is indeed confirmed by observing that $\tau = 1$ in (1.16) for $\frac{1}{2} \leq s < 1$.

(iii) *Strong angular singularity and Maxwellian molecules*. For $\gamma = 0$, we model the Boltzmann equation by (1.18). As shown above, if $\frac{1}{2} \leq s < 1$, then the toy model (1.18) will admit the smoothing effect in the ultra-analytic class $G^{\frac{1}{2s}}(\mathbb{T}_x^3 \times \mathbb{R}_v^3)$ rather than in the analytic setting. Naturally, we may expect a similar ultra-analytic smoothing effect for the Boltzmann equation when $\gamma = 0$ and $\frac{1}{2} \leq s < 1$, and this remains unknown at moment. Here we mention Barbaroux–Hundertmark–Ried–Vugalter [9], who considered the spatially homogeneous Boltzmann equation (i.e., $F = F(t, v)$ is independent of x) and established the regularization effect in the Gevrey class with sharp index $\frac{1}{2s}$ for the case of Maxwellian molecules.

1.4. Difficulties and methodologies

When exploring the analyticity of the spatially inhomogeneous Boltzmann equation, the main difficulty arises from the degeneracy in the spatial direction. Compared with elliptic equations that usually admit analytic regularity, we may only expect Gevrey regularity for general hypoelliptic equations. For the specific hypoelliptic Boltzmann equation, when performing the standard energy, the key part is the treatment of the commutator between ∂_v and the transport operator $\partial_t + v \cdot \partial_x$, since the spatial derivative ∂_x will be involved in the commutator. To overcome the degeneracy in the spatial direction, we may apply a global pseudo-differential calculus to derive the intrinsic hypoelliptic structure induced by the non-trivial interaction between the diffusion part and the transport part. This hypoellipticity enables us to conclude the smoothing effect in Gevrey space of index $1 + \frac{1}{2s}$; interested readers may refer to [2, 21] and the references therein.

Inspired by the regularization effect for the toy model (1.18), we would expect similar regularity properties for the Boltzmann equation. Recently in [11], the last two authors and Cao verified the analytic smoothing effect for the Landau equation. This equation can be regarded as a diffusive model of the Boltzmann equation, obtained as a grazing limit of the latter. Note that the linear Landau collision behaves as the differential operator Δ_v, rather than the fractional Laplacian in the Boltzmann counterpart, so the treatment of the Landau equation is usually simpler than that of the Boltzmann equation. Although less technicality is involved in the Landau collision case than the Boltzmann counterpart, the methods developed for the Landau equation may usually apply to the Boltzmann equation with technical modifications. However, the situation could be quite different if we investigate the analytic or more general Gevrey class regularity of the two equations. In fact, to obtain the Gevrey class regularity, the key and subtle part is to derive quantitative estimates with respect to the orders of derivatives, which is usually hard for the highly non-linear collision terms. To explore the analytic smoothing effect of the Landau equation, the argument therein relies crucially on some differential calculus so that Leibniz's formula may apply when handling the non-linear Landau collision part. However, there will be essential difficulties for the Boltzmann collision term if we apply a similar argument to that in the case of the Landau equation with modifications, since the Boltzmann collision behaves as a pseudo-differential rather than a differential operator; hence we have to work with pseudo-differential calculus, which prevents us applying Leibniz's formula. Precisely, the analytic smoothing effect of the Landau equation, obtained in [11], relies on the following second-order differential operator:

$$M = -\int_0^t |\partial_v + \rho\partial_x|^2 \, d\rho = -t\Delta_v - t^2\partial_x \cdot \partial_v - \frac{t^3}{3}\Delta_x,$$

which is elliptic in the x and v variables. The introduction of M is inspired by the explicit solution to the Fokker–Planck equation (i.e., a specific form of equation (1.18) with $s = 1$). We could take advantage of the strong diffusion property (i.e., the heat diffusion $-\Delta_v$) of the Landau collision part to control the commutator between M and the transport operator $\partial_t + v \cdot \partial_x$, which is

$$[M, \partial_t + v \cdot \partial_x] = \Delta_v, \tag{1.23}$$

recalling that $[\cdot,\cdot]$ stands for the commutator between two operators. Moreover, the quantitative estimates on the commutators between M^k, $k \in \mathbb{Z}_+$, and the non-linear Landau collision part is hard, but achievable with the help of a Leibniz-type formula (see [11, Lemma 4.2]). This enables us to perform quantitative estimates on $M^k f$ with $k \in \mathbb{Z}_+$ and then derive, with the help of the ellipticity of M, the analytic regularization effect of the Landau equation. Note that we cannot apply the above operator M directly to the Boltzmann equation, since the Boltzmann collision part behaves as a fractional Laplacian $(-\Delta_v)^s$, $0 < s < 1$, and the diffusion is too weak to control the commutator (1.23) between M and the transport operator. Inspired by the explicit representation (1.19), a natural attempt is to modify M as follows to save the game:

$$M_s := -\int_0^t (1 + |D_v + \rho D_x|^2)^s \, d\rho, \quad D_x = \frac{\partial_x}{i} \quad \text{and} \quad D_v = \frac{\partial_v}{i},$$

where M_s is a Fourier multiplier defined by

$$\mathcal{F}_{x,v}(M_s f)(t,m,\eta) = -\int_0^t (1 + |\eta + \rho m|^2)^s \, d\rho (\mathcal{F}_{x,v} f)(t,m,\eta).$$

Observe that

$$\int_0^t m \cdot \partial_\eta (1 + |\eta + \rho m|^2)^s \, d\rho = \int_0^t \frac{d}{d\rho}(1 + |\eta + \rho m|^2)^s \, d\rho = (1 + |\eta + tm|^2)^s - (1 + |\eta|^2)^s,$$

which implies

$$[\psi(t,m,\eta), \partial_t - m \cdot \partial_\eta] = (1 + |\eta|^2)^s, \quad \psi(t,m,\eta) := -\int_0^t (1 + |\eta + \rho m|^2)^s \, d\rho.$$

Thus the commutator

$$[M_s, \partial_t + v \cdot \partial_x] = (1 - \Delta_v)^s$$

could be controlled by the diffusive part of the Boltzmann collision. Moreover, we need to handle the commutator

$$M_s^k \Gamma(f, f) - \Gamma(f, M_s^k f), \quad k \in \mathbb{Z}_+,$$

where Γ is the non-linear Boltzmann collision operator defined by (1.7). It is not hard to control the above commutator by constants C_k depending on k. However, it is quite difficult and seems not possible to get a quantitative upper bound with respect to $k \in \mathbb{Z}_+$, say

$$C^{k+1}(k!)^r,$$

with C a constant independent of k, since M_s is a pseudo-differential rather than a differential operator, so that Leibniz's formula cannot apply. Thus, to handle the non-linear Boltzmann collision part, it seems reasonable to work with differential rather than pseudo-differential operators, so that we can take advantage of Leibniz's formula, as well as an induction argument, to derive quantitative estimates with respect to derivatives. On the other hand, the classical first-order differential operator ∂_v or ∂_x is not a good choice, since the Boltzmann equation is degenerate in the spatial variable x and the spatial derivative ∂_x will appear in the commutator between ∂_v and the transport operator.

The new idea in this text is that instead of the sole ∂_x or ∂_v, we work with the following combination of ∂_x and ∂_v with time-dependent coefficients:

$$\xi(t)\partial_{x_j} + \zeta(t)\partial_{v_j}, \quad 1 \le j \le 3,$$

such that, denoting by $\rho'(t)$ the time derivative of the function $\rho(t)$,

$$[\xi(t)\partial_{x_j} + \zeta(t)\partial_{v_j}, \partial_t + v \cdot \partial_x] = -\xi'(t)\partial_{x_j} - \zeta'(t)\partial_{v_j} + \zeta(t)\partial_{x_j} = -\zeta'(t)\partial_{v_j}. \quad (1.24)$$

As will be seen in the last two sections, the commutator above indeed can be controlled by the diffusive Boltzmann operator. The choice of $\xi(t)$ and $\zeta(t)$ is flexible, provided $\xi'(t) = \zeta(t)$. For the sake of simplicity, we choose $\xi = (1+\delta)^{-1}t^{\delta+1}$ and $\zeta = t^\delta$ and consider a family of first-order differential operators H_δ defined by

$$H_\delta = \frac{1}{\delta+1} t^{\delta+1}\partial_{x_1} + t^\delta \partial_{v_1}, \quad (1.25)$$

where δ satisfies

$$1 + \frac{1}{2s} < \delta. \quad (1.26)$$

In view of (1.24), the spatial derivatives are not involved in the commutator between H_δ and the transport operator, that is,

$$[H_\delta, \partial_t + v \cdot \partial_x] = -\delta t^{\delta-1}\partial_{v_1}. \quad (1.27)$$

More generally, we have

$$\forall k \geq 1, \quad [H_\delta^k, \partial_t + v \cdot \partial_x] = -\delta k t^{\delta-1} \partial_{v_1} H_\delta^{k-1}, \tag{1.28}$$

which can be derived by using induction on k. In fact, the validity of (1.28) for $k = 1$ follows from (1.27). Now, supposing that

$$\forall \ell \leq k-1, \quad [H_\delta^\ell, \partial_t + v \cdot \partial_x] = -\delta \ell t^{\delta-1} \partial_{v_1} H_\delta^{\ell-1}, \tag{1.29}$$

we will prove the validity of (1.29) for $\ell = k \geq 2$. To do so, we use (1.27) and (1.28), as well as the fact that

$$[T_1 T_2, T_3] = T_1 T_2 T_3 - T_3 T_1 T_2 = T_1 [T_2, T_3] + [T_1, T_3] T_2,$$

to compute

$$\begin{aligned} [H_\delta^k, \partial_t + v \cdot \partial_x] &= [H_\delta H_\delta^{k-1}, \partial_t + v \cdot \partial_x] \\ &= H_\delta [H_\delta^{k-1}, \partial_t + v \cdot \partial_x] + [H_\delta, \partial_t + v \cdot \partial_x] H_\delta^{k-1} \\ &= H_\delta (-\delta(k-1) t^{\delta-1} \partial_{v_1} H_\delta^{k-2}) - \delta t^{\delta-1} \partial_{v_1} H_\delta^{k-1} \\ &= -\delta(k-1) t^{\delta-1} \partial_{v_1} H_\delta^{k-1} - \delta t^{\delta-1} \partial_{v_1} H_\delta^{k-1} = -\delta k t^{\delta-1} \partial_{v_1} H_\delta^{k-1}. \end{aligned}$$

This gives the validity of (1.29) for $\ell = k$. Thus (1.28) holds true for all $k \geq 1$. This enables us to apply the diffusion in the velocity direction to obtain a crucial estimate of the directional derivatives $H_\delta^k f$ for the solution f. Moreover, the classical derivatives can be generated by a linear combination of H_δ for suitable δ with time-dependent coefficients, so that the desired quantitative estimate on the classical derivatives is available (see (1.32) below for the explicit formulation).

In this text let λ be an arbitrary given number satisfying (1.26), that is, $\lambda > 1 + \frac{1}{2s}$. We define δ_1 and δ_2 in terms of λ by setting

$$\delta_1 = \lambda, \quad \delta_2 = \begin{cases} 1 + 2s + (1-2s)\lambda & \text{if } 0 < s < \dfrac{1}{2}, \\ \dfrac{1}{2}\Big(\lambda + 1 + \dfrac{1}{2s}\Big) & \text{if } \dfrac{1}{2} \leq s < 1. \end{cases} \tag{1.30}$$

By virtue of the fact that $\lambda > 1 + \frac{1}{2s}$, direct computation yields that

$$\delta_1 > \delta_2 > 1 + \frac{1}{2s}. \tag{1.31}$$

So both δ_1 and δ_2 satisfy (1.26). With δ_1 and δ_2 given above, let H_{δ_1} and H_{δ_2} be defined by (1.25):

$$H_{\delta_1} = \frac{1}{\delta_1 + 1} t^{\delta_1 + 1} \partial_{x_1} + t^{\delta_1} \partial_{v_1}, \quad H_{\delta_2} = \frac{1}{\delta_2 + 1} t^{\delta_2 + 1} \partial_{x_1} + t^{\delta_2} \partial_{v_1}.$$

Then ∂_{x_1} and ∂_{v_1} can be generated by a linear combination of H_{δ_j}, $j = 1, 2$, that is,

$$\begin{cases} t^{\lambda+1}\partial_{x_1} = t^{\delta_1+1}\partial_{x_1} = \dfrac{(\delta_2+1)(\delta_1+1)}{\delta_2-\delta_1} H_{\delta_1} - \dfrac{(\delta_2+1)(\delta_1+1)}{\delta_2-\delta_1} t^{\delta_1-\delta_2} H_{\delta_2}, \\ t^{\lambda}\partial_{v_1} = t^{\delta_1}\partial_{v_1} = -\dfrac{\delta_1+1}{\delta_2-\delta_1} H_{\delta_1} + \dfrac{\delta_2+1}{\delta_2-\delta_1} t^{\delta_1-\delta_2} H_{\delta_2}. \end{cases} \tag{1.32}$$

This enables us to control the classical derivatives in terms of the directional derivatives in H_{δ_1} and H_{δ_2}.

1.5. Arrangement of the paper

The rest of this paper is arranged as follows. In Section 2 we recall a few preliminary estimates that will be used throughout the argument. Section 3 is devoted to estimating the commutator between directional derivatives and the collision operator. The proof of the main result is presented in Sections 4 and 5, where we treat, respectively, the strong angular singularity case and the mild one.

2. Preliminaries

In this section we will recall some estimates to be used later. Let $\mathcal{L}$ be the linearized Boltzmann operator in (1.6) and let $|||\cdot|||$ be the triple norm defined by (1.13). Then by the coercive estimate and identification of the triple norm (cf.[5, Propositions 2.1 and 2.2] for instance), it follows that

$$\forall f \in \mathcal{S}(\mathbb{R}^3_v), \quad c_0 |||f|||^2 \le (\mathcal{L}f, f)_{L^2_v} + \|f\|^2_{L^2_v}, \tag{2.1}$$

and that, for Maxwellian molecules and hard potential cases with $\gamma \ge 0$,

$$\forall f \in \mathcal{S}(\mathbb{R}^3_v), \quad c_0 \|f\|_{H^s_v} \le |||f|||, \tag{2.2}$$

where s is the number in (1.4), $c_0 > 0$ is a small constant and $\mathcal{S}(\mathbb{R}^3_v)$ stands for the Schwartz space in $\mathbb{R}^3_v$. Note that the above estimates still hold true for any f such that $|||f||| < +\infty$.

For simplicity of notation, in the following argument we will use C_0 to denote a generic constant which may vary from line to line by enlarging C_0 if necessary. Now we recall the trilinear estimate of the collision operator, which says that (cf. [25, Theorem 2.1]), for any $f, g, h \in \mathcal{S}(\mathbb{R}^3_v)$,

$$|(\mathcal{T}(f, g, \mu^{\frac{1}{2}}), h)_{L^2_v}| = |(\Gamma(f, g), h)_{L^2_v}| \le C_0 \|f\|_{L^2_v} \cdot |||g||| \cdot |||h|||, \tag{2.3}$$

recalling that $\mathcal{T}$ is defined in (1.10). Furthermore, we mainly employ the counterpart of the above estimate after performing a partial Fourier transform in the x variable. Then, by

[22, Lemma 3.2], the estimate

$$|(\widehat{\mathcal{T}}(\hat{f},\hat{g},\mu^{\frac{1}{2}}),\hat{h})_{L^2_v}| = |(\widehat{\Gamma}(\hat{f}(m),\hat{g}(m)),\hat{h}(m))_{L^2_v}| \\ \leq C_0 |||\hat{h}(m)||| \int_{\mathbb{Z}^3} \|\hat{f}(m-\ell)\|_{L^2_v} |||\hat{g}(\ell)||| \, d\Sigma(\ell) \tag{2.4}$$

holds true for any $m \in \mathbb{Z}^3$ and for any $f, g, h \in L^1_m(\mathcal{S}(\mathbb{R}^3_v))$. More generally, if $\omega = \omega(v)$ is a given function of the variable v satisfying the condition that there exists a constant $\widetilde{C} > 0$ such that

$$\forall v \in \mathbb{R}^3, \quad |\omega(v)| \leq \widetilde{C} \mu(v)^{\frac{1}{4}}, \tag{2.5}$$

then following the same argument for proving (2.3), with $\mu^{\frac{1}{2}}$ therein replaced by ω, gives that

$$\forall f, g, h \in \mathcal{S}(\mathbb{R}^3_v), \quad |(\mathcal{T}(f,g,\omega),h)_{L^2_v}| \leq C_0 \widetilde{C} \|f\|_{L^2_v} \cdot |||g||| \cdot |||h|||.$$

As a result, similarly to (2.4), we perform a partial Fourier transform in the x variable to conclude

$$|(\widehat{\mathcal{T}}(\hat{f},\hat{g},\omega),\hat{h}(m))_{L^2_v}| \leq C_0 \widetilde{C} |||\hat{h}(m)||| \int_{\mathbb{Z}^3} \|\hat{f}(m-\ell)\|_{L^2_v} |||\hat{g}(\ell)||| \, d\Sigma(\ell), \tag{2.6}$$

with $\widetilde{C}$ the constant in (2.5). In particular, if g in (2.6) is a function of the variable v only, then (2.6) reduces to

$$|(\widehat{\mathcal{T}}(\hat{f},g,\omega),\hat{h}(m))_{L^2_v}| \leq C_0 \widetilde{C} \|\hat{f}(m)\|_{L^2_v} |||g||| \times |||\hat{h}(m)|||.$$

This, with the fact that (cf. [5, Proposition 2.2])

$$|||g||| \leq \tilde{c} \|(1+|v|^{2+\gamma} - \Delta_v) g\|_{L^2_v}$$

for some constant $\tilde{c} > 0$, yields that, enlarging C_0 if necessary,

$$|(\widehat{\mathcal{T}}(\hat{f},g,\omega),\hat{h}(m))_{L^2_v}| \leq C_0 \widetilde{C} \|\hat{f}(m)\|_{L^2_v} \|(1+|v|^{2+\gamma} - \Delta_v) g\|_{L^2_v} |||\hat{h}(m)|||.$$

As a result, if $g = g(v) \in \mathcal{S}(\mathcal{R}^3_v)$ is any function of the variable v only, satisfying the condition that there exists a constant $\widetilde{C}_\gamma > 0$, depending only on the number γ in (1.3), such that

$$\forall v \in \mathbb{R}^3, \ \forall k \in \mathbb{Z}_+, \quad |(1+|v|^{2+\gamma} - \Delta_v) \partial^k_v g(v)| \leq \widetilde{C}_\gamma L_k \mu(v)^{\frac{1}{8}}, \tag{2.7}$$

with L_k constants depending only on k, then

$$\forall k \in \mathbb{Z}_+, \quad |(\widehat{\mathcal{T}}(\hat{f},\partial^k_v g,\omega),\hat{h}(m))_{L^2_v}| \leq C_0 \widetilde{C} L_k \|\hat{f}(m)\|_{L^2_v} |||\hat{h}(m)||| \tag{2.8}$$

by enlarging C_0 if necessary, recalling that $\widetilde{C}$ is the constant given in (2.5). Similarly, for any functions $\omega = \omega(v)$ and $g = g(v)$ of the variable v only, satisfying (2.5) and (2.7), respectively, we have

$$\forall k \in \mathbb{Z}_+, \quad |(\widehat{\mathcal{T}}(\partial^k_v g,\hat{f},\omega),\hat{h}(m))_{L^2_v}| \leq C_0 \widetilde{C} L_k |||\hat{f}(m)||| \times |||\hat{h}(m)|||. \tag{2.9}$$

Finally, we recall an estimate (cf. [21, Lemma 2.5]) that will be frequently used to control the non-linear term $\Gamma(f, g)$. For an arbitrary given integer $j_0 \geq 1$, it holds that

$$\int_{\mathbb{Z}^3}\Big[\int_0^T\Big(\int_{\mathbb{Z}^3}\sum_{1\leq j\leq j_0}\|\hat{f}_j(t,m-\ell)\|_{L^2_v}|||\hat{g}_j(t,\ell)|||\,d\Sigma(\ell)\Big)^2 dt\Big]^{\frac{1}{2}} d\Sigma(m)$$

$$\leq \sum_{j=1}^{j_0}\Big(\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\hat{f}_j(t,m)\|_{L^2_v}\,d\Sigma(m)\Big)\int_{\mathbb{Z}^3}\Big(\int_0^T |||\hat{g}_j(t,m)|||^2\,dt\Big)^{\frac{1}{2}} d\Sigma(m), \quad (2.10)$$

for any $f_j \in L^1_m L^\infty_T L^2_v$ and any g_j such that $|||g_j||| \in L^1_m L^2_T$ with $1 \leq j \leq j_0$. It can be derived directly by Minkowski's inequality and Fubini's theorem; cf. [21, Lemma 2.5] for detail.

3. Commutator estimates

This section is devoted to dealing with the commutator between the directional derivative H^k_δ and the collision part $\Gamma(g, h)$, recalling that H_δ is defined by (1.25). With the notation in Section 1.1, the results on commutator estimates can be stated as follows.

Proposition 3.1. *Assume that the cross-section satisfies* (1.3) *and* (1.4) *with* $\gamma \geq 0$ *and* $0 < s < 1$. *Recall that* H_δ *is defined by* (1.25), *with* δ *an arbitrary given number satisfying* (1.26). *Let* $k \geq 1$ *and* $T \geq 1$ *be given, and let* $f \in L^1_m L^\infty_T L^2_v$ *be any solution to the Cauchy problem* (1.5) *satisfying*

$$\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H^k_\delta f}(t,m)\|_{L^2_v}\,d\Sigma(m) + \int_{\mathbb{Z}^3}\Big(\int_0^T |||\widehat{H^k_\delta f}(t,m)|||^2\,dt\Big)^{\frac{1}{2}} d\Sigma(m) < +\infty. \quad (3.1)$$

Suppose that for any $j \leq k-1$ *we have*

$$\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H^j_\delta f}(t,m)\|_{L^2_v}\,d\Sigma(m) + \int_{\mathbb{Z}^3}\Big(\int_0^T |||\widehat{H^j_\delta f}(t,m)|||^2\,dt\Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 C_*^j (j!)^\tau}{(j+1)^2}, \quad (3.2)$$

where $\tau \geq 1$ *is given in* (1.16), *and* $\varepsilon_0, C_* > 0$ *are constants. If* $C_* \geq 4T^\delta$, *then there exists a constant* C, *depending only on the number* C_0 *in* (2.6) *but independent of* k, *such that for any* $\varepsilon > 0$ *we have*

$$\int_{\mathbb{Z}^3}\Big(\int_0^T |(\mathcal{F}_x(H^k_\delta\Gamma(f,f)), \widehat{H^k_\delta f})_{L^2_v}|\,dt\Big)^{\frac{1}{2}} d\Sigma(m)$$

$$\leq C\varepsilon^{-1}\varepsilon_0\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H^k_\delta f}(t,m)\|_{L^2_v}\,d\Sigma(m)$$

$$+ (\varepsilon + C\varepsilon^{-1}\varepsilon_0) \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H_\delta^k f}(t,m)|||^2 dt \Big)^{\frac{1}{2}} d\Sigma(m)$$
$$+ C\varepsilon^{-1}\varepsilon_0^2 \frac{C_*^k (k!)^\tau}{(k+1)^2}.$$

Remark 3.2. We impose assumption (3.1) to ensure rigorous rather than formal computations in the proof of Proposition 3.1 when performing estimates involving the term $H_\delta^k f$.

Proof of Proposition 3.1. If no confusion occurs, in the proof we will write $H = H_\delta$ for short, omitting the subscript δ. To simplify the notation, we denote by C some generic constants, which may vary from line to line and which depend only on the number C_0 in (2.3). Note that these generic constants C as below are independent of k.

In view of (1.11), it follows from the Leibniz formula that

$$H^k \Gamma(f,f) = \sum_{j=0}^{k} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} \mathcal{T}(H^{k-j} f, H^{j-p} f, H^p \mu^{\frac{1}{2}}).$$

As a result, taking a partial Fourier transform for the x variable on both sides and using the notation (1.12), we conclude that

$$\int_{\mathbb{Z}^3} \Big(\int_0^T |(\mathcal{F}_x(H^k \Gamma(f,f)), \widehat{H^k f})_{L_v^2}| \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \le \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3, \tag{3.3}$$

with

$$\begin{cases} \mathcal{J}_1 = \int_{\mathbb{Z}^3} \Big[\sum_{0\le p\le k} \binom{k}{p} \int_0^T |(\widehat{\mathcal{T}}(\hat f, \widehat{H^{k-p} f}, H^p \mu^{\frac{1}{2}}), \widehat{H^k f})_{L_v^2}| \, dt \Big]^{\frac{1}{2}} d\Sigma(m), \\ \mathcal{J}_2 = \int_{\mathbb{Z}^3} \Big[\sum_{j=1}^{k-1} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} \int_0^T |(\widehat{\mathcal{T}}(\widehat{H^{k-j} f}, \widehat{H^{j-p} f}, H^p \mu^{\frac{1}{2}}), \widehat{H^k f})_{L_v^2}| \, dt \Big]^{\frac{1}{2}} d\Sigma(m), \\ \mathcal{J}_3 = \int_{\mathbb{Z}^3} \Big(\int_0^T |(\widehat{\mathcal{T}}(\widehat{H^k f}, \hat f, \mu^{\frac{1}{2}}), \widehat{H^k f})_{L_v^2}| \, dt \Big)^{\frac{1}{2}} d\Sigma(m). \end{cases} \tag{3.4}$$

We proceed to estimate $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$ as follows.

Estimate on $\mathcal{J}_1$. We first control the term $\mathcal{J}_1$ by dividing it into two terms. That is,

$$\begin{aligned} \mathcal{J}_1 \le & \int_{\mathbb{Z}^3} \Big(\int_0^T |(\widehat{\mathcal{T}}(\hat f, \widehat{H^k f}, \mu^{\frac{1}{2}}), \widehat{H^k f})_{L_v^2}| \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \\ & + \int_{\mathbb{Z}^3} \Big[\sum_{1\le p\le k} \binom{k}{p} \int_0^T |(\widehat{\mathcal{T}}(\hat f, \widehat{H^{k-p} f}, H^p \mu^{\frac{1}{2}}), \widehat{H^k f})_{L_v^2}| \, dt \Big]^{\frac{1}{2}} d\Sigma(m) \\ := & \, \mathcal{J}_{1,1} + \mathcal{J}_{1,2}. \end{aligned} \tag{3.5}$$

Direct verification shows that

$$\forall p \ge 0, \ \forall t \in [0,T], \quad |H^p \mu^{\frac{1}{2}}| = |t^{\delta p} \partial_{v_1}^p \mu^{\frac{1}{2}}| \le (2T^\delta)^p p! \mu^{\frac{1}{4}}. \tag{3.6}$$

Then we apply (2.6) with $\omega = H^p\mu^{\frac{1}{2}}$ to control $\mathcal{J}_{1,2}$ in (3.5) as follows: for any $\varepsilon > 0$,

$$
\begin{aligned}
\mathcal{J}_{1,2} &= \int_{\mathbb{Z}^3}\Big[\sum_{1\le p\le k}\binom{k}{p}\int_0^T |(\widehat{\mathcal{T}}(\hat f, \widehat{H^{k-p}f}, H^p\mu^{\frac12}), \widehat{H^k f})_{L^2_v}|\,dt\Big]^{\frac12}\,d\Sigma(m)\\
&\le C\int_{\mathbb{Z}^3}\Big[\sum_{p=1}^{k}\binom{k}{p}(2T^\delta)^p p!\int_0^T\Big(\int_{\mathbb{Z}^3_\ell}\|\hat f(m-\ell)\|_{L^2_v}|||\widehat{H^{k-p}f}(\ell)|||\,d\Sigma(\ell)\Big)\\
&\qquad\qquad\times|||\widehat{H^k f}(m)|||\,dt\Big]^{\frac12}\,d\Sigma(m)\\
&\le \varepsilon\int_{\mathbb{Z}^3}\Big[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\Big]^{\frac12}\,d\Sigma(m)\\
&\quad+\frac{C}{\varepsilon}\int_{\mathbb{Z}^3}\Big[\int_0^T\Big\{\sum_{p=1}^{k}\binom{k}{p}(2T^\delta)^p p!\int_{\mathbb{Z}^3}\|\hat f(m-\ell)\|_{L^2_v}|||\widehat{H^{k-p}f}(\ell)|||\\
&\qquad\qquad\times d\Sigma(\ell)\Big\}^2\,dt\Big]^{\frac12}\,d\Sigma(m).
\end{aligned}
\tag{3.7}
$$

Moreover, in order to treat the last term in (3.7) we apply (2.10) to get

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\Big[\int_0^T\Big\{\sum_{p=1}^{k}\binom{k}{p}(2T^\delta)^p p!\int_{\mathbb{Z}^3_\ell}\|\hat f(m-\ell)\|_{L^2_v}|||\widehat{H^{k-p}f}(\ell)|||\,d\Sigma(\ell)\Big\}^2\,dt\Big]^{\frac12}\,d\Sigma(m)\\
&\le C\sum_{p=1}^{k}\binom{k}{p}(2T^\delta)^p p!\int_{\mathbb{Z}^3}\sup_{0<t\le T}\|\hat f(t,m)\|_{L^2_v}\,d\Sigma(m)\\
&\qquad\times\int_{\mathbb{Z}^3}\Big[\int_0^T|||\widehat{H^{k-p}f}(t,m)|||^2\,dt\Big]^{\frac12}\,d\Sigma(m)\\
&\le C\varepsilon_0^2\sum_{p=1}^{k}\frac{k!}{(k-p)!}(2T^\delta)^p\frac{C_*^{k-p}[(k-p)!]^\tau}{(k-p+1)^2}\\
&\le C\varepsilon_0^2 C_*^k(k!)^\tau\sum_{p=1}^{k}\frac{2^{-p}}{(k-p+1)^2},
\end{aligned}
\tag{3.8}
$$

where in the last line we used condition (3.2), as well as the fact that $C_* > 4T^\delta$. For the last term in (3.8) we have, denoting by $[\frac{k}{2}]$ the largest integer less than or equal to $\frac{k}{2}$,

$$
\begin{aligned}
&\sum_{p=1}^{k}\frac{2^{-p}}{(k-p+1)^2}\\
&\qquad\le\sum_{p=1}^{[\frac{k}{2}]}\frac{1}{(k-p+1)^2}2^{-p}+\sum_{p=[\frac{k}{2}]+1}^{k}\frac{1}{(k-p+1)^2}2^{-p}
\end{aligned}
$$

$$\leq C\left\{\sum_{p=1}^{[\frac{k}{2}]}\frac{1}{(k+1)^2}2^{-p}+\sum_{p=[\frac{k}{2}]+1}^{k}\frac{1}{(k+1)^2}(k+1)^2 2^{-p}\right\}$$
$$\leq \frac{C}{(k+1)^2}, \tag{3.9}$$

the last inequality using the fact that

$$\sum_{p=[\frac{k}{2}]+1}^{k}(k+1)^2 2^{-p}\leq \sum_{p=[\frac{k}{2}]+1}^{k}(k+1)^2 2^{-\frac{k}{2}}\leq (k+1)^3 2^{-\frac{k}{2}}\leq C.$$

As a result, we substitute (3.9) into (3.8) to conclude that

$$\int_{\mathbb{Z}^3}\left[\int_0^T\left\{\sum_{p=1}^{k}\binom{k}{p}(2T^\delta)^p p!\int_{\mathbb{Z}^3_\ell}\|\hat{f}(m-\ell)\|_{L^2_v}|||\widehat{H^{k-p}f}(\ell)|||\,d\Sigma(\ell)\right\}^2 dt\right]^{\frac{1}{2}} d\Sigma(m)$$
$$\leq C\varepsilon_0^2\frac{C_*^k(k!)^\tau}{(k+1)^2},$$

which with (3.7) yields

$$\mathcal{J}_{1,2}\leq \varepsilon\int_{\mathbb{Z}^3}\left[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\right]^{\frac{1}{2}}d\Sigma(m)+C\varepsilon^{-1}\varepsilon_0^2\frac{C_*^k(k!)^\tau}{(k+1)^2}. \tag{3.10}$$

Moreover, following a similar argument to above with a slight modification, we conclude that

$$\mathcal{J}_{1,1}=\int_{\mathbb{Z}^3}\left(\int_0^T|(\hat{\mathcal{T}}(\hat{f},\widehat{H^k f},\mu^{\frac{1}{2}}),\widehat{H^k f})_{L^2_v}|\,dt\right)^{\frac{1}{2}}d\Sigma(m)$$
$$\leq \varepsilon\int_{\mathbb{Z}^3}\left[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\right]^{\frac{1}{2}}d\Sigma(m)$$
$$+C\varepsilon^{-1}\left(\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\hat{f}(t,m)\|_{L^2_v}\,d\Sigma(m)\right)\int_{\mathbb{Z}^3}\left[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\right]^{\frac{1}{2}}d\Sigma(m)$$
$$\leq(\varepsilon+C\varepsilon^{-1}\varepsilon_0)\int_{\mathbb{Z}^3}\left[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\right]^{\frac{1}{2}}d\Sigma(m).$$

Here we used assumption (3.1) to ensure the right-hand side is finite. Substituting the above estimate and (3.10) into (3.5) yields that, for any $\varepsilon>0$,

$$\mathcal{J}_1\leq(\varepsilon+C\varepsilon^{-1}\varepsilon_0)\int_{\mathbb{Z}^3}\left[\int_0^T|||\widehat{H^k f}(t,m)|||^2\,dt\right]^{\frac{1}{2}}d\Sigma(m)+C\varepsilon^{-1}\varepsilon_0^2\frac{C_*^k(k!)^\tau}{(k+1)^2}. \tag{3.11}$$

Estimate on $\mathcal{J}_2$. Recall that $\mathcal{J}_2$ is given in (3.4). Following a similar argument to that in (3.7) and (3.8) yields that, for any $\varepsilon > 0$,

$$
\begin{aligned}
\mathcal{J}_2 \leq\ & \varepsilon \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \\
& + C\varepsilon^{-1} \sum_{j=1}^{k-1} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} (2T^\delta)^p p! \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^{k-j} f}(t,m)\|_{L^2_v} \, d\Sigma(m) \\
& \qquad \times \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^{j-p} f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m).
\end{aligned} \tag{3.12}
$$

Moreover, we use assumption (3.2) and then repeat the computation in (3.9), to conclude that, for any $1 \leq j \leq k-1$,

$$
\begin{aligned}
& \sum_{p=0}^{j} \binom{j}{p} (2T^\delta)^p p! \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^{j-p} f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \\
& \quad \leq \varepsilon_0 C_*^j (j!)^\tau \sum_{p=0}^{j} \frac{2^{-p}}{(j-p+1)^2} \\
& \quad \leq C\varepsilon_0 \frac{C_*^j (j!)^\tau}{(j+1)^2}.
\end{aligned}
$$

Substituting the above estimate into the last term on the right-hand side of (3.12) and using condition (3.2) again, we compute

$$
\begin{aligned}
& \sum_{j=1}^{k-1} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} (2T^\delta)^p p! \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^{k-j} f}(t,m)\|_{L^2_v} \, d\Sigma(m) \\
& \qquad \times \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^{j-p} f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \\
& \quad \leq C\varepsilon_0 \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \frac{C_*^j (j!)^\tau}{(j+1)^2} \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^{k-j} f}(t,m)\|_{L^2_v} \, d\Sigma(m) \\
& \quad \leq C\varepsilon_0^2 \sum_{j=1}^{k-1} \frac{k!}{j!(k-j)!} \frac{C_*^j (j!)^\tau}{(j+1)^2} \frac{C_*^{k-j} [(k-j)!]^\tau}{(k-j+1)^2} \\
& \quad \leq C\varepsilon_0^2 C_*^k \sum_{j=1}^{k-1} \frac{k!(j!)^{\tau-1} [(k-j)!]^{\tau-1}}{(k-j+1)^2 (j+1)^2} \\
& \quad \leq C\varepsilon_0^2 \frac{C_*^k (k!)^\tau}{(k+1)^2},
\end{aligned}
$$

the last inequality using the facts that $p!q! \leq (p+q)!$ and that $\tau \geq 1$. This, together with (3.12), yields

$$\mathcal{J}_2 \leq \varepsilon \int_{\mathbb{Z}^3} \left(\int_0^T |||\widehat{H^k f}(t,m)|||^2 \, dt \right)^{\frac{1}{2}} d\Sigma(m) + C\varepsilon^{-1}\varepsilon_0^2 \frac{C_*^k (k!)^\tau}{(k+1)^2}. \tag{3.13}$$

Estimate on $\mathcal{J}_3$. It remains to deal with $\mathcal{J}_3$, recalling that $\mathcal{J}_3$ is given in (3.4). We repeat the computation in (3.7) and (3.8) to conclude that

$$\begin{aligned}
\mathcal{J}_3 &= \int_{\mathbb{Z}^3} \left(\int_0^T |(\widehat{\mathcal{T}}(\widehat{H^k f}, \hat{f}, \mu^{\frac{1}{2}}), \widehat{H^k f})_{L^2_v}| \, dt \right)^{\frac{1}{2}} d\Sigma(m) \\
&\leq \varepsilon \int_{\mathbb{Z}^3} \left[\int_0^T |||\widehat{H^k f}(t,m)|||^2 \, dt \right]^{\frac{1}{2}} d\Sigma(m) \\
&\quad + C\varepsilon^{-1} \left(\int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k f}(t,m)\|_{L^2_v} \, d\Sigma(m) \right) \int_{\mathbb{Z}^3} \left[\int_0^T |||\hat{f}(t,m)|||^2 \, dt \right]^{\frac{1}{2}} d\Sigma(m) \\
&\leq \varepsilon \int_{\mathbb{Z}^3} \left[\int_0^T |||\widehat{H^k f}(t,m)|||^2 \, dt \right]^{\frac{1}{2}} d\Sigma(m) \\
&\quad + C\varepsilon^{-1}\varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k f}(t,m)\|_{L^2_v} \, d\Sigma(m).
\end{aligned}$$

Combining the upper bound of $\mathcal{J}_3$ above and estimates (3.11) and (3.13) with (3.3), we obtain the assertion in Proposition 3.1. The proof is completed. ■

Proposition 3.3. *Under the same assumption as in Proposition 3.1, we can find a constant C, depending only on T, δ and the number C_0 in (2.8) and (2.9) but independent of k, such that for any $\varepsilon > 0$,*

$$\begin{aligned}
&\int_{\mathbb{Z}^3} \left(\int_0^T |(\mathcal{F}_x([H_\delta^k, \mathcal{L}]f), \widehat{H_\delta^k f})_{L^2_v}| \, dt \right)^{\frac{1}{2}} d\Sigma(m) \\
&\qquad \leq \varepsilon \int_{\mathbb{Z}^3} \left(\int_0^T |||\widehat{H_\delta^k f}(t,m)|||^2 \, dt \right)^{\frac{1}{2}} d\Sigma(m) + C\varepsilon^{-1} \frac{\varepsilon_0 C_*^{k-1} (k!)^\tau}{(k+1)^2}.
\end{aligned}$$

Proof. This is just a specific case of Proposition 3.1. Recall that $\mathcal{L}$ is defined in (1.6), that is,

$$\mathcal{L}f = -\Gamma(\mu^{\frac{1}{2}}, f) - \Gamma(f, \mu^{\frac{1}{2}}) = -\mathcal{T}(\mu^{\frac{1}{2}}, f, \mu^{\frac{1}{2}}) - \mathcal{T}(f, \mu^{\frac{1}{2}}, \mu^{\frac{1}{2}}).$$

Then, denoting $H = H_\delta$, using Leibniz's formula again gives

$$\begin{aligned}
[H^k, \mathcal{L}]f &= -\sum_{j=1}^{k} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} \mathcal{T}(H^{j-p}\mu^{\frac{1}{2}}, H^{k-j} f, H^p \mu^{\frac{1}{2}}) \\
&\quad - \sum_{j=1}^{k} \sum_{p=0}^{j} \binom{k}{j} \binom{j}{p} \mathcal{T}(H^{k-j} f, H^{j-p}\mu^{\frac{1}{2}}, H^p \mu^{\frac{1}{2}}) \\
&\stackrel{\text{def}}{=} R_1(f) + R_2(f).
\end{aligned} \tag{3.14}$$

Moreover, we may write, as in (3.3),

$$\int_{\mathbb{Z}^3}\left(\int_0^T |(\mathscr{F}_x(R_2(f)), \widehat{H^k f})_{L^2_v}|\,dt\right)^{\frac12} d\Sigma(m)$$
$$\leq \int_{\mathbb{Z}^3}\Big[\sum_{j=1}^{k}\sum_{p=0}^{j}\binom{k}{j}\binom{j}{p}$$
$$\times\int_0^T |(\widehat{\mathcal{T}}(\widehat{H^{k-j}f}, H^{j-p}\mu^{\frac12}, H^p\mu^{\frac12}), \widehat{H^k f})_{L^2_v}|\,dt\Big]^{\frac12} d\Sigma(m). \quad (3.15)$$

By direct verification, it follows that, for any $p \geq 0$ and any $t \in [0,T]$,

$$\begin{aligned}|(1+|v|^{2+\gamma}-\Delta_v)H^p\mu^{\frac12}| &= |(1+|v|^{2+\gamma}-\Delta_v)t^{\delta p}\partial_{v_1}^p\mu^{\frac12}|\\ &\leq C(2T^\delta)^p p!\mu^{\frac18}.\end{aligned}$$

This, with (3.6), enables us to use (2.8) with $g=\mu^{\frac12}$ and $\omega = H^p\mu^{\frac12}$ to compute

$$\begin{aligned}&\sum_{j=1}^{k}\sum_{p=0}^{j}\binom{k}{j}\binom{j}{p}|(\widehat{\mathcal{T}}(\widehat{H^{k-j}f}, H^{j-p}\mu^{\frac12}, H^p\mu^{\frac12}), \widehat{H^k f})_{L^2_v}|\\
&\leq C\sum_{j=1}^{k}\sum_{p=0}^{j}\binom{k}{j}\binom{j}{p}(2T^\delta)^p p!\times[(2T^\delta)^{j-p}(j-p)!]\|\widehat{H^{k-j}f}(m)\|_{L^2_v}|||\widehat{H^k f}(m)|||\\
&\leq C\sum_{j=1}^{k}\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\|\widehat{H^{k-j}f}(m)\|_{L^2_v}|||\widehat{H^k f}(m)|||.\end{aligned}$$

Thus, for any $\varepsilon>0$,

$$\begin{aligned}&\int_{\mathbb{Z}^3}\Big[\sum_{j=1}^{k}\sum_{p=0}^{j}\binom{k}{j}\binom{j}{p}\int_0^T |(\widehat{\mathcal{T}}(\widehat{H^{k-j}f}, H^{j-p}\mu^{\frac12}, H^p\mu^{\frac12}), \widehat{H^k f})_{L^2_v}|\,dt\Big]^{\frac12} d\Sigma(m)\\
&\leq C\int_{\mathbb{Z}^3}\Big[\int_0^T\sum_{j=1}^{k}\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\|\widehat{H^{k-j}f}(m)\|_{L^2_v}|||\widehat{H^k f}(m)|||\,dt\Big]^{\frac12} d\Sigma(m)\\
&\leq \varepsilon\int_{\mathbb{Z}^3}\Big[\int_0^T |||\widehat{H^k f}(t,m)|||^2\,dt\Big]^{\frac12} d\Sigma(m)\\
&\quad+\frac{C}{\varepsilon}\int_{\mathbb{Z}^3}\Big[\int_0^T\Big\{\sum_{j=1}^{k}\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\|\widehat{H^{k-j}f}(m)\|_{L^2_v}\Big\}^2 dt\Big]^{\frac12} d\Sigma(m).\end{aligned}$$

As for the last term, recalling that $C_* > 4T^\delta$, we use the triangle inequality for norms to get

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\Big[\int_0^T\Big\{\sum_{j=1}^k\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\|\widehat{H^{k-j}f}(m)\|_{L^2_v}\Big\}^2dt\Big]^{\frac12}d\Sigma(m)\\
&\quad\le\int_{\mathbb{Z}^3}\sum_{j=1}^k\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\Big[\int_0^T\|\widehat{H^{k-j}f}(m)\|^2_{L^2_v}dt\Big]^{\frac12}d\Sigma(m)\\
&\quad\le T^{\frac12}\sum_{j=1}^k\frac{k!}{(k-j)!}(j+1)(2T^\delta)^j\int_{\mathbb{Z}^3}\sup_{0<t\le T}\|\widehat{H^{k-j}f}(t,m)\|_{L^2_v}d\Sigma(m)\\
&\quad\le 2\varepsilon_0T^{\frac12}T^\delta\sum_{j=1}^k\frac{k!}{(k-j)!}(j+1)(2T^\delta)^{j-1}\frac{C_*^{k-j}[(k-j)!]^\tau}{(k-j+1)^2}\le C\varepsilon_0\frac{C_*^{k-1}(k!)^\tau}{(k+1)^2},
\end{aligned}
$$

the last line using inductive assumption (3.2) and the last inequality following from a similar argument to that in (3.8) and (3.9). Combining the above estimates we conclude that

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\Big[\sum_{j=1}^k\sum_{p=0}^j\binom{k}{j}\binom{j}{p}\int_0^T|(\widehat{\mathcal{T}}(\widehat{H^{k-j}f},H^{j-p}\mu^{\frac12},H^p\mu^{\frac12}),\widehat{H^kf})_{L^2_v}|\,dt\Big]^{\frac12}d\Sigma(m)\\
&\quad\le\varepsilon\int_{\mathbb{Z}^3}\Big[\int_0^T|||\widehat{H^kf}(t,m)|||^2dt\Big]^{\frac12}d\Sigma(m)+C\varepsilon^{-1}\frac{\varepsilon_0C_*^{k-1}(k!)^\tau}{(k+1)^2},
\end{aligned}
$$

which with (3.15) yields

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\Big(\int_0^T|(\mathcal{F}_x(R_2(f)),\widehat{H^kf})_{L^2_v}|\,dt\Big)^{\frac12}d\Sigma(m)\\
&\qquad\le\varepsilon\int_{\mathbb{Z}^3}\Big[\int_0^T|||\widehat{H^kf}(t,m)|||^2dt\Big]^{\frac12}d\Sigma(m)+C\varepsilon^{-1}\frac{\varepsilon_0C_*^{k-1}(k!)^\tau}{(k+1)^2}.
\end{aligned}
$$

Similarly, using (2.9) instead of (2.8), we can verify that the above estimate still holds true with $R_2(f)$ replaced by $R_1(f)$. Thus the assertion in Proposition 3.3 follows by observing

$$
\begin{aligned}
&\int_{\mathbb{Z}^3}\Big(\int_0^T|(\mathcal{F}_x([H^k_\delta,\mathcal{L}]f),\widehat{H^k_\delta f})_{L^2_v}|\,dt\Big)^{\frac12}d\Sigma(m)\\
&\qquad\le\sum_{j=1}^2\int_{\mathbb{Z}^3}\Big(\int_0^T|(\mathcal{F}_x(R_j(f)),\widehat{H^kf})_{L^2_v}|\,dt\Big)^{\frac12}d\Sigma(m)
\end{aligned}
$$

due to (3.14). The proof is thus completed. ∎

4. Analytic smoothing effect for strong angular singularity

In this section we consider the case when the cross-section has strong angular singularity, that is, the number s in (1.4) satisfies that $\frac{1}{2} \leq s < 1$. This will yield the analytic regularity of weak solutions to the Boltzmann equation (1.5) at any positive time.

4.1. Quantitative estimate for directional derivatives

To get the analyticity of solutions at positive times, it relies on a crucial estimate on the derivatives in the direction H_δ defined in (1.25), with δ therein satisfying condition (1.26). In this section we will perform an energy estimate on the directional derivatives of regular solutions, and the treatment for the classical derivatives will be presented in the next subsection.

Theorem 4.1. *Assume that the cross-section satisfies* (1.3) *and* (1.4) *with* $\gamma \geq 0$ *and* $\frac{1}{2} \leq s < 1$. *Let* $T \geq 1$ *be arbitrarily given, and let* $f \in L^1_m L^\infty_T L^2_v$ *be any solution to the Cauchy problem* (1.5) *satisfying that, for any* $N \in \mathbb{Z}_+$ *and any* $\beta \in \mathbb{Z}^3_+$,

$$\int_{\mathbb{Z}^3} \Big(\sup_{0<t\leq T} t^{\frac{1+2s}{2s}(N+|\beta|)} |m|^N \|\partial_v^\beta \hat{f}(t,m,\cdot)\|_{L^2_v} \Big) \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T t^{\frac{1+2s}{s}(N+|\beta|)} |m|^{2N} |||\partial_v^\beta \hat{f}(t,m,\cdot)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) < +\infty. \tag{4.1}$$

Moreover, let H_δ *be defined by* (1.25) *with* δ *an arbitrary given number satisfying* (1.26). *Then there exists a sufficiently small constant* $\varepsilon_0 > 0$ *and a large constant* $L \geq 1$, *with* L *depending only on* T, δ *and the numbers* c_0, C_0 *in Section* 2, *such that if*

$$\int_{\mathbb{Z}^3} \Big(\sup_{0<t\leq T} \|\hat{f}(t,m)\|_{L^2_v} \Big) \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\hat{f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \varepsilon_0, \tag{4.2}$$

then the estimate

$$\int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H_\delta^k f}(t,m)\|_{L^2_v} \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H_\delta^k f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 L^k k!}{(k+1)^2} \tag{4.3}$$

holds true for any $k \in \mathbb{Z}_+$. *Moreover, the above estimate* (4.3) *is still true if we replace* H_δ *by*

$$\frac{1}{1+\delta} t^{\delta+1} \partial_{x_i} + t^\delta \partial_{v_i}$$

with $i = 2$ *or* 3.

Proof. To simplify the notation we will use the capital letter C to denote some generic constants, which may vary from line to line and which depend only on T, δ and the numbers c_0, C_0 in Section 2. Note that these generic constants C as below are independent of the derivative order denoted by k. If there is no confusion, in the following argument we will write $H = H_\delta$ for short, omitting the subscript δ.

We use induction on k to prove the quantitative estimate (4.3). The validity of (4.3) for $k = 0$ follows from (4.2) if we choose $L \geq 1$. Using the notation $H := H_\delta$ and supposing the estimate

$$\int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^j f}(t,m)\|_{L^2_v} \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^j f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 L^j j!}{(j+1)^2} \tag{4.4}$$

holds true for any $j \leq k-1$ with given $k \geq 1$, we will prove in the following argument that estimate (4.4) still holds true for $j = k$ provided $L \geq 4T^\delta$.

To do so we begin with the claim that the estimate

$$\int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k f}(t,m)\|_{L^2_v} \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) < +\infty \tag{4.5}$$

holds true for any $k \in \mathbb{Z}_+$. In fact, by Leibniz's formula we compute, for any $0 < t \leq T$ and for any $m \in \mathbb{Z}^3$,

$$\begin{aligned}
|||\widehat{H^k f}(t,m)||| &\leq C_{\delta,k} \sum_{j\leq k} t^{(1+\delta)j+\delta(k-j)} |m_1|^j \, |||\partial_{v_1}^{k-j} \hat{f}(t,m)||| \\
&\leq C_{\delta,k} \sum_{j\leq k} t^{\delta k + j} |m|^j \, |||\partial_{v_1}^{k-j} \hat{f}(t,m)||| \\
&\leq C_{\delta,k} (1+T)^k t^{(\delta - \frac{1+2s}{2s})k} \sum_{j\leq k} t^{\frac{1+2s}{2s}k} |m|^j \, |||\partial_{v_1}^{k-j} \hat{f}(t,m)|||,
\end{aligned}$$

and similarly,

$$\|\widehat{H^k f}(t,m)\|_{L^2_v} \leq C_{\delta,k} (1+T)^k t^{(\delta - \frac{1+2s}{2s})k} \sum_{j\leq k} t^{\frac{1+2s}{2s}k} |m|^j \, \|\partial_{v_1}^{k-j} \hat{f}(t,m)\|_{L^2_v}, \tag{4.6}$$

where $C_{\delta,k}$ is a constant depending only on k and δ. Then assertion (4.5) follows from assumption (4.1) by observing the fact that $\delta > \frac{1+2s}{2s}$.

Step (1) Applying H^k to equation (1.5) yields

$$\begin{aligned}
(\partial_t + v\cdot\partial_x + \mathcal{L}) H^k f &= -[H^k, \partial_t + v\cdot\partial_x] f - [H^k, \mathcal{L}] f + H^k \Gamma(f,f) \\
&= \delta k t^{\delta-1} \partial_{v_1} H^{k-1} f - [H^k, \mathcal{L}] f + H^k \Gamma(f,f),
\end{aligned}$$

the last equality using (1.28). Furthermore, we perform a partial Fourier transform in x and then consider the real part after taking the inner product of L^2_v with $\widehat{H^k f}$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{H^k f}\|^2_{L^2_v} &+ (\mathscr{L}\widehat{H^k f}, \widehat{H^k f})_{L^2_v}\\ &\leq \delta k t^{\delta-1}|(\partial_{v_1}\widehat{H^{k-1} f}, \widehat{H^k f})_{L^2_v}| + |(\mathscr{F}_x([H^k, \mathscr{L}]f), \widehat{H^k f})_{L^2_v}|\\ &\quad + |(\mathscr{F}_x(H^k\Gamma(f,f)), \widehat{H^k f})_{L^2_v}|.\end{aligned}$$

This with (2.1) yields that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{H^k f}\|^2_{L^2_v} &+ c_0 |||\widehat{H^k f}|||^2\\ &\leq \|\widehat{H^k f}\|^2_{L^2_v} + \delta k t^{\delta-1}|(\partial_{v_1}\widehat{H^{k-1} f}, \widehat{H^k f})_{L^2_v}|\\ &\quad + |(\mathscr{F}_x([H^k, \mathscr{L}]f), \widehat{H^k f})_{L^2_v}| + |(\mathscr{F}_x(H^k\Gamma(f,f)), \widehat{H^k f})_{L^2_v}|. \end{aligned}\tag{4.7}$$

For the second term on the right-hand side of (4.7), recalling that $\frac{1}{2} \leq s < 1$, it follows from (2.2) that

$$\begin{aligned}\delta k t^{\delta-1}|(\partial_{v_1}\widehat{H^{k-1} f}, \widehat{H^k f})_{L^2_v}| &\leq Ck\|\widehat{H^{k-1} f}\|_{H^s_v}\|\widehat{H^k f}\|_{H^s_v}\\ &\leq \frac{c_0}{2}|||\widehat{H^k f}|||^2 + Ck^2|||\widehat{H^{k-1} f}|||^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widehat{H^k f}\|^2_{L^2_v} + \frac{c_0}{2}|||\widehat{H^k f}|||^2 &\leq \|\widehat{H^k f}\|^2_{L^2_v} + Ck^2|||\widehat{H^{k-1} f}|||^2\\ &\quad + |(\mathscr{F}_x([H^k, \mathscr{L}]f), \widehat{H^k f})_{L^2_v}|\\ &\quad + |(\mathscr{F}_x(H^k\Gamma(f,f)), \widehat{H^k f})_{L^2_v}|. \end{aligned}\tag{4.8}$$

Together with Grönwall's inequality, we integrate the above estimate over $[0, t]$ for any $0 < t \leq T$; this implies that

$$\begin{aligned}\sup_{0<t\leq T}\|\widehat{H^k f}(t)\|^2_{L^2_v} &+ \int_0^T |||\widehat{H^k f}(t)|||^2\, dt\\ &\leq C\lim_{t\to 0}\|\widehat{H^k f}(t)\|^2_{L^2_v} + Ck^2\int_0^T |||\widehat{H^{k-1} f}|||^2\, dt\\ &\quad + C\int_0^T |(\mathscr{F}_x([H^k, \mathscr{L}]f), \widehat{H^k f})_{L^2_v}|\, dt\\ &\quad + C\int_0^T |(\mathscr{F}_x(H^k\Gamma(f,f)), \widehat{H^k f})_{L^2_v}|\, dt,\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{Z}^3} \sup_{0<t\le T} \|\widehat{H^k f}(t,m)\|_{L^2_v}\, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2\, dt \Big)^{\frac12} d\Sigma(m) \\
&\quad \le C \int_{\mathbb{Z}^3} \big[\lim_{t\to 0} \|\widehat{H^k f}(t,m)\|^2_{L^2_v} \big]^{\frac12} d\Sigma(m) \\
&\qquad + Ck \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^{k-1} f}(t,m)|||^2\, dt \Big)^{\frac12} d\Sigma(m) \\
&\qquad + C \int_{\mathbb{Z}^3} \Big(\int_0^T |(\mathcal{F}_x([H^k,\mathcal{L}]f), \widehat{H^k f})_{L^2_v}|\, dt \Big)^{\frac12} d\Sigma(m) \\
&\qquad + C \int_{\mathbb{Z}^3} \Big(\int_0^T |(\mathcal{F}_x(H^k\Gamma(f,f)), \widehat{H^k f})_{L^2_v}|\, dt \Big)^{\frac12} d\Sigma(m). \qquad (4.9)
\end{aligned}$$

We will proceed to deal with the terms on the right-hand side of (4.9).

Step (2) For the first term on the right-hand side of (4.9), we claim

$$\int_{\mathbb{Z}^3} \big[\lim_{t\to 0} \|\widehat{H^k f}(t,m)\|^2_{L^2_v} \big]^{\frac12} d\Sigma(m) = 0. \qquad (4.10)$$

In fact, by (4.1), we see that for each $j \in \mathbb{Z}_+$,

$$\sup_{0<t\le T} t^{\frac{1+2s}{2s}k} |m|^j \|\partial_{v_1}^{k-j} \hat f(t,m)\|_{L^2_v} \in L^1_m,$$

which implies

$$\sup_{m\in\mathbb{Z}^3} \big(\sup_{0<t\le T} t^{\frac{1+2s}{2s}k} |m|^j \|\partial_{v_1}^{k-j} \hat f(t,m)\|_{L^2_v} \big) \le C_{k,j} < +\infty,$$

where $C_{k,j}$ are constants depending only on k and j. As a result, combining the above estimate with (4.6) yields that, for any $0 < t \le T$ and any $m \in \mathbb{Z}^3$,

$$\|\widehat{H^k f}(t,m)\|_{L^2_v} \le C_{\delta,k}(1+T)^k t^{(\delta - \frac{1+2s}{2s})k} \sum_{j\le k} C_{k,j},$$

recalling that $C_{\delta,k}$ is a constant depending only on k and δ. This with condition (1.26) yields

$$\forall m \in \mathbb{Z}^3, \quad \lim_{t\to 0} \|\widehat{H^k f}(t,m)\|_{L^2_v} = 0,$$

and thus assertion (4.10) follows.

Step (3) For the second term on the right-hand side of (4.9), it follows from inductive assumption (4.4) that

$$k \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^{k-1} f}(t,m)|||^2\, dt \Big)^{\frac12} d\Sigma(m) \le \frac{\varepsilon_0 L^{k-1} k!}{k^2} \le C \frac{\varepsilon_0 L^{k-1} k!}{(k+1)^2} \qquad (4.11)$$

for $k \geq 1$. By assertion (4.5), which holds true for any $k \in \mathbb{Z}_+$, we see that condition (3.1) in Proposition 3.1 is fulfilled. Moreover, it follows from inductive assumption (4.4) that condition (3.2) holds with $C_* = L$ therein. This enables us to apply Propositions 3.1 and 3.3 to control the remaining terms on the right-hand side of (4.9); this gives that the estimate

$$\begin{aligned}
&\int_{\mathbb{Z}^3}\Big(\int_0^T |(\mathscr{F}_x([H^k,\mathscr{L}]f),\widehat{H^k f})_{L^2_v}|\,dt\Big)^{\frac12} d\Sigma(m)\\
&\qquad+\int_{\mathbb{Z}^3}\Big(\int_0^T |(\mathscr{F}_x(H^k\Gamma(f,f)),\widehat{H^k f})_{L^2_v}|\,dt\Big)^{\frac12} d\Sigma(m)\\
&\quad\leq C\varepsilon^{-1}\varepsilon_0\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H^k f}(t,m)\|_{L^2_v}\,d\Sigma(m)\\
&\qquad+(\varepsilon+C\varepsilon^{-1}\varepsilon_0)\int_{\mathbb{Z}^3}\Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2\,dt\Big)^{\frac12} d\Sigma(m)\\
&\qquad+C\varepsilon^{-1}\varepsilon_0^2\frac{L^k k!}{(k+1)^2}+C\varepsilon^{-1}\frac{\varepsilon_0 L^{k-1}k!}{(k+1)^2}
\end{aligned}\tag{4.12}$$

holds true for any $\varepsilon > 0$.

Step (4) Substituting estimates (4.10), (4.11) and (4.12) into (4.9), we obtain that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_{Z^3}\sup_{0<t\leq T}\|\widehat{H^k f}(t,m)\|_{L^2_v}\,d\Sigma(m)+\int_{Z^3}\Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2\,dt\Big)^{\frac12} d\Sigma(m)\\
&\quad\leq C\varepsilon^{-1}\varepsilon_0\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H^k f}(t,m)\|_{L^2_v}\,d\Sigma(m)\\
&\qquad+(\varepsilon+C\varepsilon^{-1}\varepsilon_0)\int_{\mathbb{Z}^3}\Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2\,dt\Big)^{\frac12} d\Sigma(m)\\
&\qquad+C\varepsilon^{-1}\varepsilon_0^2\frac{L^k k!}{(k+1)^2}+C\varepsilon^{-1}\frac{\varepsilon_0 L^{k-1}k!}{(k+1)^2}.
\end{aligned}\tag{4.13}$$

Note that ε_0 is a sufficiently small number, so we may assume without loss of generality that

$$C\varepsilon_0 \leq \frac{1}{16},\tag{4.14}$$

with $C > 0$ the constant in (4.13). Consequently, if we choose in particular $\varepsilon = \frac14$ in (4.13), then in view of (4.14) we have

$$\begin{aligned}
&\int_{Z^3}\sup_{0<t\leq T}\|\widehat{H^k f}(t,m)\|_{L^2_v}\,d\Sigma(m)+\int_{Z^3}\Big(\int_0^T |||\widehat{H^k f}(t,m)|||^2\,dt\Big)^{\frac12} d\Sigma(m)\\
&\quad\leq \frac12\frac{\varepsilon_0 L^k k!}{(k+1)^2}+8C\frac{\varepsilon_0 L^{k-1}k!}{(k+1)^2}\leq\frac{\varepsilon_0 L^k k!}{(k+1)^2},
\end{aligned}$$

provided L is large enough such that $L \geq 16C$. This yields the validity of (4.4) for $j = k$. Thus assertion (4.3) follows. The treatment for

$$\frac{1}{1+\delta} t^{\delta+1} \partial_{x_i} + t^{\delta} \partial_{v_i}, \quad i = 2 \text{ or } 3,$$

is just the same. The proof of Theorem 4.1 is completed. ∎

4.2. Proof of Theorem 1.1: Analytic regularization effect for $\frac{1}{2} \leq s < 1$

Here we prove Theorem 1.1 for the case $\frac{1}{2} \leq s < 1$, and it suffices to prove that for any $T \geq 1$ and any $\lambda > 1 + \frac{1}{2s}$, there exists a constant C, depending only on T, λ and the numbers c_0, C_0 in (2.1) and (2.3), such that

$$\forall \alpha, \beta \in \mathbb{Z}_+^3, \qquad \sup_{0<t\leq T} t^{(\lambda+1)|\alpha|+\lambda|\beta|} \|\partial_x^\alpha \partial_v^\beta f(t)\|_{L^2_{x,v}} \leq C^{|\alpha|+|\beta|+1}(|\alpha|+|\beta|)!. \quad (4.15)$$

The key part to proving (4.15) is the quantitative estimate (4.3). In the following discussion, let δ_j, $j = 1, 2$, be defined in terms of λ by (1.30). Accordingly, define H_{δ_j}, $j = 1, 2$, by (1.25).

Under the smallness condition (1.15), Duan–Liu–Sakamoto–Strain [22] obtained the global existence and uniqueness of the mild solution $f \in L^1_m L^\infty_T L^2_v$ to the Boltzmann equation (1.5), which satisfies that there exists a constant $C_1 > 0$ such that for any $T \geq 1$,

$$\int_{\mathbb{Z}^3} \Big(\sup_{0<t\leq T} \|\hat{f}(t,m)\|_{L^2_v} \Big) d\Sigma(m) + \int_{\mathbb{Z}^3} \left(\int_0^T |||\hat{f}(t,m)|||^2 \, dt \right)^{\frac{1}{2}} d\Sigma(m) \leq C_1 \epsilon. \quad (4.16)$$

Moreover, it is shown in [21] that the above mild solution admits Gevrey regularity at $t > 0$, that is, there exists a constant $C_2 > 0$ such that the estimate

$$\begin{aligned} &\int_{\mathbb{Z}^3} \Big(\sup_{0<t\leq T} \phi(t)^{\frac{1+2s}{2s}(N+|\beta|)} |m|^N \|\partial_v^\beta \hat{f}(t,m)\|_{L^2_v} \Big) d\Sigma(m) \\ &\quad + \int_{\mathbb{Z}^3} \left(\int_0^T \phi(t)^{\frac{1+2s}{s}(N+|\beta|)} |m|^{2N} |||\partial_v^\beta \hat{f}(t,m)|||^2 \, dt \right)^{\frac{1}{2}} d\Sigma(m) \\ &\leq C_2^{N+|\beta|+1} (N+|\beta|)^{\frac{1+2s}{2s}} \end{aligned}$$

holds true for any $N \in \mathbb{Z}_+$ and any $\beta \in \mathbb{Z}_+^3$, where $\phi(t) = \min\{t, 1\}$. Note that the constant C_1 in (4.16) is independent of T and the fact that

$$\forall k \in \mathbb{Z}_+, \ \forall 0 < t \leq T, \qquad t^{\frac{1+2s}{2s}k} \leq T^{\frac{1+2s}{2s}k} \phi(t)^{\frac{1+2s}{2s}k},$$

and thus conditions (4.1) and (4.2) are fulfilled by the above mild solution f, provided ϵ is small enough. This enables us to apply Theorem 4.1 to H_{δ_j}, $j = 1, 2$, given above, to conclude that for any $T \geq 1$, there exists a constant L, depending only on T, δ_1, δ_2 and the numbers c_0, C_0 in (2.1) and (2.3), such that for each $j = 1, 2$, the estimate

$$\int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k_{\delta_j} f}(t,m)\|_{L^2_v}\, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^k_{\delta_j} f}(t,m)|||^2\, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 L^k k!}{(k+1)^2} \tag{4.17}$$

holds true for any $k \in \mathbb{Z}_+$, where $\varepsilon_0 = C_1 \epsilon$ with C_1 the constant in (4.16). Observe that the discrete Lebesgue spaces ℓ^p are increasing in $p \in [1, +\infty]$, so that in particular $L^1_m \subset L^2_m$ for $m \in \mathbb{Z}^3$. Then it follows from (4.17) that, for any $k \in \mathbb{Z}_+$ and each $j = 1, 2$,

$$\begin{aligned} \sup_{0<t\leq T} \|H^k_{\delta_j} f(t)\|_{L^2_{x,v}} &= \sup_{0<t\leq T} \|\widehat{H^k_{\delta_j} f}(t)\|_{L^2_m L^2_v} \\ &\leq \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k_{\delta_j} f}(t,m)\|_{L^2_v}\, d\Sigma(m) \leq \frac{\varepsilon_0 L^k k!}{(k+1)^2}. \end{aligned} \tag{4.18}$$

Next we will deduce the estimate on classical derivatives. As a preliminary step, we first prove that, for any $k \in \mathbb{Z}_+$,

$$\|(A_1 + A_2)^k f\|_{L^2_{x,v}} \leq 2^k \|A_1^k f\|_{L^2_{x,v}} + 2^k \|A_2^k f\|_{L^2_{x,v}}, \tag{4.19}$$

where A_j, $j = 1, 2$, are two Fourier multipliers with symbols $a_j = a_j(m, \eta)$, that is,

$$\mathcal{F}_{x,v}(A_j f)(m,\eta) = a_j(m,\eta) \mathcal{F}_{x,v} f(m,\eta),$$

with $\mathcal{F}_{x,v} f$ the full Fourier transform in $(x, v) \in \mathbb{T} \times \mathbb{R}^3$. To prove (4.19) we compute

$$\begin{aligned} &|\mathcal{F}_{x,v}((A_1 + A_2)^k f)(m,\eta)|^2 \\ &\quad = |(a_1(m,\eta) + a_2(m,\eta))^k \mathcal{F}_{x,v} f(m,\eta)|^2 \\ &\quad \leq (|a_1(m,\eta)| + |a_2(m,\eta)|)^{2k} \times |\mathcal{F}_{x,v} f(m,\eta)|^2 \\ &\quad \leq 2^{2k} |a_1(m,\eta)^k \mathcal{F}_{x,v} f(m,\eta)|^2 + 2^{2k} |a_2(m,\eta)^k \mathcal{F}_{x,v} f(m,\eta)|^2 \\ &\quad \leq 2^{2k} |\mathcal{F}_{x,v}(A_1^k f)(m,\eta)|^2 + 2^{2k} |\mathcal{F}_{x,v}(A_2^k f)(m,\eta)|^2, \end{aligned}$$

the second inequality using the fact that $(p+q)^{2k} \leq (2p)^{2k} + (2q)^{2k}$ for any numbers $p, q \geq 0$ and any $k \in \mathbb{Z}_+$. As a result, we combine the above estimate with the Parseval equality, to conclude that

$$\begin{aligned} \|(A_1 + A_2)^k f\|^2_{L^2_{x,v}} &= \int_{\mathbb{Z}^3 \times \mathbb{R}^3} |\mathcal{F}_{x,v}((A_1 + A_2)^k f)(m,\eta)|^2\, d\Sigma(m)\, d\eta \\ &\leq 2^{2k} \int_{\mathbb{Z}^3 \times \mathbb{R}^3} |\mathcal{F}_{x,v}(A_1^k f)(m,\eta)|^2\, d\Sigma(m)\, d\eta \\ &\quad + 2^{2k} \int_{\mathbb{Z}^3 \times \mathbb{R}^3} |\mathcal{F}_{x,v}(A_2^k f)(m,\eta)|^2\, d\Sigma(m)\, d\eta \\ &\leq 2^{2k} \|A_1^k f\|^2_{L^2_{x,v}} + 2^{2k} \|A_2^k f\|^2_{L^2_{x,v}}. \end{aligned}$$

This gives (4.19). Now we use (1.32) and then apply (4.19) with

$$A_1 = \frac{(\delta_2+1)(\delta_1+1)}{\delta_2-\delta_1} H_{\delta_1},$$
$$A_2 = -\frac{(\delta_2+1)(\delta_1+1)}{\delta_2-\delta_1} t^{\delta_1-\delta_2} H_{\delta_2},$$

to compute that, observing $\delta_1 > \delta_2$,

$$\begin{aligned}\sup_{0<t\leq T} t^{(\lambda+1)k} \|\partial_{x_1}^k f(t)\|_{L^2_{x,v}} &= \sup_{0<t\leq T} \|(A_1+A_2)^k f(t)\|_{L^2_{x,v}} \\ &\leq 2^k \sup_{0<t\leq T} \|A_1^k f(t)\|_{L^2_{x,v}} + 2^k \sup_{0<t\leq T} \|A_2^k f(t)\|_{L^2_{x,v}} \\ &\leq C_3^k \sup_{0<t\leq T} (\|H_{\delta_1}^k f\|_{L^2_{x,v}} + \|H_{\delta_2}^k f\|_{L^2_{x,v}}),\end{aligned}$$

where C_3 is a constant depending only on T, δ_1, δ_2. Combining the above estimate with (4.18), we conclude that

$$\sup_{0<t\leq T} t^{(\lambda+1)k} \|\partial_{x_1}^k f(t)\|_{L^2_{x,v}} \leq \varepsilon_0 (2C_3L)^k k!.$$

Similarly, the above estimate is also true with ∂_{x_1} replaced by ∂_{x_2} or ∂_{x_3}. This, with the fact that

$$\forall \alpha \in \mathbb{Z}_+^3, \quad \|\partial_x^\alpha f\|_{L^2_{x,v}} \leq \sum_{1\leq j\leq 3} \|\partial_{x_j}^{|\alpha|} f\|_{L^2_{x,v}},$$

gives

$$\forall \alpha \in \mathbb{Z}_+^3, \quad \sup_{0<t\leq T} t^{(\lambda+1)|\alpha|} \|\partial_x^\alpha f(t)\|_{L^2_{x,v}} \leq \varepsilon_0 (6C_3L)^{|\alpha|} |\alpha|!.$$

In the same way we have

$$\forall \beta \in \mathbb{Z}_+^3, \quad \sup_{0<t\leq T} t^{\lambda|\beta|} \|\partial_v^\beta f(t)\|_{L^2_{x,v}} \leq \varepsilon_0 (6C_3L)^{|\beta|} |\beta|!.$$

Consequently, for any $\alpha, \beta \in \mathbb{Z}_+^3$,

$$\begin{aligned}&\sup_{0<t\leq T} t^{(\lambda+1)|\alpha|+\lambda|\beta|} \|\partial_x^\alpha \partial_v^\beta f(t)\|_{L^2_{x,v}} \\ &\quad\leq \sup_{0<t\leq T} (t^{2(\lambda+1)|\alpha|} \|\partial_x^{2\alpha} f(t)\|_{L^2_{x,v}})^{\frac{1}{2}} (t^{2\lambda|\beta|} \|\partial_v^{2\beta} f(t)\|_{L^2_{x,v}})^{\frac{1}{2}} \\ &\quad\leq \varepsilon_0 (6C_3L)^{|\alpha|+|\beta|} (|2\alpha|!|2\beta|!)^{\frac{1}{2}} \leq \varepsilon_0 (12C_3L)^{|\alpha|+|\beta|} (|\alpha|+|\beta|)!, \qquad (4.20)\end{aligned}$$

the last inequality using the fact that $p!q! \leq (p+q)! \leq 2^{p+q} p!q!$ for any $p, q \in \mathbb{Z}$. Thus the desired estimate (4.15) follows from (4.20) by choosing C large enough such that $C > 12C_3L + 1$. We have proven Theorem 1.1 for the strong angular singularity condition that $\frac{1}{2} \leq s < 1$.

5. Optimal Gevrey smoothing effect for mild angular singularity

This section focuses on the mild angular singularity case, i.e., $0 < s < \frac{1}{2}$ in (1.4). In this case, we can expect Gevrey class regularity with optimal Gevrey index $\frac{1}{2s}$.

Theorem 5.1. *Assume that the cross-section satisfies* (1.3) *and* (1.4) *with* $\gamma \geq 0$ *and* $0 < s < \frac{1}{2}$. *Let* $T \geq 1$ *be arbitrarily given, and let* $f \in L^1_m L^\infty_T L^2_v$ *be any solution to the Cauchy problem* (1.5) *satisfying* (4.1). *Moreover, let* λ *be an arbitrary given number satisfying* (1.26) *and let* H_{δ_1} *and* H_{δ_2} *be two vector fields defined by* (1.25), *with* δ_j *defined in terms of* λ *by* (1.30). *Then there exists a sufficiently small constant* $\varepsilon_0 > 0$ *and a large constant* $L \geq 1$, *with* L *depending only on* T, λ *and the numbers* c_0 *and* C_0 *in Section* 2, *such that if*

$$\int_{\mathbb{Z}^3} \big(\sup_{0<t\leq T} \|\hat{f}(t,m)\|_{L^2_v} \big) \, d\Sigma(m) + \int_{\mathbb{Z}^3} \Big(\int_0^T |||\hat{f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \varepsilon_0,$$

then the estimate

$$\sum_{1\leq j\leq 2} \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^k_{\delta_j} f}(t,m)\|_{L^2_v} \, d\Sigma(m) + \sum_{1\leq j\leq 2} \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^k_{\delta_j} f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 L^k (k!)^{\frac{1}{2s}}}{(k+1)^2} \tag{5.1}$$

holds true for any $k \in \mathbb{Z}_+$.

Sketch of the proof of Theorem 5.1. The proof is similar to that of Theorem 4.1. So for brevity we only sketch the proof, emphasizing the difference. In the following argument, we always assume that $0 < s < \frac{1}{2}$, and denote by C different generic constants, depending only on T, λ and the numbers c_0, C_0 in Section 2.

As in the previous section we use induction on k to prove (5.1). Suppose that for given $k \geq 1$, the estimate

$$\sum_{1\leq j\leq 2} \int_{\mathbb{Z}^3} \sup_{0<t\leq T} \|\widehat{H^\ell_{\delta_j} f}(t,m)\|_{L^2_v} \, d\Sigma(m) + \sum_{1\leq j\leq 2} \int_{\mathbb{Z}^3} \Big(\int_0^T |||\widehat{H^\ell_{\delta_j} f}(t,m)|||^2 \, dt \Big)^{\frac{1}{2}} d\Sigma(m) \leq \frac{\varepsilon_0 L^\ell (\ell!)^{\frac{1}{2s}}}{(\ell+1)^2} \tag{5.2}$$

holds true for any $\ell \leq k-1$. We will prove the above estimate is still valid for $\ell = k$. Repeating the argument before (4.7), we have the following estimate similar to (4.7):

$$\frac{1}{2}\frac{d}{dt} \sum_{1\leq j\leq 2} \|\widehat{H^k_{\delta_j} f}\|^2_{L^2_v} + c_0 \sum_{1\leq j\leq 2} |||\widehat{H^k_{\delta_j} f}|||^2 \leq \sum_{1\leq j\leq 2} \|\widehat{H^k_{\delta_j} f}\|^2_{L^2_v} + \sum_{1\leq j\leq 2} \delta_j k t^{\delta_j - 1} |(\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}, \widehat{H^k_{\delta_j} f})_{L^2_v}|$$

$$+ \sum_{1\leq j\leq 2} |(\mathscr{F}_x([H^k_{\delta_j}, \mathscr{L}]f), \widehat{H^k_{\delta_j} f})_{L^2_v}|$$

$$+ \sum_{1\leq j\leq 2} |(\mathscr{F}_x(H^k_{\delta_j}\Gamma(f,f)), \widehat{H^k_{\delta_j} f})_{L^2_v}|. \tag{5.3}$$

It suffices to deal with the second term on the right-hand side, since the other terms can be treated in the same way as in the previous case of $\frac{1}{2} \leq s < 1$.

For each $j = 1, 2$, and for any $\varepsilon > 0$, we have

$$\begin{aligned} & k t^{\delta_j - 1} |(\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}, \widehat{H^k_{\delta_j} f})_{L^2_v}| \\ & \leq k t^{\delta_j-1} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|_{H^{-s}_v} \|\widehat{H^k_{\delta_j} f}\|_{H^s_v} \\ & \leq \varepsilon |||\widehat{H^k_{\delta_j} f}|||^2 + C\varepsilon^{-1} k^2 t^{2(\delta_j-1)} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^{-s}_v}, \end{aligned} \tag{5.4}$$

the last inequality using (2.2). Moreover, recalling that $0 < 2s < 1$, we use the interpolation inequality

$$\forall \tilde{\varepsilon} > 0, \quad \|g\|^2_{H^{-s}_v} \leq \tilde{\varepsilon}\|g\|^2_{H^s_v} + \tilde{\varepsilon}^{-\frac{1-2s}{2s}} \|g\|^2_{H^{s-1}_v},$$

with $\tilde{\varepsilon} = \varepsilon^2 t^{2\delta_1} k^{-2} t^{-2(\delta_j-1)}$ and $g = \partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}$; this gives

$$\begin{aligned} & \varepsilon^{-1} k^2 t^{2(\delta_j-1)} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^{-s}_v} \\ & \leq \varepsilon t^{2\delta_1} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^s_v} + \varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} t^{\frac{1}{s}(\delta_j-1)} t^{-\frac{1-2s}{s}\delta_1} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^{s-1}_v} \\ & \leq \varepsilon \|t^{\delta_1} \partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^s_v} + C\varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} t^{\frac{1}{s}(\delta_j-1-(1-2s)\delta_1)} |||\widehat{H^{k-1}_{\delta_j} f}|||^2, \end{aligned} \tag{5.5}$$

the last inequality using (2.2) again. As for the last term on the right-hand side of (5.5), we use definition (1.30) of δ_j and the fact that $\delta_1 > \delta_2$ in view of (1.31), to compute, for $j = 1, 2$,

$$\delta_j - 1 - (1-2s)\delta_1 \geq \delta_2 - 1 - (1-2s)\delta_1 \geq 2s + (1-2s)\lambda - (1-2s)\lambda \geq 0,$$

which yields

$$\forall 0 < t \leq T, \quad \varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} t^{\frac{1}{s}(\delta_j-1-(1-2s)\delta_1)} |||\widehat{H^{k-1}_{\delta_j} f}|||^2 \leq C\varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} |||\widehat{H^{k-1}_{\delta_j} f}|||^2,$$

and thus, substituting the above inequality into (5.5),

$$\varepsilon^{-1} k^2 t^{2(\delta_j-1)} \|\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^{-s}_v} \leq \varepsilon \|t^{\delta_1} \partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^s_v} + C\varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} |||\widehat{H^{k-1}_{\delta_j} f}|||^2.$$

Consequently, we combine the above estimate with (5.4) to obtain that, for any $\varepsilon > 0$ and any $t \in]0, T]$,

$$\begin{aligned} & k t^{\delta_j-1} |(\partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}, \widehat{H^k_{\delta_j} f})_{L^2_v}| \\ & \leq \varepsilon |||\widehat{H^k_{\delta_j} f}|||^2 + \varepsilon \|t^{\delta_1} \partial_{v_1} \widehat{H^{k-1}_{\delta_j} f}\|^2_{H^s_v} + C\varepsilon^{\frac{s-1}{s}} k^{\frac{1}{s}} |||\widehat{H^{k-1}_{\delta_j} f}|||^2. \end{aligned} \tag{5.6}$$

As for the second term on the right-hand side of (5.6), we first use the second equation in (1.32) and then the fact that

$$\forall (m,\eta)\in\mathbb{Z}^3\times\mathbb{R}^3,\quad |a(m,\eta)b(m,\eta)^{k-1}|^2\leq |a(m,\eta)|^{2k}+|b(m,\eta)|^{2k},$$

to compute

$$\begin{aligned}\|t^{\delta_1}\partial_{v_1}\widehat{H_{\delta_j}^{k-1}f}\|_{H_v^s}^2 &= \|\mathcal{F}_x(t^{\delta_1}\partial_{v_1}H_{\delta_j}^{k-1}f)\|_{H_v^s}^2\\ &\leq C\|\mathcal{F}_x(H_{\delta_1}H_{\delta_j}^{k-1}f)\|_{H_v^s}^2+C\|\mathcal{F}_x(H_{\delta_2}H_{\delta_j}^{k-1}f)\|_{H_v^s}^2\\ &\leq C\|\widehat{H_{\delta_1}^k f}\|_{H_v^s}^2+C\|\widehat{H_{\delta_2}^k f}\|_{H_v^s}^2\leq C\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^k f}|||^2,\end{aligned}$$

the last inequality following from (2.2). Substituting the above estimate into (5.6) we conclude that, for any $\varepsilon>0$ and for each $j=1,2$,

$$kt^{\delta_j-1}|(\partial_{v_1}\widehat{H_{\delta_j}^{k-1}f},\widehat{H_{\delta_j}^k f})_{L_v^2}|\leq C\varepsilon\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^k f}|||^2+C\varepsilon^{\frac{s-1}{s}}k^{\frac1s}|||\widehat{H_{\delta_j}^{k-1}f}|||^2,$$

which with (5.3) yields that, for any $\varepsilon>0$,

$$\begin{aligned}&\frac12\frac{d}{dt}\sum_{1\leq j\leq 2}\|\widehat{H_{\delta_j}^k f}\|_{L_v^2}^2+c_0\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^k f}|||^2\\ &\quad\leq \sum_{1\leq j\leq 2}\|\widehat{H_{\delta_j}^k f}\|_{L_v^2}^2+C\varepsilon\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^k f}|||^2+C\varepsilon^{\frac{s-1}{s}}k^{\frac1s}\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^{k-1}f}|||^2\\ &\qquad+\sum_{1\leq j\leq 2}|(\mathcal{F}_x([H_{\delta_j}^k,\mathcal{L}]f),\widehat{H_{\delta_j}^k f})_{L_v^2}|+\sum_{1\leq j\leq 2}|(\mathcal{F}_x(H_{\delta_j}^k\Gamma(f,f)),\widehat{H_{\delta_j}^k f})_{L_v^2}|.\end{aligned}$$

Letting ε above be sufficiently small, we get that

$$\begin{aligned}&\frac12\frac{d}{dt}\sum_{1\leq j\leq 2}\|\widehat{H_{\delta_j}^k f}\|_{L_v^2}^2+\frac{c_0}{2}\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^k f}|||^2\\ &\quad\leq \sum_{1\leq j\leq 2}\|\widehat{H_{\delta_j}^k f}\|_{L_v^2}^2+Ck^{\frac1s}\sum_{1\leq j\leq 2}|||\widehat{H_{\delta_j}^{k-1}f}|||^2\\ &\qquad+\sum_{1\leq j\leq 2}|(\mathcal{F}_x([H_{\delta_j}^k,\mathcal{L}]f),\widehat{H_{\delta_j}^k f})_{L_v^2}|+\sum_{1\leq j\leq 2}|(\mathcal{F}_x(H_{\delta_j}^k\Gamma(f,f)),\widehat{H_{\delta_j}^k f})_{L_v^2}|.\end{aligned}$$

Note that the above estimate is quite similar to (4.8), with the factor k^2 therein replaced by $k^{\frac1s}$ here. Moreover, observe that

$$\begin{aligned}&k^{\frac{1}{2s}}\sum_{1\leq j\leq 2}\int_{\mathbb{Z}^3}\Big(\int_0^T|||\widehat{H_{\delta_j}^{k-1}f}(t,m)|||^2\,dt\Big)^{\frac12}d\Sigma(m)\\ &\quad\leq k^{\frac{1}{2s}}\frac{\varepsilon_0 L^{k-1}[(k-1)!]^{\frac{1}{2s}}}{k^2}\leq C\frac{\varepsilon_0 L^{k-1}(k!)^{\frac{1}{2s}}}{(k+1)^2},\end{aligned}$$

which just follows from inductive assumption (5.2). Thus we may repeat the argument after (4.8) and use the above estimate instead of (4.11), to conclude that

$$\sum_{1\leq j\leq 2}\Big[\int_{\mathbb{Z}^3}\sup_{0<t\leq T}\|\widehat{H_{\delta_j}^k f}(t,m)\|_{L_v^2}\,d\Sigma(m) + \int_{\mathbb{Z}^3}\Big(\int_0^T |||\widehat{H_{\delta_j}^k f}(t,m)|||^2\,dt\Big)^{\frac{1}{2}}\,d\Sigma(m)\Big] \leq \frac{\varepsilon_0 L^k (k!)^{\frac{1}{2s}}}{(k+1)^2}.$$

Then (5.2) holds for $\ell = k$, and thus (5.1) follows. The proof of Theorem 5.1 is completed. ∎

Completing the proof of Theorem 1.1: Gevrey smoothing effect for $0 < s < \frac{1}{2}$. With the help of (5.1), the Gevrey estimate (1.17) for $0 < s < \frac{1}{2}$ just follows from the same argument as that in Section 4.2. So we omit it for brevity. ∎

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