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Properties of non-equilibrium steady states for the non-linear BGK equation on the torus

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Abstract. We study the non-linear BGK model in one dimension coupled to a spatially varying thermostat. We show existence, local uniqueness, and linear stability of a steady state when the linear coupling term is large compared to the non-linear self-interaction term. This model possesses a non-explicit spatially dependent non-equilibrium steady state. For the existence and the local uniqueness we utilise a fixed point argument, reducing the study of the non-linear model to the linear BGK model, while for the stability we adapt the L^2 hypocoercivity theory, yielding existence of a spectral gap for the linearised operator around this non-equilibrium steady state.

1. Introduction

Properties of non-equilibrium steady states (NESS) in systems in contact with thermal reservoirs (thermostats) remain major open problems in statistical mechanics. Key questions include the existence, uniqueness, and structure of these states – whose analytic expression is not explicitly known – as well as the rate at which out-of-equilibrium systems converge towards them.

There are a few classical open systems at the microscopic level that have been introduced and studied to understand and derive the macroscopic Fourier law of heat conduction from microscopic Hamiltonian dynamics, for which there are some partial answers. A particular class of models that has attracted a lot of attention in the mathematical and theoretical physics community on thermal transport is the one-dimensional chain of atoms/oscillators with nearest neighbour interactions, perturbed at the boundaries with two reservoirs at different temperatures. Harmonic such crystals is a case well studied, where the non-equilibrium steady state is explicit but it corresponds to a rather unphysical scenario where Fourier's law breaks down [40]. Regarding anharmonic crystals, even though there are partial answers in certain cases on the non-equilibrium steady state – existence, uniqueness, and exponential relaxation are ensured even quantitatively in the number of particles in certain cases – very little is known on the structure of such states. Apart from microscopic oscillator chains, existence and uniqueness are also provided for

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a system of Newtonian particles with a long-range repulsive interaction potential [29]. Again, the situation is far from well understood regarding a *canonical* description of the state.

The same is true for systems in the mesoscopic level, i.e. kinetic models described by a one-particle distribution function $f(t,x,v)$, where $t, x \in \Omega \subset \mathbb{R}^d$, $v \in \mathbb{R}$ are the time, space, and velocity variables respectively. In particular, there are no explicit solutions of the stationary (time-independent) Boltzmann equation in the more realistic non-equilibrium scenario and in many cases even the existence of such states is not known.

In this article we study the non-linear BGK model of the Boltzmann equation, which is a kinetic relaxation model introduced by Bhatnagar, Gross, and Krook [9] as a toy model for Boltzmann flows. We continue our investigation as in [27], where we showed existence of a NESS for the one-dimensional model on the interval $(0,1)$ with diffusive boundary conditions playing the role of two thermal reservoirs at the boundaries.

Here we change the boundary conditions to periodic boundaries, meaning that we consider a gas of particles on the one-dimensional torus $\mathbb{T}$. Our objective is to study existence, uniqueness, and $L^2_{x,v}$ stability for the non-linear BGK model coupled to a linear BGK operator with spatially varying temperature – now being a thermostat that acts all over the space.

1.1. Description of the model

We consider a gas of particles on the one-dimensional torus $\mathbb{T}$ where the collisions among the particles are represented by the non-linear BGK operator.

$$\partial_t f + v\partial_x f = \mathcal{L} f = \frac{1}{\kappa}\big(\rho_f(\alpha \mathcal{M}_{u_f,T_f} + (1-\alpha)\mathcal{M}_{\tau(x)}) - f\big), \tag{1.1}$$

where $x \in \mathbb{T}$, $v \in \mathbb{R}$, $\tau(x)$ is a fixed function which varies with x, and $\alpha \in [0,1]$. Also, with κ we denote the Knudsen number, i.e. the ratio between the mean free path and the typical observation length.

For a stationary solution $f(x,v)$ on the phase space $\mathbb{T} \times \mathbb{R}$ we define the hydrodynamic moments, the spatial density $\rho_f(x)$, the bulk velocity $u_f(x)$, and the pressure $P_f(x)$ respectively as

$$\rho_f(x) = \int_{-\infty}^{\infty} f(x,v)\,\mathrm{d}v, \quad \rho_f(x)u_f(x) = \int_{-\infty}^{\infty} v f(x,v)\,\mathrm{d}v,$$
$$P_f(x) = \int_{-\infty}^{\infty} v^2 f(x,v)\,\mathrm{d}v = \rho_f(x)[T_f(x) + u_f(x)^2],$$

and then the local temperature profile corresponding to f is T_f. We denote by $\mathcal{M}_{u_f,T_f}(v)$ the Maxwellian with temperature T_f:

$$\mathcal{M}_{u_f,T_f}(v) = (2\pi T_f)^{-1/2}\exp\Big(-\frac{|v-u_f|^2}{2T_f}\Big).$$

Finally, we write $\mathcal{M}_{\tau(x)} := \mathcal{M}_{0,\tau(x)}$ for the Maxwellian with the spatially varying temperature $\tau(x)$.

Our main objectives are (i) to determine the range of the parameters α, κ, and $\underline{\tau} \leq \tau \leq \bar{\tau}$ for which we have existence of a non-equilibrium stationary state $g(x, v)$ to (1.1), (ii) to show uniqueness of such a state for small α, and (iii) to prove L^2 linear stability around g via adaptations of hypocoercivity methods in this non-equilibrium scenario.

1.1.1. Motivation and state of the art. The main goal of our study is to understand fundamental properties – such as existence, uniqueness, and possible stability – of non-equilibrium steady states in particle systems in kinetic theory. There are many works in this direction both for BGK and Boltzmann equations.

On BGK models: Boltzmann-type models with several thermostats acting either at the boundaries or in the whole space have received a lot of attention in recent years. In particular, in [14, 15] the authors study the non-linear BGK model on the torus with periodic boundary conditions when scatterers at two different temperatures are imposed. In the cases covered there, one can find an explicit steady state which is spatially homogeneous and which is also unique. However, our linear thermostat, in this article, is spatially non-uniform, meaning that our non-equilibrium steady state is non-explicit, spatially inhomogeneous, and there is complex behaviour of the hydrodynamic quantities.

In [27] we also started studying the non-linear BGK equation on the domain $(0, 1)$ when the boundary conditions are diffusive, meaning that whenever a particle hits one of the boundaries it is reflected back in the domain and its velocity is “thermalised” according to the temperature of the boundary. We studied the case where the boundary temperatures are sufficiently away from each other, which is an extreme non-equilibrium regime – for a fixed Knudsen number – and we showed existence of a steady state. A very similar scenario for the non-linear quadratic BGK model with large boundary data was studied in [41] also providing existence, but the prescription of the boundary data is different.

On Boltzmann models: For the Boltzmann equation on bounded domains out-of-equilibrium there are many contributions in the perturbative case around the equilibrium, that is, when the boundary temperatures are close to the equilibrium at some uniform temperature. In [22] the authors constructed a unique steady state in the kinetic regime, i.e. finite Knudsen number, in the neighbourhood of the equilibrium and proved dynamical stability, generalising and expanding existence and uniqueness results in [31, 32] on convex domains. Existence and stability results on the NESS of the Boltzmann equation expanding around a small Knudsen number include [1–3]. Moreover, existence of non-equilibrium steady states to the Boltzmann equation in the slab under diffuse reflection boundary conditions in a non-perturbative setting was proven in [4].

Regarding the spatially homogeneous Boltzmann equation in the presence of scatterers and Kac’s toy model, existence, uniqueness, and exponential relaxation towards the NESS can be found in [16] and in [26], respectively.

In the context of deriving Fourier's law, i.e. a heat conduction law stating that the heat flux vector is proportional to the gradient of the temperature with the proportionality to be the thermal conductivity of the material, one needs to let the Knudsen number to go to 0, allowing thus a large number of collisions per unit time and establishing a hydrodynamic regime. Historically, that was obtained formally by Boltzmann and Maxwell [10,37] and a rigorous proof was given in [23,24] in the slab geometry, in the close to equilibrium case. For a collection of the recent works on the stationary Boltzmann equation of bounded general domains on the perturbative close-to-equilibrium regime for both finite and close-to-0 Knudsen number, we suggest the recent survey [25].

The question of deriving Fourier's law from a deterministic particle system in the microscopic level remains a major open question in statistical mechanics [11, 19, 28, 36], and recent works include studies of harmonic as well as anharmonic oscillator chains in contact with thermal reservoirs at the boundaries [5, 6, 17, 18, 21, 33, 34, 38–40]. These works provide existence and uniqueness of a NESS for a large class of interaction potentials, exponential relaxation towards the NESS, and in some more specific cases quantitative convergence rates as a function of the number of particles. Fourier's law has been shown to hold in cases of harmonic atom chains perturbed by a conservative stochastic dynamics as considered in [8].

1.2. Notation

We write $A \lesssim B$ in order to say that $A \leq CB$ for some finite constant C independent of α. We also write $f(x) = \mathcal{O}(g(x))$ to denote that there is a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ and $f(z) = o(g(z))$ for $z \to z_0$ if there is, for any $\varepsilon > 0$, a neighbourhood U_ε of z_0 such that $|f(z)| \leq \varepsilon |g(z)|$. By $L^2_{x,v}$ we denote the space with functions that are in L^2 for both space and velocity variables x, v. Finally, for simplicity we write $\mathcal{M}_{T_f}(v)$ for $\mathcal{M}_{0,T_f}(v)$.

1.3. Plan of the paper

In the rest of this section we state the main results. In Section 2 we prove the existence and uniqueness of the steady for state the full non-linear problem. This is split into subsections with the basic steps of the proof: in Section 2.2 we prove existence and uniqueness for the linear version of the model, in Section 2.3 we provide upper and lower bounds on the moments of the NESS, and the requirements for the fixed point theorem are proved in Section 2.4. In Section 3 we prove $L^2_{x,v}$ stability of the linearised operator around the NESS, by first proving a microscopic coercivity, i.e. in the v variable (Section 3.3), a macroscopic coercivity, i.e. in the x variable (Section 3.4), and boundedness of certain operators (Section 3.5), introduced to modify the entropy and to run our hypocoercivity argument.

1.3.1. Techniques and main results. Our first main theorem gives existence and a qualified uniqueness result for a steady state.

Theorem 1 (Existence and local uniqueness). *Suppose there exist constants $\underline{\tau}$ and $\bar{\tau}$ such that $\underline{\tau} \le \tau(x) \le \bar{\tau}$ for all $x \in \mathbb{T}$. Then if $\underline{\tau}$ is sufficiently large and α is sufficiently small (depending on $\underline{\tau}$, $\bar{\tau}$), there exists a steady state $g = g(x, v)$ to the non-linear equation* (1.1)*. Moreover, there exist constants $\underline{T}$ and $\overline{T}$ depending on α, $\underline{\tau}$, $\bar{\tau}$ such that the temperature profile corresponding to this steady state $T_g(x) = \int_{-\infty}^{\infty} v^2 g(x,v)\,\mathrm{d}v / \int_{-\infty}^{\infty} g(x,v)\,\mathrm{d}v$ satisfies*

$$\underline{T} \le T_g(x) \le \overline{T}.$$

Finally, g is the unique steady state in the class of functions g satisfying this constraint.

This theorem is proved by a contraction mapping argument on the temperature. We first show that for a specific set of parameters α, $\underline{\tau}$, $\bar{\tau}$, upper and lower bounds on the temperature, pressure, and spatial density are preserved. Then in this set of parameters we show that the mapping on the temperature is in fact contractive. To construct the map we freeze the non-linearity in the equation to produce a linear equation which can be related to a Markov process. We then use this to show that the linear equation has a unique steady state for which we can define its temperature. This gives us a map from one temperature to another. Our estimates showing bounds on the temperatures rely on a representation of the steady state of this linear equation found through Duhamel's formula. This is an adaption of our techniques from [27] where we performed a Schauder fixed point argument on a similar equation.

Theorem 2 (Linear stability of the steady state). *For $\underline{\tau}$ sufficiently large so that the result of Theorem* 1 *applies and α sufficiently small, the linear operator found by linearising* (1.1) *around g has a spectral gap in $L^2(g^{-1})$.*

This theorem is proved by adapting the L^2-based hypocoercivity strategy of Dolbeault–Mouhot–Schmeiser [20]. The novelty here is that we push forward the hypocoercivity techniques to an out-of-equilibrium system, meaning that we deal with a non-explicit, spatially inhomogeneous, steady state. Therefore we work on a functional space, $L^2(g^{-1})$, with non-explicit weights. The L^2 hypocoercivity theory relies on verifying assumptions. In our case the most challenging is the *microscopic coercivity* assumption. Difficulties in verifying this assumption are related to the fact that the non-equilibrium nature of our dynamics mean there are complex behaviours relating to flow of macroscopic quantities which appear as potentially growing modes in the space spanned by g, vg, $v^2 g$. We are able to control these terms when α is small. Doing this involves bounding the normalised third and fourth moments (skewness and kurtosis) of the steady state g that we find.

Remark 1 (On the Knudsen number). Both Theorems 1 and 2 require a finite Knudsen number κ of order $\mathcal{O}(1)$, meaning that we stay on the kinetic–mesoscopic level, without passing to the hydrodynamic regime. We remark that this was also the case for the steady state we found in [27] when the model was subject to diffusive boundary conditions at two different – sufficiently large – temperatures far away from each other.

Remark 2 (Discussion for the hydrodynamic regime). The BGK operator is of special significance in kinetic theory when we are close to the hydrodynamic limit, which occurs as the Knudsen number $\kappa \to 0$. In our previous work on a bounded interval [27], this would bring us close to the situation where the steady state f approximates a Maxwellian distribution whose moments solve a stationary fluid equation with boundary conditions set at the walls. In the situation of the current paper it means we are close to $\mathcal{M}_{\tau(x)}$. In this situation we expect we could establish existence, uniqueness, and hypocoercivity arguments by expressing the solution as $f = \mathcal{M}_{\tau(x)} + \kappa h$, where κh represents a small perturbation. This approach would involve a different set of arguments from those presented in this work, likely being more straightforward since the solution would be near a Maxwellian distribution.

Remark 3 (Comparison with [27]). Regarding the stability and uniqueness of the steady state that is provided here for small α, let us remark that it is due to the thermostat at temperature $\tau(x)$ acting all over the space. This term is helping the system to be stabilised as it imposes constraints everywhere in the space. That comes in contrast with the case when the thermostats act just on the boundaries, as in [27]. This means that the steady state found for the system studied in [27] when the difference of the boundary temperatures is large, could possibly be unstable.

Remark 4 (Discussion of larger values of α). If we were to consider α larger, we are presently unsure whether or not stability would be retained. Even for values of α very close to 1, where the equation approximates the standard BGK operator, the situation is still not entirely clear. For this equation the momentum and kinetic energy are preserved and the kernel of the linearised BGK operator will contain terms not fixed by the very small thermostat term.

Nevertheless, we emphasise that the existence of a steady state is expected to hold for all $\alpha \in [0, 1]$. In this sense our statements on existence are not fully optimised, as small values of α are required for the stability argument later on in any case.

1.3.2. Discussion and open problems. There are several natural next steps from this work. For the model studied in the paper two natural next steps would be

- studying the non-linear stability of this equation by first utilising the toolbox in [30] to prove weighted L^∞ stability for the linearised equation;
- looking at the case of small Knudsen number (κ) where we expect that one should be able to use the fact that the steady state will be close to Maxwellian.

We comment that we could also look at the situation where τ does not vary very much. That is to say, $|\bar{\tau} - \underline{\tau}|$ is small and we can weight by the Maxwellian at a fixed temperature. In fact, we expect this case to be rather straightforward. We have included a brief proof of linear stability in the appendix under the assumption that the temperature of the steady state is close to a uniform temperature, close to $\bar{\tau}$, $\underline{\tau}$ in this case. We expect that this assumption could be verified by arguments similar to our contraction mapping argument to show that the temperature of the steady state is a Lipschitz function of $\tau(x)$ when τ is considered as a function in L^∞. These would be more involved than the linear stability.

We would like to be able to perform a linear stability analysis similar to the one in this paper for the equation studied in [27] with diffusive boundary conditions. For this equation it is not at all clear that the steady state found is stable when the difference between the boundary temperatures is large. This comment is also made in the related paper of Ukai [41]. Briefly, if we try and run an argument similar to the linear stability argument in this paper we are not able to verify the *microscopic coercivity assumption* and we can in fact show that it is possible for the $L^2(g^{-1})$ norm to grow for certain initial data. This is caused by complex behaviour of the hydrodynamic quantities and these terms cannot be controlled by terms coming from the thermostat since it only acts on the boundary.

2. Existence and uniqueness

2.1. Overview of the strategy

In this section we prove Theorem 1, which is split into several steps. We provide a summary of the strategy of the proof before giving all the details.

Step 1. For any $T = T(x) > 0$, it is proved that there exists a unique probability density $g = g^T$ satisfying

$$v\partial_x g = L_T g := \frac{1}{\kappa}[\rho_g(\alpha M_{T(x)} + (1-\alpha)M_{\tau(x)}) - g],$$

which is obtained through Doeblin's theorem applied to the stochastic process associated with the *linear* BGK relaxation equation

$$\partial_t f + v\partial_x f = L_T f. \tag{2.1}$$

The main technical argument is to establish Doeblin's condition, namely a positive lower bound on the solutions to (2.1).

Step 2. Given the existence of g^T from Step 1, we then continue by defining the mapping on $C(\mathbb{T})$,

$$\mathcal{F} : T \mapsto T_{g^T},$$

and proving that it is a contraction under a suitable smallness condition on α and a suitable lower bound on $\underline{\tau} \leq \tau(x)$. This means that $\mathcal{F}$ admits a unique fixed point thanks to the Banach fixed point theorem. This fixed point g is the stationary state to the BGK equation (1.1).

The proofs are based on several technical estimates established on g^T (in Lemma 1), on ρ_{g^T} (in Lemma 2), on T_{g^T} (in Lemma 3), and on $\mathcal{F}$ (in Proposition 2).

We write $L = L_T$ to simplify the notation and we write $g = g^T(x, v)$ to be the steady state of (2.1) corresponding to the temperature profile $T \in C(\mathbb{T})$. First we prove the existence and uniqueness of such a state through Doeblin's theorem.

2.2. Doeblin's argument for existence and uniqueness of a steady state to the linear equation

We need to show the existence and uniqueness of a steady state for the linear PDE (2.1). We first construct a stochastic process, the law of which is a weak solution of the PDE (2.1). Then using Doeblin's theorem from Markov process theory we show existence and uniqueness of the steady state.

To construct the stochastic process, first let us generate a Poisson process with rate $\frac{1}{\kappa}$ and call $J_1, J_2, J_3, \dots$ the jump times of the Poisson process. In order to construct the stochastic process X_t, V_t we proceed iteratively: Suppose we have it up to time J_i; then for

$$J_i < t \leq J_{i+1}, \quad \text{we set } X_t = X_{J_i} + (t - J_i)V_{J_i}$$

and

$$J_i < t < J_{i+1}, \quad V_t = V_{J_i},$$

while for $t = J_{i+1}$ we generate a new velocity independent of everything except $X_{J_{i+1}}$, drawn to have density $\alpha \mathcal{M}_{T(X_{J_{i+1}})} + (1-\alpha)\mathcal{M}_{\tau(X_{J_{i+1}})}$. It is straightforward to check that this gives the right equation: By taking a test function ϕ on $C_c^\infty((0,\infty) \times \mathbb{T} \times \mathbb{R})$, Taylor expanding the quantity $s^{-1}\mathbb{E}(\phi(t+s, X_{t+s}, V_{t+s}) - \phi(t, X_t, V_t))$, and taking the limit as $s \to 0$, we indeed recover a weak solution of our PDE (2.1).

Before applying Doeblin's theorem, let us first give a general statement of it.

Theorem 3 (Doeblin's theorem). *Let Z_t be a continuous time Markov process on a state space $\mathcal{Z}$. Let us write $f_t^{z_0}$ for the law of Z_t conditional on $Z_0 = z_0$. Suppose that there exist a time $t_* > 0$, a constant $\beta \in (0,1)$, and a probability measure ν on $\mathcal{Z}$ so that for any $z_0 \in \mathcal{Z}$ we have*

$$f_{t_*}^{z_0} \geq \beta \nu.$$

Then the Markov process has a unique steady state.

In order to prove the conditions of Doeblin's theorem for our particular Markov process, the most convenient way is to look at the PDE form of the equation.

Proposition 1. *For the linear equation with $f_0 = \delta_{x_0,v_0}$ we have $\beta \in (0,1)$ and ν a probability density such that*

$$f_t \geq \beta \nu.$$

Proof. First, since τ, T are bounded above and below we can find $c > 0$ so that

$$\alpha \mathcal{M}_{T(x)}(v) + (1-\alpha)\mathcal{M}_{\tau(x)}(v) \geq c 1_{|v| \leq 1}$$

uniformly in x. Then Duhamel's formula yields the lower bound

$$\begin{aligned} f(t,x,v) \geq {} & e^{-t/\kappa} f(0, x - vt, v) \\ & + \frac{1}{\kappa} \int_0^t e^{-(t-s)/\kappa} \int f(s, x - v(t-s), u) \, \mathrm{d}u \\ & \qquad \times (\alpha \mathcal{M}_{T(x - v(t-s))}(v) + (1-\alpha)\mathcal{M}_{\tau(x - v(t-s))}(v)) \, \mathrm{d}s. \end{aligned}$$

Then using the notation $(S_t f)(x, v) = f(x - tv, v)$, i.e. the semigroup of the transport part, and $P_c f := c1_{|v|\leq 1} \int_{\mathbb{R}} f(x, u)\,\mathrm{d}u$ we write

$$f(t, x, v) \geq e^{-t/\kappa} S_t f_0 + \frac{1}{\kappa} \int_0^t e^{-(t-s)/\kappa} S_{t-s} P_c f_s \,\mathrm{d}s.$$

This implies that $f(t, x, v) \geq e^{-t/\kappa} S_t f_0$. Now, by substituting this into the second term of the equation, we get

$$f(t, x, v) \geq \frac{1}{\kappa} \int_0^t e^{-t/\kappa} S_{t-s} P_c S_s f_0 \,\mathrm{d}s,$$

and by substituting the new lower bound in again we get

$$f(t, x, v) \geq \frac{1}{\kappa^2} \int_0^t \int_0^s e^{-t/\kappa} S_{t-s} P_c S_{s-r} P_c S_r f_0 \,\mathrm{d}r\,\mathrm{d}s.$$

Now, by our initial assumption $f_0 = \delta_{(x_0, v_0)}$, we have that

$$S_r \delta_{x_0, v_0} = \delta_{x_0 + r v_0, v_0}.$$

Therefore we have

$$\begin{aligned} P_c S_r \delta_{x_0, v_0} &= c\delta_{x_0 + r v_0} 1_{|v|\leq 1}, \\ S_{s-r} P_c S_r \delta_{(x_0, v_0)} &= c\delta_{x_0 + r v_0}(x - (s-r)v) 1_{|v|\leq 1}, \\ \text{and} \quad P_c S_{s-r} P_c S_r \delta_{(x_0, v_0)} &= \frac{c^2}{s-r} 1_{|v|\leq 1} 1_{|x_0 + r v_0 - x| \leq (s-r)1}. \end{aligned}$$

Since x is on the torus as long as $(s-r)1 \geq \sqrt{d}2$, where d is the dimension fixed here to be $d = 1$, then the last indicator function is always satisfied. So

$$P_c S_{s-r} P_c S_r \delta_{(x_0, v_0)} \geq \frac{c^2}{s-r} 1_{|v|\leq 1}$$

and hence

$$S_{t-s} P_c S_{s-r} P_c S_r \delta_{(x_0, v_0)} \geq \frac{c^2}{s-r} 1_{|v|\leq 1}$$

whenever $(s-r)1 \geq \sqrt{d}$. Finally, choose $t_* = 2\sqrt{d} = 2$, in our one-dimensional setting, and integrate over $0 < r < s < t_*$ so that indeed we have $s - r \geq \sqrt{d}$, to get our lower bound as needed. ∎

2.3. Representation formula for the steady state and moment bounds

In this subsection we proceed by getting a handy representation of our steady state for the linear BGK (2.1) through an application of Duhamel's formula. This will then allow us to estimate, below and above, the density and the pressure in the steady state.

Lemma 1. *Let g^T be a stationary solution to (2.1). We can get the following representation formula:*

$$g^T(x,v) = \int_0^1 \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})} \rho_g(x+\mathrm{sgn}(v)y) \times (\alpha \mathcal{M}_{T(x+\mathrm{sgn}(v)y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+\mathrm{sgn}(v)y)}(v))\,\mathrm{d}y.$$

Proof. To simplify the notation we write g for g^T. Let $v > 0$ and notice that Duhamel's formula yields

$$e^{t/\kappa} g(x+vt,v) - g(x,v) = \int_0^t \frac{e^{s/\kappa}}{\kappa} \rho_g(x+vs)(\alpha \mathcal{M}_{T(x+vs)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+vs)}(v))\,\mathrm{d}s.$$

Since $x \in \mathbb{T}$, by inserting $t = \frac{1}{|v|}$ and $y = vs$, we write

$$(e^{1/\kappa|v|} - 1) g(x,v) = \int_0^1 \frac{e^{y/\kappa|v|}}{\kappa|v|} \rho_g(x+y)(\alpha \mathcal{M}_{T(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))\,\mathrm{d}y,$$

or equivalently

$$g(x,v) = \int_0^1 \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})} \rho_g(x+y)(\alpha \mathcal{M}_{T(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))\,\mathrm{d}y,$$

while the same calculations for $v < 0$ give

$$g(x,v) = \int_0^1 \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})} \rho_g(x-y)(\alpha \mathcal{M}_{T(x-y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x-y)}(v))\,\mathrm{d}y.$$

This implies the stated representation formula for the steady state. ∎

We proceed by providing explicit upper and lower bounds on the density ρ_g, the pressure P_g, and consequently the temperature T_g. Our ultimate goal is, through our bounds on T_g, to determine the values of α, $\bar{\tau}$, $\underline{\tau}$ for which (i) the non-equilibrium steady state g of the linear model coincides with a steady state to the non-linear model, providing existence, and (ii) this steady state is unique.

Lemma 2. *Assuming that $\underline{T} \le \underline{\tau}$, for $\underline{\tau} \ge 1$ and $\alpha \le \frac{1-\underline{\tau}^{-1/2}}{\underline{T}^{-1/2}-\underline{\tau}^{-1/2}}$, we have uniformly in $x \in \mathbb{T}$,*

$$1 - \frac{1}{\kappa}[\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}] \lesssim \rho_g(x) \lesssim 4\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2.$$

Proof. Upper bound on the density: In order to run our computations we first establish two bounds that, while slightly fiddly, are not surprising. First, for all $x > 0$, $a \in (0,1)$ we have

$$\frac{xe^{-ax}}{1-e^{-x}} \le 1 + \frac{2}{a}. \tag{2.2}$$

To demonstrate this, we define the function

$$\eta_1(x) := 1 - e^{-x} - \frac{2}{a} e^{-(1-a/2)x} + \frac{2}{a} e^{ax/2},$$

and then we observe that

$$\eta_1'(x) = e^{-x} + \frac{(2-a)}{a} e^{-(1-a/2)x} + e^{ax/2} \geq e^{ax/2} \geq 1 \quad \text{and} \quad \eta_1(0) = 0.$$

From this it follows that $\eta_1(x) \geq x$ for all $x > 0$. We also have

$$\eta_1(x) = (1 - e^{-x}) \Big(1 + \frac{2}{a} e^{ax/2} \Big),$$

so that

$$\frac{x}{1 - e^{-x}} \leq 1 + \frac{2}{a} e^{ax/2},$$

and hence

$$\frac{x e^{-ax}}{1 - e^{-x}} \leq e^{-ax} + \frac{2}{a} e^{-ax/2} \leq 1 + \frac{2}{a},$$

thus proving the bound (2.2).

Our second bound is that for $x > 0$, $a \in (0, 1)$ there exists $C > 0$ a constant independent of a such that

$$\frac{x e^{-ax}}{1 - e^{-x}} \leq 1 + Cx. \tag{2.3}$$

To prove this, we consider

$$\eta_2(x) := \frac{x}{1 - e^{-x}},$$

whose first derivative is given by

$$\eta_2'(x) = \frac{1 - (1 + x) e^{-x}}{(1 - e^{-x})^2}.$$

We observe that $\eta_2'(x) \to 1/2$ as $x \to 0$ by L'Hopital's rule and $\eta_2'(x) \to 1$ as $x \to \infty$ and η_2' is continuous so η_2' must be bounded. This implies the existence of a C such that $\eta_2(x) \leq \eta_2(0) + Cx = 1 + Cx$. The bound (2.3) then follows from the fact that e^{-ax} is bounded by 1. Now we wish to bound a term of the form

$$\int_0^\infty \frac{1}{\kappa v} \frac{e^{-(1-y)/\kappa v}}{1 - e^{-1/\kappa v}} \mathcal{M}_\Theta(v) \, \mathrm{d}v,$$

after applying Fubini, which we are allowed to do as the integrand has positive terms. We split into bounds for small and large velocities, applying the inequalities (2.2) and (2.3) respectively:

$$\begin{aligned} \int_0^{v_*} \frac{1}{\kappa v} \frac{e^{-(1-y)/\kappa v}}{1 - e^{-1/\kappa v}} \mathcal{M}_\Theta(v) \, \mathrm{d}v &\leq \Big(1 + \frac{2}{1-y} \Big) \int_0^{v_*} \mathcal{M}_\Theta(v) \, \mathrm{d}v \\ &\leq \Big(1 + \frac{2}{1-y} \Big) (2\pi\Theta)^{-1/2} v_* \end{aligned}$$

and

$$\int_{v_*}^{\infty} \frac{1}{\kappa v} \frac{e^{-(1-y)/\kappa v}}{1-e^{-1/\kappa v}} \mathcal{M}_\Theta(v)\,\mathrm{d}v \le \int_{v_*}^{\infty} \Big(1+\frac{C}{\kappa|v|}\Big)\mathcal{M}_\Theta(v)\,\mathrm{d}v \le \Big(1+\frac{C}{\kappa|v_*|}\Big)\int_{v_*}^{\infty}\mathcal{M}_\Theta(v)\,\mathrm{d}v \le \frac{1}{2}\Big(1+\frac{C}{\kappa|v_*|}\Big).$$

Putting these together gives that for any $v_* > 0$ we have

$$\int_0^{\infty} \frac{1}{\kappa v} \frac{e^{-(1-y)/\kappa v}}{1-e^{-1/\kappa v}} \mathcal{M}_\Theta(v)\,\mathrm{d}v \le \Big(1+\frac{2}{1-y}\Big)(2\pi\Theta)^{-1/2} v_* + \frac{1}{2}\Big(1+\frac{C}{\kappa|v_*|}\Big).$$

We want to choose a v_* which balances the contribution of the two terms,

$$v_* = \Big(\frac{C}{2\kappa}\Big)^{1/2}(2\pi\Theta)^{1/4}\Big(1+\frac{2}{1-y}\Big)^{-1/2}.$$

With this choice of v_*, we obtain the bound

$$\int_0^{\infty} \frac{1}{\kappa v} \frac{e^{-(1-y)/\kappa v}}{1-e^{-1/\kappa v}} \mathcal{M}_\Theta(v)\,\mathrm{d}v \le C' \frac{1}{\sqrt{\kappa}}\Theta^{-1/4}\Big(1+\frac{2}{1-y}\Big)^{1/2}.$$

In particular, we get a bound which is integrable in y.

Applying this bound for Θ being T and τ, we get the following upper bound on the density for positive v for some $\delta \in (0,1)$:

$$\begin{aligned}
&\int_0^{\infty} g^T(x,v)\,\mathrm{d}v \\
&\quad\lesssim \int_0^1 \Big[\frac{\alpha}{\sqrt{\kappa}}\Big(1+\frac{2}{1-y}\Big)^{1/2} T(y+x)^{-1/4} \\
&\qquad\qquad + \frac{1-\alpha}{\sqrt{\kappa}}\Big(1+\frac{2}{1-y}\Big)^{1/2}\tau(y+x)^{-1/4}\Big]\rho_g(x+y)\,\mathrm{d}y \\
&\quad\lesssim \frac{(1+\frac{2}{\delta})^{1/2}}{\sqrt{\kappa}} \int_0^{1-\delta} \rho_g(y+x)[\alpha T(x+y)^{1/4} + (1-\alpha)\tau(x+y)^{1/4}]\,\mathrm{d}y \\
&\qquad + \int_{1-\delta}^1 \Big[\frac{\alpha}{\sqrt{\kappa}}\Big(\frac{1}{2}+\frac{1}{(1-y)}\Big)^{1/2} T(y+x)^{-1/4} \\
&\qquad\qquad + \frac{1-\alpha}{\sqrt{\kappa}}\Big(\frac{1}{2}+\frac{1}{(1-y)}\Big)^{1/2} T(y+x)^{-1/4}\Big]\rho_g(x+y)\,\mathrm{d}y \\
&\quad\lesssim \Big[\frac{\alpha}{\sqrt{\kappa}\inf_y T(x+y)^{1/4}} + \frac{1-\alpha}{\sqrt{\kappa}\inf_y \tau(x+y)^{1/4}}\Big] \\
&\qquad \times \Big[\Big(1+\frac{2}{\delta}\Big)^{1/2}\int_0^{1-\delta}\rho_g(x+y)\,\mathrm{d}y + \|\rho_g\|_\infty \int_{1-\delta}^1 \Big(\frac{1}{2}+\frac{1}{(1-y)}\Big)^{1/2}\mathrm{d}y\Big].
\end{aligned} \tag{2.4}$$

Expanding for δ small we notice that

$$\int_{1-\delta}^{1}\Big(\frac{1}{2}+\frac{1}{(1-y)}\Big)^{1/2}\mathrm{d}y=\mathcal{O}(i\delta\cot(1)\csc(1)/2+\sqrt{\delta(1+\delta)})=\mathcal{O}(\sqrt{\delta}).$$

Also taking into account the negative velocities, we eventually bound the last line in (2.4) as

$$\begin{aligned}\rho_g(x)&\lesssim\Big[\frac{\alpha}{\sqrt{\kappa}\underline{T}^{1/4}}+\frac{1-\alpha}{\sqrt{\kappa}\underline{\tau}^{1/4}}\Big]\Big[C_1\Big(1+\frac{2}{\delta}\Big)^{1/2}+C_2\|\rho_g\|_\infty\sqrt{\delta}\Big]\\&\lesssim\Big[\frac{\alpha}{\sqrt{\kappa}\underline{T}^{1/4}}+\frac{1-\alpha}{\sqrt{\kappa}\underline{\tau}^{1/4}}\Big]\Big[\frac{C_1}{\sqrt{\delta}}+\|\rho_g\|_\infty C_2\sqrt{\delta}\Big]\end{aligned}$$

for some finite constants C_1, C_2. Now take $\sqrt{\delta}=\frac{1}{2C_2[\frac{\alpha}{\sqrt{\kappa}\underline{T}^{1/4}}+\frac{1-\alpha}{\sqrt{\kappa}\underline{\tau}^{1/4}}]}$ to find that for all $x\in\mathbb{T}$,

$$\rho_g(x)\lesssim 4C_1C_2\Big[\frac{\alpha}{\sqrt{\kappa}\underline{T}^{1/4}}+\frac{1-\alpha}{\sqrt{\kappa}\underline{\tau}^{1/4}}\Big]^2.$$

Lower bound on the density: Let $\Lambda>0$ be sufficiently large so that $\frac{1}{(e^{1/\kappa|v|}-1)}\sim\kappa|v|$ for $|v|>\Lambda$ and write

$$\begin{aligned}&\int_0^\infty g^T(x,v)\,\mathrm{d}v\\&=\int_0^\infty\int_0^1\frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\rho_g(x+y)[\alpha\mathcal{M}_{T(x+y)}(v)+(1-\alpha)\mathcal{M}_{\tau(x+y)}(v)]\,\mathrm{d}y\,\mathrm{d}v\\&\gtrsim\int_0^1\int_\Lambda^\infty\frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\rho_g(x+y)[\alpha\mathcal{M}_{T(x+y)}(v)+(1-\alpha)\mathcal{M}_{\tau(x+y)}(v)]\,\mathrm{d}v\,\mathrm{d}y\\&\gtrsim\int_0^1\rho_g(x+y)\int_\Lambda^\infty\frac{1}{(e^{1/\kappa|v|}-1)}\frac{1}{\kappa|v|}[\alpha\mathcal{M}_{T(x+y)}(v)+(1-\alpha)\mathcal{M}_{\tau(x+y)}(v)]\,\mathrm{d}v\,\mathrm{d}y\\&\gtrsim\int_0^1\rho_g(x+y)\Big[\int_{\frac{\Lambda}{\sqrt{T(x+y)}}}^\infty\frac{\alpha}{\kappa\sqrt{T(x+y)}}\mathcal{M}_1(v)\,\mathrm{d}v\\&\qquad+\int_{\frac{\Lambda}{\sqrt{\tau(x+y)}}}^\infty\frac{1-\alpha}{\kappa\sqrt{\tau(x+y)}}\mathcal{M}_1(v)\,\mathrm{d}v\Big]\mathrm{d}y\\&\gtrsim\sqrt{\frac{\pi}{2}}-\sqrt{\frac{\pi}{2}}(\alpha T(x+y)^{-1/2}+(1-\alpha)\tau(x+y)^{-1/2})\\&\gtrsim 1-[\alpha\underline{T}^{-1/2}+(1-\alpha)\underline{\tau}^{-1/2}].\end{aligned}$$

This implies the stated lower bound on the density, which is a non-negative quantity under the constraints in the hypothesis. ∎

Now we proceed to get upper and lower bounds on the pressure $P_{g^T}(x)$ which we recall is given by

$$P_{g^T}(x)=\int_{\mathbb{R}}v^2g^T(x,v)\,\mathrm{d}v=\rho_f(x)[T_{g^T}(x)+u_{g^T}(x)^2].$$

First we notice that

$$u_{g^T}(x) = \frac{\int v g^T(x,v)\,\mathrm{d}v}{\int g^T(x,v)\,\mathrm{d}v} = 0.$$

This is since, due to mass conservation, $\partial_x(u_{g^T}(x)\rho_{g^T}(x)) = 0$, which implies

$$\int v g^T(x,v)\,\mathrm{d}v = \int v g^T(0,v)\,\mathrm{d}v = 0.$$

Moreover, the pressure is constant in x since

$$\partial_x P_{g^T}(x) = -\frac{1}{\kappa} u_{g^T}(x)\rho_{g^T}(x) = 0.$$

Lemma 3. *For α sufficiently small so that the following holds true,*

$$\begin{aligned}\alpha\Big(\sqrt{\underline{\tau}} - \sqrt{\underline{T}} + \frac{4\pi}{\kappa^2 \underline{T}^{1/4}\underline{\tau}^{1/4}}(\sqrt{\underline{\tau}} - \underline{T}^{1/4}\underline{\tau}^{1/4}) + \frac{2\pi}{\kappa^2}\alpha\Big(\frac{1}{\underline{T}^{1/4}} - \frac{1}{\underline{\tau}^{1/4}}\Big)^2\Big) \\ \le \sqrt{\underline{\tau}} - \frac{2\pi}{\kappa^2}\frac{1}{\sqrt{\underline{\tau}}},\end{aligned} \tag{2.5}$$

we have uniformly in $x \in \mathbb{T}$,

$$\begin{aligned}&\Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big] - \frac{1}{\kappa}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2 \\ &\lesssim P_{g^T}(x) \lesssim \alpha\overline{T} + (1-\alpha)\overline{\tau} + \frac{1}{\kappa}\big(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\overline{\tau}}\big).\end{aligned}$$

Proof. Upper bound on the pressure P_{g^T}: Using the representation formula for the stationary state g^T, we need to control the following for positive velocities:

$$\begin{aligned}J_+(x) := \int_0^1 &\rho_{g^T}(x+y) \\ &\times \int_0^\infty |v|\frac{e^{-(1-y)/\kappa|v|}}{\kappa(1-e^{-1/\kappa|v|})}(\alpha\mathcal{M}_{T(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))\,\mathrm{d}v\;\mathrm{d}y.\end{aligned}$$

We use the inequality

$$\frac{1}{\kappa|v|}(1-e^{-1/\kappa|v|})^{-1} \le 1 + \frac{C}{\kappa|v|}$$

that was used in the bound on the density. This gives

$$\frac{|v|}{\kappa}e^{-(1-y)/\kappa|v|}(1-e^{-1/\kappa|v|})^{-1} \le |v|^2 + \frac{C|v|}{\kappa}.$$

Integrating then gives

$$\begin{aligned}&\int_0^\infty |v|\frac{e^{-(1-y)/\kappa|v|}}{\kappa(1-e^{-1/\kappa|v|})}(\alpha\mathcal{M}_{T(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))\,\mathrm{d}v \\ &\lesssim \alpha\Big(T(x+y) + \frac{\sqrt{T(x+y)}}{\kappa}\Big) + (1-\alpha)\Big(\tau(x+y) + \frac{\sqrt{\tau(x+y)}}{\kappa}\Big).\end{aligned}$$

From this it follows that

$$J_+(x) \lesssim \int_0^1 \rho_g(x+y)\Big(\alpha\overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}})\Big)\,\mathrm{d}y$$
$$= \alpha\overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}\big(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}}\big).$$

Here we have used the fact that ρ integrates to 1. The analogous bound for negative velocities gives in total

$$P_g(x) \lesssim \alpha\overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}\big(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}}\big).$$

Lower bound on the pressure P_{g^T}: For this we bound $e^{-(1-y)/\kappa|v|} \geq 1 - \frac{1-y}{\kappa|v|}$, so that for positive velocities,

$$\int_0^\infty |v| \frac{e^{-(1-y)/\kappa|v|}}{1-e^{-1/\kappa|v|}} \mathcal{M}_{T(x+y)}(v)\,\mathrm{d}v$$
$$\gtrsim \int_0^\infty v\mathcal{M}_{T(x+y)}(v) - \int_0^\infty |v|\frac{1-y}{\kappa|v|}\mathcal{M}_{T(x+y)}(v)\,\mathrm{d}v$$
$$= \frac{\sqrt{T(x+y)}}{\sqrt{2\pi}} - \frac{1-y}{2\kappa} \gtrsim \frac{\sqrt{\underline{T}}}{2\pi} - \frac{1-y}{2\kappa}.$$

Consequently,

$$J_+(x) \gtrsim \int_0^1 \rho_{g^T}(x+y)\Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big]\,\mathrm{d}y - \int_0^1 \rho_{g^T}(x+y)\frac{1-y}{2\kappa}\,\mathrm{d}y$$
$$\gtrsim \Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big] - \frac{1}{\kappa}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2.$$

Collecting the same estimates for the negative v's we write

$$P_{g^T}(x) \gtrsim \Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big] - \frac{1}{\kappa}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2.$$

The constraint (2.5) is required to make this lower bound non-negative. ∎

2.4. Contraction of the mapping $\mathcal{F}$

Since we have ensured the uniqueness of the steady state $g = g^T$ for equation (2.1), as proved in Section 2.2, we are allowed to define the mapping

$$\mathcal{F}: C(\mathbb{T}) \to C(\mathbb{T}), \quad (\mathcal{F}T)(x) = \frac{\int_{-\infty}^{\infty} |v|^2 g(x,v)\,\mathrm{d}v}{\int_{-\infty}^{\infty} g(x,v)\,\mathrm{d}v} = \frac{P_{g^T}}{\rho_{g^T}}(x).$$

Our goal is to show that this mapping is contractive, which implies that there is a fixed point. Existence of such a fixed point, i.e. $T \in C(\mathbb{T})$ so that $T(x) = T_g(x)$ for all $x \in \mathbb{T}$, implies that the steady state found for the linear model (2.1) corresponds to a steady state in the non-linear BGK model (1.1).

Before proceeding with the statement let us define the following set that determines the Lipschitz constants for $\mathcal{F}$, in terms of α: We first define

$$\zeta_1(\alpha) = \frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}} \quad \text{and} \quad \zeta_2(\alpha) = \frac{\alpha}{\kappa \underline{T}^{1/2}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/2}}.$$

Then the set is as follows:

$$\begin{aligned}\mathcal{C}_{\overline{T},\bar{\tau},\underline{T},\underline{\tau},\kappa} = (0,1) \cap \Bigg\{\alpha : &\frac{16 C\alpha \zeta_1(\alpha)^2 [\frac{1}{2\underline{T}} + \frac{1}{\kappa \underline{T}^{3/2}}]}{1 - 4\overline{C}\sqrt{\kappa}\zeta_1(\alpha)} \\ &\times \Bigg(\frac{[\alpha \overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}})]}{[1-\zeta_2(\alpha)]^2} \\ &\quad + (1-\alpha)\Big(\bar{\tau} + \frac{\sqrt{\bar{\tau}}}{\kappa}\Big) + \alpha\Big(\overline{T} + \frac{\sqrt{\overline{T}}}{\kappa}\Big)\Bigg) \\ &+ \frac{8\alpha}{\kappa}\zeta_1(\alpha)^2 \frac{1}{\underline{T}^{1/2}}[1-\zeta_2(\alpha)]^{-1} \in [0,1)\Bigg\}.\end{aligned}$$

Proposition 2. *Let two temperature profiles $T_1, T_2 \in C(\mathbb{T})$. Assuming that*

$$\inf_{x\in\mathbb{T}} \tau(x) =: \underline{\tau} > \inf_{x\in\mathbb{T}} T(x) := \underline{T}, \quad \underline{\tau} \gtrsim \left(\frac{4}{\sqrt{\kappa}}\right)^4, \quad \textit{and} \quad \alpha \in \mathcal{C}_{\overline{T},\bar{\tau},\underline{T},\underline{\tau},\kappa},$$

the mapping $\mathcal{F}$ exhibits a contraction property

$$\|\mathcal{F}(T_1) - \mathcal{F}(T_2)\|_{L^\infty(\mathbb{T})} \le C_{\alpha,\overline{T},\bar{\tau},\underline{T},\underline{\tau},\kappa}\|T_1 - T_2\|_{L^\infty(\mathbb{T})}$$

for some explicit finite constant $C_{\alpha,\overline{T},\bar{\tau},\underline{T},\underline{\tau},\kappa} \in [0,1)$ depending on α, $\overline{T}$, $\bar{\tau}$, $\underline{T}$, $\underline{\tau}$, κ.

Proof. We fix two functions $T_1, T_2 \in C(\mathbb{T})$ and using the upper and lower bounds on the density and the upper bound on the pressure, we calculate

$$\begin{aligned}|\mathcal{F}&(T_1)(x) - \mathcal{F}(T_2)(x)| \\ &= \frac{1}{(\rho_{g^{T_1}}\rho_{g^{T_2}})(x)}|P_{g^{T_1}}(x)(\rho_{g^{T_2}} - \rho_{g^{T_1}})(x) + \rho_{g^{T_1}}(x)(P_{g^{T_1}} - P_{g^{T_2}})(x)| \\ &\le \frac{P_{g^{T_1}}}{\rho_{g^{T_1}}\rho_{g^{T_2}}}(x)|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x) + \frac{1}{\rho_{g^{T_2}}}|P_{g^{T_1}} - P_{g^{T_2}}|(x) \\ &\le C_0|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x)\frac{[\alpha\overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}})]}{[1 - \frac{1}{\kappa}[\alpha\underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]]^2} \\ &\quad + |P_{g^{T_1}} - P_{g^{T_2}}|(x)\Big[1 - \frac{1}{\kappa}[\alpha\underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]\Big]^{-1}. \end{aligned} \tag{2.6}$$

Now we estimate the difference of the densities using the representation of our steady state:

$$
\begin{aligned}
|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x) \\
= \bigg| \int_0^1 \int_0^\infty \frac{e^{y/\kappa|v|}}{\kappa|v|(e^{1/\kappa|v|}-1)} &[\rho_{g^{T_1}}(x+y)(\alpha\mathcal{M}_{T_1(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v)) \\
&- \rho_{g^{T_2}}(x+y)(\alpha\mathcal{M}_{T_2(x+y)}(v) \\
&+ (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))]\,\mathrm{d}v\,\mathrm{d}y \bigg| \\
+ \bigg| \int_0^1 \int_{-\infty}^0 \frac{e^{y/\kappa|v|}}{\kappa|v|(e^{1/\kappa|v|}-1)} &[\rho_{g^{T_1}}(x-y)(\alpha\mathcal{M}_{T_1(x-y)}(v) \\
&+ (1-\alpha)\mathcal{M}_{\tau(x-y)}(v)) \\
&- \rho_{g^{T_2}}(x-y)(\alpha\mathcal{M}_{T_2(x-y)}(v) \\
&+ (1-\alpha)\mathcal{M}_{\tau(x-y)}(v))]\,\mathrm{d}v\,\mathrm{d}y \bigg|.
\end{aligned}
$$

We look at the positive velocities and write

$$
\begin{aligned}
\int_0^1 \int_0^\infty &\frac{e^{y/\kappa|v|}}{\kappa|v|(e^{1/\kappa|v|}-1)} [\rho_{g^{T_1}}(x+y)(\alpha\mathcal{M}_{T_1(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v)) \\
&- \rho_{g^{T_2}}(x+y)(\alpha\mathcal{M}_{T_2(x+y)}(v) + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))]\,\mathrm{d}v\,\mathrm{d}y \\
= &\int_0^1 (\rho_{g^{T_1}} - \rho_{g^{T_2}})(x+y)[(1-\alpha)\mathcal{K}_1(\tau, y) + \alpha\mathcal{K}_1(T_2, y)]\,\mathrm{d}y \\
&+ \alpha \int_0^1 \rho_{g^{T_1}}(x+y)[\mathcal{K}_1(T_1, y) - \mathcal{K}_1(T_2, y)]\,\mathrm{d}y,
\end{aligned}
$$

where $\mathcal{K}_1$ is the integral kernel defined by

$$
\mathcal{K}_1(\Theta, y) = \int_0^\infty \frac{1}{\kappa|v|} e^{-(1-y)/\kappa|v|}(1 - e^{-1/\kappa|v|})^{-1}\mathcal{M}_{\Theta(y+x)}(v)\,\mathrm{d}v.
$$

Together with the negative velocities then

$$
\begin{aligned}
|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x) \\
\lesssim 2\alpha\|\rho_{g^{T_1}}\|_\infty \int_0^1 &|\mathcal{K}_1(T_1(x+y), y) - \mathcal{K}_1(T_2(x+y), y)| \\
&+ |\mathcal{K}_1(T_1(x-y), y) - \mathcal{K}_1(T_2(x-y), y)|\,\mathrm{d}y \\
+ \int_0^1 &|(1-\alpha)\mathcal{K}_1(\tau(x+y), y) + \alpha\mathcal{K}_1(T_2(x+y), y)||\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x+y)\,\mathrm{d}y \\
+ \int_0^1 &|(1-\alpha)\mathcal{K}_1(\tau(x-y), y) + \alpha\mathcal{K}_1(T_2(x-y), y)||\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x-y)\,\mathrm{d}y
\end{aligned}
$$

$$\lesssim 8\alpha\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2 \int_0^1 |\mathcal{K}_1(T_1(x+y),y) - \mathcal{K}_1(T_2(x+y),y)| + |\mathcal{K}_1(T_1(x-y),y) - \mathcal{K}_1(T_2(x-y),y)|\,\mathrm{d}y$$
$$+ \frac{4\overline{C}}{\sqrt{\kappa}} \|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}], \tag{2.7}$$

where in the last line we used the following upper bound on $\mathcal{K}_1$ from the proof of Lemma 2 on the density's upper bound:

$$\mathcal{K}_1(\Theta(x+y),y) \le \frac{C'}{\sqrt{\kappa}}\Big(1 + \frac{2}{1-y}\Big)^{1/2} \Theta(x+y)^{-1/4},$$

as well as the integrability in y, all these yielding a factor $\overline{C}$ which is independent of α, κ, and the temperatures. We notice now that $\mathcal{K}_1$ satisfies

$$|\mathcal{K}_1(T_1(y),y) - \mathcal{K}_1(T_2(y),y)| \le C|T_1(y) - T_2(y)|$$

for a constant C. Indeed, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{K}_1(\theta,y) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^\infty \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\mathcal{M}_\theta(v)\,\mathrm{d}v \\
&= \int_0^\infty \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\Big[-\frac{1}{2\theta^{3/2}}\frac{e^{-v^2/2\theta}}{\sqrt{2\pi\theta}} + \frac{v^2}{2\theta^2}\frac{e^{-v^2/2\theta}}{\sqrt{2\pi\theta}}\Big]\mathrm{d}v \\
&= \frac{1}{2\theta^2}\int_0^\infty \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\mathcal{M}_\theta(v)(-\theta + v^2)\,\mathrm{d}v \\
&\le \frac{1}{2\theta^2}\int_0^\infty \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\mathcal{M}_\theta(v)v^2\,\mathrm{d}v.
\end{aligned}$$

With similar estimates to the proof of the upper bound for the pressure, we write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{K}_1(\theta,y) &\le \frac{1}{2\theta^2}\int_0^\infty \frac{|v|e^{-(1-y)/\kappa|v|}}{\kappa(1-e^{-1/\kappa|v|})}\mathcal{M}_\theta(v)\,\mathrm{d}v \\
&\le \frac{1}{2\theta^2}\int_0^\infty \Big(v^2 + \frac{C|v|}{\kappa}\Big)\mathcal{M}_\theta(v)\,\mathrm{d}v \lesssim \frac{1}{2\theta^2}\Big(t + \frac{\sqrt{\theta}}{\kappa}\Big) \\
&= \frac{1}{2\theta} + \frac{1}{\kappa\theta^{3/2}},
\end{aligned}$$

where for the last inequality we neglected the negative terms. This implies that

$$|\mathcal{K}_1(T_1(y),y) - \mathcal{K}_1(T_2(y),y)| \le C_1\Big[\frac{1}{2\underline{T}} + \frac{1}{\kappa\underline{T}^{3/2}}\Big]|T_1(y) - T_2(y)|,$$

for a universal constant C_1. Then inserting this into (2.7) implies

$$|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x) \lesssim 8\alpha\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2 2\|T_1 - T_2\|_{L^\infty(\mathbb{T})}\int_0^1\Big[\frac{1}{2\underline{T}} + \frac{1}{\kappa\underline{T}^{3/2}}\Big]\mathrm{d}y$$
$$+ \frac{4\overline{C}}{\sqrt{\kappa}}\|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha\underline{T}^{-1/4}].$$

There is a finite constant C then so that

$$|\rho_{g^{T_1}} - \rho_{g^{T_2}}|(x) \lesssim 16C\alpha\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2\Big[\frac{1}{2\underline{T}} + \frac{1}{\kappa \underline{T}^{3/2}}\Big]\|T_1 - T_2\|_{L^\infty(\mathbb{T})} + \frac{4\bar{C}}{\sqrt{\kappa}}\|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}],$$

implying that, after taking the supremum over x in the left-hand side,

$$\|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})}\Big(1 - \frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}]\Big) \lesssim 16C\alpha\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2\Big[\frac{1}{2\underline{T}} + \frac{1}{\kappa \underline{T}^{3/2}}\Big]\|T_1 - T_2\|_{L^\infty(\mathbb{T})}$$

or, as long as $(1 - \frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}]) > 0$ which is satisfied when

$$\underline{\tau} > \underline{T}, \quad \underline{\tau} \ge \Big(\frac{4\bar{C}}{\sqrt{\kappa}}\Big)^4, \quad \text{and} \quad \alpha < \frac{\frac{\sqrt{\kappa}}{4\bar{C}} - \underline{\tau}^{-1/4}}{\underline{T}^{-1/4} - \underline{\tau}^{-1/4}},$$

we have that

$$\|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})} \lesssim \frac{16C\alpha[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}]^2[\frac{1}{2\underline{T}} + \frac{1}{\kappa \underline{T}^{3/2}}]}{(1 - \frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}])}\|T_1 - T_2\|_{L^\infty(\mathbb{T})}. \tag{2.8}$$

As a next step we estimate the difference of the pressures:

$$\begin{aligned}
&|P_{g^{T_1}} - P_{g^{T_2}}|(x) \\
&\quad = \Big|\int_0^1 \int_0^\infty \frac{|v| e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|} - 1)}[\rho_{g^{T_1}}(x+y)(\alpha \mathcal{M}_{T_1(x+y)}(v) \\
&\qquad + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v)) \\
&\qquad - \rho_{g^{T_2}}(x+y)(\alpha \mathcal{M}_{T_2(x+y)}(v) \\
&\qquad + (1-\alpha)\mathcal{M}_{\tau(x+y)}(v))]\,\mathrm{d}v\,\mathrm{d}y\Big| \\
&\quad + \Big|\int_0^1 \int_{-\infty}^0 \frac{|v| e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|} - 1)}[\rho_{g^{T_1}}(x-y)(\alpha \mathcal{M}_{T_1(x-y)}(v) \\
&\qquad + (1-\alpha)\mathcal{M}_{\tau(x-y)}(v)) \\
&\qquad - \rho_{g^{T_2}}(x-y)(\alpha \mathcal{M}_{T_2(x-y)}(v) \\
&\qquad + (1-\alpha)\mathcal{M}_{\tau(x-y)}(v))]\,\mathrm{d}v\,\mathrm{d}y\Big|.
\end{aligned}$$

Looking at the positive velocities, the terms we need to estimate are

$$\int_0^1 \Big\{\alpha \int_0^\infty \frac{|v|e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|}-1)}\mathcal{M}_{T_1(x+y)}(v)(\rho_{gT_1}-\rho_{gT_2})(x+y)\,\mathrm{d}v$$

$$+\alpha\int_0^\infty \frac{|v|e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|}-1)}\rho_{gT_2}(x+y)(\mathcal{M}_{T_1(x+y)}-\mathcal{M}_{T_2(x+y)})(v)\,\mathrm{d}v$$

$$+(1-\alpha)\int_0^\infty \frac{|v|e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|}-1)}\mathcal{M}_{\tau(x+y)}(\rho_{gT_1}-\rho_{gT_2})(x+y)\,\mathrm{d}v\Big\}\,\mathrm{d}y$$

$$=\int_0^1 \{\big((1-\alpha)\mathcal{K}_2(\tau(x+y),y)+\alpha\mathcal{K}_2(T_1(x+y),y)\big)(\rho_{gT_1}-\rho_{gT_2})(x+y)$$

$$+\alpha\rho_{gT_2}(x+y)\big(\mathcal{K}_2(T_1(x+y),y)-\mathcal{K}_2(T_2(x+y),y)\big)\}\,\mathrm{d}y,$$

where

$$\mathcal{K}_2(\Theta(x+y,y)) := \int_0^\infty \frac{|v|e^{y/\kappa|v|}}{\kappa(e^{1/\kappa|v|}-1)}\mathcal{M}_{\Theta(x+y)}(v)\,\mathrm{d}v.$$

An upper bound on this integral kernel, as was computed above by the use of inequality (2.3), is

$$\mathcal{K}_2(\Theta(x+y,y)) \le \tilde{C}\Big(\Theta(x+y)+\frac{\sqrt{\Theta(x+y)}}{\kappa}\Big)$$

for a universal constant $\tilde{C}$. We have then, also collecting the negative velocities, the following upper bound on the difference of the pressures:

$$|P_{gT_1}-P_{gT_2}|(x)$$

$$\lesssim \|\rho_{gT_1}-\rho_{gT_2}\|_{L^\infty(\mathbb{T})}\int_0^1 [|(1-\alpha)\mathcal{K}_2(\tau(x+y),y)+\alpha\mathcal{K}_2(T_1(x+y),y)|$$

$$+|(1-\alpha)\mathcal{K}_2(\tau(x-y),y)$$

$$+\alpha\mathcal{K}_2(T_1(x-y),y)|]\,\mathrm{d}y$$

$$+\alpha\|\rho_{gT_2}\|_{L^\infty(\mathbb{T})}\int_0^1 [|\mathcal{K}_2(T_1(x+y),y)-\mathcal{K}_2(T_2(x+y),y)|$$

$$+|\mathcal{K}_2(T_1(x-y),y)-\mathcal{K}_2(T_2(x-y),y)|]\,\mathrm{d}y$$

$$\lesssim \frac{16C\alpha[\frac{\alpha}{\kappa \underline{T}^{1/4}}+\frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}]^2[\frac{1}{2\underline{T}}+\frac{1}{\kappa\underline{T}^{3/2}}]}{1-\frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4}+\alpha\underline{T}^{-1/4}]}\Big[(1-\alpha)\Big(\bar{\tau}+\frac{\sqrt{\bar{\tau}}}{\kappa}\Big)+\alpha\Big(\bar{T}+\frac{\sqrt{\bar{T}}}{\kappa}\Big)\Big]$$

$$\times\|T_1-T_2\|_{L^\infty(\mathbb{T})}$$

$$+4\alpha\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}}+\frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\int_0^1 [|\mathcal{K}_2(T_1(x+y),y)-\mathcal{K}_2(T_2(x+y),y)|$$

$$+|\mathcal{K}_2(T_1(x-y),y)$$

$$-\mathcal{K}_2(T_2(x-y),y)|]\,\mathrm{d}y, \tag{2.9}$$

where for the first integral on the right-hand side we applied the upper bounds on $\mathcal{K}_2$, and on $\|\rho_{g^{T_1}} - \rho_{g^{T_2}}\|_{L^\infty(\mathbb{T})}$ from (2.8). For the next term, we applied the upper bound on the density.

We next notice that – as in the case of $\mathcal{K}_1$ – the kernel $\mathcal{K}_2$ also satisfies

$$|\mathcal{K}_2(T_1(y), y) - \mathcal{K}_2(T_2(y), y)| \leq C|T_1(y) - T_2(y)|$$

for a constant C that is integrable in $y \in \mathbb{T}$: for that we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{K}_2(\theta, y) &= \frac{1}{\theta^2}\int_0^\infty \frac{|v|e^{y/\kappa|v|}}{2\kappa(e^{1/\kappa|v|}-1)}\mathcal{M}_\theta(v)(v^2-\theta)\,\mathrm{d}v \\
&= \frac{1}{\sqrt{\theta}}\int_0^\infty \frac{v}{2\kappa}\frac{e^{y/\kappa|v|\sqrt{\theta}}}{e^{1/\kappa|v|\sqrt{\theta}}-1}\mathcal{M}_1(v)(v^2-1)\,\mathrm{d}v \\
&\lesssim \frac{1}{\sqrt{\theta}}\int_0^\infty \frac{v^3}{2\kappa}\frac{e^{y/\kappa|v|\sqrt{\theta}}}{e^{1/\kappa|v|\sqrt{\theta}}-1}\mathcal{M}_1(v)\,\mathrm{d}v \\
&\lesssim \frac{1}{2\kappa\sqrt{\theta}}\Big[v_*^4\Big(1+\frac{2}{1-y}\Big) + \sqrt{2\pi}v_*^3 e^{-v_*^2/2}\Big] \\
&\lesssim \frac{1}{2\kappa\sqrt{\theta}}\Big[v_*^4\Big(1+\frac{2}{1-y}\Big) + \frac{4\sqrt{2\pi}}{v_*}\Big],
\end{aligned}$$

where we split the integral on $[0, v_*]$, $[v_*, \infty[$. Then choosing $v_* = (\frac{4\sqrt{2\pi}}{1+\frac{2}{1-y}})^{1/5}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{K}_2(\theta, y) \lesssim \frac{1}{2\kappa\theta^{1/2}}\Big(1+\frac{2}{1-y}\Big)^{1/5},$$

which implies

$$|\mathcal{K}_2(T_1(y), y) - \mathcal{K}_2(T_2(y), y)| \lesssim \frac{1}{\kappa \underline{T}^{1/2}}\Big(1+\frac{2}{1-y}\Big)^{1/5}|T_1(y) - T_2(y)|.$$

Inserting this into (2.9),

$$\begin{aligned}
&|P_{g^{T_1}} - P_{g^{T_2}}|(x) \\
&\quad\lesssim \frac{16C\alpha[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}]^2[\frac{1}{2\underline{T}} + \frac{1}{\kappa\underline{T}^{3/2}}]}{(1 - \frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha\underline{T}^{-1/4}])}\Big[(1-\alpha)\Big(\bar{\tau} + \frac{\sqrt{\bar{\tau}}}{\kappa}\Big) + \alpha\Big(\bar{T} + \frac{\sqrt{\bar{T}}}{\kappa}\Big)\Big] \\
&\qquad \times \|T_1 - T_2\|_{L^\infty(\mathbb{T})} \\
&\qquad + \frac{8\alpha}{\kappa}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\frac{1}{\underline{T}^{1/2}}\|T_1 - T_2\|_{L^\infty(\mathbb{T})}.
\end{aligned}$$

Finally, (2.6) together with (2.8) and (2.9) yields a contraction for $\mathcal{F}$:

$$\sup_{x\in\mathbb{T}} |\mathcal{F}(T_1)(x) - \mathcal{F}(T_2)(x)| \leq C_{\alpha,\bar{T},\bar{\tau},\underline{T},\underline{\tau},\kappa}\|T_1 - T_2\|_{L^\infty(\mathbb{T})},$$

where

$$C_{\alpha,\overline{T},\bar{\tau},\underline{T},\underline{\tau},\kappa} = \frac{16C\alpha[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}]^2[\frac{1}{2\underline{T}} + \frac{1}{\kappa \underline{T}^{3/2}}]}{(1 - \frac{4\bar{C}}{\sqrt{\kappa}}[(1-\alpha)\underline{\tau}^{-1/4} + \alpha \underline{T}^{-1/4}])}$$

$$\times \Bigg\{ \frac{[\alpha \overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}})]}{[1 - \frac{1}{\kappa}[\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]]^2}$$

$$+ \left[(1-\alpha)\left(\bar{\tau} + \frac{\sqrt{\bar{\tau}}}{\kappa}\right) + \alpha\left(\overline{T} + \frac{\sqrt{\overline{T}}}{\kappa}\right)\right]\Bigg\}$$

$$+ \frac{8\alpha}{\kappa \underline{T}^{1/2}}\left[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\right]^2\left[1 - \frac{1}{\kappa}[\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]\right]^{-1}. \quad \blacksquare$$

2.5. Proof of Theorem 1

Let us define the set $S_{\underline{T},\overline{T}} := \{\underline{T} \leq T(x) \leq \overline{T}\}$. In order to apply the Banach fixed point theorem we want to ensure that the mapping $\mathcal{F}$ maps the set $S_{\underline{T},\overline{T}}$ into itself, i.e.

$$\mathcal{F}(S_{\underline{T},\overline{T}}) \subset S_{\underline{T},\overline{T}}. \tag{2.10}$$

Using the estimates from Lemmas 2 and 3, we get the upper and lower bounds on the image of $\mathcal{F}$, $\mathcal{F}(T)(x) = \frac{\int_{\mathbb{T}} |v|^2 g(x,v)\,dv}{\int_{\mathbb{T}} g(x,v)\,dv}$, stated in the following lemma.

Lemma 4. *Under the condition (2.5) on α and $\underline{\tau}$, we have uniformly in $x \in \mathbb{T}$ that*

$$\frac{1}{4}\left[\left[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\right] - \frac{1}{\kappa}\left[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\right]^2\right]\left[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\right]^{-2}$$

$$\lesssim \mathcal{F}(T)(x) \lesssim \frac{[\alpha \overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}})]}{(1 - \frac{1}{\kappa}[\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}])}.$$

Thus, in order to ensure that (2.10) holds, we need to assume that

$$\left[\alpha \overline{T} + (1-\alpha)\bar{\tau} + \frac{1}{\kappa}\left(\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\bar{\tau}}\right)\right]\left(1 - \frac{1}{\kappa}[\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]\right)^{-1} \leq \overline{T}$$

and

$$\frac{1}{4}\left[\left[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\right] - \frac{1}{\kappa}\left[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\right]^2\right]\left[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\right]^{-2} \geq \underline{T}.$$

By setting $\alpha = 0$ in the equations above we can see that this will be possible provided we first choose

$$\overline{T} > \bar{\tau}\frac{\kappa\sqrt{\bar{\tau}} + 1}{\kappa\sqrt{\bar{\tau}} - 1} \quad \text{and} \quad \underline{T} < \frac{\sqrt{\underline{\tau}}}{8\pi} - \frac{1}{\kappa}, \tag{2.11}$$

and then choose α sufficiently small depending on $\bar{\tau}$, $\underline{\tau}$, $\overline{T}$, $\underline{T}$, and κ.

Proof of Theorem 1. Since $\mathscr{F}(S_{\underline{T},\overline{T}}) \subset S_{\underline{T},\overline{T}}$ and the mapping $\mathscr{F}$ is contractive as long as $\alpha \in \mathscr{C}_{\underline{T},\underline{\tau},\kappa}$, we conclude that there is a unique fixed point, i.e. there is a unique $T \in C(\mathbb{T})$ so that $T \equiv T_g$. The uniqueness of the fixed point implies the uniqueness of the steady state found, in the class of functions satisfying the condition $\underline{T} \leq T_g(x) \leq \overline{T}$ for all $x \in \mathbb{T}$. This holds for $\underline{T}, \overline{T}$ satisfying (2.11) with α sufficiently small so that $\alpha \in \mathscr{C}_{\overline{T},\overline{\tau},\underline{T},\underline{\tau},\kappa}$, is in $(0, 1)$. ■

3. Linear stability

In this section we prove the dynamical perturbative stability in the weighted L^2 space with weight $g(x, v)^{-1}$.

Before linearising our operator around the NESS g and studying its stability, let us provide two more properties on g that are going to be needed for the basic estimates for the stability, that is, to prove bounds on higher moments.

3.1. Bounds on the third and fourth moments of the NESS

Let us define the normalised third and fourth moments of the steady state g^T.

Definition 1 (Normalised 3rd and 4th moments). Let us define the third moment of the steady state g^T corresponding to temperature T as

$$d_3(x) := \frac{1}{\rho_{g^T} T^{3/2}} \int_{\mathbb{R}} v^3 g^T(x, v)\, \mathrm{d}v$$

and the fourth moment by

$$d_4(x) := \frac{1}{\rho_{g^T} T^2} \int_{\mathbb{R}} v^4 g^T(x, v)\, \mathrm{d}v - 3.$$

Proposition 3. *For α sufficiently small and $\underline{\tau} > 1$, uniformly in $x \in \mathbb{T}$ we have for the third normalised moment of the non-equilibrium steady state g,*

$$\frac{\overline{T}^{-3/2}}{\kappa}\Big\{2\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2 (\alpha \underline{T} + (1-\alpha)\underline{\tau}) - \frac{1}{2\sqrt{2\pi}}[\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\overline{\tau}}]\Big\}$$
$$\lesssim d_3(x) \lesssim \frac{C(\overline{T}^{3/2} + \overline{\tau}^{3/2} + (\overline{T} + \overline{\tau})/\kappa)}{\underline{T}^{3/2}(1 - (\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{1/2}))},$$

while for the fourth normalised moment of g,

$$\frac{\overline{T}^{-2}}{4}\Big[\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^{-2} \sqrt{\frac{2}{\pi}} \frac{1}{\kappa}(\alpha \underline{T}^{3/2} + (1-\alpha)\underline{\tau}^{3/2}) - \frac{1}{\kappa^2}(\alpha \overline{T} + (1-\alpha)\overline{\tau})\Big] - 3$$
$$\lesssim d_4(x) \lesssim \frac{C(\overline{T}^2 + \overline{\tau}^2 + (\overline{T}^{3/2} + \overline{\tau}^{3/2})/\kappa)}{\underline{T}^2(1 - (\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{1/2}))},$$

for a finite constant C.

Proof. Upper bound on d_3: We start with the upper bound on d_3 using the representation formula for g^T. For positive velocities we write

$$\int_0^\infty v^3 g^T(x,v)\,\mathrm{d}v = \int_0^\infty \int_0^1 v^3 \frac{e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\rho_g(x+y) \times (\alpha \mathcal{M}_{T(y+x)} + (1-\alpha)\mathcal{M}_{\tau(y+x)})\,\mathrm{d}y\,\mathrm{d}v.$$

Again, we use the bound

$$\frac{1}{\kappa|v|}(1-e^{-1/\kappa|v|})^{-1} \le 1 + C/\kappa|v|,$$

for some universal constant C. Using this we have the bound

$$\int_0^\infty \frac{|v|^2 e^{-(1-y)/\kappa|v|}}{\kappa(1-e^{-1/\kappa|v|})}\mathcal{M}_T(v)\,\mathrm{d}v \le C(T^{3/2} + T/\kappa),$$

for some different universal constant C. This then implies that

$$\int_0^\infty g^T(x,v)\,\mathrm{d}v \le \int_0^1 \rho_g(x+y) C(\overline{T}^{3/2} + \bar{\tau}^{3/2} + (\overline{T} + \bar{\tau})/\kappa)\,\mathrm{d}y \le C(\overline{T}^{3/2} + \bar{\tau}^{3/2} + (\overline{T} + \bar{\tau})/\kappa).$$

Combining this with a similar bound for negative velocities and dividing by our lower bound on $\rho_g(x)$ gives

$$|d_3(x)| \le \frac{C(\overline{T}^{3/2} + \bar{\tau}^{3/2} + (\overline{T} + \bar{\tau})/\kappa)}{\underline{T}^{3/2}(1 - (\alpha \underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{1/2}))}.$$

Lower bound on d_3: Regarding the lower bound on d_3, we use the inequality $e^{-(1-y)/\kappa|v|} \ge 1 - \frac{1-y}{\kappa|v|}$ to write

$$\begin{aligned}
&\alpha \int_0^1 \rho_g(x+y)\Big(\int_0^\infty \frac{|v|^2 e^{-(1-y)/\kappa}}{\kappa(1-e^{-1/\kappa|v|})}\mathcal{M}_{T(y+x)}(v)\,\mathrm{d}v\Big)\mathrm{d}y \\
&\quad \gtrsim \alpha \int_0^1 \rho_g(x+y)\Big(\int_0^\infty \frac{|v|^2}{\kappa}\Big(1 - \frac{1-y}{\kappa|v|}\Big)\mathcal{M}_{T(y+x)}(v)\,\mathrm{d}v\Big)\mathrm{d}y \\
&\quad = \alpha \int_0^1 \rho_g(x+y)\Big[\frac{T(x+y)}{2\kappa} - \frac{1-y}{\kappa}\frac{\sqrt{T(x+y)}}{\sqrt{2\pi}}\Big]\mathrm{d}y \\
&\quad \gtrsim \alpha \frac{\underline{T}}{2\kappa} - \frac{\alpha}{2}\frac{\sqrt{\overline{T}}}{\sqrt{2\pi}\kappa} 4\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}}\Big]^2.
\end{aligned}$$

This in total gives

$$d_3(x) \gtrsim \frac{1}{\|\rho_g\|_{L^\infty}\overline{T}^{3/2}}\Big\{\alpha\Big(\frac{\underline{T}}{2\kappa} - \frac{\sqrt{\overline{T}}}{2\sqrt{2\pi}\kappa}4\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\Big)$$

$$+ (1-\alpha)\Big(\frac{\underline{\tau}}{2\kappa} - \frac{\sqrt{\overline{\tau}}}{2\sqrt{2\pi}\kappa}4\Big[\frac{\alpha}{\kappa\underline{\tau}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\Big)\Big\}$$

$$= \frac{\overline{T}^{-3/2}}{\kappa}\Big\{2\Big[\frac{\alpha}{\kappa\underline{\tau}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2(\alpha\underline{T} + (1-\alpha)\underline{\tau}) - \frac{1}{2\sqrt{2\pi}}\Big[\alpha\sqrt{\overline{T}} + (1-\alpha)\sqrt{\overline{\tau}}\Big]\Big\}.$$

The parameter α being small enough ensures the non-negativity of the lower bound.

Upper bound on d_4: We now continue with the upper bound on d_4. From the representation formula for g^T for positive velocities we write

$$\int_0^\infty v^4 g^T(x,v)\,\mathrm{d}v = \int_0^\infty\int_0^1 v^3\frac{e^{-(1-y)/\kappa}}{\kappa|v|(1-e^{-1/\kappa|v|})}\rho_g(x+y)$$
$$\times(\alpha\mathcal{M}_{T(y+x)} + (1-\alpha)\mathcal{M}_{\tau(y+x)})\,\mathrm{d}y\,\mathrm{d}v.$$

Repeating the strategy for d_3 we bound

$$\frac{|v|^4 e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})} \le |v|^4 + \frac{C}{\kappa}|v|^3,$$

for some universal constant C. This implies that we have

$$\int_0^\infty \frac{|v|^4 e^{-(1-y)/\kappa|v|}}{\kappa|v|(1-e^{-1/\kappa|v|})}\mathcal{M}_T(v)\,\mathrm{d}v \le C(T^2 + T^{3/2}/\kappa),$$

for some different universal constant C. This implies that

$$\int_0^\infty v^4 g^T(x,v)\,\mathrm{d}v \le \int_0^1 \rho_g(x+y)C(\overline{T}^2 + \overline{\tau} + (\overline{T}^{3/2} + \overline{\tau}^{3/2})/\kappa)\,\mathrm{d}y$$
$$= C(\overline{T}^2 + \overline{\tau} + (\overline{T}^{3/2} + \overline{\tau}^{3/2})/\kappa).$$

Combining this with the analogous bound for negative velocities and our lower bound on ρ_g this gives

$$|d_4(x)| \le \frac{C(\overline{T}^2 + \overline{\tau}^2 + (\overline{T}^{3/2} + \overline{\tau}^{3/2})/\kappa)}{\underline{T}^2(1 - (\alpha\underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{1/2}))}.$$

Lower bound on d_4: Finally, for the lower bound we write

$$\alpha\int_0^1 \rho_g(x+y)\Big(\int_0^\infty \frac{|v|^3 e^{-(1-y)/\kappa}}{\kappa(1-e^{-1/\kappa|v|})}\mathcal{M}_{T(y+x)}(v)\,\mathrm{d}v\Big)\,\mathrm{d}y$$
$$\gtrsim \alpha\int_0^1 \rho_g(x+y)\Big(\int_0^\infty \frac{|v|^3}{\kappa}\Big(1 - \frac{1-y}{\kappa|v|}\Big)\mathcal{M}_{T(y+x)}(v)\,\mathrm{d}v\Big)\,\mathrm{d}y$$

$$\gtrsim \alpha \int_0^1 \rho_g(x+y)\Big[\sqrt{\frac{2}{\pi}}\frac{1}{\kappa}T(x+y)^{3/2} - \frac{(1-y)}{\kappa^2}\frac{T(x+y)}{2}\Big]\,\mathrm{d}y$$
$$\gtrsim \alpha\sqrt{\frac{2}{\pi}}\frac{1}{\kappa}\underline{T}^{3/2} - \alpha\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\frac{\overline{T}}{\kappa^2}.$$

Finally, together again with the negative velocities, the $(1-\alpha)$ terms, and the upper bound on the density, we write

$$\begin{aligned}
d_4(x) \gtrsim \frac{\overline{T}^{-2}}{4}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^{-2}\Big[&\alpha\sqrt{\frac{2}{\pi}}\frac{\underline{T}^{3/2}}{\kappa} - \alpha\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\frac{\overline{T}}{\kappa^2} \\
&+ (1-\alpha)\sqrt{\frac{2}{\pi}}\frac{\underline{T}^{3/2}}{\kappa} \\
&- (1-\alpha)\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\frac{\overline{T}}{\kappa^2}\Big] - 3 \\
= \frac{\overline{T}^{-2}}{4}\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^{-2}\Big[&\sqrt{\frac{2}{\pi}}\frac{1}{\kappa}(\alpha\underline{T}^{3/2} + (1-\alpha)\underline{\tau}^{3/2}) \\
&- \Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\frac{1}{\kappa^2}(\alpha\overline{T} + (1-\alpha)\overline{\tau})\Big] - 3 \\
= \frac{\overline{T}^{-2}}{4}\Big[\Big[\frac{\alpha}{\kappa\underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^{-2}&\sqrt{\frac{2}{\pi}}\frac{1}{\kappa}(\alpha\underline{T}^{3/2} + (1-\alpha)\underline{\tau}^{3/2}) \\
- \frac{1}{\kappa^2}(\alpha\overline{T} + (1-\alpha)\overline{\tau})\Big] - 3.&
\end{aligned}$$

■

3.2. Linearising and splitting the operator

In this subsection we linearise the operator $\mathcal{L}$ around the non-equilibrium solution g: $f(t,x,v) = g(x,v) + \varepsilon h(t,x,v)$. In the following lemma we state the linearised evolution equation that the new unknown h satisfies.

Lemma 5. *The linearised equation around the steady state g is*

$$\begin{aligned}
\partial_t h = \mathfrak{L}h = -v\partial_x h + \frac{1}{\kappa}\Big(&\alpha\mathcal{M}_{T_g(x)}\int_{-\infty}^{\infty}\Big(1 + \frac{uv}{T_g} + \frac{1}{2}\Big(\frac{u^2}{T_g} - 1\Big)\Big(\frac{v^2}{T_g} - 1\Big)\Big)h(x,u)\,\mathrm{d}u \\
&+ (1-\alpha)\mathcal{M}_{\tau(x)}\int_{-\infty}^{\infty} h(x,u)\,\mathrm{d}u - h\Big).
\end{aligned} \tag{3.1}$$

Proof. We write $f = g + \varepsilon h$; then we have

$$\rho_f = \rho_g + \varepsilon\rho_h \quad \text{and} \quad \rho_f u_f = \int_{-\infty}^{\infty} v f(x,v)\,\mathrm{d}v = \rho_g u_g + \varepsilon\int_{-\infty}^{\infty} h(x,v)v\,\mathrm{d}v.$$

We recall that $u_g = 0$ (cf. the discussion just above Lemma 3), and therefore by denoting $m_h := \int_{-\infty}^{\infty} uh(x,u)\,\mathrm{d}u$,

$$u_f = \frac{\varepsilon m_h}{\rho_g(1 + \varepsilon\rho_h/\rho_g)} = \frac{\varepsilon m_h}{\rho_g} + \mathcal{O}(\varepsilon^2).$$

Regarding the temperature we write, up to first order in ε,

$$\rho_f(T_f + u_f^2) = \rho_g T_g + \varepsilon P_h,$$

and so

$$\begin{aligned} T_f &= \frac{T_g + \varepsilon P_h/\rho_g}{1+\varepsilon\rho_h/\rho_g} - u_f^2 \\ &= T_g + \varepsilon\Big(\frac{P_h}{\rho_g} - \frac{\rho_h}{\rho_g}T_g\Big) - \Big(u_g + \varepsilon\Big(\frac{m_h}{\rho_g} - \frac{\rho_h u_g}{\rho_g}\Big)\Big)^2 + o(\varepsilon) \\ &= T_g + \varepsilon\Big(\frac{P_h}{\rho_g} - \frac{\rho_h}{\rho_g}T_g\Big) + o(\varepsilon) = T_g\Big(1+\varepsilon\Big(\frac{P_h}{P_g} - \frac{\rho_h}{\rho_g}\Big)\Big) + o(\varepsilon) \\ &= T_g\Big(1 + \frac{\varepsilon}{\rho_g}\int_{-\infty}^{\infty}\Big(\frac{v^2}{T_g} - 1\Big)h(x,v)\,\mathrm{d}v\Big) + o(\varepsilon), \end{aligned}$$

where we used again that $u_g = 0$. Next we move to the linearisation of the Maxwellian. Up to first order in ε we write

$$\begin{aligned} |v-u_f|^2/2T_f &= \Big|v - \varepsilon\Big(\frac{m_h}{\rho_g}\Big)\Big|^2 \frac{1}{2T_g}\Big[1 - \frac{\varepsilon}{\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u\Big] + o(\varepsilon) \\ &= \Big[\frac{v^2}{2T_g} - \varepsilon v\Big(\frac{m_h}{T_g\rho_g}\Big)\Big]\Big[1 - \frac{\varepsilon}{\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u\Big] + o(\varepsilon) \\ &= \frac{v^2}{2T_g} - \frac{\varepsilon v^2}{2T_g\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u - \frac{\varepsilon v m_h}{T_g\rho_g} + o(\varepsilon), \end{aligned}$$

and so by expanding the exponential near 0,

$$\begin{aligned} \exp\Big(-\frac{|v-u_f|^2}{2T_f}\Big) &= \exp\Big(-\frac{v^2}{2T_g}\Big) \\ &\quad\times\Big(1 + \frac{\varepsilon v^2}{2T_g\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u + \frac{\varepsilon v m_h}{T_g\rho_g} + o(\varepsilon)\Big). \end{aligned}$$

For its normalisation we have by expanding again around 0 that

$$\begin{aligned} (2\pi T_f)^{-1/2} &= \frac{1}{\sqrt{2\pi T_g}}\Big(1 - \frac{\varepsilon}{2}\Big(\frac{P_h}{P_g} - \frac{\rho_h}{\rho_g}\Big)\Big) + o(\varepsilon) \\ &= \frac{1}{\sqrt{2\pi T_g}}\Big(1 - \frac{\varepsilon}{2\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u\Big) + o(\varepsilon). \end{aligned}$$

Therefore, altogether the infinitesimal Maxwellian is

$$\begin{aligned} \mathcal{M}_{u_f,T_f}(v) = \mathcal{M}_{0,T_g}(v)\Big(1 &+ \frac{\varepsilon v^2}{2T_g\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u + \frac{\varepsilon v}{T_g\rho_g}\int_{-\infty}^{\infty} u h(x,u)\,\mathrm{d}u \\ &- \frac{\varepsilon}{2\rho_g}\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u\Big) + o(\varepsilon) \end{aligned}$$

$$= \mathcal{M}_{0,T_g}(v)\bigg(1 + \frac{\varepsilon}{2\rho_g}\Big[\frac{v^2}{T_g} - 1\Big]\int_{-\infty}^{\infty}\Big[\frac{u^2}{T_g} - 1\Big]h(x,u)\,\mathrm{d}u + \frac{\varepsilon v}{T_g\rho_g}\int_{-\infty}^{\infty} uh(x,u)\,\mathrm{d}u\bigg) + o(\varepsilon),$$

and so the whole non-linear term is

$$\rho_f\mathcal{M}_{u_f,T_f}(v) = \rho_g\mathcal{M}_{0,T_g}(v)\bigg[1 + \frac{\varepsilon}{\rho_g}\bigg\{\int_{-\infty}^{\infty}\Big(1 + \frac{1}{2}\Big[\frac{v^2}{T_g} - 1\Big]\Big[\frac{u^2}{T_g} - 1\Big] + \frac{uv}{T_g}\Big)h(x,u)\,\mathrm{d}u\bigg\}\bigg].$$

Using then that $-v\partial_x g + \mathcal{L}g = 0$, we end up with the stated operator $\mathfrak{L}$. ■

Next we stress that our linearised operator is a non-self-adjoint operator in the weighted space $L^2(g^{-1})$. Nevertheless, we proceed by decomposing the operator $\mathfrak{L} = \mathcal{T} + \mathfrak{C}$ into a symmetric and an antisymmetric part. This is motivated by techniques in hypocoercivity (see for example [20, 35, 42]) and more recently a technique via Schur complements [7].

Lemma 6. *The linearised equation* (3.1) *has the form*

$$\partial_t f + \mathcal{T} f = \mathfrak{C} f$$

where $\mathfrak{C}$ *is symmetric in* $L^2(g^{-1})$ *and* $\mathcal{T}$ *is antisymmetric. The explicit formulas are given by*

$$\begin{aligned}\mathfrak{C} f = {}& \frac{\alpha}{2\kappa}\int_{-\infty}^{\infty}\Big(\mathcal{M}_{T_g}(v) + \frac{\mathcal{M}_{T_g}(u)}{g(x,u)}g(x,v)\Big) \\ & \times\Big(1 + \frac{uv}{T_g} + \frac{1}{2}\Big[\frac{u^2}{T_g} - 1\Big]\Big[\frac{v^2}{T_g} - 1\Big]\Big) f(x,u)\,\mathrm{d}u \\ & + \frac{1-\alpha}{2\kappa}\int_{-\infty}^{\infty}\Big(\mathcal{M}_\tau(v) + \frac{\mathcal{M}_\tau(u)}{g(x,u)}g(x,v)\Big) f(x,u)\,\mathrm{d}u \\ & - \frac{1}{2\kappa}\Big(1 + \rho_g\frac{\alpha\mathcal{M}_{T_g} + (1-\alpha)\mathcal{M}_\tau}{g(x,v)}\Big) f(x,v),\end{aligned}$$

while

$$\begin{aligned}\mathcal{T} f = {}& -v\partial_x f + \frac{\alpha}{2\kappa}\int_{-\infty}^{\infty}\Big(\mathcal{M}_{T_g}(v) - \frac{\mathcal{M}_{T_g}(u)g(x,v)}{g(x,u)}\Big) \\ & \times\Big(1 + \frac{uv}{T_g} + \frac{1}{2}\Big[\frac{u^2}{T_g} - 1\Big]\Big[\frac{v^2}{T_g} - 1\Big]\Big) f(x,u)\,\mathrm{d}u \\ & + \frac{1-\alpha}{2\kappa}\int_{-\infty}^{\infty}\Big(\mathcal{M}_{\tau(x)}(v) - \frac{\mathcal{M}_{\tau(x)}(u)g(x,v)}{g(x,u)}\Big) f(x,u)\,\mathrm{d}u \\ & - \frac{1}{2\kappa} f(x,v)\Big(1 - \frac{\rho_g(x)}{g(x,v)}(\alpha\mathcal{M}_{T_g(x)}(v) + (1-\alpha)\mathcal{M}_{\tau(x)}(v))\Big).\end{aligned}$$

Proof. We stress here that by $\mathfrak{L}$ we denote the whole operator, together with the transport part, so that

$$\mathfrak{L}h = -v\partial_x h + \frac{1}{\kappa}\bigg(\alpha\mathcal{M}_{T_g(x)}\int_{-\infty}^{\infty}\Big(1+\frac{uv}{T_g}+\frac{1}{2}\Big[\frac{u^2}{T_g}-1\Big]\Big[\frac{v^2}{T_g}-1\Big]\Big)h(x,u)\,\mathrm{d}u + (1-\alpha)\mathcal{M}_{\tau(x)}\int_{-\infty}^{\infty}h(x,u)\,\mathrm{d}u - h\bigg).$$

Calculating the adjoint of this operator, $\mathfrak{L}^*$, in $L^2(g^{-1})$ we find that it is

$$\begin{aligned}\mathfrak{L}^*h &= v\partial_x h\\ &\quad+\frac{1}{\kappa}\bigg[\alpha g(x,v)\int_{-\infty}^{\infty}\frac{\mathcal{M}_{T_g}(u)}{g(x,u)}\Big(1+\frac{uv}{T_g}+\frac{1}{2}\Big[\frac{u^2}{T_g}-1\Big]\Big[\frac{v^2}{T_g}-1\Big]\Big)h(x,u)\,\mathrm{d}u\\ &\qquad+(1-\alpha)\int_{-\infty}^{\infty}\frac{\mathcal{M}_{\tau(x)}(u)}{g(x,u)}h(x,u)\,\mathrm{d}u\, g(x,v)\bigg]\\ &\quad-\frac{\rho_g(x)}{g(x,v)}\Big(\frac{\alpha}{\kappa}\mathcal{M}_{T_g(x)}(v)+\frac{(1-\alpha)}{\kappa}\mathcal{M}_{\tau(x)}(v)\Big)h(x,v).\end{aligned} \tag{3.2}$$

Indeed, the adjoint of the transport part gives, using that g is NESS,

$$\begin{aligned}(-v\partial_x)^*h(x,v) &= v\partial_x h(x,v) - v\frac{\partial_x g(x,v)}{g(x,v)}h(x,v)\\ &= v\partial_x h(x,v)\\ &\quad - g(x,v)^{-1}\rho_g(x)(\alpha\mathcal{M}_{T_g(x)}(v)+(1-\alpha)\mathcal{M}_{\tau(x)}(v))h(x,v)-h(x,v),\end{aligned}$$

which accounts for the first and the last terms in (3.2). Then we use this to get symmetric and antisymmetric parts since $\mathcal{C} = \frac{\mathfrak{L}+\mathfrak{L}^*}{2}$ and $\mathcal{T} = \frac{\mathfrak{L}-\mathfrak{L}^*}{2}$. ∎

Typically in L^2-hypocoercivity (see [20]), the strategy is to prove that the collisional symmetric operator in $L^2(g^{-1})$ satisfies a *microscopic coercivity* property on the orthogonal complement of its null space and that the transport antisymmetric operator satisfies a *macroscopic coercivity* property exactly on the null space of the collisional operator. These conditions imply that g is in the kernel of $\mathcal{L}$. Then, after introducing a modified entropy functional, equivalent to the L^2 norm, under additional general assumptions on the decomposing operators, one can control its dissipation and conclude exponential relaxation of the semigroup to the steady state.

This general framework in L^2 to estimate relaxation rates for a general class of kinetic equations does not apply in our case as it is, because our steady state does not have an explicit formula. In [20] it is required that the global equilibrium lies in the intersection of the null spaces of the two decomposing operators, i.e. the transport and the collision operators. This is not the case in our out-of-equilibrium setting.

To our knowledge there are two articles so far dealing with such circumstances, [12] where the authors treat the problem perturbatively and [13] where the transport and collision operators are redefined.

Our approach is mainly inspired by the latter: in particular, we split the operator differently from [20], into a symmetric and a skew-symmetric part as is shown in Lemma 6. Then we show a microscopic coercivity inequality for the symmetric part and a macroscopic coercivity for the antisymmetric part.

Before we move on with proving such properties, let us stress that now one can check with explicit computations that

$$\mathcal{C}g = \mathcal{T}g = 0,$$

where $\mathcal{C}$, $\mathcal{T}$ are computed explicitly in Lemma 6. So with this decomposition the steady state belongs in the intersection of the kernels of the decomposed part of the operator.

We denote by Π the orthogonal projection to $\mathrm{Ker}(\mathcal{C})$ which is given by

$$\Pi f = \rho_f \frac{g}{\rho_g}, \qquad \text{where } \rho_f(x) = \int_{-\infty}^{\infty} f(x,v)\,\mathrm{d}v.$$

The space that we work in is

$$\mathcal{H} = \left\{ f \in L^2\left(\tfrac{\mathrm{d}x\,\mathrm{d}v}{g}\right) : \textstyle\iint_{\mathbb{T}\times\mathbb{R}} f(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0 \right\}$$

induced by the scalar product

$$\langle f_1, f_2\rangle = \langle f_1, f_2\rangle_{L^2_{x,v}(g^{-1})} = \iint f_1(x,u) f_2(x,u)/g(x,u)\,\mathrm{d}u\,\mathrm{d}x. \tag{3.3}$$

3.3. Microscopic coercivity

This subsection is devoted to the coercivity property of the symmetric operator on the orthogonal of the kernel of $\mathcal{C}$.

Proposition 4 (Microscopic coercivity). *With the above notation, we have that the operator $\mathcal{C}$ satisfies*

$$-\langle \mathcal{C}h, h\rangle \geq \lambda_m \|(I-\Pi)h\|^2$$

for some positive constant λ_m.

Proof. In this proof we will write

$$\langle f_1, f_2\rangle = \langle f_1, f_2\rangle_{L^2(g^{-1})} = \int f_1(u) f_2(u)/g(x,u)\,\mathrm{d}u,$$

which we note is a function of x.

We start by computing $\langle \mathcal{C}h, h\rangle_{L^2(g^{-1})}$. In order to do this let us rewrite

$$\begin{aligned}\mathcal{C}h = \frac{\alpha}{2\kappa}\mathcal{M}_{T_g}\Big(\langle h, g\rangle + \frac{v}{\sqrt{T_g}}\Big\langle h, \frac{u}{\sqrt{T_g}} g\Big\rangle + \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g} - 1\Big]\Big\langle h, \frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g} - 1\Big] g\Big\rangle\Big) \\ + \frac{\alpha}{2\kappa} g(x,v)\Big(\langle h, \mathcal{M}_{T_g}\rangle + \frac{v}{\sqrt{T_g}}\Big\langle h, \frac{u}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle \\ + \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g} - 1\Big]\Big\langle h, \frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big)\end{aligned}$$

$$+\frac{1-\alpha}{2\kappa}(g(x,v)\langle h,\mathcal{M}_\tau\rangle+\mathcal{M}_\tau\langle h,g\rangle)$$
$$-\frac{1}{2\kappa}\Big(1+\rho_g\frac{\alpha\mathcal{M}_{T_g}+(1-\alpha)\mathcal{M}_\tau}{g(x,v)}\Big)h(x,v).$$

Using this we write

$$\langle \mathcal{C}h,h\rangle=\frac{\alpha}{\kappa}\Big(\langle h,\mathcal{M}_{T_g}\rangle\langle h,g\rangle+\Big\langle h,\frac{u}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle\Big\langle h,\frac{u}{\sqrt{T_g}}g\Big\rangle$$
$$+\Big\langle h,\frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g}-1\Big]\mathcal{M}_{T_g}\Big\rangle\Big\langle h,\frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g}-1\Big]g\Big\rangle\Big)+\frac{1-\alpha}{\kappa}\langle h,\mathcal{M}_\tau\rangle\langle h,g\rangle$$
$$-\frac{1}{2\kappa}\int_{-\infty}^{\infty}\Big(1+\rho_g\frac{\alpha\mathcal{M}_{T_g}+(1-\alpha)\mathcal{M}_\tau}{g(x,v)}\Big)\frac{h(x,v)^2}{g(x,v)}\,\mathrm{d}v.$$

We split this into

$$\langle \mathcal{C}h,h\rangle=\frac{1}{\kappa}[\alpha\mathcal{I}+(1-\alpha)\mathcal{J}],$$

where

$$\mathcal{I}=\langle h,\mathcal{M}_{T_g}\rangle\langle h,g\rangle+\Big\langle h,\frac{u}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle\Big\langle h,\frac{u}{\sqrt{T_g}}g\Big\rangle$$
$$+\Big\langle h,\frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g}-1\Big]\mathcal{M}_{T_g}\Big\rangle\Big\langle h,\frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g}-1\Big]g\Big\rangle$$
$$-\frac{1}{2}\int_{-\infty}^{\infty}\Big(1+\rho_g(x)\frac{\mathcal{M}_{T_g}}{g}\Big)\frac{h(x,u)^2}{g(x,u)}\,\mathrm{d}u,$$

while

$$\mathcal{J}=\langle h,\mathcal{M}_\tau\rangle\langle h,g\rangle-\frac{1}{2}\int_{-\infty}^{\infty}\Big(1+\rho_g(x)\frac{\mathcal{M}_{\tau(x)}}{g}\Big)\frac{h(x,u)^2}{g(x,u)}\,\mathrm{d}u,$$

and we are going to bound them separately.

Motivated by this, we consider as a function of v (for a fixed $x\in\mathbb{T}$) the functions

$$p_1(v)g:=\frac{g}{\sqrt{\rho_g}},\quad p_2(v)g:=\frac{vg}{\sqrt{\rho_gT_g}},\quad p_3(v)g:=\frac{(\frac{v^2}{T_g}-\frac{d_3(x)v}{\sqrt{T_g}}-1)g}{\sqrt{\rho_g(2+d_4-d_3^2)}}.$$

We observe that these are orthogonal functions in $L^2_v(g^{-1})$.

Then we decompose h as the sum of $h=\hat{h}+\tilde{h}$ where $\tilde{h}$ is orthogonal in $L^2_v(g^{-1})$ to the space spanned by $p_1(v)g$, $p_2(v)g$, $p_3(v)g$, while $\hat{h}$ belongs in the space that these quantities span. We collect the above observations in the following lemma.

Lemma 7. *For fixed $x\in\mathbb{T}$, we consider the mappings $v\mapsto p_i(v)g(x,v)$ for $i=1,2,3$, where*

$$p_1(v)g:=\frac{g}{\sqrt{\rho_g}},\quad p_2(v)g:=\frac{vg}{\sqrt{\rho_gT_g}},\quad p_3(v)g:=\frac{(\frac{v^2}{T_g}-\frac{d_3(x)v}{\sqrt{T_g}}-1)g}{\sqrt{\rho_g(2+d_4-d_3^2)}}.$$

These are orthogonal functions in $L^2_v(g^{-1})$ of norm 1.

Proof. This is just a computation:

$$\int \Big(\frac{v^2}{T_g} - \frac{d_3 v}{\sqrt{T_g}} - 1\Big)^2 g \,\mathrm{d}v = \frac{\rho_g T_g^2(3+d_4)}{T_g} - \frac{2\rho_g d_3^2 T_g^{3/2}}{T_g^{3/2}} + \frac{\rho_g d_3^2 T_g}{T_g} - 2\frac{\rho_g T_g}{T_g} + \rho_g$$
$$= \rho_g(2 + d_4 - d_3^2). \qquad \blacksquare$$

We continue by writing $h = [a p_1 g + \beta p_2 g + \gamma p_3 g] + \tilde{h}$ for some constants a, β, γ and where $\tilde{h}$ is orthogonal to the space spanned by $p_1 g$, $p_2 g$, $p_3 g$ in $L^2_v(g^{-1})$. In the following lemma we collect the explicit values of each term appearing in $\mathcal{I}$, $\mathcal{J}$. One can check these by explicit computations.

Lemma 8. *Decomposing the solution of* (3.1) *as* $h = a p_1 g + \beta p_2 g + \gamma p_3 g + \tilde{h}$, *for some constants* a, β, γ, *then*

$$\langle h, g\rangle = a\sqrt{\rho_g}, \quad \Big\langle h, \frac{u}{\sqrt{T_g}} g\Big\rangle = \beta\sqrt{\rho_g},$$
$$\Big\langle h, \frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g} - 1\Big] g\Big\rangle = \gamma\sqrt{\frac{\rho_g(2 + d_4 - d_3^2)}{2}} + \beta d_3\sqrt{\rho_g}$$

and

$$\langle h, \mathcal{M}_{T_g}\rangle = \frac{a}{\sqrt{\rho_g}} + \langle \tilde{h}, \mathcal{M}_{T_g}\rangle,$$
$$\Big\langle h, \frac{u}{\sqrt{T_g}}\Big\rangle = \frac{\beta}{\sqrt{\rho_g}} - \frac{\gamma d_3}{\sqrt{\rho_g(2 + d_4 - d_3^2)}} + \Big\langle \tilde{h}, \frac{u}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle.$$

From now on we will denote by $Z = Z(x) := 2 + d_4(x) - d_3(x)^2$ and let us note that we are able to bound this from below and above uniformly in x, thanks to Proposition 3 with constants depending on α, $\underline{\tau}$, $\bar{\tau}$. Inserting these values into $\mathcal{I}$, for its first three terms we have

$$\begin{aligned}
&\langle h, \mathcal{M}_{T_g}\rangle\langle h, g\rangle + \Big\langle h, \frac{u}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle\Big\langle h, \frac{u}{\sqrt{T_g}} g\Big\rangle \\
&\qquad + \Big\langle h, \frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big\langle h, \frac{1}{\sqrt{2}}\Big[\frac{u^2}{T_g} - 1\Big] g\Big\rangle \\
&\quad = a^2 + \beta^2 + \gamma^2 \\
&\qquad + \sqrt{\rho_g}\Big[a\langle \tilde{h}, \mathcal{M}_{T_g}\rangle + \beta\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle + \frac{\gamma\sqrt{Z} + \beta d_3}{\sqrt{2}}\Big\langle \tilde{h}, \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big].
\end{aligned}$$

Moving to the fourth term in $\mathcal{I}$,we first note that we have

$$\int \frac{h^2}{g} = a^2 + \beta^2 + \gamma^2 + \int \frac{\tilde{h}^2}{g}, \tag{3.4}$$

and therefore

$$\begin{aligned}\int \frac{h^2}{g^2}\rho_g \mathcal{M}_{T_g} &= \int \frac{\hat{h}^2}{g^2}\rho_g \mathcal{M}_{T_g} + \int \frac{\tilde{h}^2}{g^2}\rho_g \mathcal{M}_{T_g} \\ &= a^2 + \beta^2 + \frac{\gamma^2(2+d_3^2)}{Z} - \frac{2\beta\gamma}{\sqrt{Z}} d_3 \\ &\quad + \sqrt{\rho_g}\Big[2a\langle \tilde{h}, \mathcal{M}_{T_g}\rangle + 2\Big(\beta - \frac{\gamma d_3}{\sqrt{Z}}\Big)\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle \\ &\qquad + 2\gamma\sqrt{\frac{2}{Z}}\Big\langle \tilde{h}, \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big] \\ &\quad + \int_{-\infty}^{\infty} \frac{\tilde{h}(x,v)^2}{g(x,v)^2}\rho_g(x)\mathcal{M}_{T_g}(v)\,\mathrm{d}v.\end{aligned}$$

Putting this together we get

$$\begin{aligned}\mathcal{I} &\le a^2 + \beta^2 + \gamma^2 + a\sqrt{\rho_g}\langle \tilde{h}, \mathcal{M}_{T_g}\rangle + 2\beta\sqrt{\rho_g}\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle \\ &\quad + \Big\langle \tilde{h}, \Big[\frac{v^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle \sqrt{\frac{\rho_g}{2}}\Big(\gamma\sqrt{\frac{Z}{2}} + \frac{\beta d_3}{\sqrt{2}}\Big) - \frac{a^2}{2} - \frac{\beta^2}{2} - \frac{\gamma^2(2+d_3^2)}{2Z} \\ &\quad - \frac{\beta\gamma}{\sqrt{Z}} d_3 - \sqrt{\rho_g}\Big(a\langle \tilde{h}, \mathcal{M}_{T_g}\rangle + \Big(\beta - \frac{\gamma d_3}{\sqrt{Z}}\Big)\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle \\ &\qquad + \gamma\sqrt{\frac{2}{Z}}\Big\langle \tilde{h}, \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big) \\ &\quad - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tilde{h}(x,v)^2}{g(x,v)^2}\rho_g(x)\mathcal{M}_{T_g}(v)\,\mathrm{d}v - \frac{a^2}{2} - \frac{\beta^2}{2} - \frac{\gamma^2}{2} - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tilde{h}(x,v)^2}{g(x,v)}\,\mathrm{d}v \\ &\le \frac{\gamma^2}{2}\Big(1 - \frac{2+d_3^2}{2+d_4-d_3^2}\Big) + \frac{\beta\gamma}{\sqrt{Z}}|d_3| \\ &\quad + \sqrt{\rho_g}\Big[\frac{\gamma d_3}{\sqrt{Z}}\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle + \Big(\frac{\gamma}{\sqrt{2}}\Big(\sqrt{\frac{Z}{2}} - \sqrt{\frac{2}{Z}}\Big) + \frac{\beta d_3}{\sqrt{2}}\Big)\Big\langle \tilde{h}, \Big[\frac{v^2}{T_g} - 1\Big]\mathcal{M}_{T_g}\Big\rangle\Big] \\ &\quad - \frac{1}{2}\int \frac{\tilde{h}^2}{g}\Big(1 + \rho_g\frac{\mathcal{M}_{T_g}}{g}\Big).\end{aligned} \tag{3.5}$$

Now regarding the terms involving $\tilde{h}$ we first estimate by Cauchy–Schwarz and then use that $\int_{-\infty}^{\infty} v^2\mathcal{M}_{T_g}(v)\,\mathrm{d}v = T_g$:

$$\begin{aligned}\sqrt{\rho_g(x)}\Big\langle \tilde{h}, \frac{v}{\sqrt{T_g}}\mathcal{M}_{T_g}\Big\rangle &= \sqrt{\rho_g(x)}\int_{-\infty}^{\infty} \tilde{h}(x,v)\frac{v}{\sqrt{T_g}}\frac{\mathcal{M}_{T_g}(v)}{g(x,v)}\,\mathrm{d}v \\ &\le \Big(\int_{-\infty}^{\infty} \frac{\tilde{h}(x,v)^2}{g(x,v)^2}\rho_g(x)\mathcal{M}_{T_g}(v)\,\mathrm{d}v\Big)^{1/2}\Big(\int_{-\infty}^{\infty} \frac{v^2}{T_g}\mathcal{M}_{T_g}(v)\,\mathrm{d}v\Big)^{1/2} \\ &= \Big(\int_{-\infty}^{\infty} \frac{\tilde{h}^2}{g^2}\rho_g(x)\mathcal{M}_{T_g}(v)\,\mathrm{d}v\Big)^{1/2},\end{aligned} \tag{3.6}$$

and similarly

$$\sqrt{\rho_g(x)}\Big\langle \tilde{h}, \frac{1}{\sqrt{2}}\Big[\frac{v^2}{T_g}-1\Big]\mathcal{M}_{T_g}\Big\rangle \leq \Big(\int_{-\infty}^{\infty}\frac{\tilde{h}^2}{g^2}\rho_g(x)\mathcal{M}_{T_g}(v)\,\mathrm{d}v\Big)^{1/2}. \tag{3.7}$$

The first line on the right-hand side of (3.5) becomes

$$\frac{\gamma^2}{2}\frac{d_4-2d_3^2}{2+d_4-d_3^2}+\frac{\beta\gamma d_3}{\sqrt{Z}}.$$

The rest of the terms are estimated by (3.6), (3.7), and Young's inequality. Altogether we get

$$\begin{aligned}\mathcal{I} &\leq \gamma^2\frac{d_4-2d_3^2}{2+d_4-d_3^2}+\frac{\beta\gamma d_3}{\sqrt{Z}}+\frac{1}{4}\Big(\frac{d_3|\gamma|}{\sqrt{Z}}+\Big|\gamma\Big(\sqrt{\frac{Z}{2}}-\sqrt{\frac{2}{Z}}\Big)+\frac{\beta d_3}{\sqrt{2}}\Big|\Big)^2\\ &\leq \varphi(d_3,d_4)(\beta^2+\gamma^2)\leq \varphi(d_3,d_4)\int_{-\infty}^{\infty}\frac{h^{\perp}(x,v)^2}{g(x,v)}\,\mathrm{d}v,\end{aligned}$$

where

$$h^{\perp} := (I-\Pi)h = h-\Big\langle h, \frac{g}{\sqrt{\rho_g}}\Big\rangle\frac{g}{\sqrt{\rho_g}} = h-\Big\langle h, p_1 g\Big\rangle\frac{g}{\sqrt{\rho_g}}.$$

Moreover, $\varphi(d_3,d_4)$ is a continuous function of d_3, d_4 which goes to 0 as $d_3, d_4 \to 0$. The last inequality above follows from equation (3.4) as we can write $h = h^{\perp}+ap_1 g$, which implies that

$$a^2+\beta^2+\gamma^2+\int\frac{\tilde{h}^2}{g}=\int\frac{h^2}{g}=\int\frac{(h^{\perp})^2}{g}+a^2;$$

cf. Lemma 7. Then $\beta^2+\gamma^2=\int\frac{(h^{\perp})^2}{g}-\int\frac{h^2}{g}$, and the claim follows.

Now we move onto $\mathcal{J}$ which is simpler as here we only want to split h into its parts that are parallel and perpendicular to g. We have

$$\begin{aligned}\mathcal{J} &= a^2+a\sqrt{\rho_g}\langle h^{\perp},\mathcal{M}_{\tau}\rangle-a^2-a\frac{1}{\sqrt{\rho_g}}\int\Big(1+\rho_g\frac{\mathcal{M}_{\tau}}{g}\Big)h^{\perp}\\ &\quad -\frac{1}{2}\int\Big(1+\rho_g\frac{\mathcal{M}_{\tau}}{g}\Big)\frac{(h^{\perp})^2}{g}\\ &= -\frac{1}{2}\int\Big(1+\rho_g\frac{\mathcal{M}_{\tau}}{g}\Big)\frac{(h^{\perp})^2}{g}.\end{aligned}$$

Therefore,

$$\langle Ch,h\rangle \leq \frac{1}{\kappa}\Big(\alpha\varphi(d_3,d_4)-\frac{1-\alpha}{2}\Big)\int\frac{(h^{\perp})^2}{g}.$$

Now d_3, d_4 change with α; however, from Proposition 3 we see that we do not have singularities in terms of α in the upper and lower bounds of d_3, d_4. Thus we can bound the moments uniformly over all $\alpha \in [0, 1]$. This implies that there is a finite constant M so that $\varphi(d_3, d_4) \leq M(\kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau}) < \infty$ for M independent of α. Then

$$\langle \mathscr{C}h, h \rangle \leq \frac{1}{\kappa}\Big(\alpha M(\kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau}) - \frac{1-\alpha}{2}\Big) \int \frac{(h^{\perp})^2}{g},$$

which yields microscopic coercivity as long as

$$\alpha > \frac{1}{2M(\kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau}) - 1}.$$

Since we can make this bound $M(\kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau})$ large, we ensure that there is a positive constant λ_m so the statement of the proposition holds true. ■

3.4. Macroscopic coercivity

We recall that if Π is the projection onto the kernel of $\mathscr{C}$ then $\Pi h = \frac{\rho_h}{\rho_g} g$. We proceed by providing a macroscopic coercivity inequality for the antisymmetric part of the linearised operator on the null space of $\mathscr{C}$.

Proposition 5. *Let the condition (2.5) on α hold, namely α is sufficiently small so that*

$$\alpha\Big(\sqrt{\underline{\tau}} - \sqrt{\underline{T}} + \frac{4\pi}{\kappa^2 \underline{T}^{1/4} \underline{\tau}^{1/4}}(\sqrt{\underline{\tau}} - \underline{T}^{1/4}\underline{\tau}^{1/4}) + \frac{2\pi}{\kappa^2}\alpha\Big(\frac{1}{\underline{T}^{1/4}} - \frac{1}{\underline{\tau}^{1/4}}\Big)^2\Big)$$
$$\leq \sqrt{\underline{\tau}} - \frac{2\pi}{\kappa^2}\frac{1}{\sqrt{\underline{\tau}}}.$$

Then with the same notation as above, for the antisymmetric operator there exists a positive constant λ_M so that

$$\|\mathcal{T}\Pi h\| \geq \lambda_M \|\Pi h\|$$

for all $h \in \mathcal{H} \cap \mathcal{D}(\mathcal{T}\Pi)$.

Proof. We first recall (cf. Lemma 6) that the antisymmetric part $\mathcal{T}$ is given by

$$\begin{aligned}\mathcal{T}h = v\partial_x h &+ \frac{\alpha}{2\kappa}\int \Big(\mathcal{M}_{T_g}(v) - \frac{\mathcal{M}_{T_g}(u) g(x,v)}{g(x,u)}\Big) \\ &\qquad \times \Big(1 + \frac{uv}{T_g} + \frac{1}{2}\Big(\frac{u^2}{T_g} - 1\Big)\Big(\frac{v^2}{T_g} - 1\Big)\Big) h(x,u)\,\mathrm{d}u \\ &+ \frac{1-\alpha}{2\kappa}\int \Big(\mathcal{M}_{\tau}(v) - \frac{\mathcal{M}_{\tau}(u) g(x,v)}{g(x,u)}\Big) h(x,u)\,\mathrm{d}u \\ &- \frac{1}{2\kappa}\Big(h - \frac{\rho_g}{g}(\alpha \mathcal{M}_{T_g}(v) + (1-\alpha)\mathcal{M}_{\tau}(v)) h\Big).\end{aligned}$$

From this we can explicitly compute, since $\mathcal{T}g = 0$, that

$$\mathcal{T}\Pi h = v\partial_x\Big(\frac{\rho_h}{\rho_g}\Big) g.$$

Consequently, with respect to the inner product defined in (3.3),

$$\begin{aligned}\|\mathcal{T}\Pi h\|^2 &= \iint_{\mathbb{T}\times\mathbb{R}} v^2 g(x,v)\Big(\partial_x\Big(\frac{\rho_h}{\rho_g}\Big)\Big)^2 \,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{T}} \rho_g(x) T_g(x)\Big(\partial_x\Big(\frac{\rho_h}{\rho_g}\Big)\Big)^2 \,\mathrm{d}x \\ &\geq \frac{1}{4}\Big[\Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big] - \frac{1}{\kappa}\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\Big] \\ &\quad\times\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^{-2} \int_{\mathbb{T}} \rho_g(x)\Big(\partial_x\Big(\frac{\rho_h}{\rho_g}\Big)\Big)^2 \,\mathrm{d}x,\end{aligned}$$

where we applied the lower bound on the temperature from Lemma 4. Therefore we conclude the macroscopic coercivity provided that $\rho_g(x)\,\mathrm{d}x$ satisfies a Poincaré inequality: Indeed, then

$$\|\mathcal{T}\Pi h\|^2 \geq \lambda_M \int_{\mathbb{T}} \rho_g(x)\Big(\frac{\rho_h}{\rho_g}\Big)^2(x)\,\mathrm{d}x = \lambda_M \iint_{\mathbb{T}\times\mathbb{R}} g(x,v)^{-1}|\Pi h|^2\,\mathrm{d}x\,\mathrm{d}v = \lambda_M\|\Pi h\|^2,$$

where $\lambda_M = \lambda_M(\alpha,\kappa,\overline{T},\underline{T},\bar{\tau},\underline{\tau})$. Now a Poincaré inequality is true since our space is the torus and ρ_g is upper and lower bounded uniformly in $x \in \mathbb{T}$: For $h \in \mathcal{H}$, i.e. $\int_{\mathbb{T}}(\frac{\rho_h}{\rho_g})(x)\,\mathrm{d}x = 0$ (this is equivalent by the upper and lower bound on the density with $\int_{\mathbb{T}}\rho_h\,\mathrm{d}x = 0$), it holds that

$$\int_{\mathbb{T}}\Big(\partial_x\Big(\frac{\rho_h}{\rho_g}\Big)\Big)^2\,\mathrm{d}x \geq \int_{\mathbb{T}}\Big(\frac{\rho_h}{\rho_g}\Big)^2\,\mathrm{d}x.$$

Hence macroscopic coercivity holds with the explicit constant

$$\begin{aligned}\lambda_M &= \frac{1}{4}\Big[\Big[\alpha\frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha)\frac{\underline{\tau}^{1/2}}{2\pi}\Big] - \frac{1}{\kappa}\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^2\Big]\Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa\underline{\tau}^{1/4}}\Big]^{-2} \\ &\quad\times[1 - [\alpha\underline{T}^{-1/2} + (1-\alpha)\underline{\tau}^{-1/2}]].\end{aligned}$$

■

3.5. Boundedness of auxiliary operators

From above we can see that

$$\Pi\mathcal{T}\Pi h = 0 \tag{3.8}$$

since $\int u g(x,u)\,\mathrm{d}u = 0$. Indeed,

$$\Pi\mathcal{T}\Pi h = \int u\partial_x\Big(\frac{\rho_h}{\rho_g}\Big)(x) g(x,u)\,\mathrm{d}u \frac{g(x,v)}{\rho_g(x)} = \partial_x\Big(\frac{\rho_h}{\rho_g}\Big)(x) u_g(x)\frac{g(x,v)}{\rho_g(x)} = 0.$$

Following the approach in [20], we define the operator

$$A := (1 + (\mathcal{T}\Pi)^*\mathcal{T}\Pi)^{-1}(\mathcal{T}\Pi)^*.$$

We will use this operator in order to define the modified entropy for the hypocoercivity. The final ingredient we will need is the boundedness of certain operators. In this subsection we record some properties of these operators. The first thing to notice is that $A = \Pi A$.

This is easy to see for example from the relation

$$Af = -\Pi\mathcal{T}f + \Pi\mathcal{T}^2\Pi Af, \quad \text{for all } f \in \mathcal{H}, \tag{3.9}$$

which follows directly from the definition of A. Then taking the inner product in (3.9) with Af we see that

$$\|Af\|^2 = \langle f, \mathcal{T}\Pi Af\rangle - \|\mathcal{T}\Pi Af\|^2 \leq \|(1-\Pi)f\|\|\mathcal{T}\Pi Af\| - \|\mathcal{T}\Pi Af\|^2$$
$$\text{or} \quad \|Af\|^2 + \|\mathcal{T}\Pi Af\|^2 \lesssim \frac{\|(1-\Pi)f\|^2}{2\varepsilon} + \frac{\varepsilon}{2}\|\mathcal{T}\Pi Af\|^2$$

for $\varepsilon > 0$, where we used (3.8). This yields the boundedness of both A and $\mathcal{T}A$ for all $f \in \mathcal{H}$.

In [20] they require a stronger hypothesis regarding the boundedness of auxiliary operators, namely that

$$\|A\mathcal{T}(I-\Pi)f\| + \|A\mathcal{C}f\| \lesssim \|(I-\Pi)f\|.$$

It is immediate on studying the proof in this paper that it is also sufficient to show that

$$\|A\mathfrak{L}(I-\Pi)f\| \lesssim \|(I-\Pi)f\|.$$

Lemma 9. *With the above notation the operators*

$$A, \ \mathcal{T}A, \ \textit{and} \ A\mathfrak{L}$$

are bounded and

$$\|A\mathfrak{L}(I-\Pi)f\| \leq C\|(I-\Pi)f\|$$

for some constant C depending on α, κ, $\overline{T}$, $\underline{T}$, $\bar{\tau}$, $\underline{\tau}$.

Proof. The operator $A\mathfrak{L}$ is bounded if and only if $\mathfrak{L}^*A^*$ is bounded. We show this is bounded by adapting an elliptic-regularity-style result as in [20] but here with some extra terms. Let us write

$$\tilde{f} := (I + (\mathcal{T}\Pi)^*(\mathcal{T}\Pi))^{-1}f$$

and define $m(x) = \frac{\rho_{\tilde{f}}}{\rho_g}(x)$. Then

$$\begin{aligned}
\mathfrak{L}^*A^*f &= \mathfrak{L}^*\mathcal{T}\Pi\tilde{f} = \mathfrak{L}^*(v(\partial_x m)g(x,v)) \\
&= v\partial_x(v(\partial_x m)g(x,v)) \\
&\quad + \frac{\partial_x m}{\kappa}\Big(\alpha g(x,v)\int u\mathcal{M}_{T_g}(u)\Big(1 + \frac{uv}{T_g} + \frac{1}{2}\Big[\frac{u^2}{T_g} - 1\Big]\Big[\frac{v^2}{T_g} - 1\Big]\Big)\,\mathrm{d}u \\
&\qquad + (1-\alpha)\int \frac{\mathcal{M}_{\tau(x)}(u)}{g(x,u)}ug(x,u)\,\mathrm{d}u\, g(x,v)\Big) \\
&\quad - \frac{\rho_g(x)}{g(x,v)}\Big(\frac{\alpha}{\kappa}\mathcal{M}_{T_g(x)}(v) + \frac{(1-\alpha)}{\kappa}\mathcal{M}_{\tau(x)}(v)\Big)vg(x,v)(\partial_x m) \\
&= v^2(\partial_x^2 m(x))g(x,v) - \frac{1-\alpha}{\kappa}(\partial_x m(x))vg(x,v),
\end{aligned}$$

where in the last line we used that g is a NESS and the cancellation of the odd Maxwellian moments. So taking the $L^2_{x,v}(g^{-1})$ norm we have

$$\|\mathfrak{L}^* A^* f\| \leq \left\| T_g \sqrt{\rho_g}[\partial_x^2 m] \int v^4 g \, \mathrm{d}v \right\|_{L^2(\mathbb{T})} + \frac{(1-\alpha)}{\kappa} \| \sqrt{\rho_g T_g} (\partial_x m(x)) \|_{L^2(\mathbb{T})}$$
$$\leq C \|m\|_{H^2(\mathbb{T})}, \tag{3.10}$$

where $C = C(\alpha, \kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau})$ and for the bounds we used the results from Lemma 3 and Proposition 3. We furthermore know that

$$(I + (\mathcal{T}\Pi)^*(\mathcal{T}\Pi))\tilde{f} = \tilde{f} + T_g(\partial_x^2 m) g = f.$$

Integrating this and dividing by ρ_g gives

$$\frac{1}{T_g(x)} m(x) - (\partial_x^2 m(x)) = \frac{1}{T_g(x)} n_g(x),$$

where $n_g(x) = \frac{\rho_f}{\rho_g}(x)$. Then since T_g is bounded above and below (see Lemma 4), $L^2 \to H^2$ elliptic regularity gives us that

$$\|m\|_{H^2(\mathbb{T})} \leq C \left\| \frac{n_g}{T_g} \right\|_{L^2(\mathbb{T})}$$
$$\leq C \Big(\frac{1}{4} \Big[\Big[\alpha \frac{\underline{T}^{1/2}}{2\pi} + (1-\alpha) \frac{\underline{\tau}^{1/2}}{2\pi} \Big] - \frac{1}{\kappa} \Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}} \Big]^2 \Big]$$
$$\times \Big[\frac{\alpha}{\kappa \underline{T}^{1/4}} + \frac{1-\alpha}{\kappa \underline{\tau}^{1/4}} \Big]^{-2} \Big)^{-1} \|n_g\|_{L^2(\mathbb{T})}.$$

This in turn yields from (3.10) that

$$\|\mathfrak{L}^* A^* f\| \leq C(\alpha, \kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau}) \|n_g\|_{L^2(\mathbb{T})} \leq C(\alpha, \kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau}) \|f\|_{L^2(g^{-1})},$$

which concludes the lemma. ∎

3.6. Proof of Theorem 2

As we have verified the main assumptions required for the abstract L^2 stability theorem – microscopic coercivity of $\mathcal{C}$, macroscopic coercivity of $\mathcal{T}$, and boundedness of certain operators – the stability of our linearised equation follows directly by considering, for A defined in (3.9), the modified entropy

$$H(f) := \frac{1}{2} \|f\|^2 + \varepsilon \langle Af, f \rangle,$$

where $\varepsilon > 0$. One then computes the entropy dissipation

$$D(f) = -\langle \mathcal{C} f, f \rangle + \varepsilon \langle A \mathcal{T} \Pi f, f \rangle + \varepsilon \langle A \mathcal{C} (I - \Pi) f, f \rangle - \varepsilon \langle \mathcal{T} A f, f \rangle$$
$$= -\langle \mathcal{C} f, f \rangle + \varepsilon \langle A \mathcal{T} \Pi f, f \rangle + \varepsilon \langle A \mathfrak{L} (I - \Pi) f, f \rangle - \varepsilon \langle \mathcal{T} A f, f \rangle.$$

The coercivity of the Dirichlet form follows thanks to Propositions 4 and 5, via which we treat the first two terms:

$$-\langle \mathcal{C} f, f\rangle + \varepsilon\langle A\mathcal{T}\Pi f, f\rangle \geq \lambda_m \wedge \left[\frac{\varepsilon\lambda_M}{1+\lambda_M}\right]\|f\|^2$$

for $\lambda_M = \lambda_M(\alpha, \kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau})$ and the same for λ_m. Then up to sufficiently small $\varepsilon \in (0,1)$, (i) the new perturbed entropy is equivalent to the L^2 norm, i.e. $\frac{1}{2}(1-\varepsilon)\|f\|^2 \leq H(f) \leq \frac{1}{2}(1+\varepsilon)\|f\|^2$ and (ii) also using the boundedness of $A\mathfrak{L}$ (cf. Lemma 9), we conclude the existence of a positive constant $\lambda = \lambda(\alpha, \kappa, \overline{T}, \underline{T}, \bar{\tau}, \underline{\tau})$ for which $D(f) \geq \lambda\|f\|^2$. From this, due to the equivalence, we get from Grönwall an exponential relaxation of the semigroup $e^{t\mathfrak{L}}$ in $H(f)$ and therefore in L^2 up to prefactors.

A. Linear stability when the temperature variation is small

Theorem 4. *Suppose that there exist a τ_* and an $\epsilon > 0$ such that $|\tau(x) - \tau| < \epsilon$ and there exists a steady state g such that $|T_g(x) - \tau_*| \leq C\epsilon$ for some C that can depend on τ_* but not on ϵ. Then the steady state g is linearly stable provided ϵ is sufficiently small in terms of τ_*.*

Proof. Let us define

$$\mathfrak{L}_{\tau_*} f = -v\partial_x f + \frac{1}{\kappa}\left(\mathcal{M}_{\tau_*}\int_{-\infty}^{\infty}\Big(1 + \alpha\frac{uv}{\tau_*} + \frac{1}{2}\Big(\frac{u^2}{\tau_*} - 1\Big)\Big(\frac{v^2}{\tau_*} - 1\Big)\Big) f(x,u)\,\mathrm{d}u - f\right).$$

Then we notice that using a standard L^2-hypocoercivity argument we can find a norm $\|\cdot\|_*$ which is equivalent to the $L^2(\mathcal{M}_{\tau_*}^{-1})$ norm for which $\mathfrak{L}_{\tau_*}$ has a spectral gap. We write

$$\langle \mathfrak{L}_{\tau_*} f, f\rangle_* \leq -\lambda_*\|f\|_*^2,$$

where λ_* depends on τ_*, κ, and the choice of norm. We also notice that

$$\begin{aligned}|\mathfrak{L} f - \mathfrak{L}_{\tau_*} f| \leq \frac{1}{\kappa}\Big(&C(\mathcal{M}_{T_g} - \mathcal{M}_{\tau_*})\Big(\rho_f + \frac{v}{\tau_* - \epsilon}m_f + \frac{1+v^2}{(\tau_*-\epsilon)^2}P_f\Big)\\ &+ C\mathcal{M}_{\tau_*}\Big(1 + \frac{1}{\tau_* - \epsilon}\Big)(1+v^2)\epsilon + (\mathcal{M}_\tau - \mathcal{M}_{\tau_*})\rho_f\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathfrak{L} f - \mathfrak{L}_{\tau_*} f\|_{L^2(\mathcal{M}_{\tau_*}^{-1})} \leq C(\tau_*)\big(&\epsilon^2 + \|(1+v^2)(\mathcal{M}_{T_g} - \mathcal{M}_{\tau_*})\|_{L^2_v(\mathcal{M}_{\tau_*}^{-1})}\\ &+ \|\mathcal{M}_\tau - \mathcal{M}_{\tau_*}\|_{L^2_v(\mathcal{M}_{\tau_*}^{-1})}\big)\|f\|.\end{aligned}$$

We can show that

$$\begin{aligned}\int_v (1+v^2)[M_T(v) - M_{T(x)}(v)]^2 M_T^{-1}dv &= \int_v (1+v^2)\Big[1 - \frac{M_{T(x)}(v)}{M_T(v)}\Big]^2 M_T(v)\,\mathrm{d}v\\ &\sim \int_v (1+v^2)M_T(v)\Big[1 - \sqrt{\frac{T}{T(x)}}e^{-\frac{|v|^2}{2}(T(x)^{-1} - T^{-1})}\Big]^2\mathrm{d}v.\end{aligned}$$

If we write $T(x) = T + \theta(x)$ with $\sup_x |\theta(x)| \le \varepsilon$,

$$\int_v (1+v^2) M_T(v) \Big[1 - \sqrt{\frac{T}{T(x)}} e^{-\frac{|v|^2}{2T}(\frac{1}{1-\theta(x)/T}-1)} \Big]^2 \, \mathrm{d}v$$

$$\sim \int_v (1+v^2) M_T(v) \Big[1 - \sqrt{\frac{T}{T(x)}} e^{\frac{|v|^2 \theta(x)}{2} + \frac{|v|^2}{2T}} \Big]^2 \, \mathrm{d}v$$

$$\lesssim \frac{\varepsilon^2}{4T^2} \int_v (1+v^2) M_T(v) e^{|v|^2 \varepsilon} \, \mathrm{d}v.$$

Applying the above for our temperatures $\tau(x)$, τ_*, T_g, we have

$$\| \mathfrak{L} f - \mathfrak{L}_{\tau_*} f \|_{L^2(\mathcal{M}_{\tau_*}^{-1})} \le C(\tau_*) \epsilon^2 \| f \|.$$

Therefore, as our other norm is equivalent we have

$$\| \mathfrak{L} f - \mathfrak{L}_{\tau_*} f \|_* \le C'(\tau_*) \epsilon^2 \| f \|.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \| f \|_*^2 \le -(\lambda_* - C'(\tau_*) \epsilon^2) \| f \|_*^2.$$

So indeed Grönwall implies that if ϵ is small enough we will have a positive spectral gap in the $*$-norm for the operator $\mathfrak{L}$. ■

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