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Traveling waves and finite gap potentials for the Calogero–Sutherland derivative nonlinear Schrödinger equation

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Abstract. We consider the Calogero–Sutherland derivative nonlinear Schrödinger equation $i\partial_t u + \partial_x^2 u \pm \frac{2}{i}\partial_x \Pi(|u|^2)u = 0$, $x \in \mathbb{T}$, where Π is the Szegő projector $\Pi(\sum_{n\in\mathbb{Z}} \hat{u}(n)\mathrm{e}^{inx}) = \sum_{n\geq 0} \hat{u}(n)\mathrm{e}^{inx}$. First, we characterize the *traveling wave* $u_0(x-ct)$ solutions to the defocusing equation (CS$^-$), and prove for the focusing equation (CS$^+$) that all the traveling waves must be either constant functions, or plane waves, or rational functions. A noteworthy observation is that the (CS) equation, which is an L^2-critical equation, is one of the few nonlinear PDEs enjoying non-trivial traveling waves with arbitrarily small and large L^2-norms. Second, we study the *finite gap potentials*, and show that they are also rational functions, containing the traveling waves, and they can be grouped into sets that remain invariant under the evolution of the system.

Contents

1. Introduction

In recent decades, the theory of traveling wave solutions has been the subject of intense research in theoretical and numerical analysis. Indeed, many nonlinear PDEs exhibit these

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types of waves [2, 10, 11]. They are important because they are explicit solutions for nonlinear PDEs, and they can sometimes provide information regarding the dynamics of the equation. However, the problem of proving the existence of these waves can be more or less challenging depending on the nonlinear part of the PDE.

In this paper, we consider a type of derivative nonlinear Schrödinger equation with a nonlocal nonlinearity, called the *Calogero–Sutherland derivative nonlinear Schrödinger equation*

$$i\partial_t u + \partial_x^2 u \pm 2D\Pi(|u|^2)u = 0, \quad x \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}), \tag{CS}$$

where $D = -i\partial_x$, and Π denotes the Szegő projector

$$\Pi\Big(\sum_{n\in\mathbb{Z}} \hat{u}(n)\mathrm{e}^{inx}\Big) := \sum_{n\geq 0} \hat{u}(n)\mathrm{e}^{inx}, \tag{1.1}$$

which is an orthogonal projector from $L^2(\mathbb{T})$ into the Hardy space

$$L^2_+(\mathbb{T}) := \big\{u \in L^2(\mathbb{T}) \;\big|\; \hat{u}(n) = 0,\ \forall n \in \mathbb{Z}_{\leq -1}\big\}. \tag{1.2}$$

We are interested in studying the traveling waves $u_0(x - ct)$ of this equation in the focusing (with sign $+$ in front of the nonlinearity) and defocusing cases (with sign $-$) in the periodic setting, namely when $x \in \mathbb{T}$. As noted in [2], the presence of the nonlocal operator $D\Pi$ appearing in the nonlinearity can make the problem of the existence of traveling waves more complicated. In this paper, the approach to characterizing the traveling waves is based on studying them, in a first stage, spectrally, i.e. by means of the spectral property of the Lax operator related to this equation (see below), before deriving, in a second stage, their *explicit formulas*.[1]

1.1. Main results

Settings and notation. In the sequel, our study takes place with potentials in the Hardy Sobolev spaces of the torus

$$H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T}), \quad s \geq 0,$$

where $L^2_+(\mathbb{T})$ is defined in (1.2) and H^s refers to the Sobolev space. We equip $L^2_+(\mathbb{T})$ with the standard inner product of $L^2(\mathbb{T})$,

$$\langle u|v\rangle = \int_0^{2\pi} u\bar{v}\,\frac{dx}{2\pi}.$$

[1] It should be noted that the idea of using the spectral theory to derive the traveling waves of (CS) draws inspiration from [15, Appendix B], where the authors provide an alternative proof to the characterization of the traveling waves for the Benjamin–Ono equation [1, 6] by first characterizing them spectrally.

We recall also, that via the isometric isomorphism

$$z \in \mathbb{D},\ \ u(z) = \sum_{k\geq 0} \hat{u}(k) z^k \quad \mapsto \quad u^*(x) := \sum_{k\geq 0} \hat{u}(k) \mathrm{e}^{ikx},\ \ x \in \mathbb{T},$$

$$\sum_{k\geq 0} |\hat{u}(k)|^2 < \infty,$$

one can interpret any element of the Hardy space as an analytic function on the open unit disc $\mathbb{D}$, whose trace on the boundary $\partial\mathbb{D}$ is in L^2.[2] We shall frequently utilize this property in various proofs. Furthermore, we denote by $\mathbb{D}$ the open unit disc on $\mathbb{C}$, $\mathbb{D}^* := \{z \in \mathbb{C};\ 0 < |z| < 1\}$. Moreover, $\mathbb{N} \equiv \mathbb{N}_{\geq 1}$ denotes the positive integers $1, 2, 3, \dots$, and for all $a \in \mathbb{N} \cup \{0\}$, $\mathbb{N}_{\geq a}$ refers to the set of integer numbers $\{n \in \mathbb{Z};\ n \geq a\}$.

First, we deal with the defocusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u - 2D\Pi(|u|^2)u = 0. \tag{CS$^-$}$$

We denote by $\mathscr{G}_1$ the set of trivial traveling waves, made up from constant functions and plane wave solutions

$$\mathscr{G}_1 = \left\{ C\mathrm{e}^{iN(x-Nt)} \;\middle|\; C \in \mathbb{C},\ N \in \mathbb{N}_{\geq 0} \right\}. \tag{1.3}$$

Theorem 1.1 (Characterization of the traveling waves of (CS$^-$)). *A potential u is a traveling wave of (CS$^-$) if and only if $u \in \mathscr{G}_1$ or*

$$u(t,x) := \mathrm{e}^{i\theta}\left(\alpha + \frac{\beta}{1 - p\mathrm{e}^{iN(x-ct)}}\right), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T}, \tag{1.4}$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N(1 + \frac{2\alpha}{\beta})$, and (α, β) are two real constants satisfying

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = -N. \tag{1.5}$$

Remark 1.1. Condition (1.5) implies that the real constants α and β must be of opposite signs.

Second, we pass to the focusing Calogero–Sutherland DNLS equation

$$i\partial_t u + \partial_x^2 u + 2D\Pi(|u|^2)u = 0. \tag{CS$^+$}$$

By changing the sign in front of the nonlinearity, the strategy adopted in the defocusing case to exhibit the traveling waves becomes significantly more complicated. However, we can ensure the existence of a larger set of traveling wave solutions for (CS$^+$) comparing to (CS$^-$), and that all the nontrivial traveling waves $u(t,x) := u_0(x - ct)$ of (CS$^+$) are also rational functions.

[2] For a simple introduction to the different definitions of Hardy spaces, we refer to [14, Chapter 3.]

Theorem 1.2. *The traveling waves $u_0(x-ct)$ of* (CS$^+$) *are either rational functions or trivial waves in $\mathcal{G}_1$. In addition, the potentials*

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1-p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},\ N \in \mathbb{N}_{\geq 1},$$

where $c = -N(1+\frac{2\alpha}{\beta})$, $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = N, \tag{1.6}$$

and the potentials

$$u(t,x) = \mathrm{e}^{i\theta}\mathrm{e}^{im(x-mt)}\Big(\alpha + \frac{\beta}{1-p\mathrm{e}^{i(x-mt)}}\Big), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},\ m \in \mathbb{N}_{\geq 1},$$

where $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = 1, \quad \beta(m-1) = 2\alpha,$$

are parts of the set of traveling waves of (CS$^+$).

Remark 1.2. It is worth noting that the condition on (α,β) appearing in (1.6) for the focusing case, allows one to obtain a *larger set* of traveling waves in comparison to the condition (1.5) of the defocusing case. Indeed, (1.6) enables, for instance, α or $\alpha+\beta$ to vanish, which leads respectively to the traveling waves

$$u(t,x) = \mathrm{e}^{i\theta}\frac{\sqrt{N(1-|p|^2)}}{1-p\mathrm{e}^{iN(x+Nt)}}$$

and

$$u(t,x) = \mathrm{e}^{i\theta}\frac{\sqrt{N(1-|p|^2)}\mathrm{e}^{iN(x-Nt)}}{1-p\mathrm{e}^{iN(x-Nt)}}.$$

Contrary to the focusing case, no traveling waves $u(t,x) := u_0(x-ct)$ with a profile $u_0(x) := \beta/(1-p\mathrm{e}^{iNx})$ or $u_0(x) := \alpha\mathrm{e}^{iNx}/(1-p\mathrm{e}^{iNx})$ can be found for the (CS$^-$) equation because otherwise, thanks to (1.5),

$$\frac{\beta^2}{1-|p|^2} = -N \quad \text{or} \quad \frac{|p|^2}{1-|p|^2}\beta^2 = -N,$$

which is clearly impossible for $p \in \mathbb{D}^*$.

Remark 1.3 (The L^2-norm and the speed of the traveling waves of (CS)).

(1) As will be established in Section 3.3 for the defocusing equation and in Section 4.2 for the focusing equation, the L^2-norm of the nontrivial traveling waves of (CS) can be arbitrarily small or large in $L^2_+(\mathbb{T})$. More rigorously, for any $r > 0$, there exists a nontrivial traveling wave $u(t,x) := u_0(x-ct)$ of (CS) where

$$\|u_0\|_{L^2} = r.$$

(2) *For the defocusing* (CS$^-$) *equation*. The nontrivial traveling waves u of the form (1.4) propagate to the right with a speed $c > N$, where N is the degree appearing in the denominator of u. In addition, when $\|u\|_{L^2} \to +\infty$, we have $c \to +\infty$, and when $\|u\|_{L^2} \to 0$ then $c \to N$. (See Remark 3.3 and Section 3.3 for the proofs).

For the focusing (CS$^+$) *equation*. Contrary to the defocusing equation, (CS$^+$)'s nontrivial traveling waves do not necessarily propagate at a relatively high speed (i.e. $c \to \infty$) when $\|u\|_{L^2}^2$ is large (i.e. $\|u\|_{L^2} \to \infty$). In fact, the speed of the traveling waves in the focusing case is independent of the size of its $L^2_+(\mathbb{T})$-norm. We refer to Remark 4.2 for an example.

In light of the previous remarks, we infer that the Calogero–Sutherland DNLS equation enjoys a significantly richer dynamic in the focusing case. In particular, one can observe that (CS$^+$) admits nontrivial *stationary waves* $u(t,x) := u_0(x)$, which is not the case for the defocusing equation. An example of nontrivial stationary waves for (CS$^+$) is

$$u(t,x) := \mathrm{e}^{i\theta}\sqrt{\frac{N(1-|p|^2)}{2(1+|p|^2)}}\Big(1 - \frac{2}{1-p\mathrm{e}^{iNx}}\Big), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},\ N \in \mathbb{N}_{\geq 1}.$$

In a second stage, we study the *finite gap potentials* of the Calogero–Sutherland DNLS equation (CS), i.e. potentials satisfying that, from a certain rank, all the gaps between the consecutive eigenvalues of the Lax operator are equal to 1 (see Section 1.2.2 for the Lax operator). It turns out that these potentials are *multiphase solutions* containing the stationary and traveling waves of (CS). The following theorem aims to characterize the finite gap potentials on $\mathbb{T}$ in the state space.

Theorem 1.3 (Characterization in the state space of (CS)'s finite gap potentials). *The finite gap potentials of* (CS) *are either the functions* $u(x) = C\mathrm{e}^{iNx}$, $C \in \mathbb{C}^*$, $N \in \mathbb{N}_{\geq 0}$, *or the rational function*

$$u(x) = \mathrm{e}^{im_0x}\prod_{j=1}^{r}\Big(\frac{\mathrm{e}^{ix}-\overline{p_j}}{1-p_j\mathrm{e}^{ix}}\Big)^{m_j-1}\Bigg(a + \sum_{j=1}^{r}\frac{c_j}{1-p_j\mathrm{e}^{ix}}\Bigg), \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j,$$

where, for $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0,\dots,N-1\}$, $m_1,\dots,m_r \in \{1,\dots,N\}$ *such that* $m_0 + \sum_{j=1}^{r} m_j = N$, *and* $(a, c_1, \dots, c_r) \in \mathbb{C}\times\mathbb{C}^r$ *satisfy for all* $j = 1,\dots,r$,

(i) *in the defocusing case,*

$$\bar{a}c_j + \sum_{k=1}^{r}\frac{c_j\overline{c_k}}{1-p_j\overline{p_k}} = -m_j,$$

(ii) *in the focusing case,*

$$\bar{a}c_j + \sum_{k=1}^{r}\frac{c_j\overline{c_k}}{1-p_j\overline{p_k}} = m_j,$$

with $a \neq 0$ *if* $m_0 \neq 0$. *Moreover, these finite gap potentials can be regrouped into sets that remain invariant under the evolution of* (CS).

In order to establish the results mentioned above, it is necessary to provide an overview regarding the integrability of the Calogero–Sutherland derivative nonlinear Schrödinger equation (CS).

1.2. About the Calogero–Sutherland DNLS equation

The Calogero–Sutherland DNLS equation (CS) has been actively studied by physicists and engineers. In particular, we cite the works of Tutiya [33], Berntson–Fagerlund [7], Stone–Anduaga–Xing [32], Polychronakos [30, 31] and Matsuno [21–28].

Mathematically, recent progress has been made with regard to this equation. In this subsection, we provide a brief overview of some established results concerning (CS).

1.2.1. Local and global well-posedness results. To the best of the author's knowledge, the first LWP result for the (CS) equation traces back to de Moura [12] who established the LWP[3] of (CS) for small initial data in $H^s(\mathbb{R})$ with $s \geq 1$, and extended his LWP result to a GWP by means of the gauge transformation. More recently, Barros–de Moura–Santos [5] presented the LWP of (CS) for small initial data in the Besov space $B_2^{1/2,1}(\mathbb{R})$.

Further, observe that the Calogero–Sutherland DNLS equation (CS) is invariant under the scaling

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda t, \lambda^2 x), \quad \lambda > 0.$$

This suggests that (CS) is L^2-critical. In the Hardy Sobolev spaces setting, i.e. in $H^s_+ := H^s \cap L^2_+$, where we recall that L^2_+ is the Hardy space defined in (1.2) in the periodic case, and as follows in the nonperiodic case

$$L^2_+(\mathbb{R}) = \left\{u \in L^2(\mathbb{R});\ \operatorname{supp} \hat{u} \subset [0,\infty)\right\},$$

Gérard–Lenzmann [16] obtained the LWP in $H^s_+(\mathbb{R})$ with $s > \frac{1}{2}$ by following the arguments of [13]. Furthermore, by virtue of a Lax pair structure associated with the Calogero–Sutherland DNLS equation (CS) (see below), they inferred the global well-posedness of the equation in all $H^k_+(\mathbb{R})$, $k \in \mathbb{N}_{\geq 1}$ for small initial data $\|u_0\|_{L^2(\mathbb{R})} < \sqrt{2\pi}$ in the focusing case. Moreover, they established the nonexistence of minimal mass blowup. Subsequently, [19] extended the flow to the critical regularity-space $L^2_+(\mathbb{R})$ and for small initial data in the focusing case. Later, [18] proved the existence of initial data regular enough in $H^\infty_+(\mathbb{R})$ satisfying $\|u_0\|^2_{L^2} = 2\pi + \varepsilon$, such that $\lim_{t\to T} \|u(t)\|_{H^s} = \infty$, $s \geq 0$, where $T \in (0, +\infty]$ is the maximal time of lifespan of the solution. Finally, [4] characterized the semi-classical limit or zero-dispersion limit of (CS) and proved that the weak limit solution can be written in terms of the branches of the multivalued solution of the Burgers equation.

Moving to the periodic setting, i.e. when $x \in \mathbb{T}$, a recent work of the author [3] shows the GWP of (CS) in all $H^s_+(\mathbb{T})$, $s \geq 0$, for small critical initial data in the focusing case,

[3] Actually, they prove the local well-posedness of a family of nonlocal nonlinear Schrödinger equations [29] that also includes the (CS) equation.

namely when $\|u_0\|_{L^2(\mathbb{T})} < 1$, and for arbitrary initial data in the defocusing case. In particular, the extension of the flow to the critical space $L^2_+(\mathbb{T})$ has been achieved after deriving the *explicit formula* for the solution of the Calogero–Sutherland DNLS equation (CS) [3, Proposition 2.5]. Moreover, under the same assumptions, the relative compactness of the trajectories has been established in $H^s_+(\mathbb{T})$, for all $s \geq 0$ [3].

1.2.2. Integrability of the (CS) equation. One of the most remarkable features of the Calogero–Sutherland DNLS equation is its integrability as a PDE on $\mathbb{R}$ and on $\mathbb{T}$. In fact, it enjoys a *Lax pair structure* in the focusing and defocusing cases [3, 16]: for any $u \in H^s_+(\mathbb{T})$, $s > \frac{3}{2}$, there exist two operators (L_u, B_u) satisfying the Lax equation

$$\frac{dL_u}{dt} = [B_u, L_u], \quad [B_u, L_u] := B_u L_u - L_u B_u,$$

where

(i) in the focusing case,

$$L_u = D - T_u T_{\bar{u}}, \quad B_u = T_u T_{\partial_x \bar{u}} - T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2, \tag{1.7}$$

(ii) in the defocusing case,

$$\tilde{L}_u = D + T_u T_{\bar{u}}, \quad \widetilde{B}_u = -T_u T_{\partial_x \bar{u}} + T_{\partial_x u} T_{\bar{u}} + i(T_u T_{\bar{u}})^2. \tag{1.8}$$

The differential operator D is $-i\partial_x$, and T_u is the Toeplitz operator of symbol u defined for any $u \in L^\infty$ by

$$T_u f = \Pi(uf), \quad \forall f \in L^2_+, \tag{1.9}$$

where Π is the Szegő projector introduced in (1.1). Note that since we are working in the Hardy space, L_u is a semi-bounded operator from below and $\tilde{L}_u$ is a nonnegative operator. In addition, as noted in [3, Proposition 2.3], the Lax operators L_u and $\tilde{L}_u$ are self-adjoint operators of domain $H^1_+(\mathbb{T})$, and are of compact resolvent. Therefore, their spectra are made up of a sequence of eigenvalues going to $+\infty$,

$$\begin{aligned} \sigma(L_u) &:= \{\nu_0(u) \leq \cdots \leq \nu_n(u) \leq \cdots\}, \quad \nu_0(u) \geq -\|u\|^2_{L^\infty}, \\ \sigma(\tilde{L}_u) &:= \{\lambda_0(u) \leq \cdots \leq \lambda_n(u) \leq \cdots\}, \quad \lambda_0(u) \geq 0. \end{aligned} \tag{1.10}$$

Recall that any Lax operator satisfies the isospectral property

$$L_{u_0} = U(t)^{-1} L_{u(t)} U(t), \tag{1.11}$$

where u_0 is the initial data, $u(t)$ is the evolution of the solution starting from u_0, and $U(t)$ is a family of operators solving the Cauchy problem

$$\begin{cases} \dfrac{d}{dt} U(t) = B_{u(t)} U(t), \\ U(0) = \mathrm{Id}\,. \end{cases}$$

The identity (1.11) implies that the spectrum of $L_{u(t)}$ is invariant by the evolution, i.e. $\nu_n(u(t)) = \nu_n(u_0)$ and $\lambda_n(u(t)) = \lambda_n(u_0)$ for all n. Therefore, in the sequel, we omit the variable u in $\nu_n(u)$ and $\lambda_n(u)$ when it does not cause confusion.

Further information regarding the spectrum of the Lax operators will be provided in Section 2.

1.2.3. Traveling waves on $\mathbb{R}$. Let us mention that the focusing Calogero–Sutherland DNLS equation (CS^+) also enjoys traveling waves and stationary waves in the *non-periodic case* (i.e. $x \in \mathbb{R}$). They are of the form

$$u(t,x) = \mathrm{e}^{i\theta}\mathrm{e}^{iv(x-vt)}\lambda^{\frac{1}{2}}\mathcal{R}(\lambda(x-2vt)+y), \quad \lambda > 0,\ y \in \mathbb{R},\ \theta \in \mathbb{T},\ v \in \mathbb{R},$$

where the profile

$$\mathcal{R}(x) = \mathrm{e}^{i\theta}\frac{\sqrt{2\mathrm{Im}\,p}}{x+p}, \quad p \in \mathbb{C}_+,\ \theta \in \mathbb{R},$$

is obtained as a ground state (minimizer) for the energy functional [16, Section 4]. Notice that all these waves are of L^2-norm equal to $\sqrt{2\pi}$. Therefore, this situation differs from the torus $\mathbb{T}$, where in the latter case, there is no L^2-threshold that would prevent the existence of small or large traveling waves in $L^2(\mathbb{T})$. Essentially, the main reason that leads to a more diverse class of traveling waves in the periodic setting compared to the nonperiodic setting is the spectral property carried by the Lax operator in both cases. Indeed, on $\mathbb{R}$, the Lax operator has an absolute continuous spectrum and a finite number of eigenvalues [16, Section 5]. In contrast with $\mathbb{T}$, the Lax operator presents only a point spectrum formed by eigenvalues [3, Section 2].

To summarize, we refer to Table 1.

	Focusing (CS^+) on $\mathbb{R}$	Defocusing (CS^-) on $\mathbb{R}$
Stationary waves	✓	
Traveling waves	✓	
Wave speed	$c \in \mathbb{R}$	
L^2-norm of traveling waves	$\lVert u\rVert_{L^2} = \sqrt{2\pi}$	
	Focusing (CS^+) on $\mathbb{T}$	Defocusing (CS^-) on $\mathbb{T}$
Nontrivial stationary waves	✓	✗
Traveling waves	✓	✓
Wave speed	$c \in \mathbb{R}$	$c \geq N$
L^2-norm of nontrivial traveling waves	$\lVert u\rVert_{L^2} \in (0,+\infty)$	$\lVert u\rVert_{L^2} \in (0,+\infty)$

Table 1. Here $N \in \mathbb{N}_{\geq 1}$ is the denominator's degree of a traveling wave of the form (1.4). In addition, by a nontrivial traveling wave, we mean a traveling wave that does not belong to $\mathcal{G}_1$, where $\mathcal{G}_1$ is the set of trivial traveling waves defined in (1.3). Moreover, by nontrivial stationary waves we mean the solutions $u(t,x) = u_0(x)$ that are not constant functions.

1.3. Outline of the paper

The paper is organized as follows. In Section 2 we present some spectral properties concerning the eigenvalues and the eigenfunctions of the Lax operators L_u and $\tilde{L}_u$. Moving on to Section 3, we focus on the traveling waves of the defocusing Calogero-Sutherland DNLS equation (CS$^-$). This section follows a two-step process. Section 3.1 provides a spectral characterization of these waves, while Section 3.2 derives their explicit formulas. Moreover, Section 3.3 includes remarks concerning the speed and L^2-norm of these traveling waves for the defocusing (CS$^-$) equation.

In Section 4 we delve into the analysis of traveling waves for the focusing Calogero–Sutherland DNLS equation (CS$^+$). Thus, we describe the set of traveling waves of (CS$^+$) in Section 4.1, and we highlight the presence of a larger set of traveling waves in the focusing case compared to the defocusing case. Similar to the defocusing case, some remarks related to the speed and the L^2-norm of the traveling waves of (CS$^+$) are discussed in Section 4.2, and in particular we establish the existence of stationary wave solutions for the focusing Calogero-Sutherland DNLS equation (CS$^+$).

Note that in order to describe the traveling waves of (CS$^+$), one needs to understand the set of finite gap potentials. To this end, Section 5 is dedicated to the study of finite gap potentials for the Calogero–Sutherland DNLS equation (CS).

Throughout this paper, we have assumed sufficient regularity on the solutions. However, in Section 6, we discuss how the same analysis can be extended to solutions with lower regularity. Lastly, in Section 7, we present some open problems for further exploration.

2. Spectral properties for the Lax operators

As mentioned in the introduction, our aim is to describe the traveling waves of the Calogero-Sutherland DNLS equation (CS). In order to accomplish this goal, our strategy relies on characterizing them first in the state space, by means of some spectral tools of the Lax operators L_u and $\tilde{L}_u$ introduced in (1.7) and in (1.8), respectively. Therefore, we need to delve deeper into the spectral properties of the Lax operators.

In the sequel, we assume, for convenience, that u is any function of the state space with enough regularity, for example, $u \in H^2_+(\mathbb{T})$. But, it is worth mentioning that the analysis can be easily extended to potentials with less regularity as well (see Section 6). Further, recall from (1.10), that the Lax operators $\tilde{L}_u$ and L_u have point spectra, bounded from below,

$$\sigma(\tilde{L}_u) := \{\lambda_0 \leq \cdots \leq \lambda_n \leq \cdots\}, \quad \lambda_0 \geq 0,$$
$$\sigma(L_u) := \{\nu_0 \leq \cdots \leq \nu_n \leq \cdots\}, \quad \nu_0 \geq -\|u\|^2_{L^\infty}.$$

The following proposition aims to give more information, regarding the multiplicity of the eigenvalues (ν_n) and (λ_n). But before that, we need to recall two useful commutator identities. We denote by S the *shift operator* defined as

$$S\colon L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T}), \quad Sh(x) = \mathrm{e}^{ix}h(x).$$

Thus, for all $u \in H^2_+(\mathbb{T})$, we have from [3, Lemma 2.3],

$$\begin{aligned} \tilde{L}_u S &= S\tilde{L}_u + S + \langle\cdot|S^*u\rangle u, \\ L_u S &= SL_u + S - \langle\cdot|S^*u\rangle u, \end{aligned} \tag{2.1}$$

where S^* denotes the adjoint operator of S,

$$S^*\colon L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T}), \quad S^*h(x) = \Pi(\mathrm{e}^{-ix}h(x)),$$

where Π is the Szegő projector defined in (1.1), and L_u and $\tilde{L}_u$ are defined in (1.7) and (1.8). In addition, we also have from the same lemma [3, Lemma 2.3],

$$\begin{aligned} [S^*, B_u] &= i(S^*L_u^2 - (L_u + \mathrm{Id})^2 S^*), \\ [S^*, \widetilde{B}_u] &= i(S^*\tilde{L}_u^2 - (\tilde{L}_u + \mathrm{Id})^2 S^*), \end{aligned} \tag{2.2}$$

where $[S^*, B_u]$ denotes the commutator $S^*B_u - B_uS^*$, and B_u and $\widetilde{B}_u$ are the two skew-adjoint operators of the Lax pairs, defined respectively in (1.7) and (1.8).

Proposition 2.1 (Multiplicity of (λ_n) and (ν_n)). *The eigenvalues of L_u and $\tilde{L}_u$ are described as follows:*

Defocusing case. *The eigenvalues (λ_n) of $\tilde{L}_u$ are all simple. More precisely,*

$$\lambda_{n+1} \geq \lambda_n + 1, \quad n \in \mathbb{N}_{\geq 0}. \tag{2.3}$$

Focusing case. *The eigenvalues (ν_n) of L_u are of multiplicity at most 2:*

$$\nu_{n+2} \geq \nu_n + 1, \quad n \in \mathbb{N}_{\geq 0}. \tag{2.4}$$

Moreover, when n is large enough, the eigenvalues of L_u are simple. More precisely,

$$\liminf_{n\to\infty} \nu_{n+1} - \nu_n \geq 1. \tag{2.5}$$

Furthermore, for all $0 \leq \alpha < 1$ such that $\|u\|^2_{L^2} < 1 - \alpha$, we have for all $n \in \mathbb{N}_{\geq 0}$,

$$\nu_{n+1} > \nu_n + \alpha. \tag{2.6}$$

Remark 2.1. We underline the following points:

(1) It should be noted that for any potential u, the eigenvalues (ν_n) of L_u cannot all be simple. For instance, take $u(x) = \mathrm{e}^{ix}$: one can easily check that for $L_u = D - T_uT_{\bar{u}}$,

$$L_u 1 = L_u e^{ix} = 0.$$

(2) Inequality (2.5) implies that as $n \gg 1$, the lower bound of the distance between two consecutive eigenvalues ν_n gets closer to 1.

Proof of Proposition 2.1. All the presented inequalities are a direct consequence of the max–min principle:

$$\lambda_n = \max_{\substack{F \subseteq L^2_+ \\ \dim F \leq n}} \min\{\langle \tilde{L}_u h | h\rangle;\ h \in F^\perp \cap H^{\frac{1}{2}}_+(\mathbb{T}),\ \|h\|_{L^2} = 1\},$$

$$\nu_n = \max_{\substack{F \subseteq L^2_+ \\ \dim F \leq n}} \min\{\langle L_u h | h\rangle;\ h \in F^\perp \cap H^{\frac{1}{2}}_+(\mathbb{T}),\ \|h\|_{L^2} = 1\}.$$

Spectrum of $\tilde{L}_u$. Let F be any subspace of $L^2_+(\mathbb{T})$ of dimension n, and consider $E := \mathbb{C}1 \oplus S(F)$, where S is the shift operator. Then

$$\lambda_{n+1} \geq \min\{\langle \tilde{L}_u h | h\rangle;\ \|h\|_{L^2} = 1,\ h \in E^\perp \cap H^{\frac{1}{2}}_+\}$$

Observe that $E^\perp = S(F^\perp)$; thus by (2.1),

$$\lambda_{n+1} \geq \min\{\langle \tilde{L}_u g | g\rangle + 1 + |\langle Sg | u\rangle|^2;\ \|g\|_{L^2} = 1,\ g \in F^\perp \cap H^{\frac{1}{2}}_+\}.$$

In addition, since $|\langle Sg|u\rangle|^2 \geq 0$, we infer for all $n \in \mathbb{N}_{\geq 0}$,

$$\lambda_{n+1} \geq \lambda_n + 1.$$

Spectrum of L_u*-inequality* (2.4). Let F be any subspace of $L^2_+(\mathbb{T})$ of dimension n, and take $G := \mathbb{C}1 \oplus S(F) + \mathbb{C}u$. Then,

$$\nu_{n+2}(u) \geq \min\{\langle L_u h | h\rangle;\ \|h\|_{L^2} = 1,\ h \in G^\perp \cap H^{\frac{1}{2}}_+\}.$$

Since $G^\perp = S(F^\perp \cap (S^*u)^\perp)$, then

$$\nu_{n+2} \geq \min\{\langle L_u Sg | Sg\rangle;\ \|g\|_{L^2} = 1,\ g \in F^\perp \cap (S^*u)^\perp \cap H^{\frac{1}{2}}_+(\mathbb{T})\}.$$

Note that $g \perp S^*u$; then by (2.1),

$$\nu_{n+2} \geq \min\{\langle L_u g | g\rangle + 1;\ \|g\|_{L^2} = 1,\ g \in F^\perp \cap (S^*u)^\perp \cap H^{\frac{1}{2}}_+(\mathbb{T})\},$$

leading to

$$\nu_{n+2} \geq \nu_n + 1.$$

Inequality (2.5). For any n, let $F_n = \operatorname{span}\{f_0, f_1, \dots, f_{n-1}\}$ be the subspace of $L^2_+(\mathbb{T})$ of dimension n made up of the first n eigenfunctions of L_u. For this choice of F_n,

$$\min\{\langle L_u h | h\rangle;\ \|h\|_{L^2} = 1,\ h \in F_n^\perp \cap H^{\frac{1}{2}}_+\} = \nu_n.$$

Let us consider the subspace $E := \mathbb{C}1 \oplus S(F_n)$ of $L^2_+(\mathbb{T})$ of dimension $n+1$. Then

$$\nu_{n+1} \geq \min\{\langle L_u g|g\rangle;\ \|g\|_{L^2} = 1,\ g \in E^\perp \cap H^{\frac{1}{2}}_+\}.$$

Note that $E^\perp \cap H^{\frac{1}{2}}_+ = S(F_n^\perp \cap H^{\frac{1}{2}}_+)$. Therefore, by (2.1),

$$\nu_{n+1} \geq \min\{\langle L_u\varphi|\varphi\rangle + 1 - |\langle S\varphi|u\rangle|^2;\ \|\varphi\|_{L^2} = 1,\ \varphi \in F_n^\perp \cap H^1_+\}.$$

It results, for all $n \in \mathbb{N}_{\geq 0}$, in

$$\nu_{n+1} \geq \nu_n + 1 - \sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1\\ \varphi\in F_n^\perp}} |\langle S\varphi|u\rangle|^2. \tag{2.7}$$

To conclude the proof, it remains to prove $\sup_{\|\varphi\|_{L^2(\mathbb{T})}=1, \varphi\in F_n^\perp} |\langle S\varphi|u\rangle|^2 \xrightarrow[n\to\infty]{} 0$.

Lemma 2.2. *Let F_n be the subspace of $L^2_+(\mathbb{T})$ defined as above. Then*

$$\sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1\\ \varphi\in F_n^\perp}} |\langle S\varphi|u\rangle| \to 0 \quad \textit{as } n \to \infty.$$

Proof. Suppose for the sake of contradiction, that for all $n \in \mathbb{N}_{\geq 0}$,

$$\sup_{\substack{\|\varphi\|_{L^2(\mathbb{T})}=1\\ \varphi\in F_n^\perp}} |\langle S\varphi|u\rangle| \geq \varepsilon, \quad \varepsilon > 0.$$

Namely, there exists $\varphi_n \in F_n^\perp$, $\|\varphi_n\|_{L^2(\mathbb{T})} = 1$ such that $|\langle S\varphi_n|u\rangle| \geq \frac{\varepsilon}{2}$. Hence, since $\|\varphi_n\|_{L^2(\mathbb{T})} = 1$, then up to a subsequence $\varphi_n \rightharpoonup \varphi$ in $L^2_+(\mathbb{T})$ as $n \to \infty$, which yields

$$|\langle S\varphi_n|u\rangle| \xrightarrow[n\to\infty]{} |\langle S\varphi|u\rangle|,$$

and so $\langle S\varphi|u\rangle \geq \frac{\varepsilon}{2}$. On the other hand, since $\varphi_n \perp F_n$ then

$$\langle \varphi_n|f_p\rangle = 0, \quad \forall 0 \leq p \leq n-1.$$

Taking $n \to \infty$, we infer

$$\langle \varphi|f_p\rangle = 0, \quad \forall p \in \mathbb{N}_{\geq 0}.$$

Note that the eigenfunctions (f_p) of the self-adjoint operator L_u form an orthonormal basis of $L^2_+(\mathbb{T})$. Therefore, we have $\varphi = 0$, which is a contradiction with $\langle S\varphi|u\rangle \geq \frac{\varepsilon}{2}$. ∎

Inequality (2.6). It is a consequence of inequality (2.7) after applying the Cauchy–Schwarz inequality and considering the fact that $\|u\|^2_{L^2} < 1 - \alpha$. ∎

In what follows, we make a slight abuse of notation by using (f_n) to denote both an orthonormal basis of $L^2_+(\mathbb{T})$ consisting of the eigenfunctions of the self-adjoint operator L_u, and an orthonormal basis of $L^2_+(\mathbb{T})$ consisting of the eigenfunctions of $\tilde{L}_u$. Nonetheless, we shall specify the context in which we are working to avoid confusion and ensure that (f_n) is understood appropriately as either the eigenfunctions of L_u or $\tilde{L}_u$.

Lemma 2.3. *Given $u \in H^2_+(\mathbb{T})$, then for all $n, p \in \mathbb{N}_{\geq 0}$,*

- Defocusing case:

$$\langle 1|u\rangle\langle u|f_n\rangle = \lambda_n\langle 1|f_n\rangle,$$
$$(\lambda_n - \lambda_p - 1)\langle Sf_p|f_n\rangle = \langle Sf_p|u\rangle\langle u|f_n\rangle.$$

- Focusing case:

$$\langle 1|u\rangle\langle u|f_n\rangle = -\nu_n\langle 1|f_n\rangle,$$
$$(\nu_n - \nu_p - 1)\langle Sf_p|f_n\rangle = -\langle Sf_p|u\rangle\langle u|f_n\rangle.$$

Proof. We prove first the identities for the *defocusing case*. By definition of $\tilde{L}_u = D + T_u T_{\bar{u}}$, we have
$$\tilde{L}_u 1 = \langle 1|u\rangle u.$$
Then taking the inner product of both sides with f_n, and using the fact that $\tilde{L}_u$ is a self-adjoint operator, leads to the first identity. For the second one, thanks to the commutator relation between $\tilde{L}_u$ and S,

$$\tilde{L}_u S f_p = S\tilde{L}_u f_p + Sf_p + \langle Sf_p|u\rangle u,$$

of equation (2.1), we infer the second identity by taking the inner product with f_n.

Further, by considering the *focusing case* with $L_u = D - T_u T_{\bar{u}}$, it follows that $L_u 1 = -\langle 1|u\rangle u$. This explains the minus sign appearing in the first statement. As for the second one, since by (2.1),
$$L_u S f_p = SL_u f_p + Sf_p - \langle Sf_p|u\rangle u,$$
then taking the inner product with f_n once more, leads to the desired identity. ∎

In light of the previous lemma and based on the commutator identities (2.1), one can investigate further information regarding the spectral data (i.e. the eigenvalues and the eigenvectors) of L_u and $\tilde{L}_u$, especially when the quantities $\langle u|f_n\rangle$ vanish. The following lemmas/propositions aim to achieve this.

For the following, we denote by E_{ν_n} the eigenspace of L_u corresponding to the eigenvalue ν_n. In addition, the notation $f \mathbin{/\!/} g$ means that the two vectors f and g are collinear in $L^2_+(\mathbb{T})$.

Proposition 2.4. *For all $n \in \mathbb{N}_{\geq 1}$, such that $\nu_n \neq 0$, we have*

$$\nu_n = \nu_{n-1} + 1 \Rightarrow [Sf_{n-1} \in E_{\nu_n}] \text{ or } [f_n \in SE_{\nu_{n-1}}],$$

Moreover, for the defocusing case,

$$\lambda_n = \lambda_{n-1} + 1 \Rightarrow Sf_{n-1} \mathbin{/\!/} f_n.$$

Remark 2.2. We have the following remarks:

(1) The condition $\nu_n \neq 0$ cannot be omitted. We refer to Appendix A.1 for an example.

(2) For the defocusing case, the condition of nonvanishing eigenvalues $\lambda_n \neq 0$ is already satisfied for all $n \in \mathbb{N}_{\geq 1}$, since $\tilde{L}_u$ is a nonnegative operator on the Hardy space, and for all $n \in \mathbb{N}_{\geq 1}$, we have by (2.3), $\lambda_n \geq \lambda_{n-1} + 1$.

Proof of Proposition 2.4. The key is to use Lemma 2.3 and the commutator identities (2.1). In view of the second identity of Lemma 2.3, we have

$$\langle Sf_{n-1}|u\rangle\langle u|f_n\rangle = 0.$$

If $\langle u|Sf_{n-1}\rangle = 0$, then by (2.1),

$$\begin{aligned} L_u Sf_{n-1} &= SL_u f_{n-1} + Sf_{n-1} \\ &= (\nu_{n-1} + 1)Sf_{n-1} \\ &= \nu_n Sf_{n-1}, \end{aligned}$$

as $\nu_n = \nu_{n-1} + 1$. Namely, $Sf_{n-1} \in E_{\nu_n}$. Let us move to the second case where $\langle u|f_n\rangle = 0$. By the first identity of Lemma 2.3,

$$\nu_n\langle f_n|1\rangle = 0.$$

Therefore, since $\nu_n \neq 0$, there exists $g_n \in H^1_+(\mathbb{T})$ such that $f_n = Sg_n$. Using the commutator identity (2.1) again, we have

$$SL_u g_n = (\nu_n - 1)Sg_n.$$

Applying S^* to both sides of the latter identity, and using the fact that $S^*S = \mathrm{Id}$, and as $\nu_n = \nu_{n-1} + 1$, we find

$$L_u g_n = \nu_{n-1} g_n.$$

That is, $g_n \in E_{\nu_{n-1}}$, and so $f_n \in SE_{\nu_{n-1}}$.

Further, note that for the *defocusing equation*, the vector spaces E_{λ_n} are of dimension 1, thanks to Proposition 2.1. Consequently, the results $[Sf_{n-1} \in E_{\lambda_n}]$ or $[f_n \in SE_{\lambda_{n-1}}]$ lead to $Sf_{n-1} \mathbin{/\!/} f_n$. ■

In the sequel, we denote by $\mathcal{I}(u)$ the set

$$\mathcal{I}(u) := \left\{n \in \mathbb{N}_{\geq 1} \;\middle|\; \langle Sf_{n-1}|f_n\rangle = 0\right\}. \tag{2.8}$$

Lemma 2.5. *The set $\mathcal{I}(u)$ is described as follows:*

Defocusing case. *For any $u \in H^2_+(\mathbb{T})$, the set $\mathcal{I}(u)$ is empty.*

Focusing case. *Given $u \in H^2_+(\mathbb{T})$, let $m \in \mathcal{I}(u)$. Assume that the eigenvalues ν_m and ν_{m-1} are simple. Then either*

$$\nu_{m-1} + 1 = \nu_{m+1}, \quad \textit{with } Sf_{m-1} \in E_{\nu_{m+1}}$$

or

$$\nu_{m-2} + 1 = \nu_m, \quad \textit{with } S^* f_m \in E_{\nu_{m-2}}$$

or

$$\nu_m = 0, \quad \textit{with } f_m \,/\!\!/\, 1.$$

Remark 2.3. We have the following remarks:

(i) Observe that in the focusing case, if $\|u\|^2_{L^2} < \frac{1}{2}$, then by inequality (2.6),

$$\nu_n > \nu_{n-1} + \frac{1}{2}, \quad \forall n \in \mathbb{N}_{\geq 1}.$$

Hence, for such u, if $m \in \mathcal{I}(u)$ then the only possible choice is to have $\nu_m = 0$ with $f_m \,/\!\!/\, 1$. In other words, for $\|u\|^2_{L^2} < \frac{1}{2}$, we have, either $\mathcal{I}(u) = \varnothing$, or $\mathcal{I}(u) = \{m\}$ and in such a case $\nu_m = 0$ and $f_m \,/\!\!/\, 1$.

(ii) For any $u \in H^2_+(\mathbb{T})$, the set $\mathcal{I}(u)$ in the focusing case is of finite cardinal, since by inequality (2.5) we have $\nu_n > \nu_{n-1} + \frac{1}{2}$ and $\nu_n \neq 0$ for all n large enough.

Proof of Lemma 2.5. We consider two cases:

Focusing case. Let $m \in \mathcal{I}(u)$. By the second identity of Lemma 2.3,

$$\langle Sf_{m-1}|u\rangle\langle u|f_m\rangle = 0.$$

If $\langle Sf_{m-1}|u\rangle = 0$, then applying the commutator identity (2.1),

$$L_u Sf_{m-1} = (\nu_{m-1} + 1)Sf_{m-1}.$$

Namely, $\nu_{m-1} + 1$ is an eigenvalue of L_u and Sf_{m-1} is the corresponding eigenfunction. Since ν_m is simple, then Sf_{m-1} cannot be collinear to f_m as $\langle Sf_{m-1}|f_m\rangle = 0$ for $m \in \mathcal{I}(u)$. Therefore, by (2.4),

$$\nu_{m-1} + 1 = \nu_{m+1}.$$

If $\langle u|f_m\rangle = 0$, then by applying the adjoint of the commutator identity (2.1),

$$S^* L_u = L_u S^* + S^* + \langle \cdot|u\rangle S^* u,$$

we infer

$$L_u S^* f_m = (\nu_m - 1)S^* f_m.$$

That is, if $S^* f_m \neq 0$, then $S^* f_m$ is an eigenfunction of L_u associated with the eigenvalue $\nu_m - 1$. Recall that we have by assumption that ν_{m-1} is simple, and since $S^* f_m$ cannot be collinear to f_{m-1} as $m \in \mathcal{I}(u)$, then

$$\nu_m - 1 = \nu_{m-2},$$

thanks to (2.4). It remains to study the case where $S^* f_m = 0$, i.e. $f_m \mathbin{/\!/} 1$. For that case we have, thanks to the first identity of Lemma 2.3, $\nu_m = 0$ as $\langle u | f_m \rangle = 0$.

Defocusing case. Suppose that there exists $m \in \mathcal{I}(u)$. Then, using the same analysis as in the focusing case, we infer that, either $\lambda_{m-1} + 1 = \lambda_{m+1}$ or $\lambda_{m-2} + 1 = \lambda_m$ or $\lambda_m = 0$. However, recall that $\lambda_n \geq \lambda_{n-1} + 1$ for all $n \in \mathbb{N}_{\geq 1}$ (inequality 2.3), thus the first two cases cannot occur. In addition, since $\tilde{L}_u$ is a nonnegative operator, where all the eigenvalues satisfy the inequality (2.3), then $\lambda_m = 0$ implies $m = 0 \notin \mathcal{I}(u)$. ∎

Corollary 2.6. *For all $n \geq 1$,*

$$\lambda_n = \lambda_{n-1} + 1 \Leftrightarrow \langle u | f_n \rangle = 0.$$

In addition,

$$\nu_n = \nu_{n-1} + 1, \ \forall n \geq N_1 \Leftrightarrow \langle u | f_n \rangle = 0, \ \forall n \geq N_2.$$

Remark 2.4. We refer to Appendix A.2 for an example that shows that N_2 is not necessarily equal to N_1.

Proof. For the *defocusing case*, suppose that $\lambda_n = \lambda_{n-1} + 1$. Then, on the one hand we have by Proposition 2.4, $S f_{n-1} \mathbin{/\!/} f_n$, and on the other hand, we infer by the second identity of Lemma 2.3,

$$\langle S f_{n-1} | u \rangle \langle u | f_n \rangle = 0.$$

That is, $\langle u | f_n \rangle = 0$. The converse is a direct consequence of the second identity of Lemma 2.3 and the previous lemma.

For the *focusing case*, the same analysis can be applied. However, it should be noted that, since not all the eigenvalues (ν_n) satisfy $\nu_n > \nu_{n-1} + \frac{1}{2}$, and $\nu_n \neq 0$, for all $n \in \mathbb{N}_{\geq 1}$, but only for large n, thanks to Proposition 2.1, then the equivalence holds for n sufficiently large. ∎

3. Traveling waves for the defocusing (CS$^-$)

3.1. Spectral characterization

One way to understand the behavior of a linear PDE's solution is to consider its Fourier transform. Specifically, on the periodic domain $\mathbb{T}$, this consists of computing the inner product with $\langle \cdot | \mathrm{e}^{inx} \rangle$ for all $n \in \mathbb{Z}$. The main idea behind this approach is to "diagonalize" the problem in the (e^{inx})-basis, which facilitates solving the equation. However, by considering the Calogero–Sutherland DNLS equation (CS$^-$), we are dealing with a nonlinear *integrable* PDE, which can also be "diagonalized" in some coordinate system (think about the Birkhoff coordinates). Thus, by imitating the idea of the linear case, we suggest taking the inner product of the (CS$^-$) equation with an appropriate orthonormal basis of $L^2_+(\mathbb{T})$.

Before proceeding, observe that the defocusing Calogero–Sutherland DNLS equation can be rewritten in terms of the Lax pair as [3, Lemmas 2.4, 5.2]

$$\partial_t u = \widetilde{B}_u u - i\tilde{L}_u^2 u. \tag{3.1}$$

This motivates the choice of the following orthonormal basis of $L^2_+(\mathbb{T})$.

Definition 3.1. Given $u \in \mathcal{C}_t H^2_+(\mathbb{T})_x$, let (g_n^t) be the evolving orthonormal basis of $L^2_+(\mathbb{T})$ defined along the curve $t \mapsto u(t)$ as

$$\begin{cases} \partial_t g_n^t = \widetilde{B}_{u(t)} g_n^t, \\ g_n^t|_{t=0} = f_n^{u_0}, \end{cases} \qquad \forall n \in \mathbb{N}_{\geq 0},$$

where $(f_n^{u_0})$ is an orthonormal basis of $L^2_+(\mathbb{T})$ made up of the eigenfunctions of $\tilde{L}_{u_0}$ at $t = 0$, and $\widetilde{B}_{u(t)}$ is the skew-adjoint operator defined in (1.8).

Remark 3.1. Note that the (g_n^t) satisfy for all $n \in \mathbb{N}_{\geq 0}$ [20, Lemma 4.1],

$$\tilde{L}_{u(t)} g_n^t = \lambda_n g_n^t.$$

Therefore, as was established in [3, Lemma 3.6], by taking the inner product of (3.1) with the g_n^t and using that $\tilde{L}_u$ is a self-adjoint operator and $\widetilde{B}_u$ is skew-symmetric, we find

$$\partial_t \langle u(t)|g_n^t\rangle = -i\lambda_n^2 \langle u(t)|g_n^t\rangle,$$

or

$$\langle u(t)|g_n^t\rangle = \langle u_0|f_n^{u_0}\rangle \mathrm{e}^{-i\lambda_n^2 t}. \tag{3.2}$$

Lemma 3.2. *For any $u \in \mathcal{C}_t H^2_+(\mathbb{T})_x$ a solution of* (CS$^-$) *and for all $n, p \in \mathbb{N}_{\geq 0}$,*

$$\begin{aligned} \langle 1|g_n^t\rangle &= \langle 1|f_n^{u_0}\rangle \mathrm{e}^{-i\lambda_n^2 t}, \\ \langle Sg_p^t|g_n^t\rangle &= \langle Sf_p^{u_0}|f_n^{u_0}\rangle \mathrm{e}^{i((\lambda_p+1)^2-\lambda_n^2)t}. \end{aligned}$$

Proof. By Definition 3.1, and since $\widetilde{B}_u$ is a skew-symmetric operator,

$$\partial_t \langle 1|g_n^t\rangle = \langle 1|\widetilde{B}_u g_n^t\rangle = -\langle \widetilde{B}_u 1|g_n^t\rangle,$$

where by (1.8),

$$\begin{aligned} \widetilde{B}_u 1 &= -T_u T_{\partial_x \bar{u}} 1 + T_{\partial_x u} T_{\bar{u}} 1 + i(T_u T_{\bar{u}})^2 1 \\ &= \langle 1|u\rangle (\partial_x u + iT_u T_{\bar{u}} u). \end{aligned}$$

Note that $\tilde{L}_u 1 = -i\partial_x 1 + T_u T_{\bar{u}} 1 = \langle 1|u\rangle u$. Therefore, $\widetilde{B}_u 1 = i\tilde{L}_u^2 1$ and

$$\partial_t \langle 1|g_n^t\rangle = -i\langle L_u^2 1|g_n^t\rangle = -i\lambda_n^2 \langle 1|g_n^t\rangle.$$

This achieves the proof of the first point. To prove the second one, we proceed in the same manner. By Definition 3.1,

$$\partial_t \langle g_n^t | S g_p^t \rangle = \langle \widetilde{B}_u g_n^t | S g_p^t \rangle + \langle g_n^t | S \widetilde{B}_u g_p^t \rangle = \langle [S^*, \widetilde{B}_u] g_n^t | g_p^t \rangle.$$

Hence, applying the commutator identity (2.2), and since $\tilde{L}_u$ is a self-adjoint operator, we infer

$$\begin{aligned} \partial_t \langle g_n^t | S g_p^t \rangle &= i \langle (S^* \tilde{L}_u^2 - (\tilde{L}_u + \mathrm{Id})^2 S^*) g_n^t | g_p^t \rangle \\ &= i(\lambda_n^2 + (\lambda_p + 1)^2) \langle g_n^t | S g_p^t \rangle. \end{aligned}$$

Therefore,

$$\langle g_n^t | S g_p^t \rangle = \langle f_n^{u_0} | S f_p^{u_0} \rangle \mathrm{e}^{i(\lambda_n^2 - (\lambda_p + 1)^2)t}. \qquad \blacksquare$$

Remark 3.2. The consideration of the evolution of $\langle u | g_n^t \rangle$, $\langle 1 | g_n^t \rangle$, and $\langle S g_p^t | g_n^t \rangle$ is motivated by the fact that any element u of the Hardy space can be written as follows.

Lemma 3.3 ([14, 15]). *For any $u \in L^2_+(\mathbb{T})$,*

$$u(z) = \langle (\mathrm{Id} - z S^*)^{-1} u | 1 \rangle, \quad z \in \mathbb{D},$$

where S^* is the adjoint operator of S in $L^2_+(\mathbb{T})$.

Therefore, by expressing the operator S^*, and the two vectors u and 1 in their matrix representations with respect to the (g_n^t)-basis, we obtain

$$u(t, z) = \langle (\mathrm{Id} - z M)^{-1} X | Y \rangle, \quad z \in \mathbb{D},$$

where X, Y are infinite column vectors and M is the infinite matrix representation:

$$X := (\langle u | g_n^t \rangle), \quad Y := (\langle 1 | g_n^t \rangle), \quad M := (\langle g_m^t | S g_n^t \rangle).$$

Proof of Lemma 3.3 ([15]). The idea is to observe that any element u of the Hardy space $L^2_+(\mathbb{T})$ can be read as an analytic function on the open unit disc $\mathbb{D}$, whose trace on the boundary $\partial\mathbb{D}$ is in L^2.[4] Thus, for any $z \in \mathbb{D}$,

$$u(z) = \sum_{k \in \mathbb{N}_{\geq 0}} \hat{u}(k) z^k = \sum_{k \in \mathbb{N}_{\geq 0}} \langle u | S^k 1 \rangle z^k = \sum_{k \in \mathbb{N}_{\geq 0}} \langle (S^*)^k u | 1 \rangle z^k.$$

As a result, by Neumann series,

$$u(z) = \langle (\mathrm{Id} - z S^*)^{-1} u | 1 \rangle. \qquad \blacksquare$$

[4] For a simple introduction to the different definitions of Hardy spaces, we refer to [14, Chapter 3].

At this stage, we consider $u(t) := u_0(x - ct)$ to be a traveling wave to the Calogero–Sutherland DNLS equation (CS^-). For all $c, t \in \mathbb{R}$, we denote by τ_{ct} the isometric linear map

$$\tau_{ct} \colon L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T}), \quad \tau_{ct} u_0(x) = u_0(x - ct).$$

Our aim for this subsection is to prove the following theorem.

Theorem 3.4. *Let $u(t) := \tau_{ct} u_0$ be a traveling wave to the (CS^-) equation. Then there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that*

$$\langle u_0 | f_N^{u_0} \rangle \neq 0.$$

Moreover, the speed c is given by

$$c = 1 + \frac{2}{N} \sum_{k=0}^{N-1} \lambda_k.$$

To this end, we shall need two key elements. First, we need Lemma 3.2 and identity (3.2). Second, we shall utilize the existence of a relationship (identity (3.3)) connecting the eigenfunctions (g_n^t) of $\tilde{L}_{u(t)}$ introduced in Definition 3.1, with the functions $(\tau_{ct} f_n^{u_0})$, where we recall that $(f_n^{u_0})$ represents the eigenfunctions of $\tilde{L}_{u_0}$.

To establish this connection, we present the following proposition, which also describes the behavior of the eigenfunctions $(f_n^{u_0})$ of $\tilde{L}_{u_0}$ under the action of the translation map on the spatial variable

$$f_n^{u_0} \mapsto \tau_{ct} f_n^{u_0}, \quad c, t \in \mathbb{R}.$$

Proposition 3.5. *Let $u(t) := \tau_{ct} u_0$ be a solution to (CS^-). There exists a sequence $(\theta_n(t)) \subseteq \mathbb{R}$, such that*

$$\tau_{ct} f_n^{u_0} = \mathrm{e}^{i\theta_n(t)} g_n^t, \quad \forall n \in \mathbb{N}_{\geq 0}. \tag{3.3}$$

In other words, the $(\tau_{ct} f_n^{u_0})$ are also eigenfunctions of $\tilde{L}_{u(t)}$.

Proof. By definition of $\tilde{L}_u = D + u\Pi(\bar{u}\cdot)$, and since $u(t) = \tau_{ct} u_0$,

$$\begin{aligned}
\tilde{L}_{u(t)} \tau_{ct} f_n^{u_0} &= D f_n^{u_0}(x - ct) + u_0(x - ct)\Pi(\bar{u}_0(x - ct) f_n^{u_0}(x - ct)), \\
&= \tau_{ct}(\tilde{L}_{u_0} f_n^{u_0}) \\
&= \lambda_n(u_0)\tau_{ct} f_n^{u_0}, \quad \forall n \in \mathbb{N}_{\geq 0}.
\end{aligned}$$

In other words, $\tau_{ct} f_n^{u_0}$ is an eigenfunction of $\tilde{L}_{u(t)}$ associated with the eigenvalue $\lambda_n(u_0)$. On the other hand, recall that all the eigenvalues $\lambda_n(u_0)$ of $\tilde{L}_{u(t)}$ are simple, as stated in Proposition 2.1. Additionally, according to Remark 3.1, the (g_n^t) are eigenfunctions of $\tilde{L}_{u(t)}$ associated to the eigenvalues $\lambda_n(u_0)$. Therefore, for all $n \in \mathbb{N}_{\geq 0}$, the two vectors $\tau_{ct} f_n^{u_0}$ and g_n^t are collinear. Since both vectors belong to an orthonormal basis of $L^2_+(\mathbb{T})$, then each one has an L^2-norm equal to 1. Thus, we infer for all $n \in \mathbb{N}_{\geq 0}$, there exists $\theta_n(t) \in \mathbb{R}$ such that for all $t \in \mathbb{R}$,

$$\tau_{ct} f_n^{u_0} = \mathrm{e}^{i\theta_n(t)} g_n^t. \qquad \blacksquare$$

Corollary 3.6. *For all $n, p \in \mathbb{N}_{\geq 0}$, and for all $t \in \mathbb{R}$, we have*

(1) *if $\langle 1 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\lambda_n^2 t,$$

(2) *if $\langle u_0 | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = -\lambda_n^2 t,$$

(3) *if $\langle S f_p^{u_0} | f_n^{u_0} \rangle \neq 0$ then*

$$\theta_n(t) = ((\lambda_p + 1)^2 - \lambda_n^2)t - ct + \theta_p(t),$$

where $\theta_n(t)$ is the angle obtained in (3.3).

Proof. By combining identity (3.2) and the two identities of Lemma 3.2 with identity (3.3) of the previous proposition, we infer

$$\begin{cases} \mathrm{e}^{i\theta_n(t)} \langle 1 | \tau_{ct} f_n^{u_0} \rangle = \langle 1 | f_n^{u_0} \rangle \mathrm{e}^{-i\lambda_n^2 t}, \\ \mathrm{e}^{i\theta_n(t)} \langle \tau_{ct} u_0 | \tau_{ct} f_n^{u_0} \rangle = \langle u_0 | f_n^{u_0} \rangle \mathrm{e}^{-i\lambda_n^2 t}, \\ \mathrm{e}^{-i\theta_p(t)} \mathrm{e}^{i\theta_n(t)} \langle S \tau_{ct} f_p^{u_0} | \tau_{ct} f_n^{u_0} \rangle = \langle S f_p^{u_0} | f_n^{u_0} \rangle \mathrm{e}^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{cases}$$

Note that $S\tau_{ct}(\cdot) = \mathrm{e}^{ict}\tau_{ct}(S\cdot)$, and since we are dealing with periodic functions, we deduce

$$\begin{cases} \mathrm{e}^{i\theta_n(t)} \langle 1 | f_n^{u_0} \rangle = \langle 1 | f_n^{u_0} \rangle \mathrm{e}^{-i\lambda_n^2 t}, \\ \mathrm{e}^{i\theta_n(t)} \mathrm{e}^{i\varphi(t)} \langle u_0 | f_n^{u_0} \rangle = \langle u_0 | f_n^{u_0} \rangle \mathrm{e}^{-i\lambda_n^2 t}, \\ \mathrm{e}^{-i\theta_p(t)} \mathrm{e}^{i\theta_n(t)} \mathrm{e}^{ict} \langle S f_p^{u_0} | f_n^{u_0} \rangle = \langle S f_p^{u_0} | f_n^{u_0} \rangle \mathrm{e}^{i((\lambda_p+1)^2 - \lambda_n^2)t}. \end{cases}$$

leading to the result. ∎

At this point, we are ready to prove the spectral characterization of the traveling waves for (CS$^-$), namely Theorem 3.4.

Proof of Theorem 3.4. The proof relies on the spectral property of $\tilde{L}_u$ discussed in Section 2 and on Corollary 3.6. Indeed, observe first by Lemma 2.5, we have $\langle S f_{n-1}^{u_0} | f_n^{u_0} \rangle \neq 0$ for all $n \in \mathbb{N}_{\geq 1}$. Hence, applying the third identity of Corollary 3.6 with $p = n - 1$ leads to the recurrence relation

$$\theta_n(t) = ((\lambda_{n-1} + 1)^2 - \lambda_n^2)t - ct + \theta_{n-1}(t), \quad n \geq 1.$$

Taking the sum of all these expressions from $n = 1$ to $n \in \mathbb{N}_{\geq 1}$, we infer

$$\theta_n(t) = \lambda_0^2 t + 2t \sum_{k=0}^{n-1} \lambda_k + nt - \lambda_n^2 t - nct + \theta_0(t). \tag{3.4}$$

Our aim is to prove that for all $n \geq 1$, $\langle u_0 | f_n^{u_0} \rangle = 0$ except for at most one n. For the sake of contradiction, suppose that there exist two integers $1 \leq n_1 < n_2$ such that $\langle u_0 | f_{n_1}^{u_0} \rangle \neq 0$ and $\langle u_0 | f_{n_2}^{u_0} \rangle \neq 0$. Then by Corollary 3.6, we infer

$$\begin{aligned} \theta_{n_1}(t) &= -\lambda_{n_1}^2 t, \\ \theta_{n_2}(t) &= -\lambda_{n_2}^2 t. \end{aligned} \tag{3.5}$$

Plugging (3.5) in (3.4) we obtain

$$\begin{aligned} n_1 ct &= n_1 t + 2t \sum_{k=0}^{n_1-1} \lambda_k + \theta_0(t) + \lambda_0^2 t, \\ n_2 ct &= n_2 t + 2t \sum_{k=0}^{n_2-1} \lambda_k + \theta_0(t) + \lambda_0^2 t. \end{aligned} \tag{3.6}$$

Further, notice that

$$\theta_0(t) = -\lambda_0^2 t. \tag{3.7}$$

Indeed, if $\langle u_0 | f_0^{u_0} \rangle \neq 0$ then by the second point of Corollary 3.6, we have the claimed identity. Otherwise, if $\langle u_0 | f_0^{u_0} \rangle = 0$ then $\langle 1 | f_0^{u_0} \rangle \neq 0$, since if it is not the case, i.e. if there exists $h \in L^2_+(\mathbb{T})$ such that $f_0^{u_0} = Sh$, then we have by the commutator relation (2.1),

$$\lambda_0 Sh = \tilde{L}_{u_0} Sh = S\tilde{L}_{u_0} h + Sh + \langle Sh | u_0 \rangle u_0,$$

implying, as $\langle Sh | u_0 \rangle = \langle f_0^{u_0} | u_0 \rangle = 0$,

$$\tilde{L}_{u_0} h = (\lambda_0 - 1) h.$$

That means h is an eigenvector of $\tilde{L}_{u_0}$ associated with an eigenvalue strictly less than λ_0, which is impossible. Therefore $\langle 1 | f_0^{u_0} \rangle \neq 0$, and so by the first identity of Corollary 3.6, we infer $\theta_0(t) = -\lambda_0^2 t$. Substituting (3.7) in (3.6), we obtain

$$c = \begin{cases} 1 + \dfrac{2}{n_1} \displaystyle\sum_{k=0}^{n_1-1} \lambda_k, \\ 1 + \dfrac{2}{n_2} \displaystyle\sum_{k=0}^{n_2-1} \lambda_k. \end{cases}$$

That is,

$$n_2 \sum_{k=0}^{n_1-1} \lambda_k = n_1 \sum_{k=0}^{n_2-1} \lambda_k,$$

or

$$(n_2 - n_1) \sum_{k=0}^{n_1-1} \lambda_k = n_1 \sum_{k=n_1}^{n_2-1} \lambda_k.$$

But recall by (2.3), $\lambda_{n+1} > \lambda_n$, for all n. Combining this fact with the last equality, we conclude

$$n_1(n_2 - n_1)\lambda_{n_1-1} > n_1(n_2 - n_1)\lambda_{n_1},$$

leading to a contradiction. As a consequence, for any traveling wave solution $u(t,x) := u_0(x - ct)$ of (CS$^-$), there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that

$$\langle u_0 | f_N^{u_0} \rangle \neq 0,$$

where $(f_n^{u_0})$ is any orthonormal basis of $L^2_+(\mathbb{T})$ consisting of the eigenfunctions of $\tilde{L}_{u_0}$. Moreover, u travels with the speed

$$c = 1 + \frac{2}{N} \sum_{k=0}^{N-1} \lambda_k. \tag{3.8}$$

■

Remark 3.3. In view of the previous theorem and Corollary 2.6, it follows that any traveling wave solution u of (CS$^-$) propagates with a speed

$$c = N + 2\lambda_0. \tag{3.9}$$

Indeed, since $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $1 \leq n < N$, then by Corollary 2.6,

$$\lambda_n = \lambda_{n-1} + 1, \quad \forall 1 \leq n < N.$$

leading to the fact that (3.8) is equivalent to (3.9). Further, since $\tilde{L}_u$ is a nonnegative operator, then $\lambda_0 \geq 0$, which implies that the speed of the traveling wave solution satisfies $c \geq N$. However, as will be observed in Section 3.3, the speed $c = N$ can only be reached by traveling waves of the form $u(t,x) = \mathrm{e}^{iN(x-Nt)}$.

3.2. Explicit formulas for the traveling waves

Recall by Remark 3.2 that any elements of the Hardy space, in particular u_0, can be written as

$$u_0(z) = \langle (\mathrm{Id} - zM)^{-1} X | Y \rangle, \tag{3.10}$$

where X, Y are infinite column vectors and M is an infinite matrix:

$$X := (\langle u_0 | f_n^{u_0} \rangle), \quad Y := (\langle 1 | f_n^{u_0} \rangle), \quad M = (\langle f_p^{u_0} | S f_n^{u_0} \rangle). \tag{3.11}$$

In the following, we denote by $\mathcal{G}_1$ the set of the *semi-trivial traveling waves*, made up from the constant and the plane wave solutions

$$\mathcal{G}_1 = \left\{ C \mathrm{e}^{iN(x-Nt)} \,\middle|\, C \in \mathbb{C},\ N \in \mathbb{N}_{\geq 0} \right\}.$$

Theorem (Theorem 1.1). *The traveling waves $u(t,x) = u_0(x - ct)$ of (CS$^-$) are the potentials $u(t,x) \in \mathcal{G}_1$ and*

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N(1 + \frac{2\alpha}{\beta})$, and (α, β) are two real constants satisfying

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = -N. \tag{3.12}$$

Proof. The proof is based on the inversion spectral formula

$$u_0(z) = \langle (\mathrm{Id} - zM)^{-1} X | Y \rangle$$

of (3.10), and on the spectral characterization of u_0 described in Theorem 3.4. In the sequence, to make the notation less cluttered, we denote $f_n := f_n^{u_0}$.

Let $u(t,x) := u_0(x - ct)$. As a first step, we prove that the infinite matrices X, Y and M reduce to finite matrices in the context of a traveling wave solution. Indeed, by Theorem 3.4, there exists at most one $N \in \mathbb{N}_{\geq 1}$, such that $\langle u_0 | f_N \rangle \neq 0$. We focus on the case where such an N exists, that is,

$$\begin{cases} \langle u_0 | f_N \rangle \neq 0, \\ \langle u_0 | f_n \rangle = 0, \forall n \in \mathbb{N}_{\geq 1} \backslash \{N\}. \end{cases} \tag{3.13}$$

The case where $\langle u | f_n \rangle = 0$ for all $n \in \mathbb{N}_{\geq 1}$ can be handled similarly, leading also to the reduction of the study to finite matrices. From now on, we suppose (3.13) holds. Therefore, it follows by Lemma 2.3 that $\lambda_n \langle 1 | f_n \rangle = 0$, implying that

$$\langle 1 | f_n \rangle = 0, \quad \forall n \in \mathbb{N} \backslash \{N\},$$

as the eigenvalues λ_n are all positive for any $n \in \mathbb{N}_{\geq 1}$ since $\tilde{L}_u$ is a nonnegative operator. Therefore, the two infinite column vectors X and Y of (3.11) reduce to

$$X = \begin{pmatrix} \langle u_0 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle u_0 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} \langle 1 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle 1 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}. \tag{3.14}$$

On the other hand, since $\langle u_0 | f_n \rangle = 0$, for all $n \in \mathbb{N} \backslash \{N\}$, then by Corollary 2.6, we have $\lambda_n = \lambda_{n-1} + 1$ for all $n \in \mathbb{N} \backslash \{N\}$. Whence, $S f_{n-1} \mathbin{/\!/} f_n$ for all $n \in \mathbb{N} \backslash \{N\}$, thanks to Proposition 2.4. More specifically,

$$\begin{cases} f_n \,/\!/\, S^n f_0, & 1 \le n \le N-1, \\ f_n \,/\!/\, S^{n-N} f_N, & n \ge N. \end{cases} \tag{3.15}$$

As a consequence, the set $\{(S^n f_0)_{n=0,\dots,N-1}, (S^n f_N)_{n\ge 0}\}$ is an orthonormal basis of $L^2_+(\mathbb{T})$ and the matrix $M = (\langle f_p | S f_n\rangle)$ reduces to

$$M = \left(\begin{array}{ccccc|ccc} 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \ddots & 1 & 0 & & & \\ \langle f_0 | S^N f_0\rangle & 0 & \dots & 0 & \langle f_N | S^N f_0\rangle & 0 & & \\ 0 & \dots & & \dots & 0 & 1 & 0 & \\ \hline 0 & \dots & & \dots & 0 & 0 & 1 & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots \end{array}\right).$$

Hence, the infinite matrices X, Y and M in formula (3.10) can be reduced to finite matrices involving only the first $N+1$ coordinates of X, Y, and M [15]. Indeed, denoting $\boldsymbol{\xi} := (\mathrm{Id} - zM)^{-1} X$, we have

$$(\mathrm{Id} - zM)\boldsymbol{\xi} = X.$$

That is,

$$\left(\begin{array}{ccccc|ccc} 1 & -z & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & & \ddots & -z & 0 & & & \\ -\langle f_0 | S^N f_0\rangle z & 0 & \dots & 1 & -\langle f_N | S^N f_0\rangle z & 0 & & \\ 0 & \dots & & \dots & 1 & -z & 0 & \\ \hline 0 & \dots & & \dots & 0 & 1 & -z & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots \end{array}\right) \cdot \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{N-1} \\ \xi_N \\ \xi_{N+1} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \langle u_0 | f_0 \rangle \\ 0 \\ \vdots \\ 0 \\ \langle u_0 | f_N \rangle \\ 0 \\ \vdots \end{pmatrix}$$

Thus, for all $n \ge N+1$, the nth coordinate of $\boldsymbol{\xi}$ is $\xi_n = z\xi_{n+1}$, i.e.

$$\xi_{N+1} = z^{n-N-1}\xi_n, \quad \forall n \ge N+1,$$

and since $\sum_{n\geq 0}|\xi_n|^2 < \infty$, then

$$\xi_n = 0, \quad \forall n \geq N+1.$$

As a result,

$$\left\langle (\mathrm{Id} - z(M_{mn})_{m,n\geq N+1})^{-1}(X_n)_{n\geq N+1} \middle| (Y_m)_{m\geq N+1}\right\rangle = 0,$$

and therefore

$$u_0(z) = \langle (\mathrm{Id} - zM_{\leq N})^{-1} X_{\leq N} | Y_{\leq N}\rangle_{\mathbb{C}^{N+1}\times\mathbb{C}^{N+1}},$$

where $M_{\leq N} := (M_{mn})_{0\leq m,n\leq N}$, $X_{\leq N} := (X_n)_{0\leq n\leq N}$, and $Y_{\leq N} := (Y_N)_{0\leq n\leq N}$. Consequently, u_0 is a rational function

$$u_0(z) = \frac{P(z)}{\det(\mathrm{Id} - zM_{\leq N})},$$

where $P(z) = Y^*_{\leq N} \cdot \mathrm{Com}(\mathrm{Id} - zM_{\leq N})^\intercal \cdot X_{\leq N}$. Computing the numerator P and the denominator of u_0 via these finite matrices, we obtain that u_0 is of the form

$$u_0(z) = \frac{az^N + b}{1 - pz^N}, \quad a, b \in \mathbb{C}, \tag{3.16}$$

where $p = \langle f_0 | S^N f_0\rangle$, $|p| < 1$.

If $p = 0$: Namely, if $\langle f_0 | S^N f_0\rangle = 0$, then

$$S^N f_0 = \sum_{n\geq n} \langle S^N f_0 | f_n\rangle f_n = \langle S^N f_0 | f_N\rangle f_N,$$

since by (3.15), the set $\{(S^n f_0)_{n=0,\dots,N-1}, \ (S^n f_N)_{N\geq 0}\}$ is an orthonormal basis of $L^2_+(\mathbb{T})$. Thus, the two vectors f_N and $S^N f_0$ are collinear, leading to the following: for all $n \in \mathbb{N}_{\geq 0}$,

$$f_n \mathbin{/\!/} S^n f_0,$$

thanks to (3.15). Consequently, $\{S^n f_0, n \in \mathbb{N}_{\geq 0}\}$ is an orthonormal basis of $L^2_+(\mathbb{T})$, which means that the vector f_0 is necessarily collinear to 1. Further, recall from (3.14) that

$$\begin{aligned}
u_0 &= \langle u_0 | f_0\rangle f_0 + \langle u_0 | f_N\rangle f_N \\
&= \langle u_0 | f_0\rangle f_0 + \langle u_0 | S^N f_0\rangle S^N f_0, \\
&= \langle u_0 | 1\rangle + \langle u_0 | \mathrm{e}^{iNx}\rangle \mathrm{e}^{iNx}
\end{aligned}$$

and, as $p = 0$, i.e. $\langle f_0 | S^N f_0\rangle = 0$, we have by the second identity of Lemma 2.3, either

$$\langle u_0 | f_0\rangle = 0 \quad \text{or} \quad \langle u_0 | S^N f_0\rangle = 0,$$

i.e.

$$\langle u_0 | 1\rangle = 0 \quad \text{or} \quad \langle u_0 | \mathrm{e}^{iNx}\rangle = 0.$$

Therefore, either $u_0(x)$ is a complex constant, or $u_0(x) = C\mathrm{e}^{iNx}$, with $C \in \mathbb{C}$, $N \in \mathbb{N}_{\geq 1}$. Taking $u(t,x) = u_0(x-ct) = C\mathrm{e}^{iN(x-ct)}$, and substituting it into the defocusing Calogero–Sutherland DNLS equation (CS$^-$), we can infer, since the nonlinearity $D\Pi(|\mathrm{e}^{iN(x-ct)}|^2)\mathrm{e}^{iN(x-ct)}$ vanishes, that

$$Nc\mathrm{e}^{iN(x-ct)} - N^2\mathrm{e}^{iN(x-ct)} = 0,$$

and thus $c = N$. As a result, if $p = 0$ then the traveling waves $u(t,x) := u_0(x-ct)$ are

$$u(t,x) = C\mathrm{e}^{iN(x-Nt)}, \quad C \in \mathbb{C},\ N \in \mathbb{N}_{\geq 0}.$$

If $p \neq 0$: The potential u_0 of (3.16) can be rewritten as

$$u_0 = \alpha + \frac{\beta}{1-pz^n}, \quad \alpha, \beta \in \mathbb{C},\ p, z \in \mathbb{D},$$

In order to find the relation between α, β and obtain the speed c, we substitute $u(t,z) := u_0(\mathrm{e}^{-ict}z)$ into the defocusing Calogero–Sutherland DNLS equation (CS$^-$). This equation can be rewritten as

$$i\partial_t u - (z\partial_z)^2 u - 2z\partial_z \Pi(|u|^2)u = 0, \tag{3.17}$$

after observing that $D = -i\partial_x$ can be expressed as $D \equiv z\partial_z$. Thus, starting from

$$u := u(t,z) = \alpha + \frac{\beta}{1-p\mathrm{e}^{-iNct}z^N},$$

and computing $i\partial_t u$ and $(z\partial_z)^2 u$, we find

$$\begin{aligned} i\partial_t u &= -c\beta N\Big(\frac{1}{1-p\mathrm{e}^{-iNct}z^N} - \frac{1}{(1-p\mathrm{e}^{-iNct}z^N)^2}\Big), \\ (z\partial_z)^2 u &= \beta N^2\Big(\frac{1}{1-p\mathrm{e}^{-iNct}z^N} - \frac{3}{(1-p\mathrm{e}^{-iNct}z^N)^2} + \frac{2}{(1-p\mathrm{e}^{-iNct}z^N)^3}\Big). \end{aligned}$$

For the nonlinear part,

$$\begin{aligned} |u|^2 = |\alpha|^2 + \alpha\bar{\beta} &+ \frac{\alpha\bar{\beta}\bar{p}\mathrm{e}^{iNct}}{z^N - \bar{p}\mathrm{e}^{iNct}} + \frac{\alpha\bar{\beta}}{1-p\mathrm{e}^{-iNct}z^N} \\ &+ \frac{|\beta|^2 z^N}{(1-p\mathrm{e}^{-iNct}z^N)(z^N - \bar{p}\mathrm{e}^{iNct})}. \end{aligned}$$

Recall that Π is an orthonormal projector into the Hardy space (in particular to a subspace of the holomorphic functions on $\mathbb{D}$). Thus, applying Π, it follows that

$$\Pi(|u|^2) = |\alpha|^2 + \alpha\bar{\beta} + \frac{\alpha\bar{\beta}}{1-p\mathrm{e}^{-iNct}z^N} + \frac{|\beta|^2}{1-|p|^2}\frac{1}{1-p\mathrm{e}^{-iNct}z^N},$$

and hence,

$$z\partial_z \Pi(|u|^2)\cdot u = A\Big(\frac{-\alpha}{1-p\mathrm{e}^{-iNct}z^N} + \frac{-\beta+\alpha}{(1-p\mathrm{e}^{-iNct}z^N)^2} + \frac{\beta}{(1-p\mathrm{e}^{-iNct}z^N)^3}\Big),$$

where

$$A = N\Big(\bar{\alpha}\beta + \frac{|\beta|^2}{1-|p|^2}\Big).$$

Substituting the expressions of $i\partial_t u$, $(z\partial_z)^2 u$ and $z\partial_z\Pi(|u|^2)u$ into (3.17), and comparing the terms $1/(1-p\mathrm{e}^{-iNct}z^N)^n$ for $n = 1,2,3$, we deduce the following:

- With $n = 3$, $A = -N^2$. That is,

$$\bar{\alpha}\beta + \frac{|\beta|^2}{1-|p|^2} = -N.$$

- With $n = 2$ and $n = 1$,

$$c = -N\Big(1 + \frac{2\alpha}{\beta}\Big).$$

As a result, for $p \neq 0$,

$$u(t,z) := \alpha + \frac{\beta}{1-p\mathrm{e}^{-iNct}z^N}, \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N(1+\frac{2\alpha}{\beta})$, and $(\alpha,\beta) \in \mathbb{C}\times\mathbb{C}$ satisfy

$$\alpha\beta + \frac{|\beta|^2}{1-|p|^2} = -N. \tag{3.18}$$

Finally, observe by (3.18) that the two complex constants (α,β) satisfy $\bar{\alpha}\beta \in \mathbb{R}$. Thus, by a slight abuse of notation on α and β, we have obtained that the traveling waves of (CS$^-$) with $p \neq 0$ are given by

$$u(t,z) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1-p\mathrm{e}^{-iNct}z^N}\Big), \quad p \in \mathbb{D}^*,\ \theta \in \mathbb{T},$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N(1+\frac{2\alpha}{\beta})$, and $(\alpha,\beta) \in \mathbb{R}\times\mathbb{R}$ satisfy

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = -N. \qquad \blacksquare$$

3.3. The L^2-norm and the speed

In this subsection we analyze how the traveling waves of (CS$^-$) behave, by providing information regarding their L^2-norms and their speed c. Recall that the set of traveling wave solutions of the defocusing Calogero–Sutherland DNLS equation are made up by the trivial solutions

$$\mathcal{G}_1 = \big\{C\mathrm{e}^{iN(x-Nt)} \,\big|\, C \in \mathbb{C},\ N \in \mathbb{N}_{\geq 0}\big\},$$

and by the set of functions

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*, \theta \in \mathbb{T}, \tag{3.19}$$

where $N \in \mathbb{N}_{\geq 1}$, $c := -N(1 + \frac{2\alpha}{\beta})$, and (α, β) are two real constants satisfying (3.12).

For $u \in \mathscr{G}_1$, it is easy to see that the L^2-norm of the semi-trivial solution can be arbitrarily small or large in $[0, +\infty)$, and its speed c is given as $c = N \in \mathbb{N}_{\geq 0}$. The following proposition aims to provide those for the nontrivial traveling waves of (CS$^-$).

Proposition 3.7 (L^2-norm of a nontrivial traveling wave and the speed). *We have the following properties:*

(i) *For any $r > 0$, there exists a nontrivial traveling wave $u(t,x) := u_0(x - ct)$ for* (CS$^-$) *with*

$$\|u_0\|_{L^2} = r.$$

In other words, the traveling waves of (CS$^-$) *can be arbitrarily small or large in $L^2_+(\mathbb{T})$.*

(ii) *Let u be a traveling wave for* (CS$^-$) *of the form* (3.19)*; then u propagates to the right with a speed $c > N$. In addition, when $\|u\|_{L^2} \to \infty$ then $c \to \infty$ and when $\|u\|_{L^2} \to 0$ then $c \to N$.*

Remark 3.4 (Nonexistence of a stationary solution for (CS$^-$)). Since for any traveling wave $u_0(x - ct)$ of the defocusing Calogero–Sutherland DNLS equation (CS$^-$) we have $c \geq N$, where N is the degree of u_0 in the denominator, then there is no stationary solution (i.e. $u(t,x) = u_0(x)$) for the (CS$^-$) equation. Another way to see this is by observing that if $c = 0$, which occurs when $\alpha = -\frac{\beta}{2}$ according to Theorem 1.1, then we have by (3.12),

$$\frac{1 + |p|^2}{1 - |p|^2}\beta^2 = -N,$$

which is impossible as $p \in \mathbb{D}^*$.

Proof of Proposition 3.7. (i) *The L^2-norm of the nontrivial traveling wave can be arbitrarily small or large.* Let u be a traveling wave of the form (3.19),

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*, \ N \in \mathbb{N}_{\geq 1},$$

where $(\alpha, \beta) \in \mathbb{R}^2$ satisfies the identity (3.12). Recall that any function u in the Hardy space can be seen as an analytic function on the open unit disc $\mathbb{D}$, whose trace on the boundary $\partial\mathbb{D}$ is in L^2. Hence,

$$\|u\|^2_{L^2} = \int_{z \in \mathcal{C}(0,1)} |u(z)|^2 \, \frac{dz}{2\pi i z},$$

where

$$
\begin{aligned}
|u(z)|^2 &= \Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{-iNct} z^N}\Big)\Big(\alpha + \frac{\beta z^N}{z^N - \bar{p}\mathrm{e}^{iNct}}\Big) \\
&= \alpha^2 + \alpha\beta + \frac{\alpha\beta \mathrm{e}^{iNct}\bar{p}}{z^N - \bar{p}\mathrm{e}^{iNct}} + \frac{\alpha\beta}{1 - p\mathrm{e}^{-iNct} z^N} \\
&\quad + \frac{\beta^2 z^N}{(1 - p\mathrm{e}^{-iNct} z^N)(z^N - \bar{p}\mathrm{e}^{iNct})}.
\end{aligned}
$$

Writing

$$
\frac{\beta^2 z^N}{(1 - p\mathrm{e}^{-iNct} z^N)(z^N - \bar{p}\mathrm{e}^{iNct})} = \frac{\beta^2}{1 - |p|^2}\Big(\frac{1}{1 - p\mathrm{e}^{-iNct} z^N} + \frac{\bar{p}\mathrm{e}^{iNct}}{z^N - \bar{p}\mathrm{e}^{iNct}}\Big),
$$

we infer

$$
\begin{aligned}
|u(z)|^2 &= \alpha^2 + \alpha\beta + \Big(\alpha\beta + \frac{\beta^2}{1 - |p|^2}\Big)\frac{1}{1 - p\mathrm{e}^{-iNct} z^N} \\
&\quad + \Big(\alpha\beta + \frac{\beta^2}{1 - |p|^2}\Big)\frac{\mathrm{e}^{iNct}\bar{p}}{z^N - \bar{p}\mathrm{e}^{iNct}}.
\end{aligned}
$$

Therefore,

$$
\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta + \alpha\beta + \frac{\beta^2}{1 - |p|^2}, \tag{3.20}
$$

since for $N \in \mathbb{N}_{\geq 1}$,

$$
\Big\langle 1 \Big| \frac{1}{z^N - \bar{p}\mathrm{e}^{iNct}} \Big\rangle = \int_{z \in \mathcal{C}(0,1)} \frac{z^N}{1 - p\mathrm{e}^{-iNct} z^N} \frac{dz}{2\pi i z} = 0.
$$

Consequently, by (3.12),[5]

$$
\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta - N.
$$

In addition, since by (3.12),

$$
\alpha = -\frac{N}{\beta} - \frac{\beta}{1 - |p|^2},
$$

then

$$
\begin{aligned}
\|u\|_{L^2}^2 &= \Big(-\frac{N}{\beta} - \frac{\beta}{1 - |p|^2}\Big)^2 + \Big(-\frac{N}{\beta} - \frac{\beta}{1 - |p|^2}\Big)\beta - N \\
&= \frac{|p|^2}{1 - |p|^2}\Big(\frac{\beta^2}{1 - |p|^2} + 2N\Big) + \frac{N^2}{\beta^2}.
\end{aligned}
$$

Observe that $\|u\|_{L^2}^2$ is a continuous function of $|p|^2$ and β^2. Moreover, by taking $\beta \to 0$ then $\|u\|_{L^2}^2 \to \infty$. And if we take $|p|^2 \to 0$ then

$$
\|u\|_{L^2}^2 \underset{|p|^2 \to 0}{\sim} \frac{N^2}{\beta^2},
$$

which can be arbitrary small when $\beta \gg 1$.

[5] As we shall see in Corollary 5.5, this corresponds to $\|u\|_{L^2}^2 = \lambda_N - N$ where $\lambda_N > N + \lambda_0 > N$.

(ii) *Speed* $c > N$. By Theorem 1.1, the speed of the traveling waves of the form (3.19) is given by $c = -N(1 + \frac{2\alpha}{\beta})$. Further, recall from (3.12), that

$$\frac{\alpha}{\beta} = -\frac{N}{\beta^2} - \frac{1}{1 - |p|^2}.$$

Substituting the latter identity in the expression For c, it follows that

$$c = N\Big(\frac{1 + |p|^2}{1 - |p|^2} + \frac{2N}{\beta^2}\Big) > N. \tag{3.21}$$

It remains to prove that

- when $\|u\|_{L^2} \to +\infty$, we have $c \to +\infty$,
- and when $\|u\|_{L^2} \to 0$ then $c \to N$.

Indeed, observe that $\|u\|_{L^2}^2 \to \infty$ when $\beta^2 \to 0$ or $|p|^2 \to 1$, and in both cases

$$c \to \infty.$$

On the other hand, $\|u\|_{L^2}^2$ is arbitrarily small when $|p|^2 \to 0$ and β is big enough. Hence, by passing to the limit $|p|^2 \to 0$ in (3.21), we infer

$$c \underset{|p|^2 \to 0}{\sim} N\Big(1 + \frac{2N}{\beta^2}\Big),$$

which can arbitrarily close to N as $\|u\|_{L^2}^2$ is arbitrarily close to 0. ∎

4. Traveling waves for the focusing (CS$^+$)

4.1. Toward the characterization of the traveling waves for (CS$^+$)

Recall that to characterize the traveling waves of the defocusing equation (CS$^-$), a spectral analysis was initially conducted, followed by the derivation of explicit formulas. Here, we aim to replicate the same strategy. But before proceeding, we shall require some analogous lemmas to the defocusing case.

Lemma 4.1 (Analog of Lemma 3.2). *Let $u \in \mathcal{C}_t H^2_+(\mathbb{T})_x$ be the solution of* (CS$^+$). *Then, for all $n, p \in \mathbb{N}_{\geq 0}$,*

$$\begin{aligned}
\langle 1 | g_n^t \rangle &= \langle 1 | f_n^{u_0} \rangle \mathrm{e}^{-i\nu_n^2 t},\\
\langle u | g_n^t \rangle &= \langle u_0 | f_n^{u_0} \rangle \mathrm{e}^{-i\nu_n^2 t}\\
\langle S g_p^t | u \rangle &= \langle S f_p^{u_0} | u_0 \rangle \mathrm{e}^{i(\nu_p + 1)^2 t},\\
\langle S g_p^t | g_n^t \rangle &= \langle S f_p^{u_0} | f_n^{u_0} \rangle \mathrm{e}^{i((\nu_p + 1)^2 - \nu_n^2) t},
\end{aligned}$$

where (g_n^t) denotes the orthonormal basis of the $L^2_+(\mathbb{T})$ solution to the Cauchy problem

$$\begin{cases} \partial_t g_n^t = B_{u(t)} g_n^t, \\ g_n^t|_{t=0} = f_n^{u_0}, \end{cases} \qquad \forall n \in \mathbb{N}_{\geq 0},$$

and $(f_n^{u_0})$ is an orthonormal basis of $L^2_+(\mathbb{T})$ made up of the eigenfunctions of L_{u_0} and $B_{u(t)}$ is the skew-adjoint operator defined in (1.8).

Proof. Since the focusing Calogero–Sutherland DNLS equation can also be rewritten in terms of its Lax operators [3, Lemma 2.4]

$$\partial_t u = B_u u - i L_u^2 u,$$

then one can repeat exactly the same proof of Lemma 3.2 and obtain the same results. ∎

Lemma 4.2 (Analog of Proposition 3.5). *Let $u := \tau_{ct} u_0$ be a traveling wave of (CS$^+$) such that the eigenvalue $\nu_n(u_0)$ is simple. Then there exists $\theta_n(t) \in \mathbb{R}$ such that*

$$\tau_{ct} f_n^{u_0} = \mathrm{e}^{i\theta_n(t)} g_n^t, \tag{4.1}$$

where the (g_n^t) denotes the orthonormal basis defined in the previous lemma.

Lemma 4.3 (Analog of Corollary 3.6 in the focusing case). *Let u_0 be a function such that the eigenvalues $(\nu_n(u_0))$ are simple. Then, for all $n, p \in \mathbb{N}_{\geq 0}$, $t \in \mathbb{R}$, we have the following:*

(1) *if $\langle 1 | f_n^{u_0} \rangle \neq 0$ then*
$$\theta_n(t) = -\nu_n^2 t,$$
(2) *if $\langle u_0 | f_n^{u_0} \rangle \neq 0$ then*
$$\theta_n(t) = -\nu_n^2 t,$$
(3) *if $\langle S f_p^{u_0} | f_n^{u_0} \rangle \neq 0$ then*
$$\theta_n(t) = ((\nu_p + 1)^2 - \nu_n^2)t - ct + \theta_p(t),$$

where $\theta_n(t)$ is the angle obtained in (4.1).

At this stage, we are equipped with the necessary tools to replicate the proof of the defocusing equation. However, it is important to emphasize two fundamental differences between the Lax operators L_u and $\tilde{L}_u$, which ultimately offer a considerably expanded set of traveling waves for (CS$^+$) in comparison to (CS$^-$):

- The gap between the eigenvalues differs between the focusing and the defocusing cases (Proposition 2.1).
- The eigenvalues λ_n of $\tilde{L}_u$ are not zero for any $n \in \mathbb{N}_{\geq 1}$.

Indeed, in the defocusing case, since all the eigenvalues satisfy $\lambda_n > \lambda_{n-1} + \frac{1}{2}$ (inequality (2.3)) and $\lambda_n \neq 0$ for all $n \in \mathbb{N}_{\geq 1}$, then we obtained in Lemma 2.5,

$$\mathcal{I}(u) = \varnothing, \quad \forall u \in H^2_+(\mathbb{T}),$$

where $\mathcal{I}(u)$ was defined in (2.8). As a consequence, we inferred that if $u(t,x) = u_0(x - ct)$ is a traveling wave of (CS$^-$), then there exists at most one $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $n \in \mathbb{N}\backslash\{N\}$. Now, for the focusing equation, recall that we have previously observed in the second point of Remark 2.3, that $\mathcal{I}(u)$ is of finite cardinal for all $u \in H^2_+(\mathbb{T})$. In particular, for $u_0 \in H^2_+(\mathbb{T})$, we denote by $m_1, \dots, m_n$ its elements

$$\mathcal{I}(u_0) = \{m_1, \dots, m_n\}.$$

Theorem 4.4 (Toward the characterization of the traveling waves of (CS$^+$)). *The traveling waves $u_0(x - ct)$ of (CS$^+$) are either rational functions or the plane waves $u(t,x) = C\mathrm{e}^{iN(x-Nt)}$. In addition, the potentials*

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*, \ \theta \in \mathbb{T}, \ N \in \mathbb{N}_{\geq 1}, \tag{4.2}$$

where $c = -N(1 + \frac{2\alpha}{\beta})$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = N, \tag{4.3}$$

and the potentials

$$u(t,x) = \mathrm{e}^{i\theta}\mathrm{e}^{im(x-ct)}\Big(\alpha + \frac{\beta}{1 - p\mathrm{e}^{i(x-ct)}}\Big), \quad p \in \mathbb{D}^*, \ \theta \in \mathbb{T}, \ m \in \mathbb{N}_{\geq 1}, \tag{4.4}$$

where $c = m$, $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\alpha\beta + \frac{\beta^2}{1 - |p|^2} = 1, \quad \beta(m-1) = 2\alpha,$$

are parts of the set of the traveling waves of (CS$^+$).

Proof. We proceed in two steps:

Step 1 (*Spectral characterization of the traveling waves of* (CS$^+$)). Let $u(t,x) := u_0(x - ct)$ be a traveling wave for (CS$^+$). Our goal is to prove that there exists $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n^{u_0} \rangle = 0$ for all $n \geq N$. Once more, in order to simplify the notation, in the following we write f_n instead of $f_n^{u_0}$. Recall by the second point of Remark 2.3 that $\mathcal{I}(u_0)$ is of finite cardinal, that is, there exists $m_1 < \dots < m_j \in \mathbb{N}_{\geq 1}$ such that

$$\begin{cases} \langle S f_{n-1} | f_n \rangle = 0, & \forall n \in \{m_1, \dots, m_j\}, \\ \langle S f_{n-1} | f_n \rangle \neq 0, & \forall n \in \mathbb{N}\backslash\{m_1, \dots, m_j\}. \end{cases}$$

Suppose that there exists an integer $\ell \gg 1$, $\ell > m_j$ such that $\langle u_0 | f_\ell \rangle \neq 0$. Otherwise, we already have what we claim to prove. Then,

- for all $n \geq \ell + 1$, the quantities $\langle S f_{n-1} | f_n \rangle \neq 0$,
- since $\ell \gg 1$, then by inequality (2.5), the eigenvalues $(\nu_n)_{n \geq \ell+1}$ are simple. This implies that Lemma 4.2 holds for $n \geq \ell + 1$.

Therefore, using the third point of Lemma 4.3, we obtain, for all $n \geq \ell + 1$,

$$\begin{aligned}
\theta_n(t) &= -\nu_n^2 t + (\nu_{n-1} + 1)^2 t - ct + \theta_{n-1}(t) \\
&= -\nu_n^2 t + (\nu_{n-1} + 1)^2 t - \nu_{n-1}^2 t + (\nu_{n-2} + 1)^2 t - 2ct + \theta_{n-2}(t) \\
&= \quad \dots \\
&= -\nu_n^2 t + (n - \ell) t + \nu_\ell^2 t - (n - \ell) ct + 2t \sum_{k=\ell}^{n-1} \nu_k + \theta_\ell(t),
\end{aligned}$$

where $\theta_n(t)$ is the angle obtained in Lemma 4.2, and $\theta_\ell(t) = -\nu_\ell^2 t$ thanks to the second point of Lemma 4.3. Hence, for all $n \geq \ell + 1$,

$$\theta_n(t) = -\nu_n^2 t + (n - \ell) t - (n - \ell) ct + 2t \sum_{k=\ell}^{n-1} \nu_k. \tag{4.5}$$

As a consequence, there exists at most one integer $N \geq \ell$ such that

$$\langle u_0 | f_N \rangle \neq 0.$$

Indeed, suppose for the sake of contradiction that there exist $n_2 > n_1 > \ell$, such that $\langle u_0 | f_{n_1} \rangle \neq 0$ and $\langle u_0 | f_{n_2} \rangle \neq 0$. Then, combining the second point of Lemma 4.3, and equation (4.5), we obtain

$$\begin{cases}
c = 1 + \dfrac{2}{n_1 - \ell} \displaystyle\sum_{k=\ell}^{n_1 - 1} \nu_k, \\
c = 1 + \dfrac{2}{n_2 - \ell} \displaystyle\sum_{k=\ell}^{n_2 - 1} \nu_k.
\end{cases}$$

Hence,

$$(n_2 - \ell) \sum_{k=\ell}^{n_1 - 1} \nu_k = (n_1 - \ell) \sum_{k=\ell}^{n_2 - 1} \nu_k,$$

or

$$(n_2 - n_1) \sum_{k=\ell}^{n_1 - 1} \nu_k = (n_1 - \ell) \sum_{k=n_1}^{n_2 - 1} \nu_k.$$

As a result,

$$(n_1 - \ell)(n_2 - n_1) \nu_{n_1 - 1} \geq (n_1 - \ell)(n_2 - n_1) \nu_{n_1},$$

leading to a contradiction, since for $k \geq \ell$, we have $\nu_{k+1} > \nu_k$. Therefore, there exists $N \in \mathbb{N}_{\geq 1}$ such that $\langle u_0 | f_n \rangle = 0$ for all $n \geq N$.

Step 2 (*They are rational functions or potentials in* $\mathcal{G}_1$). Since $\langle u_0 | f_n\rangle = 0$ for all $n \geq N$, it follows by Corollary 2.6 that $\nu_n = \nu_{n-1} + 1$ for all $n \geq N_2$. Note that the potentials satisfying $\nu_n = \nu_{n-1} + 1$ for all $n \geq N_2$ are referred to as "finite gap potentials" for (CS$^+$), and are studied deeply in Section 5. In particular, Theorem 1.3 provides a full characterization of these potentials in the state space. They are either $u(x) = C\mathrm{e}^{iNx}$, $C \in \mathbb{R}^*$, $N \in \mathbb{N}_{\geq 0}$, or rational functions

$$u(x) = \mathrm{e}^{im_0x}\prod_{j=1}^{r}\Big(\frac{\mathrm{e}^{ix} - \overline{p_j}}{1 - p_j\mathrm{e}^{ix}}\Big)^{m_j-1} \times \left(\alpha + \sum_{j=1}^{r}\frac{\beta_j}{1 - p_j\mathrm{e}^{ix}}\right), \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j, \tag{4.6}$$

where, for $N \in \mathbb{N}_{\geq 1}$, $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^r m_j = N$, and $(\alpha, \beta_1, \dots, \beta_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, r$,

$$\bar{\alpha}\beta_j + \sum_{k=1}^{r}\frac{\alpha_j\overline{\alpha_k}}{1 - p_j\overline{p_k}} = m_j.$$

It remains to verify that (4.2) and (4.4) are traveling waves for (CS$^+$). To do so, one can simply substitute them into the (CS$^+$) equation and check that they satisfy it. ■

Remark 4.1. As was observed in the previous proof all the traveling waves $u_0(x - ct)$ of (CS$^+$) are either $u(t,x) = C\mathrm{e}^{iN(x-Nt)}$ or the rational functions $u(t,x) := u_0(x - ct)$, where u_0 is defined in (4.6) and the constants α, β_j and c can be described by substituting u into the (CS$^+$) equation.

4.2. The L^2-norm and the speed

In this subsection we analyze the L^2-norm and the speed of the traveling waves of (CS$^+$) and establish the existence of stationary solutions for the focusing Calogero-Sutherland DNLS equation (CS$^+$).

Proposition 4.5. *We have the following properties regarding the* L^2 *norm and the speed of the traveling waves of* (CS$^+$)*:*

(i) *For any* $r > 0$, *there exists a nontrivial traveling wave* $u(t,x) := u_0(x - ct)$ *for* (CS$^+$) *with*
$$\|u_0\|_{L^2} = r.$$
(ii) *Let* u *be a traveling wave for* (CS$^+$) *of the form* (4.2); *then* u *can propagate to the right or to the left with any speed* $c \in \mathbb{R}$.

Remark 4.2. Contrary to the defocusing case, we do not necessarily have that the traveling wave propagates with a speed $c \to \infty$ when $\|u\|_{L^2}^2 \to \infty$. For instance, take

$$u(t,x) := \frac{N}{\beta} - \frac{\beta}{1-|p|^2} + \frac{\beta}{1-p\mathrm{e}^{iN(x-ct)}}, \quad \beta^2 := \frac{2N}{\frac{1+|p|^2}{1-|p|^2} - \frac{c}{N}}.$$

The proof of this statement will be achieved at the end of the following proof.

Proof of Proposition 4.5. We start by proving (i):

The L^2-norm. Let u be a traveling wave for (CS$^+$) of the form (4.2),

$$u(t,x) := \mathrm{e}^{i\theta}\Big(\alpha + \frac{\beta}{1-p\mathrm{e}^{iN(x-ct)}}\Big), \quad p \in \mathbb{D}^*.$$

Our goal is to prove that the L^2-norm of these traveling waves can be arbitrarily small or large. The computation of its L^2-norm has been performed in the proof of Proposition 3.7. Therefore, by identity (3.20),

$$\|u\|_{L^2}^2 = \alpha^2 + \alpha\beta + \alpha\beta + \frac{\beta^2}{1-|p|^2},$$

where the two reals (α,β) satisfies condition (4.3),

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = N.$$

That is,

$$\|u\|_{L^2}^2 = \frac{|p|^2}{1-|p|^2}\Big(\frac{\beta^2}{1-|p|^2} - 2N\Big) + \frac{N^2}{\beta^2}.$$

Like for the defocusing case, $\|u\|_{L^2}^2$ is a continuous function of β^2 and $|p|^2$. And by taking $\beta \to 0$ one has $\|u\|_{L^2}^2 \to \infty$. In addition, if $|p| \to 0$ then

$$\|u\|_{L^2}^2 \underset{|p|^2\to 0}{\sim} \frac{N^2}{\beta^2}.$$

Hence, it is sufficient to take β big enough so that $\|u\|_{L^2}$ can be arbitrarily small.

Speed: $c \in \mathbb{R}$. By Theorem 4.4, there exist traveling waves for (CS$^+$) that propagate with a speed $c = -N(1+\frac{2\alpha}{\beta})$, where $N \in \mathbb{N}_{\geq 1}$ and the two reals (α,β) satisfy

$$\alpha\beta + \frac{\beta^2}{1-|p|^2} = N, \quad 0 < |p| < 1.$$

That is,

$$\begin{aligned} c &= -N\Big(1 + \frac{2N}{\beta^2} - \frac{2}{1-|p|^2}\Big) \\ &= -N\Big(-\frac{1+|p|^2}{1-|p|^2} + \frac{2N}{\beta^2}\Big). \end{aligned} \tag{4.7}$$

By taking, for example, $\beta = |p|$, we infer

$$c = N\frac{|p|^4 + (2N+1)|p|^2 - 2N}{|p|^2(1-|p|^2)}.$$

Assume that $N = 1$, and by taking $x = |p|^2 \in (0,1)$, we infer that the continuous function

$$c(x) := \frac{x^2 + 3x - 2}{x(1-x)}$$

satisfies $\inf_{x\in(0,1)} c(x) = -\infty$ and $\sup_{x\in(0,1)} c(x) = +\infty$.

Proof of Remark 4.2. For a traveling wave u of the form (4.2),

$$u(t,x) := \alpha + \frac{\beta}{1 - pe^{iN(x-ct)}}, \quad \alpha\beta + \frac{\beta^2}{1-|p|^2} = N,$$

where $N \in \mathbb{N}_{\geq 1}$, one has by (4.7) that u propagates with a speed

$$c = -N\Big(-\frac{1+|p|^2}{1-|p|^2} + \frac{2N}{\beta^2}\Big).$$

Thus, for any $N \in \mathbb{N}_{\geq 1}$, let

$$\beta := \sqrt{\frac{2N}{\frac{1+|p|^2}{1-|p|^2} - \frac{\lambda}{N}}}, \quad p \in \mathbb{D},$$

where λ is a parameter in $\mathbb{R}$, and with $|p|^2$ big enough so that β is well defined. Hence, one computes

$$c = -N\left(-\frac{1+|p|^2}{1-|p|^2} + \frac{2N}{\frac{2N}{\frac{1+|p|^2}{1-|p|^2} - \frac{\lambda}{N}}}\right)\lambda \in \mathbb{R}.$$

That is, u can propagate with any speed in $\mathbb{R}$, regardless of the valued attained by the L^2-norm of u. ■

Corollary 4.6. *The potentials*

$$u(t,x) := e^{i\theta}\sqrt{\frac{N(1-|p|^2)}{2(1+|p|^2)}}\Big(1 - \frac{2}{1-pe^{iNx}}\Big), \quad p \in \mathbb{D}^*,\ N \in \mathbb{N}_{\geq 1},\ \theta \in \mathbb{T},$$

are stationary solutions for (CS$^+$). *Conversely, the defocusing* (CS$^-$) *equation does not exhibit stationary wave solutions except for the complex constant functions.*

Proof. Through a straightforward calculation, one can easily check that the obtained waves satisfy the (CS$^+$) equation. On the other hand, for the defocusing equation, we have already established via Remark 3.3, or the second point of Proposition 3.7, the nonexistence of stationary waves $u(t,x) = u_0(x)$ for (CS$^-$). ■

5. Finite gap potentials

This section aims to examine the finite gap potentials associated with the Calogero–Sutherland DNLS equation (CS) in both the focusing and defocusing cases. Remarkably, these potentials manifest as rational functions containing the traveling and solitary waves of (CS).

In the following, we adopt a slight abuse of notation, where for all $n \in \mathbb{N}_{\geq 1}$, we denote by

$$\gamma_n(u) := \nu_n - \nu_{n-1} - 1 \tag{5.1}$$

the *gap* between consecutive eigenvalues in the focusing context, and by

$$\gamma_n(u) := \lambda_n - \lambda_{n-1} - 1$$

the gap in the defocusing context. At this point, several observations can be made. First, recall that in the defocusing case, the (λ_n) satisfy inequality (2.3), and thus, for all $n \in \mathbb{N}_{\geq 1}$, $\gamma_n(u)$ is nonnegative in the defocusing case. Second, notice that since the eigenvalues (ν_n) and (λ_n) of the Lax operators L_u and $\tilde{L}_u$ are invariant by the evolution, then for all $n \in \mathbb{N}_{\geq 1}$,

$$\gamma_n(u(t)) = \gamma_n(u_0), \quad \forall t.$$

Definition 5.1 (Finite gap potential). A function $u \in L^2_+(\mathbb{T})$ is said to be a finite gap potential of (CS) if there exists $m \in \mathbb{N}_{\geq 1}$ such that

$$\gamma_n(u) = 0, \quad \forall n \geq m,$$

where γ_n is defined in (5.1).

Recall that any function in the Hardy space $L^2_+(\mathbb{T})$ can be seen as a holomorphic function on the unit disc $\mathbb{D}$ whose trace on the boundary $\partial\mathbb{D}$ is in L^2. Hence, in what follows, we denote by $\mathcal{B}_N$ the set of finite Blaschke products of degree N:

$$\psi(x) = \mathrm{e}^{i\theta} \prod_{k=1}^{N} \frac{\mathrm{e}^{ix} - \overline{p_k}}{1 - p_k \mathrm{e}^{ix}}, \quad \theta \in \mathbb{R},\ p_k \in \mathbb{D},$$

which can be identified as the set of functions

$$\psi(z) = \mathrm{e}^{i\theta} \frac{z^N \bar{Q}(\frac{1}{z})}{Q(z)}, \quad z \in \bar{\mathbb{D}} := \{|z| \leq 1\},\ \theta \in \mathbb{R},$$

where

$$Q(z) := \prod_{j=1}^{N} (1 - p_j z), \quad p_j \in \mathbb{D}.$$

In other words, $z^N \bar{Q}(\frac{1}{z})$ is a Schur polynomial[6] of degree N.

[6]A polynomial $q(z) = \sum_{k=1}^{N} a_k z^k$ is called a Schur polynomial if all its roots are in the open unit disc $\mathbb{D}$.

Remark 5.1. By convention, we suppose that a finite Blaschke product of degree 0 is a constant in $\mathbb{C}$.

Proposition 5.2. *Let u be a finite gap potential of* (CS$^+$). *There exist $(\nu, \psi) \in \mathbb{R} \times \mathcal{B}_n$, $n \in \mathbb{N}_{\geq 0}$ such that*

$$L_u S^k \psi = (\nu + k) S^k \psi, \quad \forall k \in \mathbb{N}_{\geq 0}. \tag{5.2}$$

In addition, the same goes for the defocusing Calogero–Sutherland DNLS equation (CS$^-$).

Proof. Let u be a finite gap potential, that is, $\nu_n = \nu_{n-1} + 1$ for all $n \geq m$. We denote by n_0 the eventual indices where ν_{n_0} may vanish. Then, by Proposition 2.4,

$$S f_{n-1} \,/\!/\, f_n, \quad \forall n \geq N := \max\{m, n_0\} + 2,$$

as the eigenvalues (ν_n) are simple for $n \geq m + 1$. Therefore, letting $\psi := f_{N-1}$, we have

$$L_u S^k \psi = (\nu_{N-1} + k) S^k \psi, \quad \forall k \in \mathbb{N}_{\geq 0}.$$

It remains to prove that ψ is a finite Blaschke product. Observe that, by taking the inner product of both sides of the previous identity with ψ,

$$\langle S^k \psi | \psi \rangle = 0, \quad \forall k \in \mathbb{N}_{\geq 1}.$$

That is,

$$\langle |\psi|^2 | \mathrm{e}^{ikx} \rangle = 0, \quad \forall k \in \mathbb{N}_{\geq 1}$$

or

$$\langle |\psi|^2 | \mathrm{e}^{ikx} \rangle = 0, \quad \forall k \in \mathbb{Z} \backslash \{0\},$$

as $|\psi|^2$ is a real value. Consequently, $|\psi|^2$ is a real constant, which can be supposed equal to 1 since we have assumed that the eigenfunctions of L_u constitute an orthonormal basis of $L^2_+(\mathbb{T})$. Thus, $|\psi| = 1$ on $\mathbb{T}$. In order to conclude, we need the following classical lemma [8, Exercise 6.12], which we prove for the convenience of the reader.

Lemma 5.3. *Let χ be an analytic function on the open unit ball that extends continuously to an inner function*[7] *on the closed unit disc. Then $\chi \in \mathcal{B}_n$.*

Proof. Given a holomorphic function χ on the open unit ball that extends continuously to the unit circle while satisfying $|\psi| = 1 > 0$ on $\mathbb{T}$, we know that its zeros are finite, isolated, and all localized inside the open unit disc $\mathbb{D}$. We denote them by $\overline{p_1}, \ldots, \overline{p_n}$. Hence, χ can be factorized as

$$\chi(z) = \upsilon(z) \cdot \prod_{k=1}^{n} \frac{z - \overline{p_k}}{1 - p_k z},$$

[7] A bounded analytic function ψ on $\mathbb{D}$ is said to be *inner* if $|\psi(\mathrm{e}^{ix})| = 1$ for almost every $x \in \mathbb{T}$. Note that a Blaschke product is a rational inner function.

where υ is a holomorphic function without zeros on $\mathbb{D}$. Therefore, $1/\upsilon$ is a holomorphic function on $\mathbb{D}$, which continuously extends to the unit circle while satisfying $|1/\upsilon| = 1$ on $\mathbb{T}$. Thus, by the maximum principle, we infer that $|1/\upsilon| \leq 1$ on $\mathbb{D}$. Using the same argument on υ instead of $1/\upsilon$, we deduce that $|\upsilon| \leq 1$ on the unit disc. As a consequence, $|\upsilon| = 1$ on the closed unit disc $\{|z| \leq 1\}$ and so

$$\chi(z) = \mathrm{e}^{i\theta} \cdot \prod_{k=1}^{n} \frac{z - \overline{p_k}}{1 - p_k z}, \quad \theta \in \mathbb{R}. \qquad \blacksquare$$

Coming back to the proof of Proposition 5.2, we denote by $\underline{\psi}$ the function obtained by the isometric isomorphism map

$$\underline{\psi}(z) = \sum_{k\geq 0} \hat{\psi}(k) z^k, \;\; z \in \mathbb{D} \quad \mapsto \quad \psi(x) := \sum_{k\geq 0} \hat{\psi}(k) \mathrm{e}^{ikx}, \;\; x \in \mathbb{T}.$$

In particular, since $\psi \in L^2_+(\mathbb{T})$ then $\underline{\psi} \in \mathbb{H}_2(\mathbb{D})$, where

$$\mathbb{H}_2(\mathbb{D}) := \left\{u \in \mathrm{Hol}(\mathbb{D});\ \sup_{0\leq r<1} \int_0^{2\pi} |u(r\mathrm{e}^{i\theta})|^2 \tfrac{d\theta}{2\pi} < \infty\right\}.$$

Hence, by [9, Theorem 4.5.3],

$$\underline{\psi}(r\mathrm{e}^{ix}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(x-\theta)\psi(\mathrm{e}^{i\theta})\, dt, \quad 0 \leq r < 1,$$

where P_r denotes the Poisson kernel

$$P_r(x-\theta) = \frac{1-r^2}{1-2r\cos(x-\theta)+r^2}.$$

Note that the function $\psi \in \mathrm{Dom}(L_u) := H^1_+(\mathbb{T})$ is continuous on $\mathbb{T}$ as

$$D\psi = L_u\psi + u\Pi(\bar{u}\psi) \in L^1_+(\mathbb{T}). \tag{5.3}$$

Therefore, the Poisson theorem [17, Theorème 30] implies that the holomorphic function $\underline{\psi}(r\mathrm{e}^{ix})$ extends continuously to $\psi(\mathrm{e}^{ix})$ as $r \to 1$. In addition, recall that $|\psi| = 1$ on $\mathbb{T}$. Thus, applying the previous lemma, we infer that $\underline{\psi}$ is a Blaschke product and so is ψ. $\blacksquare$

At this stage, we aim to characterize the finite gap potentials of (CS). To this end, we regroup them according to the following procedure: for any finite gap potential u of (CS), we denote by $\mathcal{N}(u)$ the nonnegative integer

$$\mathcal{N}(u) := \min\left\{n \in \mathbb{N}_{\geq 0} \;\middle|\; \exists\psi \in \mathcal{B}_n,\ L_u S^k\psi = (\nu + k)S^k\psi,\ \forall k \geq 0\right\}, \tag{5.4}$$

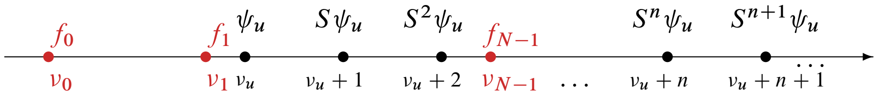
and we define, for $N \in \mathbb{N}_{\geq 0}$, the set

$$\boxed{\mathcal{U}_N := \left\{u \text{ finite gap potential},\ \mathcal{N}(u) = N\right\}.}$$

This means that, for any $u \in \mathcal{U}_N$, there exists a finite Blaschke product ψ_u of minimal degree N, satisfying

$$L_u S^k \psi_u = (\nu_u + k) S^k \psi_u, \quad \forall k \in \mathbb{N}_{\geq 0}, \tag{5.5}$$

where ν_u is the corresponding eigenvalue of ψ_u. That is, $\{S^k \psi_u \mid k \in \mathbb{N}_{\geq 0}\}$ are parts of the orthonormal basis of $L^2_+(\mathbb{T})$. Further, observe that, since $\deg \psi_u = N$, then there exist N eigenfunctions $f_0, \dots, f_{N-1}$ of L_u that generate the model space $(\psi_u L^2_+)^\perp$ which is of dimension N [14, Corollary 5.18]. We denote by $\nu_0, \dots, \nu_{N-1}$ the associated eigenvalues. Note that the latter N eigenvalues are not necessarily smaller than ν_u. We summarize this discussion by the following diagram: for any $u \in \mathcal{U}_N$,



Of course, the same goes for the defocusing equation with $\tilde{L}_u$ instead of L_u, up to the fact that the remaining N eigenvalues $\nu_0, \dots, \nu_{N-1}$ are necessarily smaller than ν_u, since the eigenvalues of $\tilde{L}_u$ satisfy property (2.3). Further, note that by taking the minimum in (5.4) we guarantee that

(1) if $u \in \{v \text{ finite gap potential, } L_v S^k \psi = (v + k) S^k \psi,\ \psi \in \mathcal{B}_N\}$, then $u \notin \{v \text{ finite gap potential, } L_v S^k \psi = (v + k) S^k \psi,\ \psi \in \mathcal{B}_{N-1}\}$,

(2) the set $\mathcal{U}_N$ is invariant under the evolution of (CS) (see Proposition 5.6).

The following theorem aims to characterize the finite gap potentials of the Calogero–Sutherland DNLS (CS) in the state space.

Theorem 5.4. *Let $N \in \mathbb{N}_{\geq 1}$. A potential u is in $\mathcal{U}_N$ if and only if $u(x) = C e^{iNx}$, $C \in \mathbb{C}^*$, or u is a rational function,*

$$\begin{aligned} u(x) = e^{im_0 x} &\prod_{j=1}^{r} \left(\frac{e^{ix} - \overline{p_j}}{1 - p_j e^{ix}} \right)^{m_j - 1} \\ &\times \left(a + \sum_{j=1}^{r} \frac{c_j}{1 - p_j e^{ix}} \right), \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j, \end{aligned} \tag{5.6}$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^{r} m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$ satisfy for all $j = 1, \dots, N - m$,

(i) *in the focusing case,*

$$\bar{a} c_j + \sum_{k=1}^{r} \frac{c_j \overline{c_k}}{1 - p_j \overline{p_k}} = m_j,$$

(ii) *in the defocusing case,*

$$\bar{a} c_j + \sum_{k=1}^{r} \frac{c_j \overline{c_k}}{1 - p_j \overline{p_k}} = -m_j,$$

with $a \neq 0$ if $m_0 \in \{1, \dots, N-1\}$. Further, if $N = 0$, then u is a complex constant function.

Remark 5.2. As we shall see in Step 4 of the proof of Theorem 5.4, if $u \in \mathcal{U}_N$, then the eigenvalue of L_u associated

- with the Blaschke product $\psi_u = \mathrm{e}^{i\theta}\mathrm{e}^{iNx}$, if $u = C\mathrm{e}^{iNx}$, is given by
  (i) $\nu_u = N - C^2$ in the focusing case,
  (ii) $\lambda_u = N + C^2$ in the defocusing case;
- with the Blaschke product

$$\psi_u = \mathrm{e}^{i\theta}\mathrm{e}^{im_0\cdot x}\prod_{j=1}^{r}\Big(\frac{\mathrm{e}^{ix}-\overline{p_j}}{1-p_j\mathrm{e}^{ix}}\Big)^{m_j}, \quad \theta \in \mathbb{R},\ p_j \neq p_k,\ j \neq k,$$

  if u is the rational function (5.6), is given by
  (i) $\nu_u = m_0 - |a|^2 - \sum_{j=1}^{r} a\overline{c_j}$ in the focusing case,
  (ii) $\lambda_u = m_0 + |a|^2 + \sum_{j=1}^{r} a\overline{c_j}$ in the defocusing case.

In order to establish this theorem, we recall a specific case of formula (3.10).

Remark 5.3. Let (f_n) be an orthonormal basis of $L^2_+(\mathbb{T})$. For any $n \geq 0$,

$$f_n(z) = \langle (\mathrm{Id} - zM)^{-1}\mathbb{1}_n | Y\rangle_{\ell^2}, \quad z \in \mathbb{D},$$

where $\mathbb{1}_n$ and Y are the column vectors

$$\mathbb{1}_n := (\delta_{pn})_{p\geq 0}, \quad Y := (\langle 1 | f_m\rangle)_{m\geq 0},$$

and M is the matrix representation of the operator S^* in the (f_m)-basis,

$$M = (M_{mp})_{mp\geq 0}, \quad M_{mp} = \langle f_p | S f_m\rangle.$$

In what follows, we denote by $\mathbb{C}_{\leq N}[X]$ the set of polynomials P in complex coefficients with degree at most N and by $\mathbb{C}_N[X]$ those of degree N.

Proof of Theorem 5.4. We present the proof for the focusing case. Note that the same arguments can be performed to deduce the result in the defocusing case. The key ingredient is the inversion spectral formula (3.10),

$$u(z) = \langle (\mathrm{Id} - zM)^{-1}X | Y\rangle, \tag{5.7}$$

where X, Y, and M are defined in (3.11). The proof will be split into five steps.

Let $u \in \mathcal{U}_N$; then by Proposition 5.2 there exists a finite Blaschke product

$$\psi_u(z) = \mathrm{e}^{i\theta}\frac{\mathrm{e}^{iNx}\overline{Q}(1/z)}{Q(z)}; \quad Q(z) := \prod_{j=1}^{N}(1-p_jz), \quad p_j \in \mathbb{D}, \tag{5.8}$$

such that (5.2) is satisfied:

$$L_u S^k\psi_u = (\nu_u + k)S^k\psi_u, \quad \forall k \in \mathbb{N}_{\geq 0}.$$

Step 1. As a first step, we prove that any $u \in \mathcal{U}_N$ must be a rational function

$$u(z) = \frac{P(z)}{Q(z)}, \quad P \in \mathbb{C}_{\leq N}[z],$$

where $Q(z)$ is the same denominator as the Blaschke product $\psi_u(z)$ associated with $u \in \mathcal{U}_N$. Indeed, first observe that combining (5.2) with the commutator identity (2.1) leads to

$$\langle u | S^k \psi_u \rangle = 0, \quad \forall k \geq 1.$$

Hence, we infer, thanks to Lemma 2.3, that the infinite matrices M, X, and Y of (5.7) written in the basis $(f_k)_{k=0}^{N-1} \cup (S^k \psi_u)_{k \geq 0}$ are of the form

$$M = \left(\begin{array}{cccc|ccc} \langle f_0|Sf_0\rangle & \dots & \langle f_{n-1}|Sf_0\rangle & \langle \psi_u|Sf_0\rangle & & & \\ \vdots & & \vdots & \vdots & & & \\ \langle f_0|Sf_{N-1}\rangle & \dots & \langle f_{N-1}|Sf_{N-1}\rangle & \langle \psi_u|Sf_{N-1}\rangle & & & \\ 0 & \dots & \dots & 0 & 1 & & \\ \hline 0 & \dots & \dots & 0 & 0 & 1 & \\ \vdots & & & & \vdots & \ddots & \ddots \end{array}\right), \quad n \in \mathbb{N},$$

$$X = \begin{pmatrix} \langle u|f_0\rangle \\ \vdots \\ \langle u|f_{N-1}\rangle \\ \langle u|\psi_u\rangle \\ 0 \\ \vdots \end{pmatrix}, \quad Y = \begin{pmatrix} \langle 1|f_0\rangle \\ \vdots \\ \langle 1|f_{N-1}\rangle \\ \langle 1|\psi_u\rangle \\ 0 \\ \vdots \end{pmatrix}.$$

Therefore, following the same procedure as presented in the proof of Theorem 1.1, one can observe that the infinite matrices M, X, and Y can be reduced to finite matrices that involve only the first $N+1$ coordinates of each of these matrices. That is,

$$u(z) = \langle (\mathrm{Id} - zM_{\leq N})^{-1} X_{\leq N} | Y_{\leq N} \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}},$$

where $M_{\leq N} = (M_{mn})_{0 \leq m,n \leq N}$, $X_{\leq N} = (X_n)_{0 \leq n \leq N}$, and $Y_{\leq N} = (Y_n)_{0 \leq n \leq N}$. As a consequence, u is a rational function,

$$u(z) = \frac{P(z)}{\det(\mathrm{Id} - zM_{\leq N})}, \quad P \in \mathbb{C}_{\leq N}[z].$$

Note that $\det(\mathrm{Id} - zM_{\leq N})$ coincides with the denominator of the eigenfunction

$$\psi_u = \mathrm{e}^{i\theta} \frac{z^N \bar{Q}(1/z)}{Q(z)},$$

since by Remark 5.3, ψ_u is also expressed via the inversion spectral formula

$$\psi_u(z) = \langle (\mathrm{Id} - zM_{\leq N})^{-1} \mathbb{1}_N | Y_{\leq N} \rangle_{\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}} = \frac{\langle \mathrm{Com}(\mathrm{Id} - zM_{\leq N})^{\intercal} \mathbb{1}_N | Y_{\leq N} \rangle}{\det(\mathrm{Id} - zM_{\leq N})},$$

and hence $\det(\mathrm{Id} - zM_{\leq N}) = Q(z)$. Thus,

$$u(z) = \frac{P(z)}{Q(z)}, \quad P \in \mathbb{C}_{\leq N}[z], \quad Q(z) = \prod_{k=1}^{N}(1 - p_k z),\ p_k \in \mathbb{D}.$$

Step 2. In this step, we prove that if $u \in \mathcal{U}_N$ then

$$|u|^2 = z\partial_z \log \psi_u - \nu_u, \quad \text{on } \partial\mathbb{D}. \tag{5.9}$$

Indeed, recall that $L_u\psi_u = \nu_u\psi_u$. Then by definition of $L_u = z\partial_z - T_u T_{\bar{u}}$,

$$z\partial_z\psi_u - u\Pi(\bar{u}\psi_u) = \nu_u\psi_u. \tag{5.10}$$

On $\partial\mathbb{D}$,

$$\bar{u}\psi_u(z) = \mathrm{e}^{i\theta}\frac{z^N\bar{P}(1/z)}{z^N\bar{Q}(1/z)} \cdot \frac{z^N\bar{Q}(1/z)}{Q(z)} = \mathrm{e}^{i\theta}\frac{z^N\bar{P}(1/z)}{Q(z)}$$

extends as a holomorphic function on $\mathbb{D}$. Hence, $\Pi(\bar{u}\psi_u) = \bar{u}\psi_u$, and so identity (5.10) can be read as

$$\frac{z\partial_z\psi_u}{\psi_u} = |u|^2 + \nu_u,$$

implying that identity (5.9) holds.

Step 3. In this step, we prove that the rational function u obtained in Step 1 can be rewritten either as $u(z) = Cz^N$, $C \in \mathbb{C}^*$, or

$$u(z) = z^{m_0}\prod_{j=1}^{r}\Big(\frac{z - \overline{p_j}}{1 - p_j z}\Big)^{m_j - 1}\frac{q(z)}{\prod_{j=1}^{r}(1 - p_j z)}, \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j,$$

where $m_0 \in \{0, \ldots, N-1\}$, $m_1, \ldots, m_r \in \{1, \ldots, N\}$,

$$m_0 + \sum_{j=1}^{r} m_j = N,$$

and such that $\deg(q) = r$ if $m_0 \neq 0$. Indeed, we write (5.8) as $\psi_u = \mathrm{e}^{i\theta}z^N$ (if all the p_k in (5.8) vanish), or

$$\psi_u = \mathrm{e}^{i\theta}z^{m_0}\prod_{j=1}^{r}\Big(\frac{z - \overline{p_j}}{1 - p_j z}\Big)^{m_j}, \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j,\ \theta \in \mathbb{R}, \tag{5.11}$$

where $m_0 \in \{0, \ldots, N-1\}$, $m_1, \ldots, m_r \in \{1, \ldots, N\}$, such that $m_0 + \sum_{j=1}^{r} m_j = N$. As a first point, we prove that when $m_0 \geq 1$, then the numerator P of u can be factorized as $P(z) = z^{m_0}P_{N-m_0}(z)$ with $P_{N-m_0} \in \mathbb{C}_{N-m_0}[z]$. Let $m_0 \geq 1$; then $\langle u|\psi_u\rangle \neq 0$, because otherwise there exists a Blaschke product $\chi_u = S^*\psi_u$ of degree $N-1$, such that by the commutator identity (2.1),

$$L_u S^k\chi_u = (\nu_k - 1 + k)S^k\chi_u, \quad k \geq 0,$$

meaning that $u \in \mathcal{U}_{N-1}$, which is a contradiction with the fact that $u \in \mathcal{U}_N$. Hence, $\langle u|\psi_u\rangle \neq 0$. This leads to

(i) the numerator P of $u(z)$ must be degree N,

(ii) $\langle 1|u\rangle = 0$.

Indeed, for (i), it is sufficient to note that

$$0 \neq \langle \psi_u|u\rangle = \int_{z\in\mathcal{C}(0,1)} \mathrm{e}^{i\theta} \frac{z^N \bar{Q}(1/z)}{Q(z)} \frac{\bar{P}(1/z)}{\bar{Q}(1/z)} \frac{dz}{2\pi i z} = \mathrm{e}^{i\theta} z^N \bar{P}(1/z)_{|z=0}.$$

For (ii), observe by Lemma 2.3,

$$\langle 1|u\rangle \underbrace{\langle u|\psi_u\rangle}_{\neq 0} = -\nu_u \langle 1|\psi_u\rangle,$$

where the right-hand side vanishes since $\psi_u = S\chi$, $\chi \in L^2_+(\mathbb{T})$ for $m_0 \geq 1$. Therefore, if $m_0 = 1$, then by (i) and (ii),

$$u(z) = \frac{zP_{N-1}(z)}{\prod_{j=1}^r (1-p_j z)^{m_j}}, \quad P_{N-1} \in \mathbb{C}_{N-1}[z], \quad p_k \in \mathbb{D}^*, \ p_j \neq p_k.$$

Now, if $m_0 = 2$, we have by (ii), $\langle u|1\rangle = 0$, that is, $u = Sv$ with $v \in L^2_+(\mathbb{T})$. Thus, by the definition of $L_u = z\partial_z - u\Pi(\bar{u}\cdot)$,

$$L_u z = z - \langle 1|v\rangle u.$$

Taking the inner product of the latter identity with ψ_u,

$$(\nu_u - 1)\langle z|\psi_u\rangle = -\langle 1|v\rangle \underbrace{\langle u|\psi_u\rangle}_{\neq 0}.$$

Note that for $m_0 = 2$ we have $\langle z|\psi_u\rangle = 0$. This implies that $\langle 1|v\rangle = 0$, leading to $u = S^2 w$ with $w \in L^2_+(\mathbb{T})$. Therefore, if $m_0 = 2$, then u can be decomposed as

$$u(z) = \frac{z^2 P_{N-2}(z)}{\prod_{j=1}^r (1-p_j z)^{m_j}}, \quad P_{N-2} \in \mathbb{C}_{N-2}[z], \quad p_k \in \mathbb{D}^*, \ p_j \neq p_k.$$

Now, if $m_0 = 3$, then by repeating the same procedure as above and taking the inner product of

$$L_u z^2 = 2z^2 - \langle 1|w\rangle u$$

with ψ_u, one obtains

$$(\nu_u - 2)\underbrace{\langle z^2|\psi_u\rangle}_{=0} = -\langle 1|w\rangle \underbrace{\langle u|\psi_u\rangle}_{\neq 0}.$$

That is, $\langle 1|w\rangle = 0$, i.e. $u = S^3 \underline{w}$ with $\underline{w} \in L^2_+(\mathbb{T})$. Therefore, for all $m \in \{0,\ldots,N-1\}$,

$$u(z) = \frac{z^{m_0} P_{N-m_0}(z)}{\prod_{j=1}^r (1-p_j z)^{m_j}}, \quad p_j \in \mathbb{D}^*, \ p_j \neq p_k,$$

where $P_{N-m_0} \in \mathbb{C}_{N-m_0}[z]$ if $m_0 \geq 1$, thanks to (i). In addition, if $m_0 = N$, i.e. all the p_j in (5.11) vanish, then $u(z) = Cz^N$, $C \in \mathbb{C}^*$. Finally, it remains to prove that $(z - \overline{p_j})^{m_j-1}$ divides the numerator of u. Indeed, by identity (5.9) of Step 2, $u(z)\bar{u}(\frac{1}{z}) = z\partial_z \log \psi_u - \nu_u$, where one computes by (5.11),

$$z\partial_z \log \psi_u = m_0 + \sum_{j=1}^{r} m_j \Big(\frac{1}{1 - p_j z} + \frac{\overline{p_j}}{z - \overline{p_j}} \Big).$$

That is, for all $m_0 \in \{0, \dots, N-1\}$,

$$\frac{P_{N-m_0}(z)}{\prod_{j=1}^r (1 - p_j z)^{m_j}} \frac{z^{N-m_0} \overline{P_{N-m_0}}(\frac{1}{z})}{\prod_{j=1}^r (z - \overline{p_j})^{m_j}} = m_0 - \nu_u + \sum_{j=1}^{r} m_j \Big(\frac{1}{1 - p_j z} + \frac{\overline{p_j}}{z - \overline{p_j}} \Big),$$

where $p_j \in \mathbb{D}^*$, $p_k \neq p_j$ for $k \neq j$. Observe that on the right-hand side, $\frac{1}{p_j}$ is a pole of multiplicity 1. Then the same should hold for the left-hand side as well. Therefore, if $m_j \geq 2$, $j = 1, \dots, r$, this implies that $\frac{1}{p_j}$ on the left-hand side must be a root of multiplicity $(m_j - 1)$ of $\overline{P_{N-m_0}}(\frac{1}{z})$. That is,

$$P_{N-m_0}(\overline{p_j}) = 0, \dots, P_{N-m_0}^{(m_j-2)}(\overline{p_j}) = 0,$$

where $P_{N-m_0}^{(m)}$ is the mth derivative of P_{N-m_0}. As a result, we see that $(z - \overline{p_j})^{m_j-1}$ divides $P_{N-m_0}(z)$, and so

$$u(z) = z^{m_0} \prod_{j=1}^{r} \Big(\frac{z - \overline{p_j}}{1 - p_j z} \Big)^{m_j-1} \frac{q(z)}{\prod_{j=1}^r (1 - p_j z)}, \quad p_j \in \mathbb{D}^*, \ p_k \neq p_j, \ k \neq j,$$

with $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, $m_0 + \sum_{j=1}^r m_j = N$, and such that $\deg(q) = r$ if $m_0 \neq 0$ thanks to (i).

Step 4. In this step, we write the rational function u obtained in Step 3 in its partial fractional decomposition

$$u(z) = z^{m_0} \prod_{j=1}^{r} \Big(\frac{z - \overline{p_j}}{1 - p_j z} \Big)^{m_j-1} \Big(a + \sum_{j=1}^{r} \frac{c_j}{1 - p_j z} \Big), \quad p_j \in \mathbb{D}^*, \ p_k \neq p_j, \ k \neq j,$$

where $a \neq 0$ if $m_0 \neq 0$, and we infer by (5.9) of Step 2 that, for all $j = 1, \dots, r$,

$$\bar{a} c_j + \sum_{k=1}^{r} \frac{\overline{c_k} c_j}{1 - \overline{p_k} p_j} = m_j. \tag{5.12}$$

Indeed, by applying Π to (5.9),

$$\Pi(|u|^2) = \Pi(z\partial_z \log \psi_u - \nu_u).$$

Observe, on the one hand, that

$$\Pi(z\partial_z \log \psi_u - \nu_u) = \sum_{j=1}^{r} \frac{m_j}{1-p_j z} + m_0 - \nu_u,$$

and on the other hand,

$$\begin{aligned}\Pi(|u|^2) &= \Pi\left(a + \sum_{j=1}^{r} \frac{c_j}{1-p_j z}\right)\\ &= |a|^2 + \sum_{j=1}^{r} \bar{a} c_j + \bar{a} \sum_{j=1}^{r} \frac{c_j}{1-p_j z} + \sum_{j=1}^{r}\sum_{k=1}^{r} \frac{c_j \overline{c_k}}{(1-p_j\overline{p_k})(1-p_j z)}.\end{aligned}$$

Therefore, for all $j = 1, \dots, r$, the requested conditions (5.12) and

$$\nu_u = m_0 - |a|^2 - \sum_{j=1}^{r} \bar{a} c_j.$$

Step 5. We prove the converse. For $N \in \mathbb{N}_{\geq 1}$, let $u = Cz^N$, $C \in \mathbb{C}^*$, or

$$u(z) = z^{m_0} \prod_{j=1}^{r} \left(\frac{z-\overline{p_j}}{1-p_j z}\right)^{m_j - 1} \left(a + \sum_{j=1}^{r} \frac{c_j}{1-p_j z}\right), \quad p_j \in \mathbb{D}^*,\ p_k \neq p_j,\ k \neq j,$$

where $m_0 \in \{0, \dots, N-1\}$, $m_1, \dots, m_r \in \{1, \dots, N\}$, such that $m_0 + \sum_{j=1}^{r} m_j = N$, and $(a, c_1, \dots, c_r) \in \mathbb{C} \times \mathbb{C}^r$, satisfy

$$\bar{a} c_j + \sum_{k=1}^{r} \frac{\overline{c_k} c_j}{1 - \overline{p_k} p_j} = m_j, \tag{5.13}$$

with $a \neq 0$ if $m \neq 0$. Our aim is to prove that $u \in \mathcal{U}_N$, that is,

- there exists $\psi \in \mathcal{B}_N$ such that $L_u S^k \psi = (\mu + k) S^k \psi$ for all $k \in \mathbb{N}_{\geq 0}$, where μ is a real constant,
- ψ is of minimal degree, i.e. there does not exist $\chi \in \mathcal{B}_\ell$ with $\ell < N$, such that χ satisfies $L_u S^k \chi = (\mu_1 + k) S^k \chi$ for all $k \in \mathbb{N}_{\geq 0}$.

For the moment, let us deal with the more complicated case, i.e. u is a rational function. We start by proving the first point. Let

$$\psi := \mathrm{e}^{i\theta} z^{m_0} \prod_{j=1}^{r} \left(\frac{z-\overline{p_j}}{1-p_j z}\right)^{m_j} \in \mathcal{B}_N, \quad \theta \in \mathbb{R},\ p_k \in \mathbb{D}^*.$$

Observe that $\bar{u}\psi$ extends as a holomorphic function on $\mathbb{D}$ as $p_k \in \mathbb{D}$. Then, by definition of L_u,

$$L_u \psi = z\partial_z \psi - |u|^2 \psi,$$

where

$$z\partial_z\psi = m_0\psi + \sum_{k=1}^{r}\Big(\frac{\overline{p_k}}{z-\overline{p_k}} + \frac{1}{1-p_kz}\Big)\psi,$$

and thanks to (5.13),

$$\begin{aligned}
|u|^2 &= |a|^2 + \bar{a}\sum_{k=1}^{r}\frac{c_k}{1-p_kz} + a\sum_{k=1}^{r}\frac{\overline{c_k}z}{z-\overline{p_k}} + \sum_{k=1}^{r}\sum_{j=1}^{r}\frac{c_k\overline{c_j}z}{(1-p_kz)(z-\overline{p_j})} \\
&= |a|^2 + a\sum_{k=1}^{r}\overline{c_k} + \sum_{k=1}^{r}\Big(\bar{a}c_k + \sum_{j=1}^{r}\frac{c_k\overline{c_j}}{1-p_k\overline{p_j}}\Big)\frac{1}{1-p_kz} \\
&\quad + \sum_{j=1}^{r}\Big(\bar{a}c_j + \sum_{k=1}^{r}\frac{c_k\overline{c_j}}{1-p_k\overline{p_j}}\Big)\frac{\overline{p_j}}{z-\overline{p_j}} \\
&= |a|^2 + a\sum_{k=1}^{r}\overline{c_k} + \sum_{k=1}^{r}\frac{m_k}{1-p_kz} + \sum_{j=1}^{r}\frac{m_j\overline{p_j}}{z-\overline{p_j}}.
\end{aligned} \tag{5.14}$$

Therefore,

$$L_u\psi = \Big(m_0 - |a|^2 - a\sum_{k=1}^{r}\overline{c_k}\Big)\psi.$$

Additionally, observe that for all $k \in \mathbb{N}_{\geq 1}$, $\langle S^k\psi|u\rangle = 0$. Hence, by applying the commutator identity (2.1), we deduce

$$L_uS^k\psi = (\mu + k)S^k\psi, \quad \forall k \in \mathbb{N}_{\geq 0}, \tag{5.15}$$

where $\mu := m_0 - |a|^2 - a\sum_{k=1}^{r}\overline{c_k}$.

It remains to prove that ψ is of minimal degree. Suppose for the sake of contradiction that there exists $\chi \in \mathcal{B}_\ell$, with $\ell < N$, such that $L_uS^j\chi = (\mu_1 + k)S^j\chi$ for all $j \in \mathbb{N}_{\geq 0}$. By comparing the latter identity to (5.15), and thanks to (2.5), we infer that there exist $k', k \in \mathbb{N}_{\geq 1}$ such that $S^k\psi = S^{k'}\chi$, i.e.

$$\chi := \mathrm{e}^{i\tilde{\theta}}z^{m_0+k-k'}\prod_{j=1}^{r}\Big(\frac{z-\overline{p_j}}{1-p_jz}\Big)^{m_j}, \quad m_0 + k - k' + \sum_{j=1}^{r}m_j = \ell < N.$$

Therefore, by repeating Steps 1 to 4, we infer that u must be of the form

$$u(z) = z^{m_0+k-k'}\prod_{j=1}^{r}\Big(\frac{z-\overline{p_j}}{1-p_jz}\Big)^{m_j-1}\Big(a + \sum_{j=1}^{r}\frac{c_j}{1-p_jz}\Big),$$

which is a contradiction. ∎

Corollary 5.5. *Given $N \in \mathbb{N}_{\geq 1}$, let $u \in \mathcal{U}_N$. Then,*

(i) *in the focusing case, $\|u\|_{L^2}^2 = N - \nu_u$,*

(ii) *in the defocusing case, $\|u\|_{L^2}^2 = \lambda_u - N$,*

where ν_u is the eigenvalue introduced in (5.5) and λ_u is the corresponding one in the defocusing case.

Remark 5.4. Based on the previous statement, one can conclude that for any potential $u \in \mathcal{U}_N$, we have

$$\begin{cases} \nu_u < N & \text{(focusing case),} \\ \lambda_u > N & \text{(defocusing case).} \end{cases}$$

Proof of Corollary 5.5. Let $u \in \mathcal{U}_N$. Then in light of the previous theorem, we have either $u = C\mathrm{e}^{iNx}$, $C \in \mathbb{C}^*$, or u is the rational function (5.6). Thus, if $u = C\mathrm{e}^{iNx}$ then the results follow easily by Remark 5.2. Now, if u is the rational function (5.6), then by computing the L^2-norm of u in the focusing case, we infer via (5.14), that

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{z \in \mathcal{C}(0,1)} \left(|a|^2 + a \sum_{k=1}^{r} \overline{c_k} + \sum_{k=1}^{r} \frac{1}{1 - p_k z} + \sum_{k=1}^{r} \frac{\overline{p_k}}{z - \overline{p_k}} \right) \frac{dz}{2\pi i z} \\ &= |a|^2 + a \sum_{k=1}^{r} \overline{c_k} + N - m, \end{aligned}$$

which is equal to $-\nu_u + N$ by the second (i) of Remark 5.2. For the defocusing case, we shall have

$$\begin{aligned} \|u\|_{L^2}^2 &= \int_{z \in \mathcal{C}(0,1)} \left(|a|^2 + a \sum_{k=1}^{r} \overline{c_k} - \sum_{k=1}^{r} \frac{1}{1 - p_k z} - \sum_{k=1}^{r} \frac{\overline{p_k}}{z - \overline{p_k}} \right) \frac{dz}{2\pi i z} \\ &= |a|^2 + a \sum_{k=1}^{r} \overline{c_k} - N + m, \end{aligned}$$

which is equal to $\lambda_u - N$ by the second (ii) of Remark 5.2. ∎

Proposition 5.6. *For any $N \in \mathbb{N}_{\geq 0}$, the set of finite gap potentials $\mathcal{U}_N$ is conserved along the flow of the (CS) equation.*

Proof. Let u_0 be a finite gap potential in $\mathcal{U}_N$, that is, there exists $\psi_{u_0} \in \mathcal{B}_N$ of minimal degree N satisfying

$$L_{u_0} S^k \psi_{u_0} = (\nu_{u_0} + k) S^k \psi_{u_0}, \quad \forall k \geq 0. \tag{5.16}$$

Our aim is to prove that there exists $\varrho(t) \in \mathcal{B}_N$ of minimal degree[8] such that

$$L_{u(t)} S^k \varrho(t) = (\nu_{u_0} + k) S^k \varrho(t), \quad \forall k \geq 0.$$

[8]In the sense, that there does not exist $\chi(t) \in \mathcal{B}_\ell$ with $\ell < N$, such that $\chi(t)$ satisfies $L_u S^k \chi(t) = (\mu_1 + k) S^k \chi(t)$ for all $k \in \mathbb{N}_{\geq 0}$.

Let $\varrho(t)$ be a solution of the Cauchy problem

$$\begin{cases} \partial_t \varrho(t) = B_{u(t)}\varrho(t), \\ \varrho(0) = \psi_{u_0}. \end{cases}$$

Hence, by Remark 3.1,

$$L_{u(t)}\varrho(t) = \nu_{u_0}\varrho(t). \tag{5.17}$$

In addition, recall that by Lemma 4.1,

$$\langle S\varrho(t)|u(t)\rangle = \langle S\psi_0|u_0\rangle \mathrm{e}^{-i\nu_{u_0}^2 t},$$

where here $\langle S\psi_0|u_0\rangle$ vanishes after combining the commutator identity (2.1) and equation (5.16). Therefore, by (2.1),

$$L_{u(t)}S\varrho(t) = (\nu_{u_0}+1)S\varrho(t). \tag{5.18}$$

This yields

$$\begin{cases} \partial_t S\varrho(t) = B_{u(t)}S\varrho(t), \\ S\varrho(0) = S\psi_{u_0}. \end{cases}$$

Indeed, by the commutator identity (2.2),

$$\begin{aligned} \partial_t S\varrho(t) &= SB_{u(t)}\varrho(t) \\ &= B_{u(t)}S\varrho(t) - i(\tilde{L}_u^2 S - S(\tilde{L}_u + \mathrm{Id})^2)\varrho(t), \end{aligned}$$

which is equal to $\partial_t S\varrho(t) = B_{u(t)}S\varrho(t)$ thanks to (5.17) and (5.18). Consequently, by repeating the same procedure, we obtain for all $k \in \mathbb{N}_{\geq 0}$,

$$L_{u(t)}S^k\varrho(t) = (\nu_{u_0}+k)S^k\varrho(t),$$

with

$$\begin{cases} \partial_t S^k\varrho(t) = B_{u(t)}S^k\varrho(t), \\ S^k\varrho(0) = S^k\psi_{u_0}. \end{cases}$$

Further, observe that $\varrho(t) \in \mathcal{B}_N$. Indeed, by applying Lemma 4.1,

$$\langle S^k\varrho(t)|\varrho(t)\rangle = \langle S^k\psi_{u_0}|\psi_{u_0}\rangle \mathrm{e}^{i((\nu_{u_0}+k)^2-\nu_{u_0}^2)t} = 0, \quad \forall k \in \mathbb{N}_{\geq 1},$$

leading to

$$\langle \mathrm{e}^{ikx}||\varrho(t)|^2\rangle = 0, \quad k \in \mathbb{Z}\backslash\{0\}.$$

Thus, following the same lines as the proof of Proposition 5.2, we deduce that $\varrho(t)$ is a finite Blaschke product. To infer that the degree of this finite Blaschke product is N, we should notice that each of the ψ_{u_0} and $\varrho(t)$ enjoys an inverse spectral formula (Remark 5.3),

$$\begin{aligned} \psi_{u_0} &= \langle (\mathrm{Id} - zM_{\leq N}(u_0))^{-1}\mathbb{1}_N | Y_{\leq N}(u_0)\rangle, \\ \varrho(t) &= \langle (\mathrm{Id} - zM_{\leq N}(u(t)))^{-1}\mathbb{1}_N | Y_{\leq N}(u(t))\rangle, \end{aligned}$$

where $M_{\leq N}(u_0)$ and $M_{\leq N}(u(t))$ are the finite matrices of order $(N+1)\times(N+1)$ obtained from the representation matrix of S^* in the L^2 basis $(h_k)_{k=0}^{N-1} \cup (S^k\psi_{u_0})_{k\geq 0}$ constituted respectively from the eigenfunctions of L_{u_0} at $t=0$, and from the eigenfunctions $(e_k(t))_{k=0}^{N-1} \cup (S^k\varrho(t))_{k\geq 0}$ of $L_{u(t)}$ at any time t. Therefore, in view of the fourth identity of Lemma 4.1, we infer

$$M_{\leq N}(u(t)) = \mathrm{Diag}(\mathrm{e}^{-i(\nu_n+1)^2t})M_{\leq N}(u_0)\,\mathrm{Diag}(\mathrm{e}^{-i\nu_n^2 t}).$$

That is,

$$|\det(M_{\leq N}(u_0))| = \big|\det\big(M_{\leq N}(u(t))\big)\big|,$$

and so,

$$\deg\big(\det\big(\mathrm{Id}-zM_{\leq N}(u(t))\big)\big) = \deg\big(\det(\mathrm{Id}-zM_{\leq N}(u_0))\big) = N.$$

As a result, $u(t)\in\mathcal{U}_n$ with $n\leq N$. It remains to show that $u(t)\notin\mathcal{U}_n$ with $n<N$. Suppose that there exists $\chi(t)\in\mathcal{B}_n$ with $n<N$ such that

$$L_{u(t)}S^k\chi(t) = (\nu_u+k)S^k\chi(t).$$

Then applying the same procedure as above, we infer that $\chi(0)\in\mathcal{B}_n$ with $n<N$ and

$$L_{u_0}S^k\chi(0) = (\lambda_u+k)S^k\chi(0),$$

leading to $u_0\in\mathcal{U}_n$, $n<N$, which is a contradiction.

Note that the same proof works in the defocusing case. ■

6. Remark on the regularity of *u*

Recall that at the beginning of Section 2, we supposed for more convenience that u is a function with enough regularity, typically in $H^2_+(\mathbb{T})$. However, the same strategy adopted to derive the traveling waves of the Calogero–Sutherland DNLS equation (CS) and to characterize the finite gap potentials can be extended to less regular spaces. *In this section, we discuss some remarks that allow the extension of the main results to the critical regularity $L^2_+(\mathbb{T})$.*

First, we recall from [3] the following theorem.

Theorem ([3]). *For any $0\leq s\leq 2$, let $u_0\in H^s_+(\mathbb{T})$. Then there exists a unique potential $u\in\mathcal{C}(\mathbb{R},H^s_+(\mathbb{T}))$ a solution of (CS$^-$) such that, for any sequence $(u_0^\varepsilon)\subseteq H^2_+(\mathbb{T})$,*

$$\|u_0^\varepsilon-u_0\|_{H^s}\xrightarrow[\varepsilon\to 0]{} 0,$$

we have for all $T>0$,

$$\sup_{t\in[-T,T]}\|u^\varepsilon(t)-u(t)\|_{H^s}\to 0,\quad \varepsilon\to 0.$$

Moreover, the L^2-norm of the limit potential u is conserved in time:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad \forall t \in \mathbb{R}.$$

Furthermore, the same holds for (CS$^+$) *under the additional condition* $\|u_0\|_{L^2} < 1$.

At the second stage, recall that Lemma 3.2, Proposition 3.5, and Corollary 3.6 were the keys to characterizing the traveling waves for the defocusing equation (CS$^-$), and Lemma 4.1, Proposition 4.2, and Corollary 4.3 for the focusing equation (CS$^+$). As a result, we need to extend these propositions/lemmas/corollaries to less regular potentials u. Hence, we recall from [3, Corollary 3.12] the following result.

Corollary ([3, Corollary 3.12]). *For any $0 \leq s \leq 2$, let $u_0 \in H^s_+(\mathbb{T})$. There exists an orthonormal basis (g^t_n) of $L^2_+(\mathbb{T})$ constituted from the eigenfunctions of $L_{u(t)}$, such that for all $n \in \mathbb{N}_{\geq 0}$,*

$$\langle u(t)|g^t_n\rangle = \langle u_0|f^{u_0}_n\rangle e^{-it\lambda^2_n(u_0)}, \quad \forall t \in \mathbb{R},$$

where $u(t)$ is the solution of (CS$^-$) *starting at u_0 at $t = 0$. Furthermore, the same holds for* (CS$^+$) *under the additional condition* $\|u_0\|_{L^2} < 1$.

Remark 6.1. Note that there is a point hidden in the previous corollary, namely, the fact that L_u is well defined with $u \in L^2_+(\mathbb{T})$. We refer the readers to [16, Appendix A] for the construction of this operator and to [3, Corollary 3.2] for a way to identify its spectrum.

By repeating the same analysis as the proof of [3, Corollary 3.12], one can establish the existence of an orthonormal basis (g^t_n) of $L^2_+(\mathbb{T})$ satisfying

$$\langle 1|g^t_n\rangle = \langle 1|f^{u_0}_n\rangle e^{-i\lambda^2_n t},$$
$$\langle S g^t_p|g^t_n\rangle = \langle S f^{u_0}_p|f^{u_0}_n\rangle e^{i((\lambda_p+1)^2-\lambda^2_n)t}.$$

Finally, in Section 5, more precisely in (5.3), we made use of the fact that the domain of the Lax operator L_u with $u \in H^2_+(\mathbb{T})$ is $H^1_+(\mathbb{T})$, in order to infer that $\Pi(\bar{u}\psi) \in L^2$. However, it should be noted that the Lax operator L_u with $u \in L^2_+(\mathbb{T})$ has as its domain a subset of $H^{1/2}_+(\mathbb{T})$ [16, Appendix A]. Hence, we need the following lemma to infer the result.

Lemma ([3, Lemma 2.7]). *Let $h \in H^{1/2}_+(\mathbb{T})$, $u \in L^2_+(\mathbb{T})$; then*

$$\|T_{\bar{u}} h\|^2_{L^2(\mathbb{T})} \leq (\langle Dh|h\rangle + \|h\|^2_{L^2(\mathbb{T})})\|u\|^2_{L^2(\mathbb{T})},$$

where we recall T_u was defined in (1.9).

7. Open problems

(1) A full characterization of the traveling waves $u_0(x - ct)$ of (CS$^+$) is still an open problem.

(2) Note that throughout this paper, we have treated the case where the traveling waves of the Calogero–Sutherland DNLS equation (CS) are of the form

$$u(t,x) := u_0(x-ct), \quad c \in \mathbb{R}.$$

But one may wonder whether there exist traveling wave solutions with a phase factor, such as

$$u(t,x) := \mathrm{e}^{i\varphi(t)} u_0(x-ct), \quad \varphi(t), c \in \mathbb{R}. \tag{7.1}$$

However, let us underline the following feature: observe that the mean $\langle u|1\rangle$ is conserved along the flow of the Calogero–Sutherland DNLS equation (CS), for any solution u in the Hardy space of the circle $\mathbb{T}$. Indeed, by applying an integration by parts and since u is in the Hardy space, then

$$i\partial_t \langle u|1\rangle = -\langle \partial_x^2 u|1\rangle \pm 2\langle D\Pi(|u|^2)|\bar{u}\rangle = 0.$$

Therefore, we have the following observations:

- If $\langle u_0|1\rangle \neq 0$, then $\varphi(t)$ in (7.1) must be a constant in time.
- With regard to the case where $\langle u_0|1\rangle = 0$, the question of the existence of traveling waves of (CS) of the form (7.1) remains an open problem. However, one can easily prove that $(\varphi(t), c)$ are related via the identity

  $$\varphi'(t) - Nc = -N^2,$$

  where N is the positive integer appearing after rewriting u_0 as $u_0 = S^N v_0$ with $\langle v_0|1\rangle \neq 0$, as $\langle u_0|1\rangle = 0$. Indeed, by writing the solution $u(t,x)$ as

  $$\begin{aligned} u(t,x) &= \mathrm{e}^{i\varphi(t)} u_0(x-ct) \\ &= \mathrm{e}^{i\varphi(t)} \mathrm{e}^{iN(x-ct)} v_0(x-ct), \end{aligned}$$

  one observes that if u satisfies (CS), then

  $$\begin{cases} -(\varphi'(t) - Nc)v_0 - N^2 v_0 + P(\partial_x v_0, \partial_x^2 v_0) \mp 2i\partial_x \Pi(|v_0|^2) v_0 = 0, \\ P(w, \widetilde{w}) := (2N-c)iw + \widetilde{w}. \end{cases}$$

  We conclude by taking the inner product of the last identity with 1, so that

  $$\varphi'(t) - Nc = -N^2.$$

A. Counterexamples

A.1. Counterexample if $\nu_n = 0$ in Proposition 2.4

The following counterexample illustrates the necessity of the condition $\nu_n \neq 0$ in order to obtain the first point in Proposition 2.4.

Consider the 0-gap potential (i.e. a potential satisfying $\gamma_n(u) = 0$ for all $n \in \mathbb{N}_{\geq 0}$, where $\gamma_n(u)$ is defined in (5.1))

$$u(z) = \frac{\sqrt{1-|p|^2}}{1-pz}, \quad p \in \mathbb{D}.$$

One can easily check that $L_u f_0 = -f_0$ for

$$f_0(z) := \frac{\sqrt{1-|p|^2}}{1-pz},$$

and that, for all $k \in \mathbb{N}_{\geq 0}$, $L_u S^k \psi = k S^k \psi$, where

$$\psi(z) := \frac{z-\bar{p}}{1-pz}.$$

Therefore, the spectrum of L_u is given by

$$\sigma(L_u) = \{-1 < 0 < 1 < 2 < \cdots < n < n+1 < \cdots\},$$

where we note that $\nu_1 = \nu_0 + 1$ and $S f_0 \not\equiv \psi$.

A.2. Integers N_1 and N_2 not necessarily equal in Corollary 2.6

In this part of the appendix we prove that the two integers N_1 and N_2 appearing in Corollary 2.6 are not necessarily equal.

Let

$$u(z) := \frac{\sqrt{2(1-|p|^4)}z}{1-p^2z^2}, \quad p \in \mathbb{D}^*.$$

For such a u, one can check that

$$\psi_u := \frac{(z-\bar{p})(z+\bar{p})}{(1-pz)(1+pz)}$$

is an eigenfunction of L_u associated with the eigenvalue 0. Additionally, for all $k \in \mathbb{N}_{\geq 0}$,

$$\langle S^k \psi_u | u \rangle = \int_{z \in \mathcal{C}(0,1)} \frac{z^k(z^2-\bar{p}^2)}{1-p^2z^2} \frac{\sqrt{2(1-|p|^4)}z}{z^2-\bar{p}^2} \frac{dz}{2\pi i z} = 0,$$

leading, by (2.1), to $L_u S^k \psi_u = k S^k \psi_u$, for all $k \in \mathbb{N}_{\geq 0}$. Note that $\deg \psi_u = 2$. Then it remains to find two eigenvectors of L_u, generating the model space[9] $(\psi_u L^2_+(\mathbb{T}))^\perp$. First, we have $L_u 1 = 0$ as $L_u 1 = -\langle 1|u\rangle 1$ and $\langle u|1\rangle = 0$. Second, by taking

$$f_0 = \frac{\sqrt{1-|p|^4}z}{1-p^2z^2},$$

[9][14, Corollary 5.18]

one has $L_u f_0 = -f_0$. Therefore, by denoting for all $k \in \mathbb{N}_{\geq 0}$, $f_{2+k} := S^k \psi_u$ and $f_1 := 1$, we have $\langle u | f_n \rangle = 0$ for all $n \geq 1$. But, on the other hand, $\nu_2 - \nu_1 - 1 = 0 - 0 - 1 \neq 0$.

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