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On distribution relations of polylogarithmic Eisenstein classes

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Abstract. We show that for Siegel modular varieties of arbitrary genus, the natural distribution relations satisfied by certain integral Eisenstein cohomology classes defined by Kings admit an adelic refinement. This generalizes the classical relations for Siegel units on modular curves.

1. Introduction

Motivic cohomology classes such as Beilinson's Eisenstein symbols have several important arithmetic applications. Kato [11] used these symbols to construct Euler systems for Galois representations attached to newforms and obtained spectacular results towards p-adic Birch–Swinnerton Dyer and Iwasawa main conjectures in these settings. The construction of Kato's Euler system makes essential use of the so-called *distribution relations* satisfied by these elements. These relations describe the behavior of Eisenstein symbols under pushforward, pullback and conjugation morphisms between modular curves. Colmez [3] later gave an adelic reformulation of these relations in the analogous setting of modular forms. The adelic version is more useful since, e.g., it aids the construction of Euler systems via representation theoretic methods.

The construction of Beilinson's symbols can be generalized to other Shimura varieties by means of polylogarithms on abelian schemes. The resulting classes are again of motivic origin and referred to as Eisenstein classes by analogy. Using an Iwasawa theoretic approach, Kings [16] showed that the p-adic étale realization of these classes satisfies a p-adic interpolation property in varying weights. In particular, they enjoy certain integrality properties.

The modest purpose of this note is to verify that the analogous adelic distribution relations hold *integrally* for polylogarithmic Eisenstein classes constructed in the cohomology Siegel modular varieties of arbitrary genus. We point out that for genus larger than one, Colmez's argument does not immediately transfer to Kings' setting owing partly to the failure of Galois descent in cohomology and requires solving a non-trivial lifting problem. The adelic relations play a pivotal role in the construction of Euler systems via pushforwards of test vectors in the cohomology of more general Shimura varieties e.g., see [20].

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The formulation presented here is also needed in two forthcoming works [2, 23]. It is our hope that the adelic extension of Kings' theory presented here for higher genus will find similar applications.

1.1. Main result

Let $(V_{\mathbb{Z}}, \psi)$ denote the standard symplectic $\mathbb{Z}$-module of rank $2n$ (see Section 3.1) and $\mathbf{G} := \mathrm{GSp}_{2n}(V_{\mathbb{Z}}, \psi)$ denote the $\mathbb{Z}$-group scheme of automorphisms of $V_{\mathbb{Z}}$ that preserve ψ up to a scalar. For any ring R, we denote $V_R := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. Let p be a rational prime and $c > 1$ an integer with $(c, p) = 1$. We denote $\mathbb{Z}_{cp} := \prod_{\ell | cp} \mathbb{Z}_\ell$, $\mathbb{A}_f^{cp}$ the group of finite rational adeles away from primes dividing cp and $G := \mathbf{G}(\mathbb{A}_f^{cp}) \times \mathbf{G}(\mathbb{Z}_{cp})$. Let $\mathcal{S}^{cp}$ denote the $\mathbb{Z}_p$-module of all locally constant compactly supported $\mathbb{Z}_p$-valued functions on $V_{\mathbb{A}_f} \setminus \{0\}$ of the form $\phi_{cp} \otimes \phi^{cp}$ where ϕ_{cp} is the characteristic function of $V_{\mathbb{Z}_{cp}}$ and ϕ^{cp} is a function on $V_{\mathbb{A}_f^{cp}} \setminus \{0\}$. Then $\mathcal{S}^{cp}$ is a smooth G-representation. For $K \subset G$ a subgroup, we denote by $\mathcal{S}^{cp}(K) \subset \mathcal{S}^{cp}$ the submodule of K-invariants. For $N \geq 1$, let $K_N \subset G$ denote the principal congruence subgroup of level N, i.e., the subgroup of $\mathbf{G}(\widehat{\mathbb{Z}})$ that acts trivially on $V_{\mathbb{Z}}/NV_{\mathbb{Z}}$. For each neat compact open subgroup $K \subset G$, let $\mathrm{Sh}(K)$ denote the Siegel modular variety over $\mathbb{Q}$ of level K and $\mathcal{A}_K \to \mathrm{Sh}(K)$ the universal abelian scheme. Let $\mathcal{H}_{\mathbb{Z}_p} = \mathcal{H}_{\mathcal{A}_K, \mathbb{Z}_p}$ denote the $\mathbb{Z}_p$-sheaf on $\mathrm{Sh}(K)$ of p-adic Tate modules of $\mathcal{A}_K$ and $\mathcal{H}_{\mathbb{Q}_p}$ the corresponding $\mathbb{Q}_p$-sheaf. For $k \geq 0$, let $\Gamma^k(\mathcal{H}_{\mathbb{Z}_p})$ (resp., $\mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})$) denote the k-th divided power (resp., k-th symmetric power) sheaf. For each torsion section $t\colon \mathrm{Sh}(K) \to \mathcal{A}_K \setminus \mathcal{A}[c]$, Kings [16] has constructed a *p-adic étale Eisenstein class*

$$ {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t) \in \mathrm{H}^{2n-1}_{\text{ét}}\big(\mathrm{Sh}(K), \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n)\big) $$

in the continuous étale cohomology [10] of $\mathrm{Sh}(K)$. For $K \subset G$ a neat compact open subgroup, let $\mathcal{E}^k(K)$ denote the $\mathbb{Z}_p$-submodule of $\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(K), \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n))$ given by the image of cohomology with coefficients in $\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n)$. The main result of [16] implies that $N^k{}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t) \in \mathcal{E}^k(K)$ if t is N-torsion for N satisfying $(N, c) = 1$. For $N \geq 3$ and $v \in V_{\widehat{\mathbb{Z}}} \setminus NV_{\widehat{\mathbb{Z}}}$, let $t_{v,N}\colon \mathrm{Sh}(K_N) \to \mathcal{A}_{K_N}$ denote the section corresponding to $v + NV_{\widehat{\mathbb{Z}}} \in V_{\mathbb{Z}}/NV_{\mathbb{Z}}$ under the universal level N structure on $\mathcal{A}_{K_N}$ and $\xi_{v,N} \in \mathcal{S}^{cp}(K_N)$ denote the characteristic function of $v + NV_{\widehat{\mathbb{Z}}}$. Finally, let Υ be the collection of all compact open subgroups of G which are G-conjugate to a subgroup of K_N for some $N \geq 3$ satisfying $(N, cp) = 1$ and which are of the form $\mathbf{G}(\mathbb{Z}_{cp})L$ for some $L \subset \mathbf{G}(\mathbb{A}_f^{cp})$.

Theorem A (Theorem 4.19). *There exists a unique collection of $\mathbb{Z}_p$-module homomorphisms*

$$ \varphi^k(K)\colon \mathcal{S}^{cp}(K) \to \mathcal{E}^k(K) $$

indexed by $K \in \Upsilon$ satisfying the following conditions:

- *for each $N \geq 3$ prime to cp and $v \in V_{\widehat{\mathbb{Z}}} \setminus NV_{\widehat{\mathbb{Z}}}$,*

$$ \varphi^k(K_N)(\xi_{v,N}) = N^k{}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,N}), $$

- *for each* $K, L \in \Upsilon$ *satisfying* $L \subset K$, *we have commutative diagrams*

$$\begin{array}{ccc} \mathcal{S}^{cp}(L) & \xrightarrow{\varphi^k(L)} & \mathcal{E}^k(L) \\ {\scriptstyle \mathrm{pr}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_*} \\ \mathcal{S}^{cp}(K) & \xrightarrow{\varphi^k(K)} & \mathcal{E}^k(K) \end{array} \qquad \begin{array}{ccc} \mathcal{S}^{cp}(L) & \xrightarrow{\varphi^k(L)} & \mathcal{E}^k(L) \\ {\scriptstyle \mathrm{pr}^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ \mathcal{S}^{cp}(K) & \xrightarrow{\varphi^k(K)} & \mathcal{E}^k(K), \end{array}$$

where pr_* *and* pr^* *denote respectively trace and inclusion maps,*

- *for each* $K \in \Upsilon$ *and* $g \in G$, *we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{S}^{cp}(K) & \xrightarrow{\varphi^k(K)} & \mathcal{E}^k(K) \\ {\scriptstyle [g]^*}\downarrow & & \downarrow{\scriptstyle [g]^*} \\ \mathcal{S}^{cp}(gKg^{-1}) & \xrightarrow{\varphi^k(gKg^{-1})} & \mathcal{E}^k(gKg^{-1}), \end{array}$$

where $[g]^*$ *denotes the (contravariant) conjugation isomorphisms.*

We refer to our result as an *integral parametrization* of Eisenstein classes by Schwartz spaces. For $n = 1$ and $k = 0$, our result recovers the distribution relations for Kummer images of Siegel units proved in [11, §1–2]. We remark that the classes ${}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,N})$ for $k \geq 1$ and $n = 1$ are closely related to the Soulé twisting construction applied to the classes for $k = 0$ [15, §4.7]. For $n \geq 2$, Lemma [19] has established that these classes are not all zero for certain weights k.

The core ideas that go into the proof of Theorem A are derived from [3, 11]. However, we must carefully address some complications not encountered in these works. One of the issues is in defining the maps $\varphi^k(K)$ at, say, principal levels $K = K_N$ for functions that are not supported on $V_{\hat{\mathbb{Z}}} \setminus \{0\}$ and constant modulo $NV_{\hat{\mathbb{Z}}}$. More precisely, there is no obvious way to attach a linear combination of torsion sections for such functions. In the setting of [3, §1], the passage from integral to rational adeles is made by defining the action of the adelic group in the limit, using which maps at finite level can be recovered by taking invariants. But since the K-invariants of the G-representation $\varinjlim_L \mathcal{E}^k(L)$ are not necessarily equal to $\mathcal{E}^k(K)$, one cannot define $\varphi^k(K)$ in this manner without potentially violating integrality, a crucial requirement in the context of Euler systems. So one has to construct for each K-invariant Schwartz function a class in $\mathcal{E}^k(K)$ which lifts the corresponding class in the limit. We show, among other things, that these lifts can indeed be constructed compatibly for all levels $K \in \Upsilon$ using Hecke correspondences, Galois descent for torsion sections and some elementary topological properties of the action of G on $V_{\mathbb{A}_f}$.

2. p-adic polylogarithms

In this section, we recall the construction and p-adic interpolation of Eisenstein classes via polylogarithms following [9, 16]. The main purpose is to establish some basic "distribution relations" (Section 2.5) that describe the effect of isogenies and base change on these classes. Except for Section 2.5, the content is based on the original works and we skip most of the proofs within these subsections.

Throughout, **Sch** denotes the category of finite type separated schemes over a fixed Noetherian regular scheme of dimension at most 1. For any $X \in \mathbf{Sch}$, we let $\mathbf{Sh}(X_{\text{ét}})$ denote the category of étale sheaves of abelian groups on the small étale site $X_{\text{ét}}$ of X and $\mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$ the category of inverse systems on $\mathbf{Sh}(X_{\text{ét}})$. The i-th right derived functors of inverse limits of global sections of $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1} \in \mathbf{Sh}(X_{\text{ét}})^{\mathbb{N}}$ is denoted $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ and referred to as continuous étale cohomology [10]. For p a rational prime invertible on X, we let $\text{Ét}(X)_{\mathbb{Z}_p}$ denote the abelian category of (constructible) $\mathbb{Z}_p$-sheaves and its isogeny category of $\mathbb{Q}_p$-sheaves by $\text{Ét}(X)_{\mathbb{Q}_p}$. For $\Lambda \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$, we let $\mathcal{D}(X)_\Lambda$ denote the bounded "derived" category of $\text{Ét}(X)_\Lambda$ in the sense of [4, Theorem 6.3][1]. There is a full six functor formalism on these and for any $\mathcal{F} \in \text{Ét}(X)_\Lambda$, we have $\mathrm{Hom}_{\mathcal{D}(X)_\Lambda}(\Lambda, \mathcal{F}[i]) = \mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ for all $i \geq 0$ by [7, Lemma 4.1].

2.1. Purity for unipotent sheaves

Let $\pi: X \to S$ be a separated morphism of finite type in **Sch**. A $\mathbb{Z}_p$-sheaf $\mathcal{F} \in \text{Ét}(X)_{\mathbb{Z}_p}$ is said to be *S-unipotent of length k* if there exists a decreasing filtration $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n \supset \mathcal{F}^{n+1} = 0$ such that the $\mathcal{F}^i/\mathcal{F}^{i+1}$ are isomorphic to $\pi^*\mathcal{G}^i$ for $\mathbb{Z}_p$-sheaves $\mathcal{G}^i \in \text{Ét}(S)_{\mathbb{Z}_p}$. We refer to $\mathcal{G}^i$ as the i-th graded piece of $\mathcal{F}$. We can similarly define unipotence of Λ-sheaves for $\Lambda = \mathbb{Q}_p$ or $\Lambda = \mathbb{Z}/p^r\mathbb{Z}$ (i.e., étale p^r-torsion sheaves).

Lemma 2.1. *Let $\Lambda \in \{\mathbb{Z}/p^r\mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}_p\}$. Suppose $\pi_i: X_i \to S$ for $i = 1, 2$ be morphisms as above such that π_i is smooth of relative dimension d_i. Let $f: X_1 \to X_2$ be any S-morphism. Then for any S-unipotent Λ-sheaf $\mathcal{F}$ (of some finite length), $f^!\mathcal{F} \simeq f^*\mathcal{F}(d_1 - d_2)[2d_1 - 2d_2]$ functorially in $\mathcal{F}$.*

2.2. The $\mathbb{Q}_p$-logarithm

Let $\pi: A \to S \in \mathbf{Sch}$ denote an abelian scheme of relative dimension d, i.e., A is a group scheme and π is a smooth proper morphism with connected geometric fibers of dimension d. The unit section is denoted by $e: S \to A$. Let p be a prime invertible on S. The *p-adic Tate module* of π is defined to be first relative homology

$$\mathcal{H}_{\mathbb{Z}_p} := \mathcal{H}om_S(R^1\pi_*\mathbb{Z}_p, \mathbb{Z}_p) = R^{2d-1}\pi_*\mathbb{Z}_p(d) \in \text{Ét}(S)_{\mathbb{Z}_p} \tag{2.2}$$

of A with respect to S. It is a lisse $\mathbb{Z}_p$-sheaf and fiberwise equals the Tate module. We let $\mathcal{H}_{\mathbb{Q}_p} := \mathcal{H}_{\mathbb{Z}_p} \otimes \mathbb{Q}_p \in \text{Ét}(S)_{\mathbb{Q}_p}$ denote the corresponding $\mathbb{Q}_p$-sheaf. For $r \geq 1$, we similarly define $\mathcal{H}_{\Lambda_r}$ where $\Lambda_r := \mathbb{Z}/p^r\mathbb{Z}$. Then

$$A[p^r] \simeq \mathcal{H}_{\Lambda_r} \simeq \mathcal{H}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^r\mathbb{Z}, \tag{2.3}$$

where $A[p^r]$ on the left denotes the associated representable sheaf of p^r torsion in A. In what follows, we denote $\mathcal{H}_{\mathbb{Z}_p}$ simply by $\mathcal{H}$ if no confusion can arise.

[1] See [8, §0] for a short explanation how this differs from an ordinary derived category. See also [6, Appendix A.1].

The low term exact sequence for the Leray spectral sequence associated with the composition $\mathcal{H}om_S(\mathbb{Z}_p, -) \circ \pi_*$ evaluated at $\pi^*\mathcal{H}$ gives

$$\begin{aligned} 0 &\to \mathrm{Ext}^1_S(\mathbb{Z}_p, \mathcal{H}) \xrightarrow{\pi^*} \mathrm{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathcal{H}) \to \mathrm{Hom}_S(\mathbb{Z}_p, R^1\pi_*\pi^*\mathcal{H}) \\ &\to \mathrm{Ext}^2_S(\mathbb{Z}_p, \mathcal{H}) \xrightarrow{\pi^*} \mathrm{Ext}^2_A(\mathbb{Z}_p, \pi^*\mathcal{H}). \end{aligned}$$

The maps π^* are necessarily injective as $e^* \circ \pi^* = (\pi \circ e)^* = \mathrm{id}$ and therefore the morphism to the second line above is 0. By the projection formula, $R^1\pi_*\pi^*\mathcal{H} \simeq R^1\pi_*\mathbb{Z}_p \otimes \mathcal{H}$. Since $R^1\pi_*\mathbb{Z}_p \simeq \mathcal{H}om_S(\mathcal{H}, \mathbb{Z}_p) =: \mathcal{H}^\vee$ and $\mathcal{H}^\vee \otimes \mathcal{H} \simeq \mathcal{H}om_S(\mathcal{H}, \mathcal{H})$, we get a split short exact sequence

$$0 \to \mathrm{Ext}^1_S(\mathbb{Z}_p, \mathcal{H}) \to \mathrm{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathcal{H}) \to \mathrm{Hom}_S(\mathcal{H}, \mathcal{H}) \to 0. \tag{2.4}$$

with left splitting given by e^*. Let $\phi\colon \mathrm{Hom}_S(\mathcal{H}, \mathcal{H}) \to \mathrm{Ext}^1(\mathbb{Z}_p, \pi^*\mathcal{H})$ denote the unique right splitting satisfying $e^* \circ \phi = 0$.

Definition 2.5. The *first logarithm sheaf* is defined to be the pair $(\mathcal{L}og^{(1)}_{\mathbb{Z}_p}, \mathbf{1}^{(1)})$ where $\mathcal{L}og^{(1)}_{\mathbb{Z}_p} = \mathcal{L}og^{(1)}_{A,\mathbb{Z}_p} \in \acute{\mathrm{E}}\mathrm{t}(A)_{\mathbb{Z}_p}$ is such that $\phi(\mathrm{id}_{\mathcal{H}}) \in \mathrm{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathcal{H})$ is represented by

$$0 \to \pi^*\mathcal{H} \to \mathcal{L}og^{(1)}_{\mathbb{Z}_p} \xrightarrow{\delta} \mathbb{Z}_p \to 0 \tag{2.6}$$

and $\mathbf{1}^{(1)}\colon \mathbb{Z}_p \to e^*\mathcal{L}og^{(1)}_{\mathbb{Z}_p}$ is a fixed right splitting of the pullback of (2.6) under identity.[2] The pair $(\mathcal{L}og^{(1)}_{\mathbb{Z}_p}, \mathbf{1}^{(1)})$ is then unique up to a unique isomorphism. We denote by $\mathcal{L}og^{(1)}_{\mathbb{Q}_p}$ the associated $\mathbb{Q}_p$-sheaf.

By definition, $\mathcal{L}og^{(1)}_{\mathbb{Z}_p}$ is S-unipotent of length one (see Section 2.1). One defines $\mathcal{L}og^{(1)}_{\Lambda_r}$ for $r \geq 1$ in the same way as $\mathcal{H}_{\Lambda_r}$. Then,

$$\mathcal{L}og^{(1)}_{\Lambda_r} = \mathcal{L}og^{(1)}_{\mathbb{Z}_p} \otimes \Lambda_r \quad \text{and} \quad \mathcal{L}og^{(1)}_{\mathbb{Z}_p} = \big(\mathcal{L}og^{(1)}_{\Lambda_r}\big)_{r \geq 1}.$$

Definition 2.7. For $k \geq 1$, the *k-th $\mathbb{Q}_p$-logarithm sheaf* is the pair $(\mathcal{L}og^{(k)}_{\mathbb{Q}_p}, \mathbf{1}^{(1)})$ where

$$\mathcal{L}og^{(k)}_{\mathbb{Q}_p} := \mathrm{Sym}^k\big(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}\big) \in \acute{\mathrm{E}}\mathrm{t}(A)_{\mathbb{Q}_p}$$

and

$$\mathbf{1}^{(k)} := \frac{1}{k!}\mathrm{Sym}^k(\mathbf{1}^{(1)})\colon \mathbb{Q}_p \to e^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$$

is the splitting map induced by $\mathbf{1}^{(1)}$ on the symmetric power.

The $\mathbb{Q}_p$-logarithms for $k \geq 1$ and their canonical splittings fit into an inverse system as follows. Let $\beta := \delta \oplus \mathrm{id}\colon \mathcal{L}og^{(1)}_{\mathbb{Q}_p} \to \mathbb{Q}_p \oplus \mathcal{L}og^{(1)}_{\mathbb{Q}_p}$ denote the diagonal map given

[2] This pullback is necessarily split since $e^* \circ \phi = 0$.

by the sum of the projection δ in (2.6) and identity. For $k \geq 2$, define *transition maps* $u^k\colon \mathcal{L}og^{(k)}_{\mathbb{Q}_p} \to \mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}$ via

$$\begin{aligned}\mathcal{L}og^{(k)}_{\mathbb{Q}_p} &= \mathrm{Sym}^k(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \xrightarrow{\mathrm{Sym}^k\beta} \mathrm{Sym}^k(\mathbb{Q}_p \oplus \mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \\ &\simeq \bigoplus_{i+j=k} \mathrm{Sym}^i(\mathbb{Q}_p) \otimes \mathrm{Sym}^j(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \\ &\to \mathrm{Sym}^1(\mathbb{Q}_p) \otimes \mathrm{Sym}^{k-1}(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \simeq \mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}.\end{aligned}$$

For $k \geq 2$, we claim that $(e^*u^k) \circ \mathbf{1}^{(k)}\colon \mathbb{Q}_p \to e^*\mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}$ is equal to $\mathbf{1}^{(k-1)}$. First note that by definition of $\mathbf{1}^{(1)}$, we have

$$(e^*\beta) \circ \mathbf{1}^{(1)} = \mathrm{id} \oplus \mathbf{1}^{(1)}$$

as maps $\mathbb{Q}_p \to \mathbb{Q}_p \oplus e^*\mathcal{L}og^{(1)}_{\mathbb{Q}_p}$. Therefore

$$\begin{aligned}\mathrm{Sym}^k(e^*\beta) \circ \mathbf{1}^k &= \frac{1}{k!}\mathrm{Sym}^{(k)}(\mathrm{id} \oplus \mathbf{1}^{(1)}) \\ &\simeq \bigoplus_{i+j=k} \frac{1}{k!}\binom{k}{i}\big(\mathrm{Sym}^i(\mathrm{id}) \otimes \mathrm{Sym}^j(\mathbf{1}^{(1)})\big) \\ &= \bigoplus_{i+j=k} \frac{1}{i!}\big(\mathrm{Sym}^i(\mathrm{id}) \otimes \mathbf{1}^{(j)}\big).\end{aligned}$$

The projection of the last sum above to the summand at $i = 1$ is equal to $\mathbf{1}^{(k-1)}$. Since u^k is obtained from $\mathrm{Sym}^k(\beta)$ by post-composition with projection to $i = 1$ summand as well, the claim follows. If we set $\mathcal{L}og^{(0)}_{\mathbb{Q}_p} := \mathbb{Q}_p$, $\mathbf{1}^{(0)}\colon \mathbb{Q}_p \to e^*\mathcal{L}og^{(0)}_{\mathbb{Q}_p}$ the identity and $u^1 := \delta$ (2.6), we still have

$$(e^*u^1) \circ \mathbf{1}^{(1)} = \mathbf{1}^{(0)}.$$

By construction, we have for each $k \geq 1$ an exact sequence

$$0 \to \pi^*\mathrm{Sym}^k\mathcal{H}_{\mathbb{Q}_p} \to \mathcal{L}og^{(k)}_{\mathbb{Q}_p} \xrightarrow{u^k} \mathcal{L}og^{(k-1)}_{\mathbb{Q}_p} \to 0 \tag{2.8}$$

whose pullback under e splits, giving an identification $e^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p} \simeq \prod_{i=0}^k \mathrm{Sym}^i\mathcal{H}_{\mathbb{Q}_p}$ such that e^*u^k is identified with the projection map

$$\prod_{i=0}^{k} \mathrm{Sym}^i\mathcal{H}_{\mathbb{Q}_p} \to \prod_{i=0}^{k-1} \mathrm{Sym}^i\mathcal{H}_{\mathbb{Q}_p}.$$

One sees by induction that $\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ is S-unipotent of length k with graded pieces given by symmetric powers of $\mathcal{H}_{\mathbb{Q}_p}$.

Definition 2.9. The $\mathbb{Q}_p$*-logarithm prosheaf* $(\mathcal{L}og_{\mathbb{Q}_p}, \mathbf{1})$ is the pro-system $(\mathcal{L}og^{(k)}_{\mathbb{Q}_p}, \mathbf{1}^{(k)})_{k\geq 0}$ of $\mathbb{Q}_p$-sheaves whose transitions maps are given by u^k for $k \geq 1$.

The logarithm prosheaf satisfies several important properties. Below we record the ones needed later on.

Proposition 2.10 (Pullback compatibility). *Suppose that $f: T \to S$ is a morphism in* **Sch**, *$A_T := A \times_S T$ denotes the pullback of A to T and $f_A: A_T \to A$ denotes the natural map. Then there are canonical isomorphisms*

$$f_A^*(\mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p}) \simeq \mathcal{L}\mathrm{og}^{(k)}_{A_T,\mathbb{Q}_p}$$

for all $k \geq 0$ such that $f_A^*(\mathbf{1}_A^{(k)})$ is identified with the splitting $\mathbf{1}^{(k)}_{A_T}$. These isomorphisms commute with transition maps.

Proposition 2.11 (Functoriality). *For any isogeny $\varphi: A \to A'$ of abelian schemes over S, there are unique isomorphisms*

$$\varphi_\#: \mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p} \to \varphi^! \mathcal{L}\mathrm{og}^{(k)}_{A',\mathbb{Q}_p} \simeq \varphi^* \mathcal{L}\mathrm{og}^{(k)}_{A',\mathbb{Q}_p}$$

for all $k \geq 0$ such that $\mathbf{1}_A^{(k)}$ is sent to $\mathbf{1}_{A'}^{(k)}$. These isomorphisms commute with transition maps and their pullbacks under identity induce $\mathrm{Sym}^k \varphi_*$ on k-th graded pieces.

Corollary 2.12 (Splitting principle). *Let $\iota: D \to A$ be a closed subscheme contained in the kernel of an isogeny $\varphi: A \to A'$ and $\pi_D: D \to S$ denote its structure map. Then there exist isomorphisms*

$$\varrho_D^k: \iota^* \mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p} \xrightarrow{\sim} \prod_{i=0}^{k} \pi_D^* \mathrm{Sym}^i(\mathcal{H}_{A,\mathbb{Q}_p})$$

for all $k \geq 0$ that commute with transition maps and are independent of the isogeny φ.

Proof. Let e' denote the identity section of A'. First assume that $(\iota, D) = (t, S)$ is a section of $\ker \varphi$ over S. We define ϱ_t^k as the composition

$$t^* \mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p} \xrightarrow{t^*\varphi_\#} (e')^* \mathcal{L}\mathrm{og}^{(k)}_{A',\mathbb{Q}_p} \xrightarrow{(e^*\varphi_\#)^{-1}} e^* \mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p} \simeq \prod_{i=0}^{k} \mathrm{Sym}^i(\mathcal{H}_{A,\mathbb{Q}_p}),$$

where the last isomorphism is induced by (2.8) as above. If $\psi: A' \to A''$ is any isogeny over S, the corresponding isomorphism defined with respect to $\varphi' := \psi \circ \varphi$ is easily seen to coincide with ϱ_t^k using the cocycle condition $\varphi'_\# = (\varphi^*\psi)_\# \circ \varphi_\#$, which holds by uniqueness of the maps involved. As any two isogenies from A that annihilate t can be refined by a common isogeny (by quasi-compactness of S), ϱ_t^k does not depend on φ.

In general, let $A_D := A \times_S D$ and $t_D: D \to A_D$ be the tautological section obtained from t by base change. Then we define ϱ_D^k as $\varrho_{t_D}^k$ after identifying

$$\iota^* \mathcal{L}\mathrm{og}^{(k)}_{A,\mathbb{Q}_p} \simeq t_D^* \mathcal{L}\mathrm{og}^{(k)}_{A_D,\mathbb{Q}_p} \quad \text{and} \quad \pi_D^* \mathcal{H}_{A,\mathbb{Q}_p} \simeq \mathcal{H}_{A_D,\mathbb{Q}_p}. \qquad \blacksquare$$

Proposition 2.13 (Vanishing of cohomology). *There exist natural isomorphisms*

$$R^{2d}\pi_*(\mathcal{L}\mathrm{og}^{(k)}_{\mathbb{Q}_p}) \simeq \mathbb{Q}_p(-d)$$

for all $k \geq 0$ *that commute with the transition maps. For* $i = 0, \dots, 2d-1$*, the induced maps* $R^i u^k \colon R^i\pi_* \mathcal{L}og^{(k)}_{\mathbb{Q}_p} \to R^i\pi_*\mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}$ *are zero for all* $k \geq 1$*. In particular,*

$$\varprojlim_k \mathrm{H}^i_{\text{ét}}\big(A, \mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d)\big) \simeq \begin{cases} 0 & \textit{if } i < 2d, \\ \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p) & \textit{if } i = 2d. \end{cases}$$

2.3. Cohomology classes

Fix $c > 1$ an integer invertible on S and let $D := A[c]$ be the group of c-torsion points of A. Then D is a finite étale group scheme over S. Let $U = U_D := A \setminus D$ denote the complement of D in A and consider the diagram.

$$\begin{array}{ccccc} U & \xrightarrow{j_D} & A & \xleftarrow{\iota_D} & D \\ & {\scriptstyle \pi_U}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_D} & \\ & & S & & \end{array} \tag{2.14}$$

where $j = j_D$ and $\iota = \iota_D$ are natural inclusions and $\pi_D := \pi \circ \iota$, $\pi_U := \pi \circ j$ are the structure maps. For any $\mathcal{F} \in \mathcal{D}(A)_{\mathbb{Q}_p}$, we have a distinguished triangle $R\iota_*\iota^!\mathcal{F} \to \mathcal{F} \to Rj_*j^*\mathcal{F} \to R\iota_*\iota^!\mathcal{F}[1] \in \mathcal{D}(A)_{\mathbb{Q}_p}$ known as the *localization* triangle. Applying $R\pi_* = R\pi_!$ to the localization triangle with $\mathcal{F} = \mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d)$ for a fixed k, we get a distinguished triangle

$$\begin{aligned} R\pi_{D,*}\iota^!\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d) \to R\pi_*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d) \to R\pi_{U,*}j^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d) \\ \to R\pi_{D,*}\iota^!\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d)[1] \in \mathcal{D}(S)_{\mathbb{Q}_p}. \end{aligned} \tag{2.15}$$

Using Lemma 2.1 and the fact that $\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ is S-unipotent for each integer k, we see that $\iota^!\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d) = \iota^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}[-2d]$ and therefore $R\pi_{D,*}\iota^!\mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d) = R\pi_{D,*}\iota^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}[-2d]$, etc. Applying $\mathrm{Hom}_{\mathcal{D}(X)_{\mathbb{Q}_p}}(\mathbb{Q}_p, -)$ and using the adjunctions $\pi^* \dashv R\pi_*$, etc., we obtain a long exact sequence [12, Theorem II.1.3]

$$\begin{aligned} \cdots \to \mathrm{H}^{2d-1}_{\text{ét}}\big(A, \mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)\big) \\ \to \mathrm{H}^0_{\text{ét}}\big(D, \iota^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}\big) \to \mathrm{H}^{2d}_{\text{ét}}\big(A, \mathcal{L}og^{(k)}_{\mathbb{Q}_p}(d)\big) \to \cdots, \end{aligned} \tag{2.16}$$

where $\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}$ denotes the pullback of $\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ to the open subset U. By construction, these sequences commute with maps induced by transition maps from $k+1$ to k. Abusing notation, we denote the inverse limit over k of each of the groups appearing in (2.16) by removing the superscript (k). Taking inverse limit of (2.16) over all k, we obtain an exact sequence

$$0 \to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{H}^0_{\text{ét}}(D, \iota^*\mathcal{L}og_{\mathbb{Q}_p}) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p) \tag{2.17}$$

by Proposition 2.13 and left exactness of inverse limit. The middle map appearing in (2.17) is denoted res_U and referred to as the *residue map*. The rightmost map in (2.17) is induced by the composition of augmentation $\pi_{D,*}\iota^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p} \to \pi_{D,*}\iota^*\mathbb{Q}_p$ followed by counit adjunction $\pi_{D,*}\iota^*\mathbb{Q}_p = \pi_{D,!}\pi_D^!\mathbb{Q}_p \to \mathbb{Q}_p$. Via the identification of Corollary 2.12, the restriction of the rightmost map above to the $k = 0$ component is the trace morphism

$$\varepsilon_D\colon \mathrm{H}^0_{\text{ét}}(D, \mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p).$$

Let $\mathbb{Q}_p[D]^0 = \ker \varepsilon_D$.

Definition 2.18. The *polylogarithm class with residue* $\alpha \in \mathbb{Q}_p[D]^0$ is the unique cohomology class

$${}_\alpha\mathrm{pol}_{\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big)$$

such that $\mathrm{res}_U({}_\alpha\mathrm{pol}_{\mathbb{Q}_p}) = \alpha$. For $k \geq 0$, we define ${}_\alpha\mathrm{pol}^k_{\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}(U, \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d))$ to be the image of ${}_\alpha\mathrm{pol}_{\mathbb{Q}_p}$.

Remark 2.19. We have a similarly defined exact sequence (of groups of inverse limits)

$$0 \to \mathrm{Ext}^{2d-1}_U\big(\mathbb{Q}_p, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{Hom}_D(\mathbb{Q}_p, \iota^*\mathcal{L}og_{\mathbb{Q}_p}) \to \mathrm{Hom}_S(\mathbb{Q}_p, \mathbb{Q}_p)$$

and a class in the Ext group corresponding to $\alpha \in \mathbb{Q}_p[D]^0$. This is the perspective taken in [9, §5.2].

Fix now an integer $N > 1$ that is invertible on S and such that $(N, c) = 1$. Let $t\colon S \to A$ be a non-zero N-torsion section. Then t factors via U as $(N, c) = 1$ and we will also denote $t\colon S \to U$. The unit adjunction $\mathrm{id} \to Rt_*t^*$ on U gives a morphism

$$\begin{aligned} R\pi_{U,*}\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d) &\to R\pi_{U,*}Rt_*t^*\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d) \\ &\xrightarrow{\sim} R(\pi_U \circ t)_*t^*\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d) = t^*\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d) \end{aligned}$$

which in turn induces the pullback

$$t^*\colon \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)\big) \tag{2.20}$$

on cohomology. By Corollary 2.12, we have a map

$$\begin{aligned} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)\big) &\xrightarrow[\sim]{\varrho^k_t} \mathrm{H}^{2d-1}_{\text{ét}}\Big(S, \prod_{i=0}^{k} \mathrm{Sym}^i(\mathcal{H}_{\mathbb{Q}_p})(d)\Big) \\ &\xrightarrow{\mathrm{pr}^k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(d)\big), \end{aligned} \tag{2.21}$$

where pr^k is the projection on the k-th symmetric power.

Definition 2.22. Let $\alpha \in \mathbb{Q}_p[D]^0$, $t\colon S \to U$ be a non-zero N-torsion section as above and $k \geq 0$ be an integer. The k-th *rational Eisenstein class*

$${}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(d)\big)$$

along t with residue α is the image of ${}_\alpha\mathrm{pol}^k_{\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}(U, \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d))$ under the composition $\mathrm{pr}^k \circ \varrho^k_t \circ t^*$ of (2.21) and (2.20).

There is a special choice of α which enjoys certain compatibility properties and which will be used in Section 4. Let $\pi_D^*\colon \mathrm{H}^0(S,\mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D,\mathbb{Q}_p)$ denote the pullback map induced by π_D and $e_*\colon \mathrm{H}^0_{\text{ét}}(S,\mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D,\mathbb{Q}_p)$ the pushforward map induced by counit adjunction $\mathbb{Q}_p = \pi_{D,*}e_!e^!\mathbb{Q}_p \to \pi_{D,*}\mathbb{Q}_p$. Let

$$\alpha_c = \alpha_{A,c} := c^{2d}e_*(1) - \pi_D^*(1), \tag{2.23}$$

where $1 \in \mathrm{H}^0_{\text{ét}}(S,\mathbb{Q}_p)$ denotes the global section given locally by $1 \in \mathbb{Q}_p$. Then $\alpha_c \in \mathbb{Q}_p[D]^0$ since $\pi_{D,*}(\alpha_c) = c^{2d}(\pi_D \circ e)_*(1) - \pi_{D,*}\pi_D^*(1) = c^{2d} - c^{2d} = 0$.

Definition 2.24. We denote by ${}_c\mathrm{pol}_{\mathbb{Q}_p}$ (resp., ${}_c\mathrm{Eis}^k(t)_{\mathbb{Q}_p}$) the polylogarithm (resp., Eisenstein) class with residue α_c.

Remark 2.25. See [15, §4] for a precise relationship between these classes in the elliptic case and Kato's Siegel units. The general construction of the polylogarithm has its origins in the work of Beilinson and Levin [1], whose ideas were later placed in a much broader framework by Wildeshaus [27] and Kings [13, 14]. When S is the Siegel modular variety of genus two of a suitable level, Faltings [5] has also constructed a "potentially motivic" Eisenstein class in $\mathrm{H}^3_{\text{ét}}(S,\mathbb{Q}_p(3))$, whereas the weight zero construction above yields a class in $\mathrm{H}^3_{\text{ét}}(S,\mathbb{Q}_p(2))$. This other class has recently been used to construct a new Euler system in [25].

2.4. Norm compatibility

We maintain the notations of Section 2.3. Suppose now that $\pi'\colon A' \to S$ is another abelian scheme with unit section $e'\colon S \to A'$ and define D', U', etc., analogously as in diagram (2.14). For notational clarity, we will denote $\mathcal{F} = \mathcal{L}og^{(k)}_{A,\mathbb{Q}_p}(d)$ and $\mathcal{G} = \mathcal{L}og^{(k)}_{A',\mathbb{Q}_p}(d)$ in this subsection.

Let $\varphi\colon A \to A'$ be an S-isogeny and $\varphi_D\colon D \to D'$ denote its restriction to D. Set $\tilde{D} := \varphi^{-1}(D')$, $\tilde{U} := \varphi^{-1}(U') \subset U$ and denote by $j_{\tilde{D}}\colon \tilde{U} \to A$, $\jmath\colon \tilde{U} \to U$ the inclusion maps. The unit adjunction $\mathrm{id} \to R\jmath_*\jmath^*$ gives a restriction transformation

$$r_{U,\tilde{U}}\colon Rj_{D,*}j_D^* \to Rj_{D,*}R\jmath_*\jmath^*j_D^* \xrightarrow{\sim} Rj_{\tilde{D},*}j_{\tilde{D}}^*.$$

Since $\tilde{\varphi} := \varphi_{|\tilde{U}}\colon \tilde{U} \to U'$ is the pullback of φ along $j_{D'}$, we have $\tilde{\varphi}^! j_{D'}^* \simeq j_{\tilde{D}}^*\varphi^!$ and a base change isomorphism $\varphi^! Rj_{D',*} \xrightarrow{\sim} Rj_{\tilde{D},*}\tilde{\varphi}^!$. Define

$$\begin{aligned}\varphi^{\natural}\colon Rj_{D,*}j_D^*\mathcal{F} &\xrightarrow{\varphi_\#} Rj_{D,*}j_D^*\varphi^!\mathcal{G} \xrightarrow{r_{U,\tilde{U}}} Rj_{\tilde{D},*}j_{\tilde{D}}^*\varphi^!\mathcal{G}\\ &\xrightarrow{\sim} Rj_{\tilde{D},*}\tilde{\varphi}^! j_{D'}^*\mathcal{G} \xrightarrow{\sim} \varphi^! Rj_{D',*}j_{D'}^*\mathcal{G}.\end{aligned}$$

By [9, Lemma 5.1.2], there is a morphism of distinguished triangles in $\mathcal{D}(A')_{\mathbb{Q}_p}$

$$\begin{array}{ccccccc} R\varphi_* R\iota_{D,*}\iota_D^!\mathcal{F} & \longrightarrow & R\varphi_*\mathcal{F} & \longrightarrow & R\varphi_* Rj_{D,*}j_D^*\mathcal{F} & \longrightarrow & R\varphi_* R\iota_{D,*}\iota_D^!\mathcal{F}[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi_\natural} & & \downarrow \\ R\iota_{D',*}\iota_{D'}^!\mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & Rj_{D',*}j_{D'}^*\mathcal{G} & \longrightarrow & R\iota_{D',*}\iota_{D'}^!\mathcal{G}[1], \end{array}$$

where $\varphi_\natural\colon R\varphi_* Rj_{D,*}j_D^*\mathcal{F} \to Rj_{D',*}j_{D'}^*\mathcal{G}$ denotes the mate of $\varphi^\natural$ under the adjunction $R\varphi_* = R\varphi_! \dashv \varphi^!$. Applying $R\pi'_! = R\pi'_*$, we obtain a diagram in $\mathcal{D}(S)_{\mathbb{Q}_p}$ where the top row is (2.15) and the bottom row is the version defined for A'. Repeating the steps of Section 2.3, we obtain a *norm map* $\mathcal{N}_\varphi$ from the short exact sequence (2.17) for A to that for A'. On $\mathrm{H}^0_{\text{ét}}(S,\mathbb{Q}_p)$, $\mathcal{N}_\varphi$ is just identity while the map $\mathrm{H}^0_{\text{ét}}(D,\mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D',\mathbb{Q}_p)$ is the trace $\varphi_{D,*}$. By uniqueness of polylogarithms with respect to residues and compatibility of adjunction morphisms, we obtain the following.

Proposition 2.26 (Norm compatibility). *$\mathcal{N}_\varphi({}_\alpha\mathrm{pol}_{A,\mathbb{Q}_p}) = {}_\beta\mathrm{pol}_{A',\mathbb{Q}_p}$ where β denotes the trace $\varphi_{D,*}(\alpha) \in \mathbb{Q}_p[D']^0$.*

We will also need the following result.

Lemma 2.27. *If φ has constant degree and* $(\deg\varphi, c) = 1$, $\varphi_{D,*}(\alpha_c) = \alpha_{A',c}$.

Proof. Since $\varphi_{D,*} \circ e_*(1) = (\varphi_D \circ e)_*(1) = e'_*(1)$, we only need to show that

$$\varphi_{D,*} \circ \pi_D^*(1) = \pi_{D'}^*(1).$$

This can be established étale locally, i.e., over a finite étale cover of S where D, D' become constant group schemes on $\Gamma := (\mathbb{Z}/c\mathbb{Z})^{2d}$. In this case, $\varphi_D\colon D \to D'$ is determined by an automorphism of Γ and $\varphi_{D,*}$ identifies with the endomorphism on $\oplus_\gamma \mathrm{H}^0_{\text{ét}}(S,\mathbb{Q}_p)$ given by identity maps between the permuted components determined by the automorphism of Γ. As $(1)_{\gamma\in\Gamma}$ is clearly preserved by such maps, the claim follows. ∎

2.5. Distribution relations

Fix α, t and k as in Definition 2.22 for all of this subsection. For the next result, we let $f\colon T \to S$ denote a fixed morphism in **Sch**. Set $A_T := A \times_S T$, $D_T := A_T[c]$ and $U_T := A_T \setminus D_T$. We denote by $f_A\colon A_T \to A$ the natural map and by $f_D\colon D_T \to D$, $f_U\colon U_T \to U$ the restriction of f_A to D_T, U_T respectively. Let $e_T\colon T \to A_T$ denote the identity section and $t_T\colon T \to U_T$ the tautological section induced by base changing t.

$$\begin{array}{ccc} U_T & \xrightarrow{\;f_U\;} & U \\ \pi_{U_T}\big\downarrow \big\uparrow t_T & & \pi_U \big\downarrow \big\uparrow t \\ T & \xrightarrow{\;f\;} & S \end{array} \tag{2.28}$$

Denote by

$$f_k^*\colon \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathcal{H}_{A,\mathbb{Q}_p})(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(T, \mathrm{Sym}^k(\mathcal{H}_{A_T,\mathbb{Q}_p})(d)\big),$$
$$f_D^*\colon \mathrm{H}^0_{\text{ét}}(D,\mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D_T,\mathbb{Q}_p)$$

the pullback maps induced by the unit adjunction $\mathrm{id} \to Rf_*f^*$, etc. More generally, this adjunction induces (via base change applied to (2.28) and Proposition 2.10) a morphism from the triangle (2.15) to Rf_* applied to the corresponding triangle for A_T. Thus we get a pullback map from the sequence (2.17) to the corresponding sequence for A_T.

Lemma 2.29. *Let β denote the image of α under f_D^*. Then*

$$f_k^*\big({}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t)\big) = {}_\beta\mathrm{Eis}^k_{\mathbb{Q}_p}(t_T).$$

Moreover, if $\alpha = \alpha_c$, we have $\beta = \alpha_{A_T,c}$.

Proof. Since $\alpha \in \mathbb{Q}_p[D]^0 = \ker \varepsilon_D$, the compatibility of (2.17) along pullbacks implies that $\beta \in \mathbb{Q}_p[D_T]^0$. Let ${}_\beta\mathrm{pol}^{(k)}_{T,\mathbb{Q}_p}$ denote the polylogarithm for A_T with residue β. Then

$$f_U^*({}_\alpha\mathrm{pol}^k_{\mathbb{Q}_p}) = {}_\beta\mathrm{pol}^k_{T,\mathbb{Q}_p}$$

by uniqueness of these classes with respect to residues. It is easily seen from the proof of Corollary 2.12 that $\varrho^k_{t_T}$ is the pullback of ϱ^k_t along f once the functorial identifications are made. Combining this with the relation $t \circ f = f_U \circ t_T$, we see that $(\oplus_{i=1}^k f_i^*) \circ \varrho_t^k \circ t^* = \varrho^k_{t_T} \circ (t_T)^* \circ f_U^*$. So

$$\begin{aligned} f_k^*\big({}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t)\big) &= f_k^* \circ \mathrm{pr}^k \circ \varrho^k_t \circ t^*({}_\alpha\mathrm{pol}^k_{\mathbb{Q}_p}) \\ &= \mathrm{pr}^k \circ \varrho^k_{t_T} \circ t_T^* \circ f_A^*({}_\alpha\mathrm{pol}^k_{\mathbb{Q}_p}) = {}_\beta\mathrm{Eis}^k_{\mathbb{Q}_p}(t_T) \end{aligned}$$

which proves the first claim. For the second claim, note that since $f_D^* \circ \pi_D^* = \pi_{D_T}^* \circ f^*$ as maps $\mathrm{H}^0_{\text{ét}}(S,\mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D,\mathbb{Q}_p)$ and $f^*(1) = 1 \in \mathrm{H}^0_{\text{ét}}(T,\mathbb{Q}_p)$, we have $f_D^*(\pi_D^*(1)) = \pi^*_{D_T}(1)$. So it suffices to show that $f_D^* \circ e_{D,*}(1) = e_{D_T,*}(1)$. This is easily seen by moving to an étale cover of S on which D is trivialized. ■

The next result is an analogue of [11, Lemma 1.7 (2)]. Let $\varphi\colon A \to A'$ be a S-isogeny and D', U', $\widetilde{U}$, etc., be as in Section 2.4. For the result below, we assume that $s := \varphi \circ t \neq e'$ and that $\ker\varphi$ is a constant S-group scheme over a finite abelian group Γ, so that the structure map $f\colon \ker\varphi \to S$ is identified with $\sqcup_{\gamma\in\Gamma}\mathrm{id}_S$. For any $\gamma \in \Gamma$, we denote $e_\gamma\colon S \to \ker\varphi$ the section indexed by γ and set $t_\gamma := t + i \circ e_\gamma$ where $i\colon\ker\varphi \to A$ denotes the inclusion. Finally, let

$$\mathrm{Sym}^k\varphi_*\colon \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathcal{H}_{A,\mathbb{Q}_p})(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathcal{H}_{A',\mathbb{Q}_p})(d)\big)$$

denote the map induced by φ.

Lemma 2.30. *Let $\beta \in \mathbb{Q}_p[D']^0$ denote the image of α under $\varphi_{D,*}$. Then*

$$ {}_\beta\mathrm{Eis}^k_{\mathbb{Q}_p}(s) = \sum_{\gamma\in\Gamma} \mathrm{Sym}^k\varphi_*\big({}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t_\gamma)\big).$$

Proof. As $s \neq e'$ is N-torsion and $(N,c) = 1$, s factors via U' and therefore each t_γ factors via $\widetilde{U}$. Let $\tau\colon\ker\varphi \to \widetilde{U}$ be the morphism which equals t_γ on the component indexed by γ. We claim that the diagram

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\ \widetilde{\varphi}\ } & U' \\ {\scriptstyle\tau}\uparrow & & \uparrow{\scriptstyle s} \\ \ker\varphi & \xrightarrow{\ f\ } & S \end{array} \tag{2.31}$$

is Cartesian in **Sch**. For this, we may replace $\widetilde{\varphi}$ with $\varphi\colon A \to A'$. So let $p\colon X \to A, q\colon X \to S$ in **Sch** be such that $\varphi \circ p = s \circ q$. Assume wlog that X is connected. Since $\pi \circ p = \pi' \circ \varphi \circ p = \pi' \circ s \circ q = q$, p is an S-morphism. Since $\varphi \circ (p - t \circ q) = s \circ q - s \circ q = 0$, there is a unique S-morphism $\kappa\colon X \to \ker\varphi$ such that $p - t \circ q = i \circ \kappa$. Since X is connected, there is a unique $\delta \in \Gamma$ such that κ factors as $X \xrightarrow{q} S \xrightarrow{e_\delta} \ker\varphi$. So $p = i \circ \kappa + t \circ q = t_\delta \circ q = \tau \circ \kappa$ and the universal property is verified.

Since (2.31) is Cartesian, we have base change isomorphisms $\sqcup_\gamma s^* = f^* s^* \simeq \tau^* \widetilde{\varphi}^!$ and $s^* R\widetilde{\varphi}_* \xrightarrow{\sim} Rf_*\tau^*$. In particular,

$$Rs_*s^*R\widetilde{\varphi}_*\widetilde{\varphi}^! \xrightarrow{\sim} R(sf)_*(sf)^* = \sqcup_\gamma Rs_*s^*.$$

Denoting $\mathcal{F}_U := \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)$ and $\mathcal{F}_{U'} := \mathcal{L}og^{(k)}_{U',\mathbb{Q}_p}(d)$, the stated isomorphisms yield a commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2d-1}_{\text{ét}}(U,\mathcal{F}_U) & \xrightarrow{\jmath^*\circ\varphi_\#} & \mathrm{H}^{2d-1}_{\text{ét}}(\widetilde{U},\widetilde{\varphi}^*\mathcal{F}_{U'}) & \xrightarrow{\widetilde{\varphi}_*} & \mathrm{H}^{2d-1}_{\text{ét}}(U',\mathcal{F}_{U'}) \\
\downarrow{\scriptstyle \oplus t_\gamma^*} & & \downarrow{\scriptstyle \oplus s^*} & & \downarrow{\scriptstyle s^*} \\
\bigoplus_\gamma \mathrm{H}^{2d-1}_{\text{ét}}(S,t_\gamma^*\mathcal{F}_U) & \xrightarrow{(t_\gamma^*\varphi_\#)_\gamma} & \bigoplus_\gamma \mathrm{H}^{2d-1}_{\text{ét}}(S,s^*\mathcal{F}_{U'}) & \xrightarrow{\Sigma} & \mathrm{H}^{2d-1}_{\text{ét}}(S,s^*\mathcal{F}_{U'}),
\end{array}$$

where maps in the right square are induced by various adjunction transformations applied to $\mathcal{L}og^{(k)}_{U',\mathbb{Q}_p}(d)$, e.g., the middle vertical arrow is induced by $R\widetilde{\varphi}_*\widetilde{\varphi}^! \to Rs_*s^*R\widetilde{\varphi}_*\widetilde{\varphi}^!$. The composition given by the top row is just $\mathcal{N}_\varphi$ by the discussion in Section 2.4. So by Proposition 2.26, it suffices to show that for each $\gamma \in \Gamma$, the composition $\varrho^k_s \circ t_\gamma^*\varphi_\# \circ (\varrho^k_{t_\gamma})^{-1}$ is equal to $\mathrm{Sym}^k\varphi_*$ on the k-th symmetric power. But this composition is easily seen to be $e^*\varphi_\#$ once the maps at the ends are written in terms of an isogeny $\psi\colon A' \to A''$ that annihilates s (see the proof Corollary 2.12). That claim then follows by Proposition 2.11. ∎

2.6. Interpolation

In this subsection, we recall the definition of integral logarithm sheaf and state Kings' result on the interpolation of Eisenstein classes. For proofs, we refer the reader to [16, §4–6].

Let $\pi\colon A \to S$ be as in Section 2.2. Recall (2.3) that for each positive integer r, $\mathcal{H}_r := \mathcal{H}_{\Lambda_r}$ is isomorphic to the representable sheaf associated with p^r-torsion subscheme $X_r := A[p^r]$. Let $\mathrm{pr}_r\colon X_r \to S$ denote the structure map and set

$$\Lambda_r[\mathcal{H}_r] = \Lambda_r[X_r] := \mathrm{pr}_{r,*}\Lambda_r \in \mathbf{Sh}(S_{\text{ét}}). \tag{2.32}$$

Then $\Lambda_r[\mathcal{H}_r]$ is a sheaf of (abelian) group algebras over Λ_r with product $\mathcal{H}_r \times \mathcal{H}_r \to \mathcal{H}_r$ induced by the group structure on X_r. More generally, let $t\colon S \to A$ be a torsion section and let $\mathrm{pr}_r\colon X_r\langle t\rangle \to S$ be defined by the pullback diagram

$$\begin{array}{ccc}
X_r\langle t\rangle & \longrightarrow & A_r \\
\downarrow{\scriptstyle \mathrm{pr}_r} & & \downarrow{\scriptstyle [p^r]} \\
S & \xrightarrow{t} & A,
\end{array}$$

where $A_r := A$ considered as a finite étale cover of A. We denote by $\mathcal{H}_r\langle t\rangle$ the representable sheaf corresponding to $X_r\langle t\rangle$ and define

$$\Lambda_r\big[\mathcal{H}_r\langle t\rangle\big] = \Lambda_r\big[X_r\langle t\rangle\big] := \mathrm{pr}_{r,*}\Lambda_r \in \mathbf{Sh}(S_{\text{ét}}). \tag{2.33}$$

Then $\Lambda_r[\mathcal{H}_r\langle e\rangle] = \Lambda_r[\mathcal{H}_r]$. Since the cover A_r is an X_r-torsor on A, $X_r\langle t\rangle$ is an X_r-torsor over S. This makes $\Lambda_r[\mathcal{H}_r\langle t\rangle]$ a sheaf of modules over $\Lambda_r[\mathcal{H}_r]$. Let

$$\lambda_r = \lambda_r\langle t\rangle\colon X_{r+1}\langle t\rangle \to X_r\langle t\rangle$$

be the map induced by the universal property of $X_r\langle t\rangle$ applied to $X_{r+1}\langle t\rangle \to A_{r+1} \xrightarrow{[p]} A_r$ and pr_{r+1}. Then $(X_r\langle t\rangle)_r$ forms a pro-system of finite étale covers of S. The adjunction $\lambda_{r,*} = \lambda_{r,!} \dashv \lambda_r^! = \lambda_r^*$ (by étaleness of λ_r) gives us a map $\lambda_{r,*}\Lambda_{r+1} = \lambda_{r,!}\lambda_r^!\Lambda_{r+1} \to \Lambda_{r+1}$ and post composing with $\mathrm{pr}_{r,*}$ gives us a map

$$\Lambda_{r+1}\big[\mathcal{H}_{r+1}\langle t\rangle\big] = \mathrm{pr}_{r,*}\lambda_{r,*}\,\Lambda_{r+1} \to \mathrm{pr}_{r,*}\,\Lambda_{r+1}.$$

Reducing modulo p^r, we obtain an induced "trace" map

$$\mathrm{Tr}_r\colon \Lambda_{r+1}\big[\mathcal{H}_{r+1}\langle t\rangle\big] \to \Lambda_r\big[\mathcal{H}_r\langle t\rangle\big] \tag{2.34}$$

which is compatible with the underlying module structures.

Definition 2.35. The *sheaf of Iwasawa algebras of* $\mathcal{H}$ *on* S is defined to be the pro-system $\Lambda(\mathcal{H}) := (\Lambda_r[\mathcal{H}_r])_{r\geq 1}$ with transition maps given by (2.34) for $t = e$. The *sheaf of Iwasawa modules associated with* t is defined to be the pro-system $(\Lambda_r[\mathcal{H}_r\langle t\rangle])_{r\geq 1}$.

For each non-negative integer k, let $\Gamma_k(\mathcal{H}_r)$ denote the sheafification of the presheaf that sends an open subscheme $U \subset S$ to the k-th divided power algebra $\Gamma_k(\mathcal{H}_r(U))$. Then the reduction maps $\mathcal{H}_{r+1} \to \mathcal{H}_r$ induce isomorphisms $\Gamma_k(\mathcal{H}_{r+1}) \otimes_{\mathbb{Z}/p^{r+1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z} \simeq \Gamma_k(\mathcal{H}_r)$ and we obtain a $\mathbb{Z}_p$-sheaf $\Gamma_k(\mathcal{H}) := (\Gamma_k(\mathcal{H}_r))_{r\geq 1}$. There is a canonical map

$$\gamma_k\colon \mathrm{Sym}^k(\mathcal{H}) \to \Gamma_k(\mathcal{H}) \tag{2.36}$$

induced by sending $m^{\otimes k} \in \mathrm{Sym}^k(\mathcal{H})$ to $k!m^{[k]}$ for m a section of $\mathcal{H}$. It induces an isomorphism $\mathrm{Sym}^k(\mathcal{H}) \otimes \mathbb{Q}_p \to \Gamma_k(\mathcal{H}) \otimes \mathbb{Q}_p \simeq \Gamma_k(\mathcal{H}_{\mathbb{Q}_p})$ between the corresponding $\mathbb{Q}_p$-sheaves.

The sheaf $\mathcal{H}_r \in \mathbf{Sh}(S_{\text{ét}})$ possesses over $X_r \in S_{\text{ét}}$ a tautological section $\tau_r \in \Gamma(X_r, \mathcal{H}_r) = \mathcal{H}_r(X_r) = \mathrm{Hom}_S(X_r, X_r)$ corresponding to the identity map $X_r \to X_r$. Its k-th divided power gives rise to a section $\tau_r^{[k]} \in \Gamma(X_r, \Gamma_k(\mathcal{H}_r))$. Let $\Gamma_k(\mathcal{H}_r)_{|X_r} := \mathrm{pr}_r^*\,\Gamma_k(\mathcal{H}_r)$ denote the restriction of $\Gamma_k(\mathcal{H}_r)$ to X_r. Then

$$\begin{aligned}\Gamma\big(X_r, \Gamma_k(\mathcal{H}_r)\big) &= \mathrm{Hom}_{X_r}\big(\Lambda_r, \Gamma_k(\mathcal{H}_r)_{|X_r}\big) \simeq \mathrm{Hom}_S\big(\mathrm{pr}_{r,!}\Lambda_r, \Gamma_k(\mathcal{H}_r)\big)\\ &\simeq \mathrm{Hom}_S\big(\Lambda_r[\mathcal{H}_r], \Gamma_k(\mathcal{H}_r)\big),\end{aligned}$$

where the penultimate isomorphism follows via the adjunction $\mathrm{pr}_{r,!} \dashv \mathrm{pr}_r^! = \mathrm{pr}_r^*$ and the last by the identification $\mathrm{pr}_{r,!} = \mathrm{pr}_{r,*}$. Thus $\tau_r^{[k]}$ corresponds to a morphism

$$\mathrm{mom}_r^k\colon \Lambda_r[\mathcal{H}_r] \to \Gamma_k(\mathcal{H}_r). \tag{2.37}$$

For fixed k and varying r, the maps mom_r^k are compatible with respect to

$$\mathrm{Tr}_r\colon \Lambda_{r+1}[\mathcal{H}_{r+1}] \to \Lambda_r[\mathcal{H}_r]$$

and the reduction maps $\Gamma_k(\mathcal{H}_{r+1}) \to \Gamma_k(\mathcal{H}_{r+1}) \otimes_{\Lambda_{r+1}} \Lambda_r \simeq \Gamma_k(\mathcal{H}_r)$ [16, Lemma 4.5.1].

Definition 2.38. The *k-th moment map* is defined to be the morphism $\mathrm{mom}^k\colon \Lambda(\mathcal{H}) \to \Gamma_k(\mathcal{H})$ of pro-systems obtained by the compatible system $(\mathrm{mom}_r^k)_{r\ge 1}$ given in (2.37).

Parallel to the construction of $\Lambda(\mathcal{H})$ is the construction of a pro-sheaf on A that is the integral analogue of the $\mathbb{Q}_p$-logarithm pro-sheaf. Let $[p^r]\colon A_r \to A$ be the X_r-torsor over A as above. Denote by $\mu_r\colon A_{r+1} \to A_r$ the transition map induced by $[p]\colon A \to A$. Then we have a pro-system $(A_r)_{r\ge 1}$ of finite étale covers on A. For $s \ge 1$, let

$$\Lambda_s[A_r] := [p^r]_*\Lambda_s \in \mathbf{Sh}(A_{\text{ét}}).$$

Then as above, we get morphisms

$$\mu_{r,s}\colon \Lambda_s[A_{r+1}] \to \Lambda_s[A_r] \quad \text{for all } s, r.$$

If $s = r$, the pre-composition of this map with reduction modulo p^r gives a “trace” map $\mathrm{Tr}_r\colon \Lambda_{r+1}[A_{r+1}] \to \Lambda_r[A_r]$ as in (2.34).

Definition 2.39. The *integral logarithm sheaf* $\mathcal{L}$ is defined to be the pro-system

$$\big(\Lambda_r[A_r]\big)_{r\ge 1} \in \mathbf{Sh}(A_{\text{ét}})^{\mathbb{N}}$$

with transition maps given by Tr_r.

Via the X_r-torsor $[p^r]\colon A_r \to A$, $\mathcal{L}$ becomes a free rank 1 module over $\pi^*\Lambda(\mathcal{H})$. The base change compatibility of such pro-systems [16, §4.4] implies that $\mathcal{L}$ is compatible with arbitrary base change. Consequently, there is an isomorphism

$$\varsigma_t\colon t^*\mathcal{L} \simeq \Lambda\big(\mathcal{H}\langle t\rangle\big) \tag{2.40}$$

of sheaf of modules over $\Lambda(\mathcal{H})$. In particular, $e^*\mathcal{L} \simeq \Lambda(\mathcal{H})$.

The integral logarithm sheaf enjoys properties similar to those of $\mathcal{L}\mathrm{og}_{\mathbb{Q}_p}$. Below we record the ones needed to state Kings' result.

Proposition 2.41 (Splitting principle). *Let n be a positive integer and $t\colon S \to A$ be an n-torsion section. Then there exists a canonical homomorphism $[n]_\#\colon t^*\mathcal{L} \to \Lambda(\mathcal{H})$ which is an isomorphism if $(n, p) = 1$.*

Corollary 2.42. *Let c be an integer prime to p and $D = A[c]$. Then there exists a canonical isomorphism $\iota_D^*\mathcal{L} \simeq \pi_D^*\Lambda(\mathcal{H})$ where ι_D, π_D are as in (2.14).*

Proposition 2.43 (Vanishing of cohomology). *Let r, s be non-negative integers. There exist natural isomorphisms $R^{2d}\pi_*(\Lambda_s[A_r]) \simeq \Lambda_s(-d)$ which are compatible with respect to $\mu_{r,s}$ and reduction modulo p^{s-1}. For each $i = 0, \ldots, 2d-1$, there exist a sufficiently large integer r' such that $R^i\pi_*\Lambda_s[A_{r'}] \to R^i\pi_*\Lambda_s[A_r]$ is zero. In particular,*

$$\mathrm{H}^i_{\text{ét}}\big(A, \mathcal{L}(d)\big) \simeq \begin{cases} 0 & \text{if } i < 2d \\ \mathrm{H}^0_{\text{ét}}(S, \mathbb{Z}_p) & \text{if } i = 2d. \end{cases}$$

Remark 2.44. By [10, Proposition 1.6] and above, $\mathrm{H}^i_{\text{ét}}(A, \mathcal{L}(d)) = \varprojlim_r \mathrm{H}^i_{\text{ét}}(A, \Lambda_r[A_r])$ where the limit involves ordinary étale cohomology groups (or continuous groups with constant pro-systems $\Lambda_r[A_r]$).

Fix as before an integer $c > 1$ that is invertible on S but which is now also prime to p. We retain the notations introduced in diagram (2.14). Repeating the same argument as in Section 2.3 and invoking Proposition 2.43, we find an exact sequence

$$0 \to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big) \xrightarrow{\text{res}} \mathrm{H}^0_{\text{ét}}(D, \iota^*\mathcal{L}) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p), \tag{2.45}$$

where res is again referred to as the *residue* map. By Corollary 2.42, we may replace $\mathrm{H}^0_{\text{ét}}(D, \iota^*\mathcal{L})$ with $\mathrm{H}^0_{\text{ét}}(D, \pi_D^*\Lambda(\mathcal{H}))$. Since we have a section $\mathbf{1}\colon \mathbb{Z}_p \to \Lambda(\mathcal{H})$ that is induced by sending $1 \in \mathbb{Z}/p^r\mathbb{Z}$ to the identity in $\Lambda_r[A_r]$, there exists a corresponding inclusion $\mathrm{H}^0(D, \mathbb{Z}_p) \to \mathrm{H}^0_{\text{ét}}(D, \pi_D^*\Lambda(\mathcal{H}))$. Let $\mathbb{Z}_p[D]^0$ denote the kernel of the trace map

$$\mathrm{H}^0_{\text{ét}}(D, \mathbb{Z}_p) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Z}_p).$$

Then $\mathbb{Z}_p[D]^0$ lies in the image of residue map for the same reasons as in Section 2.3. Note that the class α_c of (2.23) is a member of $\mathbb{Z}_p[D]^0$.

Definition 2.46. Let $\alpha \in \mathbb{Z}_p[D]^0$. The *integral étale polylogarithm with residue* α is the unique class

$$_\alpha\mathrm{pol}_{\mathbb{Z}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big)$$

such that $\mathrm{res}({}_\alpha\mathrm{pol}_{\mathbb{Z}_p}) = \alpha$.

Fix now an integer $N > 1$ prime to c and invertible on S. Let $t\colon S \to U$ be an N-torsion section. By the adjunction $\mathrm{id} \to Rt_*t^*$, isomorphism (2.40) and Proposition 2.41, we obtain a composition

$$\begin{aligned} \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big) &\xrightarrow{t^*} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}(d)\big) \underset{\sim}{\xrightarrow{\varsigma_t}} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathcal{H})\langle t\rangle(d)\big) \\ &\xrightarrow{[N]_\#} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathcal{H})(d)\big). \end{aligned} \tag{2.47}$$

Definition 2.48. The *Iwasawa-Eisenstein class*

$$_\alpha\mathcal{EI}_N(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathcal{H})(d)\big)$$

associated with t, N and α is defined to be image of $_\alpha\mathrm{pol}_{\mathbb{Z}_p}$ under the composition $[N]_\# \circ \varsigma_t \circ t^*$ of (2.47).

The following result of Kings shows that the classes just defined interpolate rational Eisenstein classes. Let $\mathrm{mom}^k_{\mathbb{Q}_p}$ denote the composition

$$\begin{aligned}\mathrm{H}^{2d-1}_{\text{ét}}\big(S,\Lambda(\mathcal{H})(d)\big) &\xrightarrow{\mathrm{mom}^k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S,\Gamma_k(\mathcal{H})(d)\big)\\ &\xrightarrow{-\otimes\mathbb{Q}_p} \mathrm{H}^{2d-1}_{\text{ét}}\big(S,\Gamma_k(\mathcal{H})(d)\big)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\\ &\xrightarrow[\sim]{\sigma_k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S,\mathrm{Sym}^k\mathcal{H}_{\mathbb{Q}_p}(d)\big),\end{aligned}$$

where the first map is induced by the moment map of Definition 2.38 and σ_k is induced by (2.36).

Theorem 2.49 ([16, Theorem 6.3.3]). *For $\alpha\in\mathbb{Z}_p[D]^0$ and $N>1$ an integer relatively prime to c and invertible on S, the k-th rational Eisenstein class ${}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t)$ along a non-zero N-torsion section $t\colon S\to A$ satisfies*

$$\mathrm{mom}^k_{\mathbb{Q}_p}\big({}_\alpha\mathcal{EI}_N(t)\big)=N^k\,{}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t).$$

In particular, $N^k{}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t)$ lies in the image of $\mathrm{H}^{2d-1}_{\text{ét}}(S,\Gamma_k(\mathcal{H})(d))$ under $\sigma_k\circ(-\otimes\mathbb{Q}_p)$.

3. Siegel modular varieties

In this section, we recall the definition and moduli interpretation of Siegel modular varieties with principal level structures. Since Eisenstein classes of Definition 2.22 are a priori only defined for scheme theoretic sections, we need to spell out the effect of various maps between moduli varieties in a non-adelic fashion.

3.1. The Shimura data

For $n\geq 1$ an integer, let I_n denote the $n\times n$ identity matrix. Let

$$J=J_n=\begin{pmatrix} & I_n\\ -I_n & \end{pmatrix}\in\mathrm{Mat}_{2n\times 2n}(\mathbb{Z})$$

be the standard symplectic matrix and $\mathbf{G}=\mathrm{GSp}_{2n}$ denote the reductive group scheme over $\mathbb{Z}$ whose R points for a ring R are given by

$$\mathbf{G}(R)=\big\{g\in\mathrm{GL}_{2g}(R)\mid g^tJg=c(g)J\text{ for }c(g)\in R^\times\big\}.$$

The induced homomorphism $c\colon\mathrm{GSp}_{2g}\to\mathbb{G}_m$ is called the *similitude*. The center of $\mathbf{G}$ is denoted by $\mathbf{Z}$ which is identified with $\mathbb{G}_m$ via diagonal matrices. Let $\mathbb{S}:=\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ denote the Deligne torus and define

$$h_{\mathrm{std}}\colon\mathbb{S}\to\mathbf{G}_{\mathbb{R}}\quad(a+b\sqrt{-1})\mapsto\begin{pmatrix}aI_n & bI_n\\ -bI_n & aI_n\end{pmatrix}.$$

Let $\mathcal{X}$ denote the $\mathbf{G}(\mathbb{R})$-conjugacy class of h_{std}. Then $(\mathbf{G}_{\mathbb{Q}}, \mathcal{X})$ satisfies axioms SV1–SV6 of [22] and in particular, constitutes a Shimura datum. Its reflex field is $\mathbb{Q}$. For a neat compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathrm{Sh}(K) = \mathrm{Sh}_{\mathbf{G}}(\mathcal{X}, K)$ denote the corresponding canonical model over $\mathbb{Q}$. It is a smooth quasi-projective variety of dimension $n(n+1)/2$ whose $\mathbb{C}$-points are identified with the double quotient $\mathbf{G}(\mathbb{Q}) \setminus [\mathcal{X} \times \mathbf{G}(\mathbb{A}_f)]/K$.

Let $(V_{\mathbb{Z}}, \psi)$ denote the standard symplectic $\mathbb{Z}$-module where $V_{\mathbb{Z}} := \mathbb{Z}^{2n}$ and $\psi\colon V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathbb{Z}$ is the pairing induced by J. We let $e_1, \dots, e_{2n}$ denote the standard basis of $V_{\mathbb{Z}}$. For any commutative ring R with unity, we denote $V_R := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ and $V_{\mathbb{A}_f}$ simply as V_f. We will view elements of V_R as column vectors and let $\mathbf{G}(R)$ act on V_R by left matrix multiplication. For each positive integer N, let

$$K_N := \left\{ g \in \mathbf{G}(\mathbb{A}_f) \mid (g-1)V_{\widehat{\mathbb{Z}}} \subset N V_{\widehat{\mathbb{Z}}} \right\}$$

be the *principal congruence subgroup* of $\mathbf{G}(\mathbb{A}_f)$ of level N. These subgroups form a base for the topology of $\mathbf{G}(\mathbb{A}_f)$ at identity and $K_N \trianglelefteq K_1$ for all $N \geq 1$. The quotient $\mathbf{G}(\mathbb{A}_f)/K_1$ is identified with the set of self-dual $\widehat{\mathbb{Z}}$-lattices in V_f by identifying the coset $gK_1 \in \mathbf{G}(\mathbb{A}_f)/K_1$ with the lattice $gV_{\widehat{\mathbb{Z}}}$. More generally, $\mathbf{G}(\mathbb{A}_f)/K_N$ is identified with the set of pairs $(\widehat{H}, \overline{\eta})$ where $\widehat{H} \subset V_f$ is a self-dual $\widehat{\mathbb{Z}}$-lattice and $\overline{\eta}\colon V_{\mathbb{Z}}/NV_{\mathbb{Z}} \xrightarrow{\sim} \widehat{H}/N\widehat{H}$ is a choice of a symplectic isomorphism.

3.2. Moduli interpretation

For $N \geq 3$, $\mathrm{Sh}(K_N)$ is the $\mathbb{Q}$-scheme representing the following moduli problem. Let $\mathbf{Sch}_{\mathbb{Q}}$ denote the category of locally Noetherian $\mathbb{Q}$-schemes and let $\mathfrak{M}_N\colon \mathbf{Sch}_{\mathbb{Q}} \to \mathbf{Sets}$ denote the contravariant functor that sends a $\mathbb{Q}$-scheme S to isomorphism classes of triples (A, λ, η) where

- A is an abelian scheme over S of relative dimension n,
- $\lambda\colon A \xrightarrow{\sim} A^\vee$ is a principal polarization,
- $\eta\colon (V_{\mathbb{Z}}/NV_{\mathbb{Z}})_S \xrightarrow{\sim} A[N]$ is a symplectic similitude of group schemes over S, i.e., η is an isomorphism of S-group schemes such that the Weil pairing on $A[N]$ corresponds to a $(\mathbb{Z}/N\mathbb{Z})^\times$-multiple of the pairing ψ after fixing an identification of $(\mathbb{Z}/N\mathbb{Z})^\times$ with the roots of unity μ_N.

The isomorphism η is also referred to as a *principal level N structure*. Given a morphism $f\colon T \to S$ of schemes, the morphism $\mathfrak{M}_N(S) \to \mathfrak{M}_N(T)$ is given by pullback of families. To say that $\mathfrak{M}_N$ is represented by $\mathrm{Sh}(K_N)$ is to say that there exists a natural isomorphism

$$\Psi_N\colon \mathfrak{M}_N \to \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{Q}}}\big(-, \mathrm{Sh}(K_N)\big)$$

of functors on $\mathbf{Sch}_{\mathbb{Q}}$. Abusing notation, we denote by $\mathrm{id}_N \in \mathfrak{M}_N(\mathrm{Sh}(K_N))$ the isomorphism class that corresponds to the identity map $\mathrm{id}_N\colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_N)$. By definition, there is an abelian scheme $\pi_N\colon \mathcal{A}_N \to \mathrm{Sh}(K_N)$ with principal polarization $\lambda_N^{\mathrm{univ}}$ and principal level N-structure η_N^{univ} such that the isomorphism class of $(\mathcal{A}_N, \lambda_N^{\mathrm{univ}}, \eta_N^{\mathrm{univ}})$ equals id_N. It is referred to as the *universal family* on $\mathrm{Sh}(K_N)$. We fix such a family once and for all for each $N \geq 3$. For $v \in V_{\widehat{\mathbb{Z}}}$, we let

$$t_{v,N}\colon \mathrm{Sh}(K_N) \to \mathcal{A}_N$$

denote the canonical N-torsion section that corresponds under η_N^{univ} to the class of v in $V_{\widehat{\mathbb{Z}}}/NV_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}}/NV_{\mathbb{Z}}$.

Let $M, N \geq 3$ be integers such that $M|N$ and let $g \in \mathbf{G}(\mathbb{A}_f)$ be an element such that $K_N \subset gK_M g^{-1}$ that we fix for the rest of this paragraph. There is a finite étale map $[g]_{N,M}\colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_M)$ of $\mathbb{Q}$-schemes given on complex points by right multiplication by g in the second component. It induces a natural transformation

$$\mathrm{Hom}_{\mathbf{Sch}_{\mathbb{Q}}}\big(-, \mathrm{Sh}(K_N)\big) \to \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{Q}}}\big(-, \mathrm{Sh}(K_M)\big),$$

corresponding to which there is a natural transformation

$${}^{g}\Phi_{N,M}\colon \mathfrak{M}_N \to \mathfrak{M}_M.$$

We explicitly describe the effect of ${}^{g}\Phi_{N,M}$ for certain g, N, M (cf. [18, p. 9]). Assume that g is such that $g(dV_{\widehat{\mathbb{Z}}}) \subset V_{\widehat{\mathbb{Z}}} \subset gV_{\widehat{\mathbb{Z}}}$ where $d := N/M$. Let $\iota\colon V_{\mathbb{Z}}/MV_{\mathbb{Z}} \hookrightarrow V_{\widehat{\mathbb{Z}}}/g(NV_{\widehat{\mathbb{Z}}})$ be the inclusion given by

$$\iota\colon V_{\mathbb{Z}}/MV_{\mathbb{Z}} = dV_{\widehat{\mathbb{Z}}}/NV_{\widehat{\mathbb{Z}}} \xrightarrow[\sim]{g\cdot} g(dV_{\widehat{\mathbb{Z}}})/g(NV_{\widehat{\mathbb{Z}}}) \hookrightarrow V_{\widehat{\mathbb{Z}}}/g(NV_{\widehat{\mathbb{Z}}})$$

and let $\gamma\colon V_{\mathbb{Z}}/NV_{\mathbb{Z}} \to V_{\mathbb{Z}}/NV_{\mathbb{Z}}$ be the symplectic endomorphism induced by $v \mapsto g^{-1}v$ for $v \in V_{\widehat{\mathbb{Z}}}$ (which is well defined since $g^{-1}V_{\widehat{\mathbb{Z}}}$ is contained in $V_{\widehat{\mathbb{Z}}}$). Note that

$$\ker\gamma = g(NV_{\widehat{\mathbb{Z}}})/NV_{\widehat{\mathbb{Z}}} \quad \text{and} \quad (V_{\mathbb{Z}}/NV_{\mathbb{Z}})/\ker\gamma \simeq V_{\widehat{\mathbb{Z}}}/g(NV_{\widehat{\mathbb{Z}}}).$$

Thus $V_{\mathbb{Z}}/MV_{\mathbb{Z}}$ embeds into the coimage of γ via ι. We note that $\ker\gamma$ is a totally isotropic subspace of $V_{\mathbb{Z}}/NV_{\mathbb{Z}}$ with respect to the induced symplectic pairing and its cardinality equals k^n for some positive integer $k = k_\gamma$. Now let $S \in \mathbf{Sch}_{\mathbb{Q}}$ and $(A_N, \lambda_N, \eta_N) \in \mathfrak{M}_N(S)$ be (a triple representing) an isomorphism class. Let C_γ be the finite flat subgroup scheme of A_N (over S) given by the image of $(\ker\gamma)_S$ under η_N and let $\overline{\eta}_N$: $V_{\widehat{\mathbb{Z}}}/g(NV_{\widehat{\mathbb{Z}}})_S \to A_N/C_\gamma$ be the embedding on quotients induced by η_N. By ι_S, we denote the morphism of constant group schemes over S determined by ι. Consider the triple $({}^{g}A_M, {}^{g}\lambda_M, {}^{g}\eta_M)$ where

- ${}^{g}A_M$ is obtained by the quotient map $\psi_\gamma\colon A_N \to A_N/C_\gamma$,
- ${}^{g}\lambda_M\colon {}^{g}A_M \to {}^{g}A^\vee$ is the unique principal polarization satisfying $[k] \circ \lambda_N = \psi_\gamma^\vee \circ {}^{g}\lambda_M \circ \psi_\gamma$ (see [21, Proposition 13.8]),
- ${}^{g}\eta_M\colon (V_{\mathbb{Z}}/MV_{\mathbb{Z}})_S \xrightarrow{\sim} {}^{g}A_M[M]$ is given by the composition $\overline{\eta}_N \circ \iota_S$.

Then $({}^{g}A_M, {}^{g}\lambda_M, {}^{g}\eta_M)$ represents the isomorphism class ${}^{g}\Phi_{N,M}(A_N, \lambda_N, \eta_N) \in \mathfrak{M}_M(S)$.

It will be useful to note a few special cases of the aforementioned description. First, if g is identity, the corresponding map on triples is given by “forgetting” the level N

structure, i.e., by restricting η_N to $dV_{\mathbb{Z}}/NV_{\mathbb{Z}}$. Second, if $g = \kappa \in K_1$ and $N = M$, the class of a triple (A_N, λ_N, η_N) is sent to the class of $(A_N, \lambda_N, \eta_N \circ \kappa)$. This induces a right action of $K_1 = \mathrm{GSp}_{2n}(\widehat{\mathbb{Z}})$ on $\mathfrak{M}_N(S)$. Third, if g is an element of the center $\mathbf{Z}(\mathbb{Q})$ (satisfying the conditions of the discussion), the isogeny $\psi_\gamma\colon A_N \to {}^g A_M$ factors via an isomorphism $A_N \simeq {}^g A_M$ since C_γ is the kernel of $[k]\colon A_N \to A_N$ and $A_N/\ker[k] \simeq A_N$. Using this, we see that the map ${}^g\Phi_{N,M}$ for $g \in \mathbf{Z}(\mathbb{Q})$ is again the forgetful one. This can also be seen directly by the complex uniformization of these varieties.

We apply the above description to universal families and record some observations. For an arbitrary $g \in \mathbf{G}(\mathbb{A}_f)$ satisfying $K_N \subset gK_M g^{-1}$, there is a triple

$$({}^g\mathcal{A}_M, {}^g\lambda_M, {}^g\eta_M) \in \mathfrak{M}_M\big(\mathrm{Sh}(K_N)\big)$$

corresponding to $[g]_{N,M} \in \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{Q}}}(\mathrm{Sh}(K_N), \mathrm{Sh}(K_M))$ under Ψ_M. By definition, this triple is obtained by pulling back the universal family on $\mathrm{Sh}(K_M)$ along $[g]_{N,M}$. But $[g]_{N,M} = \Psi_M \circ {}^g\Phi_{N,M}(\mathrm{id}_N)$. The preceding discussion implies that when $g \in \mathbf{G}(\mathbb{A}_f)$ is such that $(dV_{\widehat{\mathbb{Z}}}) \subset V_{\widehat{\mathbb{Z}}} \subset gV_{\widehat{\mathbb{Z}}}$, there is an isogeny

$$ {}^g\psi_{N,M} : \mathcal{A}_N \to {}^g\mathcal{A}_M \tag{3.1}$$

over $\mathrm{Sh}(K_N)$ such that the tautological section $[g]^*_{N,M}(t_{v,M})\colon \mathrm{Sh}(K_N) \to {}^g\mathcal{A}_M$ induced by $t_{v,M}$ is ${}^g\psi_{N,M} \circ t_{gdv,N}$. When $g = \kappa \in K_1$ in particular (so that $K_N \subset K_M$), the discussion above gives us a pullback diagram

$$\begin{array}{ccc} \mathcal{A}_N & \longrightarrow & \mathcal{A}_M \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ \mathrm{Sh}(K_N) & \xrightarrow{\kappa_{N,M}} & \mathrm{Sh}(K_M) \end{array} \tag{3.2}$$

and the tautological section of π_N induced by $t_{v,M}$ for $v \in V_{\widehat{\mathbb{Z}}}$ equals $t_{\kappa dv,N}$.

Remark 3.3. We note that a general $g \in \mathbf{G}(\mathbb{A}_f)$ can be written as $z_a h$ where $h \in \mathbf{G}(\mathbb{A}_f)$ satisfies $V_{\widehat{\mathbb{Z}}} \subset hV_{\widehat{\mathbb{Z}}}$ and $z_a \in \mathbf{Z}(\mathbb{Q}) \simeq \mathbb{Q}^\times$ is identified with a positive integer a. Given $M \geq 3$, we can find N a sufficiently large multiple of M such that $h(N/M)V_{\widehat{\mathbb{Z}}} \subset V_{\widehat{\mathbb{Z}}}$. Since $z_a\colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_N)$ is the identity morphism, the effect of ${}^g\Phi_{N,M}$ can be described by ${}^h\Phi_{N,M}$.

4. Parametrization

In this section, we prove our main result in Theorem 4.19. We first need some preliminaries however.

4.1. Symplectic Orbits

For this subsection, we use the notations introduced in Section 3.1. In particular, $\mathbf{G} = \mathrm{GSp}_{2n}$ denotes the symplectic $\mathbb{Z}$-group scheme of rank n.

Lemma 4.1. *Let R be a Euclidean domain. The map that sends an ideal of R to the $\mathbf{G}(R)$-orbit of $\alpha e_1 \in V_R$ for α a generator of the ideal establishes a bijection between the set of ideals of R and the orbit space $\mathbf{G}(R) \setminus V_R$.*

Proof. Pick a $v \in V_R$ and write $v = a_1e_1 + \cdots + a_{2n}e_{2n}$. Let $\alpha \in R$ be a generator of the ideal generated by $a_1, \ldots, a_n$. For any $x \in R$ and $1 \le i \le n$, let $A_{i,n+i}(x) \in \mathrm{Mat}_{2n\times 2n}(R)$ be the matrix that has 1's on the diagonal, x in the $(i, n+i)$ entry and 0's elsewhere. Then $A_{i,n+i}(x) \in \mathbf{G}(R)$ and the action of $A_{i,n+i}(x)$ on v replaces the i-th coordinate with $a_i + xa_{n+i}$ while keeping everything else the same. The matrices $B_{i,n+i}$ obtained by switching i-th and $(n+i)$-th row of the $2n \times 2n$ identity matrix also lie in $\mathbf{G}(R)$ and their action on v is given by switching the i-th and $(n+i)$-th coordinates. By using these matrices, we can apply the Euclidean algorithm to replace the i-th coordinate of v with a generator for (a_i, a_{n+i}) for all $1 \le i \le n$ and make the remaining entries of v equal to zero. Since $\mathbf{G}(R)$ also contains matrices of the form

$$\begin{pmatrix} A & \\ & (A^t)^{-1} \end{pmatrix}$$

for any $A \in \mathrm{GL}_n(R)$, we easily deduce that $\alpha e_1 \in \mathbf{G}(R)v$. Clearly $\alpha_1e_1, \alpha_2e_1 \in V_R$ are in the same $\mathbf{G}(R)$-orbit only if α_1 and α_2 generate the same ideal. ∎

For R as above and $v \in V_R$, let $I_v \subset R$ denote the ideal generated by its standard coordinates. For I an ideal of R, let $K_{R,I} \subset \mathbf{G}(R)$ the subgroup of elements γ such that $\gamma v \in v + IV_R$ for all $v \in V_R$.

Lemma 4.2. *Suppose that R is a discrete valuation ring and let ϖ be a uniformizer. Then for any $v \in V_R$ and ideals $I, J \subset R$,*

$$K_{R,I}v + JV_R = \begin{cases} I_vV_R \setminus (\varpi I_vV_R) & \text{if } I = R,\ I_v \supsetneq J, \\ v + (J + I\cdot I_v)V_R & \text{otherwise}. \end{cases}$$

Proof. First assume that $J = 0$. Then the case $I = R$ is Lemma 4.1, so we also assume that I is proper. Let $d_v \in I_v$ be a generator and $\kappa \in \mathbf{G}(R)$ such that $v = \kappa d_v e_1$. Since $K_{R,I}$ is normal in $\mathbf{G}(R)$, $K_{R,I}v = K_{R,I}(\kappa d_v e_1) = \kappa d_v(K_{R,I}e_1)$ and the claim is further reduced to the case $v = e_1$. But this holds since $K_{R,I}$ contains matrices whose first column is $[1 + a_1, a_2, \ldots, a_{2n}]^t$ for arbitrary $a_i \in I$. Now note that the cases for $J \neq 0$ follow easily from the corresponding ones for $J = 0$. ∎

Lemma 4.3. *Let M, N be positive integers such that $M|N$ and $v \in V_{\mathbb{Z}}$. Let $d_v \in \mathbb{Z}$ denote a generator of I_v, $b = \gcd(Md_v, N)$ and S the set of primes ℓ such that $d_v\mathbb{Z}_\ell \supsetneq N\mathbb{Z}_\ell$ and $\ell \nmid M$. Then*

$$K_Mv + NV_{\widehat{\mathbb{Z}}} = \prod_{\ell\in S}(d_vV_{\mathbb{Z}_\ell} \setminus \ell d_vV_{\mathbb{Z}_\ell}) \prod_{\ell\notin S}(v + bV_{\mathbb{Z}_\ell}).$$

Proof. Since $K_M = \prod_\ell K_{\mathbb{Z}_\ell, M\mathbb{Z}_\ell}$ and $v + NV_{\widehat{\mathbb{Z}}} = \prod_\ell(v + NV_{\mathbb{Z}_\ell})$, the claim follows by Lemma 4.2. ∎

Remark 4.4. If we write $N = Md$, the result above shows that

$$(V_{\mathbb{Z}}/NV_{\mathbb{Z}})^{K_M/K_N} = dV_{\mathbb{Z}}/NV_{\mathbb{Z}}.$$

4.2. RIC functors

The adelic distribution relations of Eisenstein classes are most conveniently described as a morphism between two *RIC functors* [24, §2] (cf. [6, §2]). To make the note more self-contained, we briefly recall the terminology and establish a basic result needed later on.

Fix for this subsection only a locally profinite group G and a non-empty collection Υ of compact open subgroups. We assume that Υ is closed under intersections, conjugation by elements of G and for every $K, L \in \Upsilon$, there exists a $K' \in \Upsilon$ such that $K' \subset L$ and $K' \triangleleft K$. For such an Υ, we associate a category $\mathcal{P}(G) = \mathcal{P}(G, \Upsilon)$ whose objects are elements of Υ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{P}(G)}(L, K) = \{g \in G \mid g^{-1}Lg \subset K\} \quad \text{for } L, K \in \Upsilon.$$

Elements of $\mathrm{Hom}_{\mathcal{P}(G)}(L, K)$ will be written as either $(L \xrightarrow{g} K)$ or $[g]_{L,K}$, and composition in $\mathcal{P}(G)$ is given by

$$(L \xrightarrow{g} K) \circ (L' \xrightarrow{h} L) = (L' \xrightarrow{h} L \xrightarrow{g} K) = (L' \xrightarrow{hg} K).$$

If e denotes the identity of G, the inclusion $(L \xrightarrow{e} K)$ will also denoted by $\mathrm{pr}_{L,K}$.

Definition 4.5. Let R be a commutative ring with identity. An *RIC functor M on* (G, Υ) *valued in R-Mod* is a pair of covariant functors

$$M^*\colon \mathcal{P}(G, \Upsilon)^{\mathrm{op}} \to R\text{-Mod}, \quad M_*\colon \mathcal{P}(G, \Upsilon) \to R\text{-Mod}$$

satisfying the following three conditions:

(C1) $M^*(K) = M_*(K)$ for all $K \in \Upsilon$. Denote this common R-module by $M(K)$.

(C2) For all $K \in \Upsilon$ and $g \in G$,

$$(gKg^{-1} \xrightarrow{g} K)^* = (K \xrightarrow{g^{-1}} gKg^{-1})_* \in \mathrm{Hom}_{R\text{-Mod}}\big(M(K), M(gKg^{-1})\big).$$

Here for a morphism $\phi \in \mathcal{P}(G)$, we denote $\phi_* := M_*(\phi)$ and $\phi^* := M^*(\phi)$.

(C3) $[\gamma]_{K,K,*}\colon M(K) \to M(K)$ is the identity map for all $K \in \Upsilon$ and $\gamma \in K$.

We will denote the RIC functor above simply as $M\colon \mathcal{P}(G, \Upsilon) \to R$-Mod. We refer to the maps ϕ^* (resp., ϕ_*) in axiom (C2) as the *pullback* (resp., *pushforward*) induced by ϕ. If moreover the element of G underlying the morphism ϕ is e, we also refer to $\phi^* = \mathrm{pr}^*$ (resp., $\phi_* = \mathrm{pr}_*$) as a *restriction* (resp., an *induction*).

Definition 4.6. A *morphism* $\varphi\colon M_1 \to M_2$ between two RIC functors M_1, M_2 is a collection of morphisms $\varphi(K)\colon M_1(K) \to M_2(K)$ for all $K \in \Upsilon$ that together constitute a natural transformation $M_{1,*} \to M_{2,*}$ and a natural transformation $M_1^* \to M_2^*$. If $\varphi(K)$ is injective for all K, we say that M_1 is a *sub-functor* of M_2.

Definition 4.7. Let $M\colon \mathcal{P}(G,\Upsilon) \to R$-Mod be an RIC functor. We say that M is

(G) *Galois* if for all $L, K \in \Upsilon$ such that $L \triangleleft K$, we have

$$\mathrm{pr}_{L,K}^*\colon M(K) \xrightarrow{\sim} M(L)^{K/L}.$$

Here the action of $\gamma \in K/L$ on $M(L)$ is via pullbacks induced by $(L \xrightarrow{\gamma} L)$ and $M(L)^{K/L}$ denotes the invariants of $M(L)$ under this action.

(Co) *cohomological* if for all $L, K \in \Upsilon$ with $L \subset K$,

$$(L \xrightarrow{e} K)_* \circ (L \xrightarrow{e} K)^* = [K : L] \cdot (K \xrightarrow{e} K)^*.$$

That is, the composition is multiplication by index $[K : L]$ on $M(K)$.

(M) *Mackey* if for all $K, L, L' \in \Upsilon$ with $L, L' \subset K$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_\gamma M(L_\gamma) & \xrightarrow{\sum \mathrm{pr}_*} & M(L) \\ {\scriptstyle \oplus[\gamma]^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M(L') & \xrightarrow[\mathrm{pr}_*]{} & M(K) \end{array} \tag{4.8}$$

where the direct sum in the top left corner is over a fixed choice of coset representatives $\gamma \in K$ of the double quotient $L \setminus K/L'$ and $L_\gamma = L \cap \gamma L' \gamma^{-1} \in \Upsilon$. The condition is then satisfied by any such choice of representatives of $L \setminus K/L'$.

If M satisfies both (M) and (Co), we will say that M is *CoMack*. If S is an R-algebra, the mapping $K \mapsto M(K) \otimes_R S$ is an S-valued RIC functor, which is cohomological or Mackey if M is so.

Remark 4.9. An RIC functor is referred to as a "cohomology functor" in [6]. We prefer the former terminology, since the standard name for the axiom (Co) [26] conflicts with the latter terminology.

Definition 4.10. Let $N\colon \mathcal{P}(G,\Upsilon) \to R$-Mod be an RIC functor and $S \subset \Upsilon$ a non-empty subset. Let $\mathcal{G} = \{B_K \mid K \in S\}$ be a collection of R-submodules $B_K \subset N(K)$ indexed by $K \in S$. We say that $\mathcal{G}$ is *compatible under pullbacks of* N if for all $L, K \in S$ and $g \in G$ satisfying $g^{-1}Lg \subset K$, the morphism $[g]_{L,K}^*\colon N(K) \to N(L)$ sends B_K to B_L. We say that $\mathcal{G}$ is *compatible under restrictions* if we only require the previous condition for L, $K \in S$ with $L \subset K$ and g the identity element. We similarly define compatibility under pushforwards and inductions.

By definition, a family $\mathcal{G}$ as above constitutes a sub-functor of N if $S = \Upsilon$ and $\mathcal{G}$ is compatible under both pullbacks and inductions (equivalently, pushforwards and restrictions).

Definition 4.11. Let $M, N\colon \mathcal{P}(G,\Upsilon) \to R$-Mod be two RIC functors, $S \subset \Upsilon$ be any non-empty subset and $\mathcal{G} = \{B_K \mid K \in S\}$ be a family of submodules $B_K \subset N(K)$ indexed by

$K \in S$ that is compatible under pullbacks of N. Let $\mathcal{F} = \{\varphi_K : B_K \to M(K) \mid K \in S\}$ be a collection of R-module homomorphisms indexed by S. We say that $\mathcal{F}$ is *compatible under pullbacks of* M if for all $L, K \in S$ and $g \in G$ satisfying $g^{-1}Lg \subset K$, we have $[g]^*_{L,K} \circ \varphi_K = \varphi_L \circ [g]^*_{L,K}$ as maps $B_K \to M(L)$. We similarly define compatibility of $\mathcal{F}$ under restrictions, pushforwards or inductions when $\mathcal{G}$ has these properties respectively.

Suppose $B_K = N(K)$ for all $K \in S$. Then a family $\mathcal{F}$ as above constitutes a morphism $\varphi : N \to M$ of RIC functors if $S = \Upsilon$ and $\mathcal{F}$ is compatible under pullbacks and inductions (equivalently, pushforwards and restrictions).

Lemma 4.12. *Let $M, N : \mathcal{P}(G, \Upsilon) \to R$-Mod be RIC functors such that M is Mackey and all restriction maps in M are injective. Let $\mathcal{F} = \{\varphi_K : N(K) \to M(K) \mid K \in \Upsilon\}$ be a family of morphisms that is compatible under pullbacks. Then $\mathcal{F}$ is compatible under inductions and thus constitutes a morphism of RIC functors.*

Proof. Let $L, K \in \Upsilon$ with $L \subset K$. Pick a $K' \in \Upsilon$ such that $K' \lhd K$, $K' \subset L$. Since $\mathrm{pr}^*_{K',K} : M(K) \to M(K')$ is injective, $\mathrm{pr}_{L,K,*} \circ \varphi_L = \varphi_K \circ \mathrm{pr}_{L,K,*}$ if and only if

$$\mathrm{pr}^*_{K',K} \circ \mathrm{pr}_{L,K,*} \circ \varphi_L = \mathrm{pr}^*_{K',K} \circ \varphi_K \circ \mathrm{pr}_{L,K,*} \tag{4.13}$$

as maps $N(L) \to M(K')$. Since $\mathcal{F}$ is compatible under restrictions, we have $\mathrm{pr}^*_{K',K} \circ \varphi_K = \varphi_{K'} \circ \mathrm{pr}^*_{K',K}$. As M is Mackey, (4.13) is equivalent to

$$\sum_{\gamma \in K/L} [\gamma]^*_{K',L} \circ \varphi_L = \sum_{\gamma \in K/L} \varphi_{K'} \circ [\gamma]^*_{K',L}$$

(see [24, Lemma 2.1.11] for details). But this clearly holds by pullback compatibility of $\mathcal{F}$. ∎

4.3. Functors for GSp_{2n}

We now resume the notations introduced in Sections 3.1 and 3.2. Fix for the rest of this note a rational prime p and an integer $c > 1$ with $(c, p) = 1$. Denote by Υ_1 the collection of all principal congruence subgroups K_N for $N \geq 3$ satisfying $(N, cp) = 1$. Let

$$G := \mathbf{G}(\mathbb{A}_f^{cp}) \times \mathbf{G}(\mathbb{Z}_{cp}),$$

where $\mathbb{Z}_{cp} := \prod_{\ell | cp} \mathbb{Z}_\ell$ and $\mathbb{A}_f^{cp}$ denote the finite rational adeles away from primes dividing cp. We will also denote $V_f^{cp} := V_{\mathbb{Z}} \otimes \mathbb{A}_f^{cp}$, $V_{\mathbb{Z}_{cp}} := V_{\mathbb{Z}} \otimes \mathbb{Z}_{cp}$, $\mathbf{Z}(\mathbb{Q})^{cp} := \mathbf{Z}(\mathbb{Q}) \cap G$ and $I^{cp} := \{a \in \mathbb{Z} \mid (a, cp) = 1\}$. For $a \in \mathbb{Q}^\times$, we denote by $z_a \in \mathbf{Z}(\mathbb{Q}) \simeq \mathbb{Q}^\times$ the corresponding element of the center given by $2n$-copies of a on the diagonal. Let Υ be the collection of all compact open subgroups of G which are contained in a G-conjugate of a group in Υ_1 and which are of the form $\mathbf{G}(\mathbb{Z}_{cp})L$ for some $L \subset \mathbf{G}(\mathbb{A}_f^{cp})$. It is easily verified that both Υ_1, Υ satisfy the conditions of the previous subsection with respect to K_1, G respectively.

Set $X := V_f \setminus \{0\}$ and view elements of $X \subset V_f = \bigoplus_{i=1}^{2n} \mathbb{A}_f e_i \simeq \mathbb{A}_f^{2n}$ as column vectors. There is a smooth right action $X \times G \to X$ given by $(v, g) \mapsto g^{-1}v$, i.e., left matrix multiplication by inverse of g. Let $\mathcal{S}^{cp}(X)$ be the set of all $\mathbb{Z}_p$-valued functions on X of the form $\phi_{cp} \otimes \phi^{cp}$ where $\phi_{cp} = \mathrm{ch}(V_{\mathbb{Z}_{cp}})$ is the characteristic function of $\mathbb{Z}_{cp}$ and ϕ^{cp} is a locally constant compactly supported function on $(V_f^{cp}) \setminus \{0\}$. Then $\mathcal{S}^{cp}(X)$ is a smooth left G-representation via its action on X. We let

$$\mathcal{S}\colon \mathcal{P}(G, \Upsilon) \to \mathbb{Z}_p\text{-Mod}$$

denote the RIC functor associated with the representation $\mathcal{S}^{cp}(X)$. That is, for any $K \in \Upsilon$, $\mathcal{S}(K) = \mathcal{S}^{cp}(X)^K$ with restriction, inductions and conjugations given in the obvious manner.

Fix now a non-negative integer k. Recall that for each $K_N \in \Upsilon$, we have fixed a choice $\pi_N\colon \mathcal{A}_N \to \mathrm{Sh}(K_N)$ of a universal abelian scheme in Section 3.2. Let $\mathcal{H}_{\mathbb{Z}_p} = \mathcal{H}_{K_N,\mathbb{Z}_p}$ be the corresponding sheaf of Tate modules. For each N satisfying $K_N \in \Upsilon_1$, we denote

$$\mathcal{E}_{N,\mathbb{Q}_p} = \mathcal{E}^k_{N,\mathbb{Q}_p} := \mathrm{H}^{2n-1}_{\text{ét}}\big(\mathrm{Sh}(K_N), \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n)\big).$$

Given two such M, N such that $M|N$, we have a restriction map $\mathrm{pr}^*_{N,M}\colon \mathcal{E}_{M,\mathbb{Q}_p} \to \mathcal{E}_{N,\mathbb{Q}_p}$ induced by unit adjunction for $\mathrm{pr}_{N,M} = [\mathrm{id}]_{N,M}\colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_M)$ and the isomorphism $\mathcal{A}_N \to \mathrm{pr}^*_{N,M}(\mathcal{A}_M)$ specified by (3.1) for $g = \mathrm{id}$. Let

$$\widehat{\mathcal{E}}_{\mathbb{Q}_p} = \widehat{\mathcal{E}}^k_{\mathbb{Q}_p} := \varinjlim_N \mathcal{E}^k_{N,\mathbb{Q}_p},$$

where the limit is taken with respect to restriction maps for $M|N$.

Lemma 4.14. *$\widehat{\mathcal{E}}_{\mathbb{Q}_p}$ is a smooth G-representation.*

Proof. Given $g \in G$ and $x \in \mathcal{E}_{M,\mathbb{Q}_p}$ we can find $z = z_a \in \mathbf{Z}(\mathbb{Q})^{cp}$ for some $a \in I^{cp}$, an element $h \in G$ and a multiple $N \in I^{cp}$ of M such that $g = zh$ and $\eta(N/M)V_{\widehat{\mathbb{Z}}} \subset V_{\widehat{\mathbb{Z}}} \subset \eta V_{\widehat{\mathbb{Z}}}$ is satisfied for $\eta \in \{z^{-1}, h\}$. The action of η is described by the pullback $[\eta]^*_{N,M}\colon \mathcal{E}_{M,\mathbb{Q}_p} \to \mathcal{E}_{N,\mathbb{Q}_p}$ induced by adjunction for $[\eta]_{N,M}\colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_M)$ and the inverse of the isomorphism between symmetric power of sheaves of Tate modules induced by the (prime-to-p) isogeny ${}^{\eta}\psi_{N,M}\colon \mathcal{A}_N \to {}^{\eta}\mathcal{A}_M$ in (3.1). Since $\psi^{\mathrm{univ}}_{z^{-1}}$ is just $[a]$, $z_a = (z_{a^{-1}})^{-1}$ acts by a^k. Then $g \cdot x$ is defined to be the element corresponding to $a^k \cdot [h]^*_{N,M}(x) \in \mathcal{E}_{N,\mathbb{Q}_p}$. This action is well defined since the isogenies $\psi^{\mathrm{univ}}_{\eta}$ for varying η satisfy an obvious cocycle condition. ∎

Since $\mathrm{pr}_{N,M}$ is Galois with Galois group K_M/K_N, we have

$$\mathrm{pr}^*_{N,M} \circ \mathrm{pr}_{N,M,*} = \sum_{\gamma \in K_M/K_N} [\gamma]^*_{N,M},$$
$$\mathrm{pr}_{N,M,*} \circ \mathrm{pr}^*_{N,M} = [K_M : K_N] \cdot \mathrm{id}.$$

Using these, one deduces that the natural map from $\mathcal{E}_{N,\mathbb{Q}_p}$ to $\widehat{\mathcal{E}}_{\mathbb{Q}_p}$ identifies the former with the K_N-invariants of the latter and that $\mathrm{pr}^*_{N,M}$ (resp., $\mathrm{pr}_{N,M,*}$) is identified with inclusion (resp. $\sum_{\gamma\in K_N/K_M}\gamma$). We let

$$\mathcal{E}_{\mathbb{Q}_p}\colon \mathcal{P}(G,\Upsilon)\to \mathbb{Q}_p\text{-Mod}$$

denote the associated RIC functor, i.e., $\mathcal{E}_{\mathbb{Q}_p}(K) := (\widehat{\mathcal{E}}_{\mathbb{Q}_p})^K$ with obvious choice for pullback and pushforward maps. It is CoMack and Galois by construction.

Next we define an integral structure on $\mathcal{E}_{\mathbb{Q}_p}$. Let $\Upsilon_2\subset\Upsilon$ be the subset of all groups that are contained in a congruence subgroup in Υ_1. For $K\in\Upsilon_2$, we let $\pi_K\colon \mathcal{A}_K\to \mathrm{Sh}(K)$ be the abelian scheme given by pulling back $\mathcal{A}_M$ along the degeneration map $\mathrm{Sh}(K)\to \mathrm{Sh}(K_M)$ for some $K_M\in\Upsilon_1$ that contains K. Then $\mathcal{A}_K$ is independent of the choice of M, since $K_{M_1}\cap K_{M_2} = K_{\mathrm{lcm}(M_1,M_2)}$. Let $\mathcal{H}_{\mathbb{Z}_p} = \mathcal{H}_{K,\mathbb{Z}_p}$ the associated $\mathbb{Z}_p$-sheaf on $\mathrm{Sh}(K)$. If we pick $K_N\in\Upsilon_1$ such that $K_N\trianglelefteq K$, the natural pullback map along $\mathrm{Sh}(K_N)\to \mathrm{Sh}(K)$ identifies $\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(K),\mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n))$ with the K/K_N-invariants of $\mathcal{E}_{N,\mathbb{Q}_p} = \mathcal{E}_{\mathbb{Q}_p}(K_N)$. Again this identification is independent of the choice of N. Let $\mathcal{E}_{\mathbb{Z}_p}(K)\subset \mathcal{E}_{\mathbb{Q}_p}(K)$ denote the image of $\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(K),\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n))$ under

$$\mathrm{H}^{2n-1}_{\text{ét}}\big(\mathrm{Sh}(K),\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n)\big)\xrightarrow{-\otimes\mathbb{Q}_p}\mathrm{H}^{2n-1}_{\text{ét}}\big(\mathrm{Sh}(K),\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n)\big)\otimes\mathbb{Q}_p$$
$$\xrightarrow[\sim]{\sigma_k}\mathrm{H}^{2n-1}_{\text{ét}}\big(\mathrm{Sh}(K),\mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n)\big)\xrightarrow{\sim}\mathcal{E}_{\mathbb{Q}_p}(K),$$

where σ_k is the isomorphism induced by the map (2.36).

Lemma 4.15. *The family $\{\mathcal{E}_{\mathbb{Z}_p}(K)\mid K\in\Upsilon_2\}$ is compatible under pullbacks and pushforwards of $\mathcal{E}_{\mathbb{Q}_p}$.*

Proof. Let $(L\xrightarrow{g}K)\in\mathcal{P}(G,\Upsilon)$ be such that $L,K\in\Upsilon_2$. We wish to show that

$$[g]^*_{L,K}\colon \mathcal{E}_{\mathbb{Q}_p}(K)\to\mathcal{E}_{\mathbb{Q}_p}(L)$$

preserves the corresponding $\mathbb{Z}_p$-submodules. Since $\mathbf{Z}(\mathbb{Q})^{cp}$ acts by invertible scalars, we may assume wlog that g is such that $V_{\widehat{\mathbb{Z}}}\subset gV_{\widehat{\mathbb{Z}}}$. Choose $K_M, K_N\in\Upsilon_1$ such that $K_M\supset K$ and $L\supset K_N$. Replacing N by a multiple, we may assume that $M|N$ and that $g(M/N)V_{\widehat{\mathbb{Z}}}\subset V_{\widehat{\mathbb{Z}}}$. Recall that the isogeny ${}^g\psi_{N,M}\colon \mathcal{A}_N\to {}^g\mathcal{A}_M$ (3.1) is given as the quotient of $\mathcal{A}_N$ by the group $C_\gamma\subset\mathcal{A}_N[N]$ corresponding to the kernel of $\gamma\colon V_{\mathbb{Z}}/NV_{\mathbb{Z}}\to V_{\mathbb{Z}}/NV_{\mathbb{Z}}$ defined by $v\mapsto g^{-1}v$. Since $g^{-1}Lg\subset K_M$, the right action of L/K_N on

$$\mathcal{A}_N\xrightarrow{\sim}\mathcal{A}_L\times_{\mathrm{Sh}(L)}\mathrm{Sh}(K_N)$$

preserves C_γ. Thus ${}^g\psi_{N,M}$ descends to an isogeny ${}^g\psi_{L,K}\colon\mathcal{A}_L\to{}^g\mathcal{A}_K$ (where ${}^g\mathcal{A}_K := [g]^*\mathcal{A}_K$) giving an isomorphism

$$\mathcal{H}_{\mathbb{Z}_p,L}\xrightarrow{\sim}[g]^*\mathcal{H}_{\mathbb{Z}_p,K}. \tag{4.16}$$

Here $[g]\colon\mathrm{Sh}(L)\to\mathrm{Sh}(K)$ denotes the map given by right multiplication by g. The pullback map $\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(K),\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n))\to\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(L),\Gamma_k(\mathcal{H}_{\mathbb{Z}_p})(n))$ defined using (4.16) induces (after tensoring with $\mathbb{Q}_p$) a map $p_g\colon\mathcal{E}_{\mathbb{Q}_p}(K)\to\mathcal{E}_{\mathbb{Q}_p}(L)$ that sends $\mathcal{E}_{\mathbb{Z}_p}(K)$ to

$\mathcal{E}_{\mathbb{Z}_p}(L)$ by construction. Since ${}^g\psi_{N,M}$ is the pullback of ${}^g\psi_{L,K}$ along $\mathrm{Sh}(K_N) \to \mathrm{Sh}(K)$, p_g is compatible with $[g]^*_{N,M}\colon \mathcal{E}_{M,\mathbb{Q}_p} \to \mathcal{E}_{N,\mathbb{Q}_p}$ and therefore equal to the map $[g]^*_{L,K}$ of $\mathcal{E}_{\mathbb{Q}_p}$. A similar argument applies to the pushforward $[g]_{L,K,*}$. ■

For $K \in \Upsilon$ arbitrary, choose $g \in G$ such that $K' := gKg^{-1} \in \Upsilon_2$ and define $\mathcal{E}_{\mathbb{Z}_p}(K) := [g]^*_{K',K}(\mathcal{E}_{\mathbb{Z}_p}(K'))$. This is independent of the choice of g. Indeed if $h \in G$ is such that $K'' := hKh^{-1} \in \Upsilon_2$, $[hg^{-1}]^*$ sends $\mathcal{E}_{\mathbb{Z}_p}(K')$ to $\mathcal{E}_{\mathbb{Z}_p}(K'')$ by Lemma 4.15. The same result implies that the family $\{\mathcal{E}_{\mathbb{Z}_p}(K) \mid K \in \Upsilon\}$ is compatible under pullbacks and push-forwards. So the association $K \mapsto \mathcal{E}_{\mathbb{Z}_p}(K)$ assembles into an RIC functor

$$\mathcal{E}_{\mathbb{Z}_p}\colon \mathcal{P}(G,\Upsilon) \to \mathbb{Z}_p\text{-Mod} \tag{4.17}$$

which is CoMack but *not* necessarily Galois.

Remark 4.18. An alternative way to define $\mathcal{E}_{\mathbb{Q}_p}$ and its integral sub-lattice is to use the notion of equivariant sheaves associated to algebraic representations of $\mathbf{G}$ along the lines of [6, §4]. It is also possible to incorporate the action of a monoid $\Sigma \subset \mathbf{G}(\mathbb{A}_f)$ as in *loc. cit.* larger than the group G that allows one to define the action of certain Hecke correspondences at the prime p as well.

4.4. The main result

Recall that for $v \in V_{\widehat{\mathbb{Z}}}$ and $N \geq 3$, we denote by $t_{v,N}\colon \mathrm{Sh}(K_N) \to \mathcal{A}_N$ the torsion section given by $\eta_N^{\mathrm{univ}}(v)$. For N such that $K_N \in \Upsilon_1$ and $v \in V_{\widehat{\mathbb{Z}}} \setminus NV_{\widehat{\mathbb{Z}}}$, we denote

$$\xi_{v,N} := \mathrm{ch}(v + NV_{\widehat{\mathbb{Z}}})\colon X \to \mathbb{Z}_p$$

the characteristic function of $v + NV_{\widehat{\mathbb{Z}}} \subset X$. Note that $v + NV_{\widehat{\mathbb{Z}}}$ is K_N-stable and equals the product $V_{\mathbb{Z}_{cp}}(v^{cp} + NV_{\widehat{\mathbb{Z}}^{cp}})$ where v^{cp} denotes the image of v under the projection $V_f \to V_f^{cp}$ and $\widehat{\mathbb{Z}}^{cp} = \widehat{\mathbb{Z}}/\mathbb{Z}_{cp}$. Thus $\xi_{v,N} \in \mathcal{S}(K_N)$. By Theorem 2.49, we have

$$N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,N}) \in \mathcal{E}_{\mathbb{Z}_p}(K_N) \quad \text{for all } v \in V_{\widehat{\mathbb{Z}}} \setminus NV_{\widehat{\mathbb{Z}}}.$$

Theorem 4.19. *There exists a unique morphism $\varphi^k\colon \mathcal{S} \to \mathcal{E}_{\mathbb{Z}_p}$ of RIC functors on $\mathcal{P}(G,\Upsilon)$ such that for each $K_N \in \Upsilon_1$ and $v \in V_{\widehat{\mathbb{Z}}} \setminus NV_{\widehat{\mathbb{Z}}}$, we have*

$$\varphi^k(K_N)(\xi_{v,N}) = N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,N}).$$

Proof. We are going to construct this morphism in several steps. Since we exclusively work with $\mathbb{Z}_p$-coefficients, we will denote $\mathcal{E}_{\mathbb{Z}_p}$ simply as $\mathcal{E}$.

Step 1. We first consider principal levels. For $K_M \in \Upsilon_1$, let $A_M \subset \mathcal{S}(K_M)$ denote the $\mathbb{Z}_p$-span of $\xi_{v,M}$ for $v \in V_{\widehat{\mathbb{Z}}} \setminus MV_{\widehat{\mathbb{Z}}}$ and let $\varphi_M\colon A_M \to \mathcal{E}(K_M)$ be the $\mathbb{Z}_p$-linear map given by $\xi_{v,M} \mapsto M^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,M})$. This is well defined since A_M is free over $\xi_{v,M}$ for v running over representatives for $(V_{\mathbb{Z}}/MV_{\mathbb{Z}}) \setminus \{0\}$. Denote by $\mathcal{S}_1, \mathcal{E}_1$ the RIC functors on $\mathcal{P}(K_1,\Upsilon_1)$ obtained by restricting the domain of $\mathcal{S}$, $\mathcal{E}$. Let

$$\mathcal{F} := \{\varphi_M\colon A_M \to \mathcal{E}(K_M) \mid K_M \in \Upsilon_1\}$$

be the collection of all φ_M. Clearly, $\{A_M \mid K_M \in \Upsilon_1\}$ is compatible under pullbacks of $\mathcal{S}_1$. We claim that $\mathcal{F}$ is compatible under pullbacks of $\mathcal{E}_1$. It suffices to check that for any $(K_N \xrightarrow{\kappa} K_M) \in \mathcal{P}(K_1, \Upsilon_1)$ and $v \in V_{\widehat{\mathbb{Z}}} \setminus M V_{\widehat{\mathbb{Z}}}$, the elements $[\kappa]^* \circ \varphi_M(\xi_{v,M}) \in \mathcal{E}(K_N)$ and $\varphi_N \circ [\kappa]^*(\xi_{v,M}) \in \mathcal{E}(K_N)$ are equal. Since $[\kappa]^*(\xi_{v,M}) = \mathrm{ch}(\kappa v + M V_{\widehat{\mathbb{Z}}})$ is the sum of $\mathrm{ch}(\kappa v + M w + N V_{\widehat{\mathbb{Z}}})$ for $w \in V_{\mathbb{Z}}/d V_{\mathbb{Z}}$, we have

$$[\kappa]^* \circ \varphi_M(\xi_{v,M}) = [\kappa]^*\big(M^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,M})\big), \tag{4.20}$$

$$\varphi_N \circ [\kappa]^*(\xi_{v,M}) = \sum_{w \in V_{\mathbb{Z}}/d V_{\mathbb{Z}}} N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{\kappa v + M w, N}). \tag{4.21}$$

Now Lemma 2.29 applied to (3.2) implies that

$$[\kappa]^*\big({}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,M})\big) = {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}\big([\kappa]^*(t_{v,M})\big) = {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{\kappa d v, N}).$$

But Lemma 2.30 (in conjunction with Lemma 2.27) applied to the multiplication by d isogeny $[d]\colon \mathcal{A}_N \to \mathcal{A}_N$ over $\mathrm{Sh}(K_N)$ implies that

$${}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{\kappa d v, N}) = \sum_w d^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{\kappa v + M w, N}).$$

Thus the equalities (4.20) and (4.21) are themselves equal.

Step 2. Next we consider the action of center. Let $\widehat{A} = \bigcup_M A_M$ and let $\widehat{\mathcal{E}}_1$ (resp., $\widehat{\mathcal{E}}$) be the inductive limit of $\mathcal{E}(K)$ over all $K \in \Upsilon_1$ (resp., $K \in \Upsilon$) with respect to restriction maps. By Step 1, we have an induced map $\widehat{\varphi}\colon \widehat{A} \to \widehat{\mathcal{E}}_1$ of smooth K_1-representations. As any element of Υ contains an element of Υ_1, $\widehat{\mathcal{E}}_1 = \widehat{\mathcal{E}}$. So the target of $\widehat{\varphi}$ is a G-representation. We show that $\widehat{\varphi}$ extends uniquely to a map

$$\widehat{\varphi}\colon \widehat{\mathcal{S}} \to \widehat{\mathcal{E}} \tag{4.22}$$

of $(\mathbf{Z}(\mathbb{Q})^{cp} K_1)$-representations as follows. First note that $\widehat{A}$ is simply the subspace of all functions in $\widehat{\mathcal{S}} = \mathcal{S}^{cp}(X)$ that are supported on $V_{\widehat{\mathbb{Z}}}$. Next recall that $\mathrm{supp}(\phi)$ for any non-zero $\phi \in \widehat{\mathcal{S}}$ is of the form $V_{\mathbb{Z}_{cp}} Y$ for $Y \subset (V_f^{cp}) \setminus \{0\}$. Since $X = \bigcup_{M \geq 1} 1/M \cdot (V_{\widehat{\mathbb{Z}}} \setminus \{0\})$, there exists a positive integer $a \in I^{cp}$ such that $a \cdot \mathrm{supp}(\phi) \subset V_{\widehat{\mathbb{Z}}}$. So $\phi = z_a^{-1} \cdot \xi$ for some $\xi \in \widehat{A}$ and the only possible extension is to set

$$\widehat{\varphi}(\phi) := a^{-k} \widehat{\varphi}(\xi).$$

For this to be well defined, we must have $\widehat{\varphi}(z_a \cdot \xi_{v,M})$ equal to $a^k \widehat{\varphi}(\xi_{v,M})$ for all $a \in I^{cp}$, M satisfying $K_M \in \Upsilon_1$ and $v \in V_{\widehat{\mathbb{Z}}} \setminus M V_{\widehat{\mathbb{Z}}}$. But this follows since $z_a \cdot \xi_{v,M} = \xi_{av,aM} \in \mathcal{S}(K_{aM})$ is mapped under φ_{aM} to $(aM)^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{av,aM})$ and this class coincides with $\mathrm{pr}^*_{aM,M}$ applied to

$$a^k \varphi_M(\xi_{v,M}) = (aM)^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,M})$$

by Lemma 2.29.

Step 3. We now enlarge the domain of each φ_M to all of $\mathcal{S}(K_M)$ for any $K_M \in \Upsilon_1$. Let $B_M \subset \mathcal{S}(K_M)$ denote the $\mathbb{Z}_p$-submodule of all finite sums $\sum_i z_{a_i} \cdot \xi_i$ where $\xi_i \in A_M$ and $a_i \in \mathbf{Z}(\mathbb{Q})^{cp}$. For any $\phi = \sum_i z_{a_i} \cdot \xi_i \in B_M$, set

$$\varphi_M(\phi) := \sum_i a_i^k \, \varphi_M(\xi_i).$$

Then $\varphi_M \colon B_M \to \mathcal{E}(K_M)$ is well defined (and uniquely determined) by Step 2 and injectivity of restrictions of $\mathcal{E}$. We claim that $B_M = \mathcal{S}(K_M)$ for all M. It suffices to show that $\mathrm{ch}(C) \in B_M$ for $C \subset V_{\widehat{\mathbb{Z}}}$ any K_M-invariant compact open subset of the form $V_{\mathbb{Z}_{cp}} Y$ for $Y \subset V_f^{cp} \setminus \{0\}$. For such C, we can pick a $N = N_C \in I^{cp}$ a multiple of M such that C is a finite disjoint union of cosets in $V_{\widehat{\mathbb{Z}}}/N V_{\widehat{\mathbb{Z}}}$. Since C is also K_M-invariant, we can write C as a finite disjoint union of sets of the form $K_M v + N V_{\widehat{\mathbb{Z}}}$ and we may also assume $v \in V_{\mathbb{Z}}$. But Lemma 4.3 implies that $\mathrm{ch}(K_M v + N V_{\widehat{\mathbb{Z}}})$ can be written as a difference of two sums of functions of the form $\mathrm{ch}(aw + MaV_{\widehat{\mathbb{Z}}})$ for $a \in I^{cp}$, $w \notin MV_{\widehat{\mathbb{Z}}}$. This completes the step.

Step 4. We define maps for levels in $K \in \Upsilon_2$. As in Step 3, we need to show that all elements of $\widehat{\varphi}(\mathcal{S}(K))$ lift to $\mathcal{E}(K)$. Fix any $K_N \in \Upsilon_1$ such that $K_N \subset K$. Recall that $K \subset K_1$ acts on (the left of) $V_{\mathbb{Z}}/N V_{\mathbb{Z}}$. For any $v \in V_{\widehat{\mathbb{Z}}} \setminus N V_{\widehat{\mathbb{Z}}}$, let $K_v \subset K$ denote the stabilizer of $v + N V_{\widehat{\mathbb{Z}}} \in V_{\mathbb{Z}}/N V_{\mathbb{Z}}$. Since pr_{K_N, K_v} is Galois and $v + N V_{\widehat{\mathbb{Z}}}$ is K_v/K_N-invariant, the section $t_{v,N} = \eta_N^{\mathrm{univ}}(v)$ descends to an N-torsion section $t_{v,K_v} \colon \mathrm{Sh}(K_v) \to \mathcal{A}_{K_v}$. Thus for any $\gamma \in K$, we have $[\gamma]^*_{K_v, K_N}(t_{v,K_v}) = t_{\gamma v, N}$. Now note that

$$\mathrm{ch}(Kv + N V_{\widehat{\mathbb{Z}}}) = \sum_{\gamma \in K/K_v} \xi_{\gamma v, N}$$

is an element of $\mathcal{S}(K)$. We define

$$\varphi_K\big(\mathrm{ch}(Kv + N V_{\widehat{\mathbb{Z}}})\big) := \mathrm{pr}_{K_v, K, *}\big(N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,K_v})\big)$$

which belongs to $\mathcal{E}(K)$ by Theorem 2.49. That this agrees with $\sum_{\gamma \in K/K_v} \widehat{\varphi}(\xi_{\gamma v, N})$ in $\widehat{\mathcal{E}}$ follows since

$$\mathrm{pr}^*_{K, K_N} \circ \mathrm{pr}_{K_v, K, *}\big(N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,K})\big) = \sum_{\gamma \in K/K_v} [\gamma]^*_{K_v, K_N}\big(N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,K_v})\big)$$

by the Mackey axiom and since $\varphi_N(\xi_{\gamma v, N}) = [\gamma]^*_{K_v, K_N}(N^k {}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,K_v}))$ for any $\gamma \in K$ by Lemma 2.29.

As in Step 3, we can now define φ_K for any finite linear combination of functions as above scaled by elements of $\mathbf{Z}(\mathbb{Q})^{cp}$. So it only remains to argue that all elements of $\mathcal{S}(K)$ are of this form. Again, it suffices to show this for characteristic functions $\mathrm{ch}(C) \in \mathcal{S}(K)$ for some $C \subset V_{\widehat{\mathbb{Z}}}$. But this follows since we can find a sufficiently large $N = N_C \in I^{cp}$ such that $K_N \subset K$ and C is a finite disjoint union of sets of the form $Kv + N V_{\widehat{\mathbb{Z}}}$.

Step 5. The final step is to show that the family $\{\varphi_K \mid K \in \Upsilon_2\}$ extends uniquely to a morphism of functors on $\mathcal{P}(G, \Upsilon)$. To this end, it suffices to establish $\widehat{\varphi}$ (4.22) is a map of G-representations. Indeed, any $K \in \Upsilon$ is such that $gKg^{-1} \in \Upsilon_2$ for some $g \in G$ and we can define $\varphi_K(\xi)$ as $[g]_{gKg^{-1},K,*} \circ \varphi_{gKg^{-1}}(g \cdot \xi)$. The resulting family of homomorphisms indexed by Υ would then be compatible under pullbacks by G-equivariance and Lemma 4.12 would give the desired claim.

Fix $g \in \mathbf{G}(\mathbb{A}_f)$ and $\xi \in \widehat{\mathcal{S}}$. We wish to show that $\widehat{\varphi}(g \cdot \xi) = g \cdot \widehat{\varphi}(\xi)$. Recall that $\widehat{\varphi}$ was shown to equivariant with respect to $\mathbf{Z}(\mathbb{Q})^{cp}$ in Step 2. Since any $\xi \in \widehat{\mathcal{S}}$ is K_M-invariant for some $K_M \in \Upsilon_1$, it suffices to restrict to the case where $\xi = \xi_{v,M}$ for some v, M and g satisfies $V_{\widehat{\mathbb{Z}}} \subset gV_{\widehat{\mathbb{Z}}}$. Let $N \in I^{cp}$ be a multiple of M such that $g(N/M)V_{\widehat{\mathbb{Z}}} \subset V_{\widehat{\mathbb{Z}}}$. Moreover, let

$$^{g}\psi \colon \mathcal{A}_N \to {}^{g}\mathcal{A}_M$$

denote the isogeny in (3.1) and ${}^{g}t_{v,M} \colon \mathrm{Sh}(K_N) \to {}^{g}\mathcal{A}_M$ denote the torsion section that is obtained as the pullback of $t_{v,M} \colon \mathrm{Sh}(K_M) \to \mathcal{A}_M$ under $[g]_{N,M} \colon \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_M)$.

$$\begin{array}{ccccc} \mathcal{A}_N & \xrightarrow{\;{}^{g}\psi\;} & {}^{g}\mathcal{A}_M & \longrightarrow & \mathcal{A}_M \\ & \searrow^{\pi_N} & \downarrow \uparrow {}^{g}t_{v,M} & & \pi_M \downarrow \uparrow t_{v,M} \\ & & \mathrm{Sh}(K_N) & \xrightarrow[{[g]_{N,M}}]{} & \mathrm{Sh}(K_M). \end{array} \tag{4.23}$$

Then the universal torsion sections of $\mathcal{A}_N$ that map to ${}^{g}t_{v,M}$ under ${}^{g}\psi$ are $t_{gdv+w,N}$ where $d := N/M$ and $w \in gNV_{\widehat{\mathbb{Z}}}$ runs over representatives of $g(NV_{\widehat{\mathbb{Z}}})/NV_{\widehat{\mathbb{Z}}} \subset V_{\mathbb{Z}}/NV_{\mathbb{Z}}$. Now $[g]^*_{N,M} \cdot \xi_{v,M} = z_d^{-1} \cdot \mathrm{ch}(gdv + gNV_{\widehat{\mathbb{Z}}})$ and the right-hand side expands as $\sum_w z_d^{-1} \cdot \mathrm{ch}(gdv + w + NV_{\widehat{\mathbb{Z}}})$ with w running over $g(NV_{\widehat{\mathbb{Z}}})/NV_{\widehat{\mathbb{Z}}}$. So we need to show that

$$[g]^*_{N,M}\big(M^k\,{}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{v,M})\big) = \sum_w \mathrm{Sym}^k({}^{g}\psi)_*\big(d^{-k}N^k\,{}_c\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{gdv+w,N})\big).$$

But this follows by Lemmas 2.29 and 2.30 as before. ∎

By Step 3 of the proof, the image of $\varphi^k(K_N)$ is the $\mathbb{Z}_p$-linear span of Eisenstein classes along non-zero universal N-torsion sections. An interesting implication of this is the following.

Corollary 4.24. *For any integer $N \geq 3$ and relatively prime to cp, the $\mathbb{Z}_p$-submodule of $\mathrm{H}^{2n-1}_{\text{ét}}(\mathrm{Sh}(K_N), \mathrm{Sym}^k(\mathcal{H}_{\mathbb{Q}_p})(n))$ spanned by ${}_c\mathrm{Eis}^{(k)}_{\mathbb{Q}_p}(t_{v,N})$ for non-zero $v \in V_{\mathbb{Z}}/NV_{\mathbb{Z}}$ is stable under the natural action of Hecke correspondences $[K_N g K_N]$ for any $g \in G$.*

Remark 4.25. Note that our parametrization result is only made for the *image* of the integral Eisenstein classes [16, Definition 6.4.3] in the $\mathbb{Q}_p$-cohomology, and not for the integral classes themselves. For $n = 1$ and c satisfying $(c, 6p) = 1$, an alternative parametrization for these integral classes can be obtained as follows. Since the units of the structure sheaf of modular curves have Galois descent, it is straightforward to define Siegel units associated with arbitrary Schwartz functions. One can then exploit [15, Theorem 4.7.1] and the compatibility of Ohta's twisting morphism with Kings' moment maps (see [17, Theorem 4.5.1 (2)]) to define "Eisenstein classes" integrally for any Schwartz function of the form specified in [20, Definition 9.1.3]. These classes then agree with the integral Eisenstein classes defined by Kings (up to a normalization factor) when the Schwartz function corresponds to a genuine N-torsion section by the results of [15]. The adelic distribution relations of these classes then immediately follow from those of the Siegel units. We are grateful to David Loeffler for sharing this observation.

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