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Realisation of linear algebraic groups as automorphism groups

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Abstract. Let G be a linear algebraic group, over a field F. We show that G is isomorphic to the automorphism group scheme of a smooth projective F-variety, defined as the blow-up of a projective space, along a suitable smooth subvariety.

1. Introduction

Let X be a projective variety over a field F. The automorphism group functor $\mathbf{Aut}(X)$ is represented by a group scheme, locally of finite type over F. This is due to Grothendieck (see also [10, Theorem 3.7]). Note that the sub-group scheme $\mathbf{Aut}^0(X) \subset \mathbf{Aut}(X)$, defined as the connected component of the identity, is then a group scheme of finite type over F, that is to say, an algebraic group over F.

Conversely, it is natural to ask the following question.

Question 1.1. Let G be an algebraic group over a field F.

Does there exist a smooth projective F-variety X, such that $G \simeq \mathbf{Aut}(X)$?

Consider the case of an abelian variety $G = A$. The natural arrow $A \xrightarrow{\sim} \mathbf{Aut}^0(A)$ is then an isomorphism. Meanwhile, Question 1.1 is non-trivial: the answer is yes, if and only if A has finitely many automorphisms, over an algebraic closure of F. For different proofs, see [1, 7, 9].

Brion and Schröer [4] recently proved that any *connected* G is isomorphic to *the connected component* $\mathbf{Aut}^0(X)$, for some projective, geometrically integral F-variety X. We refer to their paper, for an overview of the whole topic.

This paper treats Question 1.1, in the case of a *linear* algebraic F-group G, possibly non-smooth. The answer is positive in full generality; see Theorem 2.1. In the recent preprint [2], Bragg proves that every finite étale F-group scheme is isomorphic to $\mathbf{Aut}(C)$, for C a proper, smooth, geometrically integral F-curve. In our work, it is unclear whether assuming G/F étale, could lead to simpler arguments.

This paper is organised as follows. The main theorem is stated in Section 2, and proved in Section 9. The raw ideas go like this. One first proves a new and convenient structure

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result for linear algebraic F-groups: given such a G, there is a finite-dimensional F-vector space W, together with an integer $n \geq 1$ and an F-linear subspace L of its n-th divided power $\Gamma^n_F(W)$, such that G is isomorphic to $\mathrm{Stab}(L) \subset \mathrm{PGL}(W)$ (Proposition 8.3). One then defines an F-variety X, by jointly blowing-up the Veronese embedding $\mathbb{P}(W) \hookrightarrow \mathbb{P}(\Gamma^n_F(W))$ and the linear subvariety $\mathbb{P}(L) \hookrightarrow \mathbb{P}(\Gamma^n_F(W))$. It remains to carefully check, that the natural homomorphism of F-group schemes $G \to \mathbf{Aut}(X)$ is an isomorphism.

Tools and intermediate results are developed in Sections 3–8. Some of them are of independent interest. For instance, Proposition 6.1 describes the F-automorphisms of the blow-up of a smooth F-variety at a smooth center (under some extra assumptions), while Proposition 5.1 describes its infinitesimal automorphisms (in full generality). The former is achieved via explicit computations in Chow rings.

Our proofs are self-contained: they essentially do not rely on previous works cited in the bibliography. Our methods and results are especially interesting in positive characteristic. In characteristic zero, the main results are still new, but their proofs, and altogether the proof of Theorem 2.1, are then considerably simpler. Indeed, divided powers may then be replaced by symmetric powers, and most algebraic results (e.g., Lemma 8.1) become exercises. Also, whenever checking that a homomorphism $\phi \colon G \to H$ of linear algebraic F-groups is trivial (resp., injective, surjective), it then suffices to prove that $\phi(\overline{F}) \colon G(\overline{F}) \to H(\overline{F})$ has the same property, as a morphism of abstract groups. Thus, differential calculus may be dismissed entirely. Sections 4 and 5 are not needed, the length of the proof of Proposition 8.3 is halved, and that of Lemma 8.1 is reduced tenfold.

2. Statement of the theorem

Theorem 2.1. *Let F be a field, and let G be a linear algebraic F-group (= an affine F-group scheme of finite type).*

There exists a smooth projective F-variety X, such that G is isomorphic to $\mathbf{Aut}(X)$*, as a group scheme over F. More precisely, X can be picked as the blow-up of a projective space, along a suitable smooth F-subvariety.*

3. Conventions, notation

Rings and algebras over them, are commutative with unit.

Denote by F a field, with algebraic closure $\overline{F}$. Unless specified otherwise, by "F-vector space" one means "finite-dimensional F-vector space". Denote by $F[\varepsilon]$, $\varepsilon^2 = 0$, the F-algebra of dual numbers. A variety over F is a separated F-scheme of finite type. A linear algebraic group over F is an affine F-variety, equipped with the structure of a group scheme over F. Equivalently, a linear algebraic group over F is a closed F-subgroup scheme of GL_n, for some $n \geq 1$.

Let X be a variety over F. For an F-algebra A, denote by $X_A := X \times_F A$ the A-scheme obtained from X by extending scalars. Set $\overline{X} := X \times_F \overline{F}$.

The tangent sheaf $TX \to X$ is defined point-wise, for every F-algebra A, by

$$TX(A) = X\big(A[\varepsilon]\big).$$

If X is smooth over F, it is (the total space of) a vector bundle, dual to $\Omega^1(X/F)$.
A global section of the tangent sheaf is called a vector field on X.

3.1. Automorphism groups

For an F-variety X, denote by $\mathbf{Aut}(X)$ the automorphism group functor of X. For every F-algebra A, $\mathbf{Aut}(X)(A)$ is defined as the group of automorphisms of the A-scheme X_A. If X/F is projective, this functor is represented by a group scheme, locally of finite type over F.

For X/F arbitrary, by [3, Lemma 3.1], there is a canonical isomorphism

$$H^0(X, TX) \xrightarrow{\sim} \mathrm{Lie}\big(\mathbf{Aut}(X)\big).$$

Let G/F be a group scheme, locally of finite type. If G acts on the F-variety X, and for a closed subscheme $Z \subset X$, we use the notation $\mathbf{Stab}_G(Z) \subset G$ for the closed F-subgroup scheme defined by

$$\mathbf{Stab}_G(Z)(A) = \big\{g \in G(A),\ g(Z_A) = Z_A\big\},$$

for all F-algebras A. That it is representable follows from [5, II 1.3.6].

3.2. Grassmannians

Let V be an F-vector space. Pick an integer d, $0 \leq d \leq \dim(V)$. Denote by $\mathrm{Gr}(d, V)$ the Grassmannian of d-dimensional *subspaces* of V. Set $\mathbb{P}(V) = \mathrm{Gr}(1, V)$. For $v \in V - \{0\}$, denote by $(v) \in \mathbb{P}(V)(F)$ (or abusively v if no confusion arises) the line directed by v. Recall that, for $E \in \mathrm{Gr}(d, V)(F)$, the tangent space $T_E(\mathrm{Gr}(d, V))$ is naturally isomorphic to $\mathrm{Hom}_F(E, V/E)$.

3.3. Weil restriction

Recall the following important tool.

Definition 3.1. Let A be a finite F-algebra. Let Y be a quasi-projective scheme over A. Denote by $R_{A/F}(Y)$ the Weil restriction of Y. It is a quasi-projective variety over F, characterised by the formula, for every F-algebra B:

$$R_{A/F}(Y)(B) = Y(B \otimes_F A).$$

3.4. Symmetric and divided powers

Let V be a vector space over F. Define

$$V^\vee := \mathrm{Hom}_F(V, F).$$

For each $n \geq 1$, define

$$\mathrm{Sym}^n(V) := H_0(S_n, V^{\otimes n}),$$
$$\Gamma^n(V) := H^0(S_n, V^{\otimes n}),$$

where coinvariants and invariants are taken w.r.t. the natural permutation action of the symmetric group S_n on $V^{\otimes n}$. These are, respectively, the n-th symmetric power and the n-th divided power of V. For $v \in V$, set

$$[v]_n := v \otimes v \otimes \cdots \otimes v \in \Gamma^n(V).$$

These are called pure symbols. If $|F| \geq n$, they span $\Gamma^n(V)$.

There is a canonical non-degenerate pairing of F-vector spaces

$$\Gamma^n(V^\vee) \times \mathrm{Sym}^n(V) \longrightarrow F$$
$$\big([\phi]_n, x_1 x_2 \dots x_n\big) \longmapsto \phi(x_1) \dots \phi(x_n).$$

It is perfect – whence an isomorphism

$$\Gamma^n(V^\vee) \xrightarrow{\sim} \mathrm{Sym}^n(V)^\vee.$$

Denote by $\mathrm{Sym}(V) = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}^n(V)$ and $\Gamma(V) = \bigoplus_{n \in \mathbb{N}} \Gamma^n(V)$ the symmetric and divided powers algebras of V, respectively. Relations in $\Gamma(V)$, arise from the motto "$[v]_n = \frac{v^n}{n!}$". These are:

(1) $[v]_0 = 1$,

(2) $[v + v']_n = \sum_0^n [v]_i [v']_{n-i}$,

(3) $[\lambda v]_n = \lambda^n [v]_n$,

(4) $[v]_n [v]_m = \binom{n+m}{n} [v]_{n+m}$.

For details, see [12].

Let $(e_1, \dots, e_d)$ be an F-basis of V. Then, $\Gamma^n(V)$ inherits a (canonical) basis, consisting of symbols $[e_1]_{a_1} \dots [e_d]_{a_d}$, where $a_i \geq 0$ and $a_1 + \cdots + a_d = n$. Dually, $\mathrm{Sym}^n(V)$ inherits its usual monomial basis, consisting of tensors $e_1^{a_1} \dots e_d^{a_d}$.

There are two natural arrows

$$\mathrm{Sym}^n(V) \longrightarrow \Gamma^n(V)$$
$$v_1 v_2 \dots v_n \longmapsto [v_1]_1 [v_2]_1 \dots [v_n]_1,$$

and

$$\Gamma^n(V) \longrightarrow \mathrm{Sym}^n(V)$$
$$[v]_n \longmapsto v^n.$$

Their composites equal $n!\mathrm{Id}$. Hence, if $\mathrm{char}(F) = 0$ or $p > n$, they are isomorphisms.

4. Jet spaces via infinitesimal Weil restriction

"One-dimensional" jet spaces (i.e. with values in $F[X]/X^n$, for some $n \geq 1$) are a famous tool in many branches of geometry. However, they would not suffice here. In this section, we offer a self-contained exposition of what we shall actually need.

Definition 4.1. Denote by A a finite local F-algebra with residue field F, by $\mathcal{M} \subset A$ its maximal ideal, and by $\rho \colon A \to A/\mathcal{M} = F$ its residue homomorphism, which is a retraction of the inclusion $F \hookrightarrow A$.

Definition 4.2 (Jet spaces). Let $q : A \to A'$ be a homomorphism of finite local F-algebras with residue field F. For any F-variety X, q induces a morphism of F-varieties

$$q_* \colon R_{A/F}(X) \longrightarrow R_{A'/F}(X).$$

Formula: for an F-algebra B and for

$$x \in X(B \otimes_F A),$$

$q_*(x)$ is defined as

$$\big(x \circ (\mathrm{Id}_B \otimes q)\big) \in X(B \otimes_F A').$$

Let G be a contravariant group functor, on affine F-varieties.

Define a group functor $\mathbf{J}(G, \rho)$ by

$$\mathbf{J}(G, \rho)(B) := \mathrm{Ker}\,\big(G(A \otimes_F B) \xrightarrow{\rho_*} G(B)\big),$$

for every F-algebra B. For an F-variety X, if generating no confusion, we set

$$\mathbf{J}(X, \rho) := \mathbf{J}\big(\mathbf{Aut}(X), \rho\big).$$

Example 4.3. If $A = F[\varepsilon]$, then ρ_* is the tangent sheaf $TX \to X$, and

$$\mathbf{J}(X, \rho) = H^0(X, TX) = \mathrm{Lie}\,\big(\mathbf{Aut}(X)\big).$$

Lemma 4.4. *Keep notation and assumptions above. Assume moreover, that X is a smooth F-variety. Consider a diagram*

$$\begin{tikzcd} A_1 \arrow[r, "q"] \arrow[d, "\rho_1"] & A_2 \arrow[d, "\rho_2"] \\ F \arrow[r, equal] & F, \end{tikzcd}$$

of epimorphisms of finite local F-algebras with residue field F.

Denote by $\mathcal{M}_i$ the maximal ideal of A_i, and set $\mathcal{I} := \mathrm{Ker}(q)$. Assume that $\mathcal{I}\mathcal{M}_1 = 0$. Then, the morphism of F-varieties

$$q_* \colon R_{A_1/F}(X) \longrightarrow R_{A_2/F}(X)$$

is a torsor under the (pull-back via ρ_2^* of the) vector bundle $TX \otimes_F \mathcal{I}$. Thus, the morphism ρ_* is a composite of torsors under the vector bundle TX. As such, it is affine and smooth.

Proof. The assertion is local on the smooth F-variety X, so that one may assume $X = \mathrm{Spec}(R)$ affine. Then $R_{A_i/F}(X)$ $(i = 1, 2)$ is affine as well. Let B be an F-algebra. By the infinitesimal lifting criterion for smooth morphisms [13, Tag 37.11.7], one sees that the map

$$q_*(B)\colon X(A_1 \otimes_F B) \longrightarrow X(A_2 \otimes_F B)$$

is onto. Let $x_1, y_1 \in X(A_1 \otimes_F B)$ be such that $q_*(B)(x_1) = q_*(B)(x_2)$. Set

$$x_0 := (\rho_1)_*(B)(x_1) = (\rho_2)_*(B)(x_2) \in X(B).$$

Consider x_1, y_1 (resp. x_0) as homomorphisms of F-algebras $R \to A_1 \otimes_F B$ (resp. $R \to B$), and form the difference

$$\delta := (y_1 - x_1)\colon R \longrightarrow A_1 \otimes_F B.$$

This is a priori just an F-linear map. Since $q_*(B)(x_1) = q_*(B)(x_2)$, it takes values in $\mathcal{I} \otimes_F B \subset A_1 \otimes_F B$. Consider $\mathcal{I} \otimes_F B$ as an R-module via x_0, treating $\mathcal{I}$ just as an F-vector space. One then checks that

$$\delta\colon R \longrightarrow \mathcal{I} \otimes_F B$$

is an F-derivation. Conversely, assume given a homomorphism of F-algebras

$$x_1\colon R \longrightarrow A_1 \otimes_F B.$$

Denote its reduction mod $\mathcal{M}$ by $x_0\colon R \to B$. Pick a derivation

$$\delta\colon R \longrightarrow \mathcal{I} \otimes_F B,$$

where the target is considered as an R-module as above. Then

$$y_1 := (x_1 + \delta)\colon R \longrightarrow A_1 \otimes_F B$$

is a homomorphism of F-algebras, such that $q_*(B)(x_1) = q_*(B)(x_2)$. This completes the description of the torsor structure.

For the last assertion, one may assume $\mathcal{M} \neq 0$. Let $n \geq 1$ be the smallest integer such that $\mathcal{M}^n = 0$. Apply induction on $\dim_F(A)$, writing ρ as the composite

$$A = A/\mathcal{M}^n \longrightarrow A/\mathcal{M}^{n-1} \longrightarrow \cdots \longrightarrow A/\mathcal{M}^2 \longrightarrow A/\mathcal{M} = F. \qquad \blacksquare$$

Remark 4.5. Let V be an F-vector space. Consider its affine space

$$X = \mathbb{A}_F(V) := \mathrm{Spec}(\mathrm{Sym}^*(V^\vee)).$$

Treating ρ as a linear form on the F-vector space A, ρ_* is simply

$$\begin{aligned} \mathbb{A}_F(V \otimes_F A) &\longrightarrow \mathbb{A}_F(V) \\ w &\longmapsto (\mathrm{Id}_V \otimes \rho)(w). \end{aligned}$$

Thus, it is a trivial fibration in affine spaces.

Remark 4.6. If X is affine, using the preceding lemma, and vanishing of coherent cohomology over an affine base, one sees that ρ_* is a trivial $\mathbb{A}^N$-fibration, as well.

Proposition 4.7. *Let X be an F-variety. There is a functorial isomorphism between $\mathbf{J}(X,\rho)$, and the functor of sections of the morphism of F-schemes $R_{A/F}(X_A) \xrightarrow{\rho_*} X$.*

Proof. Let us describe, for every F-algebra B, a functorial bijection

$$\mathbf{J}(X,\rho)(B) \xrightarrow{\sim} \left\{s\colon X_B \to R_{A\otimes_F B/B}(X_{A\otimes_F B}),\ \rho_* \circ s = \mathrm{Id}\right\}.$$

For simplicity, we assume $B = F$; the construction actually works in general.

Giving a section $s\colon X \to R_{A/F}(X_A)$ amounts to giving a morphism of A-schemes $f\colon X \times_F A \to X \times_F A$. Assuming that $\rho_*(f)\colon X \to X$ is the identity, one then just needs to show that f is an iso. Since $\mathcal{M}$ is nilpotent, one sees that, as a homeomorphism of the topological space $X \times_F A$, f is the identity. Let (U_i) be a covering of X by open affines. From what was just said, f restricts to morphisms of A-schemes $f_i\colon U_i \times_F A \to U_i \times_F A$. By a straightforward glueing argument, one thus reduces to the case X affine. One may then use Lemma 4.10 below, applied to the homomorphism of A-algebras

$$\Phi\colon \mathcal{O}_X(X) \otimes_F A \longrightarrow \mathcal{O}_X(X) \otimes_F A,$$

which is such that $f = \mathrm{Spec}(\Phi)$. Note that Φ is regarded here as a morphism between free A-modules. Since $\phi = \mathrm{Id}$ is an iso, one concludes that Φ is an iso. Hence f is an isomorphism of A-schemes, as desired. ∎

Corollary 4.8. *Assume that X is a smooth projective F-variety. Denote by $\mathbf{Aut}(X)^0 \subset \mathbf{Aut}(X)$ the connected component of the identity. It is a group scheme, of finite type over F. Then $\mathbf{J}(X,\rho) = \mathbf{J}(\mathbf{Aut}(X)^0,\rho)$ is a smooth, connected and unipotent linear algebraic F-group. Moreover, it is F-split. In other words, it has a composition series with quotients $\mathbb{G}_a$.*

Proof. If $\mathcal{M}^2 = 0$, then the morphism of F-schemes $R_{A/F}(X_A) \xrightarrow{\rho_*} X$ is a trivial torsor under $TX \otimes_F \mathcal{M}$, as proved in Lemma 4.4. By Proposition 4.7, (and using flat base-change: $H^0(X,TX) \otimes_F B = H^0(X_B, TX_B)$, for every F-algebra B), one sees that $\mathbf{J}(X,\rho)$ is the affine space of the finite-dimensional F-vector space $H^0(X,TX) \otimes_F \mathcal{M}$, which proves the corollary. The general case follows by dévissage, as in the proof of Proposition 4.7. ∎

Lemma 4.9. *Let $\iota\colon Z \hookrightarrow Y$ be a closed immersion of smooth affine F-varieties. Denote by $\mathbf{J}(\iota,\rho) \subset \mathbf{J}(Y,\rho)$ (resp. $\mathbf{J}_0(\iota,\rho) \subset \mathbf{J}(Y,\rho)$) the sub-group functor formed by infinitesimal automorphisms f, such that $f_{|Z}$ factors through ι (resp. $f_{|Z} = \iota$). There is an exact sequence (of group functors)*

$$1 \longrightarrow \mathbf{J}_0(\iota,\rho) \longrightarrow \mathbf{J}(\iota,\rho) \xrightarrow{f \mapsto f_{|Z}} \mathbf{J}(Z,\rho) \longrightarrow 1.$$

Proof. Let us show that

$$1 \longrightarrow \mathbf{J}_0(\iota,\rho)(F) \longrightarrow \mathbf{J}(\iota,\rho)(F) \xrightarrow{f \mapsto f_{|Z}} \mathbf{J}(Z,\rho)(F) \longrightarrow 1$$

is exact, as a sequence of abstract groups. The same proof works to show exactness for points in an arbitrary F-algebra R. The only non-trivial part is surjectivity, which we check by induction on $\dim_F(A)$, using Lemma 4.4, of which we adopt notation, and the description of $\mathbf{J}(X,\rho)$ provided in Proposition 4.7. Let $f_1: Z \to R_{A_1/F}(Z)$. By induction, $f_2 := q_* \circ f_1$ extends, to $\tilde{f_2}: Y \to R_{A_2/F}(Y)$. Since Y is affine, $H^1(Y, TY \otimes_F \mathcal{I}) = 0$, so that $\tilde{f_2}$ lifts via q_*, to $g_1: Y \to R_{A_1/F}(Y)$. Consider $(g_1)_{|Z}: Z \to R_{A_1/F}(Y)$. Via q_*, it is sent to f_2. Thus, there exists a unique $\varepsilon \in H^0(Z, TY \otimes_F \mathcal{I})$, such that $(g_1)_{|Z} + \varepsilon = f_1$. Again, since Y is affine, ε extends, to $\tilde{\varepsilon} \in H^0(Y, TY \otimes_F \mathcal{I})$. Then, $\tilde{f_1} := g_1 + \tilde{\varepsilon}$ is the sought-for extension of f_1. ∎

The following result is standard. Lacking a reference, a proof is included.

Lemma 4.10 (Improved Nakayama's, for Artinian rings). *Let M, N be A-modules, and let $\Phi: M \to N$ be an A-linear map. Denote by $\phi: M \otimes_A F \to N \otimes_A F$ the induced F-linear map. The following holds.*

(1) *If ϕ is surjective, so is Φ.*

(2) *Assume that N is a free A-module, possibly of infinite rank. Then if ϕ is injective, so is Φ.*

Proof. To prove (1), proceed by induction on the smallest $k \geq 1$, such that $\mathcal{M}^k M = \mathcal{M}^k N = 0$. Case $k = 1$ is clear. Assume that $\mathcal{M}^{k+1} M = \mathcal{M}^{k+1} N = 0$, and that ϕ is onto. By induction, $(\phi/\mathcal{M}^k): M/\mathcal{M}^k M \to N/\mathcal{M}^k N$ and $\phi_{|\mathcal{M}^k M}: \mathcal{M}^k M \to \mathcal{M}^k N$ are onto. By dévissage, one then sees that Φ is onto.

Let us prove (2) by induction on the length of A. It suffices to show the following.

Let $a \neq 0 \in \mathcal{M}$ be such that $a\mathcal{M} = 0$. Assume that the (A/aA)-linear map

$$\Phi_a := (\Phi/a): M/aM \longrightarrow N/aN$$

is injective (in addition to injectivity of ϕ). Then Φ is injective.

To prove this assertion, pick $x \in \mathrm{Ker}(\Phi)$. Since Φ_a is injective, one gets $x \in aM$. Write $x = am$. Since $a\Phi(m) = \Phi(x) = 0 \in N$, since N is a free A-module, and since $A - \mathcal{M} = A^\times$, it must be the case that $\Phi(m) \in \mathcal{M}N$. Hence $\phi(m) = 0$. Injectivity of ϕ then implies $m \in \mathcal{M}M$. Thus, $x \in a\mathcal{M}M = 0$. ∎

5. Infinitesimal automorphisms of blow-ups

The following improves on [7, Lemma 4.2].

Proposition 5.1. *Let $\iota: Z \hookrightarrow Y$ be a closed immersion of smooth F-varieties. Denote by*

$$\beta: X := \mathrm{Bl}_Z(Y) \longrightarrow Y$$

the blow-up of Y along Z, and by

$$i\colon E \hookrightarrow X$$

the exceptional divisor. Let A be an F-algebra. There is a natural monomorphism

$$\phi\colon \mathrm{Stab}_{\mathbf{Aut}(Y)}(Z)(A) \longrightarrow \mathbf{Aut}(X)(A).$$

Assume that each irreducible component of Z has codimension ≥ 2 in Y. Let A be a finite local F-algebra, with residue homomorphism $\rho\colon A \to F$. Then ϕ induces an isomorphism

$$\Phi\colon \mathbf{J}\big(\mathrm{Stab}_{\mathbf{Aut}(Y)}(Z), \rho\big) \xrightarrow{\sim} \mathbf{J}(X, \rho).$$

Proof. Recall that formation of blow-ups is functorial and commutes to base-change. Precisely, let $f\colon Y \times_F A \to Y \times_F A$ be an automorphism of A-scheme, preserving the subscheme $Z \times_F A$. By functoriality of the blow-up, f induces an A-automorphism of $X \times_F A$. This provides the definition of ϕ. Assume that $f \in \mathrm{Stab}_{\mathbf{Aut}(Y)}(Z)(A)$ is such that $\phi(f) = \mathrm{Id}$. Consider the commutative diagram

$$\begin{tikzcd} X_A \arrow[r, "\beta_A"] \arrow[d, "\mathrm{Id}"] & Y_A \arrow[d, "f"] \\ X_A \arrow[r, "\beta_A"] & Y_A. \end{tikzcd}$$

Since β_A is surjective, one sees that f, as a continuous map, is the identity. Checking that $f = \mathrm{Id}$ becomes local on Y, so that one may assume $Y = \mathrm{Spec}(B)$ is affine and connected. Then B is integral, because Y is smooth over F. Denote by $I \subset B$ the ideal defining Z. Then $X := \mathrm{Proj}(R)$, where $R := \bigoplus_{n=0}^{\infty} I^n$. It is covered by the open affines $\mathrm{Spec}(R[\frac{1}{f}]_0)$, for $0 \neq f \in I$. Since B is integral, the natural arrow $B \otimes_F A \to R[\frac{1}{f}]_0 \otimes_F A$ is injective. The claim follows.

For the second assertion, note that elements of the source and target of Φ are topologically the identity. Thus, the question is local on Y, so that one may assume Y (and hence Z) affine. We use (and adopt notation of) Proposition 4.7. Note that $\mathbf{J}(\mathrm{Stab}_{\mathbf{Aut}(Y)}(Z), \rho) \subset \mathbf{J}(Y, \rho)$ is the sub-functor $\mathbf{J}(\iota, \rho)$ of Lemma 4.9. By [7, Lemma 4.2] (or more accurately, its proof), one knows that

$$H^0(X, TX) = \mathrm{Ker}\big(H^0(Y, TY) \to H^0(Z, N_{Z/Y})\big).$$

Equivalently, a vector field on X is the same thing as a vector field on Y, restricting to a vector field on Z. Via this identification, the torsor structures on both sides are automatically compatible with Φ. One can then proceed by induction on $\dim_F(A)$ again, using Lemma 4.4, which provides an exact sequence

$$0 \longrightarrow H^0(Y, TY \otimes_F \mathcal{I}) \otimes_F R \longrightarrow \mathbf{J}(Y, \rho_1)(R) \longrightarrow \mathbf{J}(Y, \rho_2)(R) \longrightarrow 1,$$

functorial in the F-algebra R. For simplicity, let us work with F-points – the case of

R-points being the same. Consider the natural diagram (of abstract groups)

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Ker}\big(H^0(Y,TY) \to & H^0(Z,N_{Z/Y})\big) \otimes_F \mathcal{I} \arrow[r, "\simeq"] \arrow[d] & H^0(X,TX) \otimes_F \mathcal{I} \arrow[d] \\
& \mathbf{J}(\iota,\rho_1)(F) \arrow[r] \arrow[d] & \mathbf{J}(X,\rho_1)(F) \arrow[d] \\
& \mathbf{J}(\iota,\rho_2)(F) \arrow[r, "\simeq"] \arrow[d] & \mathbf{J}(X,\rho_2)(F) \arrow[d] \\
& 1 & 1.
\end{tikzcd}$$

(Note that horizontal arrows are given by functoriality of the blow-up. In the right column, we used $H^0(X, TX \otimes_F \mathcal{I}) = H^0(X,TX) \otimes_F \mathcal{I}$, and a similar fact in the left column, which hold because $\dim_F(\mathcal{I}) < \infty$.) In this diagram, columns are clearly exact, except possibly at the bottom. To conclude, it remains to prove surjectivity of

$$\mathbf{J}(\iota,\rho_1)(F) \longrightarrow \mathbf{J}(\iota,\rho_2)(F).$$

Using the exact sequence of Lemma 4.9, a diagram chase reduces this to checking surjectivity of $\mathbf{J}_0(\iota,\rho_1)(F) \to \mathbf{J}_0(\iota,\rho_2)(F)$. Pick $f_2 \in \mathbf{J}_0(\iota,\rho_2)(F) \subset \mathbf{J}(Y,\rho_2)(F)$. Extend it (via q_*) to $g_1 \in \mathbf{J}(Y,\rho_1)(F)$. Then $q_*((g_1)_{|Z}) = \iota$, so that $(g_1)_{|Z} = \iota + \varepsilon$, for $\varepsilon \in H^0(Z, TY \otimes_F \mathcal{I})$. Since Y is affine, ε extends, to $\tilde{\varepsilon} \in H^0(Y, TY \otimes_F \mathcal{I})$. Then, $f_1 := g_1 - \tilde{\varepsilon}$ is the sought-for lift of f_2. ∎

6. Automorphisms of blow-ups of projective space, via Chow rings

We begin with gathering, from [8], material on blow-ups and their Chow rings.

Proposition 6.1. *Let $\iota\colon Y \hookrightarrow Z$ be a closed immersion between smooth geometrically integral F-varieties, of codimension $c \geq 2$. Denote by*

$$\beta\colon X := \mathrm{Bl}_Y(Z) \longrightarrow Z$$

the blow-up of Z along Y, and by $e\colon E \hookrightarrow X$ the exceptional divisor.

(1) *The restriction $\beta_{|X-E}\colon (X-E) \to (Z-Y)$ is an isomorphism, providing a natural arrow*

$$\phi\colon \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Z-Y) = \mathrm{Pic}(Z)$$
$$L \longmapsto L_{|X-E}.$$

(2) *The morphism*

$$\pi := \beta_{|E} : E \longrightarrow Y$$

is the projective bundle of the normal bundle $N_{Y/Z}$. *Denote by* $\mathcal{O}_E(1)$ *its twisting sheaf, and set* $\zeta := c_1(\mathcal{O}_E(1)) \in \mathrm{CH}^1(E)$. *The normal bundle* $N_{E/X}$ *is canonically isomorphic to* $\mathcal{O}_E(-1)$. *For all* $i \geq 1$,

$$[E]^i = (-1)^{i-1} e_*(\zeta^{i-1}) \in \mathrm{CH}^i(X).$$

(3) *(Projective bundle formula for* π*). The arrow*

$$\begin{aligned} \bigoplus_{j=i+1-c}^{i} \mathrm{CH}^j(Y) &\longrightarrow \mathrm{CH}^i(E) \\ (x_j) &\longmapsto \sum \pi^*(x_j).\zeta^{i-j} \end{aligned}$$

is an isomorphism, for every $i \geq 1$. *In particular, for* $i = 1$, *the natural arrow*

$$\begin{aligned} \mathrm{Pic}(Y) \oplus \mathbb{Z} &\longrightarrow \mathrm{Pic}(E) \\ (L, a) &\longmapsto \pi^*(L) + \mathcal{O}(a) \end{aligned}$$

is an isomorphism. Denote the projection on the second factor by

$$\begin{aligned} \mathrm{Pic}(E) &\longrightarrow \mathbb{Z} \\ L &\longmapsto d(L). \end{aligned}$$

(4) *The natural arrow*

$$\begin{aligned} \mathrm{Pic}(Z) \oplus \mathbb{Z} &\longrightarrow \mathrm{Pic}(X) \\ (L, a) &\longmapsto \beta^*(L) + \mathcal{O}_X(aE) \end{aligned}$$

is an isomorphism. Its inverse is given by

$$\begin{aligned} \mathrm{Pic}(X) &\longrightarrow \mathrm{Pic}(Z) \oplus \mathbb{Z} \\ L &\longmapsto \big(\phi(L), -d\big(e^*(L)\big)\big). \end{aligned}$$

(5) *More generally, for* $i \geq 1$ *there is a natural exact sequence*

$$0 \longrightarrow \mathrm{CH}^{i-c}(Y) \longrightarrow \mathrm{CH}^i(Z) \oplus \mathrm{CH}^{i-1}(E) \xrightarrow{\sigma} \mathrm{CH}^i(X) \longrightarrow 0,$$

with injection given by

$$w \longmapsto \big(-\iota_*(w), \pi^*(w).\zeta^{c-1}\big),$$

and surjection given by

$$(u, v) \longmapsto \beta^*(u) + e_*(v).$$

If $i < c$, *this boils down to an isomorphism*

$$\mathrm{CH}^i(Z) \oplus \mathrm{CH}^{i-1}(E) \xrightarrow{\sim} \mathrm{CH}^i(X).$$

(6) *Let $i \geq 1$, and $u \in \mathrm{CH}^i(Z)$. Via item (5), the product*

$$\beta^*(u).[E] \in \mathrm{CH}^{i+1}(X)$$

equals $\sigma(0, \pi^*(\iota^*(u)))$.

Proof. Items (1), and the first two assertions of (2), are standard features of blow-ups. The self-intersection formula for $[E]$ [8, Section 6.3], and the projection formula [8, Example 8.1.1], then prove the last formula of (2) by induction on i:

$$[E]^{i+1} = [E].[E]^i = (-1)^{i-1}[E].e_*(\zeta^{i-1}) = (-1)^{i-1}e_*\big(e^*\big([E]\big).\zeta^{i-1}\big) = (-1)^i e_*(\zeta^i).$$

(The starting case $i = 1$ holds by definition.)

Item (3) is [8, Theorem 3.3 (b)]. Item (5) is [8, Proposition 6.7 (e)] (note the explicit formulas in its proof). Item (4) is a particular case of (5), for $i = 1$.

Observe that $\beta \circ e = \iota \circ \pi$. Item (6) follows, using the projection formula:

$$\beta^*(u).[E] = \beta^*(u).e_*(1_E) = e_*\big(e^*\big(\beta^*(u)\big)\big) = e_*\big(\pi^*\big(\iota^*(u)\big)\big). \qquad \blacksquare$$

The content of the following two propositions is that, under suitable assumptions, the automorphism group of a blow-up in projective space, is "as naively expected."

Proposition 6.2. *For $N \geq 6$, let $Y_1, Y_2 \subset \mathbb{P}^N$ be disjoint smooth closed F-subvarieties, geometrically integral and of dimensions in $[1, N-3]$. Denote by*

$$\beta\colon X := \mathrm{Bl}_Y(\mathbb{P}^N) \longrightarrow \mathbb{P}^N$$

the blow-up of $\mathbb{P}^N$ along $Y := Y_1 \sqcup Y_2$, and by $e_i\colon E_i \hookrightarrow X$ the exceptional divisor lying above Y_i, $i = 1, 2$. By functoriality of the blow-up, there is a natural homomorphism of abstract groups

$$\Phi\colon \mathrm{Stab}_{\mathrm{Aut}(\mathbb{P}^N)}(Y_1)(F) \cap \mathrm{Stab}_{\mathrm{Aut}(\mathbb{P}^N)}(Y_2)(F) \longrightarrow \mathrm{Aut}(X)(F).$$

If the F-varieties E_1 and E_2 are not isomorphic, then Φ is an isomorphism.

Proof. That Φ is injective is straightforward. Let us check surjectivity. Let $f\colon X \to X$ be an F-automorphism. Observe that β is the composite

$$X \xrightarrow{\beta_2} X_1 \xrightarrow{\beta_1} \mathbb{P}^N,$$

where β_1 (resp. β_2) is the blow-up of $\mathbb{P}^N$ along Y_1 (resp. of X_1 along $\beta_1^{-1}(Y_2)$). By item (4) of Proposition 6.1, applied two times, one gets that $\mathrm{CH}^1(X)$ is a free $\mathbb{Z}$-module of rank 3, with basis $(\beta^*([H]), [E_1], [E_2])$, where $H \subset \mathbb{P}^N$ is a hyperplane. Since $c \geq 3$, item (5) (applied two times, to β_1 and β_2) provides a natural isomorphism

$$\mathrm{CH}^2(X) \simeq \mathbb{Z}.[\beta^*(H)]^2 \oplus \mathrm{CH}^1(E_1) \oplus \mathrm{CH}^1(E_2).$$

Assume that $f(E_i) \neq E_j$, for all $\{i, j\} \subset \{1, 2\}$. Since $f(E_i) \subset X$ is an effective divisor not contained in $E_1 \sqcup E_2$, the last formula of item (4) then yields a decomposition, for $i = 1, 2$,

$$\big[f(E_i)\big] = a_i\big[\beta^*(H)\big] - b_i[E_1] - c_i[E_2] \in \mathrm{CH}^1(X),$$

with $a_i \geq 1$, and $b_i, c_i \geq 0$. Using item (6) two times (exchanging the roles of E_1 and E_2), one gets, for $i = 1, 2$,

$$[E_1].[E_2] \in \mathrm{CH}^1(E_i) \subset \mathrm{CH}^2(X),$$

w.r.t. the direct sum decomposition above. Thus $[E_1].[E_2] = 0$. One also computes

$$\begin{aligned}\big[f(E_1)\big].\big[f(E_2)\big] &= \big(a_1\big[\beta^*(H)\big] - b_1[E_1] - c_1[E_2]\big).\big(a_2\big[\beta^*(H)\big] - b_2[E_1] - c_2[E_2]\big) \\ &= \big(a_1a_2\big[\beta^*(H)\big]^2, *, *\big),\end{aligned}$$

where it is needless to know the expression of the second and third components. It suffices to observe that $[f(E_1)].[f(E_2)] \neq 0$, whereas $[E_1].[E_2] = 0$. This is impossible, since f induces a ring automorphism of $\mathrm{CH}^*(X)$. Consequently, it must be the case that $f(E_i) = E_j$ for some $\{i, j\} \subset \{1, 2\}$. Then $i = j$, by the assumption made on E_1 and E_2. Say $i = j = 2$, so that $f(E_2) = E_2$. Assume that $f(E_1) \neq E_1$. Then, as above, one may write

$$\big[f(E_1)\big] = a_1\big[\beta^*(H)\big] - b_1[E_1] - c_1[E_2] \in \mathrm{CH}^1(X),$$

with $a_1 \geq 1$, and $b_1, c_1 \geq 0$. Compute:

$$\begin{aligned}\big[f(E_1)\big].\big[f(E_2)\big] &= \big(a_1\big[\beta^*(H)\big] - b_1[E_1] - c_1[E_2]\big).[E_2] \\ &= a_1\big[\beta^*(H)\big].[E_2] - c_1[E_2].[E_2] \in \mathrm{Pic}(E_2) \subset \mathrm{CH}^2(X),\end{aligned}$$

w.r.t. the direct sum decomposition above. Via the projection formula for the projective bundle $\pi_2 \colon E_2 \to Y_2$ (items (2) and (3) of Proposition 6.1), one gets

$$a_1\big[\beta^*(H)\big].[E_2] - c_1[E_2].[E_2] = \big(\iota_2^*\big(a_1[H]\big), c_1\big) \in \big(\mathrm{Pic}(Y_2) \oplus \mathbb{Z}\big) \simeq \mathrm{Pic}(E_2),$$

where $\iota_2 \colon Y_2 \hookrightarrow \mathbb{P}^N$ is the closed immersion, and using item (6) with $u := [H]$. Since $a_1 \geq 1$, the divisor class $\iota_2^*(a_1[H]) \in \mathrm{Pic}(Y_2)$ is ample, on the positive-dimensional projective variety Y_2, hence is non-zero. It follows that $[f(E_1)].[f(E_2)] \neq 0$, contradiction. Thus, $f(E_1) = E_1$ and $f(E_2) = E_2$. Then, f restricts to an automorphism g of $X - E_1 - E_2$, which by item (1) is an open subvariety of $\mathbb{P}^N$, with complement Y of codimension ≥ 2. By Lemma 6.4, g indeed extends to an automorphism of $\mathbb{P}^N$, which necessarily fixes Y_1 and Y_2 separately. ∎

Remark 6.3. Under the same assumptions, Proposition 6.2 can be generalised to a blow-up of any number of disjoint smooth subvarieties.

Lemma 6.4. *Let $Y \subset \mathbb{P}^N$ be a closed F-subvariety, of codimension ≥ 2. Set $U := \mathbb{P}^N - Y$. Every F-automorphism of U extends to an automorphism of $\mathbb{P}^N$.*

Proof. Recall that, on a normal F-variety, regular functions are invariant upon removing a closed subvariety of codimension ≥ 2. The same property then holds for global sections of line bundles, and one also infers that $\mathrm{Pic}(U) = \mathrm{Pic}(\mathbb{P}^N) = \mathbb{Z}.[\mathcal{O}(1)]$. One can then reproduce the classical proof that $\mathrm{Aut}(\mathbb{P}^N)(F) = \mathrm{PGL}_{N+1}(F)$, with U in place of $\mathbb{P}^N$. Here are details. Let g be an F-automorphism of U. Then $g^*([\mathcal{O}(1)])$ is ample and generates $\mathrm{Pic}(U)$; hence $g^*(\mathcal{O}(1)) \simeq \mathcal{O}(1)$. Fix such an isomorphism of line bundles, and consider the effect of g^* on

$$H^0\big(U, \mathcal{O}(1)\big) = \langle X_0, \dots, X_N \rangle = F^{N+1}.$$

This gives a well-defined $\tilde{g} \in \mathrm{PGL}_{N+1}(F)$ – the sought for extension of g. ∎

The following instructive exercise concludes this section. The proof given is by counting points over finite fields, which is more elementary than by Chow groups.

Lemma 6.5. *For $i = 1, 2$, let $a_i, m_i \geq 2$ be integers, and let V_i be a vector bundle of rank m_i over $\mathbb{P}_F^{a_i - 1}$. Denote by $\mathbb{P}(V_i) \to \mathbb{P}_F^{a_i-1}$ the corresponding projective bundles. Assume that $a_1 \neq a_2$, and that $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ are isomorphic as F-varieties. Then $m_1 = a_2$ and $m_2 = a_1$.*

Proof. By a classical "spreading out" argument, one may assume that $F = \mathbb{F}_q$ is a finite field. Indeed, there exists a sub-ring $R \subset F$, which is a $\mathbb{Z}$-algebra of finite type, such that all data in the Lemma are defined over R. More precisely, for $i = 1, 2$ there is a vector bundle $\mathcal{V}_i$ of rank m_i over $\mathbb{P}_R^{a_i-1}$, such that the projective bundles $\mathbb{P}(\mathcal{V}_1)$ and $\mathbb{P}(\mathcal{V}_2)$ are isomorphic as R-schemes. Specialising at a closed point of $\mathrm{Spec}(R)$, one gets a similar data over a finite field, as claimed. Consider the morphism of $\mathbb{F}_q$-varieties $\mathbb{P}(V_i) \to \mathbb{P}_{\mathbb{F}_q}^{a_i-1}$. It induces a surjection of finite sets

$$\mathbb{P}(V_i)(\mathbb{F}_q) \longrightarrow \mathbb{P}_{\mathbb{F}_q}^{a_i-1}(\mathbb{F}_q),$$

with fibers (non-canonically isomorphic to) $\mathbb{P}_{\mathbb{F}_q}^{m_i-1}$. Counting points, one gets

$$\mathbb{P}(V_i)(\mathbb{F}_q) = \frac{(q^{a_i} - 1)(q^{m_i} - 1)}{(q-1)^2}.$$

Since the $\mathbb{F}_q$-varieties $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ are isomorphic, one has

$$\frac{(q^{a_1} - 1)(q^{m_1} - 1)}{(q-1)^2} = \frac{(q^{a_2} - 1)(q^{m_2} - 1)}{(q-1)^2}.$$

For $n \geq 1$, extend scalars to $\mathbb{F}_{q^n}$ to get the same formula, with q^n in place of q. Thus,

$$\frac{(X^{a_1} - 1)(X^{m_1} - 1)}{(X-1)^2} = \frac{(X^{a_2} - 1)(X^{m_2} - 1)}{(X-1)^2} \in \mathbb{Q}(X),$$

and the conclusion follows. ∎

7. Divided powers to the rescue of projective geometry

One can think of the results this section, as a characteristic-free version of polarity. If $\mathrm{char}(F) = 0$, many of these boil down to facts found in [6, Chapter 1].

7.1. Veronese embedding

Here is a convenient coordinate-free definition of the Veronese embedding. Up to the choice of a basis, it agrees with the usual one.

Definition 7.1. Let V be an F-vector space. Let $n \geq 1$ be an integer. The arrow of F-varieties

$$\begin{aligned} \mathrm{Ver}_n \colon \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^n(V)\big) \\ v &\longmapsto [v]_n. \end{aligned}$$

is a closed embedding, called the n-th Veronese embedding.

Proposition 7.2. *Let V be a finite-dimensional F-vector space. Let $n \geq 1$ be an integer. Consider the n-th Veronese embedding*

$$\mathbb{P}(V) \xrightarrow{\mathrm{Ver}_n} \mathbb{P}\big(\Gamma^n(V)\big).$$

The natural arrow

$$\mathbf{Aut}\big(\mathbb{P}(V)\big) = \mathrm{PGL}(V) \longrightarrow \mathrm{PGL}\left(\Gamma^n(V)\right) = \mathbf{Aut}\big(\mathbb{P}\left(\Gamma^n(V)\right)\big)$$

induces an isomorphism of linear algebraic F-groups

$$\phi \colon \mathrm{PGL}(V) \longrightarrow \mathrm{Stab}_{\mathrm{PGL}(\Gamma^n(V))}\left(\mathbb{P}(V) \subset \mathbb{P}\big(\Gamma^n(V)\big)\right).$$

Proof. Can assume $\dim(V) \geq 2$, and F infinite. Let A be an F-algebra. Pick

$$f \in \mathrm{Stab}_{\mathbf{Aut}(\mathbb{P}(\Gamma^n(V)))}\big(\mathbb{P}(V)\big)(A).$$

Then f restricts to an automorphism of the A-scheme $\mathbb{P}(V) \times_F A$; that is, to an element $\psi(f) \in \mathrm{PGL}(V)(A)$. By Yoneda's lemma, this defines an F-morphism

$$\psi \colon \mathrm{Stab}_{\mathbf{Aut}(\mathbb{P}(\Gamma^n(V)))}\big(\mathbb{P}(V)\big) \longrightarrow \mathrm{PGL}(V),$$

which is a retraction of ϕ. Hence, ϕ is an embedding. Since its source is smooth, it suffices to show that every element

$$f \in \mathrm{Ker}(\psi)\big(F[\varepsilon]\big)$$

lies in the image of $\phi(F[\varepsilon])$. Since $F[\varepsilon]$ is local, Grothendieck–Hilbert's Theorem 90 yields $H^1(F[\varepsilon], \mathbb{G}_m) = 0$, so that f lifts to

$$f' \in \mathrm{GL}\left(\Gamma^n(V)\right)\big(F[\varepsilon]\big).$$

Since $\psi(f) = \mathrm{Id}$, there exists a morphism of F-schemes

$$\lambda\colon \big(\mathbb{A}(V) - \{0\}\big) \longrightarrow \mathbb{R}_{F[\varepsilon]/F}(\mathbb{G}_{\mathrm{m}}) \simeq \mathbb{G}_m \times_F \mathbb{A}^1$$
$$v \longmapsto \lambda_1(v) + \lambda_2(v)\varepsilon,$$

such that

$$f'\big([v]_n\big) = \lambda(v)[v]_n,$$

identically on points. Let us check that λ is constant. Since $\dim(V) \geq 2$, and the source of λ is normal, it extends to a morphism of F-varieties

$$\Lambda\colon \mathbb{A}(V) \longrightarrow \mathbb{G}_m \times_F \mathbb{A}^1 \subset \mathbb{A}^2.$$

Denote by δ the degree of Λ, as a polynomial map. From the equality

$$f'\big([v]_n\big) = \lambda(v)[v]_n,$$

also valid on functors of points, we get $n = \delta + n$, whence $\delta = 0$ and Λ is constant. Rescaling f', we can thus assume $\Lambda = 1$. Since F is infinite, pure symbols $[v]_n$ span the F-vector space $\Gamma^n(V)$, so that $f' = \mathrm{Id}$. Hence, $f = \mathrm{Id}$, as desired. ■

7.2. Characteristic-free polarity

Definition 7.3. Let W be an F-vector space, let $X \subset \mathbb{P}(W)$ be a closed F-subscheme, defined by a sheaf of ideals

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}(W)} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Define

$$E_{X,m} := H^0\big(\mathbb{P}(W), \mathcal{I}_X(m)\big) \subset H^0\big(\mathbb{P}(W), \mathcal{O}(m)\big) = \mathrm{Sym}^m(W^\vee).$$

For all sufficiently large m, it generates $\mathcal{I}_X(m)$. For brevity, denote $E_{X,m}$ by E_X. Dualizing the exact sequence F-vector spaces

$$0 \longrightarrow E_X \longrightarrow \mathrm{Sym}^m(W^\vee) \longrightarrow \mathrm{Sym}^m(W^\vee)/E_X \longrightarrow 0,$$

one gets an exact sequence, denoted by

$$0 \longrightarrow L_X \longrightarrow \Gamma^m(W) \longrightarrow E_X^\vee \longrightarrow 0.$$

Lemma 7.4. *Keep notation of Definition 7.3. For m large enough, there is a natural isomorphism of F-schemes*

$$P_X\colon X \longrightarrow \mathbb{P}(L_X) \cap \mathrm{Ver}_m\big(\mathbb{P}(W)\big)$$
$$z \longmapsto [z]_m,$$

where $\cap$ denotes scheme-theoretic intersection in $\mathbb{P}(\Gamma^m(W))$.

Proof. Let R be an F-algebra. Since $m \gg 0$, the set $X(R) \subset \mathbb{P}(W)(R)$ consists of those lines, on which all m-linear forms in E_X vanish. Using duality between $\mathrm{Sym}^m(W^\vee)$ and $\Gamma^m(W)$, this translates as

$$X(R) = \{(w) \in \mathbb{P}(W)(R),\ \phi([w]_m) = 0,\ \forall \phi \in E_W\}.$$

Via the closed immersion Ver_m, the right side of the equality coincides with $\mathbb{P}(L_X)(R) \cap \mathrm{Ver}_m(\mathbb{P}(W))(R)$. This holds for any R, whence the desired isomorphism P_X. ■

7.3. Morphisms of varieties induced by multiplication of $\Gamma(V)$

The following notion is especially important if $\mathrm{char}(F) = p$.

Definition 7.5 (F-disjointness). Let $a_1, a_2, \ldots, a_d$ be positive integers. Say that $a_1, a_2, \ldots, a_d$ are F-disjoint if the following holds.

(1) For $i = 1, \ldots, d-1$, one has $a_i + a_{i-1} + \cdots + a_1 < a_{i+1}$.

(2) If $\mathrm{char}(F) = p$, for $i = 1, \ldots, d-1$ one has $a_i + a_{i-1} + \cdots + a_1 < p^{v_p(a_{i+1})}$.

Example 7.6. Assume $\mathrm{char}(F) = p$, and $a_i = p^{r_i}$, with $0 \le r_1 < r_2 < \cdots < r_d$. Then, $a_1, a_2, \ldots, a_d$ are F-disjoint.

Remark 7.7. If $\mathrm{char}(F) = p$, then (2) implies (1) in Definition above. Thinking in base p, (2) is equivalent to the following. The position of the least non-zero digit of a_{i+1}, is strictly bigger than that of the greatest non-zero digit of $a_i + a_{i-1} + \cdots + a_1$.

Recall a well-known fact.

Lemma 7.8. *Let $a_1, \ldots, a_d$ be non-negative integers. The p-adic valuation of the multinomial coefficient $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$ is the number of carryovers, when computing the sum $a_1 + \cdots + a_d$ in base p. Hence, if $\mathrm{char}(F) = p$ and $a_1, a_2, \ldots, a_d$ are F-disjoint, $\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d}$ is prime-to-p.*

Proof. One can apply induction on $r \ge 2$, using the formula

$$\binom{a_1+\cdots+a_d}{a_1,\ldots,a_d} = \binom{a_1+\cdots+a_d}{a_1+a_2, a_3, \ldots, a_d}\binom{a_1+a_2}{a_1, a_2}.$$

The claim to prove when $r = 2$ is a classical fact, which is also a nice elementary exercise left to the reader. The second assertion readily follows. ■

Lemma 7.9. *Let V be an F-vector space. The following is true.*

(1) *Let a, b be F-disjoint integers, and let $y \in V - \{0\}$. The multiplication*

$$\begin{aligned} M_y\colon \Gamma^a(V) &\longrightarrow \Gamma^{a+b}(V) \\ x &\longmapsto x[y]_b \end{aligned}$$

is an F-linear injection.

(2) *The formula*

$$\begin{aligned}\mu\colon \mathbb{P}\big(\Gamma^a(V)\big) \times_F \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^{a+b}(V)\big)\\ (x,y) &\longmapsto x[y]_b\end{aligned}$$

defines a morphism of F-varieties, injective on $\bar{F}$-points.

(3) *Let $a_1, a_2, \ldots, a_d$ be F-disjoint integers. Then, the formula*

$$\begin{aligned}\tau\colon \mathbb{P}(V) \times_F \cdots \times_F \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^{a_1+\cdots+a_d}(V)\big)\\ (x_1, x_2, \ldots, x_d) &\longmapsto [x_1]_{a_1}[x_2]_{a_2}\ldots[x_d]_{a_d}\end{aligned}$$

defines a morphism of F-varieties, injective on $\bar{F}$-points.

Proof. Can assume $F = \bar{F}$. Let us prove item (1). Pick a basis $(e_1, \ldots, e_n)$ of V, with $e_n = y$. Work in the standard basis $[e_1]_{a_1}\ldots[e_n]_{a_n}$ of $\Gamma^a(V)$, indexed by partitions $a = a_1 + \cdots + a_n$. Similarly, work in the standard basis $[e_1]_{c_1}\ldots[e_n]_{c_n}$ of $\Gamma^{a+b}(V)$, indexed by partitions $a + b = c_1 + \cdots + c_n$. Let us compute:

$$\big([e_1]_{a_1}\ldots[e_n]_{a_n}\big)[y]_b = \binom{a_n+b}{b}[e_1]_{a_1}\ldots[e_{n-1}]_{a_{n-1}}[e_n]_{a_n+b}.$$

If $\mathrm{char}(F) = 0$, it readily follows that M_y is injective. Assume $\mathrm{char}(F) = p$. Since $a_n \leq a < p^{v_p(b)}$, computing $a_n + b$ in base p occurs without carryovers. Thanks to Lemma 7.8, $\binom{a_n+b}{b} \in F$ is non-zero. Consequently, M_y is still injective. Let us prove that (2) implies (3). If $d \geq 3$, τ factors as the composite of

$$\begin{aligned}\mathbb{P}(V) \times_F \cdots \times_F \mathbb{P}(V) \times_F \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^{a_1+\cdots+a_{d-1}}(V)\big) \times_F \mathbb{P}(V)\\ (x_1, \ldots, x_d) &\longmapsto \big([x_1]_{a_1}\ldots[x_{d-1}]_{a_{d-1}}, x_d\big)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\big(\Gamma^{a_1+\cdots+a_{d-1}}(V)\big) \times_F \mathbb{P}(V) &\xrightarrow{\mu} \mathbb{P}\big(\Gamma^{a_1+\cdots+a_d}(V)\big)\\ (x,y) &\longmapsto x[y]_{a_d}.\end{aligned}$$

By induction, item (3) thus indeed follows from (2). It remains to prove (2). That μ is well defined, follows from (1) (injectivity of M_y, for $y \neq 0$). Let us check injectivity of μ on F-points. Let $y, y' \in V - \{0\}$ and $x, x' \in \Gamma^a(V) - \{0\}$, be such that $\mu(x,y) = \mu(x',y')$. Rescaling, one can assume

$$x[y]_b = x'[y']_b \in \Gamma^{a+b}(V).$$

Suppose that $(y) \neq (y') \in \mathbb{P}(V)(F)$. Pick a basis $(e_1 = y, e_2 = y', e_3, \ldots, e_n)$ of V. Working in the monomial basis $([e_1]_{a_1}\ldots[e_n]_{a_n})$ of $\Gamma^a(V)$, and using $a < b$, one sees that $\mathrm{Im}(M_y) \cap \mathrm{Im}(M_{y'}) = \{0\}$, contradicting $M_y(x) = M_{y'}(x') \neq 0$. Hence y and y' are collinear. Rescaling them, one can assume $y = y'$. Using item (1), one concludes that $x = x'$, which finishes the proof. ∎

Remark 7.10. In general, morphisms in items (2) and (3) above, are not injective on tangent spaces: they are not closed immersions.

Remark 7.11. If $\mathrm{char}(F) = 0$, one can then replace $\Gamma^a(V)$ by $\mathrm{Sym}^a(V)$, and accordingly replace symbols $[x]_a$ by $\frac{x^a}{a!}$. Using that the polynomial F-algebra $\mathrm{Sym}(V)$ is a UFD, the proof of Lemma 7.9 is then easier.

8. Concrete Tannakian construction

The goal of this section is Proposition 8.3, a key tool for proving the main theorem. It is related to a result found in [11]: every linear algebraic F-group is isomorphic to the stabiliser of a single tensor of type $(2, 1)$ (also known as a non-associative finite-dimensional F-algebra). Indeed, both statements improve on a classical fact in Tannakian formalism: every linear algebraic F-group is isomorphic to the stabiliser of a finite number of tensors.

At first glance, the reader may think that Proposition 8.3 is a reformulation of the following standard result of Chevalley: for every embedding of linear algebraic F-groups $H \hookrightarrow G$, there exists a finite-dimensional representation V of G, together with a line $L \in \mathbb{P}(V)(F)$, such that $H = \mathrm{Stab}(L) \subset \mathrm{GL}(V)$. However, let it be clear that this resemblance is misleading. To the knowledge of the author, there is no reasonably simple way to deduce Proposition 8.3 from Chevalley's theorem.

8.1. Explicit action of $\mathbf{PGL}_d$ with trivial stabilisers

Lemma 8.1. *Let $d \geq 3$, and let V be a d-dimensional F-vector space. Let $1, a_1, a_2, \dots, a_{d+1}$ be F-disjoint integers. Choose a basis $(e_1, \dots, e_d)$ of V, and define*

$$r := a_1 + \dots + a_{d+1},$$
$$e_{d+1} := e_1 + e_2 + \dots + e_d,$$
$$x := [e_1]_{a_1} [e_2]_{a_2} \dots [e_d]_{a_d} [e_{d+1}]_{a_{d+1}} \in \Gamma^r(V).$$

Then $x \neq 0$, and

$$\mathrm{Stab}_{\mathrm{PGL}(V)}((x)) = \{1\},$$

for the natural action of $\mathrm{PGL}(V)$ *on* $\mathbb{P}(\Gamma^r(V))$.

Proof. That $x \neq 0$ follows from item (3) of Lemma 7.9. The natural extension of F-groups

$$1 \longrightarrow \mathbb{G}_m = Z\big(\mathrm{GL}(V)\big) \longrightarrow \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V) \longrightarrow 1$$

gives rise to the extension of F-groups

$$1 \longrightarrow \mu_r \longrightarrow \mathrm{Stab}_{\mathrm{GL}(V)}(x) \longrightarrow \mathrm{Stab}_{\mathrm{PGL}(V)}((x)) \longrightarrow 1.$$

(Observe that $Z(\mathrm{GL}(V)) \cap \mathrm{Stab}_{\mathrm{GL}(V)}(x) = \mu_r \subset \mathbb{G}_m$.) Its kernel is not étale if $\mathrm{char}(F) = p$. To show triviality of $\mathrm{Stab}_{\mathrm{PGL}(V)}((x))$, one may assume $F = \overline{F}$. Pick $f \in \mathrm{GL}(V)(F)$.

Assume that $f(x) = x$. Using the injectivity statement of Lemma 7.9 (3), one sees that f fixes each e_i up to scalars, implying that f is homothetic. In other words, the group $\mathrm{Stab}_{\mathrm{PGL}(V)}((x))(F)$ is trivial. If $\mathrm{char}(F) = 0$, this finishes the proof. If $\mathrm{char}(F) = p$, it remains to prove triviality of $\mathrm{Lie}(\mathrm{Stab}_{\mathrm{PGL}(V)}(x))$. In computations that will follow, one typically uses the formula

$$[x]_a[x]_b = \binom{a+b}{a}[x]_{a+b},$$

for various $a, b \in \mathbb{N}$, and one checks whether or not $\binom{a+b}{a}$ is divisible by p. Observe that p divides $a_1, \ldots, a_{d+1}$, because $1, a_1, a_2, \ldots, a_{d+1}$ are F-disjoint. Pick $u \in \mathrm{End}(V)$, with matrix $(u_{i,j})_{1\leq i,j\leq d}$ in the basis $(e_1, \ldots, e_d)$. Set $u_i := u(e_i)$. If $i \leq d$, then $u_i = \sum_{j=1}^{d} u_{j,i} e_j$, and $u_{d+1} = u_1 + \cdots + u_d$. Set

$$f := (\mathrm{Id} + \varepsilon u) \in \mathrm{GL}(V)\big(F[\varepsilon]\big).$$

Assume there exists $c \in F$, such that

$$f(x) = (1 + c\varepsilon)x.$$

To conclude the proof, one needs to show $u \in F\mathrm{Id}$. Note that $f(x)$ reads as

$$x = [e_1 + \varepsilon u_1]_{a_1}[e_2 + \varepsilon u_2]_{a_2} \ldots [e_{d+1} + \varepsilon u_{d+1}]_{a_{d+1}}.$$

Expanding divided powers, and comparing coefficients of ε, one gets

$$(E)\colon c[e_1]_{a_1} \ldots [e_{d+1}]_{a_{d+1}} = \sum_{i=1}^{d+1} [u_i]_1[e_1]_{a_1}[e_2]_{a_2} \ldots [e_i]_{a_i-1} \ldots [e_{d+1}]_{a_{d+1}}.$$

In the monomial basis of $\Gamma^r(V)$ furnished by $e_1, \ldots, e_d$, consider the coefficient of

$$M := [e_1]_{a_1-1}[e_2]_{a_2+1}[e_3]_{a_3+a_{d+1}}[e_4]_{a_4}[e_5]_{a_5} \ldots [e_d]_{a_d},$$

of both sides of this equality. It vanishes on the left side. In the sum on the right side, only $i = 1$ can contribute. Let us expand the corresponding term, reading as

$$[u_{1,1}e_1 + \ldots + u_{d,1}e_d]_1[e_1]_{a_1-1}[e_2]_{a_2}[e_3]_{a_3} \ldots [e_d]_{a_d}[e_1 + e_2 + e_3 + \cdots + e_d]_{a_{d+1}}.$$

In the decomposition

$$[e_1 + e_2 + e_3 + \cdots + e_d]_{a_{d+1}} = \sum_{b_1+\cdots+b_d=a_{d+1}} [e_1]_{b_1}[e_2]_{b_2}[e_3]_{b_3} \ldots [e_d]_{b_d},$$

the only two partitions that may contribute to a non-zero multiple of M, are

$$(b_1, b_2, b_3, b_4, \ldots, b_d) = (0, 0, a_{d+1}, 0, \ldots, 0),$$
$$(b_1, b_2, b_3, b_4, \ldots, b_d) = (0, 1, a_{d+1} - 1, 0, \ldots, 0).$$

These terms are given, respectively, by

$$u_{2,1}[e_2]_1[e_1]_{a_1-1}[e_2]_{a_2}[e_3]_{a_3}\dots[e_d]_{a_d}[e_3]_{a_{d+1}} = \binom{a_3+a_{d+1}}{a_3}(a_2+1)u_{2,1}M,$$

$$u_{3,1}[e_3]_1[e_1]_{a_1-1}[e_2]_{a_2}[e_3]_{a_3}\dots[e_d]_{a_d}[e_2]_1[e_3]_{a_{d+1}-1} = \binom{a_3+a_{d+1}}{1, a_3, a_{d+1}-1}(a_2+1)u_{3,1}M.$$

Gathering the information above, one gets

$$0 = \binom{a_3+a_{d+1}}{a_3}(a_2+1)u_{2,1} + \binom{a_3+a_{d+1}}{1, a_3, a_{d+1}-1}(a_2+1)u_{3,1}.$$

Since $1, a_1, a_2, \dots, a_{d+1}$ are F-disjoint, it follows from Lemma 7.8 that p does not divide $\binom{a_3+a_{d+1}}{a_3}(a_2+1)$, but divides $\binom{a_3+a_{d+1}}{1,a_3,a_{d+1}-1}$ (for the latter fact, observe that adding 1 and $(a_{d+1}-1)$ in base p, occurs with carryovers). Thus $u_{2,1}=0$. One can reproduce this argument, with any triple of distinct indices $\in \{1,\dots,d\}$, in place of $(1,2,3)$. One thus gets $u_{i,j}=0$ for all $i \neq j$. Thus, $u_i = \alpha_i e_i$, $i=1,\dots,d$. In (E), put the term of index $i = d+1$ on the other side of the equation. This gives

$$\begin{aligned}(E')\colon c[e_1]_{a_1}\dots[e_{d+1}]_{a_{d+1}} &- [\alpha_1 e_1 + \dots + \alpha_d e_d]_1[e_1]_{a_1}\dots[e_d]_{a_d}[e_{d+1}]_{a_{d+1}-1}\\ &= \sum_{i=1}^{d} a_i\alpha_i[e_1]_{a_1}[e_1]_{a_2}\dots[e_i]_{a_i}\dots[e_{d+1}]_{a_{d+1}} = 0.\end{aligned}$$

To finish, let us work in the following basis of V:

$$(f_1 := -e_2, f_2 := -e_3, \dots, f_{d-1} := -e_d, f_d := e_{d+1}),$$

and in the induced monomial basis of $\Gamma^r(V)$. Note that

$$e_1 = f_1 + \dots + f_d.$$

Equality (E') gives

$$c[e_1]_{a_1}[f_1]_{a_2}\dots[f_d]_{a_{d+1}} = [\alpha_1 e_1 + \dots + \alpha_d e_d]_1[e_1]_{a_1}[f_1]_{a_2}\dots[f_{d-1}]_{a_d}[f_d]_{a_{d+1}-1}.$$

Express both sides in the monomial basis, and consider the coefficient of

$$[f_1]_{a_2+1}[f_2]_{a_3}\dots[f_d]_{a_{d+1}}.$$

On the left side, it is $c(a_2+1)=c$. Since

$$[f_d]_1[f_d]_{a_{d+1}-1} = a_{d+1}[f_d]_{a_{d+1}} = 0,$$

it is 0 on the right side, so that $c=0$. Thus,

$$(E'')\colon [\alpha_1 e_1 + \dots + \alpha_d e_d]_1[e_1]_{a_1}[f_1]_{a_2}\dots[f_{d-1}]_{a_d}[f_d]_{a_{d+1}-1} = 0.$$

Let $\beta_i \in F$ be such that

$$\alpha_1 e_1 + \cdots + \alpha_d e_d = \beta_1 f_1 + \cdots + \beta_d f_d.$$

Computing the coefficient of $[f_1]_{1+a_1+a_2}[f_2]_{a_3}[f_3]_{a_4} \ldots [f_{d-1}]_{a_d}[f_d]_{a_{d+1}-1}$ in (E''), one gets

$$\beta_1 \binom{1+a_1+a_2}{1, a_1, a_2} = 0 \in F,$$

where the multinomial coefficient is prime-to-p by Lemma 7.8. Thus $\beta_1 = 0$. In the same fashion, $\beta_2 = \cdots = \beta_{d-1} = 0$. In other words: all α_i are equal to β_d, so that $u = \beta_d \mathrm{Id}$, as was to be shown. ∎

8.2. Linear algebraic groups as stabilisers of symbols

Proposition 8.2. *Let G be a linear algebraic group over F. There exists a representation $G \hookrightarrow \mathrm{GL}(W)$, such that the composite $G \hookrightarrow \mathrm{GL}(W) \to \mathrm{PGL}(W)$ is faithful, together with the following data.*

(1) *A closed subscheme $Z \subset \mathbb{P}(W)$, such that $G = \mathrm{Stab}_{\mathrm{PGL}(W)}(Z)$.*

(2) *A G-fixed rational point $(w_0) \in \mathbb{P}(W)(F) - Z(F)$.*

Proof. Pick $n \geq 2$ and a faithful representation $G \hookrightarrow \mathrm{GL}_{n-1}$. Note that the natural composite

$$G \hookrightarrow \mathrm{GL}_{n-1} \subset \mathrm{GL}_n \longrightarrow \mathrm{PGL}_n$$

is still an embedding. This way, one gets a faithful representation $G \hookrightarrow \mathrm{PGL}_n$, such that the action of G on $\mathbb{P}^{n-1}$ has (e_n) as an F-rational fixed point. Define

$$d := 2n, \quad V_1 = V_2 := F^n, \quad V := V_1 \oplus V_2.$$

Pick n large enough, so that $d - 1 > \dim(G)$. Consider the diagonal composite

$$G \hookrightarrow \mathrm{GL}(V_1) \overset{x \mapsto (x,x)}{\hookrightarrow} \mathrm{GL}(V),$$

inducing

$$G \hookrightarrow \mathrm{PGL}(V_1) \overset{x \mapsto (x,x)}{\hookrightarrow} \mathrm{PGL}(V).$$

Consider the canonical basis $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})$ of V, obtained by putting together two copies of the canonical basis of V_1. Let $(a_1, a_2, \ldots, a_{d+1})$, $r = a_1 + \cdots + a_{d+1}$ and $x \in \Gamma^r(V)$ be as in the premises of Lemma 8.1. Setting $W := \Gamma^r(V)$, this lemma states that $\mathrm{Stab}_{\mathrm{PGL}(V)}((x) \in \mathbb{P}(W)(F))$ is trivial. Since $\mathrm{PGL}(V)$ is a smooth F-group, acting on the smooth F-variety $\mathbb{P}(W)$, it is known that the $\mathrm{PGL}(V)$-orbit

$$O\big(\mathrm{PGL}(V), x\big) := \mathrm{PGL}(V).x$$

is a locally closed subscheme of $\mathbb{P}(W)$. Indeed, its closure

$$\overline{O\big(\mathrm{PGL}(V), x\big)} \subset \mathbb{P}(W)$$

equipped with its reduced induced scheme structure, is a $\mathrm{PGL}(V)$-stable closed subscheme, and the orbit $O(\mathrm{PGL}(V), x) \subset \overline{O(\mathrm{PGL}(V), x)}$ is open in its closure. Denote by $Z \subset \overline{O(\mathrm{PGL}(V), x)}$ its complement, considered with its reduced induced structure. Set $U := \mathbb{P}(W) - Z$. One has G-equivariant embeddings

$$G \overset{\text{closed}}{\hookrightarrow} \mathrm{PGL}(V) \xrightarrow[\sim]{g \to g.x} O(\mathrm{PGL}(V), x) \overset{\text{closed}}{\hookrightarrow} U \overset{\text{open}}{\hookrightarrow} \mathbb{P}(W).$$

Thus, the G-orbit map

$$\alpha \colon G \xrightarrow{g \to g.x} \mathbb{P}(W)$$

is a locally closed immersion (even though G may not be smooth). Consider its scheme-theoretic image $X \overset{\text{closed}}{\hookrightarrow} \mathbb{P}(W)$. Let us check that X is G-stable. Let R be an F-algebra. Since formation of scheme-theoretic image of a quasi-compact morphism commutes to flat base-change [13, Lemma 101.38.5], the R-scheme $X_R := X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(R)$ is the scheme-theoretic image of the G_R-orbit map α_R. Pick $\gamma \in G(R)$. Then $\gamma.X_R$ is the scheme-theoretic image of the R-morphism

$$(\gamma.\alpha_R) \colon G_R \xrightarrow{g \to (\gamma g).x} \mathbb{P}(W)_R.$$

This morphism factors as

$$G_R \xrightarrow{g \to \gamma g} G_R \xrightarrow{\alpha_R} \mathbb{P}(W)_R.$$

Since $g \mapsto \gamma g$ is an isomorphism, we conclude that $(\gamma.\alpha_R)$ and α_R share the same scheme-theoretic image. Equivalently, $\gamma.X_R = X_R$, proving that X is G-invariant. Let us check that inclusion of linear algebraic F-groups

$$G \subset \mathrm{Stab}_{\mathrm{PGL}(V)}(X)$$

is an equality. Set

$$Y := X - O(G, x) \subset \mathbb{P}(W).$$

It is a closed subset of X. Consider it as a closed subscheme of X, using the reduced induced structure (as such, it may not be G-invariant). Let R be a finite local $\bar{F}$-algebra, and let $\phi \in \mathrm{Stab}_{\mathrm{PGL}(V)}(X)(R)$. Arguing by contradiction, suppose that $\phi.x \notin O(G, x)(R)$. Denote by $\phi_0 \in \mathrm{Stab}_{\mathrm{PGL}(V)}(X)(\bar{F})$ the special fiber of ϕ. Since R is local and $O(G, x) \subset X$ is open, one has $\phi_0.x \notin O(G, x)(\bar{F})$. The monomorphism of F-schemes

$$\begin{aligned} \beta \colon G &\longrightarrow X \\ g &\longmapsto (g\phi_0).x \end{aligned}$$

would then, set-theoretically, take values in Y. Indeed, suppose that there exists $g \in G(\bar{F})$, such that $(g\phi_0).x \in O(G, x)(\bar{F})$. Because $O(G, x)$ is a principal homogeneous space of G, there exists $\gamma \in G(\bar{F})$, such that $(g\phi_0).x = \gamma.x$. Then

$$\gamma^{-1} g\phi_0 \in \mathrm{Stab}_{\mathrm{PGL}(V)}(x)(\bar{F}) = \{1\},$$

implying $\phi_0 \in G(\bar{F})$, hence $\phi_0.x \in O(G,x)(\bar{F})$, contradicting $\phi_0.x \notin O(G,x)(\bar{F})$. Thus, set-theoretically, the monomorphism β takes values in Y. This is impossible because $\dim(G) > \dim(Y)$, as Noetherian topological spaces. One concludes that $\phi.x \in O(x)(R)$, implying $\phi \in G(R)$. This proves $G = \mathrm{Stab}_{\mathrm{PGL}(V)}(X)$.

It remains to prove that X does not intersect $\mathrm{Ver}_r(\mathbb{P}(V)) \subset \mathbb{P}(W)$. Recall that $V = V_1 \oplus V_2$, $V_1 = V_2$, and that the G-action on $\mathbb{P}(V)$ occurs via the composite

$$G \hookrightarrow \mathrm{PGL}(V_1) \xrightarrow{\mathrm{diag}} \mathrm{PGL}(V).$$

By choice of the basis $(e_1, \dots, e_d)$, the orbit $O(\mathrm{PGL}(V_1), x)$ is thus contained in the image of the composite F-morphism

$$\begin{aligned}\pi\colon \mathbb{P}(V_1)^n \times_F \mathbb{P}(V_2)^n \times_F \mathbb{P}(V) \hookrightarrow \mathbb{P}(V)^n \times_F \mathbb{P}(V)^n \times_F \mathbb{P}(V) &\xrightarrow{\tau} \mathbb{P}\big(\Gamma^{a_1+\dots+a_{d+1}}(V)\big)\\ (x_1, x_2, \dots, x_{d+1}) &\mapsto [x_1]_{a_1}[x_2]_{a_2}\dots[x_{d+1}]_{a_{d+1}}.\end{aligned}$$

Here the first arrow is obtained by taking products of the natural closed immersions $\mathbb{P}(V_i) \hookrightarrow \mathbb{P}(V)$, $i = 1, 2$. The arrow τ is that of item (3) of Lemma 7.9. Observe that Ver_r is the composite F-morphism

$$\begin{aligned}\mathbb{P}(V) \xrightarrow{\mathrm{diag}} \mathbb{P}(V)^{d+1} &\xrightarrow{\tau} \mathbb{P}\big(\Gamma^{a_1+\dots+a_{d+1}}(V)\big)\\ x &\longmapsto [x]_{a_1}[x]_{a_2}\dots[x]_{a_{d+1}}.\end{aligned}$$

By item (3) of Lemma 7.9, τ is injective on $\bar{F}$-points. Since $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ intersect trivially as linear subspaces of $\mathbb{P}(V)$, it follows that

$$\mathrm{Im}(\pi)(\bar{F}) \cap \mathrm{Ver}_r\big(\mathbb{P}(V)\big)(\bar{F}) = \varnothing.$$

Since the source of π is proper, one gets

$$\overline{O\big(\mathrm{PGL}(V_1), x\big)}(\bar{F}) \subset \mathrm{Im}(\pi)(\bar{F}),$$

so that

$$\overline{O\big(\mathrm{PGL}(V_1), x\big)} \cap \mathrm{Ver}_r\big(\mathbb{P}(V)\big) = \varnothing,$$

and a fortiori

$$X \cap \mathrm{Ver}_r\big(\mathbb{P}(V)\big) = \varnothing.$$

Next, consider the closed subscheme

$$Z := X \sqcup \mathrm{Ver}_r\big(\mathbb{P}(V)\big) \subset \mathbb{P}(W),$$

which is indeed a disjoint union. Let us check that the natural embedding

$$G \hookrightarrow \mathrm{Stab}_{\mathrm{PGL}(W)}(Z)$$

is an isomorphism. To do so, let R be a finite local $\bar{F}$-algebra, and let

$$\phi \in \mathrm{Stab}_{\mathrm{PGL}(W)}(Z)(R).$$

Then $\phi(\mathrm{Ver}_r(\mathbb{P}(V))_R) \subset Z_R$ is an irreducible smooth clopen R-subscheme, of dimension $d-1 > \dim(G) = \dim(X)$. It thus intersects $X_R \subset Z_R$ trivially. In other words: $\phi(\mathrm{Ver}_r(\mathbb{P}(V))_R) = \mathrm{Ver}_r(\mathbb{P}(V))_R$, and consequently $\phi(X_R) = X_R$. By Proposition 7.2, ϕ belongs to $\mathrm{PGL}(V)(R)$. Since $G = \mathrm{Stab}_{\mathrm{PGL}(V)}(X)$, one then gets $\phi \in G(R)$, as was to be shown. Item (1) is proved. For (2), recalling that $e_n \in H^0(G, V_1)$ and $e_{2n} \in H^0(G, V_2)$, one may take

$$(w_0) := [e_n]_{a_1}[e_{2n}]_{r-a_1} = \tau(e_n, e_{2n}, e_{2n}, \ldots, e_{2n}) \in H^0\big(G, \mathbb{P}(W)(F)\big).$$

The injectivity of τ (already used above to prove $X \cap \mathrm{Ver}_r(\mathbb{P}(V)) = \varnothing$) then guarantees that $(w_0) \notin \mathrm{Ver}_r(\mathbb{P}(V))(\bar{F})$ and $(w_0) \notin \mathrm{Im}(\pi)(\bar{F})$. Hence $(w_0) \notin Z(F)$. ■

Proposition 8.3. *Let G be a linear algebraic group over F. There exists an F-vector space W, integers $n, l \geq 1$, and a linear subspace*

$$L \in \mathrm{Gr}\big(l, \Gamma^n(W)\big)(F),$$

such that

$$G \xrightarrow{\sim} \mathrm{Stab}_{\mathrm{PGL}(W)}(L),$$

as group schemes over F. Moreover, one can take L such that the closed subvarieties $\mathbb{P}(L) \overset{\mathrm{lin}}{\hookrightarrow} \mathbb{P}(\Gamma^n(W))$ and $\mathbb{P}(W) \overset{\mathrm{Ver}_n}{\hookrightarrow} \mathbb{P}(\Gamma^n(W))$ do not intersect.

Proof. Pick a G-representation W, a closed subvariety $Z \subset \mathbb{P}(W)$ and $(w_0) \in \mathbb{P}(W)(F)$ as in Proposition 8.2. For $m \in \mathbb{N}$, consider the F-subspaces $E_Z \in \mathrm{Gr}(l, \mathrm{Sym}^m(W^\vee))(F)$ and $L_Z \in \mathrm{Gr}(l, \Gamma^m(W))(F)$, introduced in Definition 7.3. These are G-stable. Fix m large enough, so that $m \neq -1 \in F$, and

$$G = \mathrm{Stab}_{\mathrm{PGL}(W)}(E_Z)$$

(see Lemma 7.4). Considering the exact sequences of Definition 7.3, one sees that

$$\mathrm{Stab}_{\mathrm{PGL}(W)}(E_Z) = \mathrm{Stab}_{\mathrm{PGL}(W)}(E_Z^\vee) = \mathrm{Stab}_{\mathrm{PGL}(W)}(L_Z),$$

so that

$$G = \mathrm{Stab}_{\mathrm{PGL}(W)}(L_Z).$$

If $\mathrm{char}(F) = 0$ (resp. $\mathrm{char}(F) = p$), set $q := m+1$ (resp. $q := p^s > m$, the smallest p-th power greater than m). Set $n := m + q$. Consider the F-linear map

$$\begin{aligned} M_{w_0}\colon \Gamma^m(W) &\longrightarrow \Gamma^n(W) \\ x &\longmapsto x[w_0]_q. \end{aligned}$$

It is injective by Lemma 7.9. Set

$$L := M_{w_0}(L_Z) \subset \Gamma^n(W).$$

Since $(w_0) \in H^0(G, \mathbb{P}(W))$, there is a natural inclusion of F-groups

$$\mathrm{Stab}_{\mathrm{PGL}(W)}(L_Z) = G \subset \mathrm{Stab}_{\mathrm{PGL}(W)}(L).$$

Let us show it is an equality. To do so, one may assume $F = \bar{F}$. Let A be a finite local F-algebra with maximal ideal $\mathcal{M}$, and let $g \in \mathrm{Stab}_{\mathrm{PGL}(W)}(L)(A)$. We need to show that $g \in \mathrm{Stab}_{\mathrm{PGL}(W)}(L_Z)(A)$. Let us first show that g fixes $(w_0) \in \mathbb{P}(W)(F) \subset \mathbb{P}(W)(A)$. Assume that $A = F$. If w_0 and $g(w_0)$ are not F-collinear, complete them into an F-basis $(w_0, w_1 = g(w_0), w_2, \dots, w_d)$ of W. By assumption, the two subspaces

$$g(L) = [w_1]_q.g(L_Z) \subset \Gamma^n(W)$$

and

$$L = [w_0]_q.L_Z \subset \Gamma^n(W)$$

are equal. Work in the natural basis of $\Gamma^n(W)$ induced by $(w_0, w_1, w_2, \dots, w_d)$. Then, elements of L are linear combinations of symbols of the shape

$$\text{(A): } [w_0]_{a_0} \dots [w_d]_{a_d},$$

with $a_0 \geq q$ and $a_0 + \dots + a_d = n < 2q$. Similarly, elements of $g(L)$ are linear combinations of symbols of the shape

$$\text{(B): } [w_0]_{b_0}[w_1]_{b_1} \dots [w_d]_{b_d},$$

with $b_1 \geq q$ and $b_0 + \dots + b_d = n < 2q$. But no symbol is of both shapes (A) and (B) – a contradiction. Hence g fixes (w_0). If $\mathrm{char}(F) = 0$, this is enough to conclude. It remains to treat the case $\mathrm{char}(F) = p$, $q = p^s > m$ and A arbitrary. Denote by $\bar{g} \in \mathrm{Stab}_{\mathrm{PGL}(W)}(L)(F)$ the residue of g. Pick $z \neq 0 \in W$, such that $(z) \in Z(F)$, so that $[z]_m \in L_Z$ and $(z) \neq (w_0)$. Define

$$w_1 := \bar{g}(z) \in W.$$

By the case $A = F$ dealt with before,

$$\big(\bar{g}(w_0)\big) = (w_0) \in \mathbb{P}(W)(F),$$

so that $(w_0) \neq (w_1)$. Complete w_0, w_1 into an F-basis $(w_0, w_1, w_2, \dots, w_d)$ of W. Rescaling g by an element of $A^\times$, one can assume

$$g(w_0) = w_0 + \varepsilon_1 w_1 + \dots + \varepsilon_d w_d,$$

where $\varepsilon_i \in \mathcal{M}$. Assume first, that $\varepsilon_i \mathcal{M} = 0$ for $i = 1, \dots, d$. There exists $\eta \in \Gamma^m(W) \otimes_F \mathcal{M}$ such that

$$\big[g(z)\big]_m = [w_1]_m + \eta \in \Gamma^m(W) \otimes_F A,$$

and a little computation gives

$$\big[g(w_0)\big]_q = [w_0]_q + \varepsilon_1 [w_0]_{q-1}[w_1]_1 + \dots + \varepsilon_d [w_0]_{q-1}[w_d]_1 \in \Gamma^q(W) \otimes_F A.$$

Developing the product, rearranging terms, one gets

$$\big[g(w_0)\big]_q\big[g(z)\big]_m = (m+1)\varepsilon_1[w_0]_{q-1}[w_1]_{m+1} + \sum_{i=2}^{d}\varepsilon_i[w_0]_{q-1}[w_1]_m[w_i]_1 + [w_0]_q E,$$

for some $E \in \Gamma^m(W)\otimes_F A$. Recall that all elements of L are linear combinations of symbols of shape (A) above. Observe that symbols $[w_0]_{q-1}[w_1]_{m+1}$ and $[w_0]_{q-1}[w_1]_m[w_i]_1$ are not of shape (A). Since $L = g(L)$, it must be the case that $\varepsilon_i = 0$ for $i = 2,\dots,d$. Since $m \neq -1 \in F$, one also has $\varepsilon_1 = 0$. It remains to remove the assumption $\varepsilon_i\mathcal{M} = 0$. This is a straightforward induction on $k \geq 1$, such that $\mathcal{M}^k = 0$. If $k = 1$ there is nothing to do. Assume that $\mathcal{M}^{k+1} = 0$. By induction applied to $A/\mathcal{M}^k$, one gets $\varepsilon_i \in \mathcal{M}^k$, so that $\varepsilon_i\mathcal{M} = 0$ and the above applies, yielding $\varepsilon_i = 0$. We have proved $g((w_0)) = (w_0) \in \mathbb{P}(W)(A)$. By item (1) of Lemma 7.9, the A-linear map

$$\begin{aligned} M_{w_0}\colon \Gamma^m(W)\otimes_F A &\longrightarrow \Gamma^{m+q}(W)\otimes_F A \\ x &\longmapsto x[w_0]_q \end{aligned}$$

is injective. Since

$$M_{w_0}(L_Z) = M_{g(w_0)}\big(g(L_Z)\big) = M_{w_0}\big(g(L_Z)\big),$$

it is then straightforward to see that $g(L_Z) = L_Z$. We have proved

$$G = \mathrm{Stab}_{\mathrm{PGL}(W)}(L_Z) = \mathrm{Stab}_{\mathrm{PGL}(W)}(L).$$

To conclude, it remains to prove that $\mathbb{P}(L)$ and $\mathrm{Ver}_n(\mathbb{P}(W))$ intersect trivially.

By item (2) of Lemma 7.9, the morphism

$$\begin{aligned} \mu\colon \mathbb{P}\big(\Gamma^m(V)\big)\times_F \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^n(V)\big) \\ (x,y) &\longmapsto x[y]_q \end{aligned}$$

is injective on $\bar{F}$-points. Introduce the graph of Ver_m,

$$\begin{aligned} \Delta\colon \mathbb{P}(V) &\longrightarrow \mathbb{P}\big(\Gamma^m(V)\big)\times_F \mathbb{P}(V) \\ v &\longmapsto \big([v]_m, v\big). \end{aligned}$$

Observe that

$$\mathbb{P}(L) = \mu\big(\mathbb{P}(L_Z)\times\{w_0\}\big)$$

and

$$\mathrm{Ver}_n\big(\mathbb{P}(W)\big) = \mu\big(\Delta\big(\mathbb{P}(V)\big)\big).$$

Because $(w_0) \notin Z(\bar{F})$, one has $[w_0]_m \notin L_Z$, so that

$$\big(\mathbb{P}(L_Z)\times\{w_0\}\big)\cap\Delta\big(\mathbb{P}(V)\big) = \varnothing.$$

Thus, $\mathbb{P}(L)\cap\mathrm{Ver}_n(\mathbb{P}(W)) = \varnothing$, as desired. ∎

Question 8.4. In Proposition 8.3, can one take $l = 1$?

Remark 8.5. We suspect that the answer to Question 8.4 is yes, and sketch an optimistic strategy to investigate it. Let W, l, n and L be furnished by Proposition 8.3. By inspection of its proof, one may assume that n is odd. Then, there is a well-defined F-linear map

$$\begin{aligned} \Psi\colon \Lambda^l\big(\Gamma^n(W)\big) &\longrightarrow \Gamma^n\big(\Lambda^l(W)\big) \\ [w_1]_n \wedge \cdots \wedge [w_l]_n &\longmapsto [w_1 \wedge \cdots \wedge w_l]_n. \end{aligned}$$

Assume that $l < \dim(W)$ (which does not at all follow from the proof above). Then Ψ is injective. Set $W' := \Lambda^l(W)$ and consider the composition of closed embeddings

$$\mathrm{Gr}\big(l, \Gamma^n(W)\big) \overset{\mathrm{Pl}}{\hookrightarrow} \mathbb{P}\big(\Lambda^l\big(\Gamma^n(W)\big)\big) \overset{\Psi}{\hookrightarrow} \mathbb{P}(\Gamma^n(W')),$$

where Pl is the Plücker embedding. Denote by $L' \in \mathbb{P}(\Gamma^n(W'))$ the image of L under this composite. One may then hope that, for a suitable choice of the data, the composite arrow

$$G \xrightarrow{\sim} \mathrm{Stab}_{\mathrm{PGL}(W)}(L) \longrightarrow \mathrm{Stab}_{\mathrm{PGL}(W')}(L')$$

is an isomorphism.

Remark 8.6. A positive answer to Question 8.4 would not simplify the proof of Theorem 2.1. It may, however, be useful in other contexts.

9. Proof of Theorem 2.1

Let W, n, L and $l > \dim(W)$ be as in Proposition 8.3. Put $w := \dim(W)$. Define

$$V := \Gamma^n(W)$$

Denote by

$$Z \subset \mathbb{P}(V)$$

the disjoint union of the closed subvarieties $\mathbb{P}(L) \simeq \mathbb{P}^{l-1}$ and $\mathrm{Ver}_n(\mathbb{P}(W)) \simeq \mathbb{P}^{w-1}$. In the proof of Proposition 8.3, W is fixed from the beginning, where w can be picked arbitrarily large. The construction then works for all n sufficiently large. It is straightforward to check that, when n goes to infinity, so do l and $\dim(V) - l$ (whereas w stays fixed). In particular, one may assume that $w \neq l$ and $w \neq (\dim(V) - l)$.

Proposition 9.1. *The natural inclusion*

$$G \hookrightarrow \mathrm{Stab}_{\mathrm{PGL}(V)}\big(Z \subset \mathbb{P}(V)\big)$$

is an isomorphism of algebraic F-groups.

Proof. We first show that

$$\mathrm{Stab}_{\mathrm{PGL}(V)}(Z) = \mathrm{Stab}_{\mathrm{PGL}(V)}\left(\mathrm{Ver}_n\left(\mathbb{P}(W)\right)\right) \cap \mathrm{Stab}_{\mathrm{PGL}(V)}\left(\mathbb{P}(L)\right).$$

Inclusion $\supset$ is clear. To get equality, as both sides are linear algebraic groups over F, it suffices to prove equality of their points, with values in a finite local F-algebra A, which reads as

$$\mathrm{Stab}_{\mathrm{PGL}(V)}(Z)(A) = \mathrm{Stab}_{\mathrm{PGL}(V)}\left(\mathrm{Ver}_n\left(\mathbb{P}(W)\right)\right)(A) \cap \mathrm{Stab}_{\mathrm{PGL}(V)}\left(\mathbb{P}(L)\right)(A).$$

Pick $f \in \mathrm{Stab}_{\mathrm{PGL}(V)}(Z)(A)$. It induces an automorphism of the A-scheme $Z_A := Z \times_F A$, which is the disjoint union of its irreducible clopen subschemes $\mathrm{Ver}(\mathbb{P}(W))_A$ and $\mathbb{P}(L)_A$. These are projective spaces of distinct dimensions, hence non-isomorphic. Thus f preserves them both (which is a purely topological fact), proving the claim.

To conclude, apply Proposition 7.2 combined to equality $G = \mathrm{Stab}_{\mathrm{PGL}(W)}(\mathbb{P}(L))$, provided by Proposition 8.3. ■

Define

$$X := \mathrm{Bl}_Z\left(\mathbb{P}(V)\right).$$

The action of G on $\mathbb{P}(V)$ stabilizes Z; hence an embedding of F-group schemes

$$\Phi\colon G \longrightarrow \mathbf{Aut}(X)$$

Proposition 9.2. *The arrow Φ is an isomorphism.*

Proof. We may assume $F = \bar{F}$. By Proposition 9.1, we know that

$$G \xrightarrow{\sim} \mathrm{Stab}_{\mathbf{Aut}(\mathbb{P}(V))}\left(Z \subset \mathbb{P}(V)\right).$$

Using Proposition 5.1, we thus know that Φ induces an isomorphism

$$\mathbf{J}(G,\rho) \longrightarrow \mathbf{J}\big(\mathbf{Aut}(X),\rho\big),$$

for every finite F-algebra A, with residue homomorphism $\rho\colon A \to F$.

To conclude, it remains to prove that

$$\Phi(F)\colon G(F) \longrightarrow \mathbf{Aut}(X)(F)$$

is onto, as a homomorphism of abstract groups. Denote by $E_1 \subset X$ (resp. $E_2 \subset X$) the exceptional divisor lying over $\mathbb{P}^{w-1}$ (resp. $\mathbb{P}^{l-1}$). Since $w \neq l$ and $w \neq (\dim(V) - l)$, Lemma 6.5 implies that E_1 and E_2 are non-isomorphic F-varieties. Proposition 6.2 then applies, concluding the proof. ■

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