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The low-dimensional homology of projective linear group of degree two

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Abstract. In this article we study the low-dimensional homology of the projective linear group $\mathrm{PGL}_2(A)$ over a commutative ring A. In particular, we prove a Bloch–Wigner type exact sequence over local domains. As application we prove that

$$H_2\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \simeq \mathrm{K}_2(A)\big[\tfrac{1}{2}\big] \quad \text{and} \quad H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \simeq \mathrm{K}_3^{\mathrm{ind}}(A)\big[\tfrac{1}{2}\big],$$

provided $|A/\mathfrak{m}_A| \neq 2, 3, 4, 8$.

1. Introduction

Let A be a commutative ring with 1. Let $\mathrm{GE}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by elementary and diagonal matrices. We say that A is a GE_2-ring if $\mathrm{GE}_2(A) = \mathrm{GL}_2(A)$. This is equivalent to the condition that $\mathrm{E}_2(A) = \mathrm{SL}_2(A)$.

A ring A is called universal for GE_2 if the unstable K-group $\mathrm{K}_2(2, A)$ is generated by Steinberg symbols (see Section 3). We say that A is a universal GE_2-ring if it is a GE_2-ring and is universal for GE_2. If G is any subgroup of $\mathrm{GL}_2(A)$ containing the central subgroup $Z = A^\times I_2$ of scalar matrices, then we will let PG denote the quotient group G/Z.

As our first main result we show that for any commutative ring A, we have the exact sequence

$$H_2\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \to A_{A^\times} \to H_1\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mathcal{G}_A \to 1, \quad (1.1)$$

where $\mathrm{C}(2, A)$ is the central subgroup of $\mathrm{K}_2(2, A)$ generated by Steinberg symbols, $\mathcal{G}_A$ is the square class group of A, i.e., $\mathcal{G}_A := A^\times/(A^\times)^2$, and $A_{A^\times} := A/\langle a - 1 : a \in A^\times \rangle$ (see Theorem 4.1). It follows from this that if A is a universal GE_2-ring, then

$$H_1\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \simeq \mathcal{G}_A \oplus A_{A^\times}.$$

As our second main result we show that if A is a universal GE_2-ring, then we have the exact sequence

$$H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \to \mathcal{P}(A) \xrightarrow{\lambda} H_2\big(\mathrm{PB}_2(A), \mathbb{Z}\big) \to H_2\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \to \mu_2(A) \to 1, \quad (1.2)$$

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where $\mathcal{P}(A)$ is the scissors congruence group of A and $\mathrm{PB}_2(A)$ is the group of upper triangular matrices in $\mathrm{PGL}_2(A)$ (for the general statement see Theorem 8.4 and Corollary 8.6).

Let A be a local ring such that $|A/\mathfrak{m}_A| \neq 2, 3, 4$. Then $H_2(\mathrm{PB}_2(A), \mathbb{Z}) \simeq A^\times \wedge A^\times$ (Proposition 9.7) and we show that the map λ is given by

$$\lambda\big([a]\big) = 2\big(a \wedge (1-a)\big)$$

(Proposition 10.1). As an application we show that if A is a local domain (local ring) such that $|A/\mathfrak{m}_A| \neq 2, 3, 4$ ($|A/\mathfrak{m}_A| \neq 2, 3, 4, 5, 8, 9, 16$), then

$$H_2\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \simeq \mathrm{K}_2(A)\big[\tfrac{1}{2}\big]. \tag{1.3}$$

Let $\mathcal{B}_E(A)$ be the kernel of λ. Then as our third main result we show that if A is a local domain such that $|A/\mathfrak{m}_A| \neq 2, 3, 4, 8$, then we obtain the sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big) \to H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \to \mathcal{B}_E(A) \to 0, \tag{1.4}$$

which is exact at every term except possibly at the term $H_3(\mathrm{PGL}_2(A), \mathbb{Z})$, where the homology of the sequence is annihilated by 4 (see Theorem 11.7 for the general statement). As an application we prove the Bloch–Wigner exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)\big[\tfrac{1}{2}\big] \to H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \to \mathcal{B}(A)\big[\tfrac{1}{2}\big] \to 0, \tag{1.5}$$

where $\mathcal{B}(A) \subseteq \mathcal{P}(A)$ is the Bloch group of A. As an application of this exact sequence we show that

$$H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \simeq \mathrm{K}_3^{\mathrm{ind}}(A)\big[\tfrac{1}{2}\big].$$

The earliest version of the celebrated Bloch–Wigner exact sequence that we found in the literature is the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\mathbb{C}), \mu(\mathbb{C})\big) \to H_3\big(\mathrm{PGL}_2(\mathbb{C}), \mathbb{Z}\big) \to \mathcal{B}(\mathbb{C}) \to 0$$

(see [5, Theorem 4.10]). The exact sequence (1.5) can be seen as a generalization of this classical result to local domains. As we will see, in general the coefficients $\mathbb{Z}[\frac{1}{2}]$ cannot be replaced with integral coefficients $\mathbb{Z}$, even over infinite fields (see for example Proposition 11.9). Moreover, we study the sequence (1.4) over quadratically closed fields, real closed fields, finite fields and non-dyadic local fields (see Propositions 11.5 and 11.9). Finally, we prove a Bloch–Wigner type exact sequence for $\mathrm{PGL}_2(\mathbb{Z})$ and $\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{2}])$.

Here we outline the organization of the present paper. In Section 2, we recall some needed results from the literature over algebraic K-groups, the scissors congruence group and the Bloch–Wigner exact sequence. In Section 3, we recall the Steinberg group $\mathrm{St}(2, A)$, the K-group $\mathrm{K}_2(2, A)$ and give some of its basic properties. In Section 4, we give a detailed account of the action of $\mathrm{PGE}_2(A)$ over $\mathrm{K}_2(2, A)$, construct the important map

$$\kappa : \left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \to A_{A^\times}$$

and prove our first main result, i.e., the exactness of the sequence (1.1). In Section 5, we study two chain complexes $Y_\bullet(A^2) \subseteq L_\bullet(A^2)$ made out of unimodular vectors in A^2 which are columns of matrices in $\mathrm{GE}_2(A)$ and $\mathrm{GL}_2(A)$, respectively, and study the connection between their homology groups. In Section 6, we study the connection between the first homology group of these complexes and the group $(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)})^{\mathrm{ab}}_{\mathrm{PGE}_2(A)}$. In Section 7, we introduce and study a spectral sequence which will be our main tool in handling the second and the third homology groups of $\mathrm{PGE}_2(A)$. In Section 8, we study certain terms of the spectral sequence and prove the exactness of the sequence (1.2). In Section 9, the homology groups of $\mathrm{PB}_2(A)$ have been studied. In Section 10, we calculate the map λ and prove the isomorphism (1.3). In Section 11, we prove our claim about the sequence (1.4) and present the proof of the Bloch–Wigner exact sequence (1.5). Moreover, we prove a Bloch–Wigner type exact sequence over finite fields, real closed fields, non-dyadic local fields and the Euclidean domains $\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{2}]$.

Notations. In this paper all rings are commutative, except possibly group rings, and have the unit element 1. For a commutative ring A let $\mathrm{GL}_2(A)$ be the group of invertible matrices of degree two. If $\mathrm{G}(A)$ is a subgroup of $\mathrm{GL}_2(A)$ which contains $A^\times I_2 = Z(\mathrm{GL}_2(A))$, by $\mathrm{PG}(A)$ we mean $\mathrm{G}(A)/A^\times I_2$. Let $\mu(A)$ denote the group of roots of unity in A, i.e.

$$\mu(A) := \{a \in A : \text{there is } n \in \mathbb{N} \text{ such that } a^n = 1\},$$

and $\mu_2(A) := \{a \in A : a^2 = 1\}$. Let $\mathcal{G}_A := A^\times/(A^\times)^2$. The element of $\mathcal{G}_A$ represented by $a \in A^\times$ is denoted by $\langle a \rangle$. If $\mathcal{B} \to \mathcal{A}$ is a homomorphism of abelian groups, by $\mathcal{A}/\mathcal{B}$ we mean $\mathrm{coker}(\mathcal{B} \to \mathcal{A})$. For an abelian group $\mathcal{A}$, by $\mathcal{A}[\frac{1}{2}]$ we mean $\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$.

2. Algebraic K-theory and scissors congruence group

Let A be a commutative ring. For any non-negative integer $n \geq 1$, we associate two type K-groups to A: Quillen's K-group $\mathrm{K}_n(A)$ and Milnor's K-group $\mathrm{K}_n^M(A)$.

Quillen's K-group $\mathrm{K}_n(A)$ is defined as the n-th homotopy group of the plus-construction of the classifying space of the stable linear group $\mathrm{GL}(A)$, with respect to the perfect elementary subgroup $\mathrm{E}(A)$:

$$\mathrm{K}_n(A) := \pi_n\big(B\,\mathrm{GL}(A)^+\big).$$

Since $B\,\mathrm{E}(A)^+$ is homotopy equivalent to the universal cover of $B\,\mathrm{GL}(A)^+$, for $n \geq 2$ we have

$$\mathrm{K}_n(A) \simeq \pi_n\big(B\,\mathrm{E}(A)^+\big).$$

The Hurewicz map in algebraic topology induces the commutative diagram (for $n \geq 2$)

$$\begin{array}{ccc} & & H_n\big(\mathrm{E}(A),\mathbb{Z}\big) \\ & \overset{h_n'}{\nearrow} & \downarrow \\ \mathrm{K}_n(A) & & \\ & \underset{h_n}{\searrow} & \\ & & H_n\big(\mathrm{GL}(A),\mathbb{Z}\big). \end{array}$$

If A' is another commutative ring, there is a natural anti-commutative product map

$$\mathrm{K}_m(A) \otimes_{\mathbb{Z}} \mathrm{K}_n(A') \to \mathrm{K}_{m+n}(A \otimes_{\mathbb{Z}} A'), \quad x \otimes y \mapsto x \star y.$$

When $A' = A$ and $\eta : A \otimes_{\mathbb{Z}} A \to A$ is given by $a \otimes b \mapsto ab$, then we have the product map

$$\mathrm{K}_m(A) \otimes_{\mathbb{Z}} \mathrm{K}_n(A) \xrightarrow{\eta_* \circ \star} \mathrm{K}_{n+m}(A), \quad x \otimes y \mapsto \eta_*(x \star y).$$

For more on these K-groups and the construction of the product map see [26, Chapter 2].

The n-th Milnor K-group $\mathrm{K}_n^M(A)$ is defined as the abelian group generated by symbols $\{a_1, \dots, a_n\}$, $a_i \in A^\times$, subject to the following relations

(i) $\{a_1, \dots, a_i a_i', \dots, a_n\} = \{a_1, \dots, a_i, \dots, a_n\} + \{a_1, \dots, a_i', \dots, a_n\}$, for any $1 \leq i \leq n$,

(ii) $\{a_1, \dots, a_n\} = 0$ if there exist i, j, $i \neq j$, such that $a_i + a_j = 0$ or 1.

Clearly we have the anti-commutative product map

$$\mathrm{K}_m^M(A) \otimes_{\mathbb{Z}} \mathrm{K}_n^M(A) \to \mathrm{K}_{m+n}^M(A),$$
$$\{a_1, \dots, a_m\} \otimes \{b_1, \dots, b_n\} \mapsto \{a_1, \dots, a_m, b_1, \dots, b_n\}.$$

It can be shown that

$$\mathrm{K}_1(A) \overset{h_1}{\simeq} H_1\big(\mathrm{GL}(A), \mathbb{Z}\big) \simeq \mathrm{GL}(A)/\,\mathrm{E}(A), \quad \mathrm{K}_2(A) \overset{h_2'}{\simeq} H_2\big(\mathrm{E}(A), \mathbb{Z}\big).$$

For $n = 1$, we have the natural homomorphism

$$\mathrm{K}_1^M(A) \to \mathrm{K}_1(A), \quad \{a\} \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

The determinant induces the isomorphism

$$\mathrm{K}_1(A) \simeq A^\times \times \mathrm{SK}_1(A) \simeq \mathrm{K}_1^M(A) \times \mathrm{SK}_1(A),$$

where

$$\mathrm{SK}_1(A) := \mathrm{SL}(A)/\,\mathrm{E}(A).$$

If A is a local ring, then $\mathrm{SK}_1(A) = 1$ and thus

$$\mathrm{K}_1(A) \simeq \mathrm{K}_1^M(A).$$

For $n = 2$ we have the natural homomorphism

$$\mathrm{K}_2^M(A) \to \mathrm{K}_2(A), \quad \{a, b\} \mapsto \eta_*\left(\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \star \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}\right).$$

The following result is well known.

Theorem 2.1 (Matsumoto–van der Kallen). *Let A be either a field or a local ring such that its residue field has more than five elements. The natural homomorphism*

$$\mathrm{K}_2^M(A) \to \mathrm{K}_2(A)$$

is an isomorphism

$$(A^\times \otimes_{\mathbb{Z}} A^\times)/\big\langle a \otimes (1-a) : a(1-a) \in A^\times \big\rangle \simeq \mathrm{K}_2^M(A) \simeq \mathrm{K}_2(A).$$

Proof. See [26, Theorem 1.14] and [18, Proposition 3.2]. ∎

Using products of K-groups, one can show that for any positive integer n, there is a natural map

$$\psi_n : \mathrm{K}_n^M(A) \to \mathrm{K}_n(A).$$

For the group $\mathrm{K}_3(A)$ we have the following general result.

Theorem 2.2 (Suslin). *For any ring A we have the exact sequence*

$$\mathrm{K}_1(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{K}_2(A) \xrightarrow{\star} \mathrm{K}_3(A) \xrightarrow{h_3'} H_3\big(\mathrm{E}(A), \mathbb{Z}\big) \to 0.$$

Proof. See [28, Corollary 5.2] ∎

Let $\mathcal{W}_A := \{a \in A : a(1-a) \in A^\times\}$. By definition, the *classical scissors congruence group* $\mathcal{P}(A)$ of A is the quotient of the free abelian group generated by symbols $[a]$, $a \in \mathcal{W}_A$, by the subgroup generated by the elements

$$[a] - [b] + \left[\frac{b}{a}\right] - \left[\frac{1-a^{-1}}{1-b^{-1}}\right] + \left[\frac{1-a}{1-b}\right],$$

where $a, b, a/b \in \mathcal{W}_A$. Let

$$S^2_{\mathbb{Z}}(A^\times) := (A^\times \otimes_{\mathbb{Z}} A^\times)/\langle a \otimes b + b \otimes a : a, b \in A^\times \rangle.$$

The map

$$\lambda : \mathcal{P}(A) \to S^2_{\mathbb{Z}}(A^\times), \quad [a] \mapsto a \otimes (1-a)$$

is well defined. The kernel of λ is called the *Bloch group* of A and is denoted by $\mathcal{B}(A)$. If A is either a field or a local ring such that its residue field has more than five elements, then we have the exact sequence

$$0 \to \mathcal{B}(A) \to \mathcal{P}(A) \to S^2_{\mathbb{Z}}(A^\times) \to \mathrm{K}_2^M(A) \to 0.$$

The group $\mathrm{K}_3(A)$ is closely related to the Bloch group of A. Over a local ring, the indecomposable part of $\mathrm{K}_3(A)$ is defined as follows:

$$\mathrm{K}_3^{\mathrm{ind}}(A) := \mathrm{K}_3(A)/\,\mathrm{K}_3^M(A).$$

Theorem 2.3 (A Bloch–Wigner exact sequence). *Let A be either a field or a local domain such that its residue field has more than 9 elements. Then there is a natural exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\sim} \to \mathrm{K}_3^{\mathrm{ind}}(A) \to \mathcal{B}(A) \to 0,$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A), \mu(A))^{\sim}$ *is the unique non-trivial extension of* $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A), \mu(A))$ *by* $\mu_2(A)$.

Proof. The case of infinite fields has been proved by Suslin in [28, Theorem 5.2] and the case of finite fields has been settled by Hutchinson in [7, Corollary 7.5]. The case of local rings has been dealt in [18, Theorem 6.1]. ■

Let A be either a field or a local domain such that its residue field has more than five elements. Then by Theorem 2.1, $\mathrm{K}_2(A) \simeq \mathrm{K}_2^M(A)$. Since $\mathrm{K}_1(\mathbb{Z}) \simeq \{\pm 1\}$ [26, Example 1.9 (vii)], we have

$$\mathrm{im}\big(\mathrm{K}_1(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{K}_2(A) \xrightarrow{\star} \mathrm{K}_3(A)\big) \subseteq \mathrm{im}\big(\mathrm{K}_3^M(A) \to \mathrm{K}_3(A)\big).$$

Let α_A be the following composite

$$H_3\big(\mathrm{SL}_2(A), \mathbb{Z}\big)_{A^\times} \to H_3\big(\mathrm{SL}(A), \mathbb{Z}\big) \simeq \mathrm{K}_3(A)/\big(\mathrm{K}_1(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{K}_2(A)\big) \to \mathrm{K}_3^{\mathrm{ind}}(A).$$

Note that over local rings $\mathrm{E}(A) = \mathrm{SL}(A)$ and $A^\times \simeq \mathrm{GL}(A)/\mathrm{E}(A)$ acts trivially on the group $H_3(\mathrm{SL}(A), \mathbb{Z})$. The following question was asked by Suslin (see [24, Question 4.4]).

Question 2.4. *For an infinite field F, is the map $\alpha_F : H_3(\mathrm{SL}_2(F), \mathbb{Z})_{F^\times} \to \mathrm{K}_3^{\mathrm{ind}}(F)$ an isomorphism?*

Hutchinson and Tao proved that α_F always is surjective [11, Lemma 5.1]. The answer of the above question is true for all finite fields except for $F = \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_8$ (see [18, Proposition 6.4, Example 6.6]). For more on the above question see [17].

Theorem 2.5. *Let A be a local domain such that $|A/\mathfrak{m}_A| \neq 2, 3, 4, 8$. Then the map*

$$\alpha_A : H_3\big(\mathrm{SL}_2(A), \mathbb{Z}[\tfrac{1}{2}]\big)_{A^\times} \to \mathrm{K}_3^{\mathrm{ind}}(A)[\tfrac{1}{2}]$$

is an isomorphism.

Proof. See [16, Theorem 3.7], [18, Theorem 5.4] and [14, Theorem 6.4]. ■

3. Elementary matrices and the Steinberg group of degree two

Let A be a commutative ring. The elementary group of degree two over A, denoted by $\mathrm{E}_2(A)$, is the subgroup of $\mathrm{GL}_2(A)$ generated by the elementary matrices

$$E_{12}(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad E_{21}(a) := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad a \in A.$$

The elementary matrices satisfy the following relations

(a) $E_{ij}(x)E_{ij}(y) = E_{ij}(x + y)$ for any $x, y \in A$,

(b) $W_{ij}(u)E_{ji}(x)W_{ij}(u)^{-1} = E_{ij}(-u^2x)$, for any $u \in A^\times$ and $x \in A$, where $W_{ij}(u) := E_{ij}(u)E_{ji}(-u^{-1})E_{ij}(u)$.

The *Steinberg group* of A, denoted by $\mathrm{St}(2, A)$, is the group with generators $x_{12}(r)$ and $x_{21}(s)$, $r, s \in A$, subject to the relations

(α) $x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$ for any $r, s \in A$,

(β) $w_{ij}(u)x_{ji}(r)w_{ij}(u)^{-1} = x_{ij}(-u^2r)$, for any $u \in A^\times$ and $r \in A$, where $w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.

The natural map

$$\phi : \mathrm{St}(2, A) \to \mathrm{E}_2(A), \quad x_{ij}(r) \mapsto E_{ij}(r)$$

is a well-defined epimorphism. The kernel of this map is denoted by $\mathrm{K}_2(2, A)$:

$$\mathrm{K}_2(2, A) := \ker(\phi).$$

Always there is a natural map

$$\mathrm{K}_2(2, A) \to \mathrm{K}_2(A),$$

which in general neither is surjective nor injective. If A is a local ring, then this map always is surjectve [27, Theorem 2.13].

For any $u \in A^\times$, let

$$h_{ij}(u) := w_{ij}(u)w_{ij}(-1).$$

It is not difficult to see that $h_{ij}(u)^{-1} = h_{ji}(u)$ [10, Corollary A.5]. For any $u, v \in A^\times$, the element

$$\{u, v\}_{ij} := h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}$$

lies in $\mathrm{K}_2(2, A)$ and in the center of $\mathrm{St}(2, A)$ [4, Section 9]. It is straightforward to check that $\{u, v\}_{ji} = \{v, u\}_{ij}^{-1}$. An element of form

$$\{v, u\} := \{v, u\}_{12} = h_{12}(uv)h_{12}^{-1}(u)h_{12}(v)^{-1}$$

is called a *Steinberg symbol* in $\mathrm{K}_2(2, A)$.

Let $\mathrm{C}(2, A)$ be the subgroup of $\mathrm{K}_2(2, A)$ generated by the Steinberg symbols $\{u, v\}$, $u, v \in A^\times$. Then $\mathrm{C}(2, A)$ is a central subgroup of $\mathrm{K}_2(2, A)$.

We say that A is *universal for* GE_2 if $\mathrm{K}_2(2, A) = \mathrm{C}(2, A)$. This definition of universal for GE_2 is equivalent to the original definition of Cohn in [3, p. 8]. For a proof of this fact see [10, Appendix A]. A commutative semilocal ring is universal for GE_2 if and only if none of the rings $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/6$ is a direct factor of $A/J(A)$, where $J(A)$ is the Jacobson radical of A [13, Theorem 2.14].

The elementary group $\mathrm{E}_2(A)$ is generated by the matrices

$$E(a) := \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \quad a \in A.$$

In fact,

$$E_{12}(a) = E(-a)E(0)^{-1}, \quad E_{21}(a) = E(0)^{-1}E(a),$$
$$E(0) = E_{12}(1)E_{21}(-1)E_{12}(1).$$

For any $a \in A^\times$, let

$$D(a) := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{D}_2(A).$$

Since $D(-a) = E(a)E(a^{-1})E(a)$, we have $D(a) \in \mathrm{E}_2(A)$. It is straightforward to check that

(1) $E(x)E(0)E(y) = D(-1)E(x+y)$,

(2) $E(x)D(a) = D(a^{-1})E(a^2x)$,

(3) $D(ab)D(a^{-1})D(b^{-1}) = 1$,

where $x, y \in A$ and $a, b \in A^\times$. Let $\mathrm{C}(A)$ be the group generated by symbols $\varepsilon(a)$, $a \in A$, subject to the relations

(i) $\varepsilon(x)\varepsilon(0)\varepsilon(y) = h(-1)\varepsilon(x+y)$ for any $x, y \in A$,

(ii) $\varepsilon(x)h(a) = h(a^{-1})\varepsilon(a^2x)$, for any $x \in A$ and $a \in A^\times$,

(iii) $h(ab)h(a^{-1})h(b^{-1}) = 1$ for any $a, b \in A^\times$,

where

$$h(a) := \varepsilon(-a)\varepsilon(-a^{-1})\varepsilon(-a).$$

Note that by (iii), $h(1) = 1$ and $h(-1)^2 = 1$. Moreover, $\varepsilon(-1)^3 = h(1) = 1$ and $\varepsilon(1)^3 = h(-1)$. There is a natural surjective map

$$\mathrm{C}(A) \to \mathrm{E}_2(A), \quad \varepsilon(x) \mapsto E(x).$$

We denote the kernel of this map by $\mathrm{U}(A)$. Thus we have the extension

$$1 \to \mathrm{U}(A) \to \mathrm{C}(A) \to \mathrm{E}_2(A) \to 1.$$

Proposition 3.1 (Hutchinson). *Let A be a commutative ring. Then the homomorphism*

$$\mathrm{St}(2, A) \to \mathrm{C}(A)$$

given by $x_{12}(a) \mapsto \varepsilon(-a)\varepsilon(0)^3$ and $x_{21}(a) \mapsto \varepsilon(0)^3\varepsilon(a)$ induces isomorphisms

$$\frac{\mathrm{St}(2, A)}{\mathrm{C}(2, A)} \simeq \mathrm{C}(A), \quad \frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)} \simeq \mathrm{U}(A).$$

Proof. See [10, Theorem A.14, Appendix A]. ■

It follows from this theorem that A is universal for GE_2 if and only if $\mathrm{U}(A) = 1$.

4. The group $\mathrm{K}_2(2, A)$ and the abelianization of $\mathrm{GE}_2(A)$

Let A be a commutative ring. Let $\mathrm{D}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by diagonal matrices and let $\mathrm{GE}_2(A)$ be the subgroup of $\mathrm{GL}_2(A)$ generated by $\mathrm{D}_2(A)$ and $\mathrm{E}_2(A)$. It is easy to see that $\mathrm{E}_2(A)$ is normal in $\mathrm{GE}_2(A)$ [3, Proposition 2.1] and the center of $\mathrm{GE}_2(A)$ is $A^\times I_2$. Observe that we have the split extensions

$$1 \to \mathrm{SL}_2(A) \to \mathrm{GL}_2(A) \xrightarrow{\det} A^\times \to 1, \quad 1 \to \mathrm{E}_2(A) \to \mathrm{GE}_2(A) \xrightarrow{\det} A^\times \to 1,$$

and thus

$$\mathrm{GL}_2(A) = \mathrm{SL}_2(A) \rtimes d(A^\times), \quad \mathrm{GE}_2(A) = \mathrm{E}_2(A) \rtimes d(A^\times),$$

where

$$d(A^\times) = \left\{d(a) := \mathrm{diag}(a, 1) : a \in A^\times\right\} \simeq A^\times.$$

We say that A is a GE_2-*ring* if $\mathrm{GE}_2(A) = \mathrm{GL}_2(A)$ (or equivalently $\mathrm{E}_2(A) = \mathrm{SL}_2(A)$). Semilocal rings and Euclidean domains are GE_2-rings [25, p. 245], [3, Section 2].

A ring A is called an *universal* GE_2-*ring* if it is a GE_2-ring and is universal for GE_2. A semilocal ring is a universal GE_2-ring if none of the rings $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/6$ is a direct factor of $A/J(A)$. In particular, any local ring is a universal GE_2-ring. For more example of GE_2-rings and rings universal for GE_2 see [3, 10].

There is a natural action of $\mathrm{PGE}_2(A)$ on $\mathrm{K}_2(2, A)$. Here we give a detailed description of this action. From the extension

$$1 \to \mathrm{K}_2(2, A) \to \mathrm{St}(2, A) \to \mathrm{E}_2(A) \to 1,$$

we see that $\mathrm{E}_2(A)$ acts naturally on $\mathrm{K}_2(2, A)$. More explicitly, $E_{12}(t)$ acts as conjugation by $x_{12}(t)$ and $E_{21}(t)$ acts as conjugation by $x_{21}(t)$. Note that $D(a) = \mathrm{diag}(a, a^{-1})$ acts as conjugation by $h_{12}(a)$. It is straightforward to check that

$$x_{12}(t)^{h_{12}(a)} = x_{12}(a^{-2}t), \quad x_{21}(t)^{h_{12}(a)} = x_{21}(a^2 t).$$

In particular, the scalar matrix $-I_2 = D(-1) \in \mathrm{E}_2(A)$ acts trivially on $\mathrm{K}_2(2, A)$.

For $a \in A^\times$, let $d(a) := \mathrm{diag}(a, 1) \in \mathrm{GE}_2(A)$. For any $t \in A$,

$$E_{12}(t)^{d(a)} = E_{12}(a^{-1}t), \quad E_{21}(t)^{d(a)} = E_{21}(at).$$

It is straightforward to verify that there is a compatible well-defined action of $d(A^\times)$ on $\mathrm{St}(2, A)$ determined by

$$x_{12}(t)^{d(a)} := x_{12}(a^{-1}t), \quad x_{21}(t)^{d(a)} := x_{21}(at).$$

One verifies easily that this proposed action preserves the two defining families of relations of $\mathrm{St}(2, A)$.

This implies that

$$\mathrm{GE}_2(A) = \mathrm{E}_2(A) \rtimes d(A^\times)$$

acts on $\mathrm{St}(2, A)$ via the conjugation formula given above. Using this action we define

$$\mathrm{GSt}(2, A) := \mathrm{St}(2, A) \rtimes d(A^\times)$$

and extend the canonical epimorphism $\phi : \mathrm{St}(2, A) \to \mathrm{E}_2(A)$ to a surjective homomorphism

$$\Phi : \mathrm{GSt}(2, A) \to \mathrm{GE}_2(A).$$

Furthermore, the inclusion $\mathrm{St}(2, A) \to \mathrm{GSt}(2, A)$ induces an isomorphism

$$\mathrm{K}_2(2, A) = \ker \phi \simeq \ker \Phi.$$

Thus we have the extension

$$1 \to \mathrm{K}_2(2, A) \to \mathrm{GSt}(2, A) \to \mathrm{GE}_2(A) \to 1.$$

Thus $\mathrm{GE}_2(A)$ acts by conjugation on $\mathrm{K}_2(2, A)$ which is compatible with the above action of $\mathrm{E}_2(A)$.

The matrix $d(a)$ acts by the formula given above. If we let $d'(a) := \mathrm{diag}(1, a)$, then $d'(a) = \mathrm{diag}(a^{-1}, a)d(a)$ which lifts to $(h_{12}(a^{-1}), d(a))$ in $\mathrm{GSt}(2, A)$. Then this matrix acts on $\mathrm{St}(2, A)$ via the formulas

$$x_{12}(t)^{d'(a)} = x_{12}(at), \quad x_{21}(t)^{d'(a)} = x_{21}(a^{-1}t).$$

It follows in turn that the scalar matrices $aI_2 = d(a)d'(a)$ act trivially. Hence the above action descends to an action of $\mathrm{PGE}_2(A)$ on $\mathrm{K}_2(2, A)$.

Since $\mathrm{C}(2, A)$ is central in $\mathrm{St}(2, A)$, the action by conjugation of $\mathrm{St}(2, A)$ on $\mathrm{C}(2, A)$ is trivial and hence $\mathrm{E}_2(A)$ acts trivially on the image, $\overline{\mathrm{C}}(2, A)$, of $\mathrm{C}(2, A)$ in $\mathrm{K}_2(2, A)^{\mathrm{ab}}$. It can be easily verified that the action of $d(A^\times)$ on $\mathrm{K}_2(2, A)$ induces an action on $\mathrm{C}(2, A)$ given by

$$\{u, v\}^{d(a)} = \{u, a^{-1}\}^{-1}\{u, a^{-1}v\}.$$

Hence the action of $\mathrm{PGE}_2(A)$ on $\mathrm{K}_2(2, A)$ induces an action on the group

$$\frac{\mathrm{K}_2(2, A)^{\mathrm{ab}}}{\overline{\mathrm{C}}(2, A)} \simeq \left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}}.$$

Let $A_{A^\times} := A/\langle a - 1 : a \in A^\times\rangle$. Since $-2 = (-1) - 1$, we have $2 = 0$ in $A_{A^\times}$. Hence we have a natural map

$$A/2A \to A_{A^\times}.$$

There is a natural homomorphism

$$f : \mathrm{St}(2, A) \to A_{A^\times}$$

which sends both $x_{12}(t)$ and $x_{21}(t)$ to (the class of) t, for $t \in A$. Observe that this map sends the elements $w_{ij}(a)$ to 1. Thus it sends $h_{ij}(a)$ to $2 = 0$. Therefore f extends to a homomorphism

$$f : \mathrm{GSt}(2, A) = \mathrm{St}(2, A) \rtimes d(A^\times) \to A_{A^\times}, \quad (x, d) \mapsto f(x).$$

For example $x_{12}(t)^{d(a)} = x_{12}(a^{-1}t)$ and both sides map to t. Now the diagram

$$\begin{array}{ccc} & \mathrm{GSt}(2, A) & \\ \swarrow^{f} & & \searrow \\ A_{A^\times} & \longrightarrow & \mathrm{PGE}_2(A)^{\mathrm{ab}} \end{array}$$

commutes: Here the map

$$A_{A^\times} \to \mathrm{PGE}_2(A)^{\mathrm{ab}}$$

sends t to $E_{12}(t)$ and observe that in $\mathrm{PGE}_2(A)^{\mathrm{ab}}$ we have

$$E_{12}(t) = E_{12}(t)^{W_{12}(1)} = E_{21}(-t) = E_{21}(-t)^{d(-1)} = E_{21}(t).$$

We denote the restriction of f to $\mathrm{K}_2(2, A)$ again by f:

$$f : \mathrm{K}_2(2, A) \to A_{A^\times}.$$

Note that since $f(h_{ij}(u)) = 0$ for all $u \in A^\times$, then $f(\{u, v\}) = 0$ for all $u, v \in A^\times$. Thus f naturally defines a map

$$\kappa = \bar{f} : \left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}} \to A_{A^\times}.$$

Using the action of $d(A^\times)$ on $\mathrm{St}(2, A)$, we can define an action of $d(A^\times)$ on $\mathrm{C}(A)$. More precisely, for any $a \in A^\times$ and $x \in A$ we have

$$E(x)^{d(a)} = D(a)E(a^{-1}x).$$

Thus the compatible action of $d(A^\times) \simeq A^\times$ on $\mathrm{C}(A)$ is determined by

$$\varepsilon(x)^{d(a)} = h(a)\varepsilon(a^{-1}x).$$

It is easy to verify that $h(b)^{d(a)} = h(b)$. If $\mathrm{GC}(A) := \mathrm{C}(A) \rtimes d(A^\times)$, then we have the extension

$$1 \to \mathrm{U}(A) \to \mathrm{GC}(A) \to \mathrm{GE}_2(A) \to 1.$$

Observe that we have the natural morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{K}_2(2, A) & \longrightarrow & \mathrm{GSt}(2, A) & \longrightarrow & \mathrm{GE}_2(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{U}(A) & \longrightarrow & \mathrm{GC}(A) & \longrightarrow & \mathrm{GE}_2(A) & \longrightarrow & 1. \end{array}$$

It is easy to check that the action of aI_2 on $\mathrm{GC}(A)$ is trivial. Note that

$$aI_2 = D(a^{-1})d(a^2).$$

Let $\mathrm{L}(A)$ be the subset of $\mathrm{GC}(A)$ consisting of the elements $(h(a^{-1}), d(a^2))$, $a \in A^\times$. Note that $(h(a^{-1}), d(a^2))$ lies in the center of $\mathrm{GC}(A)$:

$$\begin{aligned}\big(h(a^{-1}), d(a^2)\big)\big(\varepsilon(x), d(b)\big) &= \big(h(a^{-1})\varepsilon(x)^{d(a^2)}, d(a^2b)\big)\\ &= \big(h(a^{-1})h(a^2)\varepsilon(a^{-2}x), d(a^2b)\big)\\ &= \big(h(a)h(a^{-1})\varepsilon(x)h(a^{-1}), d(a^2b)\big)\\ &= \big(\varepsilon(x)h(a^{-1}), d(a^2b)\big)\\ &= \big(\varepsilon(x)h(a^{-1})^{d(b)}, d(ba^2)\big)\\ &= \big(\varepsilon(x), d(b)\big)\big(h(a^{-1}), d(a^2)\big).\end{aligned}$$

Thus $\mathrm{L}(A)$ is a central subgroup of $\mathrm{GC}(A)$. Now set

$$\mathrm{PGC}(A) := \mathrm{GC}(A)/\mathrm{L}(A).$$

Then we have the extension

$$1 \to \mathrm{U}(A) \to \mathrm{PGC}(A) \to \mathrm{PGE}_2(A) \to 1.$$

Theorem 4.1. *For any commutative ring A, we have the exact sequence*

$$H_2c\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \left(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \overset{\kappa}{\to} A_{A^\times} \to H_1\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mathcal{G}_A \to 1,$$

where the map on the right has a splitting $\mathcal{G}_A \to H_1(\mathrm{PGE}_2(A), \mathbb{Z})$.

Proof. From the Lyndon/Hochschild–Serre spectral sequence associated to the above extension we obtain the five term exact sequence

$$\begin{aligned}H_2\big(\mathrm{PGC}(A), \mathbb{Z}\big) \to H_2\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to H_1\big(\mathrm{U}(A), \mathbb{Z}\big)_{\mathrm{PGE}_2(A)}&\\ \to H_1\big(\mathrm{PGC}(A), \mathbb{Z}\big) \to H_1\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to 0&\end{aligned}$$

(see [2, Corollary 6.4, Chapter VII]). By Theorem 3.1,

$$H_1\big(\mathrm{U}(A), \mathbb{Z}\big)_{\mathrm{PGE}_2(A)} \simeq \left(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)}.$$

We prove that

$$H_1\big(\mathrm{PGC}(A), \mathbb{Z}\big) \simeq \mathcal{G}_A \oplus A_{A^\times}.$$

From the split extension $1 \to \mathrm{C}(A) \to \mathrm{GC}(A) \to A^\times \to 1$ we get the split exact sequence

$$0 \to H_1\big(\mathrm{C}(A), \mathbb{Z}\big)_{A^\times} \to H_1\big(\mathrm{GC}(A), \mathbb{Z}\big) \to A^\times \to 1.$$

We show that

$$H_1\big(\mathrm{C}(A), \mathbb{Z}\big) \simeq A/M,$$

where M is the additive subgroup of A generated by $x(a^2-1)$ and $3(b+1)(c+1)$, $x \in A$, $a, b, c \in A^\times$. Consider the map

$$\Phi : \mathrm{C}(A) \to A/M, \quad \prod \varepsilon(a_i) \mapsto \sum (a_i - 3).$$

This map is well defined: In A/M we have $a = a^{-1}$ and $12 = 0$. Hence

$$\Phi\big(h(a)\big) = -3a - 9 = -3(a-1).$$

It is straightforward to check that

$$\Phi\big(\varepsilon(x)\varepsilon(0)\varepsilon(y)\big) = \Phi\big(h(-1)\varepsilon(x+y)\big), \quad \Phi\big(\varepsilon(x)h(a)\big) = \Phi\big(h(a^{-1})\varepsilon(axa)\big),$$
$$\Phi\big(h(ab)h(a^{-1})h(b^{-1})\big) = -3(a+1)(b+1).$$

Thus Φ is a well-defined homomorphism. Hence we have the homomorphism

$$\bar{\Phi} : \mathrm{C}(A)/\big[\mathrm{C}(A), \mathrm{C}(A)\big] \to A/M, \quad \varepsilon(x) \mapsto x - 3.$$

Now define

$$\Psi : A/M \to \mathrm{C}(A)/\big[\mathrm{C}(A), \mathrm{C}(A)\big], \quad x \mapsto \varepsilon(x)\varepsilon(0)^{-1}.$$

This map is a well-defined homomorphism: Consider the items (i), (ii) and (iii) from the definition of $\mathrm{C}(A)$ (Section 3). If in (i) we put $y = -x$, then $\varepsilon(x)\varepsilon(0)\varepsilon(-x) = h(-1)\varepsilon(0)$. Thus in $\mathrm{C}(A)/[\mathrm{C}(A), \mathrm{C}(A)]$, we have $h(-1)\varepsilon(x)\varepsilon(-x) = 1$. From this we obtain

$$h(a)^2 = h(-1)h(a)h(-a) = h(-1)\varepsilon(-a)\varepsilon(-a^{-1})\varepsilon(-a)\varepsilon(a)\varepsilon(a^{-1})\varepsilon(a) = 1.$$

Therefore

$$\Psi(axa) = \varepsilon(axa)\varepsilon(0)^{-1} = h(a)\varepsilon(x)h(a)\varepsilon(0)^{-1} = \Psi(a).$$

Using (ii) for $x = 0$, in $\mathrm{C}(A)/[\mathrm{C}(A), \mathrm{C}(A)]$ we have $\varepsilon(a) = h(a)\varepsilon(a^{-1})h(a) = \varepsilon(a^{-1})$. This implies that $h(-a) = \varepsilon(a)\varepsilon(a^{-1})\varepsilon(a) = \varepsilon(a)^3$ and hence $h(a) = h(-1)\varepsilon(a)^3 = h(-1)\varepsilon(a^{-1})^3$. Furthermore, by (i), we have $\varepsilon(3x) = h(-1)\varepsilon(x)^3$. Using this formula we obtain

$$\begin{aligned}
\varepsilon\big(3(a+1)(b+1)\big) &= \varepsilon(0)\varepsilon(ab)^3\varepsilon(a)^3\varepsilon(b)^3\varepsilon(1)^3 \\
&= \varepsilon(0)\varepsilon(ab)^3\varepsilon(a)^3\varepsilon(b)^3 h(-1) \\
&= \varepsilon(0)h(-1)\varepsilon(ab)^3 h(-1)\varepsilon(a^{-1})^3 h(-1)\varepsilon(b^{-1})^3 \\
&= \varepsilon(0)h(ab)h(a^{-1})h(b^{-1}).
\end{aligned}$$

Thus

$$\Psi\big(3(a+1)(b+1)\big) = \varepsilon\big(3(a+1)(b+1)\big)\varepsilon(0)^{-1} = h(ab)h(a^{-1})h(b^{-1}).$$

This shows that Ψ is well defined. It is easy to see that Ψ is a homomorphism of groups. Moreover, one can easily show that $\bar{\Phi}$ and Ψ are mutually inverse. Thus $\bar{\Phi}$ is an isomorphism.

Now following the action of $A^\times$ on $\mathrm{C}(A)$, we see that the action of $A^\times$ on A/M, through $H_1(\mathrm{C}(A), \mathbb{Z})$, is given by

$$a.\bar{x} := -\overline{(a^{-1}x+3)(a-1)}.$$

Therefore

$$H_1\big(\mathrm{C}(A), \mathbb{Z}\big)_{A^\times} \simeq (A/M)_{A^\times} \simeq A/\big\{y(a-1) : y \in A,\ a \in A^\times\big\} = A_{A^\times}.$$

Now it follows from the above exact sequence that

$$H_1\big(\mathrm{GC}(A), \mathbb{Z}\big) \simeq A^\times \oplus A_{A^\times}.$$

From the extension $1 \to \mathrm{L}(A) \to \mathrm{GC}(A) \to \mathrm{PGC}(A) \to 1$ we obtain the exact sequence

$$H_1\big(\mathrm{L}(A), \mathbb{Z}\big) \to H_1\big(\mathrm{GC}(A), \mathbb{Z}\big) \to H_1\big(\mathrm{PGC}(A), \mathbb{Z}\big) \to 0.$$

Now under the isomorphism $H_1(\mathrm{GC}(A), \mathbb{Z}) \simeq A^\times \oplus A_{A^\times}$, we have

$$\big(h(a^{-1}), d(a^2)\big) \mapsto \big(a^2, -3(a-1)\big) = (a^2, 0).$$

Thus

$$H_1\big(\mathrm{PGC}(A), \mathbb{Z}\big) \simeq \mathcal{G}_A \oplus A_{A^\times}.$$

From the above arguments one sees that the above isomorphism is induced by the map

$$\mathrm{PGC}(A) \to \mathcal{G}_A \oplus A_{A^\times}, \quad \big(\varepsilon(x), d(a)\big) \mapsto \big(\langle a\rangle, \overline{x-1}\big).$$

Composing this with $\mathrm{PGSt}(2, A) \to \mathrm{PGC}(A)$, we get the map

$$\alpha : \mathrm{PGSt}(2, A) \to \mathcal{G}_A \oplus A_{A^\times}, \quad \big(x_{ij}(t), d(a)\big) \mapsto \big(\langle a\rangle, \bar{t}\big).$$

It follows from this that the restriction of α to $\mathrm{St}(2, A)$ is given by

$$\alpha : \mathrm{St}(2, A) \to \mathcal{G}_A \oplus A_{A^\times}, \quad x_{ij}(t) \mapsto \big(\langle 1\rangle, \bar{t}\big) = \big(\langle 1\rangle, f\big(x_{ij}(t)\big)\big).$$

This shows that the map

$$\left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \to \mathcal{G}_A \oplus A_{A^\times}$$

is given by $x \mapsto (\langle 1\rangle, \kappa(x))$. The determinant $\det : \mathrm{PGE}_2(A) \to \mathcal{G}_A$ induces the map $\det_* : H_1(\mathrm{PGE}_2(A), \mathbb{Z}) \to \mathcal{G}_A$. This map splits the composition

$$\mathcal{G}_A \to \mathcal{G}_A \oplus A_{A^\times} \to H_1\big(\mathrm{PGE}_2(A), \mathbb{Z}\big).$$

All these give the exact sequence of the theorem. ∎

Let $a, b \in A$ be any two elements such that $1 - ab \in A^\times$. We define

$$\langle a, b\rangle_{ij} := x_{ji}\left(\frac{-b}{1-ab}\right)x_{ij}(-a)x_{ji}(b)x_{ij}\left(\frac{a}{1-ab}\right)h_{ij}(1-ab)^{-1}.$$

It is easy to verify that $\langle a, b\rangle_{ij} \in \mathrm{K}_2(2, A)$. This element is called a *Dennis–Stein symbol*. If $u, v \in A^\times$, then

$$\{u, v\}_{ij} = \left\langle u, \frac{1-v}{u}\right\rangle_{ij} = \left\langle \frac{1-u}{v}, v\right\rangle_{ij}.$$

Hence Dennis–Stein symbols generalize Steinberg symbols.

Corollary 4.2. *If* $\mathrm{K}_2(2, A)$ *is generated by Dennis–Stein symbols, then*

$$H_1\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \simeq \mathcal{G}_A \oplus A_{A^\times}.$$

Proof. It is easy to check that $\kappa(\langle a, b\rangle_{ij}) = 0$. This implies that $\kappa = 0$. Now the claim follows from the above theorem. ∎

Corollary 4.3. *If* $2 \in A^\times$, *then* $H_1(\mathrm{PGE}_2(A), \mathbb{Z}) \simeq \mathcal{G}_A$.

Proof. Since $2 \in A^\times$, $1 = 2 - 1 \in \langle a - 1 : a \in A^\times\rangle$. Thus $A_{A^\times} = 0$ and the claim follows from the above theorem. ∎

Example 4.4. In this example we calculate the first homology of $\mathrm{PGL}_2(A)$ for some rings.

(i) If A is local with maximal ideal $\mathfrak{m}_A$, then $A_{A^\times} = 0$ when $|A/\mathfrak{m}_A| \neq 2$ and $A_{A^\times} = A/\mathfrak{m}_A \simeq \mathbb{F}_2$ when $|A/\mathfrak{m}_A| = 2$. Thus

$$H_1\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \simeq \begin{cases} \mathcal{G}_A & \text{if } |A/\mathfrak{m}_A| \neq 2 \\ \mathcal{G}_A \oplus \mathbb{Z}/2 & \text{if } |A/\mathfrak{m}_A| = 2. \end{cases}$$

(ii) Let A be a semilocal ring such that none of $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/6$ is a direct factor of $A/J(A)$. Then A is a universal GE_2-ring and so by the above theorem

$$H_1\big(\mathrm{PGL}_2(A), \mathbb{Z}\big) \simeq \mathcal{G}_A \oplus A_{A^\times}.$$

(iii) The ring of integers $\mathbb{Z}$ is a universal GE_2-ring [10, Example 6.12]. Since $\mathbb{Z}_{\mathbb{Z}^\times} \simeq \mathbb{Z}/2$ by the above theorem we have

$$H_1\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

(iv) Let m be a square free integer. The ring $A_m := \mathbb{Z}[\frac{1}{m}]$ is a GE_2-ring and

$$(A_m)_{A_m^\times} \simeq \begin{cases} 0 & \text{if } 2 \mid m, \\ \mathbb{Z}/2 & \text{if } 2 \nmid m. \end{cases}$$

If m is odd, then the inclusion $A_m \subseteq \mathbb{Z}_{(2)} = \{a/b : a, b \in \mathbb{Z},\ 2 \nmid b\}$ induces the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & (A_m)_{A_m^\times} & \longrightarrow & H_1\big(\mathrm{PGL}_2(A_m), \mathbb{Z}\big) & \longrightarrow & \mathcal{G}_{A_m} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathbb{Z}_{(2)})_{\mathbb{Z}_{(2)}^\times} & \longrightarrow & H_1\big(\mathrm{PGL}_2(\mathbb{Z}_{(2)}), \mathbb{Z}\big) & \longrightarrow & \mathcal{G}_{\mathbb{Z}_{(2)}} & \longrightarrow & 0. \end{array}$$

Since the left map is an isomorphism, it follows from this diagram that the map

$$(A_m)_{A_m^\times} \to H_1\big(\mathrm{PGL}_2(A_m), \mathbb{Z}\big)$$

is injective. Therefore

$$H_1\big(\mathrm{PGL}_2(A_m), \mathbb{Z}\big) \simeq \begin{cases} \mathcal{G}_{A_m} & \text{if } 2 \mid m, \\ \mathcal{G}_{A_m} \oplus \mathbb{Z}/2 & \text{if } 2 \nmid m. \end{cases}$$

This implies that the map κ is trivial and it follows from this and the above theorem that for any m, the natural map

$$H_2\big(\mathrm{PGL}_2\,\big(\mathbb{Z}[\tfrac{1}{m}]\big), \mathbb{Z}\big) \to \left(\frac{\mathrm{K}_2(2, \mathbb{Z}[\frac{1}{m}])}{\mathrm{C}(2, \mathbb{Z}[\frac{1}{m}])}\right)^{\mathrm{ab}}_{\mathrm{PGL}_2(\mathbb{Z}[\frac{1}{m}])}$$

is surjective. The K-group $\mathrm{K}_2(2, \mathbb{Z}[\frac{1}{m}])$ has been studied in many articles. For example see [8, 20] and their references.

5. The GE_2-unimodular vectors

A column vector $\boldsymbol{u} = \binom{u_1}{u_2} \in A^2$ is called *unimodular* if there exists $\boldsymbol{v} = \binom{v_1}{v_2} \in A^2$ such that $(\boldsymbol{u}, \boldsymbol{v}) := \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \in \mathrm{GL}_2(A)$ and it is called GE_2-*unimodular* if $(\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{GE}_2(A)$.

Lemma 5.1. *If $\boldsymbol{v} \in A^2$ is GE_2-unimodular and if $M = (\boldsymbol{v}, \boldsymbol{w}) \in \mathrm{GL}_2(A)$, then $M \in \mathrm{GE}_2(A)$ and $\boldsymbol{w}$ is GE_2-unimodular.*

Proof. By definition, $\boldsymbol{v}$ is GE_2-unimodular if there exists $N \in \mathrm{GE}_2(A)$ with $N\boldsymbol{e_1} = \boldsymbol{v}$, where $\boldsymbol{e}_1 := \binom{1}{0}$. Hence $\boldsymbol{v}$ is GE_2-unimodular if and only if there exists $P \in \mathrm{GE}_2(A)$ with $P\boldsymbol{v} = \boldsymbol{e_1}$. Thus

$$PM = (\boldsymbol{e_1}, P\boldsymbol{w}).$$

It is clear that a matrix of the form $(\boldsymbol{e_1}, \boldsymbol{u})$ is invertible if and only if it lies in $\mathrm{GE}_2(A)$. ■

For any non-negative integer n, let $L_n(A^2)$ be the free abelian group generated by the set of all $(n+1)$-tuples $(\langle \boldsymbol{v}_0 \rangle, \ldots, \langle \boldsymbol{v}_n \rangle)$ of unimodular vectors $\boldsymbol{v}_i \in A^2$ such that for any $i \neq j$, the matrix $(\boldsymbol{v}_i, \boldsymbol{v}_j)$ is invertible. Note that for a vector $\boldsymbol{v} \in A^2$, $\langle \boldsymbol{v} \rangle$ means the equivalence class up to multiplication by a unit, i.e., $\langle \boldsymbol{v} \rangle = \boldsymbol{v} A^\times$.

We consider $L_n(A^2)$ as a left $\mathrm{PGL}_2(A)$-module in a natural way. If necessary, we convert this action to a right action by the definition $m.g := g^{-1}m$. We define the n-th differential operator

$$\partial_n^L : L_n(A^2) \to L_{n-1}(A^2), \quad n \geq 1,$$

as an alternating sum of face operators which throws away the i-th component of generators. Then

$$L_\bullet(A^2) : \quad \cdots \to L_2(A^2) \xrightarrow{\partial_2^L} L_1(A^2) \xrightarrow{\partial_1^L} L_0(A^2) \to 0$$

is a complex. This complex has been studied in [10].

Let $Y_n(A^2)$ be the free abelian subgroup of $L_n(A^2)$ generated by the set of all $(n+1)$-tuples $(\langle \boldsymbol{v}_0 \rangle, \ldots, \langle \boldsymbol{v}_n \rangle)$ of GE_2-unimodular vectors. Thus $Y_\bullet(A^2)$ is a $\mathrm{PGE}_2(A)$-subcomplex of $L_\bullet(A^2)$. We say that $Y_\bullet(A^2)$ (resp. $L_\bullet(A^2)$) is *exact in dimension* k if $H_k(Y_\bullet(A^2)) = 0$ (resp. $H_k(L_\bullet(A^2)) = 0$).

For a subgroup H of a group G and any H-module M, let $\mathrm{Ind}_H^G M := \mathbb{Z}[G] \otimes_H M$. This extension of scalars is called *induction* from H to G.

Lemma 5.2. *The natural inclusion $Y_\bullet(A^2) \to L_\bullet(A^2)$ induces the isomorphism*

$$L_\bullet(A^2) \simeq \mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)} Y_\bullet(A^2).$$

Proof. Clearly

$$\phi_\bullet : \mathbb{Z}\big[\mathrm{PGL}_2(A)\big] \otimes_{\mathrm{PGE}_2(A)} Y_\bullet(A^2) \to L_\bullet(A^2)$$

given by

$$g \otimes \big(\langle \boldsymbol{v_0} \rangle, \langle \boldsymbol{v_1} \rangle, \ldots, \langle \boldsymbol{v_n} \rangle\big) \mapsto \big(\langle g\boldsymbol{v_0} \rangle, \langle g\boldsymbol{v_1} \rangle, \ldots, \langle g\boldsymbol{v_n} \rangle\big),$$

is a well-defined morphism of complexes of $\mathbb{Z}[\mathrm{PGL}_2(A)]$-modules. In fact, $\phi_\bullet$ is an isomorphism with the inverse morphism

$$\psi_\bullet : L_\bullet(A^2) \to \mathbb{Z}\big[\mathrm{PGL}_2(A)\big] \otimes_{\mathrm{PGE}_2(A)} Y_\bullet(A^2)$$

defined by

$$\big(\langle \boldsymbol{v_0} \rangle, \langle \boldsymbol{v_1} \rangle, \ldots, \langle \boldsymbol{v_n} \rangle\big) \mapsto g \otimes \big(\langle \boldsymbol{e_1} \rangle, \langle g^{-1}\boldsymbol{v_1} \rangle, \ldots, \langle g^{-1}\boldsymbol{v_n} \rangle\big),$$

where $g\boldsymbol{e_1} = \boldsymbol{v_0}$. Note that by Lemma 5.1, $\psi_\bullet$ is well defined. ∎

Corollary 5.3. *For any non-negative integer n,*

$$H_n\big(L_\bullet(A^2)\big) \simeq \mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)} H_n\big(Y_\bullet(A^2)\big).$$

Proof. Since the functor $\mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)}$ is exact on the category of $\mathbb{Z}[\mathrm{PGE}_2(A)]$-modules (since $\mathbb{Z}[\mathrm{PGL}_2(A)]$ is a free $\mathbb{Z}[\mathrm{PGE}_2(A)]$-module), the claim follows from the previous lemma. ∎

The group $\mathrm{SL}_2(A)$ (resp. $\mathrm{E}_2(A)$) acts transitively on the sets of generators of $L_0(A^2)$ (resp. $Y_0(A^2)$). Let

$$\boldsymbol{\infty} := \langle \boldsymbol{e}_1 \rangle, \quad \boldsymbol{0} := \langle \boldsymbol{e}_2 \rangle, \quad \boldsymbol{a} := \langle \boldsymbol{e}_1 + a\boldsymbol{e}_2 \rangle, \quad a \in A^\times,$$

where $\boldsymbol{e}_1 := \binom{1}{0}$ and $\boldsymbol{e}_2 := \binom{0}{1}$. Let $\epsilon : L_0(A^2) \to \mathbb{Z}$ be defined by $\sum_i n_i(\langle \boldsymbol{v}_{0,i} \rangle) \mapsto \sum_i n_i$. We denote the restriction $\epsilon|_{Y_0(A^2)} : Y_0(A^2) \to \mathbb{Z}$ again by ϵ.

Proposition 5.4 (Hutchinson). *For any commutative ring A, $H_0(Y_\bullet(A^2)) \overset{\epsilon}{\simeq} \mathbb{Z}$. In other words, the complex*

$$Y_1(A^2) \overset{\partial_1^Y}{\longrightarrow} Y_0(A^2) \overset{\epsilon}{\to} \mathbb{Z} \to 0$$

always is exact. Moreover, $H_0(L_\bullet(A^2)) \simeq \mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)} \mathbb{Z}$.

Proof. Here we follow the proof of [10, Theorem 3.3]. Clearly $\epsilon : Y_0(A^2) \to \mathbb{Z}$ is surjective. Let $X \in \ker(\epsilon)$. We may assume $X = (\langle \boldsymbol{u} \rangle) - (\langle \boldsymbol{v} \rangle)$. Since $\mathrm{GE}_2(A)$ acts transitively on the generators of $Y_0(A^2)$, we may assume $X = \boldsymbol{\infty} - E\boldsymbol{\infty}$, where $E \in \mathrm{GE}_2(A)$. For any $x \in A$ and $a, b \in A^\times$, we have

$$E(x)\,\mathrm{diag}(a,b) = \mathrm{diag}(b,a)E(b^{-1}xa).$$

Thus any element of $\mathrm{GE}_2(A)$ can be written as product $E'D'$, where $D' \in \mathrm{D}_2(A)$ and $E' \in \mathrm{E}_2(A)$. Since $D'\boldsymbol{\infty} = \boldsymbol{\infty}$, we may assume that $E \in \mathrm{E}_2(A)$. Let

$$E = E(a_1)^{c_1} \cdots E(a_n)^{c_n},$$

where $c_i \in \{1, -1\}$. If $E_i := E(a_1)^{c_1} \cdots E(a_i)^{c_i}$ for $1 \leq i \leq n$, and $E_0 = I_2$, then

$$Y := \sum_{i=1}^{n} (E_i\boldsymbol{\infty}, E_{i-1}\boldsymbol{\infty}) \in Y_1(A^2)$$

and $\epsilon(Y) = X$. This proves our claim. ∎

Over the class of local rings we have the following result of Hutchinson.

Proposition 5.5 (Hutchinson). *Let A be a local ring. Then the complex $Y_\bullet(A^2) \overset{\epsilon}{\to} \mathbb{Z}$ is exact in dimension $< |A/\mathfrak{m}_A|$.*

Proof. See [9, Lemma 3.21]. ∎

It follows from Lemma 5.2 and Shapiro's lemma that the inclusion $Y_\bullet(A^2) \to L_\bullet(A^2)$ induces isomorphisms of the homology groups

$$H_q\big(\mathrm{PGE}_2(A), Y_p(A^2)\big) \simeq H_q\big(\mathrm{PGL}_2(A), L_p(A^2)\big)$$

(for all p, q), which occur in the spectral sequences in Section 7 and in [10, Section 7]. For any $n \geq 0$, let

$$Z_n^{\mathrm{GE}_2}(A^2) := \ker(\partial_n^Y), \quad Z_n^{\mathrm{GL}_2}(A^2) := \ker(\partial_n^L).$$

Now from the above isomorphism and the fact that the functor $\mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)}$ is exact on the category of $\mathbb{Z}[\mathrm{PGE}_2(A)]$-modules, we obtain the isomorphism

$$Z_n^{\mathrm{GL}_2}(A^2) \simeq \mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)} Z_n^{\mathrm{GE}_2}(A^2).$$

From this we obtain the following lemma.

Lemma 5.6. *For all $p, q \geq 0$, we have the isomorphism*

$$H_q\big(\mathrm{PGE}_2(A), Z_p^{\mathrm{GE}_2}(A^2)\big) \simeq H_q\big(\mathrm{PGL}_2(A), Z_p^{\mathrm{GL}_2}(A^2)\big).$$

6. The homology group $H_1(Y_\bullet(A^2))$

Let $\Gamma(A)$ be the graph of unimodular rows introduced and studied in [10, Section 2] and let $\Gamma^{\mathrm{GE}}(A)$ be the analogous graph of GE_2-unimodular rows. Then Lemma 5.1 shows that $\Gamma^{\mathrm{GE}}(A)$ is precisely the path component of $\infty = \langle \boldsymbol{e_1} \rangle$ in $\Gamma(A)$. Furthermore, the transitive action of $\mathrm{PGL}_2(A)$ on $\Gamma(A)$ shows that it decomposes into homeomorphic path components

$$\Gamma(A) = \bigsqcup_{g \in \mathrm{PGL}_2(A)/\,\mathrm{PGE}_2(A)} g.\Gamma^{\mathrm{GE}}(A).$$

If we now let $Y(A)$ denote the clique complex of $\Gamma(A)$ as in [10, Section 2] and if we let $Y^{\mathrm{GE}}(A)$ denote the clique complex of $\Gamma^{\mathrm{GE}}(A)$, then it follows that

$$Y(A) = \bigsqcup_{g \in \mathrm{PGL}_2(A)/\,\mathrm{PGE}_2(A)} g.Y^{\mathrm{GE}}(A).$$

Taking geometric realizations it again follows that

$$\big|Y(A)\big| = \bigsqcup_{g \in \mathrm{PGL}_2(A)/\,\mathrm{PGE}_2(A)} g.\big|Y^{\mathrm{GE}}(A)\big|$$

and that $|Y^{\mathrm{GE}}(A)|$ is the path component at ∞ of $|Y(A)|$. In particular, it follows that the inclusion $|Y^{\mathrm{GE}}(A)| \to |Y(A)|$ induces the following result.

Proposition 6.1 (Hutchinson). *For any commutative ring A, we have the isomorphism*

$$\pi_1\big(\big|Y^{\mathrm{GE}}(A)\big|, \infty\big) = \pi_1\big(\big|Y(A)\big|, \infty\big) \simeq \frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}.$$

Proof. See [10, Theorem 6.9]. ∎

Since the space $|Y^{\mathrm{GE}}(A)|$ is path-connected, it follows from the above theorem that

$$H_1\big(\big|Y^{\mathrm{GE}}(A)\big|, \mathbb{Z}\big) \simeq \pi_1\big(\big|Y^{\mathrm{GE}}(A)\big|, \infty\big) \simeq \left(\frac{\mathrm{K}_2(2, A)}{\mathrm{C}(2, A)}\right)^{\mathrm{ab}}.$$

Let $\Delta^{\mathrm{GE}}(A)$ denote the standard ordered chain complex of the simplicial complex $Y^{\mathrm{GE}}(A)$. As in [10, Section 7], the complex $Y_\bullet(A^2)$ in the current article is the complex of non-degenerate ordered simplices of the simplicial complex $Y^{\mathrm{GE}}(A)$ and the natural map of complexes $Y_\bullet(A^2) \to \Delta^{\mathrm{GE}}(A)$ induces an isomorphism on first homology groups. Thus

$$H_1\big(Y_\bullet(A^2)\big) \simeq H_1\big(\Delta^{\mathrm{GE}}(A), \mathbb{Z}\big) = H_1\big(\big|Y^{\mathrm{GE}}(A)\big|, \mathbb{Z}\big) \simeq \left(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)}\right)^{\mathrm{ab}}.$$

Thus, we have the following.

Theorem 6.2 (Hutchinson). *For any commutative ring A, we have the isomorphisms*

$$H_1\big(Y_\bullet(A^2)\big) \simeq \left(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)}\right)^{\mathrm{ab}}, \quad H_1\big(L_\bullet(A^2)\big) \simeq \mathrm{Ind}_{\mathrm{PGE}_2(A)}^{\mathrm{PGL}_2(A)} \left(\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)}\right)^{\mathrm{ab}}.$$

In particular, if A is universal for GE_2, then $Y_\bullet(A^2) \xrightarrow{\epsilon} \mathbb{Z}$ and $L_\bullet(A^2) \xrightarrow{\epsilon} \mathbb{Z}$ are exact in dimension 1.

Remark 6.3. In [10, Theorem 7.2], Hutchinson states that $H_1(L_\bullet(A^2)) \simeq (\frac{\mathrm{K}_2(2,A)}{\mathrm{C}(2,A)})^{\mathrm{ab}}$. In fact, this is only valid when the space $Y(A)$ is path-connected; i.e., when A is a GE_2-ring. Theorem 6.2 above gives a corrected statement valid for all commutative rings.

7. The main spectral sequence

Let A be a commutative ring. The group $\mathrm{PGE}_2(A)$ acts naturally on $Z_i^{\mathrm{GE}_2}(A^2)$. By Proposition 5.4, the sequence of $\mathrm{PGE}_2(A)$-modules

$$0 \to Z_1^{\mathrm{GE}_2}(A^2) \xrightarrow{\mathrm{inc}} Y_1(A^2) \xrightarrow{\partial_1^Y} Y_0(A^2) \to \mathbb{Z} \to 0$$

is exact. Let $E_\bullet(A^2)$ be the sequence

$$0 \to Z_1^{\mathrm{GE}_2}(A^2) \xrightarrow{\mathrm{inc}} Y_1(A^2) \xrightarrow{\partial_1^Y} Y_0(A^2) \to 0$$

and $B_\bullet(\mathrm{PGE}_2(A)) \to \mathbb{Z}$ be the bar resolution of $\mathrm{PGE}_2(A)$ over $\mathbb{Z}$ [2, Chapter I, Section 5]. Let $D_{\bullet,\bullet}$ be the double complex

$$B_\bullet\big(\mathrm{PGE}_2(A)\big) \otimes_{\mathrm{PGE}_2(A)} E_\bullet(A^2).$$

From this double complex we obtain the first quadrant spectral sequence

$$E^1_{p,q} = \begin{cases} H_q\big(\mathrm{PGE}_2(A), Y_p(A^2)\big) & p = 0,1, \\ H_q\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) & p = 2, \\ 0 & p > 2 \end{cases} \Rightarrow H_{p+q}\big(\mathrm{PGE}_2(A), \mathbb{Z}\big)$$

(see [2, Section 5, Chapter VII]).

The group $\mathrm{PGE}_2(A)$ acts transitively on the set of generators of $Y_i(A^2)$ for $i=0,1$. We choose $(\boldsymbol{\infty})$ and $(\boldsymbol{\infty},\mathbf{0})$ as representatives of the orbit of the generators of $Y_0(A^2)$ and $Y_1(A^2)$, respectively. Then

$$Y_0(A^2)\simeq \mathrm{Ind}_{\mathrm{PB}_2(A)}^{\mathrm{PGE}_2(A)}\mathbb{Z},\quad Y_1(A^2)\simeq \mathrm{Ind}_{\mathrm{PT}_2(A)}^{\mathrm{PGE}_2(A)}\mathbb{Z}, \tag{7.1}$$

where

$$\mathrm{PB}_2(A):=\mathrm{Stab}_{\mathrm{PGE}_2(A)}(\boldsymbol{\infty})=\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}: a,d\in A^\times, b\in A\right\}/A^\times I_2,$$

$$\mathrm{PT}_2(A):=\mathrm{Stab}_{\mathrm{PGE}_2(A)}(\boldsymbol{\infty},\mathbf{0})=\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}: a,d\in A^\times\right\}/A^\times I_2.$$

Note that $\mathrm{PT}_2(A)\xrightarrow{\simeq}A^\times$, which is given by $\mathrm{diag}(a,d)\mapsto ad^{-1}$. The inverse of this map is

$$A^\times\xrightarrow{\simeq}\mathrm{PT}_2(A),\quad a\mapsto \mathrm{diag}(a,1)=\mathrm{diag}(1,a^{-1}).$$

Usually in our calculations we identify $\mathrm{PT}_2(A)$ with $A^\times$. The group $\mathrm{PB}_2(A)$ sits in the split extension

$$1\to N_2(A)\to \mathrm{PB}_2(A)\to \mathrm{PT}_2(A)\to 1,$$

where

$$N_2(A):=\left\{\begin{pmatrix} 1 & b\\ 0 & 1\end{pmatrix}: b\in A\right\}\simeq A.$$

So we have the split extension $0\to A\to \mathrm{PB}_2(A)\to A^\times\to 1$, where the action of $A^\times$ on A is given by $a.x:=ax$. This implies that

$$H_0(A^\times,A)=A_{A^\times}=A/\langle a-1: a\in A^\times\rangle.$$

By Shapiro's lemma, applied to (7.1), we have

$$E^1_{0,q}\simeq H_q\big(\mathrm{PB}_2(A),\mathbb{Z}\big),\quad E^1_{1,q}\simeq H_q\big(\mathrm{PT}_2(A),\mathbb{Z}\big).$$

In particular, $E^1_{0,0}\simeq\mathbb{Z}\simeq E^1_{1,0}$. Moreover,

$$d^1_{1,q}=\sigma_*-\mathrm{inc}_*,$$

where

$$\sigma:\mathrm{PT}_2(A)\to\mathrm{PB}_2(A)$$

is given by $\sigma(X)=wXw^{-1}$ for $w=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$. This easily implies that $d^1_{1,0}$ is trivial, $d^1_{1,1}$ is induced by the map $\mathrm{PT}_2(A)\to\mathrm{PB}_2(A)$, $X\mapsto X^{-2}$, and

$$d^1_{1,2}:H_2\big(\mathrm{PT}_2(A),\mathbb{Z}\big)\to H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big)$$

is trivial.

8. The scissors congruence group

Following Coronado and Hutchinson we define the *scissors congruence group* of A as follows:

$$\mathcal{P}(A) := H_0\big(\mathrm{PGL}_2(A), Z_2^{\mathrm{GL}_2}(A^2)\big).$$

It follows from Lemma 5.6, that

$$\mathcal{P}(A) \simeq H_0\big(\mathrm{PGE}_2(A), Z_2^{\mathrm{GE}_2}(A^2)\big).$$

Remark 8.1. Let A satisfy the condition that the complex $Y_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 4 (for example see Proposition 5.5). Then $\mathcal{P}(A)$ is isomorphic with the classical scissors congruence group defined in Section 2. In fact, from the exact sequence

$$Y_4(A^2) \to Y_3(A^2) \to Z_2^{\mathrm{GE}_2}(A^2) \to 0$$

we obtain the exact sequence

$$Y_4(A^2)_{\mathrm{PGE}_2(A)} \to Y_3(A^2)_{\mathrm{PGE}_2(A)} \to \mathcal{P}(A) \to 0.$$

The orbits of the action of $\mathrm{PGE}_2(A)$ on $Y_3(A)$ and $Y_4(A)$ are represented by

$$[x]' := (\boldsymbol{\infty}, \mathbf{0}, \mathbf{1}, \boldsymbol{x}), \quad \text{and} \quad [x, y]' := (\boldsymbol{\infty}, \mathbf{0}, \mathbf{1}, \boldsymbol{x}, \boldsymbol{y}), \quad x, y, x/y \in \mathcal{W}_A,$$

respectively. Thus $Y_3(A^2)_{\mathrm{PGE}_2(A)}$ is the free abelian group generated by the symbols $[x]'$, $x \in \mathcal{W}_A$ and $Y_4(A^2)_{\mathrm{PGE}_2(A)}$ is the free abelian group generated by the symbols $[x, y]'$, $x, y, x/y \in \mathcal{W}_A$. It is straightforward to check that

$$\overline{\partial_4^Y}\big([x, y]'\big) = [x]' - [y]' + \left[\frac{y}{x}\right]' - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right]' + \left[\frac{1 - x}{1 - y}\right]'.$$

This proves our claim.

Lemma 8.2. *If A satisfies the condition that $Y_\bullet(A^2)$ is exact in dimension one, then*

$$\mathcal{P}(A) \simeq H_1\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big).$$

Proof. Since $Y_\bullet(A^2)$ is exact in dimension one, the sequence

$$0 \to Z_2^{\mathrm{GE}_2}(A^2) \to Y_2(A^2) \to Z_1^{\mathrm{GE}_2}(A^2) \to 0$$

is exact. From this we obtain the long exact sequence

$$\begin{aligned}\cdots \to H_1\big(\mathrm{PGE}_2(A), Y_2(A^2)\big) &\to H_1\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to H_0\big(\mathrm{PGE}_2(A), Z_2^{\mathrm{GE}_2}(A^2)\big)\\ &\to H_0\big(\mathrm{PGE}_2(A), Y_2(A^2)\big) \to H_0\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to 0.\end{aligned}$$

The group $\mathrm{PGE}_2(A)$ acts transitively on the generators of $Y_2(A^2)$. We choose $(\boldsymbol{\infty}, \mathbf{0}, \mathbf{1})$ as representative of the orbit of the generators of $Y_2(A^2)$. Then

$$Y_2(A^2) \simeq \mathrm{Ind}_{\{1\}}^{\mathrm{PGE}_2(A)} \mathbb{Z}.$$

Thus by Shapiro's lemma,

$$H_q\big(\mathrm{PGE}_2(A), Y_2(A^2)\big) \simeq H_q\big(\{1\}, \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

The map $\partial_2^Y : Y_2(A^2) \to Y_1(A^2)$ induces the identity map

$$\overline{\partial_2^Y} : \mathbb{Z} = Y_2(A^2)_{\mathrm{PGE}_2(A)} \to Y_1(A^2)_{\mathrm{PGE}_2(A)} = \mathbb{Z}.$$

Since $\overline{\partial_2^Y}$ factors through

$$Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)} \quad \text{and} \quad Y_2(A^2)_{\mathrm{PGE}_2(A)} \to Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)}$$

is surjective, we conclude that the map

$$Y_2(A^2)_{\mathrm{PGE}_2(A)} \to Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)}$$

must be an isomorphism. Therefore

$$H_1\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \simeq H_0\big(\mathrm{PGE}_2(A), Z_2^{\mathrm{GE}_2}(A^2)\big) = \mathcal{P}(A). \qquad \blacksquare$$

Lemma 8.3. *The differential* $d_{2,1}^1$ *is trivial. In particular, if* $Y_\bullet(A)$ *is exact in dimension one, then* $E_{2,1}^2 \simeq \mathcal{P}(A)$.

Proof. Let $D'_{\bullet,\bullet}$ be the double complex $F_\bullet \otimes_{\mathrm{GE}_2(A)} E_\bullet(A^2)$, where $F_\bullet \to \mathbb{Z}$ is a projective resolution of $\mathrm{GE}_2(A)$ over $\mathbb{Z}$. From $D'_{\bullet,\bullet}$ we obtain the first quadrant spectral sequence

$$E'^1_{p.q} = \begin{cases} H_q\big(\mathrm{GE}_2(A), Y_p(A^2)\big) & p = 0, 1, \\ H_q\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) & p = 2, \\ 0 & p > 2 \end{cases} \Rightarrow H_{p+q}\big(\mathrm{GE}_2(A), \mathbb{Z}\big).$$

The natural map $p : \mathrm{GE}_2(A) \to \mathrm{PGE}_2(A)$ induces the morphism of spectral sequences

$$\begin{array}{ccc} E'^1_{p,q} & \Longrightarrow & H_{p+q}\big(\mathrm{GE}_2(A), \mathbb{Z}\big) \\ \big\downarrow{\scriptstyle p_*} & & \big\downarrow{\scriptstyle p_*} \\ E^1_{p,q} & \Longrightarrow & H_{p+q}\big(\mathrm{PGE}_2(A), \mathbb{Z}\big). \end{array} \tag{8.1}$$

As in case of the spectral sequence $E^1_{\bullet,\bullet}$ discussed in the previous section, we can show that

$$E'^1_{0,q} = H_q\big(\mathrm{B}_2(A), \mathbb{Z}\big), \quad E'^1_{1,q} = H_q\big(\mathrm{T}_2(A), \mathbb{Z}\big),$$

where

$$\mathrm{B}_2(A) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in A^\times, b \in A \right\}, \quad \mathrm{T}_2(A) := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in A^\times \right\}.$$

From the above morphism of spectral sequences we have the commutative diagram

$$\begin{array}{ccccc} E'^{1}_{2,1} & \xrightarrow{d'^{1}_{2,1}} & H_1\big(\mathrm{T}_2(A),\mathbb{Z}\big) & \xrightarrow{d'^{1}_{1,1}} & H_1\big(\mathrm{B}_2(A),\mathbb{Z}\big) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ E^{1}_{2,1} & \xrightarrow{d^{1}_{2,1}} & H_1\big(\mathrm{PT}_2(A),\mathbb{Z}\big) & \xrightarrow{d^{1}_{1,1}} & H_1\big(\mathrm{PB}_2(A),\mathbb{Z}\big). \end{array}$$

The differential $d'^{1}_{1,1}$ can be calculated similar to $d^{1}_{1,1}$. In fact, $d'^{1}_{1,1} = \sigma'_* - \mathrm{inc}_*$, where $\sigma' : \mathrm{T}_2(A) \to \mathrm{B}_2(A)$, $\mathrm{diag}(a,b) \mapsto \mathrm{diag}(b,a)$. It is easy to see that

$$d'^{1}_{1,1}\big(\mathrm{diag}(a,b)\big) = \mathrm{diag}(ba^{-1}, ab^{-1}).$$

It follows from this that $\ker(d'^{1}_{1,1}) = A^\times I_2$ and thus $p_* \circ d'^{1}_{2,1} = 0$. Since the vertical maps are surjective [2, Corollary 6.4, Chapter VII], the differential $d^{1}_{2,1}$ must be trivial. The second part follows from the first part and Lemma 8.2. ■

Theorem 8.4. *Let A be a commutative ring which satisfies the condition that $Y_\bullet(A^2)$ is exact in dimension one. Then $H_1(\mathrm{PGE}_2(A),\mathbb{Z}) \simeq \mathcal{G}_A \oplus A_{A^\times}$ and we have the exact sequence*

$$H_3\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to \mathcal{P}(A) \to H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big) \to H_2\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to \mu_2(A) \to 1.$$

Proof. Consider the composite

$$Y_2(A^2) \xrightarrow{\partial_2^Y} Z_1^{\mathrm{GE}_2}(A^2) \xrightarrow{\mathrm{inc}} Y_1(A^2).$$

Since $H_1(Y_\bullet(A^2))=0$, the left map is surjective. As we discussed in the proof of Lemma 8.2, the map

$$\overline{\partial_2^Y} : \mathbb{Z} \simeq Y_2(A^2)_{\mathrm{PGE}_2(A)} \to Y_1(A^2)_{\mathrm{PGE}_2(A)} \simeq \mathbb{Z}$$

is an isomorphism. This implies that the differential

$$d^1_{2,0} = \overline{\mathrm{inc}} : Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)} \to Y_1(A^2)_{\mathrm{PGE}_2(A)} = \mathbb{Z}$$

is surjective. On the other hand $Y_2(A^2)_{\mathrm{PGE}_2(A)} \xrightarrow{\partial_2^Y} Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)}$ is surjective. Thus $d^1_{2,0}$ is injective too and therefore

$$E^2_{2,0} = 0.$$

On the other hand, we have

$$E^2_{1,1} \simeq \mu_2(A), \quad E^2_{0,2} = H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big).$$

From the split extension $0 \to A \to \mathrm{PB}_2(A) \to A^\times \to 1$, we obtain the five term exact sequence

$$H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big) \to H_2(A^\times,\mathbb{Z}) \to A_{A^\times} \to H_1\big(\mathrm{PB}_2(A),\mathbb{Z}\big) \to A^\times \to 1,$$

(see [2, Corollary 6.4, Chapter VII]). Since the above extension splits, the left-side map in the above exact sequence is surjective. Thus

$$H_1\big(\mathrm{PB}_2(A),\mathbb{Z}\big) \simeq A^\times \oplus A_{A^\times}.$$

Since under the differential

$$d^1_{1,1}: A^\times \simeq H_1\big(\mathrm{PT}_2(A),\mathbb{Z}\big) \to H_1\big(\mathrm{PB}_2(A),\mathbb{Z}\big) \simeq A^\times \oplus A_{A^\times},$$

the element $a \in A^\times$ maps to $(a^{-2},0)$, we have $E^2_{0,1} \simeq \mathcal{G}_A \oplus A_{A^\times}$. Now the theorem follows from an easy analysis of the main spectral sequence. ■

Remark 8.5. Let A be a commutative ring. From the diagram with exact rows

$$\begin{array}{ccccccc} Y_2(A^2) & \xrightarrow{\partial_2^Y} & Z_1^{\mathrm{GE}_2}(A^2) & \longrightarrow & H_1\big(Y_\bullet(A^2)\big) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \partial_2^Y} & & \downarrow{\scriptstyle \mathrm{inc}} & & & & \\ Y_1(A^2) & = & Y_1(A^2) & & & & \end{array}$$

we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z} = Y_2(A^2)_{\mathrm{PGE}_2(A)} & \xrightarrow{\overline{\partial_2^Y}} & Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)} & \longrightarrow & H_1\big(Y_\bullet(A^2)\big)_{\mathrm{PGE}_2(A)} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}}=\overline{\partial_2^Y}} & & \downarrow{\scriptstyle d^1_{2,0}=\overline{\mathrm{inc}}} & & & & \\ \mathbb{Z} & = & \mathbb{Z} & & & & \end{array}$$

(see the proof of the above theorem). Thus by the Snake lemma we have

$$E^2_{2,0} \simeq H_1\big(Y_\bullet(A^2)\big)_{\mathrm{PGE}_2(A)}.$$

Since

$$E^2_{1,1} \simeq \mu_2(A), \quad E^2_{0,2} = H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big), \quad E^2_{0,1} \simeq \mathcal{G}_A \oplus A_{A^\times},$$

by an easy analysis of the main spectral sequence, we obtain the exact sequence

$$H_2\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to H_1\big(Y_\bullet(A^2)\big)_{\mathrm{PGE}_2(A)} \to \mathcal{G}_A \oplus A_{A^\times} \to H_1\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to 0.$$

Combining this with Theorem 6.2 we obtain the exact sequence

$$H_2\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to \left(\frac{\mathrm{K}_2(2,A)}{C(2,A)}\right)^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \to A_{A^\times} \to H_1\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) \to \mathcal{G}_A \to 1.$$

We believe that this exact sequence coincides with the exact sequence of Theorem 4.1. It seems very difficult to describe the map $(\frac{\mathrm{K}_2(2,A)}{C(2,A)})^{\mathrm{ab}}_{\mathrm{PGE}_2(A)} \to A_{A^\times}$ in the above exact sequence using the differentials of the spectral sequence, while it was reasonably easy to describe a similar map in Theorem 4.1.

Corollary 8.6. *If A is universal for* GE_2*, then we have the exact sequence*

$$H_3\big(\mathrm{PGE}_2(A),\mathbb{Z}\big)\to \mathcal{P}(A)\to H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big)\to H_2\big(\mathrm{PGE}_2(A),\mathbb{Z}\big)\to \mu_2(A)\to 1.$$

Proof. Since A is universal for GE_2, by Theorem 6.2 we have

$$H_1\big(Y_\bullet(A^2)\big)\simeq\left(\frac{\mathrm{K}_2(2,A)}{C(2,A)}\right)^{\mathrm{ab}}=0.$$

Now the claim follows from the above theorem. ∎

Example 8.7. Let A be a semilocal ring such that none of $\mathbb{Z}/2\times\mathbb{Z}/2$ and $\mathbb{Z}/6$ is a direct factor of $A/J(A)$. Then A is a universal GE_2-ring and thus by the above corollary we have the exact sequence

$$H_3\big(\mathrm{PGL}_2(A),\mathbb{Z}\big)\to \mathcal{P}(A)\to H_2\big(\mathrm{PB}_2(A),\mathbb{Z}\big)\to H_2\big(\mathrm{PGL}_2(A),\mathbb{Z}\big)\to \mu_2(A)\to 1.$$

9. The homology groups of $\mathrm{PB}_2(A)$

Let study the Lyndon/Hochschild–Serre spectral sequence associated to the split extension

$$1\to N_2(A)\to \mathrm{PB}_2(A)\to \mathrm{PT}_2(A)\to 1.$$

This is the extension $0\to A\to \mathrm{PB}_2(A)\to A^\times\to 1$. Thus we have the spectral sequence

$$\mathcal{E}^2_{p,q}=H_p\big(A^\times,H_q(A,\mathbb{Z})\big)\Rightarrow H_{p+q}\big(\mathrm{PB}_2(A),\mathbb{Z}\big).$$

We showed in the proof of Theorem 8.4 that

$$H_1\big(\mathrm{PB}_2(A),\mathbb{Z}\big)\simeq A^\times\oplus A_{A^\times}\simeq H_1\big(\mathrm{PT}_2(A),\mathbb{Z}\big)\oplus A_{A^\times}.$$

Recall that $A_{A^\times}=H_0(A^\times,A)=A/\langle a-1:a\in A^\times\rangle$.

Lemma 9.1. *Let G be an abelian group, A a commutative ring, M an A-module and $\varphi: G\to A^\times$ a homomorphism of groups which turns A and M into G-modules. If $H_0(G,A)=0$, then for any $n\geq 0$, $H_n(G,M)=0$.*

Proof. See [21, Lemma 1.8]. ∎

Corollary 9.2. *If $A_{A^\times}=0$, then $H_n(A^\times,A)=0$ for any $n\geq 0$.*

Proof. Use the above lemma by considering $\varphi=\mathrm{id}_{A^\times}:A^\times\to A^\times$. ∎

Corollary 9.3. *If $2\in A^\times$, then $H_n(A^\times,A)=0$ for any $n\geq 0$.*

Proof. If $2\in A^\times$, then $A_{A^\times}=0$. Now use the previous corollary. ∎

Lemma 9.4. *If $A_{A^\times} = 0$, then* $H_2(\mathrm{PB}_2(A), \mathbb{Z}) \simeq H_2(\mathrm{PT}_2(A), \mathbb{Z}) \oplus H_2(A, \mathbb{Z})_{A^\times}$.

Proof. By Corollary 9.2, we have $H_1(A^\times, A) = 0$. Now the claim follows from an easy analysis of the above spectral sequence. ∎

Proposition 9.5. *If A is a subring of $\mathbb{Q}$, then for any $n \geq 0$,*

$$H_n\big(\mathrm{PB}_2(A), \mathbb{Z}\big) \simeq H_n\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \oplus H_{n-1}(A^\times, A).$$

In particular, if $2 \in A^\times$, then $H_n(\mathrm{PT}_2(A), \mathbb{Z}) \simeq H_n(\mathrm{PB}_2(A), \mathbb{Z})$.

Proof. It is well known that any finitely generated subgroup of $\mathbb{Q}$ is cyclic. Thus the additive group A is a direct limit of infinite cyclic groups. Since $H_n(\mathbb{Z}, \mathbb{Z}) = 0$ for any $n \geq 2$ [2, p. 58] and since homology commutes with direct limit [2, Exercise 6, Section 5, Chapter V], we have $H_n(A, \mathbb{Z}) = 0$ for $n \geq 2$. Now the claim follows from an easy analysis of the above Lyndon/Hochschild–Serre spectral sequence. The second claim follows from Corollary 9.3. ∎

Example 9.6. (i) Since $\mathbb{Z}^\times$ act on $\mathbb{Z}$ by $(-1).n = -n$, we have $(\mathbb{Z})_{\mathbb{Z}^\times} \simeq \mathbb{Z}/2$. Moreover, using the structure of the homology of finite cyclic groups [2, p. 58–59], we have

$$H_k(\mathbb{Z}^\times, \mathbb{Z}) \simeq \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \mathbb{Z}/2 & \text{if } k \text{ is even.} \end{cases}.$$

Therefore, by the above proposition,

$$\begin{aligned} H_n\big(\mathrm{PB}_2(\mathbb{Z}), \mathbb{Z}\big) &\simeq \begin{cases} H_n\big(\mathrm{PT}_2(\mathbb{Z}), \mathbb{Z}\big) & \text{if } n \text{ is even,} \\ H_n\big(\mathrm{PT}_2(\mathbb{Z}), \mathbb{Z}\big) \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd} \end{cases} \\ &\simeq \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{if } n \text{ is even,} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(ii) Let $A_m := \mathbb{Z}[\frac{1}{m}]$, where m is a square free integer. We calculated $(A_m)_{A_m^\times}$ in Example 4.4(iv). Now if $2 \mid m$, then by Corollary 9.3, for any non-negative integer n, $H_n(A_m^\times, A_m) = 0$. Thus by Proposition 9.5

$$H_n\big(\mathrm{PB}_2(A_m), \mathbb{Z}\big) \simeq H_n\big(\mathrm{PT}_2(A_m), \mathbb{Z}\big).$$

Let p be an odd prime. Then $(A_p)_{A_p^\times} \simeq \mathbb{Z}/2$. Note that $A_p^\times \simeq \{\pm 1\} \times \langle p \rangle \simeq \mathbb{Z}/2 \times \mathbb{Z}$. By an easy analysis of the spectral sequence

$$\mathcal{E}'^2_{r,s} = H_r\big(\{\pm 1\}, H_s\big(\langle p \rangle, A_p\big)\big) \Rightarrow H_{r+s}(A_p^\times, A_p)$$

and the calculation of the homology of cyclic groups [2, P. 58–59], one can show that for any $n \geq 0$,

$$H_n(A_p^\times, A_p) \simeq \mathbb{Z}/2.$$

Thus by Proposition 9.5, $H_n(\mathrm{PB}_n(A_p), \mathbb{Z}) \simeq H_n(\mathrm{PT}_2(A_p), \mathbb{Z}) \oplus \mathbb{Z}/2, n \geq 1$. Finally, by the Künneth formula applied to

$$H_n\big(\mathrm{PT}_2(A_p), \mathbb{Z}\big) \simeq H_n\big(\{\pm 1\} \times \langle p \rangle, \mathbb{Z}\big),$$

we obtain

$$H_n\big(\mathrm{PB}_2(A_p), \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } n = 1, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \geq 2. \end{cases}$$

The following result is known for the inclusion of groups $\mathrm{T}_2(A) \subseteq \mathrm{B}_2(A)$.

Proposition 9.7. *Let $\psi_q : H_q(\mathrm{PT}_2(A), \mathbb{Z}) \to H_q(\mathrm{PB}_2(A), \mathbb{Z})$ be induced by the natural inclusion* $\mathrm{PT}_2(A) \subseteq \mathrm{PB}_2(A)$.

(i) *Let A be a semilocal ring such that for any maximal ideal $\mathfrak{m}$, $|A/\mathfrak{m}| \neq 2, 3, 4$. Then ψ_q is isomorphism for $q \leq 2$.*

(ii) *Let A be a semilocal domain such that for any maximal ideal $\mathfrak{m}$ either $A/\mathfrak{m}$ is infinite or if $|A/\mathfrak{m}| = p^d$, then $q < (p-1)d$. Then ψ_q is an isomorphism.*

(iii) *Let A be a semilocal ring such that for any maximal ideal $\mathfrak{m}$ either $A/\mathfrak{m}$ is infinite or if $|A/\mathfrak{m}| = p^d$, then $q < (p-1)d - 2$. Then ψ_q is an isomorphism.*

Proof. This can be proved as in [18, Section 2], which the case of local rings is treated. ■

10. The second homology of PGE$_2$

Let A be a commutative ring. Recall that $\mathcal{W}_A = \{a \in A : a(a-1) \in A^\times\}$. The differential $\partial_3^Y : Y_3(A) \to Z_2^{\mathrm{GE}_2}(A^2) \subseteq Y_2(A)$ induces the map

$$\overline{\partial_3^Y} : H_0\big(\mathrm{PGE}_2(A), Y_3(A^2)\big) \to \mathcal{P}(A).$$

We choose $X_a := (\boldsymbol{\infty}, \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{a})$, $a \in \mathcal{W}_A$, as representatives of the orbits of the generators of $Y_3(A^2)$ and set

$$[a] := \overline{\partial_3^Y}(X_a) \in \mathcal{P}(A).$$

Proposition 10.1. *Let A be a ring such that $Y_\bullet(A^2)$ is exact in dimension* 1. *Then under the composite*

$$d_{2,1}^2 : \mathcal{P}(A) \xrightarrow{d_{2,1}^2} H_2\big(\mathrm{PB}_2(A), \mathbb{Z}\big) \to H_2\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \simeq A^\times \wedge A^\times$$

$[a] \in \mathcal{P}(A)$ maps to $2(a \wedge (1-a))$.

Proof. It is proved in [15, Lemma 3.2] that $E'^2_{2,1} \simeq \mathcal{P}(A)$ and the composite

$$\mathcal{P}(A) \xrightarrow{d'^2_{2,1}} \frac{H_2\big(\mathrm{B}_2(A), \mathbb{Z}\big)}{(\sigma_* - \mathrm{inc}_*)\big(H_2(\mathrm{T}_2(A), \mathbb{Z})\big)} \to \frac{H_2\big(\mathrm{T}_2(A), \mathbb{Z}\big)}{(\sigma_* - \mathrm{inc}_*)\big(H_2(\mathrm{T}_2(A), \mathbb{Z})\big)}$$

is given by

$$[a] \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1-a & 0 \\ 0 & (1-a)^{-1} \end{pmatrix}$$

(see [15, Lemma 4.1] and its proof). We introduced and studies the spectral sequence

$$E'^{1}_{p,q} \Rightarrow H_{p+q}\big(\mathrm{GE}_2(A), \mathbb{Z}\big)$$

in the proof of Lemma 8.3. Observe that in [15] the author works over rings with many units, which satisfy the condition of this proposition. But [15, Lemma 4.1] (used above) is also valid in our more general setting here. Now from the commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}(A) & \xrightarrow{d'^{2}_{2,1}} & \frac{H_2(\mathrm{B}_2(A),\mathbb{Z})}{\mathrm{im}(\sigma_*-\mathrm{inc}_*)} & \longrightarrow & \frac{H_2(\mathrm{T}_2(A),\mathbb{Z})}{\mathrm{im}(\sigma_*-\mathrm{inc}_*)} \\
\| & & \downarrow p_* & & \downarrow p_* \\
\mathcal{P}(A) & \xrightarrow{d^{2}_{2,1}} & H_2\big(\mathrm{PB}_2(A), \mathbb{Z}\big) & \longrightarrow & H_2\big(\mathrm{PT}_2(A), \mathbb{Z}\big)
\end{array}$$

we obtain the desired result. ■

Let A satisfy the condition that $Y_\bullet(A^2)$ is exact in dimension 1. Then by Lemma 8.3, $E^2_{2,1} \simeq \mathcal{P}(A)$. We denote the kernel of the differential

$$d^2_{2,1} : \mathcal{P}(A) \to H_2\big(\mathrm{PB}_2(A), \mathbb{Z}\big)$$

with $\mathcal{B}_E(A)$ and we call it the GE_2-*Bloch group* of A. Hence we have

$$E^{\infty}_{2,1} \simeq \mathcal{B}_E(A). \tag{10.1}$$

Corollary 10.2. *Let A satisfy the condition that $Y_\bullet(A^2)$ is exact in dimension* 1. *If*

$$H_k\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \simeq H_k\big(\mathrm{PB}_2(A), \mathbb{Z}\big) \quad \textit{for } k \leq 2,$$

then we have the exact sequence

$$\frac{A^\times \wedge A^\times}{\langle 2\big(a \wedge (1-a)\big) : a \in \mathcal{W}_A \rangle} \to H_2\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mu_2(A) \to 1.$$

Moreover, if $Y_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 3, then we have the exact sequence

$$0 \to \frac{A^\times \wedge A^\times}{\langle 2\big(a \wedge (1-a)\big) : a \in \mathcal{W}_A \rangle} \to H_2\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mu_2(A) \to 1.$$

Proof. This follows from Theorem 8.4 and Proposition 10.1. The second part follows from the first part and the fact that the natural map

$$\overline{\partial^Y_3} : H_0\big(\mathrm{PGE}_2(A), Y_3(A^2)\big) \to \mathcal{P}(A)$$

is surjective. ■

Theorem 10.3. *Let A be a local domain (local ring) such that*

$$|A/\mathfrak{m}_A| \neq 2,3,4 \quad \big(|A/\mathfrak{m}_A| \neq 2,3,4,5,8,9,16\big).$$

Then

$$H_2\big(\mathrm{PGL}_2(A), \mathbb{Z}[\tfrac{1}{2}]\big) \simeq \mathrm{K}_2(A)[\tfrac{1}{2}].$$

Proof. Since $|A/\mathfrak{m}_A| \neq 2,3,4$ ($|A/\mathfrak{m}_A| \neq 2,3,4,5,8,9,16$ for the case of local ring), by Proposition 9.7,

$$H_k\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \simeq H_k\big(\mathrm{PB}_2(A), \mathbb{Z}\big)$$

for $k \leq 2$. Moreover, note that A is a GE_2-ring and by Proposition 5.5 the complex $Y_\bullet(A^2) \to \mathbb{Z}$ is exact in dimension < 4. Thus by the above corollary,

$$H_2\big(\mathrm{PGL}_2(A), \mathbb{Z}[\tfrac{1}{2}]\big) \simeq \frac{H_2\big(A^\times, \mathbb{Z}[\frac{1}{2}]\big)}{\big\langle 2\big(a \wedge (1-a)\big) : a \in \mathcal{W}_A\big\rangle}.$$

Now it is easy to see that

$$\frac{H_2\big(A^\times, \mathbb{Z}[\frac{1}{2}]\big)}{\big\langle 2\big(a \wedge (1-a)\big) : a \in \mathcal{W}_A\big\rangle} \simeq \frac{S^2_{\mathbb{Z}}(A^\times)[\frac{1}{2}]}{\big\langle a \otimes (1-a) : a \in A^\times\big\rangle} \simeq \mathrm{K}_2^M(A)[\tfrac{1}{2}] \simeq \mathrm{K}_2(A)[\tfrac{1}{2}]$$

(see Theorem 2.1). Recall that

$$S^2_{\mathbb{Z}}(A^\times) \simeq (A^\times \otimes_{\mathbb{Z}} A^\times)/\langle a \otimes b + b \otimes a : a, b \in A^\times\rangle. \qquad \blacksquare$$

Example 10.4. The ring $\mathbb{Z}$ is a universal GE_2-ring [10, Example 6.12]. Since $\mathbb{Z}_{\mathbb{Z}^\times} \simeq \mathbb{Z}/2$ and $H_2(\mathrm{PB}_2(\mathbb{Z}), \mathbb{Z}) = 0$ (Example 9.6), by Corollary 8.6 we have

$$H_2\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big) \simeq \mathbb{Z}/2.$$

Example 10.5. Let $p = 2, 3$. Then $A_p := \mathbb{Z}[\frac{1}{p}]$ is a universal GE_2-ring [10, Example 6.13]. By Example 9.6, we have

$$H_2\big(\mathrm{PB}_2(A_p), \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z}/2 & \text{if } p = 2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } p = 3. \end{cases}$$

By Corollary 8.6, we have the exact sequence

$$H_3\big(\mathrm{PGL}_2(A_p), \mathbb{Z}\big) \to \mathcal{P}(A_p) \xrightarrow{\lambda} H_2\big(\mathrm{PB}_2(A_p), \mathbb{Z}\big) \to H_2\big(\mathrm{PGL}_2(A_p), \mathbb{Z}\big) \to \mu_2(A_p) \to 1.$$

From these we obtain the exact sequences

$$\mathbb{Z}/2 \to H_2\big(\mathrm{PGL}_2(A_2), \mathbb{Z}\big) \to \mu_2(A_2) \to 1,$$
$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \to H_2\big(\mathrm{PGL}_2(A_3), \mathbb{Z}\big) \to \mu_2(A_3) \to 1.$$

11. The third homology of PGE_2 and a Bloch–Wigner type theorem

Lemma 11.1. *If A satisfies the condition that $Y_\bullet(A^2)$ is exact in dimension 1, then*

$$E^2_{1,2} \simeq \frac{A^\times \wedge A^\times}{2(A^\times \wedge A^\times)} \simeq \mathcal{G}_A \wedge \mathcal{G}_A.$$

Proof. To prove our claim we must show that the image of the differential

$$d^1_{2,2} : H_2\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to H_2\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \simeq A^\times \wedge A^\times$$

is $2(A^\times \wedge A^\times)$. The morphism of spectral sequences ${E'}^1_{p,q} \to E^1_{p,q}$ (see diagram (8.1) in the proof of Lemma 8.3) gives us the commutative diagram

$$\begin{array}{ccccc}
H_2\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) & \xrightarrow{{d'}^1_{2,2}} & H_2\big(\mathrm{T}_2(A), \mathbb{Z}\big) & \xrightarrow{{d'}^1_{1,2}} & H_2\big(\mathrm{B}_2(A), \mathbb{Z}\big) \\
\downarrow & & \downarrow{\scriptstyle p_*} & & \downarrow \\
H_2\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) & \xrightarrow{d^1_{2,2}} & H_2\big(\mathrm{PT}_2(A), \mathbb{Z}\big) & \xrightarrow{d^1_{1,2}=0} & H_2\big(\mathrm{PB}_2(A), \mathbb{Z}\big).
\end{array}$$

From the five term exact sequence associated to the Lyndon/Hochschild–Serre spectral of central extension $1 \to A^\times I_2 \to \mathrm{GE}_2(A) \to \mathrm{PGE}_2(A) \to 1$ [2, Corollary 6.4, Chapter VII], with coefficients in $Z_1^{\mathrm{GE}_2}(A^2)$ and Lemma 8.2 we obtain the exact sequence

$$H_2\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to H_2\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big)$$
$$\to H_1\big(A^\times, Z_1^{\mathrm{GE}_2}(A^2)\big)_{\mathrm{PGE}_2(A)} \to H_1\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to \mathcal{P}(A) \to 0.$$

Since $\mathrm{PGE}_2(A)$ acts trivially on $A^\times I_2$, we have

$$H_1\big(A^\times, Z_1^{\mathrm{GE}_2}(A^2)\big)_{\mathrm{PGE}_2(A)} \simeq H_1(A^\times, \mathbb{Z}) \otimes_{\mathbb{Z}} Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)}.$$

In the proof of Lemma 8.2, we have proved that $Z_1^{\mathrm{GE}_2}(A^2)_{\mathrm{PGE}_2(A)} \simeq \mathbb{Z}$. Thus

$$H_1\big(A^\times, Z_1^{\mathrm{GE}_2}(A^2)\big)_{\mathrm{PGE}_2(A)} \simeq A^\times.$$

The composite $Y_2(A^2) \overset{\partial_2}{\twoheadrightarrow} Z_1^{\mathrm{GE}_2}(A^2) \overset{\mathrm{inc}}{\longrightarrow} Y_1(A^2)$ gives us the composite

$$A^\times \simeq H_1\big(\mathrm{GE}_2(A), Y_2(A^2)\big) \to H_1\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to H_1\big(\mathrm{GE}_2(A), Y_1(A^2)\big) \simeq A^\times$$

which coincide with the identity map $\mathrm{id}_{A^\times}$. This shows that the natural map

$$A^\times \to H_1\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big)$$

appearing in the above five term exact sequence is injective. Therefore the map

$$H_2\big(\mathrm{GE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big) \to H_2\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big)$$

is surjective, which appears as the left vertical map in the above diagram.

By the Künneth formula,

$$H_2(\mathrm{T}_2(A), \mathbb{Z}) \simeq H_2(A^\times, \mathbb{Z}) \oplus H_2(A^\times, \mathbb{Z}) \oplus A^\times \otimes_{\mathbb{Z}} A^\times.$$

A direct calculation shows that $\ker(d'^1_{1,2})$ is generated by the elements of the form

$$(x, x, a \otimes b - b \otimes a), \quad x \in H_2(A^\times, \mathbb{Z}), \quad a, b \in A^\times.$$

It is easy to see that $p_*(\ker d'^1_{1,2}) = 2H_2(\mathrm{PT}(A), \mathbb{Z})$. From the above commutative diagram it follows that $\mathrm{im}(d^1_{2,2}) \subseteq 2H_2(\mathrm{PT}(A), \mathbb{Z})$. Finally, it is straightforward to check that

$$Y := \left(\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right]\right) \otimes \big((\boldsymbol{\infty}, \mathbf{0}) + (\mathbf{0}, \boldsymbol{\infty})\big)$$
$$\in H_2\big(\mathrm{PGE}_2(A), Z_1^{\mathrm{GE}_2}(A^2)\big)$$

and

$$d^1_{2,2}(Y) = 2(a \wedge b).$$

This shows that $\mathrm{im}(d^1_{2,2}) = 2H_2(\mathrm{PT}(A), \mathbb{Z})$. The final isomorphism follows from the following lemma applied two $A = \mathbb{Z} \to \mathbb{Z}/2 = B$ and $M = A^\times$. ■

Lemma 11.2 (Base change). *If $A \to B$ is a homomorphism of commutative rings and if M is any A-module, then the natural map*

$$\Big(\bigwedge_A^n M\Big) \otimes_A B \to \bigwedge_B^n (M \otimes_A B)$$

is an isomorphism.

Proof. See [6, Proposition A2.2]. ■

Let $\mathcal{A}$ be an abelian group. Let $\sigma_1 : \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A}) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A})$ be obtained by interchanging the copies of $\mathcal{A}$. This map is induced by the involution $\mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{A}$, $a \otimes b \mapsto -b \otimes a$ (for this use [1, Proposition 3.5]). Let $\Sigma'_2 = \{1, \sigma'\}$ be the symmetric group of order 2 and consider the following action of this group on $\mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A})$:

$$(\sigma', x) \mapsto -\sigma_1(x).$$

We say that an abelian group $\mathcal{A}$ is an ind-cyclic group if $\mathcal{A}$ is direct limit of its finite cyclic subgroups.

Proposition 11.3. *Let $\mathcal{A}$ be an abelian group and $T_{\mathcal{A}}$ be its torsion subgroup. Then*

(i) *We always have the exact sequence*

$$0 \to \bigwedge_{\mathbb{Z}}^3 \mathcal{A} \to H_3(\mathcal{A}, \mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}})^{\Sigma'_2} \to 0.$$

(ii) *If $T_{\mathcal{A}}$ is an ind-cyclic group, then Σ_2' acts trivially on* $\mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}})$ *and the exact sequence*

$$0 \to \bigwedge_{\mathbb{Z}}^3 \mathcal{A} \to H_3(\mathcal{A}, \mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}}) \to 0$$

splits naturally.

(iii) *For any integer $m \in \mathbb{Z}$, let $m : \mathcal{A} \to \mathcal{A}$ be given by $a \mapsto ma$. Then the map $m_* : H_3(\mathcal{A}, \mathbb{Z}) \to H_3(\mathcal{A}, \mathbb{Z})$ induces multiplication by m^3 on $\bigwedge_{\mathbb{Z}}^3 \mathcal{A}$ and multiplication by m^2 on* $\mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}})^{\Sigma_2'}$.

Proof. (i) By [28, Lemma 5.5] or [1, Section 6] we have the exact sequence

$$0 \to \bigwedge_{\mathbb{Z}}^3 \mathcal{A} \to H_3(\mathcal{A}, \mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A})^{\Sigma_2'} \to 0.$$

Since

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A}) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}}),$$

we obtain the first exact sequence.

(ii) Now let $T_{\mathcal{A}}$ be an ind-cyclic group. Since

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \simeq \mathbb{Z}/n,$$

the action of Σ_2' on $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n)$ is trivial. Now by passing to the limit, we see that the action of Σ_2' on $\mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}})$ is trivial. For the last part note that since $T_{\mathcal{A}}$ is ind-cyclic,

$$\bigwedge_{\mathbb{Z}}^3 T_{\mathcal{A}} = 0.$$

Now applying the first part to the inclusion $T_{\mathcal{A}} \hookrightarrow \mathcal{A}$, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & & & H_3(T_{\mathcal{A}}, \mathbb{Z}) & \xrightarrow{\simeq} & \mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}}) & & \\ & & & & \downarrow & & \downarrow\simeq & & \\ 0 & \longrightarrow & \bigwedge_{\mathbb{Z}}^3 \mathcal{A} & \longrightarrow & H_3(\mathcal{A}, \mathbb{Z}) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{A}, \mathcal{A}) & \longrightarrow & 0. \end{array}$$

Now from this diagram we obtain a natural splitting map

$$\mathrm{Tor}_1^{\mathbb{Z}}(T_{\mathcal{A}}, T_{\mathcal{A}}) \simeq H_3(T_{\mathcal{A}}, \mathbb{Z}) \to H_3(\mathcal{A}, \mathbb{Z}).$$

(iii) This part follows from (i) and (ii). ■

Lemma 11.4. *If $H_3(\mathrm{PT}_2(A), \mathbb{Z}) \simeq H_3(\mathrm{PB}_2(A), \mathbb{Z})$, then $E_{0,3}^2$ sits in the exact sequence*

$$\bigwedge_{\mathbb{Z}}^3 \mathcal{G}_A \to E_{0,3}^2 \to \mathrm{Tor}_1^{\mathbb{Z}}\left(\mu(A), \mu(A)\right)^{\Sigma_2'} \to 0.$$

Proof. By Proposition 11.3, $H_3(\mathrm{PT}_2(A), \mathbb{Z})$ sits in the exact sequence

$$0 \to \bigwedge_{\mathbb{Z}}^3 A^\times \to H_3\big(\mathrm{PT}_2(A), \mathbb{Z}\big) \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\Sigma_2'} \to 0.$$

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge_{\mathbb{Z}}^3 A^\times & \longrightarrow & H_3\big(\mathrm{PT}_2(A), \mathbb{Z}\big) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\Sigma_2'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d_{1,3}^1} & & \downarrow & & \\ 0 & \longrightarrow & \bigwedge_{\mathbb{Z}}^3 A^\times & \longrightarrow & H_3\big(\mathrm{PT}_2(A), \mathbb{Z}\big) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\Sigma_2'} & \longrightarrow & 0. \end{array}$$

It is straightforward to see that $d_{1,3}^1$ induces multiplication by 2 on the left vertical map and 0 on the right vertical map (use Proposition 11.3 for $m = 1$ and $m = -1$). Thus by the Snake lemma we have the exact sequence

$$\Big(\bigwedge_{\mathbb{Z}}^3 A^\times\Big)/2 \to E_{0,3}^2 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\Sigma_2'} \to 0.$$

Now the claim follows from this and Lemma 11.2. ∎

Over algebraically closed fields of characteristic zero the following result is called the classical Bloch–Wigner exact sequence.

Proposition 11.5 (Classical Bloch–Wigner exact sequence). *Let F be either a quadratically closed field, real closed field or a finite field, where $|F| \neq 2, 3, 4, 8$. Then we have the exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big) \to H_3\big(\mathrm{PGL}_2(F), \mathbb{Z}\big) \to \mathcal{B}_E(F) \to 0.$$

Proof. First note that $Y_\bullet(F^2) \to \mathbb{Z}$ is exact in dimension < 4 (Proposition 5.5). Second, by Proposition 9.7,

$$H_n\big(\mathrm{PT}_2(F), \mathbb{Z}\big) \simeq H_n\big(\mathrm{PB}_2(F), \mathbb{Z}\big)$$

for $n \leq 3$. Since $|\mathcal{G}_F| \leq 2$, we have

$$E_{1,2}^2 = 0, \quad E_{0,3}^2 \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big)$$

(see Lemmas 11.1 and 11.4). Now by an easy analysis of the main spectral sequence we obtain the exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big) \to H_3\big(\mathrm{PGL}_2(F), \mathbb{Z}\big) \to \mathcal{B}_E(F) \to 0.$$

Let $\bar{F}$ be the algebraic closure of F. Since

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big) \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\bar{F}), \mu(\bar{F})\big)$$

is injective, it is sufficient to prove the claim for $\bar{F}$. The isomorphism

$$\mathrm{PSL}_2(\bar{F}) \xrightarrow{\simeq} \mathrm{PGL}_2(\bar{F})$$

gives us the morphism of spectral sequences

$$\begin{array}{ccc} {E''}^1_{p,q} & \Longrightarrow & H_{p+q}\big(\mathrm{PSL}_2(\bar{F}),\mathbb{Z}\big) \\ \downarrow & & \downarrow \\ E^1_{p,q} & \Longrightarrow & H_{p+q}\big(\mathrm{PGL}_2(\bar{F}),\mathbb{Z}\big). \end{array}$$

This morphism gives us the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\tilde{\mu}(\bar{F}),\tilde{\mu}(\bar{F})\big) & \longrightarrow & H_3\big(\mathrm{PSL}_2(\bar{F}),\mathbb{Z}\big) & \longrightarrow & \mathcal{P}(\bar{F}) & \xrightarrow{[a]\mapsto\frac{1}{2}(a\wedge(1-a))} & \bigwedge_{\mathbb{Z}}^2 \bar{F}^\times \\ \downarrow 4. & & \downarrow\simeq & & \| & & \downarrow 4. \\ \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\bar{F}),\mu(\bar{F})\big) & \longrightarrow & H_3\big(\mathrm{PGL}_2(\bar{F}),\mathbb{Z}\big) & \longrightarrow & \mathcal{P}(\bar{F}) & \xrightarrow{[a]\mapsto 2(a\wedge(1-a))} & \bigwedge_{\mathbb{Z}}^2 \bar{F}^\times \end{array}$$

where $\tilde{\mu}(\bar{F}) = \mu(\bar{F})/\mu_2(\bar{F})$. For the upper exact sequence, see [5, Appendix A]. Note that the right and the left vertical maps are induced by

$$\bar{F}^\times \xrightarrow{(.)^2} \bar{F}^\times$$

and both are isomorphism. Therefore the map

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\bar{F}),\mu(\bar{F})\big) \to H_3\big(\mathrm{PGL}_2(\bar{F}),\mathbb{Z}\big)$$

is injective. This completes the proof of the proposition. ■

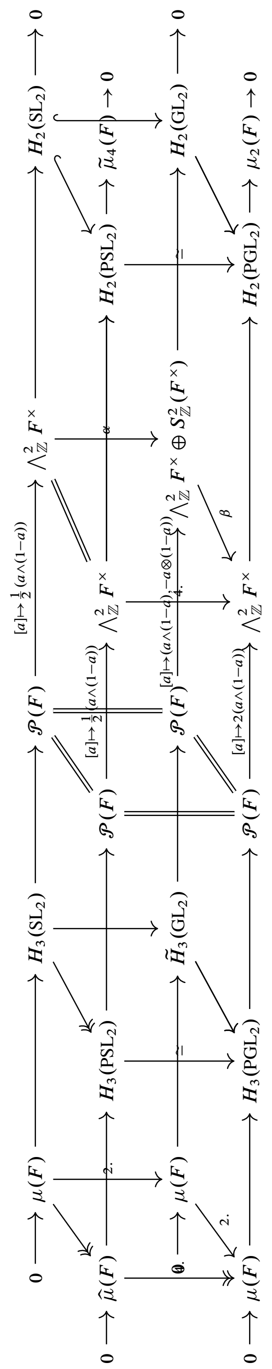
Remark 11.6. Let F be a quadratically closed field. All the groups of the commutative diagram

$$\begin{array}{ccc} \mathrm{SL}_2(F) & \twoheadrightarrow & \mathrm{PSL}_2(F) \\ \downarrow & & \downarrow\simeq \\ \mathrm{GL}_2(F) & \twoheadrightarrow & \mathrm{PGL}_2(F). \end{array}$$

act on the complex $L_\bullet(F^2)$ (see Section 5). So from the above diagram we obtain the diagram of morphisms of spectral sequences

$$\begin{array}{ccccc} & {E'''}^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{SL}_2(F),\mathbb{Z}) & \\ \swarrow & \downarrow & \swarrow & \downarrow & \\ {E''}^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{PSL}_2(F),\mathbb{Z}) & & \\ \downarrow & {E'}^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{GL}_2(F),\mathbb{Z}) & \\ \downarrow\ \swarrow & & \downarrow\ \swarrow & & \\ E^1_{p,q} & \Longrightarrow & H_{p+q}(\mathrm{PGL}_2(F),\mathbb{Z}) & & \end{array}$$

By studying the spectral sequence in the above diagram, we obtain the following commutative diagram with exact rows



where

$\alpha(a \wedge b) = 2(a \wedge b, a \otimes b),\ \beta(a \wedge b, c \otimes d) = a \wedge b - c \wedge d,\ \widehat{\mu}(F) := \mu(F)/\mu_4(F)$

and $\widetilde{H}_3(\mathrm{GL}_2)$ is the cokernel of the composite

$$\bigwedge_{\mathbb{Z}}^{3} \mathrm{T}_2(F) \to H_3\big(\mathrm{T}_2(F), \mathbb{Z}\big) \to H_3\big(\mathrm{GL}_2(F), \mathbb{Z}\big).$$

In the above diagram, the exact sequences corresponding to $\mathrm{SL}_2(F)$ and $\mathrm{PSL}_2(F)$ are proved in [5, Appendix A]. For these see also [7, 19]. For the exact sequence related to $\mathrm{GL}_2(F)$ see [15, 28]. The exact sequence related to $\mathrm{PGL}_2(F)$ is the topic of the current article (but also see [22, Appendix C, (C.3)]). The maps on the right vertical square sit in the following exact sequences

$$0 \to H_2\big(\mathrm{SL}_2(F), \mathbb{Z}\big) \to H_2\big(\mathrm{PSL}_2(F), \mathbb{Z}\big) \to \mu_2(F) \to 1,$$
$$0 \to H_2\big(\mathrm{SL}_2(F), \mathbb{Z}\big) \to H_2\big(\mathrm{GL}_2(F), \mathbb{Z}\big) \xrightarrow{\det} H_2(F^\times, \mathbb{Z}) \to 0,$$
$$F^\times \otimes_{\mathbb{Z}} F^\times \to H_2\big(\mathrm{GL}_2(F), \mathbb{Z}\big) \to H_2\big(\mathrm{PGL}_2(F), \mathbb{Z}\big) \to F^\times \xrightarrow{(.)^2} F^\times \to \mathcal{G}_F \to 1.$$

These exact sequences can be obtained by analysis of the Lyndon/Hochschild–Serre Spectral sequences associated to the group extensions $1 \to \mu_2(F) \to \mathrm{SL}_2(F) \to \mathrm{PSL}_2(F) \to 1$, $1 \to \mathrm{SL}_2(F) \to \mathrm{GL}_2(F) \xrightarrow{\det} F^\times \to 1$ and $1 \to F^\times I_2 \to \mathrm{GL}_2(F) \to \mathrm{PGL}_2(F) \to 1$, respectively.

Theorem 11.7. *Let A be a domain satisfying the condition that $Y_\bullet(A)$ is exact in dimension 1 and $H_3(\mathrm{PT}_2(A), \mathbb{Z}) \simeq H_3(\mathrm{PB}_2(A), \mathbb{Z})$. Then we have the sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big) \to H_3\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mathcal{B}_E(A) \to 0,$$

which is exact at every term except possibly at the term $H_3(\mathrm{PGL}_2(A), \mathbb{Z})$, where the homology of the complex is annihilated by 4.

Proof. The main spectral sequence gives us a filtration

$$0 \subseteq F_0H_3 \subseteq F_1H_3 \subseteq F_2H_3 \subseteq F_3H_3 = H_3\big(\mathrm{PGE}_2(A), \mathbb{Z}\big),$$

where $E^\infty_{p,3-p} \simeq F_pH_3/F_{p-1}H_3$, $0 \le p \le 3$. Since $E^1_{3,0} = 0$, we have $F_2H_3 = F_3H_3$. From $E^\infty_{2,1} \simeq \mathcal{B}_E(A)$ (see (10.1)) we obtain the exact sequence

$$0 \to F_1H_3 \to H_3\big(\mathrm{PGE}_2(A), \mathbb{Z}\big) \to \mathcal{B}_E(A) \to 0.$$

By Lemma 11.4 and Proposition 11.3 (ii), $E^2_{0,3}$ sits in the exact sequence

$$\bigwedge_{\mathbb{Z}}^{3} \mathcal{G}_A \to E^2_{0,3} \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big) \to 0.$$

Let $\mathcal{T}_2 = \mathrm{im}(\bigwedge_{\mathbb{Z}}^3 \mathcal{G}_A \to E_{0,3}^2)$. From the natural inclusion $\mu(A) \hookrightarrow A^\times \simeq \mathrm{PT}_2(A)$, we obtain the diagram with exact rows

$$\begin{array}{ccccccccc} & & & & H_3\big(\mu(A),\mathbb{Z}\big) & \xrightarrow{\simeq} & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) & & \\ & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{T}_2 & \longrightarrow & E_{0,3}^2 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) & \longrightarrow & 0 \end{array}$$

This shows that the bottom exact sequence splits and thus we have the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) \to E_{0,3}^2 \to \mathcal{T}_2 \to 0.$$

Now from the surjective map $E_{0,3}^2 \twoheadrightarrow E_{0,3}^\infty \simeq F_0H_3$, we obtain an exact sequence of the form

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) \to F_0H_3 \to \mathcal{T}_2' \to 0,$$

where $\mathcal{T}_2'$ is a 2-torsion group. Let α be the composite $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)) \to F_0H_3 \hookrightarrow F_1H_3$. From the commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) & = & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) & & & & \\ & & \downarrow & & \downarrow{\scriptstyle\alpha} & & & & \\ 0 & \longrightarrow & F_0H_3 & \longrightarrow & F_1H_3 & \longrightarrow & E_{1,2}^\infty & \longrightarrow & 0 \end{array}$$

we obtain the exact sequence

$$0 \to \mathcal{T}_2' \to \mathrm{coker}(\alpha) \to E_{1,2}^\infty \to 0.$$

But by Lemma 11.1, $E_{1,2}^\infty \simeq \mathcal{G}_A \wedge \mathcal{G}_A$. This implies that $\mathcal{T}_4 := \mathrm{coker}(\alpha)$ is a 4-torsion group. To complete the proof of the theorem we need to prove that the composite

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) \xrightarrow{\alpha} F_1H_3 \hookrightarrow H_3\big(\mathrm{PGE}_2(A),\mathbb{Z}\big)$$

is injective. Let F be the quotient field of A and $\bar{F}$ the algebraic closure of F. By Proposition 11.5, we have the classical Bloch–Wigner exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\bar{F}),\mu(\bar{F})\big) \to H_3\big(\mathrm{PGL}_2(\bar{F}),\mathbb{Z}\big) \to \mathcal{B}_E(\bar{F}) \to 0.$$

Now from the commutative diagram

$$\begin{array}{ccccccccc} & & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A),\mu(A)\big) & \longrightarrow & H_3\big(\mathrm{PGE}_2(A),\mathbb{Z}\big) & & & & \\ & & \big\downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\bar{F}),\mu(\bar{F})\big) & \longrightarrow & H_3\big(\mathrm{PGL}_2(\bar{F}),\mathbb{Z}\big) & \longrightarrow & \mathcal{B}_E(\bar{F}) & \longrightarrow & 0 \end{array}$$

we obtain the injectivity of the map $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A),\mu(A)) \to H_3(\mathrm{PGE}_2(A),\mathbb{Z})$. This completes the proof of the theorem. ∎

Corollary 11.8. *Let A be a semilocal domain such that for any maximal ideal $\mathfrak{m}$, $|A/\mathfrak{m}| \neq 2, 3, 4, 8$. Then we have the Bloch–Wigner exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)\big[\tfrac{1}{2}\big] \to H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}\big[\tfrac{1}{2}\big]\big) \to \mathcal{B}(A)\big[\tfrac{1}{2}\big] \to 0.$$

Moreover, $H_3(\mathrm{PGL}_2(A), \mathbb{Z}[\frac{1}{2}]) \simeq \mathrm{K}_3^{\mathrm{ind}}(A)[\frac{1}{2}]$.

Proof. First note that A is a GE_2-ring and thus $\mathrm{GE}_2(A) = \mathrm{GL}_2(A)$. Second, by Proposition 9.7, $H_n(\mathrm{PT}_2(A), \mathbb{Z}) \simeq H_n(\mathrm{PB}_2(A), \mathbb{Z})$ for $n \leq 3$. Consider the commutative diagram with exact rows

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{\ \lambda\ } & S^2_{\mathbb{Z}}(A^\times) \\ \| & & \downarrow{\scriptstyle \gamma} \\ \mathcal{P}(A) & \xrightarrow{\ d^2_{2,1}\ } & H_2\big(\mathrm{PT}_2(A), \mathbb{Z}\big), \end{array}$$

where $\gamma(\overline{a \otimes b}) = 2\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \wedge \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Note that $\mathcal{B}(A) := \ker(\lambda)$ is the Bloch group of A (see Section 2). Since $S^2_{\mathbb{Z}}(A^\times)[\frac{1}{2}] \simeq H_2(\mathrm{PT}_2(A), \mathbb{Z}[\frac{1}{2}])$, we have $\mathcal{B}(A)[\frac{1}{2}] \simeq \mathcal{B}_E(A)[\frac{1}{2}]$. Now the first claim follows from the above theorem.

The natural map $\mathrm{GL}_2(A) \hookrightarrow \mathrm{PGL}_2(A)$, induces the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)[\frac{1}{2}]\big) & \longrightarrow & H_3\big(\mathrm{SL}_2(A), \mathbb{Z}[\frac{1}{2}]\big)_{A^\times} & \longrightarrow & \mathcal{B}(A)[\frac{1}{2}] & \longrightarrow & 0 \\ & & \downarrow{\simeq} & & \downarrow & & \downarrow{\simeq} & & \\ 0 & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)[\frac{1}{2}]\big) & \longrightarrow & H_3\big(\mathrm{PGL}_2(A), \mathbb{Z}[\frac{1}{2}]\big) & \longrightarrow & \mathcal{B}_E(A)[\frac{1}{2}] & \longrightarrow & 0 \end{array}$$

Note that the first exact sequence can be proved as [15, Corollary 5.4]. By the Snake lemma, $H_3(\mathrm{SL}_2(A), \mathbb{Z}[\frac{1}{2}])_{A^\times} \simeq H_3(\mathrm{PGL}_2(A), \mathbb{Z}[\frac{1}{2}])$. Now the second claim follows from this and the isomorphism $H_3(\mathrm{SL}_2(A), \mathbb{Z}[\frac{1}{2}])_{A^\times} \simeq \mathrm{K}_3^{\mathrm{ind}}(A)[\frac{1}{2}]$ (see Theorem 2.5). ∎

Proposition 11.9. *For any non-dyadic local field F we have the exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big)^{\approx} \to H_3\big(\mathrm{PGL}_2(F), \mathbb{Z}\big) \to \mathcal{B}_E(F) \to 0,$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\approx}$ is an extension of $\mathbb{Z}/2$ by $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$.

Proof. First note that $\mathcal{G}_F \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ (see [12, Theorem 2.2, Chapter VI]). Thus $E^\infty_{1,2} \simeq \mathcal{G}_F \wedge \mathcal{G}_F \simeq \mathbb{Z}/2$. Moreover, since $\bigwedge^3_{\mathbb{Z}} \mathcal{G}_F = 0$, we have $E^2_{0,3} \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$. Now by an easy analysis of the main spectral sequences we obtain the exact sequences

$$0 \to \mathcal{K} \to H_3\big(\mathrm{PGL}_2(F), \mathbb{Z}\big) \to \mathcal{B}_E(F) \to 0,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(F), \mu(F)\big) \to \mathcal{K} \to \mathbb{Z}/2 \to 0.$$

The injectivity of $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F)) \to \mathcal{K}$ follows from Theorem 11.7. This completes the proof of the proposition. ∎

Proposition 11.10. *For the rings* $\mathbb{Z}$ *and* $\mathbb{Z}[\frac{1}{2}]$ *we have:*

(i) *For the ring of integers* $\mathbb{Z}$*,* $\mathcal{B}_E(\mathbb{Z}) = \mathcal{P}(\mathbb{Z})$ *and we have the exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\mathbb{Z}), \mu(\mathbb{Z})\big) \oplus \mathbb{Z}/2 \to H_3\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big) \to \mathcal{B}_E(\mathbb{Z}) \to 0.$$

(ii) *For the ring* $\mathbb{Z}[\frac{1}{2}]$*,* $\mathcal{B}_E(\mathbb{Z}[\frac{1}{2}]) = \mathcal{P}(\mathbb{Z}[\frac{1}{2}])$ *and we have the exact sequence*

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\mathbb{Z}[\tfrac{1}{2}]), \mu(\mathbb{Z}[\tfrac{1}{2}])\big)^{\approx} \to H_3\big(\mathrm{PGL}_2\big(\mathbb{Z}[\tfrac{1}{2}]\big), \mathbb{Z}\big) \to \mathcal{B}_E\big(\mathbb{Z}[\tfrac{1}{2}]\big) \to 0,$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(\mathbb{Z}[\frac{1}{2}]), \mu(\mathbb{Z}[\frac{1}{2}]))^{\approx}$ *is an extension of* $\mathbb{Z}/2$ *by* $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(\mathbb{Z}[\frac{1}{2}]), \mu(\mathbb{Z}[\frac{1}{2}]))$.

Proof. (i) First observe that $E^1_{0,2} = H_2(\mathrm{PB}_2(\mathbb{Z}), \mathbb{Z}) \simeq H_2(\mathrm{PT}_2(\mathbb{Z}), \mathbb{Z}) = 0$ (see Example 9.6). Thus $\mathcal{B}_E(\mathbb{Z}) = \mathcal{P}(\mathbb{Z})$. By Example 9.6 and Proposition 11.3

$$H_3\big(\mathrm{PB}_2(\mathbb{Z}), \mathbb{Z}\big) \simeq H_3\big(\mathrm{PT}_2(\mathbb{Z}), \mathbb{Z}\big) \oplus \mathbb{Z}/2 \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\mathbb{Z}), \mu(\mathbb{Z})\big) \oplus \mathbb{Z}/2.$$

Note that $E^1_{1,2} = H_2(\mathrm{PT}_2(\mathbb{Z}), \mathbb{Z}) = 0$. Now by an easy analysis of the main spectral sequence we obtain the exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(\mathbb{Z}), \mu(\mathbb{Z})\big) \oplus \mathbb{Z}/2 \to H_3\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big) \to \mathcal{P}(\mathbb{Z}) \to 0. \tag{11.1}$$

Let us study the associated Lyndon/Hochschild–Serre spectral sequence of the split extension $1 \to \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z}) \to \mathcal{G}_{\mathbb{Z}} \to 1$:

$$\mathcal{E}^2_{r,s} = H_r\big(\mathcal{G}_{\mathbb{Z}}, H_s\big(\mathrm{PSL}_2(\mathbb{Z}), \mathbb{Z}\big)\big) \Rightarrow H_{r+s}\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big).$$

Since $\mathrm{PSL}_2(\mathbb{Z})$ is the free product of $\mathbb{Z}/2$ and $\mathbb{Z}/3$, i.e., $\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2 * \mathbb{Z}/3$, we have

$$H_n\big(\mathrm{PSL}_2(\mathbb{Z}), \mathbb{Z}\big) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(see [2, Corollary 7.7, Chapter II]). Therefore for any $r \geq 0$, $\mathcal{E}^2_{r,2} = 0$. Moreover,

$$\mathcal{E}^2_{r,0} = H_r(\mathcal{G}_{\mathbb{Z}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, \\ \mathbb{Z}/2 & \text{if } r \text{ is odd}, \\ 0 & \text{if } r \text{ is even}. \end{cases}$$

It is known that the isomorphism $\mathbb{Z}/6 \to H_1(\mathrm{PSL}_2(\mathbb{Z}), \mathbb{Z})$ is induced by $\bar{1} \mapsto E_{12}(1)$ [3, Theorem 9.3]. The conjugate action of $\langle -1 \rangle \in \mathcal{G}_{\mathbb{Z}}$ on $\mathrm{PSL}_2(\mathbb{Z})$ is given by $\langle -1 \rangle.E_{12}(1) = E_{12}(-1)$. Thus if we replace $\mathbb{Z}/6$ by $H_1(\mathrm{PSL}_2(\mathbb{Z}), \mathbb{Z})$, we see that $\langle -1 \rangle \in \mathcal{G}_{\mathbb{Z}}$ acts on $\mathbb{Z}/6$ by $\langle -1 \rangle.\bar{r} := -\bar{r}$. Now by the known calculation of the homology of finite cyclic groups ([2, p. 58–59]) we have

$$\mathcal{E}^2_{r,1} = H_r(\mathcal{G}_{\mathbb{Z}}, \mathbb{Z}/6) \simeq \mathbb{Z}/2.$$

Since the extension splits, $d^2_{3,0} : \mathcal{E}^2_{3,0} \to \mathcal{E}^2_{1,1}$ is trivial. Now by an easy analysis of the spectral sequence we see that

$$\left|H_3\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big)\right| \le 24.$$

On the other hand, we know that $\mathrm{PGL}_2(\mathbb{Z})$ is the free product with amalgamation of the dihedral group D_2 of order 4 and the dihedral group D_3 of order 6 amalgamated along the subgroup D_1 of order 2:

$$\mathrm{PGL}_2(\mathbb{Z}) \simeq D_2 *_{D_1} D_3, \tag{11.2}$$

(see the proof of [23, Lemma 2]). Note that $D_1 \simeq \mathbb{Z}/2$, $D_2 \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ and $D_3 \simeq S_3$. Since

$$H_2(D_1, \mathbb{Z}) = 0, \quad H_3(D_1, \mathbb{Z}) \simeq \mathbb{Z}/2, \quad H_3(D_2, \mathbb{Z}) \simeq (\mathbb{Z}/2)^3, \quad H_3(D_3, \mathbb{Z}) \simeq \mathbb{Z}/6,$$

from the Mayer–Vietoris exact sequence associated to (11.2) [2, Section 9, Chapter VII] we obtain the exact sequence

$$\mathbb{Z}/2 \to (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/6 \to H_3\big(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}\big) \to 0.$$

It follows that $|H_3(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z})| \ge 24$. Therefore $H_3(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z})$ has 24 elements and in fact $H_3(\mathrm{PGL}_2(\mathbb{Z}), \mathbb{Z}) \simeq (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/6$. Thus the left hand side map of the exact sequence (11.1) must be injective. This proves our claim.

(ii) Let $A_2 = \mathbb{Z}[\frac{1}{2}]$. Then by Lemma 9.5, for any $n \ge 0$ we have

$$H_n\big(\mathrm{PB}_2(A_2), \mathbb{Z}\big) \simeq H_n\big(\mathrm{PT}_2(A_2), \mathbb{Z}\big).$$

Since $\mathcal{G}_{A_2} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, we have $E^\infty_{1,2} \simeq \mathbb{Z}/2$ (Lemma 11.1) and $E^\infty_{0,3} \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mu(A_2), \mu(A_2))$ (Lemma 11.4). Now from the main spectral sequence, we obtain the exact sequences

$$0 \to \mathcal{K} \to H_3\big(\mathrm{PGL}_2(A_2), \mathbb{Z}\big) \to \mathcal{B}_E(A_2) \to 0,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A_2), \mu(A_2)\big) \to \mathcal{K} \to \mathbb{Z}/2 \to 0.$$

As in the proof of Theorem 11.7, one can show that the map

$$\mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A_2), \mu(A_2)\big) \to \mathcal{K} \subseteq H_3\big(\mathrm{PGL}_2(A_2), \mathbb{Z}\big)$$

is injective. This completes the proof of the proposition. ∎

Remark 11.11. Let A be a domain. Then, up to isomorphism, there are at most two extension of $\mathbb{Z}/2$ by $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(A), \mu(A))$: the split and the non-split extensions. This follows from the isomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}}\big(\mathbb{Z}/2, \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)\big) \simeq \begin{cases} 0 & \text{if } \mu_{2^\infty}(F) \text{ is infinite or } \mathrm{char}(F) = 2, \\ \mathbb{Z}/2 & \text{if } \mu_{2^\infty}(F) \text{ is finite and } \mathrm{char}(F) \neq 2, \end{cases}$$

where F is the quotient field of A and $\mu_{2^\infty}(F)$ is the 2-power roots of unity in F. We believe the extension

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big) \to \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(A), \mu(A)\big)^{\approx} \to \mathbb{Z}/2 \to 0$$

appearing in the above two propositions is the non-split extension. But at the moment we do not know how to prove this.

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