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Polynomial Cauchy functional equations: A report

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Seit A. M. Legendre, C. F. Gauss und A. L. Cauchy weiss man, dass jede stetige Lösung $f: \mathbb{R}^n \to \mathbb{R}$ der Cauchy Funktionalgleichung

$$f(x+y) - f(x) - f(y) = 0, \quad x, y \in \mathbb{R}^n,$$

die Form $x \mapsto a \cdot x$, $x \in \mathbb{R}^n$, mit einem Vektor $a \in \mathbb{R}^n$ hat. Verzichtet man auf die Regularitätsforderung „stetig“, so gibt es viele andere Lösungen, die man unter Benutzung einer Hamel Basis erhalten kann. Ähnlich kann man nach allen Lösungen der *polynomialen Cauchy Funktionalgleichung*

$$f(x+y) - f(x) - f(y) = p(x, y), \quad x, y \in \mathbb{R},$$

fragen, wo p ein gegebenes Polynom in x, y ist. Man wird sicherlich erwarten, dass unter gewissen Regularitätsannahmen für eine Lösung f, diese wieder ein Polynom ist. Diesem Themenbereich widmen sich die Autoren des vorliegenden Artikels ausführlich. Präsentiert wird auch eine notwendige und hinreichende Bedingung, wann es stetige Lösungen gibt. Zum Beispiel gibt es keine stetige Lösung, wenn man $p(x, y) = x^2 y^2$ betrachtet. Es bleibt die Frage, für welche Polynome $p(x, y)$ es überhaupt Lösungen gibt. Dies ist etwa nicht der Fall für $p(x, y) = q(x) + q(y)$ mit einem Polynom q in einer Variable, dessen Grad mindestens eins ist.

1 Introduction

One of the first functional equations considered is the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad (x, y, \in \mathbb{R}), \tag{1.1}$$

whose only continuous solutions are the linear functions $x \mapsto ax$ for some $a \in \mathbb{R}$ (see e.g. [6]). In other words, continuous solutions f to the equation

$$f(x+y) - f(x) - f(y) = 0 \quad (x, y, \in \mathbb{R})$$

are automatically polynomials. In this note, we discuss a more general question, namely, what happens if the function 0 on the right side is substituted by a polynomial $p(x, y)$.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $\mathbb{R}[\mathbf{x}]$ denote the ring of all real-valued polynomials. Considered are the following functional equations (which we call the *polynomial Cauchy functional equations*):

$$f(x+y) - f(x) - f(y) = p(x, y),$$

where $p(x, y) \in \mathbb{R}[x, y]$, or

$$p(x, y) \in \mathbb{R}[x] \text{ for each fixed } y \in \mathbb{R} \quad \text{and} \quad p(x, y) \in \mathbb{R}[y] \text{ for each fixed } x \in \mathbb{R}.$$

It will be shown in Section 2 that, under each of these conditions, assuming f is continuous or approximately continuous, f itself is a polynomial[1]. In Section 3, we relax the condition of continuity in different ways, and we discuss the set of all solutions. The latter will be derived in a very simple way from the result on the structure of all solutions to the classical Cauchy functional equation (1.1), which are commonly called the *additive functions*. Finally, in Section 4, we show that any continuous solution f to

$$f(x+y) - f(x) \in \mathbb{R}[x] \quad \text{for } y \in Y$$

is already a polynomial whenever the set Y contains two y's with irrational quotient, a best possible result.

2 Polynomial Cauchy functional equations

To start with, we first consider functions which are infinitely often differentiable on $\mathbb{R}$, as in this case, we can give a proof which is entirely different from the one in case where f is merely continuous. To this end, we define[2] the degree of the zero polynomial to be 0. Recall that $\mathbb{N} := \{0, 1, 2, \ldots\}$.

Proposition 2.1. *Let* $f \in C^\infty(\mathbb{R})$. *Suppose that* $p(x, y) := f(x+y) - f(x) - f(y) \in \mathbb{R}[x, y]$. *Then* f *is a polynomial.*

[1] For the case of continuous functions, this was submitted by us as a solution to [12, Problem 12326], but was not chosen for publication.

[2] This avoids discussing subcases in our formulas below.

Proof. Write $f(x+y) - f(x) =: p_{d(y)}(x)$, where $d(y)$ is the degree of the polynomial $x \mapsto p(x, y)$. As the set $D := \{d(y) : y \in \mathbb{R}\} \subseteq \mathbb{N}$ is countable, there must exist $d \in D$ and uncountably many $y \in \mathbb{R}$, say $y \in Y \subseteq \mathbb{R}$, such that $d(y) = d$ for $y \in Y$.

Differentiating $d + 1$ times gives 0, so for all $y \in Y$ and $x \in \mathbb{R}$,

$$f^{(d+1)}(x+y) = f^{(d+1)}(x) =: H(x).$$

Thus H is a periodic function with period y. Note that H is independent of $y \in Y$. Hence there exist uncountably many periods for H. A non-constant continuous function, though, cannot have uncountably many periods (see [2]). Consequently, $H = f^{(d+1)}$ is constant, and so f is a polynomial of degree less than or equal to $d + 1$. ∎

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If, for each variable x and y separately, the function $p(x, y) := f(x+y) - f(x) - f(y)$ is a polynomial, then f is a polynomial*[3].

Proof. First we note that the assumption "$p(\,\cdot\,, y)$ and $p(x, \cdot\,)$ are polynomials in their variables separately" implies that $p(x, y)$ itself is a polynomial, so $p \in \mathbb{R}[x, y]$ (see [1] or [4, Corollary 5.3.4]).

Case 1: f is differentiable on $\mathbb{R}$. Write

$$p(x, y) = \sum_{i,j=0}^{n} a_{i,j} x^i y^j$$

with symmetrical coefficients and $a_{0,0} = -f(0)$ (the sum being finite of course). If we take $y = 0$, then for all x,

$$-f(0) = f(x+0) - f(x) - f(0) = a_{0,0} + \sum_{i=1}^{n} a_{i,0} x^i.$$

Hence $a_{i,0} = 0$ for all $i \geq 1$. By symmetry, we also have $a_{0,j} = 0$ for all $j \geq 1$. Thus we have only coefficients $a_{i,j}$ for $i, j \geq 1$. Consequently,

$$\frac{f(x+y) - f(x) - (f(y) - f(0))}{y} = \sum_{i,j=1}^{n} a_{i,j} x^i y^{j-1}.$$

As f is assumed to be differentiable, we may take $y \to 0$ and get

$$f'(x) - f'(0) = \sum_{i=1}^{n} a_{i,1} x^i.$$

This implies that f' is continuous. Integration yields

$$f(x) - f(0) - x f'(0) = \sum_{i=1}^{n} a_{i,1} \frac{x^{i+1}}{i+1}. \tag{2.1}$$

Thus f is a polynomial.

[3]Due to the symmetry, the assumption of the theorem could as well be $x \mapsto f(x+y) - f(x)$ is a polynomial for all y.

Case 2: $f \in C(\mathbb{R})$. Let $F(x) := \int_0^x f(t)\,dt$ be a primitive of f. Then, with

$$G(x,y) := F(x+y) - F(x) - F(y),$$

we have

$$\begin{aligned} G(x,y) &= \int_0^{x+y} f(t)\,dt - \int_0^x f(t)\,dt - \int_0^y f(t)\,dt \\ &\underset{t=y+s}{=} \int_{-y}^{x} f(y+s)\,ds - \int_0^x f(t)\,dt - \int_0^y f(t)\,dt \\ &= \int_{-y}^{0} f(y+s)\,ds + \int_0^x \big(f(y+s) - f(s)\big)\,ds - \int_0^y f(t)\,dt \\ &\underset{t=y+s}{=} \int_0^y f(t)\,dt + \int_0^x \big(f(y+s) - f(s)\big)\,ds - \int_0^y f(t)\,dt \\ &= \int_0^x p(y,s)\,ds + f(y)x, \end{aligned}$$

which is a polynomial in x. Again, by symmetry, and the Carroll argument (see [1]), G is a polynomial. Hence, by Case 1, F is a polynomial and so is $f = F'$. ■

For completeness, let us mention that a related result to Theorem 2.2 appears in [7, Lemma 2.5] (restricting the degree of the polynomials), and that was the base for [12, Problem 12326]. We extend that result in Section 4.

Next we present a nice result we found on Mathematics Stack Exchange [15], and which yields the possibility to deduce Theorem 2.2 immediately from Proposition 2.1.

Proposition 2.3. *Let f be a continuous solution to*

$$f(x+y) - f(x) - f(y) \in \mathbb{R}[x,y].$$

Then $f \in C^\infty(\mathbb{R})$.

Proof. Suppose that

$$f(x+y) - f(x) - f(y) =: p(x,y), \quad f \in C(\mathbb{R}). \tag{2.2}$$

Consider the smooth kernel

$$k(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ \exp\big(-\frac{1}{1-|x|^2}\big) & \text{if } |x| < 1, \end{cases}$$

and the function $\rho = k / \int_{\mathbb{R}} k(x)\,dx$. The convolution function $F := f * \rho$ given by

$$F(x) := \int_{\mathbb{R}} f(x-y)\rho(y)\,dy$$

is well defined and $C^\infty(\mathbb{R})$ (see e.g. [8, p. 186]). Since, by (2.2),

$$f(x-y)\rho(y) = f(x)\rho(y) + f(-y)\rho(y) + p(x,-y)\rho(y),$$

we obtain that

$$F(x) = f(x) \int_{\mathbb{R}} \rho(y)\, dy + \int_{\mathbb{R}} f(-y)\rho(y)\, dy + \int_{\mathbb{R}} p(x,-y)\rho(y)\, dy.$$

Hence

$$f(x) = F(x) - C + g(x),$$

where C is a constant and $g \in C^{\infty}(\mathbb{R})$. We conclude that $f \in C^{\infty}(\mathbb{R})$. ■

An analysis of the proof of Theorem 2.2 shows that the next assertion holds, too, since "approximately continuous functions" admit primitives (see [5, Theorem 2.4.1]). Recall that $f:\mathbb{R} \to \mathbb{R}$ is approximately continuous at x_0 if, for every $\varepsilon > 0$, the set

$$\{x \in \mathbb{R} : |f(x) - f(x_0)| < \varepsilon\}$$

has density 1 at x_0 (see [5, Definition 2.3.1]). A very nice feature of these "approximately continuous functions" is that a function $f:\mathbb{R} \to \mathbb{R}$ is measurable if and only if it is almost everywhere[4] approximately continuous (see [5, Theorem 2.3.13]).

Proposition 2.4. *Suppose that $f:\mathbb{R} \to \mathbb{R}$ is everywhere approximately continuous and locally bounded, and that f satisfies the polynomial Cauchy functional equation. Then f is a polynomial.*

Proof. Just re-use Case 2 in the proof of Theorem 2.2 and the comment preceding the statement of the present proposition on the existence of primitives. ■

3 A discussion on all solutions and further refinements on the presumed "smoothness" of the underlying class

Next we discuss the set of all solutions in case the polynomial Cauchy functional equation admits at least one continuous solution. Since[5] the operator $T:\mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}^2}$ given by

$$T(f)(x,y) = f(x+y) - f(x) - f(y)$$

is linear, the structure of the set $\mathcal{S}$ of solutions to the inhomogeneous equation $Tf = q$ has the canonical form $\mathcal{S} = f_0 + \ker T$, where f_0 is a special solution satisfying $Tf_0 = q$. In our present situation, this reads as follows.

Proposition 3.1. *Let $p(x,y) \in \mathbb{R}[x,y]$. Suppose that the equation*

$$f(x+y) - f(x) - f(y) = p(x,y)$$

has a continuous solution. Then the set of all solutions $f:\mathbb{R} \to \mathbb{R}$ to the equation is given by $\{p_0 + h : h \in \mathfrak{C}\}$, where $\mathfrak{C}$ is the set of all additive functions (the so called Cauchy functions) and where p_0 is a special (polynomial) solution.

[4] Abbreviated in the sequel by a.e.

[5] Recall that Y^X denotes the set of all functions from X to Y.

Proof. If $f:\mathbb{R}\to\mathbb{R}$ and $g\in C(\mathbb{R})$ satisfy the equation, then with $h := f-g$, we get that

$$h(x+y)-h(x)-h(y)=p(x,y)-p(x,y)=0.$$

Hence h is a solution to the classical Cauchy functional equation, that is, h is additive. By Theorem 2.2, g is a polynomial, say $g =: p_0$. Hence $f = p_0 + h$, with $h \in \mathfrak{C}$. The converse is obvious. ∎

Corollary 3.2. *Let $p(x,y)\in\mathbb{R}[x,y]$. The set of continuous solutions f to the equation*

$$f(x+y)-f(x)-f(y)=p(x,y)$$

is either empty or is given by

$$\{p_0 + c\,\mathrm{id} : c\in\mathbb{R}\},$$

where $\mathrm{id}(x)=x$ is the identity function and where p_0 is a special (polynomial) solution.

Proof. This follows from Proposition 3.1 and the fact that any continuous additive function h has the form $h(x)=cx$ for some $c\in\mathbb{R}$ (see [6]). ∎

Next we discuss under which conditions that are weaker than continuity a solution to the polynomial Cauchy equation is still a polynomial.

Proposition 3.3. *Let $p(x,y)\in\mathbb{R}[x,y]$. Suppose that the equation*

$$f(x+y)-f(x)-f(y)=p(x,y)$$

has a continuous solution p_0 (necessarily a polynomial) and let $f:\mathbb{R}\to\mathbb{R}$ be any other solution. Then, under each of the following conditions, f is a polynomial:

(1) *f is continuous at some point,*

(2) *f is bounded on some interval,*

(3) *the graph $\{(x,f(x)) : x\in\mathbb{R}\}$ is not dense in $\mathbb{R}^2$,*

(4) *f is measurable.*

Proof. This follows from Proposition 3.1 and the facts (given e.g. in [6]) on the structure of the non-linear solution to $c(x+y)=c(x)+c(y)$. In fact, suppose that f is a discontinuous solution to $f(x+y)-f(x)-f(y)=p(x,y)$. Then $h := f-p_0$ is a discontinuous additive function. Hence its graph is dense in $\mathbb{R}^2$ (see [6, p. 306] and [3]). As p_0 is continuous, the graph of f itself is dense in $\mathbb{R}^2$, contradicting assumptions (3), (2) and (1). Hence, f is continuous, and so f is a polynomial by Theorem 2.2. Now suppose that f is measurable. By Proposition 3.1, $f = p_0 + c$, where c is an additive function, which is measurable, too. By a theorem due to Fréchet, c is continuous (see [6, p. 241], or the proofs below). Hence f is continuous, and so, again, f is a polynomial by Theorem 2.2. ∎

An nice proof of Fréchet's result goes as follows[6].

[6]We do not know the original reference, but the proof is mentioned on Mathematics Stack Exchange [16].

Suppose that c is a Lebesgue measurable additive real-valued function (defined everywhere). Note that $c(0) = 0$. Consider the function[7] $u(x) := e^{ic(x)}$. Then u is measurable since it is the left composition of a continuous function with a measurable one (see e.g. [10, Theorem 1.7, p. 10]). Moreover, $|u| = 1$ on $\mathbb{R}$. Since c is additive, $u(x+y) = u(x)u(y)$. Choose a bounded measurable set $E \subseteq \mathbb{R}$ of finite measure such that $\int_E u(x)\,dx \neq 0$ (this is possible since u is not a.e. the zero function; just use [10, Theorem 1.39, p. 30] applied to $X = [0, 1]$). Note that

$$\int_E u(x+y)\,dx = u(y) \int_E u(x)\,dx. \tag{3.1}$$

We claim that

$$\lim_{y\to 0} \int_E u(x+y)\,dx = \int_E u(x)\,dx. \tag{3.2}$$

In fact, since $E + [-1, 1]$ is bounded, there is a compact interval I with $E + [-1, 1] \subseteq I$. Now put

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in I, \\ 0 & \text{if } x \in \mathbb{R} \setminus I. \end{cases}$$

Then $\tilde{u} \in L^1(\mathbb{R})$ (as it has compact support), and for $|y| \le 1$,

$$\begin{aligned} \left| \int_E u(x+y)\,dx - \int_E u(x)\,dx \right| &\le \int_E |u(x+y) - u(x)|\,dx \\ &= \int_E |\tilde{u}(x+y) - \tilde{u}(x)|\,dx \\ &\le \int_{\mathbb{R}} |\tilde{u}(x+y) - \tilde{u}(x)|\,dx \\ &= \|\tilde{u}(\cdot + y) - \tilde{u}\|_1 \to 0 \quad \text{as } y \to 0, \end{aligned}$$

where the latter assertion is a classical result in the theory of the space $L^1(\mathbb{R})$ (see [11, p. 74])[8]

Consequently, by combining (3.2) and (3.1), $\lim_{y\to 0} u(y) = 1 = u(0)$. Hence u is continuous[9] at 0. Thus the cluster set of c at 0 is contained in the set $\{2k\pi : k \in \mathbb{Z}\}$, and so the graph of c cannot be dense in $\mathbb{R}^2$. Hence, as already mentioned, c is continuous on $\mathbb{R}$. ■

Another proof, without integration theory, but based on the Steinhaus theorem [13] from measure theory, runs as follows. Let c be an everywhere defined Lebesgue measurable

[7] Since c itself may neither be locally bounded, nor locally L^1, we must introduce an auxiliary function u which allows us to integrate.

[8] Note that this convergence in the norm implies that, for every sequence $(y_n) \to 0$, there is a subsequence (y_{n_k}) such that $u(x + y_{n_k}) \to u(x)$ for almost every x (see [8, p. 1484]). It seems, though, that we do not have that a.e. $u(x + y) \to u(x)$ as $y \to 0$.

[9] Attention: we cannot immediately deduce that c is continuous at 0; just consider $\tilde{c}(x) = 2\pi$ if $x > 0$, $\tilde{c}(x) = -2\pi$ if $x < 0$ and $\tilde{c}(0) = 0$. Then $u = e^{i\tilde{c}} \equiv 1$.

additive real-valued function. We claim that c is continuous at 0. In fact, there exists $N \in \mathbb{N}$ such that

$$E := \{x \in [-N, N] : |c(x)| \leq N\}$$

has positive measure. Then, by Steinhaus's theorem, there is $r > 0$ such that $[-r, r] \subseteq E - E$. Given $\varepsilon > 0$, choose $m \in \mathbb{N}$ with $2N/m < \varepsilon$ and let $|x| < r/m$. Then $xm \in E - E$, that is, $xm = a - b$ for $a, b \in E$. Hence

$$|c(mx)| \leq |c(a)| + |c(b)| \leq 2N.$$

Since c is $\mathbb{Q}$-linear,

$$|c(x)| = \frac{1}{m}|c(mx)| \leq \frac{2N}{m} < \varepsilon. \qquad \blacksquare$$

Proposition 3.3 shows in particular that the non-continuous solutions to the polynomial Cauchy equation are very wild functions (provided one continuous solution exists). Recall that the class of non-continuous additive functions can only be shown to exist with the help of the axiom of choice by using so called Hamel bases ($\mathbb{Q}$-vector space basis for $\mathbb{R}$ viewed as a vector space over $\mathbb{Q}$). See [6].

Next we give a necessary and sufficient condition for the existence of *continuous* solutions to $f(x + y) - f(x) - f(y) = q(x, y)$. Obviously, the symmetry of $q(x, y)$ with respect to x and y is a necessary condition.

Proposition 3.4. *Let $q \in \mathbb{R}[x, y]$. Consider the polynomial functional equation*

$$f(x + y) - f(x) - f(y) = q(x, y). \tag{3.3}$$

Then (3.3) has a continuous solution if and only if

$$q(x, y) = \sum_{j=0}^{n} a_j[(x + y)^j - x^j - y^j] \tag{3.4}$$

for some $a_j \in \mathbb{R}$ and $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. A solution is given by

$$p_0(x) := \sum_{j=0}^{n} a_j x^j. \tag{3.5}$$

Note that, for $n \geq 1$, the degree of the polynomial $x \mapsto q(x, y)$ is at most $n - 1$.

Proof. It is straightforward to check that p_0 is a solution. Now let $f \in C(\mathbb{R})$ be a solution for (3.3). By Theorem 2.2, we know that f is a polynomial, say

$$f(x) = \sum_{j=0}^{n} a_j x^j.$$

Then

$$q(x, y) = f(x + y) - f(x) - f(y) = \sum_{j=0}^{n} a_j[(x + y)^j - x^j - y^j]. \qquad \blacksquare$$

Remark 3.5. Are formulas (3.5) and (2.1) compatible with each other? Let us have a look:

$$q(x,y) = \sum_{j=0}^{n} a_j[(x+y)^j - x^j - y^j] = -a_0 + \sum_{j=2}^{n} a_j \sum_{k=1}^{j-1} \binom{j}{k} x^{j-k} y^k$$
$$=: \sum_{i,j=0}^{n-1} a_{ij} x^i y^j .$$

In view of formula (2.1), it suffices to calculate the coefficients $a_{i,1}$. Putting $k = 1$, these can be determined from the sum

$$\sum_{j=2}^{n} a_j \binom{j}{1} x^{j-1} = \sum_{j=2}^{n} j a_j x^{j-1} = \sum_{i=1}^{n-1} (i+1) a_{i+1} x^i .$$

We conclude that $a_{i,1} = (i+1)a_{i+1}$. Consequently, the polynomial solutions f (modulo the linear term) satisfy

$$f(x) = a_0 + \sum_{j=2}^{n} a_j x^j = -a_{00} + \sum_{i=1}^{n-1} \frac{a_{i,1}}{i+1} x^{i+1} = -a_{00} + \sum_{j=2}^{n} \frac{a_{j-1,1}}{j} x^j .$$

We finish this section with the following natural question.

Problem 3.6. Do there exist symmetric polynomials $q \in \mathbb{R}[x, y]$ not of this form (3.4), but for which (3.3) nevertheless has a solution, necessarily non-continuous? For instance, what happens for $q(x, y) = x^2 y^2$?

Note that if $q(x, y) = p(x) + p(y)$ for a polynomial p of degree at least one, then the equation $f(x + y) - f(x) - f(y) = p(x) + p(y)$ has no solution. Just take $y = 0$.

4 Further refinements: Just two y's with irrational quotient are needed

The following result is a special case of a result that was mentioned (without proof) by the editorial board of the problem session in [12] and is due to Omran Kouba. The proof below is ours and generalizes the one of Proposition 2.1.

Proposition 4.1. *Let $f \in C^\infty(\mathbb{R})$ satisfy the following conditions:*[10]

$$p(x, y_j)\colon x \mapsto f(x + y_j) - f(x) - f(y_j) \text{ is a polynomial}$$

for two real numbers y_j ($j = 1, 2$), linearly independent in the $\mathbb{Q}$ vector space ${}^{\mathbb{Q}}\mathbb{R}$ of $\mathbb{R}$ over $\mathbb{Q}$. Then f is a polynomial.

[10] Of course, the constant term $f(y_j)$ is superfluous.

Proof. Write $f(x+y_j)-f(x) =: p_{d_j}(x)$, where d_j is the degree of the polynomial $x \mapsto p(x, y_j)$. Let $d := \max\{d_1, d_2\}$. Differentiating $d+1$ times gives 0, so for $j = 1, 2$ and $x \in \mathbb{R}$,

$$f^{(d+1)}(x+y_j) = f^{(d+1)}(x) =: H(x).$$

Thus H is a periodic function with periods y_j. Note that H is independent of j. Since a periodic continuous function on $\mathbb{R}$ cannot have two periods y_j with $y_1/y_2 \in \mathbb{R} \setminus \mathbb{Q}$ (see [2]), $H = f^{(d+1)}$ is constant, and so f is a polynomial of degree less than or equal to $d+1$. ∎

Next we give a short proof of an extension of [7, Lemma 2.5]. Let $\mathcal{P}_N$ be the set of polynomials of degree less than or equal to N (including the zero polynomial).

Proposition 4.2. *Fix $N \in \mathbb{N}$. Let $f \in C(\mathbb{R})$ satisfy the following conditions:*

$$x \mapsto f(x+y_j) - f(x) \in \mathcal{P}_N$$

for two real numbers y_j $(j = 1, 2)$, linearly independent in the $\mathbb{Q}$ vector space ${}^{\mathbb{Q}}\mathbb{R}$ of $\mathbb{R}$ over $\mathbb{Q}$. Then f is a polynomial of degree less than $N+1$.

Proof. Consider the set

$$G := \{y \in \mathbb{R} : x \mapsto f(x+y) - f(x) \in \mathcal{P}_N\}.$$

Then G is an additive subgroup of $\mathbb{R}$, as with $u, v \in G$, we also have $u - v \in G$. Moreover, $y_1\mathbb{Z} + y_2\mathbb{Z} \subseteq G$. Since $y_1/y_2 \notin \mathbb{Q}$, we deduce from Kronecker's theorem [8, Corollary 35.7] that G is dense in $\mathbb{R}$. Now let $u \in \mathbb{R}$ and choose $u_n \in G$ with $u_n \to u$. Note that $f(x+u_n) - f(x)$ converges locally uniformly[11] to $f(x+u) - f(x)$. As $(C(\mathbb{R}), d)$ endowed with the topology of local uniform convergence is a complete linear metric space, its $(N+1)$-dimensional subspace $\mathcal{P}_N$ is closed with respect to local uniform convergence, as it is linear-isomorphic and homeomorphic to $\mathbb{R}^{N+1}$ (see e.g. [9, 1.21, 1.44]). See also [14] for a different proof immediately adaptable to our topology here. Hence $x \mapsto f(x+u) - f(x) \in \mathcal{P}_N$. Consequently, G is closed and the denseness of G now implies that $G = \mathbb{R}$. Hence, by Theorem 2.2, f is a polynomial, which we denote by p_0.

By Corollary 3.2, the set of all continuous solutions is given by $p_0 + c\,\mathrm{id}$, $c \in \mathbb{R}$. By Proposition 3.4, there is a solution of degree at most $N+1$. Hence $\deg f \leq N+1$. ∎

We conclude with the most general version of our class of results.

Theorem 4.3. *Let $f \in C(\mathbb{R})$ satisfy the following conditions:*

$$x \mapsto f(x+y_j) - f(x) \in \mathbb{R}[x]$$

for two real numbers y_j $(j = 1, 2)$, linearly independent in the $\mathbb{Q}$ vector space ${}^{\mathbb{Q}}\mathbb{R}$ of $\mathbb{R}$ over $\mathbb{Q}$. Then f is a polynomial.

Proof. Let $p_j(x) := f(x+y_j) - f(x)$ and let $N := \max\{\deg p_1, \deg p_2\}$. Then $p_j \in \mathcal{P}_N$. By Proposition 4.2, $f \in \mathbb{R}[x]$. ∎

[11] This is a consequence of the fact that the function $H(x, u) := f(x+u) - f(x)$ is continuous on $\mathbb{R} \times \mathbb{R}$.

Let us point out that the condition $y_1/y_2 \in \mathbb{R} \setminus \mathbb{Q}$ is of course necessary to obtain the desired assertion. In fact, let f be the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $p\colon x \mapsto f(x+y) - f(x) = 0$ for every rational number y; hence p is the zero-polynomial, but f is discontinuous (everywhere), so definitely no polynomial.

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