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Short note

On an Erdős inscribed triangle inequality revisited

Cezar Lupu and Ştefan Spătaru

Abstract. In this note, we give a refinement of an inequality of Torrejon between the area of a triangle and that of an inscribed triangle. Our approach is based on using complex numbers and some elementary facts on geometric inequalities.

Dedicated to the memory of Octavian Ganea, a big enthusiast of complex numbers in geometry

1 Introduction and statement of the main result

Let us consider a triangle ABC. On each of the sides BC, CA and AB, fix arbitrary points A_1, B_1 and C_1, respectively. As pointed out in [7,9], the Erdős–Debrunner inequality

$$\min\{\text{area}(AC_1B_1), \text{area}(C_1BA_1), \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1) \tag{1}$$

is a topic with a long history. Later, Janous [5] generalized inequality (1) by proving

$$\mathcal{M}_{-1}\{\text{area}(AC_1B_1), \text{area}(C_1BA_1), \text{area}(B_1AC)\} \leq \text{area}(A_1B_1C_1),$$

where $\mathcal{M}_{-1}$ denotes the harmonic mean of three positive numbers. Moreover, Janous formulated a more general question which was extended and solved by Mascioni [7,8]. Using a different method, Frenzen, Ionaşcu and Stănică [4] proved Janous' conjecture independently of Mascioni. Yet a different method was used by Blatter in [2] for his proof of Janous' conjecture.

The purpose of this note is to extend the result of Torrejon [9] regarding the areas of triangles $A_1B_1C_1$ and ABC when the points A_1, B_1, C_1 satisfy a certain metric property.

Our main result is given by the following.

Theorem 1.1. *Let ABC be a triangle and let A_1, B_1 and C_1 be on BC, CA and AB, respectively, with none of A_1, B_1 and C_1 coinciding with a vertex of ABC. If*

$$\frac{AB + BA_1}{AC + CA_1} = \frac{BC + CB_1}{AB + AB_1} = \frac{AC + AC_1}{BC + BC_1} = \alpha,$$

then

$$\begin{aligned}&\operatorname{area}(A_1B_1C_1)\\ &\quad\le \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)}\left(\operatorname{area}(ABC)+s^4\left(\frac{\alpha-1}{\alpha+1}\right)^2\operatorname{area}(ABC)^{-1}\right),\end{aligned}$$

where s is the semi-perimeter of triangle ABC.

For $\alpha = 1$, we obtain ([6])

$$\frac{\operatorname{area}(A_1B_1C_1)}{\operatorname{area}(ABC)} \le \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)}.$$

As the arithmetic-geometric mean inequality shows $(a+b+c)(a^2+b^2+c^2) \ge 9abc$, Theorem 1.1 implies the result ([9])

$$4\operatorname{area}(A_1B_1C_1) \le \operatorname{area}(ABC) + s^4\left(\frac{\alpha-1}{\alpha+1}\right)^2\operatorname{area}(ABC)^{-1}.$$

For $\alpha = 1$, this inequality reduces to ([1])

$$4\operatorname{area}(A_1B_1C_1) \le \operatorname{area}(ABC).$$

Our approach in computing the area of the triangle $A_1B_1C_1$ will be different from the one used by Torrejon in [9]. It is based on complex numbers in the plane.

Recall that the area of a positively oriented triangle ABC whose vertices have affixes z_A, z_B and z_C is given by

$$\operatorname{area}(ABC) = \frac{1}{2}\operatorname{Im}(\overline{z_A}z_B + \overline{z_B}z_C + \overline{z_C}z_A).$$

2 Proof of Theorem 1.1

First of all, we prove the following equality:

$$\operatorname{area}(A_1B_1C_1) = 2F\cdot\frac{(s-a)(s-b)(s-c)+s^3\cdot\left(\frac{\alpha-1}{\alpha+1}\right)^2}{abc},$$

where a, b, c denote the side lengths of triangle ABC and F its area. Let furthermore z_P be the (complex) affix of an arbitrary point P.

Firstly, $2s = a+b+c = (AB+AB_1)+(BC+CB_1) = (\alpha+1)(c+AB_1)$, and consequently, $AB_1 = \frac{2s}{\alpha+1} - c$ and

$$CB_1 = CA - AB_1 = b - \frac{2s}{\alpha+1} + c = 2s - a - \frac{2s}{\alpha+1} = \frac{2s\alpha}{\alpha+1} - a.$$

Analogously, we have

$$BC_1 = \frac{2s}{\alpha+1} - a,\quad CA_1 = \frac{2s}{\alpha+1} - b,\quad BA_1 = \frac{2s\alpha}{\alpha+1} - c \quad\text{and}\quad AC_1 = \frac{2s\alpha}{\alpha+1} - b.$$

Therefore, the affixes of A_1, B_1, and C_1 are given by

$$z_{A_1} = \frac{\left(\frac{2s}{\alpha+1} - b\right)z_B + \left(\frac{2s\alpha}{\alpha+1} - c\right)z_C}{a},$$
$$z_{B_1} = \frac{\left(\frac{2s}{\alpha+1} - c\right)z_C + \left(\frac{2s\alpha}{\alpha+1} - a\right)z_A}{b},$$
$$z_{C_1} = \frac{\left(\frac{2s}{\alpha+1} - a\right)z_A + \left(\frac{2s\alpha}{\alpha+1} - b\right)z_B}{c},$$

respectively. Before computing the area of triangle $A_1B_1C_1$, we note that the expressions $z_P\overline{z_P}$ and $z_P\overline{z_Q} + \overline{z_P}z_Q$ are real numbers for arbitrary points P and Q. Therefore,

$$\begin{aligned}
&2\,\text{area}(A_1B_1C_1)\\
&\quad= \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_{A_1}}z_{B_1}\Big)\\
&\quad= \operatorname{Im}\Big(\sum_{\text{cyc}} \frac{\left(\frac{2s}{\alpha+1} - b\right)\overline{z_B} + \left(\frac{2s\alpha}{\alpha+1} - c\right)\overline{z_C}}{a} \cdot \frac{\left(\frac{2s}{\alpha+1} - c\right)z_C + \left(\frac{2s\alpha}{\alpha+1} - a\right)z_A}{b}\Big)\\
&\quad= \frac{1}{abc}\cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_B}z_C\Big[c\Big(\frac{2s}{\alpha+1} - b\Big)\Big(\frac{2s}{\alpha+1} - c\Big)\\
&\qquad\qquad + b\Big(\frac{2s\alpha}{\alpha+1} - c\Big)\Big(\frac{2s\alpha}{\alpha+1} - b\Big)\Big]\Big)\\
&\qquad + \frac{1}{abc}\cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_C}z_B a\Big(\frac{2s\alpha}{\alpha+1} - b\Big)\Big(\frac{2s}{\alpha+1} - c\Big)\Big)\\
&\quad= \frac{1}{abc}\cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_B}z_C\Big[c\Big(\frac{2s}{\alpha+1} - b\Big)\Big(\frac{2s}{\alpha+1} - c\Big)\\
&\qquad\qquad + b\Big(\frac{2s\alpha}{\alpha+1} - c\Big)\Big(\frac{2s\alpha}{\alpha+1} - b\Big)\\
&\qquad\qquad - a\Big(\frac{2s\alpha}{\alpha+1} - b\Big)\Big(\frac{2s}{\alpha+1} - c\Big)\Big]\Big)\\
&\quad= \frac{1}{abc}\cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_B}z_C\Big[b\Big(\frac{s(1-\alpha)}{\alpha} + s - b\Big)\Big(\frac{s(1-\alpha)}{\alpha} + s - c\Big)\\
&\qquad\qquad + c\Big(\frac{s(\alpha-1)}{\alpha} + s - b\Big)\Big(\frac{s(\alpha-1)}{\alpha} + s - c\Big)\\
&\qquad\qquad - a\Big(\frac{s(\alpha-1)}{\alpha} + s - b\Big)\Big(\frac{s(1-\alpha)}{\alpha} + s - c\Big)\Big]\Big)\\
&\quad= \frac{1}{abc}\cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_B}z_C\Big[2(s-a)(s-b)(s-c)\\
&\qquad\qquad + s\cdot\Big(\frac{1-\alpha}{1+\alpha}\Big)(ab - ac + ac - ab)\\
&\qquad\qquad + s^2\cdot\Big(\frac{\alpha-1}{\alpha+1}\Big)^2(a+b+c)\Big]\Big).
\end{aligned}$$

Now it can be checked without much difficulty that the coefficient of $\overline{z_B} z_C$ equals

$$2(s-a)(s-b)(s-c) + 2s^3 \cdot \Big(\frac{\alpha-1}{\alpha+1}\Big)^2.$$

This results in

$$\operatorname{area}(A_1B_1C_1) = \frac{(s-a)(s-b)(s-c) + s^3 \cdot \big(\frac{\alpha-1}{\alpha+1}\big)^2}{abc} \cdot \operatorname{Im}\Big(\sum_{\text{cyc}} \overline{z_B} z_C\Big).$$

This and $\operatorname{Im}(\sum_{\text{cyc}} \overline{z_B} z_C) = 2F$ readily lead to the stated formula for $\operatorname{area}(A_1B_1C_1)$.

Therefore, we get

$$\frac{abc \cdot s}{2} \cdot \operatorname{area}(A_1B_1C_1) = F \cdot \left(s(s-a)(s-b)(s-c) + s^4 \cdot \Big(\frac{\alpha-1}{\alpha+1}\Big)^2 \right),$$

that is (via Heron's formula),

$$\frac{abc \cdot s}{2} \cdot \operatorname{area}(A_1B_1C_1) = F^3 + s^4 \cdot \Big(\frac{\alpha-1}{\alpha+1}\Big)^2 \cdot F.$$

At this point, we are only left to prove the inequality

$$\frac{sabc}{2} \geq 4F^2 \cdot \frac{(a+b+c)(a^2+b^2+c^2)}{9abc}, \tag{2}$$

that is,

$$a^2b^2c^2 \geq 16F^2 \frac{a^2+b^2+c^2}{9}.$$

Using $abc = 4RF$, we get the equivalent inequality

$$9R^2 \geq a^2 + b^2 + c^2.$$

However, this inequality is evident since the distance between the circumcenter O and the centroid G is given by the Leibniz identity $OG^2 = 9R^2 - (a^2 + b^2 + c^2)$.

Finally, we conclude

$$F^3 + s^4 \Big(\frac{\alpha-1}{\alpha+1}\Big)^2 F \geq 4F^2 \cdot \frac{(a+b+c)(a^2+b^2+c^2)}{9abc} \operatorname{area}(A_1B_1C_1),$$

which is equivalent to the inequality of our theorem. ■

Remark. It is worth noting that inequality (2) is equivalent to

$$F \leq \frac{3}{4} \cdot \frac{abc}{\sqrt{a^2+b^2+c^2}}$$

and thus improves the result ([3, Item 4.13])

$$F \leq \frac{3\sqrt{3}}{4} \cdot \frac{abc}{a+b+c}.$$

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Cezar Lupu
Beijing Institute of Mathematical Sciences and Applications (BIMSA)
101408 Beijing
Yau Mathematical Sciences Center (YMSC)
Tsinghua University
100084 Beijing, P. R.China
lupucezar@gmail.com, lupucezar@bimsa.cn

Ştefan Spătaru
Harvard University
Cambridge, MA-02138, USA
sspataru96@gmail.com