Elemente der Mathematik 80 (2025), 128–131
DOI 10.4171/EM/539

Short note

Trirectangular equable tetrahedra with integer face areas

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Abstract. We prove that there are exactly 9 tetrahedra with one solid right angle, all faces of integral area, whose sum corresponds to the volume of the tetrahedron.

1 Motivation

The Pythagorean triangles $6, 8, 10$ and $5, 12, 13$ are said to be *equable* because their area has the same value as their perimeter. One can easily prove that they are the only equable Pythagorean triangles [4]. Extending the concept of equable Pythagorean triangles to space leads us naturally to consider tetrahedra with a solid right angle (i.e., trirectangular) and faces of integer area whose sum equals its volume (i.e., equable) [1]. The identification of all such tetrahedra is conjectured and almost proved in [2]. Hereunder, we offer a complete proof which is inspired by the appendix of [4] and requires very few calculations compared to those in [2].

2 De Gua's theorem and the main result

Consider a trirectangular tetrahedron $OABC$ labeled as in Figure 1, where

$$OA = a, \quad OB = b, \quad OC = c$$

are its legs, K_X is the area of the face opposite to X for $X = A, B, C, O$ and $V = abc/6$ denotes its volume. For completeness, we give a one-line proof of the generalisation of the Pythagorean theorem, $K_O^2 = K_A^2 + K_B^2 + K_C^2$:

$$\begin{aligned} 4K_O^2 = AB^2CH^2 &= (OA^2 + OB^2)(OH^2 + OC^2) \\ &= 4\big(K_A^2 + K_B^2 + OH^2\underbrace{(OA^2 + OB^2)}_{AB^2}\big). \end{aligned}$$

The above result, proved in [5], is also called de Gua's theorem, even if it was known more than a century before by both J. Faulhaber and R. Descartes [3].

Theorem. *There are only 9 trirectangular equable tetrahedra with integer face areas.*

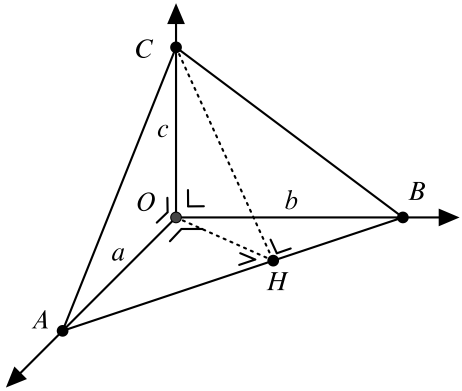

Figure 1. A trirectangular tetrahedron with trirectangular vertex at the origin and legs of length a, b, c.

Proof. Notice first that dividing V by the area of any right-angled face implies that a, b, c are rational. Now, equability of $OABC$ and de Gua's theorem give

$$ab + ac + bc + \sqrt{a^2b^2 + a^2c^2 + b^2c^2} = \frac{abc}{3}. \tag{1}$$

Isolating the root, squaring, reducing and dividing by abc implies

$$abc - 6(ab + ac + bc) + 18(a + b + c) = 0. \tag{2}$$

We recall from [6] that, for a prime p, the *p-adic valuation* of a natural number n, denoted $\nu_p(n)$, is the greatest power of p that divides n. If $n = 0$, then by convention, $\nu_p(0) = \infty$. By extension, for a rational $x = \frac{m}{n}$, we let $\nu_p(x) = \nu_p(m) - \nu_p(n)$. One can easily prove the two basic properties below of ν_p for rationals x and y:

(i) $\nu_p(x \cdot y) = \nu_p(x) + \nu_p(y)$ and

(ii) $\nu_p(x \pm y) \geq \min\{\nu_p(x), \nu_p(y)\}$, where equality holds if $\nu_p(x) \neq \nu_p(y)$.

In our case, for a prime p, if the p-adic valuations of the legs are $i \leq j \leq k$, then redefine a, b, c in order to have $\nu_p(a) = i$, $\nu_p(b) = j$ and $\nu_p(c) = k$. Since the areas of the three perpendicular triangular faces are integers, then we necessarily have the property that $j, k \geq -i$, and so $j, k \geq 0$. Now rewrite (2) as

$$\frac{abc}{2} - 3(ab + ac + bc) + 9(b + c) = -9a, \tag{3}$$

and suppose first that $p = 2$. By the above property, and because of the factor $\frac{1}{2}$ of the area of a triangle, we must even have $j, k \geq -i + 1$. Taking this into account in (3) implies $\nu_2(a) \geq 1$ since the 2-adic valuation of each term on the left-hand side of (3) is positive. Thus, a, b, c are even in the sense that their 2-adic valuation is at least equal to 1. Next, suppose $p \geq 5$ and prime. Then, applying ν_p on both sides of (3), properties (i) and (ii) imply $i \geq 0$. Indeed,

$$0 \leq i + j = \min\{i + j + k, i + j, i + k, j + k, j, k\} \leq \nu_p(-9a) = \nu_p(a) = i.$$

Hence, for all $p \neq 3$, the p-adic valuation of a, b, c is positive or zero. Thus, each of a, b, c is either an even integer or one term is a rational having a power of 3 as denominator. Finally, let $p = 3$ and suppose $i \leq -2$. When applying ν_3 on both sides of (3), then properties (i) and (ii) imply

$$\begin{aligned} i + j + 1 &= \min\{i + j + k, i + j + 1, i + k + 1, j + k + 1, j + 2, k + 2\} \\ &\leq i + 2 \end{aligned}$$

and hence the contradiction $j \leq 1$ since $j \geq -i$. So if a is a non-integer, then $i = -1$ and $j = 1 \leq k$. Otherwise, if $i = 0$, then (2) implies that 3 divides b or c. In short, either a, b, c are all even integers, or two are even integers and one is the third of an even integer. Therefore, multiplying (1) by 9 and replacing $3a$ by r, $3b$ by s, and $3c$ by t gives

$$rs + rt + st + \sqrt{r^2s^2 + r^2t^2 + s^2t^2} = \frac{rst}{9}$$

and verifies the following.

Condition (C). All three values r, s, t are even and two are divisible by 3.

Once again and after some calculations, we can verify the following:

$$0 = rst - 18(rs + rt + st) + 162(r + s + t). \tag{4}$$

By symmetry, we suppose $2 \leq r \leq s \leq t$. Routine elementary verifications that we illustrate by an example below – where we leave the other cases to the reader – allow us to eliminate empty sets of solutions for $r = 2, 4, \ldots, 18$: for example, if $r = 2$, then (4) gives

$$t = \frac{3(25s + 54)}{2s - 75} \geq s$$

which induces $s^2 - 75s - 81 \leq 0$, since $t \geq s \geq 0$, giving $38 \leq s \leq 72$. Testing the 17 even values gives no integer solution.

Getting back to our main route, in (4), we replace r by $x + 18$, s by $y + 18$, and t by $z + 18$ and obtain

$$xyz - 162(x + y + z) - 2916 = 0,$$

where $2 \leq x \leq y \leq z$, and verify the Condition (C). Solving the preceding equation in z gives

$$y \leq z = \frac{162(x + y + 18)}{xy - 162}, \tag{5}$$

which, since $2 \leq x \leq y \leq z$, implies that $xy > 162$ and hence gives the inequality

$$xy^2 - 324y - 162x - 2916 \leq 0.$$

Once again, the solution produces the inequality

$$x \leq y \leq \frac{9(18 + \sqrt{2x^2 + 36x + 324})}{x}. \tag{6}$$

Rearranging, squaring and reducing gives $x^4 - 486x^2 - 2916x \leq 0$ which factorises into $x(x+18)(x^2 - 18x - 162)$, implying $x \leq 9 + 9\sqrt{3} \cong 24.6$. Since the quotient in (6) is decreasing, taking $x = 2$ gives us $y \leq 170$. Hence, testing all $2 \leq x \leq 24$ and $x \leq y \leq 170$ in (5) that verify Condition (C) and finally evaluating their associated (a, b, c) in (1) produces six integer and three rational solutions

$$(8, 16, 168) \quad (8, 18, 66) \quad (8, 24, 32) \quad (10, 12, 54) \quad (12, 12, 24) \quad (12, 14, 18)$$
$$(20/3, 36, 336) \quad (28/3, 12, 144) \quad (32/3, 12, 36). \qquad \blacksquare$$

Acknowledgements. Very many thanks to the anonymous referee, Grant Cairns and Phil Lawson from Gardenstown, Scotland for their very constructive criticisms.

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