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Short note

A short proof of a variant of the round-robin scheduling problem

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Abstract. Suppose $2n$ people participate in a sports tournament that consists of multiple rounds. In each round, two teams of n people are formed to play against each other. We require that every two players play at least once in opposing teams and once in the same team. For this variant of the round-robin scheduling problem, an explicit formula for the minimum number of rounds needed to satisfy both conditions has recently been published. In this short note, an alternative and short proof of this is given.

1 Introduction

Scheduling problems play an important role in combinatorics, operations research, and combinatorial optimization. Examples are the round-robin tournament problem which has applications in computer science (see [2, 3, 5]) or the traveling tournament problem (see [4, 6]) in graph theory. In this article, we will address the following problem. A group of $2n$ individuals are participating in a sports tournament that consists of multiple rounds. In each round, two new teams of n people are formed to play against each other. In this tournament, the following conditions need to be met:

($\mathfrak{s}$) each two players have been at least once in the *same* team, and

($\mathfrak{o}$) each two players have played at least once in *opposing* teams.

In [1], the conditions ($\mathfrak{s}$) and ($\mathfrak{o}$) are first analysed separately, and then the case where both conditions must be fulfilled simultaneously is considered. For all three scenarios, the minimum number of rounds that a tournament must have under the respective conditions is determined as a function of n. In addition, the optimal game schedules are generated. The case in which ($\mathfrak{s}$) and ($\mathfrak{o}$) must be fulfilled simultaneously is particularly delicate. The aim of this short note is to present an alternative and simple proof of the following.

Theorem ([1]). *Let $f_{\mathfrak{s},\mathfrak{o}}(n)$ be the minimum number of rounds of a tournament with $2n$ players such that both conditions ($\mathfrak{s}$) and ($\mathfrak{o}$) are satisfied simultaneously. For $n \in \mathbb{N}$, $n \geq 2$, the following holds:*

$$f_{\mathfrak{s},\mathfrak{o}}(n) = \begin{cases} \lceil \log_2(n) \rceil + 3 & \text{if } n = 2^m - 1, \\ \lceil \log_2(n) \rceil + 2 & \text{otherwise}. \end{cases}$$

2 A short proof of the theorem

Let $N = 2n$ be an even number of regular players with $2^{k-1} < N \leq 2^k$, i.e.,

$$k = \lceil \log_2 N \rceil = \lceil \log_2 n \rceil + 1.$$

The players play against each other in two teams of n players called team 0 and team 1. We can view a competition schedule of m games as an $m \times 2n$ matrix **S** of entries $s_{ij} \in \{0, 1\}$, which tells us that, in the i-th round, player j plays in team s_{ij}. We assume the schedule **S** to be admissible if each player played at least once with and against all the other players. Clearly, $m \geq k$ since the columns of **S** consist of distinct binary numbers. Pick a submatrix **T** consisting of k rows of **S**. The columns in **T** are k-digit binary numbers from 0 to $2^k - 1$. These binary numbers come in 2^{k-1} complementary pairs, complementary meaning that all the bits have been swapped. We can represent this as a hotel with 2^{k-1} floors and each floor consisting of two rooms: the two rooms on each floor correspond to a pair of complementary binary numbers. We now put each player into a room corresponding to his number in **T**. Since $N > 2^{k-1}$, there is at least one floor that accommodates more than one player. These will yet have to play against each other (if they are room sharers) or with each other (if they are floor sharers). Thus, **T** is not an admissible schedule and $m \geq k + 1$.

We can get the better lower bound $m \geq k + 2$ in case that $N = 2^k - 2$. Assume that $m = k + 1$ (i.e., the hotel has $n + 1$ floors) and lead it to contradiction. If three players share the same floor, then we clearly need at least two extra games. So we place at most two players in each floor. At least $n - 1$ floors contain a pair of players. Could there be two loners staying alone on a floor? No: as n is odd, the Nim-sum of the columns of **T**, i.e., the sum of all columns of **T** modulo-2, written as a row is $111 \cdots 1$. Now, clearly room sharers add nothing to the Nim-sum, whereas floor sharers add $111 \cdots 1$. The two loners clearly would spoil the overall Nim-sum of **T**. There is hence an even number of pairs of room sharers and an odd number of pairs of floor sharers. Room sharers need an extra game to play against each other; floor sharers need one to play once in the same team. By parity consideration, this cannot be done in just one extra game. (Note that n is odd.)

The bounds can be reached as follows.

(1) $N = 2^k$. Fill the hotel by placing one player in each room. We need just one extra game to combine the pairs of floor sharers in the two teams.

(2) $N = 2^k - 2$. Place the players, one per room, in n floors. We need more than one extra game as seen above, but two clearly suffice.

(3) Let $2^{k-1} < N \leq 2^k - 4$. We construct **T** as follows: start with four columns like

$$\begin{matrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

and, if necessary, fill the missing columns with complementary pairs. Then we have enough elbow room and need only one extra game for these complementary pairs.

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References

[1] C. Busenhart, N. Hungerbühler, and W. Xu, A variant of the Round–Robin scheduling problem. *Ars Combin.* **158** (2024), 81–92 MR 4726301

[2] D. Cozzo and E. Giunta, Round-robin is optimal: lower bounds for group action based protocols. In *Theory of cryptography. Part IV*, pp. 310–335, Lecture Notes in Comput. Sci. 14372, Springer, Cham, 2023 MR 4697197

[3] C. C. Ribeiro, S. Urrutia, and D. de Werra, A tutorial on graph models for scheduling round-robin sports tournaments. *Int. Trans. Oper. Res.* **30** (2023), no. 6, 3267–3295 Zbl 07745327 MR 4617599

[4] M. R. Siemann and M. Walter, A polyhedral study for the cubic formulation of the unconstrained traveling tournament problem. *Discrete Optim.* **46** (2022), article no. 100741, 33 Zbl 1509.90077 MR 4496833

[5] T. Tuffaha, B. Çavdaroğlu, and T. Atan, Round–Rrobin scheduling with regard to rest differences. *TOP* **31** (2023), no. 2, 269–301 MR 4598609

[6] J. Zhao, M. Xiao, and C. Xu, Improved approximation algorithms for the Traveling Tournament Problem. In *47th International Symposium on Mathematical Foundations of Computer Science*, article no. 83, 15, LIPIcs. Leibniz Int. Proc. Inform. 241, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022 Zbl 1398.90144 MR 4482001

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