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Stable $(r+1)$-th capillary hypersurfaces

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Abstract. In this paper, we propose a new definition of stable $(r+1)$-th capillary hypersurfaces from variational perspective for any $1 \leq r \leq n-1$. More precisely, we define stable $(r+1)$-th capillary hypersurfaces to be smooth local minimizers of a new energy functional under volume-preserving and contact angle-preserving variations. Using this new concept of stable $(r+1)$-th capillary hypersurfaces, we generalize the stability results of Souam (2023) in a Euclidean half-space, and Guo, Wang and Xia (2022) in a horoball in hyperbolic space for capillary hypersurfaces to the $(r+1)$-th capillary hypersurface case.

1. Introduction

A classical result for constant mean curvature (CMC) hypersurfaces, proved by Barbosa and do Carmo [5], and Barbosa, do Carmo and Eschenburg [6], states that *"any stable immersed closed CMC hypersurface in a space form is a geodesic sphere"*. Here "stable" means that the second variation of the area functional is nonnegative for any volume-preserving variations. The following analogous result for stable immersed closed hypersurfaces with constant higher-order mean curvature in space forms has been proved by Alencar, do Carmo and Colares [2], Alencar, do Carmo and Rosenberg [3], and Barbosa and Colares [7].

Theorem 1.1 ([2, 3, 7]). *Let $0 \leq r \leq n-1$. An immersed n-dimensional closed constant $(r+1)$-th mean curvature hypersurface in space forms is stable if and only if it is a geodesic sphere.*

(We regard an open hemi-sphere as a spherical space form in this paper.)

We also mention that Palmer [32] and the second author with He [23] proved analogous result for hypersurfaces with constant $(r+1)$-th anisotropic mean curvature.

The study of capillary hypersurfaces has attracted a lot of attention in the last decades. In fluid mechanics, a capillary surface models the interface between two fluids in the absence of gravity. In fact, the free surface of the fluids locally minimizes the free energy functional under a volume constraint. We refer to the book of Finn [13] for more physical problems about capillary surfaces. From the geometric variational point of view,

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a capillary hypersurface in a domain B is a stationary point of the free energy functional for volume-preserving variations whose boundary freely moves on ∂B. By the first variational formula, it is a CMC hypersurface with boundary which intersects ∂B at a constant angle. There are plenty of important works on the existence, regularity and their min-max theory for free boundary or capillary minimal hypersurfaces, see, for example, [12, 15–17, 19, 24, 25, 28, 36] and the references therein.

The study on the classification for stable capillary hypersufaces has been initiated by Ros and Vergasta [34] for the free boundary case, and by Ros and Souam [33] for the general capillary case. When B is a Euclidean unit ball, the classification has been recently completed by Nunes [31] for the free boundary case in two dimensions and eventually by the third author with Wang [39] for the general capillary case in all dimensions, by using a new Minkowski formula involving no boundary term. When B is a Euclidean half-space, the classification has been recently settled by Souam [35].

Theorem 1.2 ([35]). *A compact immersed capillary hypersurface in a Euclidean half-space is stable if and only if it is a spherical cap.*

The anisotropic version in a half-space has been proved by the first and the third author [22].

Motivated by the concept of higher-order mean curvatures and also the capillary theory, it is natural to ask for a higher-order capillary theory. In [10, 11], Damasceno and Elbert introduced a notion of stable capillary hypersurfaces with constant higher-order mean curvature in terms of the associated stability operator, instead of that given by means of a variational problem.

In this paper, we propose a new notion of stability for higher-order capillary hypersurfaces from the variational perspective. For any $1 \le r \le n-1$, an n-dimensional $(r+1)$-th capillary hypersurface in B is a hypersurface with constant $(r+1)$-th mean curvature H_{r+1} and with boundary intersecting ∂B at a constant angle. It is known that the first variation of a higher-order mean curvature integral involves curvature terms in the boundary integral, which violates the capillary boundary condition. To overcome this difficulty, we make a restriction on the variation class. Precisely, we define a new $(r+1)$-th energy functional $\mathcal{E}_{r+1}$ and show that an $(r+1)$-th capillary hypersurface is a stationary point of $\mathcal{E}_{r+1}$ for any volume-preserving and angle-preserving variations. We say that an $(r+1)$-th capillary hypersurface is stable if the second variation of $\mathcal{E}_{r+1}$ is nonnegative for any volume-preserving and angle-preserving variations. We emphasize that comparing with the classical capillary theory ($r=0$), we only allow angle-preserving variations in the higher-order case. Equivalently speaking, we require that the normal components of variational fields stay in

$$\mathcal{F} = \left\{ \varphi \in C^\infty(M) \mid \int_M \varphi \, dA = 0 \text{ and } \nabla_\mu \varphi = q\varphi \text{ on } \partial M \right\}.$$

This is the key point for this new notion of stability (see Proposition 3.2 below for details).

Our first main result in this paper is the following classification for stable $(r+1)$-th capillary hypersurfaces in an $(n+1)$-dimensional Euclidean half-space $\overline{\mathbb{R}}^{n+1}_+$.

Theorem 1.3. *Let $1 \le r \le n-1$. A compact immersed $(r+1)$-th capillary hypersurface in $\overline{\mathbb{R}}^{n+1}_+$ is stable if and only if it is a spherical cap.*

The proof of Theorem 1.3 is based on the following higher-order Minkowski-type formula in $\overline{\mathbb{R}}^{n+1}_{+}$:

$$\int_M [H_r(1-\cos\theta\langle E_{n+1},\nu\rangle) - H_{r+1}\langle x,\nu\rangle]\,dA = 0 \quad \text{for any } 0 \le r \le n-1. \tag{1.1}$$

Formula (1.1) has been proved by Wang, Weng and the third author in [37], where was used to prove Alexandrov–Fenchel inequalities for embedded hypersurfaces with capillary boundary in $\overline{\mathbb{R}}^{n+1}_{+}$. Note that (1.1) offers an admissible test function which also satisfies $\nabla_\mu\varphi = q\varphi$ on ∂M.

When B is a $(n+1)$-dimensional horoball in hyperbolic space $\mathbb{H}^{n+1}$, we see that its boundary ∂B is a horosphere, that is, a non-compact totally umbilical hypersurface with all principal curvatures equal to 1. In the next part, we study a stability problem for $(r+1)$-th capillary hypersurfaces supported on a horosphere. For $r=0$, the classification of stable capillary hypersurfaces supported on a horosphere has been proved by the first and the third authors with Wang in [20].

Theorem 1.4 ([20]). *A compact immersed capillary hypersurface supported on a horosphere in $\mathbb{H}^{n+1}$ is stable if and only if it is totally umbilical.*

We now establish the following result for $(r+1)$-th capillary hypersurfaces.

Theorem 1.5. *Let $1 \le r \le n-1$. A compact immersed $(r+1)$-th capillary hypersurface supported on a horosphere in $\mathbb{H}^{n+1}$ with at least one elliptic point is stable if and only if it is totally umbilical and not totally geodesic.*

Here, elliptic point means that all the principal curvatures at this point are positive. The existence of elliptic point guarantees the ellipticity of operator L_r (see Proposition 2.3). When $r=0$, $L_0=\Delta$ is elliptic automatically. If the hypersurface intersects a horosphere orthogonally, then there must be an elliptic point. Therefore, we have the following classification for stable free boundary constant $(r+1)$-th mean curvature hypersurfaces.

Corollary 1.6. *Let $1 \le r \le n-1$. A compact immersed free boundary constant $(r+1)$-th mean curvature hypersurface supported on a horosphere in $\mathbb{H}^{n+1}$ is stable if and only if it is totally umbilical and not totally geodesic.*

The proof of Theorem 1.5 is based on the following higher-order Minkowski-type formula in a horoball in $\mathbb{H}^{n+1}$:

$$\int_M [H_r(V_{n+1} - \cos\theta\,\bar{g}(x,\nu)) - H_{r+1}\,\bar{g}(X_{n+1},\nu)]\,dA = 0 \quad \text{for any } 0 \le r \le n-1, \tag{1.2}$$

see [8, 20]. This formula induces an admissible test function $\varphi_{n+1} \in \mathcal{F}$ (see Proposition 5.13 below). By utilizing the Killing property of position vector field in the hyperbolic space $\mathbb{H}^{n+1}$, we obtain the desired rigidity result.

This paper is organized as follows. In Section 2, we collect some basic properties for elementary symmetric functions. In Section 3, we calculate the first and second variational formulae of the $(r+1)$-th energy functional $\mathcal{E}_{r+1}$ and introduce a new definition of stable $(r+1)$-th capillary hypersurface in space forms for any $1 \le r \le n-1$. In Section 4, we consider a rigidity of stable $(r+1)$-th capillary hypersurfaces in a Euclidean half-space and then prove Theorem 1.3 by the higher-order Minkowski-type formula (1.1). In

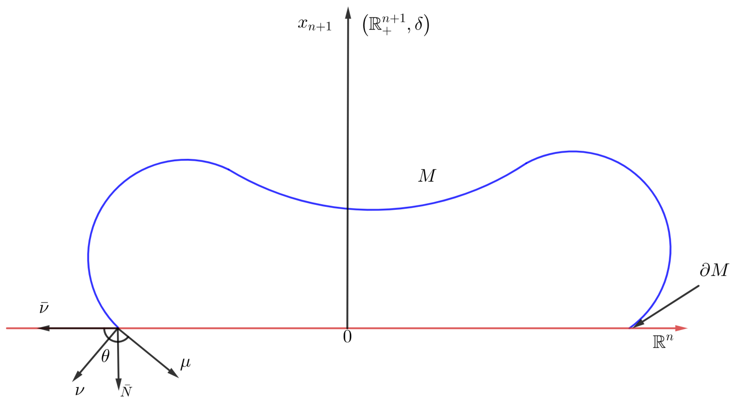

Figure 1. Hypersurface M with contact angle θ in the half-space $\mathbb{R}^{n+1}_+$.

Section 5, we focus on the stability of $(r+1)$-th capillary hypersurface supported on a horosphere in hyperbolic space. We prove some useful and powerful geometric formulae for the $(r+1)$-th capillary hypersurfaces supported on a horosphere. By the higher-order Minkowski-type formula (1.2), we finally construct an admissible test function and prove Theorem 1.5 and hence Corollary 1.6.

2. Preliminaries

Let $(\bar{M}^{n+1}, \bar{g})$ be an oriented $(n+1)$-dimensional Riemannian manifold and let B be a domain in $\bar{M}$ with smooth boundary ∂B in $\bar{M}$. Let $x\colon (M^n, g) \to (\bar{M}, \bar{g})$ be an isometric immersion of an orientable n-dimensional compact manifold M with boundary ∂M satisfying $x|_{\partial M}\colon \partial M \to \partial B$. We say this immersion $x(M)$ is supported on ∂B. For convenience, we do not distinguish M with its image $x(M)$ and ∂M with $x(\partial M)$, respectively, through all computations are carried out on M by using the pull-back of x.

We denote by $\bar{\nabla}$, $\bar{\Delta}$ and $\bar{\nabla}^2$ the gradient, the Laplacian and the Hessian on $\bar{M}$ with respect to $\bar{g}$, respectively, while ∇, Δ and ∇^2 denote the gradient, the Laplacian and the Hessian on M with respect to its induced metric, respectively. We will use the following terminology for four normal vector fields. We choose one of the unit normal vector field along x and denote it by ν. We denote by $\bar{N}$ the unit outward normal to ∂B in B and by μ the unit outward normal to ∂M in M. Let $\bar{\nu}$ be the unit normal to ∂M in ∂B such that the bases $\{\bar{\nu}, \bar{N}\}$ and $\{\nu, \mu\}$ have the same orientation in the normal bundle of $\partial M \subset \bar{M}$. See Figure 1, where $B = \mathbb{R}^{n+1}_+$ and $\partial B = \mathbb{R}^n$, n-dimensional Euclidean space. We denote by h and $h^{\partial B}$ the second fundamental form of M and ∂B in $\bar{M}$, respectively.

Under this convention, along ∂M, the angle between μ and $\bar{\nu}$ or equivalently between ν and $-\bar{N}$ is equal to θ. Precisely, in the normal bundle of ∂M, we have the following relations:

$$\mu = \sin\theta \bar{N} + \cos\theta\, \bar{\nu}, \tag{2.1}$$

$$\nu = -\cos\theta \bar{N} + \sin\theta\, \bar{\nu}. \tag{2.2}$$

Equivalently,

$$\bar{N} = \sin\theta \mu - \cos\theta\, \nu, \tag{2.3}$$

$$\bar{\nu} = \cos\theta \mu + \sin\theta\, \nu. \tag{2.4}$$

Let $\kappa = (\kappa_1, \dots, \kappa_n)$ be the vector of principal curvatures of M. The r-th normalized mean curvature, for any $1 \le r \le n$, is defined by

$$H_r := \binom{n}{r}^{-1} \sigma_r = \binom{n}{r}^{-1} \sum_{1 \le i_1 < \cdots < i_r \le n} \kappa_{i_1} \cdots \kappa_{i_r}, \quad \text{where } \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

It is convenient to set $\sigma_0(\kappa) = 1$ and $\sigma_r(\kappa) = 0$ for $r > n$.

The Newton tensors are inductively defined to be

$$P_0 = I \quad \text{and} \quad P_r = \sigma_r I - P_{r-1} h. \tag{2.5}$$

The following is a collection of basic properties about Newton tensors.

Lemma 2.1 ([18]). *For any $0 \le r \le n-1$, we have*

(1) *P_r is divergence-free, i.e., $\sum_j \nabla_j P_r^{ij} = 0$,*
(2) $\mathrm{tr}_g(P_r) = \sum_{i=1}^n P_r^{ii} = (n-r)\sigma_r$,
(3) $\mathrm{tr}_g(P_r h) = \sum_{i,j=1}^n P_r^{ij} h_{ij} = (r+1)\sigma_{r+1}$,
(4) $\mathrm{tr}_g(P_r h^2) = \sum_{i,j,k=1}^n P_r^{ij} h_i^k h_{kj} = \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}$.

Let Γ_l^+ be the Gårding cone defined by

$$\Gamma_l^+ := \{\kappa \in \mathbb{R}^n \mid \sigma_i(\kappa) > 0, 1 \le i \le l\}.$$

The Newton–MacLaurin inequalities are as follows.

Lemma 2.2 ([18]). *For any $\kappa \in \Gamma_l^+$, we have*

$$H_{k-1}(\kappa) H_l(\kappa) \le H_k(\kappa) H_{l-1}(\kappa), \quad 1 \le k < l \le n. \tag{2.6}$$

Equality holds if and only if $\kappa = c(1, \dots, 1)$ for any constant $c > 0$.

For any $0 \le r \le n-1$, we define a second-order operator $L_r : C^\infty(M) \to C^\infty(M)$ as

$$L_r f := \mathrm{div}_g(P_r \nabla f) = P_r \circ \nabla^2 f, \tag{2.7}$$

where the divergence free property of P_r has been used. If P_r is positive definite on each point of M, then L_r is an elliptic operator. In particular, the Laplacian operator is $\Delta = L_0$ is elliptic automatically. On the other hand, we have the following sufficient condition for the ellipticity of L_r.

Proposition 2.3 (Proposition 3.2 in [7]). *If H_{r+1} is positive and there exists an elliptic point on M, then for any $0 \le j \le r$,*

(i) *each operator $L_j = \mathrm{div}_g(P_j \nabla \cdot)$ is elliptic,*

(ii) *each j-th mean curvature H_j is positive.*

The following proposition is an elementary fact when M is with capillary boundary on ∂B, and ∂B is totally umbilical in $\bar{M}$.

Proposition 2.4 (Proposition 2.1 in [39]). *Assume ∂B is totally umbilical in $\bar{M}$. Let $x\colon M \to \bar{M}$ be an immersion whose boundary $x(\partial M)$ intersects ∂B at a constant angle $\theta \in (0, \pi)$. Then μ is a principal direction of ∂M in M. Namely, $h(e, \mu) = 0$ for any $e \in T(\partial M)$. In particular, for any $0 \le r \le n-1$,*

$$P_r(e, \mu) = 0 \quad \text{for all } e \in T(\partial M). \tag{2.8}$$

3. $(r+1)$-th capillary hypersurfaces and stabilities

In this section, we introduce a new notion of stability of $(r+1)$-th capillary hypersurfaces. Let $\bar{M} = \mathbb{M}^{n+1}(K)$ be a complete simply-connected $(n+1)$-dimensional Riemannian manifold with constant sectional curvature K and let ∂B be a totally umbilical hypersurface in $\mathbb{M}^{n+1}(K)$ with constant principal curvature $\kappa \in \mathbb{R}$. By a choice of orientation, we can assume $\kappa \in [0, \infty)$. It is a well-known fact that in the Euclidean space and the spherical space form, geodesic spheres ($\kappa > 0$) and totally geodesic hyperplanes ($\kappa = 0$) are all complete totally umbilical hypersurfaces, while in the hyperbolic space, the family of all complete totally umbilical hypersurfaces includes geodesic spheres ($\kappa > 1$), totally geodesic hyperplanes ($\kappa = 0$), horospheres ($\kappa = 1$) and equidistant hypersurfaces ($0 < \kappa < 1$) (see, e.g., [29, 30]), among which the horospheres and the equidistant hypersurfaces are non-compact ones.

Let $x\colon (-\varepsilon, \varepsilon) \times M \to \mathbb{M}^{n+1}(K)$ be a differentiable map such that $x(t, \cdot)\colon M \to \mathbb{M}^{n+1}(K)$ is an immersion satisfying $x(t, \partial M) \subset \partial B$ for every $t \in (-\varepsilon, \varepsilon)$ and $x(0, \cdot) = x$. We call $x(t, M)$ an admissible variation of $x(0, M) = x(M)$.

We define the r-th area functional $A_r\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, for any $0 \le r \le n-1$, by

$$A_r(t) = \int_M \sigma_r \, dA_t,$$

and the volume functional $V\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$V(t) = \int_{[0,t] \times M} x^* \, dV,$$

where dA_t is the area element of M with respect to the metric induced by $x(t, \cdot)$ and dV is the volume element of $\mathbb{M}^{n+1}(K)$. A variation is said to be volume-preserving if $V(t) = V(0) = 0$ for each $t \in (-\varepsilon, \varepsilon)$.

Let Y be an admissible variational vector field of x with normal vector field $f\nu$, i.e.,

$$\frac{\partial x}{\partial t} := Y = Y^T + f\nu,$$

where Y^T is tangent to M. From (A.12) in Appendix A, we have the first variation formulae of $A_r(t)$ and $V(t)$ as follows:

$$A_r'(t) = (r+1)\int_M \sigma_{r+1} f \, dA_t - K(n-r+1)\int_M \sigma_{r-1} f \, dA_t + \int_{\partial M} \Big(\sigma_r \bar{g}(Y,\mu) - \frac{\partial \sigma_r}{\partial h^i_\mu} \nabla_i f\Big) \, ds_t, \tag{3.1}$$

$$V'(t) = \int_M f \, dA_t. \tag{3.2}$$

In particular, when $r = 0$, we see that

$$A_0'(t) = \int_M \sigma_1 f \, dA_t + \int_{\partial M} \bar{g}(Y,\mu) \, ds_t.$$

It follows that $A_0'(0) = 0$ with volume-preserving variation if and only if M is a CMC hypersurface with boundary intersecting ∂B orthogonally, which is exactly a free boundary CMC hypersurface.

However, when $r \geq 1$, we cannot characterize constant $(r+1)$-th mean curvature hypersurface with a constant perpendicular contact angle only by (3.1) and (3.2) directly, because the integral boundary terms in (3.1) contain the r-th mean curvature and its derivative. Therefore, the key point of this problem is how to define higher-order capillary hypersurfaces by the variational method reasonably. In the following, we will give a naturally geometric variational definition for higher-order capillary hypersurfaces.

We define the r-th wetting area functional $W_r\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ inductively by

$$W_0(t) := \int_{\partial M \times [0,t]} x^* \, dA_{\partial B}, \quad W_1(t) := \frac{1}{n}\int_{\partial M} ds_t,$$

and for $2 \leq r \leq n-1$,

$$W_r(t) := \frac{1}{n}\int_{\partial M} H_{r-1}^{\partial M} \, ds_t + \frac{r-1}{n-r+2}(K+\kappa^2) W_{r-2}(t),$$

where $dA_{\partial B}$ is the area element of ∂B and ds_t is the area element of ∂M with respect to the metric induced by $x|_{\partial M}(t,\cdot)$, and $H_{r-1}^{\partial M}$ is the normalized $(r-1)$-th mean curvature of ∂M in ∂B.

For fixed $\theta \in (0,\pi)$, we define the $(r+1)$-th energy functional $\mathcal{E}_{r+1}\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ inductively by

$$\mathcal{E}_0(t) := (n+1)V(t), \quad \mathcal{E}_1(t) := A_0(t) - \cos\theta \, W_0(t),$$

and for $1 \leq r \leq n-1$,

$$\mathcal{E}_{r+1}(t) := Q_{r+1}(t) + \frac{rK}{n+2-r}\mathcal{E}_{r-1}(t),$$

where

$$Q_{r+1}(t) := \binom{n}{r}^{-1} A_r(t) - \cos\theta \sin^r\theta\, W_r(t)$$
$$- \cos^{r-1}\theta \sum_{l=0}^{r-1} \frac{(-1)^{r+l}}{n-l} \kappa^{r-l} \binom{r}{l} [(n-r)\cos^2\theta + (r-l)] \tan^l\theta\, W_l(t).$$

The definition for the higher-order energy functional is motivated by the following first variational formula.

Theorem 3.1. *Let $x(\cdot,t)\colon M \to \mathbb{M}^{n+1}(K)$, $t \in (-\varepsilon,\varepsilon)$, be a family of immersion supported on ∂B at a constant contact angle $\theta \in (0,\pi)$. Assume*

$$(\partial_t x)^\perp = f\nu \quad \text{for } f \in C^\infty(M).$$

Then
$$\frac{d}{dt}\mathcal{E}_{r+1}(t) = (n-r)\int_M H_{r+1} f\, dA_t.$$

This variational formula has been derived by Wang, Weng and the third author in the Euclidean half-space and ball, see [37, 40]. We postpone the proof of Theorem 3.1 to Appendix A.

Note that we assume all the immersions $x(\cdot,t)$ in the variational class intersect ∂B at a constant angle. We call such variation is an angle-preserving variation.

Now, we show an existence theorem for volume-preserving and angle-preserving admissible variations. We define the following function space:

$$\mathcal{F} := \left\{\varphi \in C^\infty(M) \;\middle|\; \int_M \varphi\, dA = 0 \text{ and } \nabla_\mu \varphi = q\varphi \text{ on } \partial M\right\}, \tag{3.3}$$

where

$$q = \kappa \csc\theta + \cot\theta h(\mu,\mu). \tag{3.4}$$

Considering a volume-preserving and angle-preserving admissible variation with variational field having $\varphi\nu$ as its normal part, one can see from (3.2) that $\int_M \varphi\, dA = 0$. For the capillary boundary condition $\bar g(\nu, \bar N \circ x) = -\cos\theta$, we can get from (3.6) and (3.7) below that
$$\nabla_\mu \varphi - q\varphi = -\csc\theta\, \partial_t \bar g(\nu, \bar N \circ x) = 0 \quad \text{along } \partial M.$$

Therefore, $\varphi \in \mathcal{F}$.

Conversely, we have that $\varphi \in \mathcal{F}$ induces a volume-preserving and angle-preserving admissible variation:

Proposition 3.2. *Let $x\colon M \to \bar M = \mathbb{M}^{n+1}(K)$ be an immersion such that its boundary $x(\partial M)$ intersects ∂B at a constant angle $\theta \in (0,\pi)$. Then, for a given $\varphi \in \mathcal{F}$, there exists an admissible volume-preserving and contact angle-preserving variation of x with the variational vector field having $\varphi\nu$ as its normal part.*

Proof. We argue as in Lemma 2.2 in [6] (see also Proposition 2.1 in [1]). We first assume that $x\colon M \to \bar{M}$ is embedded. For each point $p \in \partial M$, let $\nu_0 = \nu + \cos\theta \bar{N}$ be the projection of ν on $T_{x(p)}(\partial B)$. Denote

$$\gamma = \frac{1}{\bar{g}(\nu, \nu_0)}\nu_0 - \nu,$$

which is tangential to $x(M)$ along ∂M. We extend γ to a smooth vector field on $x(M)$ and still denote it by γ. We let $\eta = \gamma + \nu$ and extend η smoothly to a vector field on U, which is a δ-neighbourhood of $x(M)$ in $\bar{M}$ such that η is tangential to $T(\partial B)$ along $\partial B \cap \bar{U}$. By our construction, we see $\bar{g}(\eta, \nu) = 1$. Consider the local flow ζ_t of η in $\bar{U}$ satisfying $\frac{\partial}{\partial t}\zeta_t = \eta$. Let $\Theta\colon (-\varepsilon, \varepsilon) \times M \to \bar{M}$ be given by $\Theta(t, \cdot) = \zeta_t$. We shall find a function $\rho\colon (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ such that

$$\tilde{\Theta}(t, \cdot) = \Theta(\rho(t, \cdot), \cdot)$$

is the desired deformation.

First, since ζ_t is the local flow of η and η is tangential to $T(\partial B)$ along $\partial B \cap \bar{U}$, we know that $\tilde{\Theta}(t, \partial M) \subset \partial B$. Second, since

$$\tilde{\Theta}^* dV_{\bar{M}} = \frac{\partial \rho}{\partial t}\Theta^* dV_{\bar{M}} = \frac{\partial \rho}{\partial t} E(\rho(t, \cdot), \cdot)\, dt\, dA_M,$$

where $E(\rho(t, \cdot), \cdot) = \det(d\Theta|_{(\rho(t,\cdot),\cdot)})$, we have

$$V(\tilde{\Theta}(t, \cdot)) = \int_{[0,t]\times M} \tilde{\Theta}^* dV_{\bar{M}} = \int_M \int_0^t \frac{\partial \rho}{\partial t} E(\rho(t, \cdot), \cdot)\, dt\, dA_M.$$

Let $\rho(t, \cdot)\colon (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ be the local solution of the following initial value problem:

$$\frac{\partial \rho}{\partial t} = \frac{\varphi}{E(\rho(t, \cdot), \cdot)}, \quad \rho(0, \cdot) = 0 \quad \text{in } M.$$

It follows from the condition $\int_M \varphi\, dA = 0$ that $V(\tilde{\Theta}(t, \cdot)) = 0$, that is, $\tilde{\Theta}(t, \cdot)$ is a volume preserving admissible deformation. Now we can easily check that

$$Y := \frac{\partial}{\partial t}\Big|_{t=0} \tilde{\Theta}(t, \cdot) = \frac{\partial \rho}{\partial t}\Big|_{t=0} \cdot \eta(0, \cdot) = \varphi(\gamma + \nu) := Y^T + \varphi\nu,$$

which means the variational vector field of $\tilde{\Theta}(t, \cdot)$ has $\varphi\nu$ as its normal part.

In the immersion case, we shall first construct an admissible variation $\tilde{x}\colon (-\varepsilon, \varepsilon)\times M \to \bar{M}$ and endow $(-\varepsilon, \varepsilon) \times M$ with the pull-back metric $\tilde{x}^*(\bar{g})$, and then it is enough to prove the result for $(-\varepsilon, \varepsilon) \times M$ endowed with $\tilde{x}^*(\bar{g})$, which is the embedded case.

Finally, since $\tilde{\Theta}(t, \cdot)$ is an admissible variation, from the appendix of [33] (see also (A.7)–(A.8) in Appendix A), along ∂M, we have

$$Y = Y^T + \varphi\nu := Y^{\partial M} + \cot\theta\, \varphi\mu + \varphi\nu = Y^{\partial M} + \frac{\varphi}{\sin\theta}\bar{\nu}. \tag{3.5}$$

Here $Y^{\partial M}$ denotes the tangent part of Y to ∂M.

By (2.1), (2.2) and the fact that $\partial_t \nu = -\nabla\varphi + d\nu \circ Y^T$, we have

$$\begin{aligned}
\partial_t \bar{g}(\nu, \bar{N}) &= \bar{g}(\partial_t \nu, \bar{N}) + \bar{g}(\nu, \partial_t \bar{N}) \\
&= \bar{g}(\partial_t \nu, \sin\theta\mu - \cos\theta\nu) + \bar{g}(\sin\theta\bar{\nu} - \cos\theta\bar{N}, \partial_t \bar{N}\rangle \\
&= \sin\theta\, \bar{g}(\partial_t \nu, \mu) + \sin\theta h^{\partial B}(Y, \bar{\nu}) \\
&= \sin\theta(-\nabla_\mu \varphi + h(Y^T, \mu)) + h^{\partial B}(Y, \bar{\nu})) \\
&= \sin\theta\Big(-\nabla_\mu\varphi + \cot\theta h(\mu,\mu)\varphi + \frac{1}{\sin\theta} h^{\partial B}(\bar{\nu}, \bar{\nu})\varphi\Big),
\end{aligned} \tag{3.6}$$

where in the last equality we used (3.5), Proposition 2.4 and the fact that ∂B is totally umbilical.

By the assumption $\varphi \in \mathcal{F}$, we get

$$\partial_t \bar{g}(\nu, \bar{N}) = \sin\theta(-\nabla_\mu \varphi + q\varphi) = 0. \tag{3.7}$$

Therefore, the boundary contact angle is preserved along the local flow ζ_t. ∎

Next, we define, for any $0 \le r \le n-1$, the $(r+1)$-th capillary hypersurface and its stability.

Definition 3.3. An immersion is said to be $(r+1)$-th capillary if it is a critical point of the $(r+1)$-th energy functional $\mathcal{E}_{r+1}$ for any volume-preserving and angle-preserving variation of x.

In view of Theorem 3.1, we see that an $(r+1)$-th capillary hypersurface has constant $(r+1)$-th mean curvature H_{r+1} and constant contact angle along its boundary. In particular, when the contact angle is $\pi/2$, the hypersurface is called a free boundary constant $(r+1)$-th curvature hypersurface.

Definition 3.4. An $(r+1)$-th capillary hypersurface is called stable if $\mathcal{E}''_{r+1}(0) \ge 0$ for all volume-preserving and angle-preserving admissible variations.

For a volume-preserving and angle-preserving admissible variation with variational field having $\varphi\nu$ as its normal part, we see from Proposition A.1 in Appendix A (see also Proposition 4.1 in [7]) that

$$\partial_t \sigma_{r+1} = -L_r\varphi - \operatorname{tr}(P_r h^2)\varphi - K\operatorname{tr}(P_r)\varphi + \nabla_{(\partial x/\partial t)^T}\sigma_{r+1}.$$

Here P_r is defined by (2.5). Thus, the second variational formula of $\mathcal{E}_{r+1}$ is given by

$$\begin{aligned}
\mathcal{E}''_{r+1}(0) &= (n-r)\binom{n}{r+1}^{-1}\Big[\int_M \sigma'_{r+1}\varphi\, dA + \int_M \sigma_{r+1}\frac{d}{dt}\Big|_{t=0}(\varphi\, dA_t)\Big] \\
&= (n-r)\binom{n}{r+1}^{-1}\Big[\int_M \sigma'_{r+1}\varphi\, dA + \sigma_{r+1}V''(0)\Big] \\
&= -(n-r)\binom{n}{r+1}^{-1}\int_M \varphi[L_r\varphi + \operatorname{tr}(P_r h^2)\varphi + K\operatorname{tr}(P_r)\varphi]\, dA,
\end{aligned} \tag{3.8}$$

where we used that σ_{r+1} is constant and x is volume-preserving.

From the above calculation, we have the following proposition.

Proposition 3.5. *An $(r+1)$-th capillary hypersurface is stable if and only if*

$$-\int_M \varphi[L_r\varphi + \mathrm{tr}(P_r h^2)\varphi + K\,\mathrm{tr}(P_r)\varphi]\,dA \geq 0 \quad \textit{for all } \varphi \in \mathcal{F}, \tag{3.9}$$

where $\mathcal{F}$ is the functional space given by (3.3).

Remark 3.6. We have no boundary integral term in the second variation formula because we restrict to the angle-preserving variations. For the classical capillary theory, that is, $r = 0$, the second variation formula involves boundary integral because there is no such restriction (see, e.g., [4, 9, 26, 27, 38]).

4. Rigidity for stable $(r+1)$-th capillary hypersurfaces in a half-space

In this section we consider the case B is a Euclidean half-space $\overline{\mathbb{R}}^{n+1}_+$, where

$$\mathbb{R}^{n+1}_+ = \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}.$$

Proposition 4.1. *Any spherical caps in the half-space $\overline{\mathbb{R}}^{n+1}_+$ are stable $(r+1)$-th capillary hypersurfaces.*

Proof. Let Σ be a spherical cap in $\overline{\mathbb{R}}^{n+1}_+$ whose principal curvatures are all equal to $\Lambda > 0$. Then

$$\sigma_r = \binom{n}{r}\Lambda^r.$$

It follows from Lemma 2.1 that

$$L_r\varphi = \binom{n-1}{r}\Lambda^r\Delta\varphi$$

and

$$\mathrm{tr}(P_r h^2) = \sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2} = n\binom{n-1}{r}\Lambda^{r+2}.$$

Hence, from (3.8), for any $\varphi \in \mathcal{F}$, we have

$$\begin{aligned}\mathcal{E}''_{r+1}(0) &= -(n-r)\binom{n}{r+1}^{-1}\int_\Sigma (L_r\varphi + \mathrm{tr}(P_r h^2)\varphi)\varphi\,dA \\ &= -\frac{(r+1)(n-r)}{n}\Lambda^r\int_\Sigma(\Delta\varphi + n\Lambda^2\varphi)\varphi\,dA = \frac{(r+1)(n-r)}{n}\Lambda^r\mathcal{E}''_1(0).\end{aligned}$$

Recall that a spherical cap in $\overline{\mathbb{R}}^{n+1}_+$ is a stable capillary hypersurface and minimize the energy functional $\mathcal{E}_1$ (see [14]). Therefore, $\mathcal{E}''_1(0) \geq 0$. It follows that $\mathcal{E}''_{r+1}(0) \geq 0$ and Σ is a stable $(r+1)$-th capillary hypersurface. The proof is complete. ∎

We need the following higher-order Minkowski-type formula in $\overline{\mathbb{R}}^{n+1}_+$ from [37].

Proposition 4.2 (Proposition 2.6 in [37]). *Let $x\colon M \to \overline{\mathbb{R}}^{n+1}_+$ be an immersed hypersurface whose boundary intersects $\partial\mathbb{R}^{n+1}_+$ at a constant contact angle $\theta \in (0,\pi)$. Then*

$$\int_M [(1-\cos\theta\langle E_{n+1},\nu\rangle)H_r - \langle x,\nu\rangle H_{r+1}]\,dA = 0 \quad \textit{for any } 0 \le r \le n-1, \tag{4.1}$$

where $E_{n+1} = (0,\dots,0,1) \in \overline{\mathbb{R}}^{n+1}_+$ and x is position vector field in $\overline{\mathbb{R}}^{n+1}_+$.

Next we derive the equations for several geometric quantities. We denote the r-th Jacobi operator by

$$J_r f := L_r f + \operatorname{tr}(P_r h^2) f \quad \text{for any } 0 \le r \le n-1.$$

Proposition 4.3. *Let $x\colon M \to \overline{\mathbb{R}}^{n+1}_+$ be an immersed hypersurface. Then the following identities hold on M:*

$$J_r(\langle E_{n+1},\nu\rangle) = \langle E_{n+1},\nabla\sigma_{r+1}\rangle \quad \textit{and} \quad J_r(\langle x,\nu\rangle) = \langle x,\nabla\sigma_{r+1}\rangle + (r+1)\sigma_{r+1}.$$

Proof. The above formulae have been shown in [7]. For the convenience of reader, we give a direct computation.

For a fixed point $p \in M$, let $\{e_i\}_{i=1}^n$ be a local orthonormal basis at p such that $\nabla_{e_i} e_j|_p = 0$. In the following, we calculate at p. We have

$$\begin{aligned}
J_r(\langle\nu,E_{n+1}\rangle) &= L_r(\langle\nu,E_{n+1}\rangle) + \langle\nu,E_{n+1}\rangle\operatorname{tr}(P_r h^2)\\
&= (P_r^{ij} h_{jk}\langle e_k,E_{n+1}\rangle)_{,i} + \langle\nu,E_{n+1}\rangle P_r^{ij} h^2_{ij}\\
&= [(\sigma_{r+1}\delta_{ik} - P_{r+1}^{ik})\langle e_k,E_{n+1}\rangle]_{,i} + \langle\nu,E_{n+1}\rangle(\sigma_{r+1}\delta_{ik} - P_{r+1}^{ik})h_{ik}\\
&= \sigma_{r+1,i}\langle e_i,E_{n+1}\rangle + \sigma_{r+1}(\langle\bar\nabla_{e_i}e_i,E_{n+1}\rangle + \langle\nu,E_{n+1}\rangle\operatorname{tr}(h))\\
&\quad - (P_{r+1}^{ik}\langle\bar\nabla_{e_i}e_k,E_{n+1}\rangle + \langle\nu,E_{n+1}\rangle\operatorname{tr}(P_{r+1}h))\\
&= \langle\nabla\sigma_{r+1},E_{n+1}\rangle,
\end{aligned}$$

where in the third equality we used the relation $P_{r+1} = \sigma_{r+1}I - P_r \circ h$ and Lemma 2.1.

For the second identity, we have

$$\begin{aligned}
J_r(\langle x,\nu\rangle) &= L_r(\langle x,\nu\rangle) + \langle x,\nu\rangle\operatorname{tr}(P_r h^2) = (P_r^{ij} h_{jk}\langle x,e_k\rangle)_{,i} + \langle x,\nu\rangle P_r^{ij} h^2_{ij}\\
&= [(\sigma_{r+1}\delta_{ik} - P_{r+1}^{ik})\langle x,e_k\rangle]_{,i} + \langle x,\nu\rangle(\sigma_{r+1}\delta_{ik} - P_{r+1}^{ik})h_{ik}\\
&= \sigma_{r+1,i}\langle x,e_i\rangle + \sigma_{r+1}[(\langle x,e_i\rangle)_{,i} + \langle x,\nu\rangle\operatorname{tr}(h)]\\
&\quad - [P_{r+1}^{ik}(\langle x,e_k\rangle)_{,i} + \langle x,\nu\rangle\operatorname{tr}(P_{r+1}h)].
\end{aligned}$$

Since

$$\begin{aligned}
P_{r+1}^{ik}(\langle x,e_k\rangle)_i + \langle x,\nu\rangle\operatorname{tr}(P_{r+1}h) &= P_{r+1}^{ik}(\delta_{ik} - h_{ik}\langle x,\nu\rangle) + \langle x,\nu\rangle\operatorname{tr}(P_{r+1}h)\\
&= \operatorname{tr}(P_{r+1}) = (n-r-1)\sigma_{r+1},
\end{aligned}$$

we see that

$$J_r(\langle x,\nu\rangle) = \langle x,\nabla\sigma_{r+1}\rangle + n\sigma_{r+1} - (n-r-1)\sigma_{r+1} = \langle x,\nabla\sigma_{r+1}\rangle + (r+1)\sigma_{r+1}. \quad \blacksquare$$

Next we check the boundary equations of corresponding geometric quantities.

Proposition 4.4 ([22]). *Let $x: M \to \overline{\mathbb{R}}^{n+1}_{+}$ be an isometric immersion. Assume that M intersects $\partial\mathbb{R}^{n+1}_{+}$ at a constant contact angle $\theta \in (0,\pi)$. Then, along ∂M, we have*

$$\nabla_\mu \langle x, \nu\rangle = q\langle x, \nu\rangle, \quad \nabla_\mu(1-\cos\theta\langle E_{n+1},\nu\rangle) = q(1-\cos\theta\langle E_{n+1},\nu\rangle),$$

where $q = \cot\theta\, h(\mu,\mu)$.

Proof. By choosing $F \equiv 1$ in Proposition 2.2 of [22], we get this proposition. ∎

For the ease of notation, we denote

$$\omega := 1 - \cos\theta\langle E_{n+1},\nu\rangle.$$

Since $\theta \in (0,\pi)$, we have $\omega > 0$ on M. Motivated by the higher-order Minkowski-type formula (4.1), we let

$$\varphi := \alpha\omega - H_{r+1}\langle x,\nu\rangle, \tag{4.2}$$

where

$$\alpha = \Big(\int_M \omega\, dA\Big)^{-1}\int_M \omega H_r\, dA.$$

Proposition 4.5. *Let $x: M \to \overline{\mathbb{R}}^{n+1}_{+}$ be a compact immersed $(r+1)$-th capillary hypersurface with a contact angle $\theta \in (0,\pi)$. Then*

$$J_r\varphi = \binom{n}{r+1}\big[\alpha(nH_1H_{r+1} - (n-r-1)H_{r+2}) - (r+1)H_{r+1}^2\big], \tag{4.3}$$

$$\nabla_\mu\varphi = q\varphi, \tag{4.4}$$

$$\int_M \varphi\, dA = 0. \tag{4.5}$$

Proof. Equations (4.3) and (4.4) follow from Propositions 4.3 and 4.4, respectively. Equation (4.5) follows from (4.2) and (4.1). ∎

Now we prove a rigidity theorem for a stable $(r+1)$-th capillary hypersurface in a half-space.

Theorem 4.6. *Let M be a compact immersed $(r+1)$-th capillary hypersurface in $\overline{\mathbb{R}}^{n+1}_{+}$ with a constant contact angle $\theta \in (0,\pi)$. If M is stable, then it is a spherical cap.*

Proof. The proof is similar to that in [23], by the second author with He, for the closed hypersurface case. We prove it here for reader's convenience. Since M is compact hypersurface in $\overline{\mathbb{R}}^{n+1}_{+}$ with contact angle $\theta \in (0,\pi)$, there exists a point in M where all the principal curvatures are positive. Thus, H_{r+1} is positive constant. From Proposition 2.3, we know that each operator L_i is elliptic and $H_i > 0$ for any $i \in \{1,\dots,r\}$. Hence, α is positive constant.

From (4.4) and (4.5), we obtain that φ is an admissible test function in (3.9). Therefore, by (4.3), we have

$$
\begin{aligned}
0 &\le -\int_M \varphi J_r \varphi \, dA \\
&= -\binom{n}{r+1} \int_M \varphi[\alpha(nH_1H_{r+1} - (n-r-1)H_{r+2}) - (r+1)H_{r+1}^2] \, dA \\
&= -\alpha \binom{n}{r+1} \int_M \varphi(nH_1H_{r+1} - (n-r-1)H_{r+2}) \, dA,
\end{aligned} \tag{4.6}
$$

where in the last equality we used the fact that H_{r+1} is constant and (4.5). From (4.2),

$$
\begin{aligned}
&\varphi(nH_1H_{r+1} - (n-r-1)H_{r+2}) \\
&\quad = \alpha\omega(nH_1H_{r+1} - (n-r-1)H_{r+2}) \\
&\qquad + (H_{r+1}\langle x, \nu\rangle)(nH_1H_{r+1} - (n-r-1)H_{r+2}) \\
&\quad = \alpha(n-r-1)\omega(H_1H_{r+1} - H_{r+2}) + \alpha(r+1)\omega H_1 H_{r+1} \\
&\qquad + (H_{r+1}\langle x, \nu\rangle)(nH_1H_{r+1} - (n-r-1)H_{r+2}).
\end{aligned} \tag{4.7}
$$

Putting (4.7) into (4.6), we get

$$
\begin{aligned}
0 &\ge \int_M \varphi(nH_1H_{r+1} - (n-r-1)H_{r+2}) \, dA \\
&= \alpha(n-r-1) \int_M \omega(H_1H_{r+1} - H_{r+2}) \, dA + \alpha(r+1) \int_M \omega H_1 H_{r+1} \, dA \\
&\quad + \int_M (H_{r+1}\langle x, \nu\rangle)(nH_1H_{r+1} - (n-r-1)H_{r+2}) \, dA.
\end{aligned}
$$

From Lemma 2.2, we have

$$
\begin{aligned}
0 &\ge \alpha(r+1) \int_M \omega H_1 H_{r+1} \, dA \\
&\quad + \int_M (H_{r+1}\langle x, \nu\rangle)(nH_1H_{r+1} - (n-r-1)H_{r+2}) \, dA \\
&= \alpha(r+1) \int_M \omega H_1 H_{r+1} \, dA + (r+1)H_{r+1}^2 \int_M \omega \, dA \\
&= \alpha(r+1)H_{r+1}^2 \Big[\frac{1}{H_{r+1}} \int_M \omega H_1 \, dA - \frac{1}{\alpha} \int_M \omega \, dA\Big].
\end{aligned} \tag{4.8}
$$

Here in the second equality we used the higher-order Minkowski-type formula (4.1) twice.

By the Hölder inequality and the Newton–MacLaurin inequality (2.6), we obtain

$$
\Big(\int_M \omega \, dA\Big)^2 \le \int_M \frac{1}{H_1} \omega \, dA \int_M \omega H_1 \, dA \le \int_M \frac{H_r}{H_{r+1}} \omega \, dA \int_M \omega H_1 \, dA. \tag{4.9}
$$

Hence, from (4.9) and the definition of α, we have

$$
\begin{aligned}
&\frac{1}{H_{r+1}} \int_M \omega H_1 \, dA - \frac{1}{\alpha} \int_M \omega \, dA \\
&\quad = \frac{1}{H_{r+1}} \Big[\int_M \omega H_1 \, dA - \Big(\int_M \frac{H_r}{H_{r+1}} \omega \, dA\Big)^{-1} \Big(\int_M \omega \, dA\Big)^2\Big] \ge 0.
\end{aligned} \tag{4.10}
$$

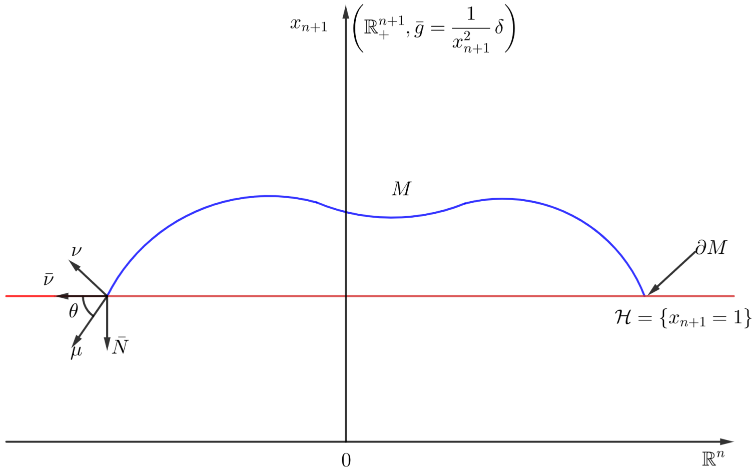

Figure 2. Hypersurface M supported on horosphere $\mathcal{H}$.

Combining (4.8) and (4.10), we see the above inequality is in fact an equality. It follows that $H_r = H_{r+1}H_1$ on M and, in turn, M is totally umbilical in $\overline{\mathbb{R}}^{n+1}_+$, i.e., M is a spherical cap. ■

5. $(r+1)$-th capillary hypersurfaces supported on a horosphere

In this section we focus on the stability of $(r+1)$-capillary hypersurfaces with boundary supported on a horosphere in hyperbolic space.

Let $(\mathbb{H}^{n+1}, \bar{g})$ be a complete simply-connected Riemannian manifold with constant sectional curvature -1. We use the upper half-space model for $\mathbb{H}^{n+1}$, which is denoted by

$$\mathbb{H}^{n+1} = \{x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}_+ : x_{n+1} > 0\}, \quad \bar{g} = \frac{1}{x_{n+1}^2}\delta,$$

where δ is a Euclidean metric in $\mathbb{R}^{n+1}$.

A horosphere, a "sphere" in $\mathbb{H}^{n+1}$ whose centre lies at $\partial_\infty \mathbb{H}^{n+1}$, up to a hyperbolic isometry, is written by the horizontal plane

$$\mathcal{H} = \{x \in \mathbb{R}^{n+1}_+ : x_{n+1} = 1\}.$$

We choose $\bar{N} = -E_{n+1} = (0, \dots, 0, -1)$. Then all principal curvatures of a horosphere are $\kappa = 1$. By Gauss' equation, namely, $R_{ijij} = -1 + \kappa_i\kappa_j = 0$ for any $i \neq j$, we know that a horosphere is isometric to the n-dimensional Euclidean space $\mathbb{R}^n$.

Let $x\colon (M^n, g) \to (\mathbb{H}^{n+1}, \bar{g})$ be an isometric immersion of an orientable n-dimensional compact manifold M with boundary ∂M satisfying $x|_{\partial M}\colon \partial M \to \mathcal{H}$. Such an immersion is called an immersion supported on a horosphere $\mathcal{H}$ (see Figure 2).

We denote by x the position vector in $\mathbb{H}^{n+1}$ and by $\bar\nabla$ the Levi-Civita connection of $\mathbb{H}^{n+1}$. We use $\langle\cdot,\cdot\rangle$ and $\bar g$ to denote the inner product of $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$, respectively, and D and $\bar\nabla$ to denote the Levi-Civita connection of $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$, respectively. Let $\{E_A\}_{A=1}^{n+1}$ be the canonical basis of $\mathbb{R}^{n+1}$ and $\bar E_A = x_{n+1}E_A$. Then $\{\bar E_A\}_{A=1}^{n+1}$ is an orthonormal basis of $\mathbb{H}^{n+1}$ with respect to $\bar g$.

The relationship of $\bar\nabla$ and D is given by

$$\bar\nabla_Y Z = D_Y Z - Y(\ln x_{n+1})Z - Z(\ln x_{n+1})Y + \langle Y,Z\rangle D(\ln x_{n+1}).$$

It is easy to check that

$$\bar\nabla_Y x = -\bar g(Y,\bar E_{n+1})x + \bar g(Y,x)\bar E_{n+1}, \tag{5.1}$$

$$\bar\nabla_Y E_i = -\bar g(Y,\bar E_{n+1})E_i + \bar g(Y,\bar E_i)E_{n+1} \quad \text{for all } i=1,2,\dots,n, \tag{5.2}$$

$$\bar\nabla_Y E_{n+1} = -\frac{1}{x_{n+1}}Y, \tag{5.3}$$

for any vector field Y in $\mathbb{H}^{n+1}$.

The following propositions play a crucial role in this section.

Proposition 5.1 (Proposition 2.2 in [20]).

(i) *The vector fields x and $\{E_i\}_{i=1}^n$ are Killing vector fields in $\mathbb{H}^{n+1}$, i.e.,*

$$\frac12(\bar g(\bar\nabla_A x, E_B) + \bar g(\bar\nabla_B x, E_A)) = \frac12(\bar g(\bar\nabla_A E_i, E_B) + \bar g(\bar\nabla_B E_i, E_A)) = 0. \tag{5.4}$$

(ii) *E_{n+1} is a conformal Killing vector field in $\mathbb{H}^{n+1}$, i.e.,*

$$\frac12(\bar g(\bar\nabla_A E_{n+1}, E_B) + \bar g(\bar\nabla_B E_{n+1}, E_A)) = -\frac{1}{x_{n+1}}\bar g_{AB}. \tag{5.5}$$

Here $\bar\nabla_A = \bar\nabla_{E_A}$ and $\bar g_{AB} = \bar g(E_A,E_B)$.

Now we recall a conformal Killing vector field X_{n+1} and a function V_{n+1} in $\mathbb{H}^{n+1}$ from [21] that we will use later. Denote

$$X_{n+1} = x - E_{n+1} \quad \text{and} \quad V_{n+1} = \frac{1}{x_{n+1}}.$$

From Proposition 5.1, we have the following important properties.

Proposition 5.2 (Proposition 2.1 in [21]).

(i) *X_{n+1} is a conformal Killing vector field with $\frac12\mathcal{L}_{X_{n+1}}\bar g = V_{n+1}\bar g$, namely,*

$$\frac12[\bar g(\bar\nabla_{E_A}X_{n+1}, E_B) + \bar g(\bar\nabla_{E_B}X_{n+1}, E_A)] = V_{n+1}\bar g_{AB}. \tag{5.6}$$

(ii) *$X_{n+1}|_{\mathcal{H}}$ is a tangential vector field on $\mathcal{H}$, i.e.,*

$$\bar g(X_{n+1},\bar N) = 0 \quad \text{on } \mathcal{H}. \tag{5.7}$$

Proposition 5.3 (Proposition 2.2 in [21]). *The function V_{n+1} satisfies the following properties*:

$$\bar{\nabla}^2 V_{n+1} = V_{n+1}\bar{g} \quad in\ \mathbb{H}^{n+1}, \qquad \partial_{\bar{N}} V_{n+1} = V_{n+1} \quad on\ \mathcal{H}.$$

Proposition 5.4. *Assume $0 \le r \le n-1$. Then any totally umbilical hypersurface supported on a horosphere $\mathcal{H}$ is a stable $(r+1)$-th capillary hypersurface.*

Proof. Let $\tilde{\Sigma}$ be a totally umbilical hypersurface supported on $\mathcal{H}$ with a contact angle θ. Suppose the principal curvatures of $\tilde{\Sigma}$ are all equal to a certain nonnegative constant $\tilde{\Lambda}$. It is easy to check that

$$\sigma_i = \binom{n}{i}\tilde{\Lambda}^i \quad \text{for any } i = 0, 1, \dots, n.$$

It follows from Lemma 2.1 that

$$L_r\varphi = \binom{n-1}{r}\tilde{\Lambda}^r\Delta\varphi, \quad \operatorname{tr}(P_r h^2) = n\binom{n-1}{r}\tilde{\Lambda}^{r+2}, \quad \operatorname{tr}(P_r) = (n-r)\binom{n}{r}\tilde{\Lambda}^r.$$

Hence, for any $\varphi \in \mathcal{F}$, by (3.8), we have

$$\begin{aligned}
\mathcal{E}''_{r+1}(0) &= -(n-r)\binom{n}{r+1}^{-1}\int_{\tilde{\Sigma}} \varphi[L_r\varphi + \operatorname{tr}(P_r h^2)\varphi - \operatorname{tr}(P_r)\varphi]\,dA \\
&= -\frac{(r+1)(n-r)}{n}\tilde{\Lambda}^r\int_{\tilde{\Sigma}} \varphi(\Delta\varphi + n\tilde{\Lambda}^2\varphi - n\varphi)\,dA \\
&= \frac{(r+1)(n-r)}{n}\tilde{\Lambda}^r\mathcal{E}''_1(0).
\end{aligned}$$

It follows from Proposition 2.5 in [20] that any totally umbilical capillary hypersurface supported on the horosphere $\mathcal{H}$ is stable. Therefore, we have that $\tilde{\Sigma}$ is a stable $(r+1)$-th capillary hypersurface. ∎

5.1. Key formulae for $(r+1)$-th capillary hypersurfaces supported on a horosphere

In this subsection we show some useful facts about $(r+1)$-th capillary hypersurfaces supported on a horosphere that we will use later. Let $P_r^{\mu\mu} = \sigma_r(h\,|\,h_{\mu\mu})$ be the r-th mean curvature deleting $h(\mu,\mu)$ component from the second fundamental form h. To simplify the notation, we will omit writing the volume form dA on M and the area form ds on ∂M.

Proposition 5.5. *Let $x\colon M \to \mathbb{H}^{n+1}$ be an isometric immersion supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. Then, for any $0 \le r \le n-1$,*

$$-(r+1)\int_M \bar{g}(x,\nu)\sigma_{r+1}\,dA = \int_{\partial M} P_r^{\mu\mu}(\cos\theta\,\bar{g}(x,\bar{\nu}) - \sin\theta)\,ds. \tag{5.8}$$

Proof. Let

$$x^T := x - \bar{g}(x,\nu)\nu = \sum_{i=1}^n (x^T)^i e_i,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of M. Thus,

$$(x^T)^i = \bar{g}(x - \bar{g}(x,\nu)\nu, e_i) = \bar{g}(x, e_i).$$

Since x is Killing vector field in $\mathbb{H}^{n+1}$, we see from Lemma 2.1 that

$$\mathrm{div}_M(P_r \circ x^T) = P_r^{ij}\nabla_j(\bar{g}(x,e_i)) = -\bar{g}(x,\nu)\,\mathrm{tr}(P_r h) = -(r+1)\sigma_{r+1}\,\bar{g}(x,\nu).$$

Integration by parts gives

$$\begin{aligned}\int_M \mathrm{div}_M(P_r \circ x^T) &= \int_{\partial M} P_r(x^T,\mu) = \int_{\partial M} P_r^{\mu\mu}\,\bar{g}(x,\mu)\\ &= \int_{\partial M} P_r^{\mu\mu}\,\bar{g}(x, \cos\theta\bar{\nu} + \sin\theta\bar{N})\\ &= \int_{\partial M} P_r^{\mu\mu}(\cos\theta\,\bar{g}(x,\bar{\nu}) - \sin\theta),\end{aligned}$$

where we have used (2.8), (2.2) and $\bar{g}(x,\bar{N}) = -1$ on ∂M. ■

Next we will derive another important integral identity.

Proposition 5.6. *Let $x\colon M \to \mathbb{H}^{n+1}$ be an isometric immersion supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. Then, for any $0 \le r \le n$,*

$$(n-r)\int_M \sigma_r \bar{g}(x,\nu)\,dA = \int_{\partial M} P_r^{\mu\mu}\bar{g}(x,\bar{\nu})\,ds. \tag{5.9}$$

In particular, for $r = 0$, we have

$$n\int_M \bar{g}(x,\nu)\,dA = \int_{\partial M} \bar{g}(x,\bar{\nu})\,ds. \tag{5.10}$$

Proof. In order to prove (5.9), we consider a vector field Z on M as follows:

$$Z = \bar{g}(x,\nu)\bar{E}_{n+1} - \bar{g}(\bar{E}_{n+1},\nu)x.$$

Recall that $\bar{E}_{n+1} = x_{n+1}E_{n+1}$. Along ∂M, we have

$$\begin{aligned}\bar{g}(Z,\mu) &= \bar{g}(x,\nu)\,\bar{g}(\bar{E}_{n+1},\mu) - \bar{g}(\bar{E}_{n+1},\nu)\,\bar{g}(x,\mu)\\ &= -\bar{g}(x,\nu)\,\bar{g}(\bar{N},\mu) + \bar{g}(\bar{N},\nu)\,\bar{g}(x,\mu)\\ &= -\sin\theta\,\bar{g}(x,\nu) - \cos\theta\,\bar{g}(x,\mu) = -\bar{g}(x,\bar{\nu}),\end{aligned}$$

where we have used (2.3), (2.4) and the fact that $\bar{N} = -\bar{E}_{n+1}$ on ∂M. By integrating by parts, we obtain

$$-\int_{\partial M} P_r^{\mu\mu}\bar{g}(x,\bar{\nu}) = \int_{\partial M} P_r^{\mu\mu}\bar{g}(Z,\mu) = \int_{\partial M} P_r(Z^T,\mu) = \int_M \mathrm{div}_M(P_r \circ Z^T).$$

Now we claim that

$$\mathrm{div}_M(P_r \circ Z^T) = -(n-r)\sigma_r\bar{g}(x,\nu). \tag{5.11}$$

Thus, Proposition 5.6 follows from claim (5.11). Next we will show the claim (5.11). First we observe that Z is tangential, i.e., $\bar{g}(Z,\nu)=0$, which implies $\mathrm{div}_M(P_r\circ Z^T)=\mathrm{div}_M(P_r\circ Z)$. From (5.1), we see that Z can be expressed as $Z=\bar{\nabla}_\nu x$.

Let $\{e_i\}_{i=1}^n$ be an othonormal basis of M. By constant sectional curvature of $\mathbb{H}^{n+1}$ being -1, we have

$$\begin{aligned}
&\bar{g}(\bar{\nabla}_{e_i}(\bar{\nabla}_\nu x),e_j)=\bar{g}(\bar{\nabla}_\nu(\bar{\nabla}_{e_i}x),e_j)-\bar{g}(\bar{\nabla}_{[\nu,e_i]}x,e_j)-\bar{g}(\bar{R}(\nu,e_i)x,e_j)\\
&\quad=\bar{\nabla}_\nu(\bar{g}(\bar{\nabla}_{e_i}x,e_j))-\bar{g}(\bar{\nabla}_{e_i}x,\bar{\nabla}_\nu e_j)-\bar{g}(\bar{\nabla}_{[\nu,e_i]}x,e_j)-\delta_{ij}\bar{g}(x,\nu)\\
&\quad=\bar{\nabla}_\nu(\bar{g}(\bar{\nabla}_{e_i}x,e_j))-\bar{g}(\bar{\nabla}_{e_i}x,\bar{\nabla}_{e_j}\nu+[\nu,e_j])-\bar{g}(\bar{\nabla}_{[\nu,e_i]}x,e_j)-\delta_{ij}\bar{g}(x,\nu)\\
&\quad=\bar{\nabla}_\nu(\bar{g}(\bar{\nabla}_{e_i}x,e_j))-h_{jk}\bar{g}(\bar{\nabla}_{e_i}x,e_k)-\bar{g}(\bar{\nabla}_{e_i}x,[\nu,e_j])-\bar{g}(\bar{\nabla}_{[\nu,e_i]}x,e_j)-\delta_{ij}\bar{g}(x,\nu).
\end{aligned}$$

By utilizing that x is Killing vector field, we obtain

$$\begin{aligned}
P_r^{ij}\bar{g}(\bar{\nabla}_{e_i}Z,e_j)&=-P_r^{ij}h_{jk}\bar{g}(\bar{\nabla}_{e_i}x,e_k)-P_r^{ij}\mathcal{L}_x\bar{g}([\nu,e_i],e_j)-\mathrm{tr}(P_r)\bar{g}(x,\nu)\\
&=-(n-r)\sigma_r\bar{g}(x,\nu),
\end{aligned}\tag{5.12}$$

where we used the fact that $P_r^{ij}h_{jk}=P_r^{kj}h_{ji}$ from (2.5). Thus, we have claim (5.11) and the proof is completed. ■

As a consequence, we get the following significant integral identity.

Corollary 5.7. *Let $x\colon M\to\mathbb{H}^{n+1}$ be an immersed constant $(r+1)$-th mean curvature hypersurface supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta\in(0,\pi)$. Then, for any $0\le r\le n-1$,*

$$\int_{\partial M}P_r^{\mu\mu}(-\sin\theta+\cos\theta\,\bar{g}(x,\bar{\nu})+\bar{g}(x,\bar{\nu})h(\mu,\mu))\,ds=0.\tag{5.13}$$

Proof. By (5.8) and (5.10), we have

$$\begin{aligned}
\int_{\partial M}P_r^{\mu\mu}(-\sin\theta+\cos\theta\bar{g}(x,\bar{\nu}))\,ds&=-(r+1)\int_M\sigma_{r+1}\bar{g}(x,\nu)\,dA\\
&=-\frac{r+1}{n}\sigma_{r+1}\int_{\partial M}\bar{g}(x,\bar{\nu})\,ds.
\end{aligned}\tag{5.14}$$

Utilizing the fact that $\sigma_{r+1}=P_{r+1}^{\mu\mu}+P_r^{\mu\mu}h(\mu,\mu)$ from (2.5), we get

$$\begin{aligned}
&\int_{\partial M}P_r^{\mu\mu}h(\mu,\mu)\bar{g}(x,\bar{\nu})\,ds\\
&\quad=\int_{\partial M}(\sigma_{r+1}-P_{r+1}^{\mu\mu})\bar{g}(x,\bar{\nu})\,ds\\
&\quad=\sigma_{r+1}\int_{\partial M}\bar{g}(x,\bar{\nu})\,ds-(n-r-1)\int_M\sigma_{r+1}\bar{g}(x,\nu)\,dA,
\end{aligned}\tag{5.15}$$

where we used (5.9) in the last equality. Combining (5.14) and (5.15), we obtain

$$\begin{aligned}
&\int_{\partial M}P_r^{\mu\mu}(-\sin\theta+\bar{g}(x,\bar{\nu})\cos\theta+\bar{g}(x,\bar{\nu})h(\mu,\mu))\,ds\\
&\quad=\frac{n-r-1}{n}\sigma_{r+1}\Big[\int_{\partial M}\bar{g}(x,\bar{\nu})\,ds-n\int_M\bar{g}(x,\nu)\,dA\Big]=0,
\end{aligned}$$

where in the last equality we used (5.10) again. ■

Now we can use the conformal Killing vector field X_{n+1} to establish a higher-order Minkowski-type formula, which is very crucial for the study of stable $(r+1)$-th capillary hypersurfaces supported on a horosphere.

Proposition 5.8. *Let $x\colon M \to \mathbb{H}^{n+1}$ be an isometric immersion supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. Then, for any $0 \le r \le n-1$,*

$$\int_M [(V_{n+1} - \cos\theta\, \bar g(x,\nu))H_r - \bar g(X_{n+1},\nu)H_{r+1}]\, dA = 0. \tag{5.16}$$

Proof. The proof can be found in Proposition 4 of [8]; we include it here for the sake of completeness. Let $X_{n+1}^T := X_{n+1} - \bar g(X_{n+1},\nu)\nu = \sum_{i=1}^n (X_{n+1}^T)^i e_i$, where $\{e_i\}_{i=1}^n$ is an orthonormal basis of M. Then

$$(X_{n+1}^T)^i = \bar g(X_{n+1} - \bar g(X_{n+1},\nu)\nu, e_i) = \bar g(X_{n+1}, e_i).$$

From Lemma 2.1 and (5.6), we find

$$\begin{aligned} \mathrm{div}_M(P_r \circ X_{n+1}^T) &= P_r^{ij}\nabla_j(\bar g(X_{n+1},e_i)) \\ &= \mathrm{tr}(P_r)V_{n+1} - \bar g(X_{n+1},\nu)\,\mathrm{tr}(P_r h) \\ &= (n-r)\sigma_r V_{n+1} - (r+1)\sigma_{r+1}\bar g(X_{n+1},\nu). \end{aligned} \tag{5.17}$$

Then, by integration by parts, we get

$$\begin{aligned} \int_M \mathrm{div}_M(P_r \circ X_{n+1}^T) &= \int_{\partial M} P_r(X_{n+1}^T,\mu) = \int_{\partial M} P_r^{\mu\mu}\bar g(X_{n+1},\mu) \\ &= \cos\theta \int_{\partial M} P_r^{\mu\mu}\bar g(X_{n+1},\bar\nu) = \cos\theta\int_{\partial M} P_r^{\mu\mu}\bar g(x,\bar\nu), \end{aligned} \tag{5.18}$$

where we used (2.8), (2.1), (5.7) and $\bar g(E_{n+1},\bar\nu) = 0$ along ∂M. Combining (5.17), (5.18) and (5.9), we complete the proof. ■

Denote the r-th Jacobi operator to be $J_r := L_r + \mathrm{tr}(P_r h^2) - \mathrm{tr}(P_r)$, where L_r is given by (2.7). Recall the conformal Killing vector $X_{n+1} = x - E_{n+1}$. In order to investigate stability of $(r+1)$-th capillary hypersurface by the above higher-order Minkowski-type formula (5.16). We next calculate differential equations for $\bar g(x,\nu)$, $\bar g(E_{n+1},\nu)$, $\bar g(X_{n+1},\nu)$ and V_{n+1}.

Proposition 5.9. *Let $x\colon M \to \mathbb{H}^{n+1}$ be a constant $(r+1)$-th mean curvature hypersurface. Then, for any $0 \le r \le n-1$,*

$$J_r\bar g(x,\nu) = 0, \tag{5.19}$$

$$J_r\bar g(E_{n+1},\nu) = -(r+1)\sigma_{r+1}V_{n+1} - (n-r)\sigma_r\bar g(E_{n+1},\nu), \tag{5.20}$$

$$J_r\bar g(X_{n+1},\nu) = (r+1)\sigma_{r+1}V_{n+1} + (n-r)\sigma_r\bar g(E_{n+1},\nu), \tag{5.21}$$

$$J_r V_{n+1} = (r+1)\sigma_{r+1}\bar g(E_{n+1},\nu) + (\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2})V_{n+1}. \tag{5.22}$$

Proof. It is clear that (5.21) follows from (5.19) and (5.20). Therefore, we only need to show (5.19), (5.20) and (5.22) one by one.

For a fixed point $p \in M$, let $\{e_i\}_{i=1}^n$ be a local orthonormal basis at p such that $\nabla_{e_i} e_j|_p = 0$. By (5.4), we calculate at p:

$$e_i \bar{g}(x, \nu) = \bar{g}(x, \bar{\nabla}_{e_i} \nu) + \bar{g}(\bar{\nabla}_{e_i} x, \nu) = \bar{g}(x, \bar{\nabla}_{e_i} \nu) - \bar{g}(\bar{\nabla}_\nu x, e_i).$$

It follows that

$$\begin{aligned}
(5.23)\qquad L_r \bar{g}(x,\nu) &= P_r^{ij} \bar{g}(x,\nu)_{,ij} \\
&= P_r^{ij}\big[\bar{g}(\bar{\nabla}_{e_j} x, \bar{\nabla}_{e_i}\nu) + \bar{g}(x, \bar{\nabla}_{e_j}(\bar{\nabla}_{e_i}\nu)) - \bar{g}(\bar{\nabla}_{e_j}(\bar{\nabla}_\nu x), e_i) - \bar{g}(\bar{\nabla}_\nu x, \bar{\nabla}_{e_j} e_i)\big] \\
&= P_r^{ij}\big[h_{ik}\bar{g}(\bar{\nabla}_{e_j} x, e_k) + \bar{g}(x, \nabla h_{ij} - h_{ij}^2 \nu) - \bar{g}(\bar{\nabla}_{e_j}(\bar{\nabla}_\nu x), e_i) + h_{ij}\bar{g}(\bar{\nabla}_\nu x, \nu)\big] \\
&= \bar{g}(x, \nabla\sigma_{r+1}) - \mathrm{tr}(P_r h^2)\bar{g}(x,\nu) - P_r^{ij}\bar{g}(\bar{\nabla}_{e_j}(\bar{\nabla}_\nu x), e_i), \\
&= -\mathrm{tr}(P_r h^2)\bar{g}(x,\nu) - P_r^{ij}\bar{g}(\bar{\nabla}_{e_j}(\bar{\nabla}_\nu x), e_i),
\end{aligned}$$

where we have used (5.4) and the fact that σ_{r+1} is constant. In (5.12), we have proved that

$$(5.24)\qquad -P_r^{ij}\bar{g}(\bar{\nabla}_{e_j}(\bar{\nabla}_\nu x), e_i) = (n-r)\sigma_r \bar{g}(x,\nu).$$

Now (5.19) follows from (5.23) and (5.24). Using (5.2) and (5.5), we can directly check that

$$\begin{aligned}
(5.25)\qquad L_r \bar{g}(E_{n+1},\nu) &= P_r^{ij}\bar{g}(E_{n+1},\nu)_{,ij} \\
&= P_r^{ij}[e_j(\bar{g}(\bar{\nabla}_{e_i} E_{n+1},\nu) + \bar{g}(E_{n+1}, \bar{\nabla}_{e_i}\nu))] \\
&= P_r^{ij}(e_j \bar{g}(E_{n+1}, \bar{\nabla}_{e_i}\nu)) \\
&= P_r^{ij}(\bar{g}(\bar{\nabla}_{e_j} E_{n+1}, \bar{\nabla}_{e_i}\nu) + \bar{g}(E_{n+1}, \bar{\nabla}_{e_j}(\bar{\nabla}_{e_i}\nu))) \\
&= P_r^{ij} h_{ik}\bar{g}(\bar{\nabla}_{e_j} E_{n+1}, e_k) + P_r^{ij}\bar{g}(E_{n+1}, \nabla h_{ij} - h_{ij}^2\nu) \\
&= -\mathrm{tr}(P_r h)V_{n+1} + \bar{g}(E_{n+1}, \nabla\sigma_{r+1}) - \mathrm{tr}(P_r h^2)\bar{g}(E_{n+1},\nu), \\
&= -\mathrm{tr}(P_r h)V_{n+1} - \mathrm{tr}(P_r h^2)\bar{g}(E_{n+1},\nu),
\end{aligned}$$

which implies (5.20). From Proposition 5.3, we see that

$$\begin{aligned}
(5.26)\qquad L_r V_{n+1} = P_r^{ij}(V_{n+1})_{,ij} &= P_r^{ij}(\bar{\nabla}_{ij} V_{n+1} - h_{ij}\bar{\nabla}_\nu V_{n+1}) \\
&= \mathrm{tr}(P_r)V_{n+1} + \mathrm{tr}(P_r h)\bar{g}(E_{n+1},\nu),
\end{aligned}$$

which obtains (5.22). The proof is complete. ∎

Now we compute the boundary equations of the corresponding geometric quantities.

Proposition 5.10 (Proposition 3.6 in [20]). *Let $x\colon M \to \mathbb{H}^{n+1}$ be an isometric immersion supported on $\mathcal{H}$. Assume $x(M)$ meets $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. Then, along ∂M, we have*

$$(5.27)\qquad \nabla_\mu(V_{n+1} - \cos\theta\, \bar{g}(E_{n+1},\nu)) = q(V_{n+1} - \cos\theta\, \bar{g}(E_{n+1},\nu)),$$

$$(5.28)\qquad \nabla_\mu \bar{g}(X_{n+1},\nu) = q\bar{g}(X_{n+1},\nu),$$

$$(5.29)\qquad \nabla_\mu \bar{g}(x,\nu) = \bar{g}(x,\bar{\nu}) + h(\mu,\mu)\bar{g}(x,\mu),$$

where q is defined by (3.4).

5.2. Rigidity for stable $(r+1)$-th capillary hypersurfaces supported on a horosphere

In this subsection we will show a uniqueness result for stable $(r+1)$-th capillary hypersurfaces supported on a horosphere $\mathcal{H}$. For notation simplicity, we denote

$$u := V_{n+1} - \cos\theta \bar{g}(x, \nu). \tag{5.30}$$

Proposition 5.11. *Assume $0 \le r \le n-1$. Let $x\colon M \to \mathbb{H}^{n+1}$ be a constant $(r+1)$-th mean curvature hypersurface with boundary supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta \in (0, \pi)$. Then u satisfies*

$$J_r u = (r+1)\sigma_{r+1}\bar{g}(E_{n+1}, \nu) + (\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2})V_{n+1} \quad \text{in } M, \tag{5.31}$$

$$\nabla_\mu u = qu \quad \text{on } \partial M. \tag{5.32}$$

Proof. The equations (5.31) and (5.32) follow from Propositions 5.9 and 5.10, respectively. ∎

Proposition 5.12. *Assume $0 \le r \le n-1$. Let $x\colon M \to \mathbb{H}^{n+1}$ be an $(r+1)$-th capillary hypersurface with boundary supported on $\mathcal{H}$ at a constant contact angle $\theta \in (0, \pi)$. If M is stable and there exists at least one elliptic point, then $\int_M u \, dA \neq 0$.*

Proof. Arguing by contradiction, suppose that $\int_M u \, dA = 0$. Combining with (5.32), we know that $u \in \mathcal{F}$. Now we choose u as an admissible test function in (3.9). Thus, by (5.30), we have

$$\begin{aligned} 0 \le -\int_M uJ_r u &= -\int_M (V_{n+1} - \cos\theta \bar{g}(x, \nu))J_r u \\ &= -\int_M V_{n+1}J_r u + \cos\theta \int_M \bar{g}(x, \nu)J_r u. \end{aligned} \tag{5.33}$$

We compute the last term of (5.33) using Green's formula. From (5.19), (2.8) and (5.32), we obtain

$$\begin{aligned} \int_M \bar{g}(x, \nu)J_r u &= \int_M uJ_r\bar{g}(x, \nu) + \int_{\partial M} [\bar{g}(x, \nu)P_r(\nabla u, \mu) - uP_r(\nabla \bar{g}(x, \nu), \mu)] \\ &= \int_{\partial M} P_r^{\mu\mu}u(q \cdot \bar{g}(x, \nu) - \nabla_\mu \bar{g}(x, \nu)). \end{aligned} \tag{5.34}$$

By (5.29), (2.2) and (2.3), we see, along ∂M,

$$\begin{aligned} & q \cdot \bar{g}(x, \nu) - \nabla_\mu \bar{g}(x, \nu) \\ &= (\csc\theta + \cot\theta\, h(\mu, \mu))\bar{g}(x, \nu) - (\bar{g}(x, \bar{\nu}) + h(\mu, \mu)\bar{g}(x, \mu)) \\ &= -\cot\theta\, \bar{g}(x, \bar{N}) - \csc\theta\, \bar{g}(x, \bar{N})h(\mu, \mu) = \cot\theta + \csc\theta\, h(\mu, \mu), \end{aligned} \tag{5.35}$$

where in the last equality, we have used $\bar{g}(x, \bar{N}) = -1$ on ∂M. Substituting (5.34)–(5.35) into (5.33), we get

$$\int_M V_{n+1}J_r u - \cos\theta \int_{\partial M} P_r^{\mu\mu}u(\cot\theta + \csc\theta h(\mu, \mu)) \le 0. \tag{5.36}$$

Now we introduce an auxiliary function

$$\Phi := -\bar{g}(E_{n+1}, \nu).$$

By (5.25), we obtain

$$L_r \Phi = -L_r \bar{g}(E_{n+1}, \nu) = V_{n+1} \operatorname{tr}(P_r h) + \bar{g}(E_{n+1}, \nu) \operatorname{tr}(P_r h^2). \tag{5.37}$$

From (2.2) and (5.3), note that

$$\Phi|_{\partial M} = -\cos\theta \quad \text{and} \quad \nabla_\mu \Phi = \sin\theta h(\mu, \mu). \tag{5.38}$$

Inserting (5.37)–(5.38) into the identity

$$\int_M [\Phi L_r \Phi + P_r(\nabla\Phi, \nabla\Phi)] = \int_M \frac{1}{2} L_r \Phi^2 = \int_{\partial M} \Phi P_r(\nabla\Phi, \mu),$$

we get an integral identity:

$$\begin{aligned} &\int_M (-V_{n+1} \operatorname{tr}(P_r h) - \bar{g}(E_{n+1}, \nu) \operatorname{tr}(P_r h^2)) \bar{g}(E_{n+1}, \nu) + \int_M P_r(\nabla\Phi, \nabla\Phi) \\ &\qquad = -\cos\theta \sin\theta \int_{\partial M} P_r^{\mu\mu} h(\mu, \mu). \end{aligned} \tag{5.39}$$

Here we used μ is a principal direction by (2.8). Using (2.2), on ∂M, we have

$$\begin{aligned} u = V_{n+1} - \cos\theta\, \bar{g}(x, \nu) &= V_{n+1} - \cos\theta(\cos\theta + \sin\theta\, \bar{g}(x, \bar{\nu})) \\ &= \sin\theta(\sin\theta - \cos\theta\, \bar{g}(x, \bar{\nu})). \end{aligned} \tag{5.40}$$

By adding (5.39) to (5.36) and applying (5.40) and (5.31), we get

$$\begin{aligned} 0 \geq &\int_M [\bar{g}(E_{n+1}^T, E_{n+1}^T) \operatorname{tr}(P_r h^2) + P_r(\nabla\Phi, \nabla\Phi)] \\ &- \cos^2\theta \int_{\partial M} P_r^{\mu\mu}(\sin\theta - \cos\theta\, \bar{g}(x, \bar{\nu}) - \bar{g}(x, \bar{\nu}) h(\mu, \mu)) \\ = &\int_M [\bar{g}(E_{n+1}^T, E_{n+1}^T) \operatorname{tr}(P_r h^2) + P_r(\nabla\Phi, \nabla\Phi)], \end{aligned} \tag{5.41}$$

where in the last equality we used (5.13).

Since there exists an elliptic point on M, we know that $H_{r+1} > 0$. From Proposition 2.3, one can see that L_i is elliptic and $H_i > 0$ for each $i = 1, 2, \ldots, r$.

Thus, by Lemma 2.2, we have

$$\begin{aligned} &\operatorname{tr}(P_r h^2) = \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2} \\ &= (r+1)\binom{n}{r+1} H_1 H_{r+1} + (n-r-1)\binom{n}{r+1}(H_1 H_{r+1} - H_{r+2}) > 0. \end{aligned} \tag{5.42}$$

From (5.41), (5.42) and (5.37), we obtain that Φ is a constant on M, i.e.,

$$L_r \Phi = (r+1)\sigma_{r+1} V_{n+1} + (\sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}) \bar{g}(E_{n+1}, \nu) = 0. \tag{5.43}$$

Since M is a compact hypersurface with boundary supported on $\mathcal{H}$, we have

$$-\Phi = \bar{g}(E_{n+1}, \nu) = \cos\theta > 0 \quad \text{on } M.$$

It follows from (5.43) that

$$\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2} < 0.$$

We get a contradiction by (5.42). Therefore, we conclude that

$$\int_M u\, dA \neq 0. \qquad \blacksquare$$

In the following part we are ready to prove the classification for stable $(r+1)$-th capillary hypersurfaces supported on a horosphere $\mathcal{H}$. Inspired by the higher-order Minkowski-type formula (5.16) and Proposition 5.12, we have an admissible test function defined by

$$\varphi_{n+1} := \lambda u - \bar{g}(X_{n+1}, \nu) H_{r+1},$$

where

$$\lambda := \Big(\int_M u\, dA\Big)^{-1} \int_M u H_r\, dA$$

is constant and u is given by (5.30).

For convenience, we denote

$$\begin{aligned} \phi &:= \lambda(r+1)\sigma_{r+1} - (n-r)\sigma_r H_{r+1},\\ \psi &:= \lambda(\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}) - (r+1)\sigma_{r+1}H_{r+1}. \end{aligned}$$

In particular, for $r=0$, we have $\lambda = 1$ and $\phi = 0$, and $\psi = |h|^2 - nH_1^2$.

Proposition 5.13. *Let $x\colon M \to \mathbb{H}^{n+1}$ be a constant $(r+1)$-th mean curvature hypersurface with boundary supported on $\mathcal{H}$. Assume $x(M)$ intersects $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. Then φ_{n+1} satisfies*

$$J_r\varphi_{n+1} = \phi\bar{g}(E_{n+1}, \nu) + \psi V_{n+1}, \tag{5.44}$$

$$\nabla_\mu \varphi_{n+1} = q\varphi_{n+1}, \tag{5.45}$$

$$\int_M \varphi_{n+1}\, dA = 0. \tag{5.46}$$

Proof. The first equation, (5.44), follows from (5.31) and (5.21); the second, (5.45), from (5.32) and (5.28); and the last one, (5.46), from (5.16) and the definition of λ. ■

Now we are going to prove the rigidity result for stable $(r+1)$-th capillary hypersurfaces with boundary supported on a horosphere in $\mathbb{H}^{n+1}$ as follows.

Theorem 5.14. *Assume $0 \le r \le n-1$. Let $x\colon M \to \mathbb{H}^{n+1}$ be an $(r+1)$-th capillary hypersurface supported on $\mathcal{H}$ at a constant contact angle $\theta \in (0,\pi)$. If M is stable and there exists at least one elliptic point, then M is totally umbilical.*

Proof. From Proposition 5.12, we know that $\int_M u\,dA \neq 0$. Thus, by (5.45) and (5.46), we can choose φ_{n+1} as an admissible test function in (3.9). Therefore,

$$
\begin{aligned}
(5.47)\qquad 0 &\le -\int_M \varphi_{n+1} J \varphi_{n+1} \\
&= -\int_M (\lambda V_{n+1} - \lambda\cos\theta\,\bar g(x,\nu) - \bar g(x - E_{n+1},\nu) H_{r+1}) J_r \varphi_{n+1} \\
&= -\int_M (\lambda V_{n+1} + \bar g(E_{n+1},\nu) H_{r+1}) J_r \varphi_{n+1} \\
&\quad + (H_{r+1} + \lambda\cos\theta)\int_M \bar g(x,\nu) J_r \varphi_{n+1}.
\end{aligned}
$$

We compute the second term in the right-hand side of (5.47) using Green's formula. By (5.19) and (5.45), we have

$$
\begin{aligned}
&\int_M \bar g(x,\nu) J_r \varphi_{n+1} \\
&\quad = \int_M J_r \bar g(x,\nu)\varphi_{n+1} + \int_{\partial M} [\bar g(x,\nu) P_r(\nabla\varphi_{n+1},\mu) - \varphi_{n+1} P_r(\nabla \bar g(x,\nu),\mu)] \\
&\quad = \int_{\partial M} P_r^{\mu\mu}(\bar g(x,\nu)\nabla_\mu \varphi_{n+1} - \varphi_{n+1}\nabla_\mu \bar g(x,\nu)) \\
(5.48)\qquad &\quad = \int_{\partial M} P_r^{\mu\mu}\varphi_{n+1}(q\cdot \bar g(x,\nu) - \nabla_\mu \bar g(x,\nu)).
\end{aligned}
$$

From (5.29), (2.2) and (2.3), we find, along ∂M,

$$
\begin{aligned}
(5.49)\qquad & q\cdot \bar g(x,\nu) - \nabla_\mu \bar g(x,\nu) \\
&= (\csc\theta + \cot\theta\, h(\mu,\mu))\,\bar g(x,\nu) - (\bar g(x,\bar\nu) + h(\mu,\mu)\,\bar g(x,\mu)) \\
&= -\cot\theta\,\bar g(x,\bar N) - \csc\theta\,\bar g(x,\bar N)\, h(\mu,\mu) = \cot\theta + \csc\theta\, h(\mu,\mu),
\end{aligned}
$$

where we used $\bar g(x,\bar N) = -1$ on ∂M. Substituting (5.48) and (5.49) into (5.47), we get

$$
\begin{aligned}
(5.50)\qquad & \int_M (\lambda V_{n+1} + \bar g(E_{n+1},\nu) H_{r+1}) J_r \varphi_{n+1} \\
&\quad - (H_{r+1} + \lambda\cos\theta)\int_{\partial M} P_r^{\mu\mu}\varphi_{n+1}(\cot\theta + \csc\theta\, h(\mu,\mu)) \le 0.
\end{aligned}
$$

Next we introduce a powerful auxiliary function to eliminate the integral boundary term of (5.50). Let

$$
(5.51)\qquad \Psi := -H_{r+1} V_{n+1} - \lambda \bar g(E_{n+1},\nu).
$$

By (5.26) and (5.25), we obtain

$$
(5.52)\qquad L_r \Psi = \phi V_{n+1} + \psi \bar g(E_{n+1},\nu).
$$

From (2.2), we have

$$
(5.53)\qquad \Psi|_{\partial M} = -H_{r+1} - \lambda\cos\theta.
$$

Using (5.3), we can directly calculate

$$\nabla_\mu \Psi = -\sin\theta(H_{r+1} - \lambda h(\mu,\mu)), \tag{5.54}$$

where we have used $\bar{g}(\bar{\nabla}_\mu E_{n+1}, \nu) = 0$ and $\bar{g}(E_{n+1}, \mu) = -\sin\theta$ on ∂M.

Inserting (5.51)–(5.54) into the integral identity

$$\int_M [\Psi L_r \Psi + P_r(\nabla\Psi, \nabla\Psi)] = \int_M \frac{1}{2} L_r \Psi^2 = \int_{\partial M} \Psi P_r(\nabla\Psi, \mu),$$

we get that

$$\begin{aligned}\int_M (-H_{r+1}V_{n+1} - \lambda\bar{g}(E_{n+1},\nu))(\phi V_{n+1} + \psi\bar{g}(E_{n+1},\nu)) + \int_M P_r(\nabla\Psi,\nabla\Psi) \\ = (H_{r+1} + \lambda\cos\theta)\sin\theta \int_{\partial M} P_r^{\mu\mu}(H_{r+1} - \lambda h(\mu,\mu)).\end{aligned} \tag{5.55}$$

Putting (5.55) into (5.50) and applying (5.44), we have

$$\begin{aligned} 0 \geq \int_M \bar{g}(E_{n+1}^T, E_{n+1}^T)(\psi\lambda - \phi H_{r+1}) + \int_M P_r(\nabla\Psi,\nabla\Psi) \\ - (H_{r+1} + \lambda\cos\theta)\int_{\partial M} P_r^{\mu\mu}[\sin\theta(H_{r+1} - \lambda h(\mu,\mu)) \\ + (\cot\theta + \csc\theta\, h(\mu,\mu))\varphi_{n+1}]. \end{aligned} \tag{5.56}$$

Next we will show that the boundary term in the right-hand side of (5.56) is zero. Indeed, by (2.2) and (5.7), along ∂M,

$$\begin{aligned}\varphi_{n+1} &= \lambda V_{n+1} - \lambda\cos\theta\bar{g}(x,\nu) - \bar{g}(X_{n+1},\nu)H_{r+1} \\ &= \lambda - \lambda\cos\theta(\cos\theta + \sin\theta\,\bar{g}(x,\bar{\nu})) - \sin\theta\,\bar{g}(x - E_{n+1},\bar{\nu})H_{r+1} \\ &= \sin\theta(\lambda\sin\theta - \lambda\cos\theta\,\bar{g}(x,\bar{\nu}) - \bar{g}(x,\bar{\nu})H_{r+1}).\end{aligned}$$

It follows that

$$\begin{aligned}\sin\theta(H_{r+1} - \lambda h(\mu,\mu)) + (\cot\theta + \csc\theta h(\mu,\mu))\varphi_{n+1} \\ = (H_{r+1} + \lambda\cos\theta)(\sin\theta - \cos\theta\,\bar{g}(x,\bar{\nu}) - \bar{g}(x,\bar{\nu})h(\mu,\mu)).\end{aligned} \tag{5.57}$$

From (5.57) and (5.13), we see that

$$\int_{\partial M} P_r^{\mu\mu}[\sin\theta(H_{r+1} - \lambda h(\mu,\mu)) + (\cot\theta + \csc\theta h(\mu,\mu))\varphi_{n+1}] = 0. \tag{5.58}$$

Putting (5.58) into (5.56), we get

$$0 \geq \int_M [\bar{g}(E_{n+1}^T, E_{n+1}^T)(\psi\lambda - \phi H_{r+1}) + P_r(\nabla\Psi,\nabla\Psi)]. \tag{5.59}$$

Since there exists an elliptic point on M, we have H_{r+1} = constant > 0. From Proposition 2.3, the operator L_i is elliptic and $H_i > 0$ for each $i = 1, 2, \ldots, r$.

By the Newton–Maclaurin inequality (2.6), we have

$$
\begin{aligned}
(5.60)\quad \psi\lambda - \phi H_{r+1} &= \lambda^2(\sigma_1\sigma_{r+1} - (r+2)\sigma_{r+2}) - 2\lambda(r+1)\sigma_{r+1}H_{r+1} + (n-r)\sigma_r H_{r+1}^2 \\
&= \binom{n}{r+1}[\lambda^2(n-r-1)(H_1H_{r+1} - H_{r+2}) + (r+1)\lambda^2 H_1 H_{r+1}] \\
&\quad + \binom{n}{r+1}(r+1)[-2\lambda H_{r+1}^2 + H_r H_{r+1}^2] \\
&= \lambda^2(n-r-1)\binom{n}{r+1}(H_1H_{r+1} - H_{r+2}) \\
&\quad + (r+1)\binom{n}{r+1}H_{r+1}^2\Big(\frac{H_1}{H_{r+1}}\Big(\lambda - \frac{H_{r+1}}{H_1}\Big)^2 + H_r - \frac{H_{r+1}}{H_1}\Big) \geq 0.
\end{aligned}
$$

Combining (5.60) and (5.59), we obtain

$$
\bar{g}(E_{n+1}^T, E_{n+1}^T)(\psi\lambda - \phi H_{r+1}) = 0 \quad \text{on } M, \tag{5.61}
$$

and Ψ is a constant on M, that is,

$$
\Psi = -H_{r+1}V_{n+1} - \lambda\bar{g}(E_{n+1}, \nu) = -H_{r+1} - \lambda\cos\theta \quad \text{on } M. \tag{5.62}
$$

We claim that

$$
\psi\lambda - \phi H_{r+1} = 0 \quad \text{on } M. \tag{5.63}
$$

In fact, the open set $U := \{p \in M \mid \psi\lambda - \phi H_{r+1} \neq 0\}$ is empty. If not, we have that $\bar{g}(E_{n+1}^T, E_{n+1}^T) = 0$ on U from (5.61). By the fact that $\bar{g}(\bar{E}_{n+1}, \bar{E}_{n+1}) = 1$,

$$
\bar{g}(E_{n+1}, \nu) = \pm V_{n+1} \quad \text{on } U. \tag{5.64}
$$

Since $\theta \in (0, \pi)$ and $H_{r+1} > 0$, we combine (5.64) and (5.62) to obtain that V_{n+1} is a positive constant c_1 on U, which means U is lying on the horosphere $\{V_{n+1} = c_1\}$. On the other hand, from (5.60) we know that U^c is a part of the totally umbilical hypersurface. By the smoothness of M, we imply that V_{n+1} is constant on the whole M. Thus, M lies on a horosphere in $\mathbb{H}^{n+1}$. Using (5.60) again, we have

$$
\psi\lambda - \phi H_{r+1} = 0 \quad \text{on } M.
$$

We get a contradiction, so the claim (5.63) is true. From (5.63), (5.60) and Lemma 2.2, we obtain that M is a totally umbilical hypersurface. ∎

A. Proof of the first variational formula

The appendix is devoted to computing the first variational formula of the $(r+1)$-th energy functional $\mathcal{E}_{r+1}$ and to proving Theorem 3.1. Let $\mathbb{M}^{n+1}(K)$ be a complete simply-connected $(n+1)$-dimensional Riemannian manifold with constant sectional curvature K. We first study the evolution equations for several useful geometric quantities under the following flow in $\mathbb{M}^{n+1}(K)$:

$$
\partial_t x = f\nu + T, \quad \text{where } T \in TM_t. \tag{A.1}
$$

Proposition A.1. *Along the general flow* (A.1)*, the following hold*:

(1) $\partial_t g_{ij} = 2fh_{ij} + \nabla_i T_j + \nabla_j T_i$,
(2) $\partial_t dA_t = (fH + \mathrm{div}T)\, dA_t$,
(3) $\partial_t \nu = -\nabla f + h(e_i, T)e_i$,
(4) $\partial_t h_{ij} = -\nabla^2_{ij} f + f(h_{ik}h^k_j - K\bar{g}_{ij}) + \nabla_T h_{ij} + h^k_j \nabla_i T_k + h^k_i \nabla_j T_k$,
(5) $\partial_t h^i_j = -\nabla^i \nabla_j f - f(h^k_j h^i_k + K\bar{g}^i_j) + \nabla_T h^i_j$,
(6) $\partial_t H = -\Delta f - (nK + |h|^2) f + \nabla_T H$,
(7) $\partial_t F = -F^j_i \nabla^i \nabla_j f - f(F^j_i h^k_j h^i_k + KF^j_i \bar{g}^i_j) + \nabla_T F$ *for* $F = F(h^j_i)$*, where* $F^i_j := \partial F/\partial h^j_i$,
(8) $\partial_t \sigma_r = -\frac{\partial \sigma_r}{\partial h^j_i} \nabla^i \nabla_j f - (\sigma_1 \sigma_r - (r+1)\sigma_{r+1}) f - K(n-r+1)\sigma_{r-1} f + \nabla_T \sigma_r$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $T_p M$ for some point p. Denote $e_i(t) := (x(t,\cdot))_*(e_i)$ and $Y(t) := \partial_t x(t,\cdot)$. Then we have $\bar{g}(e_i(t), \nu(t)) = 0$ and $[e_i(t), Y(t)] = 0$. Recall the Gauss–Weingarten formula as follows:

$$\bar{\nabla}_{e_i} e_j = \nabla_{e_i} e_j - h_{ij}\nu, \quad \bar{\nabla}_{e_i}\nu = h_{ik} e_k.$$

We calculate that

$$\begin{aligned}
\partial_t \bar{g}_{ij} &= \partial_t \bar{g}(e_i(t), e_j(t)) = \bar{g}(\bar{\nabla}_Y e_i, e_j) + \bar{g}(e_i, \bar{\nabla}_Y e_j) \\
&= \bar{g}(\bar{\nabla}_{e_i} Y, e_j) + \bar{g}(e_i, \bar{\nabla}_{e_j} Y) = \bar{g}(\bar{\nabla}_{e_i}(f\nu + T), e_j) + \bar{g}(e_i, \bar{\nabla}_{e_j}(f\nu + T)) \\
&= 2fh_{ij} + \nabla_i T_j + \nabla_j T_i.
\end{aligned}$$

It follows that

$$\begin{aligned}
\partial_t\, dA_t &= \frac{1}{2}\bar{g}^{ij}\partial_t \bar{g}_{ij}\, dA_t \\
&= \frac{1}{2}\bar{g}^{ij}(2fh_{ij} + \nabla_i T_j + \nabla_j T_i)\, dA_t = (fH + \mathrm{div}T)\, dA_t.
\end{aligned}$$

Since $\bar{g}(\partial_t \nu, \nu) = 0$, we have

$$\begin{aligned}
\partial_t \nu &= \bar{g}(\partial_t \nu, e_i(t))e_i(t) = -\bar{g}(\nu, \bar{\nabla}_Y e_i)e_i \\
&= -\bar{g}(\nu, \bar{\nabla}_{e_i} Y)e_i = -\bar{g}(\nu, \bar{\nabla}_{e_i}(f\nu + T))e_i \\
&= -(\bar{\nabla}_{e_i} f)e_i + h(e_i, T)e_i = -\nabla f + h(e_i, T)e_i.
\end{aligned}$$

We next compute

$$\begin{aligned}
\partial_t h_{ij} &= \partial_t \bar{g}(\bar{\nabla}_{e_i(t)}\nu(t), e_j(t)) \\
&= \bar{g}(\bar{\nabla}_Y \bar{\nabla}_{e_i}\nu, e_j) + \bar{g}(\bar{\nabla}_{e_i}\nu, \bar{\nabla}_Y e_j) \\
&= \bar{g}(\bar{\nabla}_{e_i}\bar{\nabla}_Y \nu, e_j) + \bar{g}(\bar{\nabla}_{[Y,e_i]}\nu, e_j) + \bar{g}(\bar{R}(Y, e_i)\nu, e_j) + \bar{g}(\bar{\nabla}_{e_i}\nu, \bar{\nabla}_{e_j} Y) \\
&= \bar{g}(\bar{\nabla}_{e_i}(\partial_t \nu), e_j) + \bar{R}(e_j, \nu, Y, e_i) + \bar{g}(\bar{\nabla}_{e_i}\nu, \bar{\nabla}_{e_j}(f\nu + T)) \\
&= \bar{g}(\bar{\nabla}_{e_i}(-\nabla f + h(e_k, T)e_k), e_j) - K\bar{g}(Y, \nu)\bar{g}_{ij} + fh^k_i h_{kj} + h^k_i \nabla_j T_k \\
&= -\nabla^2_{ij} f - Kf\bar{g}_{ij} + fh^k_i h_{kj} + \bar{g}(\bar{\nabla}_{e_i}(h(e_k, T)e_k), e_j) + h^k_i \nabla_j T_k.
\end{aligned} \tag{A.2}$$

Using the Codazzi equation in a space form, we have

$$\begin{aligned}\text{(A.3)}\quad & \bar g(\bar\nabla_{e_i}(h(e_k,T)e_k),e_j)\\ &= [\nabla_{e_i}h(e_k,T)+h(\nabla_{e_i}e_k,T)+h(e_k,\nabla_{e_i}T)]\bar g_{kj}+h(e_k,T)\,\bar g(\nabla_{e_i}e_k,e_j)\\ &= \nabla_{e_i}h(e_j,T)+h(\nabla_{e_i}e_j,T)+h_j^k\nabla_i T_k-h(e_k,T)\,\bar g(e_k,\nabla_{e_i}e_j)\\ &= \nabla_T h(e_j,e_i)+h_j^k\nabla_i T_k.\end{aligned}$$

Combining (A.2) and (A.3), we get (4). It follows that

$$\begin{aligned}\partial_t h_j^i = \partial_t(\bar g^{ik}h_{kj}) &= -2fh_k^i h_j^k-\nabla^i\nabla_j f+fh^{ik}h_{kj}-Kf\bar g_j^i+\nabla_T h_j^i\\ &= -\nabla^i\nabla_j f-fh^{ik}h_{kj}-Kf\bar g_j^i+\nabla_T h_j^i.\end{aligned}$$

The last three assertions (6)–(8) follow directly from (5); we only need the following fact:

$$\frac{\partial\sigma_r}{\partial h_j^i}h_j^k h_k^i=\sigma_1\sigma_r-(r+1)\sigma_{r+1}\quad\text{and}\quad \frac{\partial\sigma_r}{\partial h_j^i}\bar g_j^i=(n-r+1)\sigma_{r-1}. \qquad\blacksquare$$

Next we let ∂B be a totally umbilical hypersurface in $\mathbb{M}^{n+1}(K)$ with constant principal curvature $\kappa\in\mathbb{R}$. Let $x\colon M\to\mathbb{M}^{n+1}(K)$ be an immersed hypersurface with boundary ∂M supported on ∂B. Assume the contact angle $\theta\in(0,\pi)$ is constant along ∂M.

From Proposition 3.2, we choose $f\in\mathcal{F}$. Then there exists an admissible volume-preserving and contact angle-preserving variation of x with the variational vector field Y having $f\nu$ as its normal part. Namely,

$$\text{(A.4)}\qquad Y=T+f\nu\quad\text{on } M,$$

$$\text{(A.5)}\qquad \bar g(\nu,\bar N\circ x)=-\cos\theta\quad\text{on }\partial M,$$

where T is the tangent part of the variational vector field Y.

Applying (2.3), (A.4) and the admissible condition, we get

$$0=\bar g(Y,\bar N)=\bar g(Y,\sin\theta\mu-\cos\theta\nu)=\sin\theta\,\bar g(Y,\mu)-f\cos\theta\quad\text{on }\partial M.$$

Therefore,

$$\text{(A.6)}\qquad \bar g(T,\mu)=\bar g(Y,\mu)=\cot\theta f\quad\text{on }\partial M.$$

So, we can define

$$\text{(A.7)}\qquad Y=T+f\nu:=Y^{\partial M}+\cot\theta f\mu+f\nu,$$

where $Y^{\partial M}$ denotes the tangent part of Y to ∂M.

On the other hand, from (2.4) we see that Y can be also expressed as follows:

$$\text{(A.8)}\qquad Y=Y^{\partial M}+\frac{f}{\sin\theta}(\cos\theta\mu+\sin\theta\nu)=Y^{\partial M}+\frac{f}{\sin\theta}\bar\nu.$$

Recall that the r-th wetting area functional $W_r\colon(-\varepsilon,\varepsilon)\to\mathbb{R}$ is inductively given by

$$W_0(t):=\int_{\partial M\times[0,t]}x^*\,dA_{\partial B},\quad W_1(t):=\frac{1}{n}\int_{\partial M}ds_t,$$

and, for $2 \le r \le n-1$,

$$W_r(t) = \frac{1}{n}\int_{\partial M} H_{r-1}^{\partial M}\, ds_t + \frac{r-1}{n-r+2}(K+\kappa^2)W_{r-2}(t), \tag{A.9}$$

where $H_{r-1}^{\partial M}$ is the $(r-1)$-th normalized mean curvature of the closed hypersurface ∂M in ∂B.

Lemma A.2. *For any $0 \le r \le n-1$, we have the first variational formula of $W_r(t)$:*

$$\frac{d}{dt}W_r(t) = \binom{n}{r}^{-1}\int_{\partial M}\sigma_r^{\partial M}\bar{g}(Y,\bar{\nu})\, ds_t = \frac{1}{\sin\theta}\binom{n}{r}^{-1}\int_{\partial M}\sigma_r^{\partial M} f\, ds_t, \tag{A.10}$$

where $\sigma_r^{\partial M}(\tilde{h})$ is the r-th mean curvature of ∂M in ∂B.

Proof. This is obviously true for $r=0$ and $r=1$. We next only consider the case $r \ge 2$. Let ∂M be an immersed closed hypersurface in ∂B with a normal speed $\tilde{f}\bar{\nu}$. From (A.8), we see

$$\tilde{f} = \frac{1}{\sin\theta}f.$$

By the Gauss equation, we know that ∂B has intrinsic constant sectional curvature $\tau = K+\kappa^2$. From Proposition A.1, we get

$$\begin{aligned}\frac{d}{dt}\int_{\partial M}\sigma_{r-1}^{\partial M}\, ds_t &= \int_{\partial M}\Big(-\frac{\partial\sigma_{r-1}^{\partial M}(\tilde{h})}{\partial\tilde{h}_\alpha^\beta}\tilde{\nabla}^2_{\alpha\beta}\tilde{f} - (\sigma_1^{\partial M}\sigma_{r-1}^{\partial M} - r\sigma_r^{\partial M})\tilde{f}\Big)\, ds_t \\ &\quad + \int_{\partial M}(-\tau(n-r+1)\sigma_{r-2}^{\partial M}\tilde{f} + \sigma_{r-1}^{\partial M}\sigma_1^{\partial M}\tilde{f})\, ds_t \\ &= r\int_{\partial M}\sigma_r^{\partial M}\tilde{f}\, ds_t - \tau(n-r+1)\int_{\partial M}\sigma_{r-2}^{\partial M}\tilde{f}\, ds_t.\end{aligned} \tag{A.11}$$

By induction, we assume it is true for $r-2$ in (A.10). Applying (A.9) and (A.11), we have

$$\begin{aligned}\frac{d}{dt}W_r(t) &= \frac{1}{n}\binom{n-1}{r-1}^{-1}\frac{d}{dt}\Big(\int_{\partial M}\sigma_{r-1}^{\partial M}\, ds_t\Big) + \frac{\tau(r-1)}{n-r+2}\frac{d}{dt}W_{r-2}(t) \\ &= \frac{1}{n}\binom{n-1}{r-1}^{-1}\Big(r\int_{\partial M}\sigma_r^{\partial M}\tilde{f}\, ds_t - \tau(n-r+1)\int_{\partial M}\sigma_{r-2}^{\partial M}\tilde{f}\, ds_t\Big) \\ &\quad + \frac{\tau(r-1)}{n-r+2}\binom{n}{r-2}^{-1}\int_{\partial M}\sigma_{r-2}^{\partial M}\tilde{f}\, ds_t \\ &= \frac{r}{n}\binom{n-1}{r-1}^{-1}\int_{\partial M}\sigma_r^{\partial M}\tilde{f}\, ds_t \\ &= \frac{1}{\sin\theta}\binom{n}{r}^{-1}\int_{\partial M}\sigma_r^{\partial M} f\, ds_t.\end{aligned}$$

The proof is completed. ∎

Proof of Theorem 3.1. The case $r=0$ can be found in [33]. In the following proof, we only consider the case $1 \le r \le n-1$. By Proposition A.1, using integration by parts and the fact that μ is a principal direction from Proposition 2.4, we obtain

$$\begin{aligned}
\text{(A.12)} \qquad \frac{d}{dt}\int_M \sigma_r \, dA_t &= \int_M \Big[-\frac{\partial \sigma_r}{\partial h_i^j} \nabla^i \nabla_j f - f(\sigma_1\sigma_r - (r+1)\sigma_{r+1}) \\
&\quad - K(n-r+1) f \sigma_{r-1} + \nabla_T \sigma_r \Big] \, dA_t \\
&\quad + \int_M \sigma_r (f\sigma_1 + \mathrm{div}_M T) \, dA_t \\
&= (r+1)\int_M \sigma_{r+1} f dA_t - K(n-r+1)\int_M \sigma_{r-1} f \, dA_t \\
&\quad + \int_{\partial M} (\sigma_r \bar{g}(T,\mu) - \sigma_r^{\mu\mu}\nabla_\mu f) \, ds_t.
\end{aligned}$$

Utilizing (A.6), (3.3) and the fact that the principal curvature of ∂B is κ, we get, along ∂M,

$$\begin{aligned}
\text{(A.13)} \qquad \sigma_r \bar{g}(T,\mu) - \sigma_r^{\mu\mu}\nabla_\mu f &= \sigma_r f \cot\theta - q f \sigma_r^{\mu\mu} \\
&= f\Big(\cot\theta(\sigma_r - \sigma_r^{\mu\mu} h_{\mu\mu}) - \frac{\kappa}{\sin\theta}\sigma_r^{\mu\mu}\Big) \\
&= f\Big(\cot\theta \sigma_r(h|h_{\mu\mu}) - \frac{\kappa}{\sin\theta}\sigma_{r-1}(h|h_{\mu\mu})\Big).
\end{aligned}$$

Here we used the fact that

$$\sigma_r^{\mu\mu} = P_{r-1}^{\mu\mu} = \sigma_{r-1}(h|h_{\mu\mu}) \quad \text{and} \quad \sigma_r(h) = \sigma_r(h|h_{\mu\mu}) + h_{\mu\mu}\sigma_{r-1}(h|h_{\mu\mu}).$$

By (2.2), we see, along ∂M,

$$h_{\alpha\beta} = -\bar{g}(\bar{\nabla}_{e_\alpha} e_\beta, \nu) = -\bar{g}(\bar{\nabla}_{e_\alpha} e_\beta, \sin\theta\,\bar{\nu} - \cos\theta\,\bar{N}) = \sin\theta\,\hat{h}_{\alpha\beta} - \kappa\cos\theta\,\delta_{\alpha\beta},$$

for an orthonormal frame $\{e_\alpha\}_{\alpha=1}^{n-1}$ of $T(\partial M)$. Thus,

$$\text{(A.14)} \qquad \sigma_r(h|h_{\mu\mu}) = \sigma_r(\sin\theta\,\hat{h} - \kappa\cos\theta\, I_{n-1}),$$

where I_{n-1} is the $(n-1)$-th identity matrix. In general, for a $(n-1)\times(n-1)$ symmetric matrix B, we know

$$\sigma_r(I_{n-1} + B) = \sum_{l=0}^{r} \binom{n-l-1}{n-r-1} \sigma_l(B).$$

When $\kappa = 0$, from (A.14), we have

$$\cot\theta\,\sigma_r(h|h_{\mu\mu}) = \cot\theta\,\sigma_r(\sin\theta\,\hat{h}) = \cos\theta\,\sin^{r-1}\theta\,\sigma_r(\hat{h}).$$

When $\kappa \neq 0$, from (A.14), we get

$$\begin{aligned}
\sigma_r(h|h_{\mu\mu}) &= \sigma_r(\sin\theta\,\hat{h} - \kappa\cos\theta\, I_{n-1}) \\
&= (-\kappa\cos\theta)^r \sum_{l=0}^{r} \binom{n-l-1}{n-r-1} \Big(-\frac{\tan\theta}{\kappa}\Big)^l \sigma_l(\hat{h})
\end{aligned}$$

and

$$\sigma_{r-1}(h\,|\,h_{\mu\mu}) = \sigma_{r-1}(\sin\theta\,\hat{h} - \kappa\cos\theta\, I_{n-1})$$
$$= (-\kappa\cos\theta)^{r-1}\sum_{l=0}^{r-1}\binom{n-l-1}{n-r}\Big(-\frac{\tan\theta}{\kappa}\Big)^{l}\sigma_l(\hat{h}).$$

Therefore, for any $\kappa\in\mathbb{R}$, we have

$$\cot\theta\,\sigma_r(h|h_{\mu\mu}) - \frac{\kappa}{\sin\theta}\sigma_{r-1}(h|h_{\mu\mu}) = \cos\theta\sin^{r-1}\theta\,\sigma_r(\hat{h})$$
$$+\frac{\cos^{r-1}\theta}{\sin\theta}\sum_{l=0}^{r-1}(-1)^{r+l}\kappa^{r-l}\Big[\cos^2\theta\binom{n-l-1}{n-r-1}+\binom{n-l-1}{n-r}\Big]\tan^l\theta\,\sigma_l(\hat{h}). \tag{A.15}$$

Putting (A.15) and (A.13) into (A.12), we see

$$\frac{d}{dt}\int_M \sigma_r\, dA_t = (r+1)\int_M \sigma_{r+1} f\, dA_t - K(n-r+1)\int_M \sigma_{r-1} f\, dA_t$$
$$+\cos\theta\sin^{r-1}\theta\int_{\partial M} f\sigma_r(\hat{h})\, ds_t$$
$$+\frac{\cos^{r-1}\theta}{\sin\theta}\sum_{l=0}^{r-1}(-1)^{r+l}\kappa^{r-l}\Big[\cos^2\theta\binom{n-l-1}{n-r-1}$$
$$+\binom{n-l-1}{n-r}\Big]\tan^l\theta\int_{\partial M} f\sigma_l(\hat{h})\, ds_t.$$

Since $\hat{h}$ is the second fundamental form of ∂M as a closed hypersurface in ∂B, we have $\sigma_r(\hat{h}) = \sigma_r^{\partial M}$ on ∂M. From (A.10), we have the first variational formula of the r-th wetting area functional W_r as follows:

$$\frac{d}{dt}W_r(t) = \frac{1}{\sin\theta}\binom{n}{r}^{-1}\int_{\partial M} f\sigma_r(\hat{h})\, ds_t.$$

We conclude that for $1\le r\le n-1$,

$$\frac{d}{dt}\Big\{\int_M \sigma_r dA_t - \binom{n}{r}\cos\theta\sin^r\theta W_r(t)$$
$$-\cos^{r-1}\theta\sum_{l=0}^{r-1}(-1)^{r+l}\kappa^{r-l}\binom{n}{l}\Big[\cos^2\theta\binom{n-l-1}{n-r-1}+\binom{n-l-1}{n-r}\Big]\tan^l\theta W_l(t)\Big\}$$
$$= (r+1)\int_M \sigma_{r+1} f\, dA_t - K(n-r+1)\int_M \sigma_{r-1} f\, dA_t. \tag{A.16}$$

By the combination relationships

$$\binom{n}{l}\cdot\binom{n-l-1}{n-r-1} = \binom{n}{r}\cdot\binom{r}{l}\frac{n-r}{n-l}$$

and
$$\binom{n}{l}\cdot\binom{n-l-1}{n-r}=\binom{n}{r}\cdot\binom{r}{l}\frac{r-l}{n-l},$$
and from (A.16), we obtain
$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_M \sigma_r\,dA_t-\binom{n}{r}\cos\theta\sin^r\theta\, W_r(t)\\
&\qquad -\cos^{r-1}\theta\sum_{l=0}^{r-1}\frac{(-1)^{r+l}K^{r-l}}{n-l}\binom{r}{l}[(n-r)\cos^2\theta+(r-l)]\tan^l\theta\binom{n}{r}W_l(t)\Big\}\\
&= (r+1)\int_M \sigma_{r+1} f\,dA_t-K(n-r+1)\int_M\sigma_{r-1}f\,dA_t.
\end{aligned}\tag{A.17}$$
Recall that
$$\begin{aligned}
Q_{r+1}(t)=&\int_M H_r\,dA_t-\cos\theta\sin^r\theta W_r(t)\\
&-\cos^{r-1}\theta\sum_{l=0}^{r-1}\frac{(-1)^{r+l}K^{r-l}}{n-l}\binom{r}{l}[(n-r)\cos^2\theta+(r-l)]\tan^l\theta\, W_l(t).
\end{aligned}$$
Therefore, by (A.17), we have
$$\frac{d}{dt}Q_{r+1}(t)=(n-r)\int_M H_{r+1}f\,dA_t-rK\int_M H_{r-1}f\,dA_t.\tag{A.18}$$
Let
$$\mathcal{E}_{r+1}(t)=Q_{r+1}(t)+\frac{rK}{n+2-r}\mathcal{E}_{r-1}(t).\tag{A.19}$$
One can readily check that for any $-1\le s\le n-1$,
$$\frac{d}{dt}\mathcal{E}_{s+1}(t)=(n-s)\int_{M_t}H_{s+1}f\,dA.$$
In fact, it is true for $s=-1$ and $s=0$. By induction, we assume it is true for $s=r-2$, and then we calculate using (A.18) and (A.19):
$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_{r+1}(t)&=\frac{d}{dt}Q_{r+1}(t)+\Big(\frac{rK}{n+2-r}\Big)\frac{d}{dt}\mathcal{E}_{r-1}(t)\\
&=(n-r)\int_M H_{r+1}f\,dA_t-rK\int_M H_{r-1}f\,dA_t+\frac{rK}{n+2-r}(n+2-r)\int_M H_{r-1}f\,dA_t\\
&=(n-r)\int_M H_{r+1}f\,dA_t.
\end{aligned}$$
The proof of Theorem 3.1 is complete. ∎

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References

[1] Ainouz, A. and Souam, R.: Stable capillary hypersurfaces in a half-space or a slab. *Indiana Univ. Math. J.* **65** (2016), no. 3, 813–831. Zbl 1347.53009 MR 3528820

[2] Alencar, H., do Carmo, M. and Colares, A. G.: Stable hypersurfaces with constant scalar curvature. *Math. Z.* **213** (1993), no. 1, 117–131. Zbl 0792.53057 MR 1217674

[3] Alencar, H., do Carmo, M. and Rosenberg, H.: On the first eigenvalue of the linearized operator of the rth mean curvature of a hypersurface. *Ann. Global Anal. Geom.* **11** (1993), no. 4, 387–395. Zbl 0816.53031 MR 1246197

[4] Barbosa, E.: On CMC free-boundary stable hypersurfaces in a Euclidean ball. *Math. Ann.* **372** (2018), no. 1-2, 179–187. Zbl 1397.53013 MR 3856810

[5] Barbosa, J. L. and do Carmo, M.: Stability of hypersurfaces with constant mean curvature. *Math. Z.* **185** (1984), no. 3, 339–353. Zbl 0513.53002 MR 731682

[6] Barbosa, J. L., do Carmo, M. and Eschenburg, J.: Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. *Math. Z.* **197** (1988), no. 1, 123–138. Zbl 0653.53045 MR 917854

[7] Barbosa, J. L.M. and Colares, A. G.: Stability of hypersurfaces with constant r-mean curvature. *Ann. Global Anal. Geom.* **15** (1997), no. 3, 277–297. Zbl 0891.53044 MR 1456513

[8] Chen, Y. and Pyo, J.: Some rigidity results on compact hypersurfaces with capillary boundary in the hyperbolic space. Preprint 2023, arXiv:2206.09062v2.

[9] Choe, J. and Koiso, M.: Stable capillary hypersurfaces in a wedge. *Pacific J. Math.* **280** (2016), no. 1, 1–15. Zbl 1347.49068 MR 3441213

[10] Damasceno, L. and Elbert, M.: Stable free boundary surfaces with constant extrinsic curvature in 3-dimensional space forms. Preprint 2023, arXiv:2305.01093v1.

[11] Damasceno, L. and Elbert, M. F.: Stability of capillary hypersurfaces with constant higher order mean curvature. *J. Geom. Anal.* **34** (2024), no. 12, article no. 377, 37 pp. Zbl 1553.53058 MR 4812473

[12] De Philippis, G. and Maggi, F.: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.* **216** (2015), no. 2, 473–568. Zbl 1321.35158 MR 3317808

[13] Finn, R.: *Equilibrium capillary surfaces*. Grundlehren Math. Wiss. 284, Springer, New York, 1986. Zbl 0583.35002 MR 816345

[14] Gonzalez, E., Massari, U. and Tamanini, I.: Existence and regularity for the problem of a pendent liquid drop. *Pacific J. Math.* **88** (1980), no. 2, 399–420. Zbl 0415.49028 MR 607987

[15] Grüter, M.: Optimal regularity for codimension one minimal surfaces with a free boundary. *Manuscripta Math.* **58** (1987), no. 3, 295–343. Zbl 0609.35012 MR 893158

[16] Grüter, M. and Jost, J.: Allard type regularity results for varifolds with free boundaries. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **13** (1986), no. 1, 129–169. Zbl 0615.49018 MR 863638

[17] Grüter, M. and Jost, J.: On embedded minimal disks in convex bodies. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), no. 5, 345–390. Zbl 0617.49017 MR 868522

[18] Guan, P.: Curvature measures, isoperimetric type inequalities and fully nonlinear PDEs. In *Fully nonlinear PDEs in real and complex geometry and optics*, pp. 47–94. Lecture Notes in Math. 2087, Springer, Cham, 2014. Zbl 1288.35002 MR 3203559

[19] Guang, Q., Li, M. M.-c., Wang, Z. and Zhou, X.: Min-max theory for free boundary minimal hypersurfaces II: general Morse index bounds and applications. *Math. Ann.* **379** (2021), no. 3-4, 1395–1424. Zbl 1466.53074 MR 4238268

[20] Guo, J., Wang, G. and Xia, C.: Stable capillary hypersurfaces supported on a horosphere in the hyperbolic space. *Adv. Math.* **409** (2022), article no. 108641, 25 pp. Zbl 1506.53066 MR 4473639

[21] Guo, J. and Xia, C.: A partially overdetermined problem in domains with partial umbilical boundary in space forms. *Adv. Calc. Var.* **17** (2024), no. 1, 11–31. Zbl 1530.35161 MR 4685058

[22] Guo, J and Xia, C.: Stable anisotropic capillary hypersurfaces in the half-space. *Indiana Univ. Math. J.* **74** (2025), no. 2, 251–277.

[23] He, Y. and Li, H.: Stability of hypersurfaces with constant $(r+1)$-th anisotropic mean curvature. *Illinois J. Math.* **52** (2008), no. 4, 1301–1314. Zbl 1181.53052 MR 2595769

[24] Jost, J.: Embedded minimal surfaces in manifolds diffeomorphic to the three-dimensional ball or sphere. *J. Differential Geom.* **30** (1989), no. 2, 555–577. Zbl 0682.53056 MR 1010172

[25] Li, C., Zhou, X. and Zhu, J. J.: Min-max theory for capillary surfaces. *J. Reine Angew. Math.* **818** (2025), 215–262. Zbl 07962755 MR 4846024

[26] Li, H. and Xiong, C.: Stability of capillary hypersurfaces with planar boundaries. *J. Geom. Anal.* **27** (2017), no. 1, 79–94. Zbl 1362.53017 MR 3606545

[27] Li, H. and Xiong, C.: Stability of capillary hypersurfaces in a Euclidean ball. *Pacific J. Math.* **297** (2018), no. 1, 131–146. Zbl 1401.49063 MR 3864232

[28] Li, M. M.-C. and Zhou, X.: Min-max theory for free boundary minimal hypersurfaces I. Regularity theory. *J. Differential Geom.* **118** (2021), no. 3, 487–553. Zbl 1480.53074 MR 4285846

[29] López, R.: Symmetry of stationary hypersurfaces in hyperbolic space. *Geom. Dedicata* **119** (2006), 35–47. Zbl 1116.53008 MR 2247646

[30] López, R.: *Constant mean curvature surfaces with boundary*. Springer Monogr. Math., Springer, Heidelberg, 2013. Zbl 1278.53001 MR 3098467

[31] Nunes, I.: On stable constant mean curvature surfaces with free boundary. *Math. Z.* **287** (2017), no. 1-2, 473–479. Zbl 1375.53017 MR 3694683

[32] Palmer, B.: Stability of the Wulff shape. *Proc. Amer. Math. Soc.* **126** (1998), no. 12, 3661–3667. Zbl 0924.53009 MR 1473676

[33] Ros, A. and Souam, R.: On stability of capillary surfaces in a ball. *Pacific J. Math.* **178** (1997), no. 2, 345–361. Zbl 0930.53007 MR 1447419

[34] Ros, A. and Vergasta, E.: Stability for hypersurfaces of constant mean curvature with free boundary. *Geom. Dedicata* **56** (1995), no. 1, 19–33. Zbl 0912.53009 MR 1338315

[35] Souam, R.: On stable capillary hypersurfaces with planar boundaries. *J. Geom. Anal.* **33** (2023), no. 6, article no. 196, 10 pp. Zbl 1520.53052 MR 4576390

[36] Struwe, M.: The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.* **160** (1988), no. 1-2, 19–64. Zbl 0646.53005 MR 926524

[37] Wang, G., Weng, L. and Xia, C.: Alexandrov–Fenchel inequalities for convex hypersurfaces in the half-space with capillary boundary. *Math. Ann.* **388** (2024), no. 2, 2121–2154. Zbl 1543.53088 MR 4700391

[38] Wang, G. and Xia, C.: Rigidity of free boundary CMC hypersurfaces in a ball. *Surveys Geom. Anal.* **2** (2018), 138–153.

[39] Wang, G. and Xia, C.: Uniqueness of stable capillary hypersurfaces in a ball. *Math. Ann.* **374** (2019), no. 3-4, 1845–1882. Zbl 1444.53040 MR 3985125

[40] Weng, L. and Xia, C.: Alexandrov–Fenchel inequality for convex hypersurfaces with capillary boundary in a ball. *Trans. Amer. Math. Soc.* **375** (2022), no. 12, 8851–8883. Zbl 1512.53039 MR 4504655

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