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Bochner–Riesz means for the twisted Laplacian in $\mathbb{R}^2$

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Abstract. We study the Bochner–Riesz problem for the twisted Laplacian $\mathcal{L}$ on $\mathbb{R}^2$. For $p \in [1,\infty] \setminus \{2\}$, it has been conjectured that the Bochner–Riesz means $S_\lambda^\delta(\mathcal{L}) f$ of order δ converge in L^p for every $f \in L^p$ if and only if $\delta > \max(0, |(p-2)/p| - 1/2)$. We prove the conjecture by obtaining uniform L^p bounds on $S_\lambda^\delta(\mathcal{L})$ up to the sharp summability indices.

1. Introduction

The twisted Laplacian $\mathcal{L}$ on $\mathbb{R}^{2d}$ is a second order differential operator given by

$$\mathcal{L} = -\sum_{j=1}^{d} \Big(\Big(\frac{\partial}{\partial x_j} - \frac{1}{2} i y_j \Big)^2 + \Big(\frac{\partial}{\partial y_j} + \frac{1}{2} i x_j \Big)^2 \Big), \quad x, y \in \mathbb{R}^d.$$

The operator $\mathcal{L}$ is self-adjoint and it has a discrete spectrum, which is given by the set $2\mathbb{N}_0 + d := \{2k + d : k \in \mathbb{N}_0\}$. Here $\mathbb{N}_0$ denotes the set of all natural numbers including 0. For $\mu \in 2\mathbb{N}_0 + d$, let Π_μ denote the spectral projection operator to the eigenspace with the eigenvalue μ. One important property of the projection operators Π_μ is that they allow a spectral decomposition of L^2 ([19]). That is to say,

$$f = \sum_{\mu \in 2\mathbb{N}_0 + d} \Pi_\mu f, \quad \text{for all } f \in L^2(\mathbb{R}^{2d}).$$

Let $\delta \geq 0$ and $\lambda > 0$. By the spectral decomposition, the Bochner–Riesz mean $S_\lambda^\delta(\mathcal{L})$ for $\mathcal{L}$ is defined by

$$S_\lambda^\delta(\mathcal{L}) f = \sum_{\mu \in 2\mathbb{N}_0 + d} \Big(1 - \frac{\mu}{\lambda} \Big)_+^\delta \Pi_\mu f.$$

The problem known as the Bochner–Riesz problem is to determine the optimal summability index δ for $p \in [1,\infty]$ such that $S_\lambda^\delta(\mathcal{L}) f$ converges to f in L^p for every $f \in L^p$.

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Of course, this kind of problem was considered first for the Laplacian $-\Delta$ on $\mathbb{R}^n$, $n \geq 2$, and the problem has been extensively studied by numerous authors. It has been conjectured that the classical Bochner–Riesz mean $S_\lambda^\delta(-\Delta)f$ converges in $L^p(\mathbb{R}^n)$ if and only if

$$\delta > \delta_\circ(p,n) := \max\left(0, n\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}\right)$$

for $p \in [1,\infty] \setminus \{2\}$. (When $p = 2$, the convergence holds true if and only if $\delta \geq 0$ by Plancherel's theorem.) The conjecture was verified in two dimensions by Carleson and Sjölin [2]. However, in higher dimensions, it still remains open and partial results are known. For the readers who are interested in recent progress on the conjecture, we refer to [1,4–6,13,21] and references therein.

After this brief digression, we turn back to the Bochner–Riesz problem for $\mathcal{L}$. By the uniform boundedness principle, the problem is equivalent to characterizing $\delta = \delta(p)$ such that the estimate

$$\|S_\lambda^\delta(\mathcal{L})f\|_{L^p(\mathbb{R}^{2d})} \leq C\|f\|_{L^p(\mathbb{R}^{2d})} \tag{1.1}$$

holds with a constant C independent of λ and $f \in \mathcal{S}(\mathbb{R}^{2d})$. In analogue with the classical Bochner–Riesz problem, it is natural to conjecture that (1.1) holds if and only if $\delta > \delta_\circ(p, 2d)$ when $p \neq 2$. The necessity part follows by a transplantation theorem due to Kenig–Stanton–Tomas [11], and the necessary condition for L^p bound on the classical Bochner–Riesz operator $S_\lambda^\delta(-\Delta)$.

Concerning the sufficiency part, it has been proved by Thangavelu (see [19]) and by Ratnakumar–Rawat–Thangavelu (see [16]) that (1.1) holds if $\delta > \delta_\circ(p, 2d)$ on a certain range of p. The range of p was later extended by Stempak and Zienkiewicz [18] for $\max(p, p') > p_*(d) := 2(2d+1)/(2d-1)$. All those previous works rely on a common strategy due to Fefferman and Stein [3], which makes it possible to derive L^p bound on $S_\lambda^\delta(\mathcal{L})$ (up to the sharp exponent $\delta_\circ(p, 2d)$) from the L^2-L^p estimate for Π_μ:

$$\|\Pi_\mu f\|_{L^p(\mathbb{R}^{2d})} \leq C\,\mu^{d(1/p-1/2)-1/2}\,\|f\|_{L^2(\mathbb{R}^{2d})}. \tag{1.2}$$

The estimate (1.2) is optimal in that the exponent on μ cannot be improved. However, the same strategy does not work any longer if $\max(p, p') < p_*(d)$. Koch and Ricci [12], in fact, showed that the estimate (1.2) holds if and only if $p_*(d) \leq p \leq \infty$. (See also [9] for L^p-L^q bounds on Π_λ.)

Other methodologies than the aforementioned have not been exploited until recently in the context of L^p boundedness of $S_\lambda^\delta(\mathcal{L})$. The second and third named authors [15] studied the problem in a local setting where $L^p(\mathbb{R}^{2d})$ is replaced by $L^p(K)$ for a compact set $K \subset \mathbb{R}^{2d}$, and extended the previously known range for the local L^p bound ([20]) to $\max(p, p') > 2(3d+1)/(3d-1)$. Even though the results are local in their nature, they are more involved than the global bounds on the classical operator $S_\lambda^\delta(-\Delta)$. The local L^p bounds on $S_\lambda^\delta(\mathcal{L})$, in fact, imply the corresponding global bounds on $S_\lambda^\delta(-\Delta)$ (see [15,20]) by virtue of the transplantation theorem ([11]). Remarkably, in $\mathbb{R}^2$, the result in [15] gives the local L^p bounds on the optimal range of p, δ, that is to say, it verifies the Bochner–Riesz conjecture for $\mathcal{L}$ in a local setting. However, the conjecture without such a local assumption has remained open.

The objective of this article is to prove the Bochner–Riesz conjecture for $\mathcal{L}$ in $\mathbb{R}^2$ by obtaining global L^p boundedness of $S^\delta_\lambda(\mathcal{L})$. For the rest of the article, fixing $d = 1$, we denote $\delta_\circ(p) := \delta_\circ(p, 2)$.

Theorem 1.1. *Let $d = 1$ and $1 \le p \le \infty$. If $\delta > \delta_\circ(p)$, then the estimate* (1.1) *holds.*

For a given operator T, we denote the kernel of T by $T(z, z')$. To prove Theorem 1.1, we basically follow the strategy in [15], that is based on kernel expressions of the associated multiplier operators (for example, see (2.5) below). The local results in [15] were obtained by combining asymptotic expansion of the kernel $S^\delta_\lambda(\mathcal{L})(z, z')$ and estimates for the oscillatory integral operator satisfying *Carleson–Sjölin* and ellipticity conditions ([6, 14]). More precisely, it was shown that $K_\lambda(z, z') := S^\delta_\lambda(\mathcal{L})(\lambda^{1/2}z, \lambda^{1/2}z')$ gives rise to an oscillatory integral operator satisfying those conditions under the assumption that $|z - z'| < 2 - c$ for a constant $c > 0$. However, when (z, z') is near the set

$$\mathfrak{S} := \big\{(z, z') \in \mathbb{R}^2 \times \mathbb{R}^2 : |z - z'| = 2\big\},$$

the kernel K_λ exhibits a different behavior since the critical points of the phase function $\mathcal{P}(\cdot, z, z')$ (see (2.4)) are no longer nondegenerate if $(z, z') \in \mathfrak{S}$.

To deal with the matter concerning the degeneracy, we take an approach inspired by the authors' recent work [10]. We make a dyadic decomposition of the kernel away from the set $\mathfrak{S}$ such that the consequent kernels are supported in the regions $\{(z, z') : ||z - z'| - 2| \sim 2^{-j}\}$. Then, we further break the kernels along the angle of $(z - z')/|z - z'|$ so that each of the decomposed kernels is localized in a set where $z - z'$ is contained in a $2^{-j} \times 2^{-j/2}$ rectangle. Unexpectedly, it turns out that interactions between those angularly decomposed operators are not significant. After an appropriate change of variables, we observe that the operators given by those kernels are the oscillatory integral operators satisfying the *Carleson–Sjölin condition*. We combine this observation with the classical result due to Carleson–Sjölin [2] to obtain the sharp estimates.

Organization. In Section 2, we break down the proof of Theorem 1.1 to establishing Proposition 2.1, which contains the key L^4 estimate. The subsequent sections are devoted to proving Proposition 2.1. In Section 3 we further reduce the proof so that we only have to deal with the oscillatory integral operators with kernels supported near $\mathfrak{S}$ (Proposition 3.1). In Section 4, we complete the proof by proving Proposition 3.1 via angular decomposition and scaling.

Notations. For given non-negative quantities A and B, by $A \lesssim B$ we means that there exists a constant $C > 0$ such that $A \le CB$. We occasionally write $A \lesssim_\varepsilon B$ to indicate that the implicit constant depends on $\varepsilon > 0$. We write $A \sim B$ if $A \gtrsim B$ and $A \lesssim B$. For an operator T, $\|T\|_{p\to q}$ denotes the norm of T from L^p to L^q.

2. Reduction to a key L^4 estimate

In this section, we make several steps of reduction for the proof of Theorem 1.1 and single out its core part which is Proposition 2.1 below.

To prove Theorem 1.1, it is sufficient to show the estimate (1.1) only for $p = 4$ and $\delta > 0$. Indeed, the estimate (1.1) for two cases $p = 2$, $\delta \ge 0$ and $p = \infty$, $\delta > \delta(\infty)$ are well known ([19]). Interpolation with the desired L^4 estimate gives (1.1) for $2 \le p \le \infty$ and $\delta > \delta_\circ(p)$. The case $1 \le p < 2$ follows by duality.

2.1. Dyadic decomposition

Let $\psi \in C_c^\infty([1/4, 1])$ be such that $\sum_{\ell\in\mathbb{Z}} \psi(2^\ell t) = 1$ for $t > 0$. For $\delta > 0$ and $\ell \ge 1$, set

$$\psi_\ell^\delta(t) = (2^{-\ell} t)^\delta \, \psi(2^{-\ell} t) \quad \text{and} \quad \psi_0^\delta(t) = t_+^\delta \sum_{\ell\ge 0} \psi(2^\ell t),$$

so that

$$t_+^\delta = \sum_{1\le 2^\ell \le 4\lambda} 2^{\delta\ell}\, \psi_\ell^\delta(t)$$

if $0 < t \le \lambda$. Since $S_\lambda^\delta(\mathcal{L}) = \lambda^{-\delta}(\lambda - \mathcal{L})_+^\delta$, we have

$$S_\lambda^\delta(\mathcal{L}) = \lambda^{-\delta} \sum_{1\le 2^\ell \le 4\lambda} 2^{\delta\ell}\, \psi_\ell^\delta(\lambda - \mathcal{L}).$$

Therefore, for the estimate (1.1) for $p = 4$ and $\delta > 0$, it is sufficient to show

$$\|\psi_\ell^\delta(\lambda - \mathcal{L})\|_{4\to 4} \lesssim_\varepsilon (\lambda 2^{-\ell})^\varepsilon, \quad \forall \varepsilon > 0. \tag{2.1}$$

By the Fourier inversion, we note

$$\psi_\ell^\delta(\lambda - \mathcal{L}) = \frac{1}{2\pi} \int \widehat{\psi}_\ell^\delta(t)\, e^{it(\lambda-\mathcal{L})}\, dt. \tag{2.2}$$

The kernel of the propagator $e^{-it\mathcal{L}}$ is given by

$$e^{-it\mathcal{L}}(z, z') = c(\sin t)^{-1}\, e^{i(\mathcal{P}(t,z,z')-t)}, \quad z, z' \in \mathbb{R}^2, \tag{2.3}$$

(see [9, 19]), where c is a complex number and

$$\mathcal{P}(t, z, z') := t + \frac{|z - z'|^2 \cos t}{4 \sin t} + \frac{z_2 z_1' - z_1 z_2'}{2}. \tag{2.4}$$

For $\eta \in C^\infty(\mathbb{R})$, let $[\eta]^\lambda$ be the operator whose kernel is given by

$$[\eta]^\lambda(z, z') = \int \eta(t)\, (\sin t)^{-1}\, e^{i\lambda\mathcal{P}(t,z,z')}\, dt. \tag{2.5}$$

From (2.2) and (2.3), note that

$$\psi_\ell^\delta(\lambda - \mathcal{L})(\lambda^{1/2} z, \lambda^{1/2} z') = c(2\pi)^{-1}\, [\widehat{\psi}_\ell^\delta]^\lambda(z, z').$$

By scaling, the estimate (2.1) is equivalent to

$$\|[\widehat{\psi}_\ell^\delta]^\lambda\|_{4\to 4} \lesssim_\varepsilon \lambda^{-1}(\lambda 2^{-\ell})^\varepsilon, \quad \forall \varepsilon > 0. \tag{2.6}$$

We reduce the proof of (2.6) to those of the following two propositions.

Proposition 2.1. *Let $0 < \varrho < \pi/2$. Suppose that η is a compactly supported smooth function such that $\mathrm{supp}(\eta) \subset [\varrho, \pi - \varrho]$. Then, for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, \varrho, \|\eta\|_{C^{100}(\mathbb{R})})$ such that*

$$\|[\eta]^\lambda\|_{4\to4} \le C\lambda^{-1+\varepsilon}.$$

Proposition 2.2. *Let $1 \le 2^\ell \le 4\lambda$ and $\eta \in C_c^\infty((-2^{-5}, 2^{-5}))$. Then, for $n \in \mathbb{Z}$ and $\varepsilon > 0$, we have*

$$\|[\eta\widehat{\psi}_\ell^\delta(\cdot - n\pi)]^\lambda\|_{4\to4} \lesssim_\varepsilon \lambda^{-1}(\lambda 2^{-\ell})^\varepsilon (1 + 2^\ell|n|)^{-9}. \tag{2.7}$$

Proposition 2.1 is the main new contribution of this work, which we prove in the next section, while Proposition 2.2 is a consequence of the local result in [15] (see Proposition 2.3 below). Assuming Proposition 2.1 and 2.2 for the moment, we prove (2.6).

Proof of (2.6). We choose $\eta_0 \in C_c^\infty((-2^{-5}, 2^{-5}))$ and $\eta_1 \in C_c^\infty((2^{-6}, \pi - 2^{-6}))$ such that both η_0 and $\eta_1(\cdot + \pi/2)$ are symmetric with respect to $t = 0$, and

$$\eta_0(t) + \eta_1(t) + \eta_0(t - \pi) = 1 \quad \text{for } t \in [0, \pi].$$

These functions allow us to decompose

$$[\widehat{\psi}_\ell^\delta]^\lambda = \sum_{n\in\mathbb{Z}} \big([\eta_0(\cdot + n\pi)\widehat{\psi}_\ell^\delta]^\lambda + [\eta_1(\cdot + n\pi)\widehat{\psi}_\ell^\delta]^\lambda\big). \tag{2.8}$$

Changing variables $t \to t - n\pi$ gives

$$[\eta_\kappa(\cdot + n\pi)\widehat{\psi}_\ell^\delta]^\lambda = c[\eta_\kappa\widehat{\psi}_\ell^\delta(\cdot - n\pi)]^\lambda, \quad \kappa = 0, 1,$$

with $|c| = 1$. By Proposition 2.2, we have

$$\|[\eta_0(\cdot + n\pi)\widehat{\psi}_\ell^\delta]^\lambda\|_{4\to4} \lesssim_\varepsilon \lambda^{-1}(\lambda 2^{-\ell})^\varepsilon (1 + 2^\ell|n|)^{-9}. \tag{2.9}$$

Concerning $[\eta_1\widehat{\psi}_\ell^\delta(\cdot - n\pi)]^\lambda$, we note that

$$\mathrm{supp}(\eta_1\widehat{\psi}_\ell^\delta(\cdot - n\pi)) \subset [\varrho, \pi - \varrho],$$

with $\varrho = 2^{-6}$. Since

$$\Big|\Big(\frac{d}{dt}\Big)^k \widehat{\psi}_\ell^\delta(t)\Big| \lesssim 2^{\ell(1+k)}(1 + 2^\ell|t|)^{-M} \quad \text{for any } M \text{ and } k \ge 0,$$

we have $\|\eta_1\widehat{\psi}_\ell^\delta(\cdot - n\pi)\|_{C^{100}(\mathbb{R})} \le C_0 B$ with a constant C_0, where $B = 2^{-9\ell}(\varrho + |n|)^{-9}$. Applying Proposition 2.1 to

$$\eta = B^{-1}\eta_1\widehat{\psi}_\ell^\delta(\cdot - n\pi),$$

we obtain

$$\big\|[\eta_1(\cdot + n\pi)\widehat{\psi}_\ell^\delta]^\lambda\big\|_{4\to4} \lesssim_{\varepsilon,\varrho,C_0} \lambda^{-1+\varepsilon} 2^{-9\ell}(\varrho + |n|)^{-9}$$

for any $\varepsilon > 0$. By (2.8) and the triangle inequality, using this and the estimate (2.9), we get (2.6). ∎

2.2. Proof of Proposition 2.2

We use the local estimates for $[\eta]^\lambda$ with a cut-off function η supported near the origin, which were obtained in [15].

Proposition 2.3 (Theorem 3.3 in [15]). *Let $0<\rho<\pi-2^{-5}$ and $0<c_0<2$, and let $\eta_\rho\in C_c^\infty([2^{-2}\rho,\rho]\cup[-\rho,-2^{-2}\rho])$ satisfy $|(\frac{d}{dt})^m\eta_\rho|\lesssim\rho^{-m}$ for $0\le m\le 100$. Suppose $E,F\subset\mathbb{R}^2$ are compact sets such that $|z-z'|\le 2-c_0$ for all $(z,z')\in E\times F$. Then for $p>4$, we have*

$$\|\chi_E[\eta_\rho]^\lambda\chi_F\|_{p\to p}\lesssim\lambda^{-1}\rho\max\{1,(\lambda\rho)^{\delta_\circ(p)}\}\tag{2.10}$$

Although the proposition does not include the case $p=4$, interpolation with an easy L^2 estimate yields

$$\|\chi_E[\eta_\rho]^\lambda\chi_F\|_{4\to4}\lesssim_\varepsilon\lambda^{-1}\rho\max\{1,(\lambda\rho)^{\varepsilon}\}\tag{2.11}$$

for $\varepsilon>0$. In fact, we note that $\|[\eta_\rho]^\lambda\|_{2\to2}=\lambda^{-1}\|\int\eta_\rho(t)e^{it(\lambda-\mathcal{L})}dt\|_{2\to2}$. Thus, it follows that $\|[\eta_\rho]^\lambda\|_{2\to2}\le\lambda^{-1}\|\eta_\rho\|_1\lesssim\lambda^{-1}\rho$. Therefore, $\|\chi_E[\eta_\rho]^\lambda\chi_F\|_{2\to2}\lesssim\lambda^{-1}\rho$. Thus, interpolation with (2.10) gives (2.11) (taking p arbitrarily close to 4 when $\rho>\lambda^{-1}$).

We are now ready to prove Proposition 2.2. Let $\eta\in C_c^\infty((-2^{-5},2^{-5}))$ and let $\mathfrak{Q}=\{Q\}$ be a tiling of $\mathbb{R}^2$ such that $Q\in\mathfrak{Q}$ is a square of side length $1/2$. We say $Q\sim Q'$ if $\mathrm{dist}(Q,Q')=0$, and $Q\nsim Q'$ otherwise. Thus, we have

$$[\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)]^\lambda=\mathcal{I}_1+\mathcal{I}_2,$$

where

$$\mathcal{I}_1=\sum_{Q\sim Q'}\chi_Q[\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)]^\lambda\chi_{Q'}\quad\text{and}\quad\mathcal{I}_2=\sum_{Q\nsim Q'}\chi_Q[\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)]^\lambda\chi_{Q'}.$$

The desired estimate (2.7) follows if we show

$$\|\mathcal{I}_1\|_{4\to4}\lesssim\lambda^{-1}(\lambda2^{-\ell})^\varepsilon(1+2^\ell|n|)^{-9},\tag{2.12}$$

$$\|\mathcal{I}_2\|_{4\to4}\lesssim\lambda^{-N}(1+2^\ell|n|)^{-9}.\tag{2.13}$$

To show (2.12) and (2.13), we make an additional decomposition. For $j\ge3$, we define

$$\varphi_j(t):=\psi(2^jt)\quad\text{and}\quad\widetilde{\varphi}_j(t):=\psi(2^j|t|).$$

Then we write

$$[\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)]^\lambda=\sum_{j\ge3}[\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)\widetilde{\varphi}_j]^\lambda.$$

By definition, we have that for $m\in\mathbb{N}_0$ and any M,

$$\left|\left(\frac{d}{dt}\right)^m\widehat{\psi}_\ell^\delta(t)\right|\lesssim2^{(m+1)\ell}(1+2^\ell|t|)^{-M}\quad\text{and}\quad\left|\left(\frac{d}{dt}\right)^m\widetilde{\varphi}_j(t)\right|\lesssim2^{mj}.$$

Using these bounds together with the chain rule gives

$$|(\tfrac{d}{dt})^m(\eta\widehat{\psi}_\ell^\delta(\cdot-n\pi)\widetilde{\varphi}_j)|\lesssim\begin{cases}2^\ell2^{\max(\ell,j)m}(2^\ell|n|)^{-M}, & n\ne0,\\ 2^\ell2^{\max(\ell,j)m}(1+2^{\ell-j})^{-M}, & n=0.\end{cases}\tag{2.14}$$

Note that $|z-z'| \le \sqrt{2}$ if $(z,z') \in Q \times Q'$ and $Q \sim Q'$. Now, we define

$$\eta_{n,j,\ell} = \begin{cases} 2^{-\ell}(1+2^\ell|n|)^9\, 2^{100(\ell-j)}\, \eta \widehat{\psi}^\delta_\ell(\cdot - n\pi)\widetilde{\varphi}_j, & \ell > j, \\ 2^{-\ell}(1+2^\ell|n|)^9\, \eta \widehat{\psi}^\delta_\ell(\cdot - n\pi)\widetilde{\varphi}_j, & \ell \le j, \end{cases}$$

for $j,n \in \mathbb{Z}$, and $\ell \in \mathbb{N}_0$ such that $j \ge 3$, $1 \le 2^\ell \le 4\lambda$. Using (2.14) with M chosen so that $M \ge 200$, we have $|(\frac{d}{dt})^m \eta_{n,j,\ell}| \lesssim 2^{jm}$ for all n, j, ℓ, and $0 \le m \le 100$. Moreover, $\operatorname{supp}(\eta_{n,j,\ell}) \subset [2^{-2}\rho, \rho] \cup [-\rho, -2^{-2}\rho]$ with $\rho = 2^{-j}$. Thus, applying the estimate (2.11) to $\eta_{n,j,\ell}$ and considering the cases $\lambda \le 2^j$, $2^\ell \le 2^j < \lambda$, and $2^j \le 2^\ell$ separately, we obtain

$$\|\chi_Q [\eta\widehat{\psi}^\delta_\ell(\cdot - n\pi)\widetilde{\varphi}_j]^\lambda \chi_{Q'}\|_{4\to 4} \lesssim \frac{2^{\ell-j}\lambda^{-1}}{(1+2^\ell|n|)^9} \begin{cases} 1, & \lambda \le 2^j, \\ (\lambda 2^{-j})^\varepsilon, & 2^\ell \le 2^j < \lambda, \\ (\lambda 2^{-j})^\varepsilon\, 2^{100(j-\ell)}, & 2^j \le 2^\ell, \end{cases}$$

provided that $Q \sim Q'$. Summation over j yields

$$\|\chi_Q [\eta\widehat{\psi}^\delta_\ell(\cdot - n\pi)]^\lambda \chi_{Q'}\|_{4\to 4} \le C\lambda^{-1}(\lambda 2^{-\ell})^\varepsilon (1+2^\ell|n|)^{-9}$$

with a constant C, independent of λ, ℓ, n, and Q, Q' whenever $Q \sim Q'$. Hence, this gives (2.12) because for each $Q \in \mathfrak{Q}$ there are only nine $Q' \in \mathfrak{Q}$ such that $Q' \sim Q$.

Now we consider (2.13). Recall (2.5). The kernel $[\eta\widehat{\psi}^\delta_\ell(\cdot - n\pi)\widetilde{\varphi}_j]^\lambda(z,z')$ is expressed as an oscillatory integral with the phase $\mathcal{P}(t,z,z')$. Note that

$$\partial_t \mathcal{P}(t,z,z') = 1 - \frac{|z-z'|^2}{4\sin^2 t}, \tag{2.15}$$

and $|z-z'| \ge 1/2$ for $(z,z') \in Q \times Q'$ if $Q \nsim Q'$. Thus, we have

$$|\partial_t \mathcal{P}(t,z,z')| \sim 2^{2j}|z-z'|^2, \quad (t,z,z') \in \operatorname{supp}(\eta\widetilde{\varphi}_j) \times Q \times Q'$$

if $Q \nsim Q'$. Moreover,

$$|\partial_t^{m+1}\mathcal{P}(t,z,z')| \lesssim 2^{j(m+2)}|z-z'|^2 \quad \text{for any } m \in \mathbb{N}_0.$$

Now, let $L_{\mathcal{P}}$ be a differential operator defined by

$$L_{\mathcal{P}} F(t) = \frac{\partial}{\partial t}\Big(\frac{F(t)}{-i\,\partial_t \mathcal{P}_\circ(t,z,z')}\Big),$$

where $\mathcal{P}_\circ = 2^{-2j}|z-z'|^{-2}\mathcal{P}$. Denoting the adjoint of $L_{\mathcal{P}}$ by $L^*_{\mathcal{P}}$, we note that

$$L^*_{\mathcal{P}}(e^{i\lambda\mathcal{P}(t,z,z')}) = \lambda_\circ e^{i\lambda\mathcal{P}(t,z,z')},$$

with $\lambda_\circ = \lambda 2^{2j}|z-z'|^2$. Thus, repeated integration by parts based on $L_{\mathcal{P}}$ yields the expression

$$[\eta\widehat{\psi}^\delta_\ell(\cdot - n\pi)\widetilde{\varphi}_j]^\lambda(z,z') = c\lambda_\circ^{-N}\mathbf{B}_{n,j,\ell}\int L^N_{\mathcal{P}}\big((\sin t)^{-1}\eta_{n,j,\ell}(t)\big)e^{i\lambda_\circ\mathcal{P}_\circ(t,z,z')}\,dt, \tag{2.16}$$

where $c \in \mathbb{C}$, and

$$\mathbf{B}_{n,j,\ell} = \begin{cases} 2^{\ell}(1+2^{\ell}|n|)^{-9}\, 2^{100(j-\ell)}, & \ell > j, \\ 2^{\ell}(1+2^{\ell}|n|)^{-9}, & \ell \le j. \end{cases}$$

From the above inequalities for $\partial_t \mathcal{P}$ and its derivatives, it immediately follows

$$|\partial_t \mathcal{P}_\circ(t,z,z')| \sim 1 \quad \text{and} \quad |\partial_t^{m+1} \mathcal{P}_\circ(t,z,z')| \lesssim 1$$

for $(t,z,z') \in \operatorname{supp}(\eta \widetilde{\varphi}_j) \times Q \times Q'$, if $Q \nsim Q'$.

Combining this with $|(\frac{d}{dt})^m \eta_{n,j,\ell}| \lesssim 2^{jm}$ in the above, we obtain

$$|L_{\mathcal{P}}^N((\sin t)^{-1}\eta_{n,j,\ell}(t))| \lesssim 2^{(N+1)j}.$$

We also use that $\operatorname{supp}(\eta_{n,j,\ell}) \subset [2^{-j-2}, 2^{-j}] \cup [-2^{-j}, -2^{-j-2}]$. Hence, the right-hand side of (2.16) is bounded by

$$K_j(z,z') := \frac{C 2^{\ell}}{(1+2^{\ell}|n|)^9} \begin{cases} 2^{100(j-\ell)} (\lambda 2^j |z-z'|^2)^{-N}, & \ell > j, \\ (\lambda 2^j |z-z'|^2)^{-N}, & \ell \le j, \end{cases}$$

for any N and M if $(z,z') \in Q \times Q'$ and $Q \nsim Q'$. Thus, applying Young's inequality, we get

$$\Big\| \sum_{Q \nsim Q'} \chi_Q [\eta \widehat{\psi}_\ell^\delta(\cdot - n\pi) \widetilde{\varphi}_j]^\lambda \chi_{Q'} \Big\|_{4\to 4} \lesssim \frac{2^{\ell}}{(1+2^{\ell}|n|)^9} \begin{cases} 2^{100(j-\ell)} (\lambda 2^j)^{-N}, & \ell > j, \\ (\lambda 2^j)^{-N}, & \ell \le j. \end{cases}$$

for any N and M. Here, we use the fact that $|z-z'| \ge 1/2$ if $(z,z') \in Q \times Q'$ and $Q \nsim Q'$. Hence, (2.13) follows by the triangle inequality and summation over j.

3. Dyadic decomposition away from $\mathfrak{S}$

In this section, we prove the key L^4 estimate in Proposition 2.1. Throughout this section, we assume that $\eta \in C_c^\infty(\mathbb{R})$ satisfying $\operatorname{supp}(\eta) \in [\varrho, \pi - \varrho]$ with $0 < \varrho < \pi/2$. Since ϱ will be fixed throughout the rest of the article, we omit its dependence for simplicity. The first step of the proof is to decompose the kernel of $[\eta]^\lambda$ near the set $\mathfrak{S}$ dyadically.

Recall $\varphi_j = \psi(2^j \cdot)$. For $j \in \mathbb{Z}$ satisfying $0 \le j \le j_0 := [\log \lambda^{2/3}]$, we define

$$\chi_j(z,z') = \varphi_{j-2}(2 - |z-z'|),$$
$$\chi^\circ(z,z') = \sum_{j > j_0} \widetilde{\varphi}_{j-2}(2 - |z-z'|).$$

Thus, $\chi^\circ(z,z') + \sum_{0 \le j \le j_0} \chi_j(z,z') = 1$ if $|z-z'| \le 2$. We also set

$$\chi^e(z,z') := 1 - \Big(\sum_{0 \le j \le j_0} \chi_j(z,z') + \chi^\circ(z,z') \Big).$$

Consequently, $\chi^\circ + \chi^e + \sum_{0\le j\le j_0} \chi_j = 1$ on $\mathbb{R}^4$. Thus,

$$[\eta]^\lambda = \sum_{0\le j\le j_0} [\eta]^\lambda_j + [\eta]^{\lambda,\circ} + [\eta]^{\lambda,e}, \tag{3.1}$$

where $[\eta]^\lambda_j$, $[\eta]^{\lambda,\circ}$, and $[\eta]^{\lambda,e}$ are the operators whose kernels are given by

$$[\eta]^\lambda_j(z,z') = [\eta]^\lambda(z,z')\chi_j(z,z') \quad \text{and} \quad [\eta]^{\lambda,\kappa}(z,z') = [\eta]^\lambda(z,z')\chi^\kappa(z,z'),\ \ \kappa\in\{\circ,e\}.$$

As will be seen later, the operators $[\eta]^{\lambda,\circ}$ and $[\eta]^{\lambda,e}$ are much easier to handle.

We first prove $\| \sum_{0\le j\le j_0} [\eta]^\lambda_j \|_{4\to4} \lesssim_\varepsilon \lambda^{-1+\varepsilon}$, while the bounds on the other operators are to be shown near the end of this section. Since $j_0 \lesssim \log\lambda$, it suffices to show, for $0\le j\le j_0$,

$$\| [\eta]^\lambda_j \|_{4\to4} \lesssim_\varepsilon \lambda^{-1+\varepsilon}. \tag{3.2}$$

3.1. Estimate for $[\eta]^\lambda_j$

When $j < C$ for a constant C, the desired estimate (3.2) is easy to show by using Proposition 2.3. Indeed, we decompose

$$[\eta]^\lambda_j = \sum_{-2\le k} [\eta\varphi_k]^\lambda_j + \sum_{-2\le k} [\eta\varphi_k(\pi-\cdot)]^\lambda_j + [\eta\varphi_\circ]^\lambda_j,$$

where

$$\varphi_\circ = 1 - \sum_{-2\le k} \big(\varphi_k + \varphi_k(\pi-\cdot)\big).$$

Since $j < C$, $|z-z'| \le 2 - c_0$ for a constant $c_0 > 0$ if $(z,z')\in \operatorname{supp}\chi_j$. To proceed, we claim that

$$\| [\eta\varphi_k]^\lambda_j \|_{4\to4} \lesssim \sup_{B,B'} \| \chi_B\, [\eta\varphi_k]^\lambda_j\, \chi_{B'} \|_{4\to4},$$

where the supremum is taken over the balls B and B' of radius $c_0/4$ satisfying that $\operatorname{dist}(B,B') \le 2 - c_0/2$. This is not hard to prove. For $\mathfrak{n} = (\mathfrak{n}_1,\mathfrak{n}_2)\in\mathbb{Z}^2$, let $B_\mathfrak{n}$ be a ball of radius $c_0/4$ centered at $(c_0\mathfrak{n}_1/4, c_0\mathfrak{n}_2/4)$. Clearly, $\mathcal{B} = (B_\mathfrak{n})$ is a cover of $\mathbb{R}^2$ with each ball having at most 4 overlap. For each $\mathfrak{n}\in\mathbb{Z}^2$, let

$$\widetilde{\mathcal{B}}_\mathfrak{n} = \{B'\in\mathcal{B} : \operatorname{dist}(B_\mathfrak{n}, B') \le 2 - c_0/2\}.$$

Note that $|\widetilde{\mathcal{B}}_\mathfrak{n}| \le C$ with a constant $C = C(c_0) > 0$ for all $\mathfrak{n}\in\mathbb{Z}^2$. By the above discussion, $\chi_{B_\mathfrak{n}}[\eta\varphi_k]^\lambda_j\chi_{B'} \equiv 0$ if $B'\notin \mathcal{B}_\mathfrak{n}$. Thus,

$$\begin{aligned}
\| [\eta\varphi_k]^\lambda_j f\|^4_{L^4} &\le \Big\| \Big(\sum_{\mathfrak{n}\in\mathbb{Z}^2} \chi_{B_i}\Big) [\eta\varphi_k]^\lambda_j f\Big\|^4_{L^4} \le C \sum_{\mathfrak{n}\in\mathbb{Z}^2} \Big\| \chi_{B_i} [\eta\varphi_k]^\lambda_j \Big(\sum_{B'\in\mathcal{B}_\mathfrak{n}} \chi_{B'} f\Big)\Big\|^4_{L^4} \\
&\le C \sup_{B,B'\in\mathcal{B}} \|\chi_B [\eta\varphi_k]^\lambda_j \chi_{B'}\|^4_{4\to4} \sum_{\mathfrak{n}\in\mathbb{Z}^2}\sum_{B'\in\mathcal{B}_\mathfrak{n}} \|\chi_{B'} f\|^4_{L^4} \\
&\le C \sup_{B,B'\in\mathcal{B}} \|\chi_B [\eta\varphi_k]^\lambda_j \chi_{B'}\|^4_{4\to4} \|f\|^4_{L^4}
\end{aligned}$$

for a constant $C > 0$. We use that the balls $B_\mathfrak{n}$ have finite overlap. This proves the claim.

Thus, by (2.11) we have $\|[\eta\varphi_k]_j^\lambda\|_{4\to 4}\lesssim_\varepsilon \lambda^{-1+\varepsilon}$. Moreover, $[\eta\varphi_k]_j^\lambda\equiv 0$ if $2^k\ge \varrho^{-1}$. Therefore, we obtain

$$\Big\|\sum_{-2\le k}[\eta\varphi_k]_j^\lambda\Big\|_{4\to 4}\lesssim_\varepsilon \lambda^{-1+\varepsilon}. \tag{3.3}$$

The same argument shows $\|[\eta\varphi_\circ]_j^\lambda\|_{4\to 4}\lesssim_\varepsilon \lambda^{-1+\varepsilon}$. To handle $[\eta\varphi_k(\pi-\cdot)]_j^\lambda$, we use a symmetric property. Considering

$$\mathbf{L}z := 2^{-1/2}(z_1+z_2, z_1-z_2),$$

we observe that

$$\mathcal{P}(\pi-t,\mathbf{L}z,\mathbf{L}z')=\pi-\mathcal{P}(t,z,z').$$

Recalling (2.5) and changing variables $t\to\pi-t$, we see

$$[\eta\varphi_k(\pi-\cdot)]_j^\lambda(\mathbf{L}z,\mathbf{L}z')=C\,\overline{[\eta(\pi-\cdot)\varphi_k]_j^\lambda(z,z')}$$

for a constant C with $|C|=1$. Thus,

$$\|[\eta\varphi_k(\pi-\cdot))]_j^\lambda\|_{4\to 4}=\|[\eta(\pi-\cdot)\varphi_k]_j^\lambda\|_{4\to 4}.$$

Repeating the previous argument used for (3.3), we see $\|[\eta\varphi_k(\pi-\cdot)]_j^\lambda\|_{4\to 4}\lesssim \lambda^{-1+\varepsilon}$. Combining all the estimates, we get the bound (3.2) for $j<C$.

Therefore, it is reduced to proving (3.2) for $j\ge C$ with a large constant C. For the rest of this subsection, we assume $j\ge C$.

Further decomposition of the kernel. Note that $\sin\varrho\le \sin t\le 1$ for $t\in \operatorname{supp}\eta$. Thus, from (2.15) we have

$$|\partial_t\mathcal{P}(t,z,z')|\sim |(2-|z-z'|)(2+|z-z'|)-4\cos^2 t| \tag{3.4}$$

for $(t,z,z')\in \operatorname{supp}\eta\times\operatorname{supp}\chi_j$. Thus, we are naturally led to decompose dyadically (in t) away from $\pi/2$.

Let C_0 be a constant large enough. Recalling $\widetilde\varphi_l=\psi(2^l|\cdot|)$ and $(2-|z-z'|)\sim 2^{-l}$, we decompose

$$[\eta]_j^\lambda=[\eta]_{j,0}^\lambda+[\eta]_{j,1}^\lambda$$

where

$$[\eta]_{j,0}^\lambda=\sum_{|2l-j|\le C_0}[\eta\widetilde\varphi_l(\pi/2-\cdot)]_j^\lambda \quad\text{and}\quad [\eta]_{j,1}^\lambda=\sum_{|2l-j|>C_0}[\eta\widetilde\varphi_l(\pi/2-\cdot)]_j^\lambda.$$

Estimate for $[\eta]_{j,1}^\lambda$. The estimate $\|[\eta]_{j,1}^\lambda\|_{4\to 4}\lesssim_\varepsilon \lambda^{-1+\varepsilon}$ is easy to obtain. Indeed, we show this by estimating for the kernel of $[\eta\widetilde\varphi_l(\pi/2-\cdot)]_j^\lambda$. Note that

$$[\eta\widetilde\varphi_l(\pi/2-\cdot)]^\lambda(z,z')=\int \mathbf{A}(t)\,e^{i\lambda\mathcal{P}(t,z,z')}\,dt,$$

where

$$\mathbf{A}(t) = C\eta(t)\,\widetilde{\varphi}_l(\pi/2 - t)(\sin t)^{-1}.$$

Since $|2l - j| > C_0$ for a large C_0, by (3.4) we get

$$|\lambda \partial_t \mathcal{P}(t, z, z')| \gtrsim \lambda \max(2^{-j}, 2^{-2l})$$

for $t \in \operatorname{supp} \mathbf{A}$ and $(z, z') \in \operatorname{supp} \chi_j$. It is clear that $|(d/dt)^n \mathbf{A}(t)| \lesssim 2^{nl}$ for any $n \in \mathbb{N}_0$. Thus, repeated integration by parts yields

$$|[\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]_j^\lambda(z, z')| \lesssim b_l := 2^{-l}\,(1 + \lambda\, 2^{-l} \max(2^{-j}, 2^{-2l}))^{-N}$$

for any $N \in \mathbb{N}$. Consequently, we see

$$\sup_z \left\| [\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]_j^\lambda(z, \cdot) \right\|_1, \sup_{z'} \left\| [\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]_j^\lambda(\cdot, z') \right\|_1 \lesssim 2^{-j} b_l.$$

Young's inequality and the triangle inequality give

$$\| [\eta]_{j,1}^\lambda \|_{4\to 4} \lesssim \sum_{|2l-j|>C_0} 2^{-j} b_l \lesssim \lambda^{-1+\varepsilon}.$$

Estimate for $[\eta]_{j,0}^\lambda$. To complete the proof of (3.2), it remains to show

$$\| [\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]_j^\lambda \|_{4\to 4} \lesssim \lambda^{-1+\varepsilon} \quad \text{for } 1 \ll 2^j \le \lambda^{2/3} \text{ and } 2^l \sim 2^{j/2}.$$

Moreover, as before, we note

$$\| [\eta\varphi_l(\pi/2 - \cdot)]_j^\lambda \|_{4\to 4} = \| [\eta\varphi_l(\cdot - \pi/2)]_j^\lambda \|_{4\to 4}$$

from the symmetric property of the kernel. Therefore, the matter is reduced to showing

$$\| [\eta\varphi_l(\pi/2 - \cdot)]_j^\lambda \|_{4\to 4} \lesssim_\varepsilon \lambda^{-1+\varepsilon} \tag{3.5}$$

when $1 \ll 2^j \le \lambda^{2/3}$ and $2^l \sim 2^{j/2}$. For the purpose, we now consider the stationary point $S_c(z, z') \in (0, \pi/2)$ of the phase function $t \to \mathcal{P}(t, z, z')$, which is given by

$$\sin S_c(z, z') = \frac{|z - z'|}{2}. \tag{3.6}$$

Note that $\sin(\pi/2) - \sin S_c(z, z') \sim 2^{-j}$ if $(z, z') \in \operatorname{supp} \chi_j$. Thus, we have $S_c(z, z') \in [\pi/2 - c_2 2^{-j/2}, \pi/2 - c_1 2^{-j/2}]$ for some $c_1 < c_2$ if $(z, z') \in \operatorname{supp} \chi_j$.

To prove (3.5), we make further decomposition of the kernel $[\eta\varphi_l(\pi/2 - \cdot)]_j^\lambda$ so that $S_c(z, z')$ lies within an interval of length $\ll 2^{-j/2}$ and the integral for the associated kernel (for example, (2.5)) is also taken over a small interval of length $\ll 2^{-j/2}$. This can be easily achieved by finite decomposition and discarding some part of the operator which has an acceptable L^4 bound.

Let $\varepsilon_0 > 0$ be a sufficiently small constant, which we choose later. Recall $\chi_j(z, z') = \psi(2^j(2 - |z - z'|))$. Breaking $\psi(2^j \cdot)$ into smooth functions supported in finitely overlapping intervals of length $c\,\varepsilon_0 2^{1-j}$ with a small constant $c > 0$, we write $\chi_j = \sum \tilde{\chi}$, where

$$\tilde{\chi}(z, z') = \psi(2^j(2 - |z - z'|))\, \theta\Big(\frac{a - |z - z'|}{c\varepsilon_0\, 2^{-j}}\Big) \tag{3.7}$$

for some a satisfying $2-a \in (2^{-2-j}, 2^{-j})$, and $\theta \in C_c^\infty((-1,1))$. Consequently, taking c small enough, we have

$$S_c(z,z') \in J(t_0, \varepsilon_0 2^{-j/2}] := [t_0 - \varepsilon_0 2^{-j/2}, t_0 + \varepsilon_0 2^{-j/2}] \tag{3.8}$$

for some t_0 with $\pi/2 - t_0 \sim 2^{-j/2}$ if $(z,z') \in \operatorname{supp} \tilde\chi$. Let $\rho \in C_c^\infty((-2,2))$ such that $\rho = 1$ on the interval $[-1, 1]$. Set

$$\rho_0(t) = \rho(2^{j/2}(t-t_0)/\varepsilon_0).$$

Write

$$[\eta\varphi_l(\pi/2-\cdot)]^\lambda \tilde\chi = [\rho_0\,\eta\varphi_l(\pi/2-\cdot)]^\lambda \tilde\chi + [(1-\rho_0)\,\eta\varphi_l(\pi/2-\cdot)]^\lambda \tilde\chi.$$

Here, as before, $[\eta\varphi_l(\pi/2-\cdot)]^\lambda\tilde\chi$ denotes the operator whose kernel is given by a product of the kernel $[\eta\varphi_l(\pi/2-\cdot)]^\lambda$ and the function $\tilde\chi$. The other operators are also defined in the same manner. The operator $[(1-\rho_0)\eta\varphi_l(\pi/2-\cdot)]^\lambda\tilde\chi$ can be easily handled. Indeed, note from (3.4) that

$$|\partial_t \mathcal{P}(t,z,z')| \sim |\sin S_c(z,z') - \sin t| \sim_{\varepsilon_0} 2^{-j} \quad \text{if } (t,z,z') \in \operatorname{supp}(1-\rho_0) \times \operatorname{supp} \chi.$$

Since $2^{2l} \sim 2^j$, by the same argument as before, we have

$$|[(1-\rho_0)\eta\varphi_l(\pi/2-\cdot)]^\lambda\tilde\chi| \lesssim 2^{-j/2}(\lambda^{-1}2^{3j/2})^N.$$

Young's inequality yields

$$\|[(1-\rho_0)\eta\varphi_l(\pi/2-\cdot)]^\lambda\tilde\chi\|_{4\to4} \lesssim \lambda^{-1}.$$

Therefore, since there are only as many as $O(1/\varepsilon_0)$ of these $\tilde\chi$, the desired estimate follows if we show

$$\|[\rho_0\,\eta\varphi_l(\pi/2-\cdot)]^\lambda\tilde\chi\|_{4\to4} \lesssim_\varepsilon \lambda^{-1+\varepsilon}.$$

More generally, we prove

$$\|[\eta]^\lambda\tilde\chi\|_{4\to4} \lesssim \lambda^{-1+\varepsilon} \tag{3.9}$$

under the following assumption:

$$|(d/dt)^m\eta| \lesssim 2^{mj/2}, \quad \forall m; \tag{3.10}$$

$$\operatorname{supp}\eta \subset J(t_0, \varepsilon_0 2^{1-j/2}], \tag{3.11}$$

for some t_0 such that $\pi/2 - t_0 \sim 2^{-j/2}$.

Asymptotic expansion of the kernel. We make a change of variables in order that the t-derivatives of η and $\mathcal{P}$ are bounded uniformly in λ and j. Let us set

$$\tau(t,z,z') = S_c(z,z') + 2^{-j/2}t$$

and

$$\tilde\eta(t,z,z') = \eta(\tau(t,z,z'))(\sin\tau(t,z,z'))^{-1}, \quad \tilde{\mathcal{P}}(t,z,z') = 2^{3j/2}\mathcal{P}(\tau(t,z,z'),z,z').$$

Changing variables $t \to \tau(t, z, z')$, we have

$$([\eta]^\lambda \tilde{\chi})(z, z') = 2^{-j/2}\, \tilde{\chi}(z, z') \int \tilde{\eta}(t, z, z')\, e^{i\lambda 2^{-3j/2} \tilde{\mathcal{P}}(t,z,z')}\, dt.$$

Note that supp $\tilde{\eta}(\cdot, z, z')$ is contained in a small interval of length $\lesssim \varepsilon_0$ containing the zero. We also have

$$|\partial_t^m \tilde{\eta}(t, z, z')| \le C_m \quad \text{and} \quad |\partial_t^m \big(\tilde{\mathcal{P}}(t, z, z') - \tilde{\mathcal{P}}(0, z, z')\big)| \le C_m$$

for any $m \in \mathbb{N}_0$ if $(t, z, z') \in \operatorname{supp}(\tilde{\eta} \otimes \tilde{\chi})$. The former inequality follows from (3.10). The latter inequality for $m \ge 3$ is clear, and one can show the inequality for $m = 1, 2$ using (3.4) and

$$\partial_t^2 \mathcal{P}(t, z, z') = \frac{|z - z'|^2 \cos t}{2 \sin^3 t}. \tag{3.12}$$

The case $m = 0$ follows from that for $m = 1$ via the mean value theorem. Furthermore, since

$$\partial_t^2 \tilde{\mathcal{P}}(t, z, z') = 2^{j/2}\, \partial_t^2 \mathcal{P}(S_c(z, z') + 2^{-j/2} t, z, z'),$$

from (3.12) and (3.8) we also note that

$$\partial_t^2 \tilde{\mathcal{P}}(t, z, z') \sim 1, \quad (t, z, z') \in \operatorname{supp}(\tilde{\eta} \otimes \tilde{\chi}).$$

Since $\partial_t \tilde{\mathcal{P}}(0, z, z') = 0$, the function $t \to \tilde{\mathcal{P}}(t, z, z')$ has a nondegenerate critical point at 0. Taking ε_0 small enough, we apply the stationary phase method (Theorem 7.7.5 in [8]) to obtain the following:

$$([\eta]^\lambda \tilde{\chi})(z, z') = \lambda^{-1/2}\, 2^{j/4}\, \frac{\tilde{\chi}(z, z')\, \tilde{\eta}(0, z, z')}{(\partial_t^2 \tilde{\mathcal{P}}(0, z, z')/2\pi)^{1/2}}\, e^{i\lambda 2^{-3j/2} \tilde{\mathcal{P}}(0,z,z')} + E(z, z') \tag{3.13}$$

where $|E(z, z')| \lesssim \lambda^{-3/2}\, 2^{7j/4}\, |\tilde{\chi}(z, z')|$. Note

$$\|E(\cdot, z')\|_1, \|E(z, \cdot)\|_1 \lesssim \lambda^{-3/2}\, 2^{3j/4} \lesssim \lambda^{-1},$$

since $2^j \lesssim \lambda^{2/3}$. Thus, Young's inequality shows $\|E\|_{4\to 4} \lesssim \lambda^{-1}$.

Using (3.12) and (3.6), we get

$$\partial_t^2 \tilde{\mathcal{P}}(0, z, z') = 2^{1+j/2}\, \frac{\cos S_c(z, z')}{\sin S_c(z, z')}.$$

Additionally, note that

$$\tilde{\eta}(0, z, z') = \eta(S_c(z, z'))(\sin S_c(z, z'))^{-1} \quad \text{and} \quad 2^{-3j/2}\, \tilde{\mathcal{P}}(0, z, z') = \mathcal{P}(S_c(z, z'), z, z').$$

We set

$$\Phi(z, z') = \mathcal{P}(S_c(z, z'), z, z'), \tag{3.14}$$

$$A(z, z') = 2^{-j/4}\, \tilde{\chi}(z, z')\, \eta(S_c(z, z'))\, (\sin S_c(z, z') \cos S_c(z, z'))^{-1/2}. \tag{3.15}$$

For $\mathfrak{p} \in C^\infty(\mathbb{R}^4)$ and $\mathfrak{a} \in C_c^\infty(\mathbb{R}^4)$, we denote

$$\mathcal{T}_\lambda[\mathfrak{p}, \mathfrak{a}] f(z) = \int e^{i\lambda \mathfrak{p}(z,z')}\, \mathfrak{a}(z,z')\, f(z')\, dz'.$$

Now, by (3.13) the estimate (3.9) follows if we show the next proposition.

Proposition 3.1. *Let* $1 \ll 2^j \lesssim \lambda^{2/3}$ *and let* η *satisfy* (3.10) *and* (3.11). *Then,*

$$\|\mathcal{T}_\lambda[\Phi, A]\|_{4\to 4} \lesssim_\varepsilon \lambda^{\varepsilon - 1/2}\, 2^{-j/4}, \quad \textit{for any } \varepsilon > 0.$$

We postpone the proof of Proposition 3.1 until the next section. Before closing this section, we obtain the desired bounds on the operators $[\eta]^{\lambda,\circ}$ and $[\eta]^{\lambda,e}$ (see (3.1)).

3.2. Estimates for $[\eta]^{\lambda,\circ}$ and $[\eta]^{\lambda,e}$

In this subsection, we show

$$\|[\eta]^{\lambda,\circ}\|_{4\to 4} \lesssim \lambda^{-1} \quad \text{and} \quad \|[\eta]^{\lambda,e}\|_{4\to 4} \lesssim \lambda^{-1}.$$

To obtain the above bounds, we use estimates for the kernels.

We first consider $[\eta]^{\lambda,\circ}$. Setting

$$\varphi_\lambda = \sum_{2^{-l} \le C\lambda^{-1/3}} \widetilde{\varphi}_l(\pi/2 - \cdot)$$

for a large positive constant C, we decompose

$$[\eta]^{\lambda,\circ} = \sum_{2^{-l} > C\lambda^{-1/3}} [\eta \widetilde{\varphi}_l(\pi/2 - \cdot)]^{\lambda,\circ} + [\eta\varphi_\lambda]^{\lambda,\circ}. \tag{3.16}$$

The operator $[\eta\varphi_\lambda]^{\lambda,\circ}$ is easy to handle. Since $|[\eta\varphi_\lambda]^{\lambda,\circ}(z,z')| \lesssim \lambda^{-1/3}$, it follows that

$$\|[\eta\varphi_\lambda]^{\lambda,\circ}(\cdot, z')\|_1, \|[\eta\varphi_\lambda]^{\lambda,\circ}(z, \cdot)\|_1 \lesssim \lambda^{-1}.$$

Consequently,

$$\|[\eta\varphi_\lambda]^{\lambda,\circ}\|_{4\to 4} \lesssim \lambda^{-1}.$$

As for $\sum_{2^{-l} > C\lambda^{-1/3}} [\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]^{\lambda,\circ}$, we recall (2.5). Since $2^{-l} > C\lambda^{-1/3}$, note from (3.4) that

$$|\partial_t \mathcal{P}(t,z,z')| \gtrsim 2^{-2l} \quad \text{if } (t,z,z') \in \operatorname{supp} \widetilde{\varphi}_l(\pi/2 - \cdot) \times \operatorname{supp} \chi^\circ.$$

We also have

$$\left| \Big(\frac{d}{dt}\Big)^n (\eta(t)\widetilde{\varphi}_l(\pi/2 - t)(\sin t)^{-1}) \right| \lesssim 2^{nl} \quad \text{for any } n.$$

Hence, routine integration by parts gives

$$|[\eta\widetilde{\varphi}_l(\pi/2 - \cdot)]^{\lambda,\circ}(z,z')| \lesssim 2^{-l}(\lambda^{-1}2^{3l})^N.$$

Thus, we obtain

$$\|[\eta\widetilde{\varphi}_l(\pi/2-\cdot)]^{\lambda,\circ}\|_{4\to 4} \lesssim \lambda^{-2/3}2^{-l}(\lambda^{-1}2^{3l})^N$$

by the same argument as before. Taking sum over l gives

$$\Big\|\sum_{2^{-l}>C\lambda^{-1/3}}[\eta\widetilde{\varphi}_l(\pi/2-\cdot)]^{\lambda,\circ}\Big\|_{4\to 4} \lesssim \lambda^{-1}.$$

We now turn to $[\eta]^{\lambda,e}$. Note $|z-z'|>2$ for $(z,z')\in \operatorname{supp}\chi^e$. Thus, from (3.4) we have

$$|\partial_t\mathcal{P}(t,z,z')| \gtrsim |z-z'|^2-4 \quad \text{for } (t,z,z')\in \operatorname{supp}\eta\times\operatorname{supp}\chi^e.$$

Recalling (2.5), by integration by parts we obtain

$$|[\eta]^{\lambda,e}(z,z')| \lesssim \big(1+\lambda(|z-z'|^2-4)\big)^{-N}\chi^e(z,z')$$

for any $N\in\mathbb{N}$. Hence, we obtain

$$\|[\eta]^{\lambda,e}(\cdot,z')\|_1,\ \|[\eta]^{\lambda,e}(z,\cdot)\|_1 \lesssim \lambda^{-1}.$$

Therefore, we see

$$\|[\eta]^{\lambda,e}\|_{4\to 4} \lesssim \lambda^{-1}$$

by Young's inequality.

4. L^4 bounds near the set $\mathfrak{S}$

In this section, we prove Proposition 3.1. We begin by decomposing $\mathcal{T}_\lambda[\Phi,A]$ by breaking the amplitude function A along the angle of $(z-z')/|z-z'|$.

For each positive integer j, let $\Lambda_j\subset\mathbb{S}^1$ be a collection of $\varepsilon_0 2^{-j/2}$-separated points such that $\mathbb{S}^1\subset\bigcup_{\nu\in\Lambda_j}B(\nu,\varepsilon_0 2^{1-j/2})$. Let $\{\tilde{\varrho}_j^\nu\}_{\nu\in\Lambda_j}$ be a partition of unity on $\mathbb{S}^1$ subordinated to $\{B(\nu,\varepsilon_0 2^{1-j/2})\cap\mathbb{S}^1\}_{\nu\in\Lambda_j}$. Let

$$A^\nu(z,z') = A(z,z')\varrho^\nu(z,z'), \quad \text{with} \quad \varrho^\nu(z,z') = \tilde{\varrho}_j^\nu\Big(\frac{z-z'}{|z-z'|}\Big). \tag{4.1}$$

Consequently, we have

$$\mathcal{T}_\lambda[\Phi,A] = \sum_{\nu\in\Lambda_j}\mathcal{T}_\lambda[\Phi,A^\nu].$$

Since $|\Lambda_j|\lesssim 2^{j/2}$, Proposition 3.1 follows once we prove the next.

Proposition 4.1. *Let $1\ll 2^j\le\lambda^{2/3}$. Then, for $\nu\in\Lambda_j$ we have that for every $\varepsilon>0$,*

$$\|\mathcal{T}_\lambda[\Phi,A^\nu]\|_{4\to 4} \lesssim_\varepsilon \lambda^{\varepsilon-1/2}2^{-3j/4}. \tag{4.2}$$

4.1. Reduction

We first make some observations about the operator $\mathcal{T}_\lambda[\Phi, A]$. Note $S_c(z,z') = h(|z-z'|)$ for a function h, so $A(z,z') = a(|z-z'|)$ for a function a. Thus, the amplitude function A is invariant under simultaneous rotation (i.e., $A(z,z') = A(Rz, Rz')$ for any rotation R). It is easy to see the phase Φ is also invariant under simultaneous rotation. Indeed, by (3.14) and (2.4) we have

$$\Phi(z,z') = S_c(z,z') + \cos S_c(z,z') \sin S_c(z,z') + \mathbf{S}(z,z'), \tag{4.3}$$

where $\mathbf{S}(z,z') = 2^{-1}(z_2 z_1' - z_1 z_2')$. Note $\mathbf{S}(z,z') = \mathbf{S}(Rz, Rz')$ for any rotation R.

Therefore, changing variables, it is clear that

$$\|\mathcal{T}_\lambda[\Phi, A^\nu]\|_{4\to4} = \|\mathcal{T}_\lambda[\Phi(R\cdot, R\cdot), A^\nu(R\cdot, R\cdot)]\|_{4\to4}.$$

As a result, to prove (4.2) we may assume

$$\nu = e_1.$$

However, due to the term $\mathbf{S}(z,z')$ in (4.3), $\Phi(z,z')$ is not invariant under simultaneous translation, $(z,z') \to (z+v, z'+v)$. Nevertheless, this does not cause any problem in the perspective of the operator norm. Indeed, note that

$$\mathbf{S}(z+v, z'+v') = \mathbf{S}(z,z') + \mathbf{S}(v,z') + \mathbf{S}(z,v') + \mathbf{S}(v,v').$$

The second, third, and fourth terms in the phase functions can be disregarded since they do not have any effect on the operator norm. More generally, we denote

$$\Phi_1(z,z') \simeq \Phi_2(z,z')$$

if $\Phi_1(z,z') = \Phi_2(z,z') + a(z) + b(z')$ for some functions a and b. It is clear that $\|\mathcal{T}_\lambda[\Phi_1, A^{e_1}]\|_{4\to4} = \|\mathcal{T}_\lambda[\Phi_2, A^{e_1}]\|_{4\to4}$ if $\Phi_1(z,z') \simeq \Phi_2(z,z')$. Using this observation and a standard argument, we can reduce the estimate (4.2) to a local estimate.

Let $\vartheta \in C_c^\infty((-1,1)^2)$ such that $\sum_{\mathbf{k}\in\mathbb{Z}^2} \vartheta(\cdot - \mathbf{k}) = 1$, and set

$$\vartheta_{\mathbf{k}}(z) = \vartheta(\varepsilon_0^{-1} 2^{j+3} z_1 - k_1, \varepsilon_0^{-1} 2^{(j+3)/2} z_2 - k_2) \quad \text{for each } \mathbf{k} = (k_1, k_2).$$

Consequently, we have

$$\mathcal{T}_\lambda[\Phi, A^{e_1}] f = \sum_{\mathbf{k},\mathbf{k}'} \vartheta_{\mathbf{k}} \mathcal{T}_\lambda[\Phi, A^{e_1}]\, \vartheta_{\mathbf{k}'} f.$$

We also note that

$$\operatorname{supp} A^{e_1} \subset \left\{(z_1, z_2) : |z_1 - z_1' - a| < c\varepsilon_0 2^{1-j}, |z_2 - z_2'| < \varepsilon_0^2 2^{-j/2}\right\} \tag{4.4}$$

with a constant a satisfying $2 - a \in (2^{-2-j}, 2^{-j})$. Thus, we see that $\vartheta_{\mathbf{k}} \mathcal{T}_\lambda[\Phi, A^{e_1}]\vartheta_{\mathbf{k}'} = 0$ if $|k_2 - k_2'| > 3$ and $|k_1 - k_1' - \varepsilon_0^{-1} 2^{j+3} a| > 3$. Similarly as in the proof of Proposition 2.2, the estimate (4.2) follows if we show

$$\|\vartheta_{\mathbf{k}} \mathcal{T}_\lambda[\Phi, A^{e_1}]\vartheta_{\mathbf{k}'} f\|_4 \lesssim_\varepsilon \lambda^{\varepsilon - 1/2} 2^{-3j/4} \|f\|_4$$

for each pair $(\mathbf{k}, \mathbf{k}')$ satisfying $|k_2 - k_2'| \le 2$ and $|k_1 - k_1' - \varepsilon_0^{-1} 2^{j+3} a| \le 2$.

To prove the estimate, thanks to the above discussion, we may use translation $(z,z') \to (z+v, z'+v)$ for some v. Therefore, we may replace, respectively, $\vartheta_{\mathbf{k}}$ and $\vartheta_{\mathbf{k}'}$ with cutoff functions $\mathfrak{a}$ and $\mathfrak{a}'$ such that

$$\begin{aligned} \operatorname{supp} \mathfrak{a} &\subset \{(z_1,z_2) : |z_1 - a| < \varepsilon_0 2^{-j},\ |z_2| < \varepsilon_0 2^{-j/2}\}, \\ \operatorname{supp} \mathfrak{a}' &\subset \{(z_1,z_2) : |z_1| < \varepsilon_0 2^{-j},\ |z_2| < \varepsilon_0 2^{-j/2}\}, \end{aligned} \tag{4.5}$$

and $\partial^\alpha \mathfrak{a}$, $\partial^\alpha \mathfrak{a}' = O(2^{\alpha_1 j} 2^{\alpha_2 j/2})$. Let us set

$$\mathcal{A}(z,z') = \mathfrak{a}(z) A^{e_1}(z,z') \mathfrak{a}'(z').$$

Therefore, the estimate (4.2) follows from the next result.

Proposition 4.2. *Let $1 \ll 2^j \le \lambda^{2/3}$. Then, $\|\mathcal{T}_\lambda[\Phi, \mathcal{A}]\|_{4\to4} \lesssim_\varepsilon \lambda^{\varepsilon-1/2} 2^{-3j/4}$.*

Note that $\mathcal{A}$ is supported in a product of two rectangle of dimension $2^{-j} \times 2^{-j/2}$. We perform change of variables. Set

$$L_j(z,z') = (2^{-j} z_1 + 2, 2^{-j/2} z_2, 2^{-j} z_1', 2^{-j/2} z_2'),$$

and $\Phi_j = \Phi \circ L_j$ and $\mathcal{A}_j = \mathcal{A} \circ L_j$. Changing variables $(z,z') \to L_j(z,z')$, we have

$$\|\mathcal{T}_\lambda[\Phi, \mathcal{A}]\|_{4\to4} = 2^{-3j/2} \|\mathcal{T}_{2^{-3j/2}\lambda}[2^{3j/2}\Phi_j, \mathcal{A}_j]\|_{4\to4}. \tag{4.6}$$

Let $b = 2^j(2-a)$ so that $b \in (2^{-2}, 1)$. From (4.5), note that

$$\operatorname{supp} \mathcal{A}_j \subset U := \{(z,z') : |z_1 + b|, |z_1'| < \varepsilon_0,\ |z_2|, |z_2'| < \varepsilon_0\}. \tag{4.7}$$

4.2. Scaling

To obtain an estimate for $\mathcal{T}_{2^{-3j/2}\lambda}[2^{3j/2}\Phi_j, \mathcal{A}_j]$, we use the known estimate for the oscillatory integral operator satisfying Carleson–Sjölin condition [2, 7]. For the purpose, we need to take a close look at the scaled functions $\mathcal{A}_j$ and Φ_j. Recalling (3.14) and (3.15), we first consider S_c and $|z - z'|$ under L_j.

Note that $1 - \cos\sigma = g(\sigma^2)$ for an analytic function g with $g(0) = 0$ and $g'(0) = 1/2$, so g has an analytic inverse function near the origin. Consequently, we may write $g^{-1}(t) = 2t(1 + 2t\mathcal{E}(2t))$ for an analytic function $\mathcal{E}$ on a neighborhood of the origin. Let us set

$$\tilde{S}(z,z') = \frac{\pi}{2} - S_c(z,z') \quad \text{and} \quad \tilde{t}(z,z') = 2 - |z - z'|.$$

From (3.6) we have $1 - \cos\tilde{S}(z,z') = \tilde{t}(z,z')/2$. Now, recalling that $|\tilde{S}(z,z')| \lesssim 2^{-j/2}$, $\tilde{t}(z,z') \sim 2^{-j}$, and $j > C$ for a large C, from the discussion above we have $\tilde{S}^2(z,z') = g^{-1}(\tilde{t}(z,z')/2)$. Thus, we obtain

$$\tilde{S}(z,z') = \sqrt{\tilde{t}(z,z')(1 + \tilde{t}(z,z')\mathcal{E}(\tilde{t}(z,z')))}. \tag{4.8}$$

This shows that $\tilde{S}$ becomes singular on the set $\{(z,z') : \tilde{t}(z,z') = 0\}$. However, the singularity does not appear thanks to our decomposition. In fact, changing variables $(z,z') \to L_j(z,z')$, we can show the consequent scaled function $2^{j/2}\tilde{S} \circ L_j$ has bounded derivatives on U.

Indeed, writing

$$|z - z'| = (z_1 - z_1') \Big(1 + \frac{(z_2 - z_2')^2}{(z_1 - z_1')^2}\Big)^{1/2}$$

for $(z, z') \in \operatorname{supp} \mathfrak{a} \times \operatorname{supp} \mathfrak{a}'$ and using power series expansion, we have

$$\tilde t(z, z') = 2 - (z_1 - z_1') - \frac{(z_2 - z_2')^2}{2(z_1 - z_1')}(1 + O(|z_2 - z_2'|^2)), \quad (z, z') \in \operatorname{supp} \mathcal{A}.$$

Thus, it follows that

$$\tilde t_j(z, z') := 2^j\, \tilde t(L_j(z, z')) = \mathfrak{P}(z, z') + O(2^{-j}|z_2 - z_2'|^4) \tag{4.9}$$

for $(z, z') \in U$, where

$$\mathfrak{P}(z, z') = z_1' - z_1 - \frac{(z_2 - z_2')^2}{2(2 + 2^{-j}(z_1 - z_1'))}. \tag{4.10}$$

In particular, we note $\tilde t_j \sim 1$ and $\mathfrak{P} \sim 1$ on U. Combining (4.9) and (4.8) gives

$$\tilde S_j(z, z') := 2^{j/2}\, \tilde S(L_j(z, z')) = \mathfrak{P}^{1/2}(z, z') + \mathcal{E}(z, z'), \tag{4.11}$$

where $\mathcal{E}$ is an analytic function satisfying

$$\sup_{(z,z')\in U} |\partial^\alpha_{z,z'} \mathcal{E}(z, z')| \lesssim_\alpha 2^{-j}. \tag{4.12}$$

We now claim that

$$\sup_{(z,z')\in U} |\partial^\alpha_{z,z'} \mathcal{A}_j| \le C_\alpha. \tag{4.13}$$

For this, it is sufficient to show that the same bound holds for $A^{e_1} \circ L_j$, $\mathfrak{a}_1 \circ L_j$, and $\mathfrak{a}_2 \circ L_j$. Those for $\mathfrak{a}_1 \circ L_j$ and $\mathfrak{a}_2 \circ L_j$ are clear. By (4.1) and (3.15), we need only to show uniform bounds on the derivatives of $\varrho^{e_1} \circ L_j$,

$$\tilde\chi \circ L_j, \quad \mathfrak{b}_0 := (\eta \sin^{-1/2}) \circ S_c \circ L_j, \quad \text{and} \quad \mathfrak{b}_1 := 2^{-j/4} \cos^{-1/2} \circ S_c \circ L_j$$

over the set U. The bounds on $\varrho^{e_1} \circ L_j$ are easy. To handle $\partial^\alpha \tilde\chi \circ L_j$, we note from (4.9) that $\sup_{(z,z')\in U} |\partial^\alpha \tilde t_j(z, z')| \lesssim_\alpha 1$ for any α. Thus, from this and (3.7) the desired bounds follow. For the bounds on $\partial^\alpha \mathfrak{b}_0$ and $\partial^\alpha \mathfrak{b}_1$, by (4.11) we observe that

$$\sup_{(z,z')\in U} |\partial^\alpha \tilde S_j(z, z')| \lesssim_\alpha 1. \tag{4.14}$$

Thus, using (3.10), (3.11), and the fact $|\sin S_c \circ L_j(z, z')| \sim 1$, one can easily see

$$\sup_{(z,z')\in U} |\partial^\alpha \mathfrak{b}_0(z, z')| \lesssim_\alpha 1.$$

Finally, for $\mathfrak{b}_1$, we write $\mathfrak{b}_1 = 2^{-j/4} \sin^{-1/2}(\tilde S \circ L_j)$ using an elementary trigonometric identity. Denote $\varkappa(s) = (s/\sin s)^{1/2}$, which is analytic on $(-\pi, \pi)$. We write

$$\mathfrak{b}_1 = 2^{-j/4} \sin^{-1/2}(2^{-j/2} \tilde S_j) = \tilde S_j^{-1/2}\, \varkappa(2^{-j/2} \tilde S_j).$$

By (4.11) we see $\tilde{S}_j \sim 1$ on U. Therefore,

$$\sup_{(z,z')\in U} |\partial^\alpha \mathfrak{b}_1(z,z')| \lesssim_\alpha 1.$$

This proves the claim (4.13).

We now consider

$$p(z,z') := S_c + \cos S_c \sin S_c.$$

Note that $p(z,z') = \pi/2 - \tilde{S} + 2^{-1} \sin 2\tilde{S}$. Expanding in power series gives

$$p(z,z') = \frac{\pi}{2} - \frac{2}{3}\tilde{S}^3(1 + O(\tilde{S}^2)).$$

By (4.11), we get

$$\frac{\pi}{2} - p(L_j(z,z')) = \frac{2}{3} 2^{-3j/2} \big(\mathfrak{P}^{3/2}(z,z') + \mathcal{E}(z,z')\big), \tag{4.15}$$

where $\mathcal{E}$ is an analytic error satisfying (4.12). Note

$$\mathbf{S} \circ L_j(z,z') = 2^{-3j/2}\, \mathbf{S}(z,z') - 2^{-j/2} z_2'.$$

By (4.3) we see

$$2^{3j/2}\, \Phi_j(z,z') \simeq \Phi_j^*(z,z') := -\frac{2}{3}\mathfrak{P}^{3/2}(z,z') + \mathbf{S}(z,z') + \mathcal{E}(z,z'), \tag{4.16}$$

where $\mathcal{E}$ is a smooth function satisfying condition (4.12). From (4.16), it is easy to see that $\sup_{(z,z')\in U} |\partial^\alpha_{z,z'} \Phi_j^*| \le C_\alpha$.

4.3. Carleson–Sjölin argument

To estimate the right-hand side of (4.6), we follow the classic argument due to Carleson and Sjölin [2]. Similarly as before, for $\mathfrak{p}' \in C^\infty(\mathbb{R}^3)$ and $\mathfrak{a}' \in C_c^\infty(\mathbb{R}^3)$, we denote

$$\mathcal{C}_\lambda[\mathfrak{p}',\mathfrak{a}']g(z) = \int e^{i\lambda \mathfrak{p}'(z,s)}\, \mathfrak{a}'(z,s)\, g(s)\, ds, \quad (z,s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Setting $\Phi_j^{*,z_1'}(z,s) = \Phi_j^*(z,z_1',s)$ and $\mathcal{A}_j^{z_1'}(z,s) = \mathcal{A}_j(z,z_1',s)$, we observe

$$\mathcal{T}_{2^{-3j/2}\lambda}[\Phi_j^*, \mathcal{A}_j] f = \int \mathcal{C}_{2^{-3j/2}\lambda}[\Phi_j^{*,z_1'}, \mathcal{A}_j^{z_1'}]\, f(z_1', \cdot)\, dz_1'.$$

Since $2^j \le \lambda^{2/3}$, thanks to (4.6), the desired estimate in Proposition 4.2 follows via the Minkowski inequality if we show $\|\mathcal{C}_\lambda[\Phi_j^{*,z_1'}, \mathcal{A}_j^{z_1'}]\|_{4\to 4} \lesssim_\varepsilon \lambda^{\varepsilon - 1/2}$ for $\lambda \ge 1$.

For simplicity, we make an additional harmless change of variables $z_1 \to z_1' - z_1$, so that we can replace $\Phi_j^*(z,z_1',s)$ and $\mathcal{A}_j(z,z_1',s)$ with

$$\Phi^*_{j,z_1'}(z,s) := \Phi_j^*(z_1' - z_1, z_2, z_1', s) \quad \text{and} \quad \mathcal{A}_{j,z_1'}(z,s) := \mathcal{A}_j(z_1' - z_1, z_2, z_1', s),$$

respectively. Recalling (4.7), we note that

$$\operatorname{supp} \mathcal{A}_{j,z_1'} \subset \tilde{U} := \{(z,s) : |z_1 - b| \le 2\varepsilon_0,\ |z_2|, |s| \le \varepsilon_0\}. \tag{4.17}$$

The matter is reduced to showing the uniform bound, for $\lambda \ge 1$,

$$\|\mathcal{C}_\lambda[\Phi^*_{j,z_1'}, \mathcal{A}_{j,z_1'}]\|_{4\to 4} \lesssim_\varepsilon \lambda^{\varepsilon - 1/2}. \tag{4.18}$$

We are now ready to complete the proof of Proposition 4.2 by obtaining the estimate (4.18). As already mentioned, we use the well-known result regarding the oscillatory integral operator satisfying the Carleson–Sjölin condition [2, 7]. The derivatives of $\mathcal{A}_{j,z_1'}$ and $\Phi^*_{j,z_1'}$ are, as seen above, uniformly bounded. Thus, for the purpose we only have to show that the phase $\Phi^*_{j,z_1'}$ satisfies the Carleson–Sjölin condition in a uniform manner provided that ε_0 is small enough and j is large enough.

We set $e(z,s) = (z_2 - s)^2/(2z_1(2 - 2^{-j}z_1))$. Since $b \in (2^{-2}, 1)$, recalling (4.17) we note that $e = O((z_2 - s)^2) = O(\varepsilon_0^2)$ on $\operatorname{supp} \mathcal{A}_{j,z_1'}$. We also note from (4.10) that

$$\mathfrak{P}^{3/2}(z_1' - z_1, z_2, z_1', s) = z_1^{3/2}(1 - e(z,s))^{3/2}.$$

Expending $(1-e)^{3/2}$ in power series gives

$$\mathfrak{P}^{3/2}(z_1' - z_1, z_2, z_1', s) \simeq -\frac{3}{2} z_1^{3/2} e(z,s) + O(|z_2 - s|^4).$$

Consequently, since $e(z,s) = (z_2 - s)^2/(4z_1) + O(2^{-j}|z_2 - s|^2)$, we obtain

$$-\frac{2}{3}\mathfrak{P}^{3/2}(z_1' - z_1, z_2, z_1', s) \simeq \frac{1}{4} z_1^{1/2}(z_2 - s)^2 + \mathcal{E}_j(z,s),$$

where $\mathcal{E}_j$ is a smooth function on $\operatorname{supp} \mathcal{A}_{j,z_1'}$ such that

$$\mathcal{E}_j(z,s) = O(|z_2 - s|^4) + O(2^{-j}|z_2 - s|^2).$$

As for $\mathbf{S}(z, z_1', s) = 2^{-1}(z_2 z_1' - z_1 s)$, discarding the harmless term, we only need to consider $-z_1 s/2$. Therefore, recalling (4.16), to obtain (4.18) we only have to consider

$$\tilde{\Phi}^*_{j,z_1'}(z,s) = \phi(z,s) + \mathcal{E}(z, z_1', s) + \mathcal{E}_j(z,s) \tag{4.19}$$

instead of $\Phi^*_{j,z_1'}$ (here we slightly abuse the notation $\simeq$) where

$$\phi(z,s) = 4^{-1}(2(z_1' - z_1) + 2z_1^{1/2} z_2, z_1^{1/2}) \cdot (-s, s^2).$$

To show the uniform bound (4.18), we use the Carleson–Sjölin estimate [2] for the oscillatory integral operator $\mathcal{C}_\lambda[\tilde{\Phi}^*_{j,z_1'}, \mathcal{A}_{j,z_1'}]$ (also see [7] and [17], pp. 412–414) and its stability under small smooth perturbation of the phases and amplitude functions.

For a function $(z,s) \mapsto \zeta(z,s)$, we denote

$$\mathcal{M}(\zeta)(z,s) = \begin{pmatrix} \nabla_z \partial_s \zeta(z,s) \\ \nabla_z \partial_s^2 \zeta(z,s) \end{pmatrix}.$$

A computation shows that

$$\mathcal{M}(\phi)(z,s) = \frac{1}{4}\begin{pmatrix} -1 & 2s \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 + z_1^{-1/2} z_2 & 2z_1^{1/2} \\ z_1^{-1/2}/2 & 0 \end{pmatrix}$$

so $\det \mathcal{M}(\phi) = 1/8$. Therefore, by the Carleson–Sjölin estimate (see [2]), it follows that $\|\mathcal{C}_\lambda[\phi, \mathcal{A}\|_{4\to 4} \lesssim_\varepsilon \lambda^{\varepsilon - 1/2}$ for a smooth function $\mathcal{A}$ supported in $\tilde{U}$.

We now recall (4.19). Since $|z_2 - s| \le 2\varepsilon_0$ and $j > C$ for a large positive constant C and $\mathcal{E}$ satisfies (4.12), taking ε_0 small enough and $j > C$ large enough, we may regard $\tilde{\Phi}^*_{j,z_1'}$ as a small smooth perturbation of ϕ. Note that

$$\mathcal{M}(\tilde{\Phi}^*_{j,z_1'})(z,s) = \mathcal{M}(\phi)(z,s) + O(2^{-j}) + O(|z_2 - s|^3).$$

Therefore,

$$\mathcal{M}(\tilde{\Phi}^*_{j,z_1'}) - \mathcal{M}(\phi)(z,s) = O(2^{-j} + \varepsilon_0).$$

Combining this with the fact that $\tilde{\Phi}^*_{j,z_1'}, \mathcal{A}_{j,z_1'}$ has uniformly bounded derivatives, we conclude that the uniform bound (4.18) holds if ε_0 is small enough and j is large enough.

Under the above circumstance, uniformity of the estimates (i.e., stability of L^p bound on) for the oscillatory integral operator of Carleson–Sjölin type in $\mathbb{R}^2 \times \mathbb{R}$ is evident in its proof [2] (also see [7] and [17], pp. 412–414).

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