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An inverse problem in Pólya–Schur theory. I. Non-degenerate and degenerate operators

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Abstract. Given a linear ordinary differential operator T with polynomial coefficients, we study the class of closed subsets of the complex plane such that T sends any polynomial (respectively, any polynomial of degree exceeding a given positive integer) with all roots in a given subset to a polynomial with all roots in the same subset or to 0. Below we discuss some general properties of such invariant subsets, as well as the problem of existence of the minimal under inclusion invariant subset.

If a new result is to have any value, it must unite elements long since known, but till then scattered and seemingly foreign to each other, and suddenly introduce order where the appearance of disorder reigned. Then it enables us to see at a glance each of these elements at a place it occupies in the whole.

— H. Poincaré, Science and Hypothesis

1. Introduction

In 1914, generalizing some earlier results of E. Laguerre, G. Pólya and I. Schur [18] created a new branch of mathematics now referred to as the Pólya–Schur theory. The main result of [18] is a complete characterization of linear operators acting diagonally in the monomial basis of $\mathbb{R}[x]$ and sending any polynomial with all real roots to a polynomial with all real roots (or to 0). Without the requirement of diagonality of the action, a characterization of such linear operators was obtained by the second author jointly with late J. Borcea [7].

The main question considered in the Pólya–Schur theory [12] can be formulated as follows.

Problem 1.1. *Given a subset $S \subset \mathbb{C}$ of the complex plane, describe the semigroup of all linear operators $T: \mathbb{C}[z] \to \mathbb{C}[z]$ sending any polynomial with roots in S to a polynomial with roots in S (or to 0).*

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Definition 1.2. If an operator T has the latter property, then we say that S is a T*-invariant set*, or that T *preserves* S.

So far, Problem 1.1 has only been solved for the circular domains (i.e., images of the unit disk under Möbius transformations), their boundaries [7], and more recently for strips [10]. Even a very similar case of the unit interval is still open at present. It seems that for a somewhat general class of subsets $S \subset \mathbb{C}$, Problem 1.1 is out of reach of all currently existing methods.

In this paper, we consider an inverse problem in the Pólya–Schur theory which seems both natural and more accessible than Problem 1.1. We will restrict ourselves to consideration of *closed* T-invariant subsets.

Problem 1.3. *Given a linear operator* $T : \mathbb{C}[x] \to \mathbb{C}[x]$*, characterize all closed* T*-invariant subsets of the complex plane. Alternatively, find a sufficiently large class of* T*-invariant sets.*

For example, if $T = d^j/dx^j$, then a closed subset $S \subseteq \mathbb{C}$ is T-invariant if and only if it is convex. Although it seems too optimistic to hope for a complete solution of Problem 1.3 for an arbitrary linear operator T, we present below a number of relevant results valid for linear ordinary differential operators of finite order. (Note that an arbitrary linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ can be represented as a formal linear differential operator with polynomial coefficients, i.e., $T = \sum_{j=0}^{\infty} Q_j(x)\frac{d^j}{dx^j}$, where each $Q_j(x)$ is a polynomial, see [17].) To move further, we need to introduce some basic notions.

Definition 1.4. Given a linear ordinary differential operator

$$T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j} \tag{1.1}$$

of finite order $k \geq 1$ with polynomial coefficients, define its *Fuchs index* as

$$\rho_T = \max_{0 \leq j \leq k} (\deg(Q_j) - j).$$

Alternatively, the Fuchs index can be defined as the maximal difference between the output and input polynomial, when acted upon by T:

$$\rho_T = \max_{p \in \mathbb{C}[x]} (\deg(T(p)) - \deg(p)).$$

An operator T is called *non-degenerate* if $\rho_T = \deg(Q_k) - k$, and *degenerate* otherwise. In other words, T is non-degenerate if ρ_T is realized by the leading coefficient of T. We say that T is *exactly solvable* if its Fuchs index is zero.

A few operators illustrating the situation are shown in Table 1, with some of their properties listed.

Definition 1.5. Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, we denote by $\mathcal{I}_n^T$ the collection of all closed subsets $S \subset \mathbb{C}$ such that for every polynomial of degree n with roots in S, its image $T(p)$ is either 0 or has all roots in S. In this situation, we say that S *belongs to the class* $\mathcal{I}_n^T$ or, equivalently, that S is T_n*- invariant.*

Similarly, a *closed* set S belongs to the *class* $\mathcal{I}_{\geq n}^T$ if for every polynomial of degree *at least* n with roots in S, its image $T(p)$ is either 0 or has all roots in S. In this case, we say

Operator	**Fuchs index**	**Properties**
$(x^3+2x)\frac{d^3}{dx^3}+x\frac{d^2}{dx^2}+1$	0	Exactly solvable, non-degenerate
$(x+1)\frac{d^3}{dx^3}+x^4\frac{d^2}{dx^2}+2x$	2	Degenerate
$x^2\frac{d^3}{dx^3}+4\frac{d^2}{dx^2}$	-1	Non-degenerate

Table 1. Three examples of differential operators.

that S is $T_{\geq n}$*-invariant*. By definition, the class $\mathcal{I}^T_{\geq 0}$ coincides with the class of all T-invariant sets. We say that a set $S \in \mathcal{I}^T_n$ (respectively, $S \in \mathcal{I}^T_{\geq n}$) is *minimal* if there is no closed proper non-empty subset of S belonging to $\mathcal{I}^T_n$ (respectively, to $\mathcal{I}^T_{\geq n}$).

Remark 1.6. Obviously, for any T and any n, the whole complex plane $\mathbb{C}$ is a trivial example of a set belonging to both $\mathcal{I}^T_n$ and $\mathcal{I}^T_{\geq n}$. On the one hand, in case when the operator T preserves the space of polynomials of degree n, it is more natural to study the class $\mathcal{I}^T_n$. In particular, any exactly solvable operator preserves the degree of polynomials it acts upon (except for possibly finitely many exceptions in low degrees). Thus, for an exactly solvable operator, it makes sense to consider the class $\mathcal{I}^T_n$ and its elements for all (sufficiently large) n and study their behavior when $n \to \infty$. On the other hand, for an arbitrary linear operator T, it is more natural to consider non-trivial subsets of $\mathbb{C}$ belonging to $\mathcal{I}^T_{\geq n}$, where n is any non-negative integer. Observe that families of sets belonging to $\mathcal{I}^T_n$ (respectively, $\mathcal{I}^T_{\geq n}$) are closed under taking the intersection.

In the present paper (which is the first part of two), we study the class $\mathcal{I}^T_{\geq n}$ for an arbitrary T of the form (1.1). The sequel article [2] is devoted to the study of the class $\mathcal{I}^T_n$ and also of the so-called Hutchinson invariant sets for exactly solvable operators and their relation to the classical complex dynamics. A recent paper [1] contains the results of the first and the third authors jointly with N. Hemmingsson, D. Novikov, and G. Tahar on a similar topic, where we provide many details about the so-called continuously Hutchinson invariant sets for operators T of order 1.

The structure of the paper is as follows. In Section 2, we present and prove some general results about $\mathcal{I}^T_{\geq n}$ for an arbitrary operator T (with non-constant leading term). In Section 3, we prove all results related to non-degenerate operators. In Section 4 and Section 6, we prove all results related to degenerate differential operators including operators with constant leading term. In Section 5, we provide preliminary information about the asymptotic root behavior for bivariate polynomials used in Section 6. In Section 7, we discuss several natural set-ups and problem formulations similar to that of the current paper. Finally, Section 8 contains a number of open problems connected to the presented results.

2. General properties of invariant sets

Definition 2.1. Given an operator T of the form (1.1) with $Q_k(x)$ different from a constant, denote by $\mathrm{Conv}(Q_k) \subset \mathbb{C}$ the convex hull of the zero locus of $Q_k(x)$. We refer to $\mathrm{Conv}(Q_k)$ as the *fundamental polygon* of T.

The next proposition contains basic information about invariant sets in $\mathcal{I}^T_{\geq n}$.

Theorem 2.2. *The following facts hold*:

1. *for any operator T as in* (1.1) *and any non-negative integer n, every $S \in \mathcal{I}^T_{\geq n}$ is convex*;
2. *for any operator T as in* (1.1) *and any non-negative integer n, if S is an unbounded closed set belonging to $\mathcal{I}^T_{\geq n}$, then S is T-invariant, i.e., S belongs to $\mathcal{I}^T_{\geq 0}$*;
3. *for any T as in* (1.1) *with $Q_k(x)$ different from a constant and any non-negative integer n, every $S \in \mathcal{I}^T_{\geq n}$ contains the fundamental polygon* $\mathrm{Conv}(Q_k)$;
4. *for any T as in* (1.1) *with $Q_k(x)$ different from a constant and any non-negative integer n, the set $\mathcal{I}^T_{\geq n}$ has a unique minimal* (*under inclusion*) *element.*

Proof. (1) Fix $S \in \mathcal{I}^T_{\geq n}$ and choose $x_1, x_2 \in S$. Take $p(x) = (x-x_1)^m(x-x_2)^m$ for sufficiently large m, and consider $p^{(\ell)}(x)$. Then

$$p^{(\ell)}(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{m!}{(m-j)!} \frac{m!}{(m+j-\ell)!} (x-x_1)^{m-j}(x-x_2)^{m+j-\ell},$$

which implies that

$$q_\ell(x) := \frac{p^{(\ell)}(x)}{(x-x_1)^{m-\ell}(x-x_2)^{m-\ell}} = \sum_{j=0}^{\ell} \binom{\ell}{j} (m)_j\,(m)_{\ell-j}\,(x-x_1)^{\ell-j}(x-x_2)^j,$$

where $(m)_i := m!/(m-i)!$ is the Pochhammer symbol. Dividing both sides by m^ℓ and expanding the Pochhammer symbols, we see that

$$\begin{aligned} m^{-\ell} q_\ell(x) &= \Big(\sum_{j=0}^{\ell} \binom{\ell}{j} (x-x_1)^{\ell-j}(x-x_2)^j \Big) + \frac{R_1(x)}{m} + \frac{R_2(x)}{m^2} + \cdots \\ &= ((x-x_1)+(x-x_2))^\ell + O(m^{-1})R(x). \end{aligned}$$

Using the latter expression, we obtain

$$p^{(\ell)} = m^\ell((x-x_1)(x-x_2))^{m-\ell}\big((2x-x_1-x_2)^\ell + O(m^{-1})R(x)\big).$$

Therefore,

$$\begin{aligned} \frac{T(p(x))}{m^k} &= Q_k(x)((x-x_1)(x-x_2))^{m-k}\,\big((2x-x_1-x_2)^k + O(m^{-1})R(x)\big) \\ &\quad + \sum_{j=1}^{k} \frac{Q_{k-j}((x-x_1)(x-x_2))^{m-k+j}}{m^j}\,\big((2x-x_1-x_2)^{k-j} + O(m^{-1})R_j(x)\big). \end{aligned}$$

All terms in the above sum approach 0 as m gets large, implying that the roots of $T(p(x))$ are close to that of

$$Q_k(x)((x-x_1)(x-x_2))^{m-k}\,(2x-x_1-x_2))^k\,.$$

Since $(x_1+x_2)/2$ is a root of the latter polynomial, the original set S is convex.

(2) Assume that S is an unbounded set belonging to $\mathcal{I}^T_{\geq n}$ for some positive n. Take some polynomial p of degree less that n with roots in S. Consider a 1-parameter family of polynomials of degree n of the form

$$P_t := (x - \alpha(t))^{n-\deg p} p(x), \quad t \in [0, +\infty),$$

where $\alpha(t)$ is a variable point in S which continuously depends on t and escapes to ∞ when $t \to +\infty$. (Such a family obviously exists since S is convex and unbounded.) Consider the polynomial family $T(P_t)$. Since $S \in \mathcal{I}^T_{\geq n}$, the roots of $T(P_t)$ belong to S for any finite t and continuously depend on t. Since S is closed the same holds for the limit of the roots of $T(P_t)$ which do not escape to infinity. Notice that the set of finite limiting roots exactly coincides with the set of roots of $T(p)$, which finishes the proof of item (2).

(3) Take an arbitrary T with $Q_k(x)$ different from a constant, any non-negative integer n, and an arbitrary set $S \in \mathcal{I}^T_{\geq n}$. Set $p(x) = (x - \alpha)^m$, where $\alpha \in S$. Then

$$\frac{T(p(x))}{(m)_k} = \sum_{j=0}^{k} Q_j(x) \frac{(m)_j}{(m)_k} (x - \alpha)^{m-j}.$$

If $m \to \infty$, then $(m)_j/(m)_k \to 0$ for fixed $j < k$. Hence, the roots of $T(p(x))$ approach those of $Q_k(x)(x - \alpha)^{m-k}$ as m grows.

(4) Observe that for any differential operator T as above, the set $\mathcal{I}^T_{\geq n}$ is non-empty since it at least contains the whole $\mathbb{C}$. Now notice that by items (1)–(2), the intersection of all sets in $\mathcal{I}^T_{\geq n}$ is non-empty. Indeed, each of them contains all roots of $Q_k(x)$. Since this intersection is convex, it also contains the convex hull $\mathrm{Conv}(Q_k)$ of the roots of $Q_k(x)$. Since $\mathcal{I}^T_{\geq n}$ consists of closed convex sets with a non-empty common intersection, there is the unique minimal set in $\mathcal{I}^T_{\geq n}$. ∎

Let us denote by $\mathrm{M}^T_{\geq n}$ the unique minimal element in $\mathcal{I}^T_{\geq n}$ whose existence is guaranteed by item (4) of Theorem 2.2 . The following consequence of Theorem 2.2 is straightforward.

Corollary 2.3. (i) *Under the assumption that $Q_k(x)$ is not constant, one has the sequence of inclusions of closed convex sets*

$$\mathrm{M}^T_{\geq 0} \supseteq \mathrm{M}^T_{\geq 1} \supseteq \cdots. \tag{2.1}$$

(ii) *Under the same assumption, if for some integer n there exists a compact set $S \in \mathcal{I}^T_{\geq n}$, then $\mathrm{M}^T_{\geq m}$ is compact for all $m \geq n$ and there exists a well-defined limit*

$$\mathrm{M}^T_{\infty} := \lim_{n\to\infty} \mathrm{M}^T_{\geq n}. \tag{2.2}$$

Obviously, M^T_{∞} is a closed convex compact set.

Remark 2.4. The assumption that $Q_k(x)$ is different from a constant is important for the existence of the unique minimal under inclusion element in $\mathcal{I}^T_{\geq n}$. Many operators with a constant leading term violate this property. For example, for $T = d/dx$, every convex closed subset of $\mathbb{C}$ belongs to $\mathcal{I}^T_{\geq n}$ for every non-negative integer n. In fact, every point in $\mathbb{C}$ is a minimal set for $T = d/dx$. More details about operators with a constant leading term can be found in Section 6.

Remark 2.5. Corollary 3.7 of the next section implies that for a non-degenerate T, the minimal set $\mathrm{M}^T_{\geq n}$ is compact for any sufficiently large n. However, this compactness property might fail for small n. Theorem 3.14 below claims that M^T_{∞} coincides with the fundamental polygon $\mathrm{Conv}(Q_k)$.

On the other hand, as we will show in Proposition 4.5 of Section 4, for any degenerate operator T and non-negative integer n, every set in $\mathcal{I}^T_{\geq n}$ and, in particular, $\mathrm{M}^T_{\geq n}$ is unbounded implying that compact invariant sets exist if and only if T is a non-degenerate operator. Together with item (2) of Theorem 2.2, this implies that for any degenerate T and any positive integer n, $\mathcal{I}^T_{\geq n} = \mathcal{I}^T_{\geq 0}$, and if either at least one of $Q_k(x)$ or $Q_0(x)$ has positive degree, then

$$\mathrm{M}^T_{\geq 0} = \mathrm{M}^T_{\geq 1} = \mathrm{M}^T_{\geq 2} = \cdots, \tag{2.3}$$

which is a very essential difference between the cases of non-degenerate and degenerate operators. (The fact that every T-invariant set S contains all the roots of $Q_0(x)$ follows from the trivial identity $T(1) = Q_0(x)$, and that S contains all roots of $Q_k(x)$ is shown in item (3) of Theorem 2.2).

3. Non-degenerate operators

The main result of this section is Corollary 3.7, claiming that for a fixed non-denegerate differential operator T, there exists a non-negative integer n such that $\mathcal{I}^T_{\geq n}$ contains all sufficiently large disks. This implies compactness of the minimal set $\mathrm{M}^T_{\geq n}$ for large n. Unfortunately, at present we do have an explicit description the boundary of $\mathrm{M}^T_{\geq n}$ for a given T and n. Our best result in this direction is Theorem 3.14, which claims that the limit set M^T_{∞} coincides with the fundamental polygon $\mathrm{Conv}(Q_k)$.

The next example shows that Corollary 3.7 is the best we can hope for, as there exist non-degenerate exactly solvable operators for which $\mathrm{M}^T_{\geq n}$ is non-compact for small values of n.

Example 3.1. Consider the non-degenerate exactly solvable operator given by

$$T = \left(-\frac{x^2}{4} + \frac{x}{4}\right)\frac{d^2}{dx^2} + \left(\frac{x}{4} - \frac{1}{2}\right)\frac{d}{dx} + 1. \tag{3.1}$$

We have chosen T in such a way that for every $z \in \mathbb{C}$,

$$T[(x-z)^2] = (x-2z)\left(x - \left(\frac{z}{2} + \frac{1}{2}\right)\right). \tag{3.2}$$

Take any closed subset $S \in \mathcal{I}^T_{\geq 2}$. The first factor in (3.2) ensures that if $z \in S$, then we also have $2z \in S$. The second factor ensures that if $z \in S$, then $\frac{1}{2}(z+1) \in S$. These two facts imply that S must contain the interval $[1, \infty)$ of the real axis. In particular, the minimal set $\mathrm{M}^T_{\geq 2}$ cannot be bounded.

Moreover, the image of $(x-1)^4$ has -3 as root. This then implies that the entire real line lies in $\mathrm{M}^T_{\geq 2}$. Finally, the image of $(x+1)^2(x-1)^2$ has two complex (conjugate) roots, and this then implies that $\mathrm{M}^T_{\geq 2}$ is in fact the entire $\mathbb{C}$.

3.1. Existence of invariant disks

In this subsection, we will show that for any non-degenerate operator T, the collection $\mathcal{I}_{\geq n}^T$ of its n-invariant sets contains large disks centered at 0 for all sufficiently large n.

Define the n^{th} *Fuchs index* of a linear operator $T\colon \mathbb{C}_n[x] \to \mathbb{C}[x]$ as

$$\rho = \rho_n = \max_{0 \leq j \leq n} (\deg T(x^j) - j), \tag{3.3}$$

and call T *non-degenerate* if $\deg T(x^n) - n = \rho_n$. Set

$$G_T(x, y) := T[(1 + xy)^n]$$

and note that there exist polynomials P_ℓ^n, $\ell = -n, \dots, \rho$, of degree at most n, such that

$$G_T(x, y) = \sum_{-n \leq \ell \leq \rho} x^\ell P_\ell^n(xy). \tag{3.4}$$

Thus T is non-degenerate if and only if the degree of P_ρ^n is n. If $T = \sum_{j=0}^k Q_j(x) \frac{d^j}{dx^j}$ is a differential operator of order k, then

$$G_T(x, y) = \sum_{j=0}^{k} j!\, x^{-j} Q_j(x) \binom{n}{j} (xy)^j (1 + xy)^{n-j}, \tag{3.5}$$

and it follows that

$$P_\ell^n(x) = \sum_{j=0}^{k} j!\, a_{\ell,j} \binom{n}{j} x^j (1 + x)^{n-j}, \tag{3.6}$$

where $a_{\ell,j}$ is the coefficient of $x^{j+\ell}$ in $Q_j(x)$. Define

$$f_\ell^n(x) = \sum_{j=0}^{k} j!\, a_{\ell,j} \binom{n}{j} x^j. \tag{3.7}$$

In what follows, D_R denotes the open disk $\{x \in \mathbb{C} : |x| < R\}$, and $\overline{D}_R$ is the closure of D_R. We also define Ω_R as the open set $\{(x, y) \in \mathbb{C}^2 : |x| > R \text{ and } |y| > 1/R\}$.

Proposition 3.2 (Theorem 7 in [7]). *Let $T\colon \mathbb{C}_n[x] \to \mathbb{C}[x]$ be a linear operator of rank greater than one. The disk $\overline{D}_R$ is T_n-invariant if and only if $G_T(x, y) \neq 0$ for all $(x, y) \in \Omega_R$.*

Theorem 3.3. *Suppose $T\colon \mathbb{C}_n[x] \to \mathbb{C}[x]$ is a non-degenerate linear operator with n^{th} Fuchs index ρ. Let $g(x)$ be the greatest common divisor of $\{P_\ell^n(x)\}_\ell$. Then the closed disk $\overline{D}_R = \{x : |x| \leq R\}$ is T_n-invariant for all sufficiently large $R > 0$ if and only if*

(1) *all zeros of $g(x)$ lie in $\{x : |x| \leq 1\}$;*
(2) *all zeros of $P_\rho^n(x)/g(x)$ lie in $\{x : |x| < 1\}$.*

Proof. Suppose $T\colon \mathbb{C}_n[x] \to \mathbb{C}[x]$ is a non-degenerate linear operator. We first prove that conditions (1) and (2) are sufficient for T_n-invariance. Indeed assume that (1) and (2) hold. Since $\deg P_\ell^n \leq \deg P_\rho^n = n$ for all j, and since the zeros of $P_\rho^n(x)/g(x)$ lie in the open unit disk, there is a positive constant C such that $|P^n_\ell(x)/P^n_\rho(x)| < C$ for all $|x| \geq 1$ and all ℓ. Hence, for sufficiently large R, if $(x, y) \in \Omega_R$, then

$$\left|\frac{G_T(x,y)}{x^\rho P^n_\rho(xy)} - 1\right| = \left|\sum_{\ell=-n}^{\rho-1} x^{\ell-\rho}\frac{P^n_\ell(xy)}{P^n_\rho(xy)}\right| \leq \sum_{\ell=-n}^{\rho-1} R^{-(\rho-\ell)} C < \frac{C}{R-1} < 1. \tag{3.8}$$

For such R, the disk $\overline{D}_R = \{x : |x| \leq R\}$ is T_n-invariant by Proposition 3.2.

Suppose $\overline{D}_R = \{x : |x| \leq R\}$ is n-invariant for R sufficiently large. If $g(x)$ has a zero in $\{x : |x| > 1\}$, then $G_T(x, y) = 0$ for some $(x, y) \in \Omega_R$, and by Proposition 3.2, the disk $\overline{D}_R$ is not T_n-invariant. To get a contradiction, suppose $(P^n_\rho/g)(y) = 0$, where $|y| \geq 1$. Consider a sequence $\{y_j\}_{j=1}^\infty$, where $P^n_\rho(y_j) \neq 0$, $|y_j| > 1$, and $\lim_{j\to\infty} y_j = y$. Let

$$B_j(x) := x^n\frac{G_T(x, y_j/x)}{g(y_j)} = \sum_{\ell\leq\rho} x^{\ell+n}(P^n_\ell/g)(y_j).$$

Since $(P^n_\rho/g)(y) = 0$ and $P^n_\ell(y) \neq 0$ for some ℓ, we see that at least one zero, say x_j, of $B_j(x)$ tends to ∞ as $j \to \infty$. Hence for

$$R_j := \frac{1}{2}|x_j|\left(1 + \frac{1}{|y_j|}\right),$$

we have that $(x_j, y_j/x_j) \in \Omega_{R_j}$, while $G_T(x_j, y_j/x_j) = 0$. By Proposition 3.2, D_{R_j} is not T_n-invariant for any R_j. ■

Recall that the Möbius map $x \mapsto x/(1+x)$ sends the set $\{x \in \mathbb{C} : \mathrm{Re}(x) \geq -1/2\}$ to the unit disk.

Theorem 3.4. *For $T = \sum_{j=0}^k Q_j(x)\frac{d^j}{dx^j}$ and $n \in \mathbb{N}$, assume that T is non-degenerate as a linear operator $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$. Let ρ be the n^{th} Fuchs index of T, and let $a_{\rho,j}$ be the coefficient of $x^{\rho+j}$ in Q_j. Then the closed disk $\overline{D}_R$ is T_n-invariant for all sufficiently large R if and only if*

(1) *all zeros of the polynomial $h := \gcd(f^n_{-n}, f^n_{-n+1}, \dots, f^n_\rho)$ have real part greater than or equal to $-1/2$; equivalently, there is no β with* Re $\beta > 1/2$ *such that*

$$\sum_{j=0}^k x^{-j} j!\binom{n}{j} Q_j(x)\beta^j \equiv 0;$$

(2) *and all zeros of the polynomial f^n_ρ/h have real part greater than $-1/2$.*

Proof. We want to translate conditions (1) and (2) of Theorem 3.3 into this setting. This is done by (3.5), (3.6), (3.7) and suitable Möbius transformations. ■

Example 3.5. Let $T = Q_1(x)\frac{d}{dx} + Q_0(x)$ be a non-degenerate linear operator of order 1. Suppose first that

$$Q_0(x) + n\beta x^{-1}Q_1(x) \equiv 0 \tag{3.9}$$

for some β. We have

$$T = P \cdot (\beta n - xD),$$

where $P = P(x)$ is some polynomial. But then

$$T(x-R)^n = nP\cdot(x-R)^{n-1}((\beta-1)x-\beta R),$$

has a zero outside $\{x : |x| \le R\}$ if and only if $|\beta/(\beta-1)| > 1$, which is equivalent to $\operatorname{Re}\beta > 1/2$. This explains condition (1) in Theorem 3.4.

Suppose that (3.9) is not satisfied for any β. If $\deg Q_1 > \deg Q_0 + 1$, then (2) is always satisfied. Suppose that $\deg Q_1 = \deg Q_0 + 1$, and let a_i be the leading coefficient of Q_i. The polynomial in (2) equals $a_0 + na_1x$. Hence condition (2) is equivalent to $\operatorname{Re}(a_0/a_1) < n/2$.

Proposition 3.6. *Let $T\colon \mathbb{C}_n[x] \to \mathbb{C}_n[x]$ be a diagonal operator, i.e.,*

$$T(x^i) = \lambda_i x^i, \quad 0 \le i \le n.$$

The following conditions are equivalent:

(1) *there is a compact non-empty T_n-invariant set $K \ne \{0\}$,*

(2) *$\overline{D}_1$ is T_n-invariant,*

(3) *$\overline{D}_R$ is T_n-invariant for all $R > 0$,*

(4) *all zeros of the polynomial*

$$\sum_{i=0}^{n} \lambda_i \binom{n}{i} x^i$$

lie in $\overline{D}_1$.

Proof. Since the symbol of T is given by

$$G_T(x,y) = T[(1+xy)^n] = \sum_{i=0}^{n} \lambda_i \binom{n}{i} (xy)^i,$$

we see that the disk $\overline{D}_R$ is T_n-invariant if and only if all zeros of the polynomial

$$\sum_{i=0}^{n} \lambda_i \binom{n}{i} x^i$$

lie in $\overline{D}_1$. This proves the equivalence of (2), (3) and (4).

Suppose that (1) holds for some K, but not (2). Let $\zeta \in K$ be of maximal modulus. Since, the polynomial in (3) has zero outside the unit disk, the polynomial

$$T((x-\zeta)^n) = (-\zeta)^n \sum_{i=0}^{n} \lambda_i \binom{n}{i} \left(-\frac{x}{\zeta}\right)^i$$

has a zero outside K, a contradiction. ∎

Corollary 3.7. *If T is a non-degenerate differential operator, then there are an integer N_0 and a positive number R_0 such that the disk $D_R := D(0,R)$ is T_n-invariant whenever $n \ge N_0$ and $R \ge R_0$.*

Proof. Note that the zeros of f_n^T approach 0 as $n \to \infty$. Since $a_{\rho,k} \neq 0$, we see by (3.6) that there exist positive numbers N_0 and C such that

- $|P_\ell^n(x)/P_\rho^n(x)| < C$ for all $|z| \geq 1$, all ℓ, and all $n \geq N_0$,
- $\sum_{j=0}^k j!\, a_{\rho,j} \binom{n}{j} \neq 0$,
- the zeros of f_n^T are in $D(0, 1/2)$.

Therefore, the estimate in (3.8) can be made uniform in n. Indeed, we can choose $R_0 = C + 1$. ■

Remark 3.8. Note that by item (2) of Theorem 2.2, if T is a linear operator and $\Omega \subseteq \mathbb{C}$ is closed and unbounded, then Ω is T_n-invariant if and only if it is T_ℓ-invariant for all $\ell \leq n$. Indeed, if $f(z)$ has degree $\ell \leq n$, we may take a sequence $\{w_j\}_{j=1}^\infty$ in Ω for which $|w_j| \to \infty$ as $j \to \infty$. Then the zeros of

$$T(f) = \lim_{j \to \infty} T[(1 - x/w_j)^{n-\ell} f(z)]$$

are in Ω, by Hurwitz' theorem.

The following important notion can be found in Definition 1 of [7].

Definition 3.9. A polynomial $f(z_1, \dots, z_\ell) \in \mathbb{C}[z_1, \dots, z_\ell]$ is called *stable* if for all ℓ-tuples $(z_1, \dots, z_\ell) \in \mathbb{C}^\ell$ with $\operatorname{Im}(z_j) > 0$, $1 \leq j \leq \ell$, one has $f(z_1, \dots, z_\ell) \neq 0$.

Proposition 3.10. *Take a closed half-plane given by* $H = \{ax + b : \operatorname{Im} x \leq 0\}$, *where* $(a, b) \in \mathbb{C}^2$, $a \neq 0$, *and let* $T = \sum_{j=0}^k Q_j(x) \frac{d^j}{dx^j}$ *be a differential operator. Then the following facts are equivalent*:

(1) *the set of positive integers n for which H is T_n-invariant is unbounded,*

(2) *H is T_n-invariant for all $n \geq 0$,*

(3) *the polynomial* $\sum_{j=0}^k Q_j(ax + b)(-y/a)^j$, *considered as an element in* $\mathbb{C}[x, y]$, *is a stable polynomial in* (x, y).

Proof. By Remark 3.8, we see that (1) and (2) are equivalent. Now, (2) is equivalent to the fact that the operator $S\colon \mathbb{C}[x] \to \mathbb{C}[x]$ defined by

$$S(f)(x) = T(f(\phi^{-1}(x)))(\phi(x)),$$

where $\phi(x) = ax + b$, preserves stability. The operator S is again a differential operator, so the equivalence of (2) and (3) now follows from Theorem 1.2 in [8]. ■

Example 3.11. Consider the operator $T\colon \mathbb{C}_n[x] \to \mathbb{C}[x]$ given by

$$T = (x^2 - x^3) \frac{d^3}{dx^3} + (x + x^2) \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 6. \tag{3.10}$$

When $n = 3$, we have that for every $z \in \mathbb{C}$,

$$T[(x - z)^3] = 12(x - z^2)(x - z/2). \tag{3.11}$$

In particular, if z lies in a T_3-invariant set, then z^2 is also in the set. Thus, there are no large T_3-invariant disks. However, this does not violate Theorem 3.3, since the 3rd Fuchs index of T is 0, but $P_\rho^n(x) = -6(1+2x)$. Hence, the operator T is degenerate for $n = 3$ and Theorem 3.3 does not apply.

3.2. Description of the limiting minimal set M_∞^T.

Recall that in Corollary 2.3, we proved that whenever the leading coefficient $Q_k(x)$ of an operator T is has positive degree, then there is a minimal invariant set M_∞^T containing the convex hull of the roots of $Q_k(x)$. Furthermore, if T is non-degenerate, Corollary 3.7 implies that M_∞^T is compact. The next result of the third author is the main motivation for Theorem 3.14.

Theorem A (Theorem 9 in [19]). *Given a non-degenerate operator T as in (1.1) and $\varepsilon > 0$, there exists a positive integer n_ε such that for any $n > n_\varepsilon$ and any polynomial p of degree n with all roots in* $\mathrm{Conv}(Q_k)$*, all roots of $T(p)$ lie in the ε-neighborhood of* $\mathrm{Conv}(Q_k)$.

The main technical tool in the proof of Theorem 3.14 is Theorem 3.13, which is of independent interest. It extends the previous Theorem 3.3. For the proof, we will make use of the following alternative “symbol theorem”, which follows from Theorem 7 in [7].

Proposition 3.12. *Let $T : \mathbb{C}_n[x] \to \mathbb{C}[x]$ be a linear operator of rank greater than one, and let D be a closed disk in $\mathbb{C}$. Then D is T_n-invariant if and only if $G_n(x, y) \neq 0$ whenever $x \in D^c$ and $y \in D$, where*

$$G_n(x, y) = T\big((x-y)^n\big) = \sum_{i=0}^{n} \binom{n}{i} T(x^i)\, y^{n-i}.$$

Theorem 3.13. *Given a non-degenerate operator*

$$T = Q_k(x)\frac{d^k}{dx^k} + Q_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \cdots + Q_0(x),$$

let D be any closed disk that contains $\mathrm{Conv}(Q_k)$ *and such that the distance between* $\mathrm{Conv}(Q_k)$ *and the boundary of D is positive. Then D is T_n-invariant for all sufficiently large degrees n.*

Proof. By Proposition 3.12, D is T_n-invariant if the polynomial

$$G_n(x, y) = \sum_{j=0}^{k} (n)_j \cdot Q_j(x) \cdot (x-y)^{n-j}$$

is nonzero whenever $x \in D^c$ and $y \in D$.

For fixed $j < k$ and $y \in D$, the polynomial (in x)

$$\frac{Q_j(x) \cdot (x-y)^{k-j}}{Q_k(x)}$$

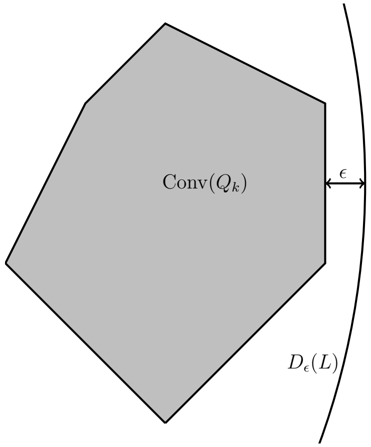

Figure 1. Illustration to the proof of Theorem 3.14.

is uniformly bounded on D^c. This is because the degree of the numerator is less than or equal to the degree of the denominator, and the zeros of $Q_k(x)$ have positive distance to D^c. By compactness of D, there is a constant C such that

$$\left|\frac{Q_j(x)\cdot(x-y)^{k-j}}{Q_k(x)}\right| \leq C, \qquad \text{for all } x \in D^c, y \in D.$$

Hence, there is a constant L, independent of n, such that

$$\left|\frac{G_n(x,y)}{Q_k(x)\cdot(x-y)^{n-k}\cdot(n)_k} - 1\right| < \frac{L}{n}, \qquad \text{for all } x \in D^c, y \in D.$$

It follows that for n sufficiently large, $G_n(x,y)$ is nonzero whenever $x \in D^c$ and $y \in D$. ■

Theorem 3.14. *If T is non-degenerate, then $M_\infty^T = \mathrm{Conv}(Q_k)$.*

Proof. We assume $\mathrm{Conv}(Q_k)$ is not a line or a point. The proofs for those cases are similar.

Take $\varepsilon > 0$. For each side L of the polygon $\mathrm{Conv}(Q_k)$, let $D_\varepsilon(L)$ be a disk containing $\mathrm{Conv}(Q_k)$ such that the distance between L and the boundary of $D_\varepsilon(L)$ is at most ε and at least $\varepsilon/2$, see Figure 1. By Theorem 3.13, $D_\varepsilon(L)$ is T_n-invariant for all $n \geq N(L,\varepsilon)$, where $N(L,\varepsilon)$ is a positive integer. But then

$$K_\varepsilon = \bigcap_L D_\varepsilon(L)$$

is T_n-invariant for all $n \geq N(\varepsilon)$, where $N(\varepsilon) = \max_L N(L,\varepsilon)$. Clearly K_ε converges to $\mathrm{Conv}(Q_k)$ when $\varepsilon \to 0$. ■

Let us now describe a special class of non-degenerate operators for which all $\mathrm{M}_{\geq n}^T$, $n = 0, 1, \ldots$, coincide with each other and with the fundamental polygon $\mathrm{Conv}(Q_k)$.

Proposition 3.15. *Take a non-degenerate operator of the form*

$$T = Q_k(x)\,\frac{d^k}{dx^k} + Q_{k-1}(x)\,\frac{d^{k-1}}{dx^{k-1}}$$

satisfying the condition

$$\frac{Q_{k-1}(x)}{Q_k(x)} = \sum_{i=1}^{\deg Q_k} \frac{\kappa_i}{x - x_i}, \tag{3.12}$$

where $\kappa_i \geq 0$ and $\{x_1, \dots, x_{\deg Q_k}\}$ is the set of all roots of $Q_k(x)$. Then,

$$\mathrm{M}^T = \mathrm{M}^T_{\geq 1} = \mathrm{M}^T_{\geq 2} = \cdots = \mathrm{M}^T_{\infty} = \mathrm{Conv}(Q_k).$$

Proof. By item (3) of Theorem 2.2, it suffices to show that under our assumptions on T, $\mathrm{Conv}(Q_k)$ is a T-invariant set. Moreover, by the Gauss–Lucas theorem, for

$$T = Q_k(x)\,\frac{d^k}{dx^k} + Q_{k-1}(x)\,\frac{d^{k-1}}{dx^{k-1}}$$

satisfying (3.12), it suffices to show that $\mathrm{Conv}(Q_k)$ is $\widetilde{T}$-invariant, where

$$\widetilde{T} = Q_k(x)\,\frac{d}{dx} + Q_{k-1}(x).$$

Assume now that $p(x)$ is an arbitrary polynomial of some degree n whose roots $r_1, \dots, r_n$ lie in $\mathrm{Conv}(Q_k)$ and consider $q = \widetilde{T}(p)$. We want to show that $q(z) \neq 0$ for any $x \in \mathbb{C} \setminus \mathrm{Conv}(Q_k)$. Assume $q(x) = 0$, which is equivalent to

$$Q_k(x)p'(x) + Q_{k-1}(x)\,p(x) = 0 \iff \frac{p'(x)}{p(x)} = -\frac{Q_{k-1}(x)}{Q_k(x)}. \tag{3.13}$$

The latter expression is equivalent to

$$\sum_{j=1}^{n} \frac{1}{x - r_j} = -\sum_{i=1}^{\deg Q_k} \frac{\kappa_i}{x - x_i},$$

where $\{x_1, \dots, x_{\deg Q_1}\}$ is the set of roots of Q_k and $\kappa_i \geq 0$. Assuming that $x \notin \mathrm{Conv}(Q_k)$, choose a line L separating z from $\mathrm{Conv}(Q_k)$. By our assumptions, L separates x from all r_j's and all x_i's. Because of that and taking into account the signs, one can easily conclude that the left-hand side of the latter expression is a complex number pointing from x to the half-plane not containing x and the right-hand side does the opposite. Therefore, (3.13) cannot hold if $x \notin \mathrm{Conv}(Q_k)$. ∎

A special case of Proposition 3.15 when $Q_k(x)$ is a real-rooted polynomial follows from more general results of [9].

4. Exactly solvable and degenerate operators: basic facts

4.1. Preliminaries on exactly solvable operators

In this section, we will need the following information, see e.g. [5].

Given an exactly solvable operator T, observe that for each non-negative integer j,

$$T(x^j) = \lambda_j^T x^j + \text{ lower order terms.} \tag{4.1}$$

Define the *spectrum* of an exactly solvable T as the sequence $\Lambda^T := \{\lambda_j^T\}_{j=0}^{\infty}$ of complex numbers.

Definition 4.1. We say that an exactly solvable operator $T = \sum_{j=0}^{k} Q_j(x)\frac{d^j}{dx^j}$ is *non-trivial* if the maximal value j_0 of the index for which $\deg Q_j(x) = j$ is strictly positive. (An example of a trivial exactly solvable operator is $T = d/dx + 1$.)

Lemma A (See [16]). *For any non-trivial exactly solvable operator T and any sufficiently large positive integer n, there exists a unique (up to a constant factor) eigenpolynomial $p_n^T(x)$ of T of degree n. Additionally, the eigenvalue of p_n^T equals λ_n^T, where λ_n^T is given by* (4.1).

One can easily show that for any non-trivial exactly solvable operator T, there exists m_T such that the sequence $\{|\lambda_m^T|\}_{m=m_T}^{\infty}$ is monotone increasing to $+\infty$, which implies that for any sufficiently large positive integer m, $|\lambda_j^T| < |\lambda_m^T|$ for $0 \le j < m$. (For a trivial exactly solvable operator T, all λ_j^T are equal.)

Remark 4.2. Unfortunately, in [16] the condition of non-triviality has been overlooked, but the proof suggested there requires it. Indeed, if we consider the trivial exactly solvable operator $T = 1 + d/dx$, then it has not eigenpolynomials except for the constant, and this case has to be considered separately.

Remark 4.3. In addition to Lemma A, observe that for any exactly solvable operator T as in (1.1) and any non-negative integer n, T has a basis of eigenpolynomials in the linear space $\mathbb{C}_n[x]$ consisting of all univariate polynomials of degree at most n. This follows immediately from, e.g., the fact that T is triangular in the monomial basis $\{1, x, \dots, x^n\}$. In other words, even if T has a multiple eigenvalue it has no Jordan blocks. However, the eigenpolynomial in the respective degree is no longer unique. A simple example of such situation occurs for $T = x^k \frac{d^k}{dx^k}$, in which case any polynomial of degree less than k lies in the kernel.

In what follows, we will use the following result.

Proposition 4.4. *Given an exactly solvable operator T as in* (1.1) *and any invariant set $S \in \mathcal{I}_{\ge n}^T$, one has that S must contain the union of all roots of the eigenpolynomial p_m^T satisfying two conditions: $n \le m$ and $|\lambda_j^T| < |\lambda_m^T|$, where $0 \le j < m$. The latter fact implies that S contains the union of all roots of all eigenpolynomials of sufficiently large degrees.*

Proof. Indeed, as we mentioned above, the sequence $\{|\lambda_n^T|\}$ will be strictly increasing to $+\infty$ starting from some positive integer m_T. Choose some $m \ge n$ such that $m \ge m_T$, which implies that $|\lambda_m^T| > |\lambda_j^T|$ for $0 \le j < m$ and that $\{p_0^T, p_1^T, \dots, p_m^T\}$ is a basis in the space $\mathbb{C}_m[x]$ of all polynomials of degree at most m.

Pick a polynomial q of degree m whose roots belong to S and expand it as

$$q(x) = \sum_{j=0}^{m} a_j p_j^T(x), \quad \text{with } a_m \neq 0.$$

Repeated application of T to q gives

$$T^{\circ \ell}(q) = \sum_{j=0}^{m} a_j \lambda_j^{\ell} p_j^T(x) = \lambda_m^{\ell} \sum_{j=0}^{m} a_j \left(\frac{\lambda_j}{\lambda_m}\right)^{\ell} p_j^T(x). \tag{4.2}$$

Since $S \in \mathcal{I}_{\geq n}^T$, all roots of $T^{\circ \ell}(q)$ belong to S. By our assumption and disregarding the common factor λ_m^{ℓ}, the polynomial in the right-hand side of (4.2) equals $a_m p_m^T(x)$ plus some polynomial of degree smaller than m whose coefficients tend to 0 as $\ell \to \infty$. Since $a_m \neq 0$, the roots of the polynomials in the right-hand side of (4.2) tend to those of p_m^T, implying that the latter roots must necessarily belong to S. ■

4.2. Preliminaries on degenerate operators

An important although not very complicated result about degenerate operator, which partially follows from our previous considerations, is as follows.

Proposition 4.5. *If T is a degenerate operator, then for any non-negative n, every set in $\mathcal{I}_{\geq n}^T$ is unbounded, and therefore is T-invariant.*

Proof. We only need to show the unboundedness, since T-invariance follows from the unboundedness by item (2) of Theorem 2.2. Let us start with the special case of degenerate exactly solvable operators. (These operators and their invariant sets are the main object of study of our sequel paper [2].)

Any exactly solvable operator T preserves the degree of a generic polynomial it acts upon and has a unique (up to a constant factor) eigenpolynomial $p_n^T(x)$ of any sufficiently large degree n, see Lemma A and Lemma 1 in [5]. Moreover, if r_n denotes the maximum of the absolute value of the roots of $p_n(x)$, then for any degenerate exactly solvable T, $\lim_{n\to\infty} r_n = +\infty$, see Theorem 1 in [5].

By Proposition 4.4, for any exactly solvable operator T, any set $S \in \mathcal{I}_{\geq n}^T$ must contain the union of all roots of all eigenpolynomials $p_m^T(x)$ for all sufficiently large m; we conclude that any such S is necessarily unbounded.

Assume now that T has a positive Fuchs index $\rho := \rho_T > 0$. Consider the operator $T' = \frac{d^{\rho}}{dx^{\rho}} \circ T$. If T is degenerate, then T' is a degenerate exactly solvable operator. By the Gauss–Lucas theorem, every $S \in \mathcal{I}_{\geq n}^T$ belongs to $\mathcal{I}_{\geq n}^{T'}$. Since every subset $S' \in \mathcal{I}_{\geq n}^{T'}$ is unbounded by the above argument, we have settled the case $\rho > 0$.

Assume, finally, that T is a degenerate operator with $\rho < 0$. Consider a family of operators

$$T_a' = (x-a)^{-\rho} \cdot T,$$

where $a \in \mathbb{C}$. Since under our assumptions, $-\rho$ is a positive integer, T_a is a degenerate exactly solvable operator for any a. Given $S \in \mathcal{I}_{\geq n}^T$, choose $a \in S$. Then $S \in \mathcal{I}_{\geq n}^{T_a'}$ and is therefore unbounded by the previous reasoning. ■

5. (Tropical) algebraic preliminaries and three types of Newton polygons

In our study of invariant sets for degenerate operators, we will need some classical results about root asymptotics of bivariate polynomials in the spirit of modern tropical geometry, see Section 38 of [11] and Chapter 4 of [20]. These results will be used in Section 6.

We start by introducing the domination partial order on points in $\mathbb{R}^2$, Namely, we say that a point $p = (u, v) \in \mathbb{R}^2$ *dominates* a point $p' = (u', v')$ if $u \geq u'$ and $v \geq v'$. Given a subset $S \subseteq \mathbb{R}^2$, we call by its *northeastern border* $\mathbf{NE}_S$ the set of all points in S which are not dominated by other points in S. Observe that $\mathbf{NE}_S$ can be empty if S is non-compact, but for compact S, $\mathbf{NE}_S$ is always non-empty. Furthermore, if S is both compact and convex then $\mathbf{NE}_S$ is contractible.

Given a bivariate polynomial $R(u, v) = \sum_{(i,j) \in \Theta} a_{i,j} u^i v^j$ with $a_{i,j} \neq 0$ for $(i, j) \in \Theta$, denote by $\mathrm{Conv}(R) \subset \mathbb{R}^2$ its *Newton polygon*, i.e., the convex hull of the set of exponents $(i, j) \in \Theta$. The northeastern border of $\mathrm{Conv}(R)$ will be denoted by $\mathbf{NE}_R$, see examples in Figure 2 and Figure 3. By the above, $\mathbf{NE}_R$ is connected and contractible. The point of $\mathbf{NE}_R$ with the maximal value of u will be called the *eastern vertex* and denoted by V_e and the point of $\mathbf{NE}_R$ with the maximal value of v will be called the *northern vertex* and denoted by V_n. The set $\mathbf{NE}_R$ coincides with a point if and only if $V_e = V_n$. Notice that every edge of the boundary of $\mathrm{Conv}(R)$ included in $\mathbf{NE}_R$ has a negative slope. Finally, denote by $R^{\mathrm{ne}}(u, v)$ the restriction of $R(u, v)$ to the subset $\Theta^{\mathrm{ne}} \subseteq \Theta$ consisting of all monomials whose exponents are the vertices of $\mathbf{NE}_R$. We will call $R^{\mathrm{ne}}(u, v)$ the *northeastern part* of $R(u, v)$.

Remark 5.1. Observe that for any bivariate $R(u, v)$ and $\alpha, \beta \in \mathbb{C}$, the change of variables of the form $u = \tilde{u} + \alpha, v = \tilde{v} + \beta$ does not change neither $\mathbf{NE}_R$ nor $R^{\mathrm{ne}}(u, v)$.

Given an arbitrary bivariate polynomial

$$R(u, v) = \sum_{(i,j) \in \Theta} a_{i,j}\, u^i v^j = \sum_{j=0}^{m} R_j(v) u^j$$

and some number $w \in \mathbb{C}$, denote by $\mathcal{U}_R(w)$ the set of zeros of the equation $R(u, w) = 0$ in the variable u considered as the divisor in $\mathbb{C}$, i.e., zeros counted with multiplicities. Here m is the degree of R with respect to u. Assume that the parameter w runs over the portion of the positive half-axis $[\kappa, +\infty)$ which contains no root of $R_m(v)$; one can always choose κ sufficiently large so that the latter condition is satisfied. (Obviously, for all $w \in [\kappa, +\infty)$, the degree of the divisor $\mathcal{U}_R(w)$ equals m.) We define the subdivisor $\mathcal{U}_R^\infty(w) \subset \mathcal{U}_R(w)$ as the set of all roots $u(w)$ whose absolute values tend to ∞ when w tends to $+\infty$ along the positive half-axis. Notice that $\mathcal{U}_R^\infty(w)$ is well-defined for all sufficiently large positive $\widetilde{\kappa} > \kappa$, since there exists $\widetilde{\kappa}$ such that for any $w \in [\widetilde{\kappa}, +\infty)$, the absolute value of every root in $\mathcal{U}_R^\infty(w)$ will be strictly larger than the absolute value of any root in the complement $\mathcal{U}_R(w) \setminus \mathcal{U}_R^\infty(w)$.

Our next goal is to describe $\mathcal{U}_R^\infty(w)$ in terms of $R^{\mathrm{ne}}(u, v)$. In what follows, we will frequently use the following statement.

Given an arbitrary bivariate polynomial $R(u, v)$ whose $\mathbf{NE}_R$ is not a single point, decompose $\mathbf{NE}_R$ into the (disjoint) union of consecutive edges $\mathbf{NE}_R = \bigcup_{s=1}^{h} e_s$ covering $\mathbf{NE}_R$ from north to east. That is, e_1 starts at V_n, e_h ends at V_e, and each e_s is adjacent to e_{s+1}, see Figure 2. The absolute values of the slopes of $e_1, \dots, e_h$ are strictly increasing. The following statement can be easily deduced from the known results of [11] (see Theorems 63–66 in Section 38) and [20] (see Sections 3 and 4 of Chapter 4). (To use the latter results, one has to substitute u and v by u^{-1} and v^{-1}, respectively.)

Proposition 5.2. *The degree of the divisor $\mathcal{U}_R^\infty(w)$ is equal to $i_e - i_n$, where $V_e = (i_e, j_e)$ and $V_n = (i_n, j_n)$. In other words,* deg $\mathcal{U}_R^\infty(w)$ *equals the length of the projection of $\mathbf{NE}_R$ onto the u-axis.*

Additionally, $\mathcal{U}_R^\infty(w)$ splits into h subdivisors $\mathcal{U}_1^\infty(w), \dots, \mathcal{U}_h^\infty(w)$ corresponding to the edges $e_1, \dots, e_h$, respectively; the degree of $\mathcal{U}_s^\infty(w)$, $s = 1, \dots, h$ equals the length of the projection of e_s on the u-axis. All zeros in the divisor $\mathcal{U}_s^\infty(w)$ have the asymptotic growth $u \sim \varepsilon w^{\mathrm{sl}_s}$ where sl_s is the absolute value of the slope of e_s.

Possible values of ε can be found by substituting $\varepsilon w^{\mathrm{sl}_s}$ in the restriction of $R(u, v)$ to the monomials contained in the edge e_s and finding the non-vanishing roots of this restriction.

Definition 5.3. Given an arbitrary bivariate polynomial $R(u, v)$ whose northeastern border $\mathbf{NE}_R$ is not a single point, we will call the slopes of edges in $\mathbf{NE}_R$ the *characteristic exponents* of $R(u, v)$. For a given edge $e_s \in \mathbf{NE}_R$, all possible values of ε corresponding to the restriction of $R(u, v)$ to this edge will be called the *leading constants* corresponding to (the characteristic exponent of) e_s. The union of all leading constants of $R(u, v)$ will be denoted by Υ_R.

Example 5.4. To illustrate Proposition 5.2 and Definition 5.3, take

$$R(u, v) = u^8 + u^7 v^2 + u^5 v^4 + (5 + 7\sqrt{-1}) u^3 v^6 - 23 u v^7.$$

One can easily check that all monomials in $R(u, v)$ belong to $\mathbf{NE}_R$ which consists of three edges e_1, e_2 and e_3 connecting $(1, 7)$ with $(3, 6)$, $(3, 6)$ with $(7, 2)$, and $(7, 2)$ with $(8, 0)$ respectively. (The exponent $(5, 4)$ of the second monomial belongs to e_2.) The degree of $\mathcal{U}_R^\infty(w)$ equals $8 - 1 = 7$. The restriction $R_1(u, v)$ of $R(u, v)$ to e_1 is given by $(5 + 7\sqrt{-1}) u^3 v^6 - 23 u v^7$. Its non-trivial zeros with respect to the variable u are given by $(5 + 7\sqrt{-1}) u^2 - 23 w = 0$. Thus for two roots from $\mathcal{U}_1^\infty(w)$, $u \sim \varepsilon w^{1/2}$, where ε are the two roots of the equation $(5 + 7\sqrt{-1}) \varepsilon^2 - 23 = 0$. They are approximately equal to -1.45392 ± 0.748212. (The absolute value of the slope of e_1 equals $1/2$.) The restriction $R_2(u, v)$ of $R(u, v)$ to e_2 is given by $u^7 v^2 + u^5 v^4 + (5 + 7\sqrt{-1}) u^3 v^6$. Its non-trivial zeros with respect to the variable u are given by $u^4 + u^2 w^2 + (5 + 7\sqrt{-1}) w^4 = 0$; we have substituted w instead of v here to keep our notation. Thus for four different roots belonging to $\mathcal{U}_2^\infty(w)$, we have $u \sim \varepsilon w$, where ε are the four roots of the equation $\varepsilon^4 + \varepsilon^2 + 5 + 7\sqrt{-1} = 0$. These are approximately equal to $-1.22651 \pm 0.961446\sqrt{-1}$ and $-0.809831 \pm 1.58673\sqrt{-1}$. (The absolute value of the slope of e_2 equals 1.) Finally, the restriction $R_3(u, v)$ of $R(u, v)$ to e_3 is given by $u^8 + u^7 v^2$ which gives $u = -v^2$. (The absolute value of the slope of e_3 equals 2.) Summarizing, we get that Υ_R consists of six complex numbers approximately given by $\{-1, -1.22651 \pm 0.961446\sqrt{-1}, -0.809831 \pm 1.58673\sqrt{-1}, -1.45392 \pm 0.748212\}$. Its convex hull contains 0 as its interior point.

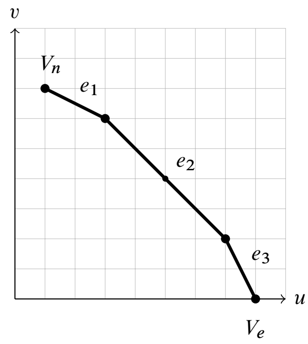

Figure 2. The northeastern border of the Newton polygon of $R(u,v) = u^8 + u^7v^2 + u^5v^4 + (5 + 7\sqrt{-1})u^3v^6 - 23uv^7$, see Example 5.4. (The Newton polygon itself is obtained by adding an edge connecting V_n with V_e.)

Corollary 5.5. *In the above notation, for a given bivariate polynomial $R(u,v)$, the family of convex hulls of $\mathcal{U}_R^\infty(w)$ converges to $\mathbb{C}$ when $w \to +\infty$ if and only if the convex hull of Υ_R contains 0 as its interior point.*

Proof. (Sketch) This statement is rather obvious, since if 0 is an interior point of the convex hull of Υ_R, then the roots in $\mathcal{U}_R^\infty(w)$ will be asymptotically moving to infinity when $w \to +\infty$ in the directions prescribed by all values of $\varepsilon \in \Upsilon_R$, and their convex hull will contain the disk of any given radius centered at 0 for sufficiently large w. ■

Let us fix a connected contractible piecewise linear curve $\mathbf{NE} \subset \mathbb{R}^2$ with integer vertices consisting of pairwise non-dominating points, see Figure 2. In other words, $\mathbf{NE}$ is a piecewise linear path with integer vertices whose edges have negative slopes whose absolute values increase when moving down along the path. Denote by $\mathrm{Pol}(\mathbf{NE})$ the set of all bivariate polynomials whose northeastern border coincides with a given $\mathbf{NE}$. (In particular, we assume that all coefficients at the corners/endpoints of $\mathbf{NE}$ are non-vanishing. $\mathrm{Pol}(\mathbf{NE})$ is a Zariski-open subset of a finite-dimensional linear space of bivariate polynomials.) Recall that the *integer length* of a closed straight interval $I \subset \mathbb{R}^2 \supset \mathbb{Z}^2$ is the number of points from $\mathbb{Z}^2$ contained in I, i.e., the number of integer points belonging to I.

Definition 5.6. Given $\mathbf{NE} \subset \mathbb{R}^2$ as above, we call it

(i) *defining* if there exists an edge in $\mathbf{NE}$ with the slope $-\alpha/\beta$ where α and β are coprime positive integers and $\beta \geq 3$;

(ii) *almost defining* if there are no edges as in (i), but there either

  (a) exists at least one edge in $\mathbf{NE}$ with the slope $-\alpha/2$ and whose integer length is larger than 2, or

  (b) there exist at least two edges with the slope $-\alpha/2$ and integer length at least 2;

(iii) *non-defining* in the remaining case i.e., when either all edges of $\mathbf{NE}$ have negative integer slopes or all edges but one have negative integer slopes and the remaining edge has a negative half integer slope and integer length 2.

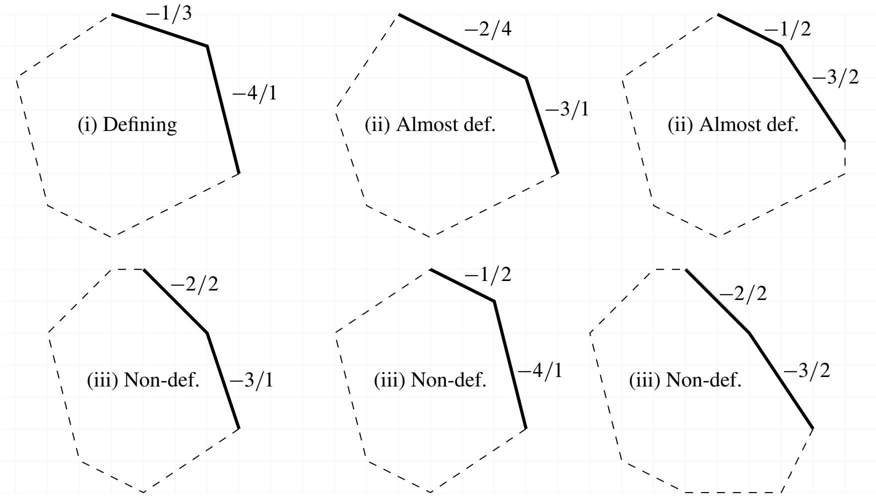

Figure 3. Examples of defining/almost defining/non-defining Newton polygons, see Definition 5.6. The slopes of the edges of the northeasten boundary are shown as fractions, such that the length of the projection is the respective denominator.

Definition 5.7. A Newton polygon $N \subset \mathbb{R}^2$ is called *defining/almost defining/non-defining* if its northeastern border contains at least one edge and is *defining/almost defining/non-defining*, respectively.

In Figure 3, we show examples of Newton polytopes illustrating Definition 5.6 and Definition 5.7.

Proposition 5.8. *Given* $\mathbf{NE} \subset \mathbb{R}^2$ *as above, the convex hull of* $\mathcal{U}_R^\infty(w)$ *converges to* $\mathbb{C}$*, when* $w \to +\infty$*,*

(i) *for any* $R \in \mathrm{Pol}(\mathbf{NE})$ *if* $\mathbf{NE}$ *is defining*;

(ii) *for generic* $R \in \mathrm{Pol}(\mathbf{NE})$ *if* $\mathbf{NE}$ *is almost defining*;

(iii) *if* $\mathbf{NE}$ *is non-defining, there is a full-dimensional subset of* $\mathrm{Pol}(\mathbf{NE})$ *for which the convex hull of* $\mathcal{U}_R^\infty(w)$ *converges to* $\mathbb{C}$ *when* $w \to +\infty$ *and the complement of the latter set in* $\mathrm{Pol}(\mathbf{NE})$ *is also full-dimensional.*

Remark 5.9. In case (ii), the condition of nongenericity is given by the fact that all $\varepsilon \in \Upsilon_R$ are real proportional to each other (i.e., they lie on the same real line in $\mathbb{C}$ passing through the origin);

In case (iii), if one forces the next to the leading coefficient for some edge with integer slope and length of projection larger than 2 to vanish, i.e., one forces the sum of the respective ε to be equal to 0, then the conclusion of Corollary 5.5 will be valid for a generic choice of the remaining coefficients at the vertices belonging to this edge.

If the convex hull of $\mathcal{U}_R^\infty(w)$ does not tend to $\mathbb{C}$, but $i_n > 0$, which means that $\mathcal{U}_R(w) \setminus \mathcal{U}_R^\infty(w)$ is non-empty, then the convex hull of $\mathcal{U}_R(w)$ will tend to the convex cone with apex at 0 spanned by the elements of Υ_R.

Proof of Proposition 5.8. By Corollary 5.5, we need to prove that the convex hull of Υ_R contains 0 as its interior point

(i) for any $R \in \mathrm{Pol}(\mathbf{NE})$ if **NE** is defining;

(ii) for generic $R \in \mathrm{Pol}(\mathbf{NE})$ if **NE** is almost defining;

(iii) if **NE** is non-defining, polynomials $R \in \mathrm{Pol}(\mathbf{NE})$ for which Υ_R contains 0 as an interior point form a full-dimensional set with the full-dimensional complement.

Indeed, assume that **NE** is defining. Then it contains an edge e_s with the slope $-\alpha/\beta$, where α and β are coprime positive integers and $\beta \geq 3$. Take now any polynomial $R(u,v) \in \mathrm{Pol}(\mathbf{NE})$ and denote by $R_s(u,v)$ the restriction of R to e_s. Substituting $u = \varepsilon v^{\alpha/\beta}$ in the equation $R_s(u,v) = 0$ and factoring out a power of v, we get a univariate algebraic equation for ε which only involves powers of ε which are multiples of $b \geq 3$. Since every non-vanishing ε appears in Υ_R together with all $\varepsilon \cdot e^{2\pi\sqrt{-1}\ell/b}$ for $\ell = 1,\dots,b-1$, one obtains that 0 lies in the interior of the convex hull of Υ_R.

Assume now that **NE** is almost defining. Then it either contains an edge e_s with the slope $-\alpha/2$ and length greater than 2 or two edges with half integer slopes and length 2 each. (All the remaining edges have integer slopes.) In the former case, the algebraic equation satisfied by ε has an even degree exceeding 2 and contains only even powers of ε. Its non-vanishing solutions come in pairs of numbers of the form $(\alpha, -\alpha)$. If at least two such pairs are non-proportional over $\mathbb{R}$ (which happens generically), then 0 is the inner point of Υ_R. Similarly, in the latter case we have two second order equations without linear terms defining ε. Again typically their pairs of solutions are non-proportional over $\mathbb{R}$ and the result follows.

Finally, assume that **NE** is non-defining. Then all edges, but possibly one, have integer slopes, which means that the corresponding equations for ε will have all possible monomials present and their non-trivial roots can either contain 0 inside their convex hull or lie in a half-plane of $\mathbb{C}$ bounded by a real line passing through the origin, in which case 0 is outside (on the boundary of) this convex hull. If there is one edge of length 2 and half-integer slope in $\mathbf{NE}_R$, then it produces one pair of opposite values for ε. ∎

6. Application of algebraic results to invariant sets of degenerate operators

In what follows, we need to consider the action of

$$T = \sum_{j=0}^{k} Q_k(x)\,\frac{d^j}{dx^j}$$

on polynomials of the form $(x-t)^n$ for sufficiently large n. One has

$$T(x-t)^n = (x-t)^{n-k} \sum_{j=0}^{k} (n)_j\,(x-t)^{k-j}\,Q_j(x) = (x-t)^{n-k}\,\psi_T(x,n,t),$$

where $\psi_T(x,n,t)$ is a trivariate polynomial. The important circumstance is that the essential part $\psi_T^+(x,n)$ of $\psi_T(x,n,t)$ is independent of t, see the beginning of Section 5. We will apply to $\psi_T^+(x,n)$ the results of the previous section and discuss how its zeros with respect to x behave when $n \to +\infty$. Denote by $a_j x^{d_j}$ the leading monomial of $Q_j(x)$ and consider the polynomial

$$\widetilde{\psi}_T(x,n) = \sum_{j=0}^{k} a_j n^j x^{d_j+k-j}.$$

(It contains much fewer monomials than $\psi_T(x,n,t)$, but with exactly the same coefficients.) Notice that the essential part $\psi_T^+(x,n)$ is obtained from $\widetilde{\psi}_T(x,n)$ by removing those monomials which do not belong to $\mathbf{NE}(\psi_T)$.

Taking the symbol polynomial $G_T(x,y) = \sum_{j=0}^{k} Q_k(x) y^j$ of T, we introduce its truncation $\widetilde{G}_T(x,y) = \sum_{j=0}^{k} a_j y^j x^{d_j}$ and observe that $\widetilde{\psi}_T(x,n)$ is obtained from $\widetilde{G}_T(x,y)$ by substituting y by n and adding $k-j$ to the powers d_j of x of the respective monomial. Thus the Newton polygon of $\widetilde{\psi}_T(x,n)$ is obtained from the Newton polygon of $\widetilde{G}_T(x,y)$ by the affine transformation A sending (i,j) to $(i+k-j,j)$. Therefore $\mathbf{NE}(\psi_T)$ is obtained from the part of the boundary of the Newton polygon of $\widetilde{\psi}_T(x,y)$ under the latter affine transformation, see Figure 4 for an example.

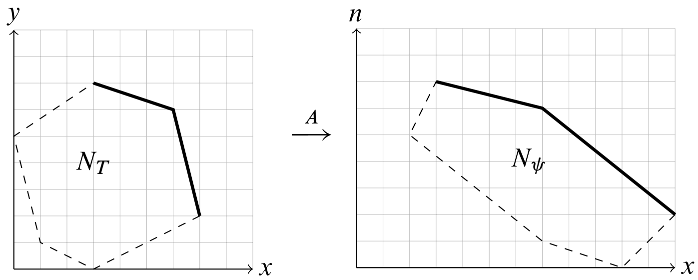

Figure 4. The affine transformation A sending N_T to N_ψ. Here $T = (x^3+\cdots)d^7/dx^7 + (x^6+\cdots)d^6/dx^6 + d^5/dx^5 + (x^7+\cdots)d^2/dx^2 + (x+\cdots)d/dx + (x^3+\cdots)$, $\widetilde{G}_T(x,y) = x^3y^7 + x^6y^6 + y^5 + x^7y^2 + xy + x^3$ and $\widetilde{\psi}_T(x,n) = n^7x^3 + n^6x^7 + n^5x^2 + n^2x^{12} + x^{10}$.

Denote the Newton polygon of $\widetilde{G}_T(x,y)$ by N_T and the Newton polygon of $\widetilde{\psi}_T(x,y)$ by N_ψ. We have that $N_\psi = A \circ N_T$. The relation between the slopes of edges before and after the affine transformation A is as follows.

If the slope sl of an edge of N_T equals $\mathrm{sl} = \mu/\nu$, where μ and ν are coprime integers and $\nu > 0$, then the slope of its image, denoted by asl, is given by $\mathrm{asl} = \mu/(\nu-\mu)$, which implies that $\mathrm{asl} = \mathrm{sl}/(1-\mathrm{sl})$ or, equivalently, $\mathrm{sl} = \mathrm{asl}/(1+\mathrm{asl})$. Therefore if asl is a negative integer, then we get

$$\mathrm{asl} = -J,\ J > 0 \quad \Longleftrightarrow \quad \mathrm{sl} = \frac{J}{J-1}\cdot$$

Obviously, any sl of the above form is positive (or $+\infty$).

Analogously, if asl is a negative half-integer, then we get

$$\mathrm{asl} = -\frac{J}{2}, \; J > 0 \text{ and odd} \quad \Longleftrightarrow \quad \mathrm{sl} = \frac{J}{J-2}.$$

Any sl of the above form is positive, with the only exception $J = 1$, for which $\mathrm{sl} = -1$.

It is easy to describe $A^{-1}(\mathbf{NE}_\psi)$ as the part of the boundary N_T starting at V_n and going southeast till we either reach the lowest point of the polygon or till the slope of the next edge becomes smaller than or equal to 1. Denote $A^{-1}(\mathbf{NE}_\psi)$ as $\mathfrak{B}_T$ and call it the *shifted northeastern border* of N_T.

One can easily check that for $T = \sum_{j=0}^{k} Q_k(x)\frac{d^j}{dx^j}$, the corresponding $\mathbf{NE}(\psi_T)$ is a single point if and only if T is non-degenerate. So for any degenerate T, its $\mathbf{NE}(\psi_T)$ contains at least one edge. Additionally, $\mathrm{asl} < 0$ if and only if $1\mathrm{sl} < 1$, which means that either $\mathrm{sl} < 0$ or $\mathrm{sl} > 1$.

Observe that the vertex V_n of $\widetilde{\psi}$ coincides with that of $\widetilde{G}$. The following notion is important for the rest of the paper.

Definition 6.1. A degenerate operator T is called *defining/almost defining/non-defining* if its Newton polygon N_ψ is defining/almost defining/non-defining, respectively, see Definition 5.6. In terms of the Newton polygon N_T, this means that its shifted northeastern border $\mathfrak{B}_N$ is not a single point, and in the *defining* case, it contains an edge with the slope of the form $J/(J-\beta)$, with $\beta \geq 3$; in the *almost defining* case, all edges of $\mathfrak{B}_N$ have slopes $J/(J-1)$, but there exists either one edge with slope $J/(J-2)$, J odd and length greater than 2, or two such edges with length 2; and in the *non-defining* case, $\mathfrak{B}_N$ contains edges of arbitrary integer length with slopes $J/(J-1)$, J being a positive integer, except for possibly one edge of integer length 2 whose slope is $J/(J-2)$, J odd.

The following result is an easy consequence of our previous considerations.

Theorem 6.2. *For any non-negative integer n and (almost) any degenerate operator T whose N_T is (almost) defining, the only set contained in $\mathcal{I}^T_{\geq n}$ is $\mathbb{C}$.*

6.1. Degenerate operators with non-defining Newton polygons

As we have seen above, the convex hull of the set Υ_T of all leading constants for (almost) every degenerate T with (almost) defining N_T contains 0 as its interior point.

For degenerate T with non-defining N_T, whose northeastern border we denote by $\mathbf{NE}_T$, it might still happen that 0 is the interior point of the latter convex hull, in which case the conclusion of Theorem 6.2 holds. However, for a full-dimensional subset of $\mathrm{Pol}(\mathbf{NE})$ with a given non-defining $\mathbf{NE}$, their leading constants belong to some half-plane in $\mathbb{C}$ bounded by a line passing through 0, and therefore 0 lies on the boundary of their convex hull. In this situation, the conclusion of Theorem 6.2 fails; we will discuss this case below.

Definition 6.3. Given a finite set $\mathcal{U} = \{u_1, \dots, u_k\}$ of (not necessarily distinct) complex numbers, we define the cone $\mathcal{C}^+\mathcal{U} \subseteq \mathbb{C}$ generated by $\mathcal{U}$ as given by

$$\mathcal{C}^+\mathcal{U} := \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_\ell u_\ell\}, \quad \text{where } \alpha_j \geq 0, \; j = 1, \dots, \ell.$$

We say that a set $S \subseteq \mathbb{C}$ is *closed with respect to* $\mathcal{C}^+\mathcal{U} \subseteq \mathbb{C}$ if for any complex number $z \in S$ and any $v \in \mathcal{C}^+\mathcal{U}$, $z + v$ belongs to S.

Obviously, 0 is the interior point of the convex hull of $\mathcal{U} = \{u_1, \dots, u_\ell\}$ if and only if $\mathcal{C}^+\mathcal{U} = \mathbb{C}$.

Given a degenerate operator T with non-defining Newton polygon N_T, denote by $\Upsilon_T := \{\varepsilon_1, \dots, \varepsilon_m\}$ the collection of all its leading constants and set $\mathcal{C}_T^+ := \mathcal{C}^+(\Upsilon_T)$. As we mentioned above, if $\mathcal{C}_T^+ = \mathbb{C}$, then the conclusion of Theorem 6.2 holds. Let us assume now that $\mathcal{C}_T^+$ is a closed sector in the plane with positive angle $\leq \pi$. (We are then missing two remaining cases: $\mathcal{C}_T^+$ being a line through the origin, and $\mathcal{C}_T^+$ being a half-line through the origin.)

Remark 6.4. Recall that by item (2) of Theorem 2.2, any set $S \in \mathcal{I}_{\geq n}^T$ is unbounded and belongs to $\mathcal{I}_{\geq 0}^T$, i.e., is unbounded and T-invariant.

Lemma 6.5. *In the above notation, any T-invariant set S is closed with respect to $\mathcal{C}_T^+$.*

Proof. Indeed, take a point $t \in S$ and consider the sequence of polynomials $T(x-t)^n$ when n increases. For $n \to \infty$, the roots of $T(x-t)^n$ whose absolute values tend to infinity will be spreading out to infinity approaching some rays whose directions are given by the elements of Υ_T. Since every S must be convex, the result follows. ■

Corollary 6.6. *In the above notation, if the product of the leading coefficient $Q_k(x)$ and the constant term $Q_0(x)$ of the operator T is not a constant, then any T-invariant set S contains the Minkowski sum $\operatorname{Conv}(Q_k Q_0) \oplus \mathcal{C}_T^+ \subset \mathbb{C}$ of $\mathcal{C}_T^+$ and $\operatorname{Conv}(Q_k Q_0)$; the latter set being the convex hull of the union of all roots of $Q_k(x)$ and $Q_0(x)$.*

Proof. It is easy to see that if any T-invariant set S must contain the zero loci of $Q_k(x)$ and of $Q_0(x)$ then by convexity of S it should contain $\operatorname{Conv}(Q_k Q_0)$. Applying Lemma 6.5, we get the required result. ■

Let us now present some conditions guaranteeing the existence of non-trivial T-invariant sets for a degenerate operators T.

6.2. Degenerate operators with non-defining Newton polygon and constant leading term

In this section, we discuss the case of a constant leading term. One can easily check that the class of degenerate operators

$$T = \frac{d^k}{dx^k} + Q_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}} + \cdots + Q_k(x)$$

with non-defining N_T splits into two subclasses:

(A) operators with constant coefficients;

(B) operators satisfying the following three conditions:

  (i) $\deg Q_{k-1} = 1$;

  (ii) $\deg Q_j \leq 1$ for $j = 0, \dots, k-2$;

  (iii) if $j_{\min}$ is the smallest value of j for which $\deg Q_j = 1$, then Q_ℓ must vanish for all $\ell \leq j_{\min} - 2$.

For the more interesting subclass (B), the northeastern border of such operator T can consist of one, two or three edges, see Figure 5 below. If it consists of one edge, then after an affine change of x, we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\,\frac{d^{k-1}}{dx^{k-1}} + \alpha\,\frac{d^{k-2}}{dx^{k-2}}, \quad \alpha \in \mathbb{C}.$$

If it consists of two edges, then after an affine change of x, we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\Big(\frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell}\alpha_i\,\frac{d^{k-1-i}}{dx^{k-1-i}}\Big) + \sum_{i=1}^{\ell}\beta_i\,\frac{d^{k-2-i}}{dx^{k-2-i}},$$

where $\ell \le k-1$ is a positive integer and all α_i and β_i are arbitrary complex numbers with the only restriction $\alpha_\ell \neq 0$.

Finally, if it consists of three edges, then after an affine change of x, we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x\Big(\frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell}\alpha_i\,\frac{d^{k-1-i}}{dx^{k-1-i}}\Big) + \sum_{i=1}^{\ell}\beta_i\,\frac{d^{k-2-i}}{dx^{k-2-i}} + \beta_{\ell+1}\,\frac{d^{k-3-\ell}}{dx^{k-3-\ell}},$$

where $\ell \le k-3$ is a positive integer, all α_i and β_i are arbitrary complex numbers with the restrictions $\alpha_\ell \neq 0$ and $\beta_{\ell+1} \neq 0$.

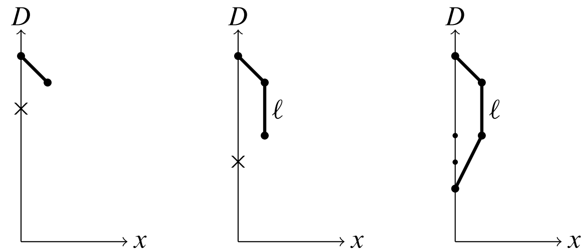

Figure 5. NE borders of the three sub-cases in subclass (B). Here, × denotes a monomial that might be present, but all monomials below × must be absent, i.e., have vanishing coefficients.

6.2.1. Subclass A, i.e., linear differential operators with constant coefficients.

Observe that in the case of constant coefficients, if S is a T-invariant set, then for any $a \in \mathbb{C}$, $S_a := S + a$ is a T-invariant set as well. (Similarly for $T_{\ge n}$-invariant sets).

Proposition 6.7. *Let*

$$T = a_k\,\frac{d^k}{dx^k} + a_{k-1}\,\frac{d^{k-1}}{dx^{k-1}} + \cdots + a_0, \quad a_k \neq 0, \tag{6.1}$$

be a linear differential operator with constant coefficients. Let $\Lambda_T^{-1} = \{\lambda_1^{-1}, \dots, \lambda_k^{-1}\}$ *be the set of the inverses of characteristic exponents (not necessarily distinct), where*

$$a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0 = a_k(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_k).$$

(*We use the convention that if* $\lambda_j = 0$ *for some* j, *which happens if* $a_0 = 0$, *then we do not consider its inverse in the list* Λ_T^{-1}). *Then a convex set* $S \subseteq \mathbb{C}$ *is* T*-invariant if and only if* S *is closed with respect to* $\mathcal{C}\Lambda_T^{-1}$.

Remark 6.8. Further notice that if $\mathcal{C}\Lambda_T^{-1} = \mathbb{C}$, which happens in the open (in the usual topology) subset of linear differential operators of the form (6.1) of any given order $k \geq 3$, the only T-invariant $S \subseteq \mathbb{C}$ is the whole $\mathbb{C}$.

Proof of Proposition 6.7. To prove the implication $\Rightarrow$, we invoke Lemma 6.5 and the observation that $\mathcal{C}\Lambda_T^{-1} = \mathcal{C}_T^{+}$.

To prove the converse implication, we proceed by induction on k, whose base is the following statement.

Lemma 6.9. *For an operator* $T = d/dx - \lambda$, *a convex set* $S \subseteq \mathbb{C}$ *is* T*-invariant if and only if for any* $x \in S$ *and* $\tau > 0$, *the number* $x - \tau\lambda$ *belongs to* S, *which is equivalent to* S *being closed with respect to* $\mathcal{C}\Lambda_T^{-1}$.

Proof. In case $\lambda = 0$, any convex set S is T-invariant by the Gauss–Lucas theorem. For $\lambda \neq 0$, using the rescaling of x we can reduce T to the special case $d/dx + 1$. Observe that for any polynomial $p(x)$, the zeros of $p'(x) + p(x)$ coincide with those of $e^{-x}(p(x)e^x)'$. Recall that $e^x = \lim_{n\to\infty}(1 + x/n)^n = \lim_{n\to\infty}(x + n)^n/n^n$. By translation invariance, we can additionally assume that either all roots of $p(x)$ are real, or among these roots there is at least one with a positive imaginary part and at least one with the negative imaginary part. For any natural n, all roots of $((x + n)^n p(x))'$ lie in the convex hull of all roots of p appended with $-n$. When $n \to \infty$, we get the required statement. In other words, all roots of $p'(x) + p(x) = e^{-x}(p(x)e^x)'$ lie in the infinite polygon (or half-line) formed by the parallel translation of the convex hull of all roots of p to infinity in the direction -1. ■

To continue our proof by induction, notice that the operator (6.1) factorizes as

$$T = a_k\Big(\frac{d}{dx} - \lambda_1\Big)\Big(\frac{d}{dx} - \lambda_2\Big)\cdots\Big(\frac{d}{dx} - \lambda_k\Big) = \Big(\frac{d}{dx} - \lambda_1\Big)\widetilde{T},$$

where $\widetilde{T}$ has order $k - 1$. Observe that the factors in the above expansion commute. By inductive hypothesis, S is a $\widetilde{T}$-invariant subset if and only if it is closed with respect to $\mathcal{C}\Lambda_{\widetilde{T}}^{-1}$.

Assume that $S \subset \mathbb{C}$ is closed with respect to $\mathcal{C}\Lambda_T^{-1}$ and let $p(x)$ be a polynomial with all roots in S. We need to show that $T(p)$ has all roots in S. We have that $T(p) = (d/dx - \lambda_1)\widetilde{T}(p)$. Since $S \subset \mathbb{C}$ is closed with respect to $\mathcal{C}\Lambda_T^{-1}$, it is also closed with respect to $\mathcal{C}\Lambda_{\widetilde{T}}^{-1}$, which implies that all roots of $\widetilde{T}(p)$ lie in S. Using Lemma 6.9 again and the fact that $\mathcal{C}\Lambda_T^{-1}$ contains $\mathcal{C}\lambda_1^{-1}$, we get that all roots of $T(p) = (d/dx - \lambda_1)\widetilde{T}(p)$ lie in S as well. ■

6.2.2. Subclass B, i.e., operators with constant leading term and deg $Q_{k-1} = 1$. In this case, we currently have only a number of sporadic results.

Let us start with operators of order 1. After an affine change of x, we only need to consider one single operator $T = d/dx - x$. The following statement holds.

Lemma 6.10. *For $T = d/dx - x$, its minimal T-invariant set $\mathrm{M}^T_{\geq 0}$ is the real axis.*

Proof. It is easy to check, using Theorem 1.3 in [8], that T is a *hyperbolicity preserver*, i.e., that T sends every real-rooted polynomial to a real-rooted polynomial (or 0).

Recall that the *symbol* $F_T(x,y)$ of the differential operator $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ is by definition given by

$$F_T(x,y) := \sum_{j=0}^{k} Q_j(x)\, y^j.$$

The above mentioned criterion claims that T is a hyperbolicity preserver if and only if the real algebraic *symbol curve* $\Gamma_F \subset \mathbb{R}^2$ given by $F_T(x,y) = 0$ must intersect each affine line with negative slope in all real points. (The real plane $\mathbb{R}^2$ is equipped with coordinates (x,y)). In other words, this number of real intersection points counting multiplicity must be equal to the degree of $F_T(x,y)$. In the case under consideration, the symbol $F_T(x,y)$ of $T = d/dx - x$ equals $y - x$ and its symbol curve has one real intersection point with each real affine line except for those parallel to $x = y$. One can also check that no subinterval of $\mathbb{R}$ is a T-invariant set. Indeed, applying T to $x - \alpha$, $\alpha \in \mathbb{R}$, we get

$$T(x-\alpha) = -x(x-\alpha) + 1 = -(x^2 - \alpha x - 1),$$

whose roots are $\alpha/2 \pm \sqrt{(\alpha/2)^2 + 1}$. These roots are the endpoints of a real interval containing α. ∎

The next results describe which operators T belonging to the class B preserve a given half-plane in $\mathbb{C}$. As a consequence, we characterize hyperbolicity preserving T in this class.

Observe that for any operator T belonging to the class B, its symbol $F_T(x,y)$ is of the form $U(y) - xV(y)$ where $U(y) = y^k + \cdots$ and $V(y) = y^{k-1} + \cdots$. (Here, $\cdots$ stands for lower degree terms in y).

Lemma 6.11. *Let $H \subset \mathbb{C}$ be an open half-plane represented as*

$$H = \{az + b : \operatorname{Im} z \leq 0\},$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, and let

$$T = U\left(\frac{d}{dx}\right) + x\,V\left(\frac{d}{dx}\right),$$

where U and V are polynomials. Then the following are equivalent:

(1) *H is T_n-invariant for all n.*

(2) *The bivariate polynomial $U(-y/a) + bV(-y/a) + aV(-y/a)x$ is stable in (x,y).*

(3) *Either $V \equiv 0$ and $U(-y/a)$ is stable, or the rational map*

$$z \longmapsto \frac{1}{a}\frac{U(-z/a)}{V(-z/a)} + \frac{b}{a}$$

maps the open upper half-plane to the closed upper half-plane.

(For the notions of stability and T_n-invariance, see Definitions 1.5 and 3.9).

Proof. By Proposition 3.10, the first two statements are equivalent. The polynomial in (2) is stable if whenever z is in the upper half-plane and

$$U(-z/a) + bV(-z/a) + aV(-y/a)w = 0,$$

then w is in the closed lower half-plane. Solving for w gives the equivalence of statements (2) and (3). ■

We recall the following version of the Hermite–Biehler theorem from [6].

Lemma 6.12. *Let $f, g \in \mathbb{R}[x]$. The following statements are equivalent*:

- *the univariate polynomial $f(x) + ig(x)$ is stable,*
- *the bivariate polynomial $f(x) + yg(x)$ is stable,*
- *f and g are real-rooted, their zeros interlace, and*

$$W(f, g) = f'(x)g(x) - f(x)g'(x) \geq 0, \quad \textit{for all } x \in \mathbb{R}.$$

Also, if the zeros of f and g interlace, then either $W(f, g) \geq 0$ for all x or $W(g, f) \geq 0$ for all x.

Corollary 6.13. *Let*

$$T = U\Big(\frac{d}{dx}\Big) + xV\Big(\frac{d}{dx}\Big),$$

where $U(y), V(y) \in \mathbb{R}[y]$. Then $\mathbb{R}$ is T_n-invariant for all n if and only if

- *there is a nonzero constant $\xi \in \mathbb{C}$ such that $\xi U(y), \xi V(y) \in \mathbb{R}[y]$, and*
- *the zeros of $\xi U(y)$ and $\xi V(y)$ are real and interlacing, and*

$$W(\xi V(y), \xi U(y)) \geq 0, \quad \textit{for all } x \in \mathbb{R}.$$

Proof. If $\mathbb{R}$ is T_n-invariant for all n, then there is a nonzero $\xi \in \mathbb{C}$ such that $\xi T : \mathbb{R}[x] \to \mathbb{R}[x]$, see Section 4 of [6].

Moreover, for any differential operator T with real coefficients, $\mathbb{R}$ is T_n-invariant for all n if and only if the closed lower half-plane is T_n-invariant for all n, see Theorem 1.2 and Theorem 1.3 in [8]. Hence the result follows from Lemma 6.11 and Lemma 6.12. ■

7. Variations of the original set-up

Above we have mainly concentrated on invariant sets for roots of polynomials of degree at least n. Currently, we neither have a description of the minimal invariant sets whose existence we have established nor a numerically stable procedure which will construct them or their approximations in specific examples.

The goal of this section is to present some interesting variations of our basic notion of invariant sets together with numerical examples illustrating the other types of invariant sets introduced below. These notions are of independent interest and might be easier to study.

Variation 1: invariant sets for roots of polynomials of a fixed degree

Instead of looking for a set which is invariant for roots of polynomials of degree at least n, we can relax the requirement and ask that a set is only invariant for roots of polynomials of degree exactly n. We call this property T_n-invariance, see Definition 1.5.

Given T and n, we denote by $\mathcal{I}_n^T$ the family of T_n-invariant sets, and we denote by M_n^T the corresponding unique minimal closed invariant set (if it exists), see the introduction. Note that $\mathrm{M}_n^T \subseteq \mathrm{M}_{\geq n}^T$. It is natural to study M_n^T for exactly solvable operators T, since in this case they preserve the degrees of polynomials they act upon.

One can observe that in many cases M_n^T can have a complicated structure – in particular, it does not need to be convex, and it can be a fractal, etc. An illustration can be found in Example 7.1 and Figure 6. We plan to carry out the detailed study of T_n-invariant sets in the sequel paper [2].

Example 7.1. The minimal invariant set M_1^T for the differential operator $T = (x^2 - x + i)\,d/dx + 1$ coincides with the classical Julia set associated with $f(x) = x^2 + i$.

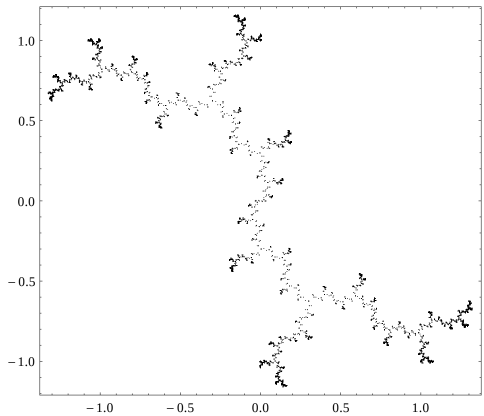

Figure 6. The minimal set M_1^T for the operator $(x^2 - x + i)\,d/dx + 1$. This set has the property that if $t \in \mathrm{M}_1^T$, then $\pm\sqrt{t-i}$ is also in M_1^T.

Variation 2: Hutchinson-invariant sets

A set $S \subset \mathbb{C}$ is called *Hutchinson-invariant in degree n* if every polynomial of the form $P(x) = (x-t)^n$, with $t \in S$, has the property that $T(P)$ has all roots in S (or is constant). In particular, a T_1-invariant set is a Hutchinson-invariant set in degree 1 and vice versa. However, for $n > 1$, T_n-invariant sets and Hutchinson-invariant sets in degree n in general do not coincide. We denote by $\mathcal{H}_n^T$ the collection of all Hutchinson-invariant sets in degree n and by $\mathrm{HM}_n^T \in \mathcal{H}_n^T$ the unique minimal under inclusion closed Hutchinson-invariant set in degree n (if it exists). Notice that

$$\mathrm{HM}_n^T \subseteq \mathrm{M}_n^T \subseteq \mathrm{M}_{\geq n}^T.$$

In particular, if HM_n^T exists, then M_n^T and $\mathrm{M}_{\geq n}^T$ exist as well.

To explain our choice of terminology, recall that a *Hutchinson operator* is defined by a finite collection of univariate functions $\phi_1, \dots, \phi_m$, and its invariant sets were introduced and studied in [15] as well as a large number of follow-up papers. In our situation, let us assume that the action of T on $(x-t)^n$ factorizes as

$$T((x-t)^n) = (x - (a_1 t + b_1)) \cdots (x - (a_m t + b_m)), \tag{7.1}$$

see, e.g., (3.2). Then we have that if $S \subset \mathbb{C}$ is Hutchinson-invariant in degree n, then $f_i(S) \subseteq S$ for all $i = 1, 2, \dots, m$, where $f_i(t) = a_i t + b_i$. If all these f_i are contractions, that is, $|a_i| < 1$, one can show that there is a unique minimal non-empty closed Hutchinson-invariant set S, and it is exactly the invariant set associated with the *Hutchinson operator* defined by $f_1, \dots, f_m$, see [15]. (One can also consider other types of factorizations similar to (7.1) with, e.g., polynomial or rational factors.) This observation implies that one can obtain many classical fractal sets such as the Sierpiński triangle, the Cantor set, the Lévy curve and the Koch snowflake as Hutchinson-invariant sets, see Example 7.2. In particular, M_n^T does not have to be connected.

Julia sets associated with rational functions can also be realized as Hutchinson-invariant sets of appropriately chosen operators T, see [2]. Let us illustrate the situation with Example 7.2 and Example 7.3.

Example 7.2. For the differential operator $T = x(x+1)\frac{d^2}{dx^2} + i\frac{d}{dx} + 2$, the set HM_2^T is a Lévy curve. The roots of $T((x-t)^2)$ are given by

$$x = \frac{1+i}{2} t \quad \text{and} \quad x = \frac{1-i}{2}(t-i).$$

The two maps

$$t \mapsto \frac{1+i}{2} t \quad \text{and} \quad t \mapsto \frac{1-i}{2}(t-i) \tag{7.2}$$

are both affine contractions which together produce a fractal Lévy curve as their invariant set, see Figure 7. In particular, every member of $\mathcal{I}_2^T$ must contain HM_2^T given by the latter curve which also implies that M_2^T exists.

Example 7.3. The differential operator $T = x(x-1)d/dx + 1$ admits *two* minimal[1] sets HM_1^T, one of which is the one-point set $\{0\}$ and the other is the unit circle. This fact is in line with known properties of the Julia sets; some very special rational functions admit several completely invariant sets containing one or two points. The reason why the above case is exceptional is that T maps the polynomial x to x^2, which has the same zeros as x. In general, such exceptional invariant sets only show up in the situation when there exists some t such that $T(x-t) = c(x-t)^k$, see [4].

Remark 7.4. There are at least two advantages in studying Hutchinson-invariant sets compared to the set-up of the present paper. The first one is that the occurring types of fractal sets have already been extensively studied which connects this topic to the existing

[1] Minimal here means that no proper closed subset is an invariant set.

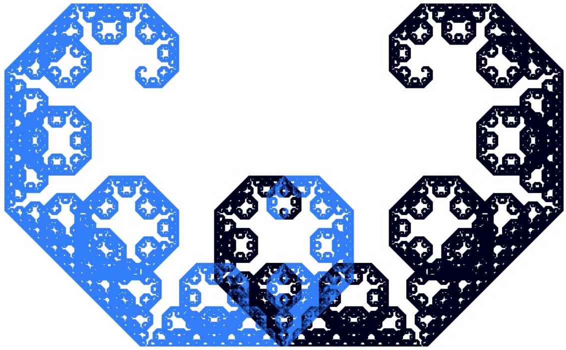

Figure 7. The Hutchinson-invariant set HM_2^T for the operator $T = x(x+1)\frac{d^2}{dx^2} + i\frac{d}{dx} + 2$. The two colors indicate the image of the set under the two maps in (7.2).

classical complex dynamics, see, e.g., [3, 13]. The second advantage is that there exists a stable Monte–Carlo-type method for producing a good approximation of HM_n^T, whenever the latter set is compact. Namely,

(1) start with some $z_0 \in \mathbb{C}$;

(2) for $j = 0, 1, 2, \dots$, pick randomly a root of $T((x - z_j)^n)$ with equal probability, and denote it by z_{j+1};

(3) plot z_{j+1} and iterate step 2 until a picture emerges.

Our experiments show that about 100 iterations per final pixel gives a clear picture. This algorithm was used to create Figure 6. The set of points z_j rapidly converge to the set HM_n^T, and the initial choice of z_0 statistically will not matter.

Further information about Hutchinson-invariant sets can be found in [14].

Variation 3: Continuously Hutchinson-invariant sets

Given T and n as above, consider

$$\psi(x, t, n) := T((x-t)^n)/(x-t)^{n-k},$$

where k is the order of the operator T. Then $\psi(x, t, n)$ is a polynomial in $\mathbb{C}[x, t, n]$. Given $n_0 \geq 0$, we say that a set S is *continuously Hutchinson-invariant with parameter* $\geq n_0$ if for every *real* number $n \geq n_0$, we have that

$$\psi(x, t_0, n) = 0 \quad \text{(considered as a polynomial in } \mathbb{C}[x])$$

has all roots in S, whenever $t_0 \in S$. We denote by $\mathcal{CH}^T_{\geq n_0}$ the collection of all continuously Hutchinson-invariant with parameter $\geq n_0$ and by $\mathrm{CHM}^T_{\geq n_0}$ the minimal non-empty closed such set S (it if exists). It is easy to verify that, for all integers $m \geq 1$,

$$\mathrm{HM}_m^T \subseteq \mathrm{CHM}^T_{\geq m} \subseteq \mathrm{CHM}^T_{\geq 0}.$$

Properties of the minimal continuously Hutchinson invariant set $\mathrm{CHM}^T_{\geq n_0}$ seem to substantially depend on whether $n_0 = 0$ or $n_0 > 0$: namely, the boundary of $\mathrm{CHM}^T_{\geq 0}$ looks rectifiable, while the boundary of $\mathrm{CHM}^T_{\geq 1}$ seem to have a fractal (and non-rectifiable) character. However, in contrast with Hutchinson-invariant sets which can be fractal, $\mathrm{CHM}^T_{\geq n_0}$ always has a finite number of connected components. For operators T of order 1, continuously Hutchinson invariant sets with positive parameter have been studied in detail in [1].

In general, it is unclear what the relation between $\mathrm{CHM}^T_{\geq n}$ and $\mathrm{M}^T_{\geq n}$ is, but for large n, we expect the inclusion $\mathrm{CHM}^T_{\geq n} \subseteq \mathrm{M}^T_{\geq n}$, since extending the domain of n from the set of large integers to the set of large real numbers does not seem to make a big difference. Note that Theorem 3.14 and Proposition 7.5 suggest that these sets coincide in the limit $n \to \infty$.

The following result shows that as n_0 grows, the minimal continuously Hutchinson-invariant set converges to the zero locus of the leading coefficient Q_k of T.

Proposition 7.5 (Convergence to the zero locus of Q_k). *Given a non-degenerate operator $T = \sum_{j=0}^{k} Q_j \frac{d^j}{dx^j}$, $R > 0$ and $\delta > 0$, then there exists $n_0 = n_0(R, \delta)$ such that for all $t \in \mathbb{C}$ with $|t| < R$, we have that each root of*

$$T[(x-t)^n] = 0$$

different from t lies at a distance at most δ from some root of $Q_k(x)$.

In particular, for any $\delta > 0$, there exists an $n_0 = n_0(\delta)$ such that the δ-neighborhood of the union of roots of $Q_k(x)$ is Hutchinson-invariant in degree n, for all $n \geq n_0$. The same holds for the continuously Hutchinson-invariant sets with parameter exceeding n.

Proof. Fix $R > 0$ and $\delta > 0$. A straightforward calculation shows that

$$\frac{\psi(x,t,n)}{n(n-1)\cdots(n-k+1)} = Q_k(x) + \sum_{j=1}^{k} \frac{Q_{k-j}(x)(x-t)^j}{(n-k+1)(n-k+2)\cdots(n-k+j)}.$$

Hence, the zeros $\psi(x,t,n) = 0$ tend to the zeros $Q_k(x)$ as $n \to \infty$, provided that $|t| < R$. Thus, for some $n_0 := n_0(\delta)$, all roots of $\psi(x,t,n) = 0$ lie at a distance at most δ from the fundamental polygon of T. ∎

Variation 4: two-point continuously Hutchinson invariant sets

Our last variation of the notion of invariant sets is inspired by the convexity property of invariant sets from $\mathcal{I}^T_{\geq n}$.

Set

$$P(x) := (x-t_1)^{n_1}(x-t_2)^{n_2}$$

and consider

$$\phi(x,t_1,n_1,t_2,n_2) := \frac{T(P)}{(x-t_1)^{n_1-k}(x-t_2)^{n_2-k}},$$

where k is the order of the operator T. Again, $\phi(x,t_1,n_1,t_2,n_2)$ is a polynomial in $\mathbb{C}[x,t_1,n_1,t_2,n_2]$. Given $n_0 \geq 0$, a set $S \subset \mathbb{C}$ is called *two-point continuously Hutchinson invariant with parameters* $\geq n_0$ if for every pair of real number $n_1, n_2 \geq n_0$, we have that

$$\phi(x,t_1,n_1,t_2,n_2) = 0 \quad \text{(considered as a polynomial in } x\text{)}$$

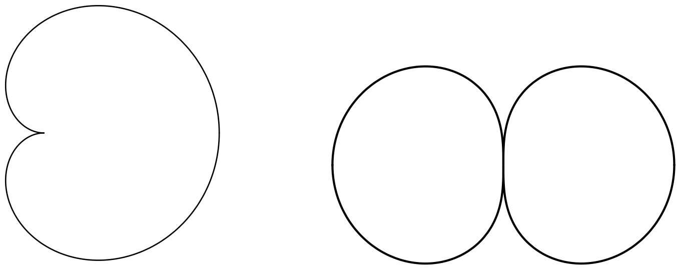

Figure 8. The boundaries of the minimal continuously Hutchinson-invariant sets $\mathrm{CHM}^T_{\geq 0}$ for the operators $T = z^2 d/dz + (z-1)$ (left), and $T = z^3 d/dz + (z+1)(z-1)$ (right). The first curve is parameterized by $r(\theta) = \sin\theta/\theta$ in polar coordinates, while the second is given by the equation $r^2(\theta) = \sin 2\theta/(2\theta)$. Proofs of these facts can be found in [1].

has all roots in S, whenever $t_1, t_2 \in S$. We denote by $\mathrm{C_2HM}^T_{\geq n_0}$ the minimal under inclusion non-empty closed set S which is two-point Hutchinson invariant with parameters $\geq n_0$ (if it exists).

Obviously, $\mathrm{CHM}^T_{\geq n_0} \subseteq \mathrm{C_2HM}^T_{\geq n_0}$. Moreover, we can apply the same technique as in Theorem 2.2, to show that two-point continuous invariant sets are *convex*.

Remark 7.6. The linear operators which factor as in (7.1) allow us to produce a large class of fractal sets associated with Hutchinson operators, where each map is an affine contraction from $\mathbb{C}$ to $\mathbb{C}$. These minimal invariant sets HM^T_n are fractals, and therefore might be difficult to study. It is highly plausible that continuously Hutchinson invariant set $\mathrm{CHM}^T_{\geq 0}$ or its larger convex cousin $\mathrm{C_2HM}^T_{\geq 0}$ have piecewise analytic boundary. For operators of order 1, discussions of analyticity of the boundary of the former set can be found in [1]. Remember that we have the set of inclusions

$$\mathrm{HM}^T_n \subseteq \mathrm{CHM}^T_{\geq 0} \subseteq \mathrm{C_2HM}^T_{\geq 0},$$

so a simple description of $\mathrm{CHM}^T_{\geq 0}$ may provide some additional insight in the nature of HM^T_n.

8. Some open problems

Here we present a very small sample of unsolved questions directly related to the results of this paper.

(1) The major open problem is whether it is possible to describe the boundary of $\mathrm{M}^T_{\geq n}$ for non-degenerate or degenerate operators with non-defining Newton polygons and Q_k different from a constant. At the moment, we only have some information what happens with $\mathrm{M}^T_{\geq n}$ when $n \to \infty$. Already for non-degenerate operators of order 1, this problem seems to be quite non-trivial, cf. [1].

(2) Another important issue is how $\mathrm{M}^T_{\geq n}$ depend on the coefficients of operator T. It seems that even in the case when T is non-degenerate and n is such that $\mathrm{M}^T_{\geq n}$ is compact, it might loose compactness under small deformation of T with the space of non-degenerate operators of the same order. Even for operators of order one the question is non-trivial. For example, consider the space of pairs of polynomials $(Q_1(x), Q_0(x))$ where $\deg Q_1(x) = k$, $\deg Q_0(x) = k-1$ and $T = Q_1(x)d/dx + Q_0(x)$. Fixing a positive integer n, is it possible to describe the space of such pairs $(Q_1(x), Q_0(x))$ for which $\mathrm{M}^T_{\geq n}$ is compact?

(3) Is it possible to characterize the invariant sets for Case B, i.e., operators with constant leading term and $\deg Q_{k-1} = 1$, see end of Section 6.

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