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On the nonlinear thin obstacle problem

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Abstract. The thin obstacle problem, or n-dimensional Signorini problem, is a classical variational problem with roots in elasticity theory and wide-ranging applications. The vast literature concerns mostly quadratic energies, whereas only partial results have been proved in the nonlinear case. In this paper, we consider the thin boundary obstacle problem for a general class of nonlinearities and we prove the optimal $C^{1,1/2}$-regularity of the solutions in any space dimension.

1. Introduction

We are interested in the boundary (thin) obstacle problems for a class of nonlinear functionals of the type

$$\min_{u\in\mathcal{A}_g}\int_{B_1^+} f(\nabla u)\,\mathrm{d}x, \tag{1.1}$$

where B_1 is the unit ball in $\mathbb{R}^n$, with $n\geq 2$, and for any subset $E\subset\mathbb{R}^n$,

$$E^+ = E\cap\{x_n>0\} \quad\text{and}\quad E' := E\cap\{x_n=0\}.$$

- The class of competitor functions is

$$\mathcal{A}_g := \{u\in W_0^{1,\infty}(B_1^+) : u_{|B_1'}\geq 0,\ u_{|(\partial B_1)^+} = g_{|(\partial B_1)^+}\},$$

  where $g\in C^2(\mathbb{R}^n)$ prescribes the boundary values and satisfies $g_{|\mathbb{R}^{n-1}\times\{0\}}\geq 0$.
- The nonlinear (non-quadratic) energy density $f:\mathbb{R}^n\to\mathbb{R}$ is of the form

$$f(p) = h(|p|) \quad\text{for all } p\in\mathbb{R}^n, \tag{1.2}$$

  with $h\in C^2(\mathbb{R})$ satisfying

$$h(0)=h'(0)=0,\quad h''(t)=1+O(t)\quad\text{for } t\to 0^+. \tag{1.3}$$

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- The function f is convex and the matrix $\nabla_p^2 f(p)$ is uniformly positive definite in compact subsets, i.e., fulfills the following local ellipticity condition:

$$\forall M > 0, \exists[5]\lambda = \lambda(M) > 0 : \langle \nabla_p^2 f(p)\xi, \xi \rangle \geq \lambda|\xi|^2,\ \forall |p| \leq M,\ \forall \xi \in \mathbb{R}^n. \tag{1.4}$$

This class of problems contains the linear case of the Dirichlet energy $h(t) = t^2/2$ and the geometric case of minimal surfaces $h(t) = \sqrt{1+t^2} - 1$, where the constant 1 is clearly irrelevant in the minimization problem. The existence and uniqueness of a solution u in the class $\mathcal{A}_g$ under suitable growth conditions on f can be established following classical results by Giaquinta–Modica and Giusti [25, 28]. The solution to (1.1) can be characterized as the weak solution to the system

$$\operatorname{div}(\nabla_p f(\nabla u)) = 0 \quad \text{in } B_1^+, \tag{1.5a}$$
$$u \nabla_p f(\nabla u) \cdot e_n = 0 \quad \text{on } B_1', \tag{1.5b}$$
$$-\nabla_p f(\nabla u) \cdot e_n \geq 0 \quad \text{on } B_1', \tag{1.5c}$$
$$u \geq 0 \quad \text{on } B_1', \tag{1.5d}$$
$$u = g \quad \text{on } (\partial B_1)^+, \tag{1.5e}$$

where e_n denotes the last vector of the standard basis of $\mathbb{R}^n$.

1.1. Previous results

The minimization problem (1.1) with the Dirichlet integral $\int |\nabla u|^2$ stems from the pioneering works of Signorini in elasticity theory [38]. Since then, lower dimensional obstacle problems naturally appeared in several fields, with applications, for instance, in fluid mechanics when describing osmosis through semi-permeable membranes, as well as in the boundary heat control problem, and many other contexts (see, e.g., the book by Duvaut-Lions [13] and the survey papers [15, 35, 37] for an extensive bibliography and further applications).

Given for granted the existence of Lipschitz solutions, which in great generality have been shown to exist in the works by Giaquinta–Modica [25] and Giusti [28], the main questions concern two aspects of the solutions u:

- the regularity of u up to B_1', where the boundary conditions are not prescribed but are implicitly determined by the solution itself through its free boundary,
- the regularity of the free boundary, that is the subset of B_1' where the solution ceases to saturate the one-side constraint and takes the natural boundary conditions.

Most of the results in the literature deal with the linear case $\nabla_p f(p) = p$, for which both the optimal regularity of the solution u and the structure of the free boundary are known. As far as the first aspect is concerned, we recall the results by Lewy [31] and Richardson [34] for $n = 2$, Caffarelli [6], Kinderlehrer [30], Ural'tseva [40] for the $C^{1,\alpha}$ regularity, for some $\alpha > 0$; whereas the optimal regularity of the solutions is due to Athanasopoulos–Caffarelli in [4], where the authors show that the solution to (1.1) with $\nabla_p f(p) = p$ are $C^{1,1/2}_{\mathrm{loc}}$ in $B_1^+ \cup B_{1'}$, and that in general cannot be more regular.

On the other hand, the nonlinear case is much less understood, despite its obvious relevance in the applications. One of the first occurrences of such instance can be found in a paper by Nitsche [33] for the case of the minimal surface operator, for which the existence of Lipschitz solutions has been first proven by Giusti [27]. Since then, only partial results have been established for the nonlinear case, which turned out to be considerably more complicated than the linear counterpart. The first important contributions are given by Frehse [20, 21], where, under very general assumptions, the author shows the continuity of the first derivatives along the directions parallel to B_1' in any dimension $n \geq 3$, and the continuity of the normal derivative in the case $n = 2$ for the Lipschitz solutions to nonlinear elliptic variational inequalities. More refined results, regarding both the optimal regularity of the solutions and their free boundaries, have been found by Athanasopoulos [3] for the minimal surface case in dimension $n = 2$, and only recently more precise results for some nonlinear cases in general dimension appeared. We recall the case of minimal surfaces with flat obstacles treated by Focardi and the third named author [18], where the optimal Hölder continuity of the derivatives is proven, together with a study of the structure and the properties of the free boundary; as well as the study of a general class of nonlinear variational inequalities by Di Fazio and the third named author in [12], where for the boundary obstacle problem the non-optimal $C^{1,\alpha}$ regularity in $B_1^+ \cup B_1'$ is shown. We point out also the nonlinear thin obstacle problems considered in [11] in relation with the boundary Bernoulli problem in dimension $n = 2$, where the optimal $C^{1,1/2}$ regularity and the structure of the free boundary are derived with the use of complex analysis techniques and the connected literature on the fully nonlinear case (see, e.g., [8, 14, 32, 36]), where analogous results are obtained with completely different methods, in particular, in [8] the $C^{1,1/2}$ regularity is proven under the assumption of rotational invariant fully nonlinear operators.

Our understanding is even more limited concerning the properties of the free boundary of the solutions. Indeed, if for the linear problem many details on the free boundary have been investigated (structure of regular points [7, 16], singular points [22], non-regular and non-singular points [2, 17, 19], to mention only few of the most relevant works), for the nonlinear case much less is known, left aside the aforementioned results in the very special case of two dimensions and for the minimal surface case considered in [18].

1.2. Main result: Optimal $C^{1,1/2}$ regularity

The aim of this paper is to establish the optimal $C^{1,1/2}$-regularity for the general class of problems (1.1). The starting point is the $C^{1,\alpha}$-regularity established in [12].

Theorem 1.1. *Let u be the solution to the boundary obstacle problem* (1.1) *or equivalently* (1.5). *Then for every $r < 1$, there exists a constant $C > 0$, depending on r,* Lip(u) *and the nonlinear energy density f, such that*

$$\|u\|_{C^{1,1/2}(B_r^+ \cup B_r')} \leq C \|u\|_{L^2(B_1^+)}.$$

The idea of the proof is very much influenced by the work on stationary two-valued graphs done by Simon–Wickramasekera [39]. This connection has been already used in the case of the minimal surface operator in [18], whose $C^{1,\alpha}$ solutions fulfilled the hypotheses (suitably formulated) of [39]. Here instead we show how the proof of [39] can be adapted in order to cover the more general cases under examination.

We proceed in several steps. First, we introduce a new frequency function tailored to the nonlinear operator $\nabla_p f(\nabla u)$ that is a modification of the famous *Almgren frequency* *function* used for Dirichlet energy minimizers [1] (see also [10]). In order to show its (almost) monotonicity, we need to use the $C^{1,\alpha}$-regularity established in [12] and to exploit a dichotomy argument first used in [39]. The monotonicity formula for this modified frequency allows then to study the limit of the blow-up sequence and leads to the optimal $C^{1,1/2}$-regularity as in the classical case of quadratic energies.

A byproduct of our main result Theorem 1.1 is that the results for the free boundary of the solutions to thin obstacle problems for a linear differential operator with Hölder continuous coefficients apply (see [23]), and moreover, if one has that $h(t) = t^2/2 + O(t^4)$ (as in the minimal surface case, for instance), then the optimal regularity in Theorem 1.1 implies that the solution u solves a thin obstacle problem for a linear operator with Lipschitz coefficients, and more refined results on the structure of the singular set can be derived (see [2, 18, 24]). Moreover, we expect that the techniques developed in the present paper, and in particular the search for a form of the frequency functions tailored to nonlinear problems, can be useful in related questions in the geometric calculus of variations (see, e.g., the general nonlinear energies for multiple valued functions [9]).

The more general nonlinear thin obstacle problems, starting from the question raised by Nitsche on minimal surfaces with non-constant unilateral thin constraints, remain still open, and more refined techniques need to be developed.

2. Frequency function

2.1. Preliminary estimates

The starting point of our analysis is the $C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$-regularity of the solutions u of (1.1) established in [12]. For our aims, we need to use the following Schauder and $W^{2,2}$ estimates: for any $\varrho \in (0, \bar\varrho)$,

$$\|u\|_{L^\infty(B^+_{\varrho/2})} + \varrho\|\nabla u\|_{L^\infty(B^+_{\varrho/2})} + \varrho^{1+\alpha}[\nabla u]_{\alpha,B^+_{\varrho/2}} \le k\Big(\varrho^{-n}\int_{B^+_\varrho} u^2\Big)^{1/2} \tag{2.1}$$

and

$$\|\nabla^2 u\|_{L^2(B^+_{\varrho/2})} \le \beta\varrho^{-2}\|u\|_{L^2(B^+_\varrho)}, \tag{2.2}$$

where the constants $\bar\varrho, k, \beta > 0$ depend on the dimension n, $\mathrm{Lip}(u)$ and the nonlinear energy density f, and

$$[\nabla u]_{\alpha,B^+_\varrho} := \sup_{x\neq y\in B^+_\varrho} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\alpha}$$

is the usual Hölder seminorm. These estimates, although not explicitly derived in [12], follow straightforwardly from the proofs therein, and we give the details in Appendix A.

2.2. Frequency function

We use the notation Λ_u for the contact set:

$$\Lambda_u := \{u = 0\} \cap B_1',$$

and Γ_u for the free boundary

$$\Gamma_u := \{x \in B'_1 : B'_r(x) \cap \Lambda_u \neq \emptyset \text{ and } B'_r(x) \setminus \Lambda_u \neq \emptyset \text{ for all } r > 0\}.$$

In the sequel of this section we will always assume that the origin is a point of the free boundary:

$$0 \in \Gamma_u.$$

Recall that by the $C^{1,\alpha}$-regularity, this implies that

$$u(0) = 0 = |\nabla u(0)|.$$

For any $\varrho \in (0,1)$, we introduce the following nonlinear modification of the standard frequency function of u:

$$N(\varrho) := \frac{D(\varrho)}{H(\varrho)}, \tag{2.3}$$

where

$$D(\varrho) := \varrho^{2-n} \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}x, \quad H(\varrho) := \varrho^{1-n} \int_{(\partial B_\varrho)^+} u^2 \, \mathrm{d}\sigma_x.$$

For convenience, we set the notation

$$\|u\|_\varrho = \left(\varrho^{-n} \int_{B_\varrho^+} u^2 \, \mathrm{d}x\right)^{1/2}.$$

The main result about the frequency is that, under the assumption of a *doubling condition* for u, a quasi-monotonicity formula holds.

Proposition 2.1. *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B'_1)$ be the solution to the boundary obstacle problem (1.5), and assume that $0 \in \Gamma_u$ and (2.1) and (2.2) hold with constants $\bar{\varrho}$, k and β. If there are $\gamma > 1$ and $\sigma \in (0, 1/2]$ such that*

$$\|u\|_\sigma > 0, \quad \|u\|_\varrho \le \gamma \|u\|_{\varrho/2} \quad \text{for any } \varrho \in (0,\sigma], \tag{2.4}$$

then there exists $\varrho_0 = \varrho_0(\bar{\varrho}, \sigma) \in (0,1)$ such that for all $\varrho \in (0, \varrho_0)$, the frequency function $N(\varrho)$ is well defined, bounded and

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}[\exp(\alpha^{-1} C \varrho^\alpha) N(\varrho)] \ge 0 \quad \text{for any } \varrho \in (0, \varrho_0),$$

where $C = C(n, \alpha, \gamma, \beta, k, [\nabla u]_\alpha) > 0$.

Remark 2.2. Let us observe that the doubling condition (2.4) can be written as

$$\int_{B_\varrho^+} u^2 \, \mathrm{d}x \le 2^n \gamma^2 \int_{B_{\varrho/2}^+} u^2 \, \mathrm{d}x \quad \text{for any } \varrho \in (0,\sigma], \tag{2.5}$$

and, in particular, $\|u\|_\sigma > 0$ implies that

$$\|u\|_\varrho > 0 \quad \text{for any } \varrho \in (0,\sigma].$$

In order to prove Proposition 2.1, we need some technical results.

Remark 2.3. Note that if h satisfies (1.3), then there exist $\bar{t} > 0$ and functions ω_1 and ω_2 such that

$$h''(t) = 1 + \omega_2(t) \quad \text{and} \quad h'(t) = t(1 + \omega_1(t)),$$

with the functions ω_1 and ω_2 satisfying, for all $t \in (0, \bar{t})$,

$$|\omega_2(t)| + |\omega_1(t)| + t|\omega_1'(t)| \le Ct, \quad \omega_2(t) = \omega_1(t) + t\omega_1'(t).$$

Moreover, formula (1.2) yields

$$\nabla_p f(p) = h'(|p|)\, \frac{p}{|p|} = (1 + \omega_1(|p|))p \tag{2.6}$$

and

$$\begin{aligned} \nabla_p^2 f(p) &= \frac{h'(|p|)}{|p|}\, \mathrm{Id} + \Big(h''(|p|) - \frac{h'(|p|)}{|p|}\Big) \frac{p \otimes p}{|p|^2} \\ &= (1 + \omega_1(|p|))\, \mathrm{Id} + \omega_1'(|p|)\, \frac{p \otimes p}{|p|}. \end{aligned} \tag{2.7}$$

Therefore, straightforward computations imply the following estimates:

$$|\nabla u - \nabla_p f(\nabla u)| \le C|\nabla u|^2, \tag{2.8}$$

$$|\Delta u - \mathrm{div}(\nabla_p f(\nabla u))| \le C|\nabla u|\,|\nabla^2 u|, \tag{2.9}$$

where the constant C is independent of u.

The following version of the Poincaré inequality is needed (see also Section 6 of [39]).

Lemma 2.4. *Let $u \in C^{1,\alpha}(B_1^+)$ with $u(0) = 0$ fulfilling the Schauder estimates (2.1). Then*

$$\int_{B_\varrho^+} u^2\, \mathrm{d}x \le C\varrho^2 \int_{B_\varrho^+} |\nabla u|^2\, \mathrm{d}x \quad \textit{for any } \varrho \in (0, \bar{\varrho}), \tag{2.10}$$

with $C > 0$ depending on the dimension n, the constants in the Schauder estimates, and in the doubling condition.

Proof. Let $\varrho \in (0, \bar{\varrho})$, $\theta \in (0, 1/2)$ and

$$\lambda := \fint_{B_{\theta\varrho}^+} u\, \mathrm{d}x.$$

Then the appropriate version of the Poincaré inequality (see Lemmas 7.12 and 7.16 of [26]) implies

$$\int_{B_\varrho^+} |u - \lambda|^2\, \mathrm{d}x \le C\varrho^2 \int_{B_\varrho^+} |\nabla u|^2\, \mathrm{d}x, \quad \text{with } C = C(n, \theta) > 0.$$

Therefore,

$$\int_{B_\varrho^+} u^2 \,\mathrm{d}x \le 2\int_{B_\varrho^+} (|u-\lambda|^2+\lambda^2)\,\mathrm{d}x \le 2C\varrho^2 \int_{B_\varrho^+} |\nabla u|^2\,\mathrm{d}x + \omega_n \varrho^n \lambda^2. \tag{2.11}$$

Since there exists $y \in B_{\theta\varrho}^+$ with $u(y)=\lambda$, and since $u(0)=0$, it follows that

$$\lambda \le \theta\varrho \sup_{B_{\theta\varrho}^+} |\nabla u|.$$

Employing also the L^∞-estimate for ∇u in (2.1), from (2.11), we get that

$$\int_{B_\varrho^+} u^2\,\mathrm{d}x \le C\varrho^2 \int_{B_\varrho^+} |\nabla u|^2\,\mathrm{d}x + k^2\omega_n\theta^2 \int_{B_\varrho^+} u^2\,\mathrm{d}x,$$

where k is the constant in (2.1). This in turn gives (2.10) if θ is sufficiently small, depending on k. ∎

In order to establish the boundedness of the frequency function we need the following auxiliary lemma.

Lemma 2.5. *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$ be the solution of the boundary obstacle problem (1.1) fulfilling the Schauder estimates (2.1) and (2.2) with constants $\bar\varrho$, k and β, the doubling condition (2.5) with constants γ and σ, and $0 \in \Gamma(u)$. Then*

$$\int_{(\partial B_\varrho)^+} u^2\,\mathrm{d}\sigma_x \le \frac{C_1}{\varrho}\int_{B_\varrho^+} u^2\,\mathrm{d}x \quad \textit{for all } \varrho < \frac{1}{2}\min\{\bar\varrho,\sigma\}, \tag{2.12}$$

with $C_1 = C(n,k,\gamma) > 0$, and there exists $\tilde\varrho \in (0,\bar\varrho)$ such that

$$\int_{B_\varrho^+} u^2\,\mathrm{d}x \le \varrho \int_{(\partial B_\varrho)^+} u^2\,\mathrm{d}\sigma_x + C_2\varrho^{1+\alpha}\int_{B_\varrho^+}|u||\nabla u|\,\mathrm{d}x \quad \textit{for all } \varrho \in (0,\tilde\varrho), \tag{2.13}$$

with $C_2 = C_2([\nabla u]_\alpha) > 0$.

Proof. We start by proving (2.12). Employing the Schauder estimates (2.1) and the doubling condition (2.5), we have that for any $0 < s < \varrho < \frac{1}{2}\min\{\bar\varrho,\sigma\}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big(\int_{(\partial B_s)^+} u^2\,\mathrm{d}\sigma_x\Big) &= \frac{n-1}{s}\int_{(\partial B_s)^+} u^2\,\mathrm{d}\sigma_x + 2\int_{(\partial B_s)^+} u\nabla u\cdot\eta\,\mathrm{d}\sigma_x \\
&\le \frac{(n-1)k^2}{s\varrho^n}|(\partial B_s)^+|\int_{B_{2\varrho}^+} u^2\,\mathrm{d}x + \frac{2k^2}{\varrho^{n+1}}|(\partial B_s)^+|\int_{B_{2\varrho}^+} u^2\,\mathrm{d}x \\
&\le (n-1)n\,2^n\,\omega_n\,s^{n-2}\frac{k^2\gamma}{\varrho^n}\int_{B_\varrho^+} u^2\,\mathrm{d}x + n\omega_n 2^n s^{n-1}\frac{k^2\gamma}{\varrho^{n+1}}\int_{B_\varrho^+} u^2\,\mathrm{d}x \\
&\le C(n,k,\gamma)\frac{1}{\varrho^2}\int_{B_\varrho^+} u^2\,\mathrm{d}x,
\end{aligned}$$

where $\eta(x) = x/|x|$ is the outer normal vector and $C(n,k,\gamma) > 0$ is a constant. Integrating with respect to $s \in (0,\varrho)$, with $\varrho < 1/2\min\{\bar\varrho,\sigma\}$, (2.12) follows.

Turning to (2.13), we notice that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\int_{(\partial B_s)^+} u^2\,\mathrm{d}\sigma_x\Big) = (n-1)\,s^{n-2}H(s) + 2\int_{(\partial B_s)^+} u\nabla u\cdot\eta\,\mathrm{d}\sigma_x \geq 2\int_{(\partial B_s)^+} u\nabla u\cdot\eta\,\mathrm{d}\sigma_x.$$

Since u fulfills (1.5), using the divergence theorem and (2.6), we deduce that

$$\begin{aligned}\int_{(\partial B_s)^+} u\nabla u\cdot\eta\,\mathrm{d}\sigma_x &= \int_{(\partial B_s)^+} u(\nabla u - \nabla_p f(\nabla u))\cdot\eta\,\mathrm{d}\sigma_x \\ &\quad + \int_{(\partial B_s)^+} u\nabla_p f(\nabla u)\cdot\eta\,\mathrm{d}\sigma_x \\ &= -\int_{(\partial B_s)^+} u\omega_1(|\nabla u|)\nabla u\cdot\eta\,\mathrm{d}\sigma_x \\ &\quad + \int_{B_s^+}\mathrm{div}(u\nabla_p f(\nabla u))\,\mathrm{d}x + \int_{B_s'} u\nabla_p f(\nabla u)\cdot e_n\,\mathrm{d}x'.\end{aligned} \tag{2.14}$$

Recall that $|\omega_1(t)| \leq Ct$ for $t \leq \bar{t}$. Hence, considering that $\nabla u(0) = 0$, we find a radius $\tilde{\varrho}$ such that

$$|\nabla u(x)| \leq \bar{t} \quad \text{for all } x \in B^+_{\tilde{\varrho}}.$$

This implies that

$$\Big|\int_{(\partial B_s)^+} u\omega_1(|\nabla u|)\nabla u\cdot\eta\,\mathrm{d}\sigma_x\Big| \leq C\int_{(\partial B_s)^+} |u||\nabla u|^2\,\mathrm{d}\sigma_x \quad \text{for all } s < \tilde{\varrho}.$$

As far as the third integral in (2.14) is concerned, we notice that by the Signorini boundary condition (1.5), we infer that

$$\int_{B_s'} u\nabla_p f(\nabla u)\cdot e_n\,\mathrm{d}x' = 0.$$

Finally, the second integral in (2.14) is positive for small enough radii. Indeed,

$$\begin{aligned}\int_{B_s^+}\mathrm{div}(u\nabla_p f(\nabla u))\,\mathrm{d}x &= \int_{B_s^+}\nabla u\cdot\nabla_p f(\nabla u)\,\mathrm{d}x + \int_{B_s^+} u\,\mathrm{div}(\nabla_p f(\nabla u))\,\mathrm{d}x \\ &= \int_{B_s^+}\nabla u\cdot\nabla_p f(\nabla u)\,\mathrm{d}x \\ &= \int_{B_s^+}|\nabla u|^2(1+\omega_1(\nabla u))\,\mathrm{d}x \geq 0,\end{aligned}$$

if $\tilde{\varrho}$ is small enough to ensure that $|\nabla u(x)| \leq C^{-1}$ for $x \in B_s^+$.

We can then estimate as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\int_{(\partial B_s)^+} u^2\,\mathrm{d}\sigma_x\Big) \geq -C\int_{(\partial B_s)^+} |u||\nabla u|^2\,\mathrm{d}\sigma_x \geq -Cs^{\alpha}\int_{(\partial B_s)^+} |u||\nabla u|\,\mathrm{d}\sigma_x,$$

where we employed the Hölder continuity of ∇u. Integrating the last inequality for $s \in (\tau, \varrho)$, with $\varrho < \tilde{\varrho}$, we obtain

$$\int_\tau^\varrho \frac{\mathrm{d}}{\mathrm{d}s}\Big(\int_{(\partial B_s)^+} u^2 \,\mathrm{d}\sigma_x\Big)\,\mathrm{d}s \geq -C\varrho^\alpha \int_\tau^\varrho \int_{(\partial B_s)^+} |u||\nabla u| \,\mathrm{d}\sigma_x \,\mathrm{d}s,$$

which gives

$$\int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x + C\varrho^\alpha \int_{B_\varrho^+} |u||\nabla u| \,\mathrm{d}x \geq \int_{(\partial B_\tau)^+} u^2 \,\mathrm{d}\sigma_x.$$

Integrating again with $\tau \in (0, \varrho)$,

$$\varrho \int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}x + C\varrho^{\alpha+1} \int_{B_\varrho^+} |u||\nabla u| \,\mathrm{d}x \geq \int_{B_\varrho^+} u^2 \,\mathrm{d}x,$$

which is (2.13). ■

2.3. Boundedness of the frequency.

If $0 \in \Gamma_u$, then u is a nonzero solution of the thin obstacle problem and this implies that $\int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x > 0$ for every $\varrho > 0$.

By virtue of (2.12) and the Poincaré inequality (2.10), it follows that

$$\int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x \leq \frac{C}{\varrho} \int_{B_\varrho^+} u^2 \,\mathrm{d}x \leq C\varrho \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x \quad \text{for all } \varrho \in (0, \min\{\sigma, \bar{\varrho}\}/2).$$

On the other hand, by the Schauder estimates (2.1) and the doubling condition (2.5), we get

$$\varrho^2 \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x \leq k^2\Big(\frac{1}{\varrho^n} \int_{B_{2\varrho}^+} u^2 \,\mathrm{d}x\Big)|B_\varrho^+| \leq C \int_{B_\varrho^+} u^2 \,\mathrm{d}x, \tag{2.15}$$

where $C = C(n, k, \gamma)$. Next, (2.13) and the Cauchy–Schwarz inequality imply that

$$\begin{aligned}\int_{B_\varrho^+} u^2 \,\mathrm{d}x &\leq \varrho \int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x + C\varrho^{1+\alpha} \int_{B_\varrho^+} |u|\,|\nabla u| \,\mathrm{d}x \\ &\leq \varrho \int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x + C\varepsilon\varrho^2 \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x + \frac{C\varrho^{2\alpha}}{\varepsilon} \int_{B_\varrho^+} |u|^2 \,\mathrm{d}x\end{aligned}$$

for any $\varepsilon > 0$. Then there exists $\varrho_0(\varepsilon) \in (0, \tilde{\varrho})$ such that

$$\Big(1 - \frac{C\varrho^{2\alpha}}{\varepsilon}\Big) \int_{B_\varrho^+} u^2 \,\mathrm{d}x \leq \varrho \int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x + C\varepsilon\varrho^2 \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x$$

for any $\varrho \in (0, \varrho_0(\varepsilon))$ (with positive left-hand side). Going back into (2.15), we have

$$\varrho^2 \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x \leq C\varrho \int_{(\partial B_\varrho)^+} u^2 \,\mathrm{d}\sigma_x + C\varepsilon\varrho^2 \int_{B_\varrho^+} |\nabla u|^2 \,\mathrm{d}x,$$

and thus choosing $\varepsilon > 0$ sufficiently small, we conclude

$$\varrho \int_{B_\varrho^+} |\nabla u|^2 \, dx \le C \int_{(\partial B_\varrho)^+} u^2 \, dx. \tag{2.16}$$

Therefore, using that

$$0 < c_1 \le 1 + \omega_1(|\nabla u|) \le c_2 \quad \text{for all } x \in B_{\varrho_0}^+ \text{ with } \varrho_0 \text{ sufficiently small,}$$

we can conclude that if $0 \in \Gamma_u$ and the doubling condition (2.5) holds, then there exist positive constants C_1, C_2 and ϱ_0 (depending on n, k, γ, and $[\nabla u]_\alpha$) such that

$$C_1 \le N(\varrho) \le C_2 \quad \text{for any } \varrho \in (0, \varrho_0]. \tag{2.17}$$

2.4. Quasi-monotonicity of the frequency

Here we prove Proposition 2.1.

Deriving (2.3), we obtain

$$N'(\varrho) = \frac{D'(\varrho)H(\varrho) - D(\varrho)H'(\varrho)}{H(\varrho)^2},$$

with

$$D'(\varrho) = (2-n)\varrho^{1-n} \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, dx + \varrho^{2-n} \int_{(\partial B_\varrho)^+} \nabla_p f(\nabla u) \cdot \nabla u \, d\sigma_x$$

and

$$H'(\varrho) = 2\varrho^{1-n} \int_{(\partial B_\varrho)^+} u u_\varrho \, d\sigma_x, \tag{2.18}$$

where $u_\varrho = \nabla u \cdot x/\varrho$ denotes the radial derivative. Now we can rewrite $N'(\varrho)$ as follows:

$$\begin{aligned} N'(\varrho) = {} & \frac{1}{(H(\varrho))^2}\Big[\Big(2\varrho^{2-n} \int_{(\partial B_\varrho)^+} |u_\varrho|^2 \, d\sigma_x\Big) H(\varrho) - \Big(\varrho^{2-n} \int_{(\partial B_\varrho)^+} u u_\varrho \, d\sigma_x\Big) H'(\varrho)\Big] \\ & + \frac{1}{(H(\varrho))^2}\Big[\Big(D'(\varrho) - 2\varrho^{2-n} \int_{(\partial B_\varrho)^+} |u_\varrho|^2 \, d\sigma_x\Big) H(\varrho) \\ & - \Big(D(\varrho) - \varrho^{2-n} \int_{(\partial B_\varrho)^+} u u_\varrho \, d\sigma_x\Big) H'(\varrho)\Big]. \end{aligned}$$

Then the Cauchy–Schwarz inequality implies that

$$\begin{aligned} N'(\varrho) \ge {} & \frac{1}{(H(\varrho))^2}\Big[\Big(D'(\varrho) - 2\varrho^{2-n} \int_{(\partial B_\varrho)^+} |u_\varrho|^2 \, d\sigma_x\Big) H(\varrho) \\ & - \Big(D(\varrho) - \varrho^{2-n} \int_{(\partial B_\varrho)^+} u u_\varrho \, d\sigma_x\Big) H'(\varrho)\Big]. \end{aligned} \tag{2.19}$$

We now estimate each term in the right-hand side of (2.19), beginning with

$$E_1 := \Big| D'(\varrho) - 2\varrho^{2-n} \int_{(\partial B_\varrho)^+} |u_\varrho|^2 \, d\sigma_x \Big|.$$

Using the divergence theorem first and $x \cdot e_n = 0$ for every $x \in B'_\varrho$, we deduce that

$$
\begin{aligned}
(2.20)\qquad \varrho \int_{(\partial B_\varrho)^+} & \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}\sigma_x \\
&= \int_{B_\varrho^+} \operatorname{div}(\nabla_p f(\nabla u) \cdot \nabla u \, x) \, \mathrm{d}x + \int_{B'_\varrho} \nabla_p f(\nabla u) \cdot \nabla u \, x \cdot e_n \, \mathrm{d}x \\
&= n \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}x + \int_{B_\varrho^+} (D^2 u \nabla_p f(\nabla u)) \cdot x \, \mathrm{d}x \\
&\quad + \int_{B_\varrho^+} (\partial_{p_l} \nabla_p f(\nabla u) \cdot \nabla u)(\partial_l \nabla u \cdot x) \, \mathrm{d}x.
\end{aligned}
$$

First, let us consider the second integral on the right-hand side and integrate by parts (with respect to the j-th variable in $D^2 u = \partial_i \partial_j u$):

$$
\begin{aligned}
\int_{B_\varrho^+} & (D^2 u \nabla_p f(\nabla u)) \cdot x \, \mathrm{d}x \\
&= \int_{(\partial B_\varrho)^+} \nabla u \cdot x \, \nabla_p f(\nabla u) \cdot \nu \, \mathrm{d}\sigma_x - \int_{B'_\varrho} \nabla u \cdot x \, \nabla_p f(\nabla u) \cdot e_n \, \mathrm{d}x \\
&\quad - \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}x - \int_{B_\varrho^+} (\nabla u \cdot x) \, (D_p^2 f(\nabla u) : D^2 u) \, \mathrm{d}x \\
&= \int_{(\partial B_\varrho)^+} \nabla u \cdot x \, \nabla_p f(\nabla u) \cdot \nu \, \mathrm{d}\sigma_x - \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}x \\
&\quad - \int_{B_\varrho^+} (\nabla u \cdot x)(D_p^2 f(\nabla u) : D^2 u) \, \mathrm{d}x,
\end{aligned}
$$

where we employed the Signorini boundary condition (1.5b) on B'_ϱ in the boundary integral on the right-hand side. Now, using (2.6) and (2.7), we get

$$
\begin{aligned}
\int_{B_\varrho^+} & (D^2 u \nabla_p f(\nabla u)) \cdot x \, \mathrm{d}x \\
&= \int_{(\partial B_\varrho)^+} (1 + \omega_1(|\nabla u|)) \nabla u \cdot x \, \nabla u \cdot \nu \, \mathrm{d}\sigma_x - \int_{B_\varrho^+} \nabla_p f(\nabla u) \cdot \nabla u \, \mathrm{d}x \\
&\quad - \int_{B_\varrho^+} \nabla u \cdot x \, (1 + \omega_1(|\nabla u|)) \Delta u \, \mathrm{d}x - \int_{B_\varrho^+} \nabla u \cdot x \, \omega_1'(|\nabla u|) \frac{\nabla u \otimes \nabla u}{|\nabla u|} : D^2 u \, \mathrm{d}x.
\end{aligned}
$$

Next we consider the third term on the right-hand side of (2.20):

$$
\begin{aligned}
\int_{B_\varrho^+} & (\partial_{p_l} \nabla_p f(\nabla u) \cdot \nabla u)(\partial_l \nabla u \cdot x) \, \mathrm{d}x = \int_{B_\varrho^+} \partial^2_{p_l p_j} f(\nabla u) \partial_j u \, \partial^2_{li} u \, x_i \, \mathrm{d}x \\
&= \int_{B_\varrho^+} (1 + \omega_1(|\nabla u|))(D^2 u \nabla u) \cdot x \, \mathrm{d}x + \int_{B_\varrho^+} \omega_1'(|\nabla u|) |\nabla u| (D^2 u \nabla u) \cdot x \, \mathrm{d}x =: A + B.
\end{aligned}
$$

We integrate by parts in A with respect to the jth variable in $D^2u = \partial_i\partial_j u$, and use (1.5b) on B'_ϱ:

$$A = \int_{B_\varrho^+} (1+\omega_1(|\nabla u|))\partial_{ij}u\,\partial_j u\,x_i\,\mathrm{d}x \tag{2.21}$$
$$= \int_{(\partial B_\varrho)^+} (1+\omega_1(|\nabla u|))\nabla u\cdot\nu\,\nabla u\cdot x\,\mathrm{d}\sigma_x$$
$$- \int_{B_\varrho^+} \nabla_p f(\nabla u)\cdot\nabla u\,\mathrm{d}x - \int_{B_\varrho^+} \nabla u\cdot x(1+\omega_1(|\nabla u|))\Delta u\,\mathrm{d}x$$
$$- \int_{B_\varrho^+} \nabla u\cdot x\,\omega_1'(|\nabla u|)\frac{\nabla u\otimes\nabla u}{|\nabla u|} : D^2u\,\mathrm{d}x.$$

Combining (2.20) and (2.21), we realize that

$$\varrho\int_{(\partial B_\varrho)^+} \nabla_p f(\nabla u)\cdot\nabla u\,\mathrm{d}\sigma_x = (n-2)\int_{B_\varrho^+}\nabla_p f(\nabla u)\cdot\nabla u\,\mathrm{d}x$$
$$+ 2\int_{(\partial B_\varrho)^+}(1+\omega_1(|\nabla u|))\nabla u\cdot\nu\,\nabla u\cdot x\,\mathrm{d}\sigma_x$$
$$- 2\int_{B_\varrho^+}\nabla u\cdot x(1+\omega_1(|\nabla u|)\Delta u\,\mathrm{d}x$$
$$- 2\int_{B_\varrho^+}\nabla u\cdot x\,\omega_1'(|\nabla u|)\frac{\nabla u\otimes\nabla u}{|\nabla u|}\cdot D^2u\,\mathrm{d}x + B.$$

Therefore,

$$D'(\varrho) - 2\varrho^{2-n}\int_{(\partial B_\varrho)^+}\left|\nabla u\cdot\frac{x}{\varrho}\right|^2\mathrm{d}\sigma_x$$
$$= 2\varrho^{1-n}\int_{(\partial B_\varrho)^+}\omega_1(|\nabla u|)\nabla u\cdot x\,\nabla u\cdot\nu\,\mathrm{d}\sigma_x$$
$$- 2\varrho^{1-n}\int_{B_\varrho^+}\nabla u\cdot x(1+\omega_1(|\nabla u|)\Delta u\,\mathrm{d}x$$
$$- 2\varrho^{1-n}\int_{B_\varrho^+}\nabla u\cdot x\,\omega_1'(|\nabla u|)\frac{\nabla u\otimes\nabla u}{|\nabla u|}\cdot D^2u\,\mathrm{d}x$$
$$+ \varrho^{1-n}\int_{B_\varrho^+}\omega_1'(|\nabla u|)|\nabla u|(D^2u\nabla u)\cdot x\,\mathrm{d}x =: \sum_{i=1}^4 A_i,$$

and hence

$$E_1 := \left|D'(\varrho) - 2\varrho^{2-n}\int_{(\partial B_\varrho)^+}|u_r|^2\,\mathrm{d}\sigma_x\right| \le \sum_{i=1}^4 |A_i|.$$

Let us discuss each term $|A_i|$. We have

$$|A_1| \le C\varrho^{2-n+\alpha}\|\nabla u\|_\infty^2|\partial B_\varrho| \overset{(2.1)}{\le} C\varrho^{-1-n+\alpha}\int_{B_{2\varrho}^+}u^2\,\mathrm{d}x \overset{(2.10)}{\le} C\varrho^{1-n+\alpha}\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x$$

and

$$|A_2| \le C\varrho^{1-n}(1+C\varrho^\alpha)\int_{B_\varrho^+}|\nabla u||x||\Delta u - \operatorname{div}\nabla_p f(\nabla u)|\,\mathrm{d}x$$
$$\overset{(2.9)}{\le} C\varrho^{2-n+\alpha}\int_{B_\varrho^+}|D^2u||\nabla u|\,\mathrm{d}x \le C\varrho^{2-n+\alpha}\Big(\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{B_\varrho^+}|D^2u|^2\,\mathrm{d}x\Big)^{1/2}$$
$$\overset{(2.2)}{\le} C\varrho^{-n+\alpha}\Big(\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{B_{2\varrho}^+}|u|^2\,\mathrm{d}x\Big)^{1/2} \overset{(2.5),(2.10)}{\le} C\varrho^{1-n+\alpha}\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x.$$

Similarly,

$$|A_3|+|A_4| \le \varrho^{1-n}\int_{B_\varrho^+}|\omega_1'(|\nabla u|)||D^2u||\nabla u|^2|x|\,\mathrm{d}x$$
$$\le C\varrho^{2-n+\alpha}\int_{B_\varrho^+}|D^2u||\nabla u|\,\mathrm{d}x \le C\varrho^{1-n+\alpha}\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x.$$

Therefore,

$$E_1 \le C\varrho^{1-n+\alpha}\int_{B_\varrho^+}|\nabla u|^2\,\mathrm{d}x \le C\varrho^{\alpha-1}D(\varrho). \tag{2.22}$$

Now, let us consider

$$E_2 := \Big|D(\varrho) - \varrho^{2-n}\int_{(\partial B_\varrho)^+} u u_\varrho\,\mathrm{d}\sigma_x\Big|. \tag{2.23}$$

Integrating by parts and employing (1.5b), we get

$$\begin{aligned}\int_{B_\varrho^+}\nabla_p f(\nabla u)\cdot\nabla u\,\mathrm{d}x &= \int_{\partial B_\varrho^+} u\nabla_p f(\nabla u)\cdot\nu\,\mathrm{d}\sigma_x - \int_{B_\varrho^+} u\,\partial_j(\partial_{p_j} f(\nabla u))\,\mathrm{d}x\\ &= \int_{(\partial B_\varrho)^+} u(1+\omega_1(|\nabla u|))\nabla u\cdot\nu\,\mathrm{d}\sigma_x\\ &\quad - \int_{B_\varrho^+} uD^2 f(\nabla u) : D^2u\,\mathrm{d}x\\ &= \int_{(\partial B_\varrho)^+} u(1+\omega_1(|\nabla u|))\nabla u\cdot\nu\,\mathrm{d}\sigma_x\\ &\quad - \int_{B_\varrho^+}(1+\omega_1(|\nabla u|))\,u\,\Delta u\,\mathrm{d}x\\ &\quad - \int_{B_\varrho^+} u\omega_1'(|\nabla u|)\frac{\nabla u\otimes\nabla u}{|\nabla u|} : D^2u\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}E_2 = \Big|&\varrho^{2-n}\int_{(\partial B_\varrho)^+} u\omega_1(|\nabla u|)\nabla u\cdot\nu\,\mathrm{d}\sigma_x - \varrho^{2-n}\int_{B_\varrho^+}(1+\omega_1(|\nabla u|))\,u\,\Delta u\,\mathrm{d}x\\ &-\varrho^{2-n}\int_{B_\varrho^+} u\omega_1'(|\nabla u|)\frac{\nabla u\otimes\nabla u}{|\nabla u|} : D^2u\,\mathrm{d}x\Big|\end{aligned} \tag{2.24}$$

Then, using that $u \in C^{1,\alpha}(B_\varrho^+)$,

$$E_2 \le \varrho^{3-n+\alpha} \|\nabla u\|_\infty^2 |\partial B_\varrho| + C\varrho^{2-n}(1 + C\varrho^\alpha) \int_{B_\varrho^+} |u||\Delta u - \operatorname{div} D_p f(\nabla u)| \, \mathrm{d}x$$
$$+ C\varrho^{2-n} \int_{B_\varrho^+} |u||D^2 u||\nabla u| \, \mathrm{d}x,$$

and employing the estimate (2.9), the Hölder inequality, the Schauder estimates (2.2) together with the doubling assumption and the Poincaré inequality (2.10), we conclude that

$$E_2 \le C\varrho^{2-n+\alpha} \int_{B_\varrho^+} |\nabla u|^2 \, \mathrm{d}x \le C\varrho^\alpha D(\varrho). \tag{2.25}$$

Moreover, by virtue of (2.18) and (2.23), we can deduce that

$$|H'(\varrho)| \le \left| H'(\varrho) - \frac{2}{\varrho} D(\varrho) \right| + \frac{2}{\varrho} |D(\varrho)| \le C\varrho^{-1}(E_2 + |D(\varrho)|). \tag{2.26}$$

Putting together (2.17), (2.22), (2.25) and (2.26) into (2.19), we conclude that

$$N'(\varrho) \ge -C\varrho^{\alpha-1} N(\varrho).$$

Whence we infer that

$$\frac{\mathrm{d}}{\mathrm{d}\varrho}(\exp(\alpha^{-1} C\varrho^\alpha) N(\varrho)) \ge 0,$$

thus concluding the proof of Proposition 2.1.

3. Blowup sequence

Here we prove that the rescalings at a free boundary point of the solutions to the nonlinear obstacle problems (1.1) converge to the solutions of the classical Signorini problem for the Dirichlet energy.

Proposition 3.1. *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$ be the solution to the boundary obstacle problem (1.5), $0 \in \Gamma_u$, and let (2.1) and (2.2) hold with constants $\bar\varrho$, k and β. Assume that there are $\gamma > 1$ and $\sigma \in (0, 1/2]$ such that*

$$\|u\|_\sigma > 0, \quad \|u\|_\varrho \le \gamma \|u\|_{\varrho/2} \quad \textit{for any } \varrho \in (0, \sigma]. \tag{3.1}$$

Then, for every sequence of positive real numbers ϱ_j such that $\varrho_j \searrow 0$, there is a subsequence $\varrho_{j'}$ such that

$$\frac{\varrho_{j'}^{n/2} u(\varrho_{j'} x)}{\|u\|_{L^2(B^+_{\varrho_{j'}})}} \to \varphi(x) \quad \textit{in } C^1 \textit{ locally on } \{x_n \ge 0\},$$

where $\varphi \in C^{1,1/2}(\{x_n > 0\})$ is a homogeneous solution to the Signorini problem

$$\int_{(\partial B_1)^+} |\nabla \varphi|^2 \, \mathrm{d}x \le \int_{(\partial B_1)^+} |\nabla \psi|^2 \, \mathrm{d}x \quad \textit{for all } \psi|_{(\partial B_1)^+} = \varphi|_{(\partial B_1)^+}, \ \psi|_{B_1'} \ge 0.$$

Moreover,

$$e^{C\varrho^\alpha} N_u(\varrho) \geq N_u(0) = N_\varphi(0) \geq \frac{3}{2} \quad \textit{for any } \varrho \in (0, \varrho_0),$$

and

$$\|u\|_\varrho \leq C\Big(\frac{\varrho}{\sigma}\Big)^{3/2} \|u\|_\sigma \quad \textit{for any } \varrho \in (0, \sigma], \tag{3.2}$$

with $C = C(n, k, \gamma, [\nabla u]_\alpha) > 0$ *and* ϱ_0 *as in Proposition* 2.1.

Proof. Let ϱ_j be a sequence of real numbers such that $\varrho_j \searrow 0$ and let u_j be the sequence of functions defined as $u_j(x) = \lambda_j u(\varrho_j x)$, with

$$\lambda_j := \frac{\varrho_j^{n/2}}{\|u\|_{L^2(B^+_{\varrho_j})}}$$

chosen so that $\|u_j\|_{L^2(B_1^+)} = 1$ for any $j \in \mathbb{N}$. By virtue of (2.1) and (3.1) (see also (2.5)), we have that $\|u_j\|_{C^{1,\alpha}(B_R^+)} \leq C(n, R, \gamma)$ for all $R > 0$ and for j sufficiently large. Thus, there is a subsequence $u_{\varrho_{j'}}$ that converges in C^1 locally on $\{x_n \geq 0\}$ to a function $\varphi = \varphi(x) \in C^{1,\alpha}$. With an abuse of notation, we keep denoting the subsequence with u_{ϱ_j}.

Let us show that φ is solution of the Signorini problem. For any suitable test function ψ with compact support in B_1^+, we have

$$\int_{B_1^+} \nabla u_j \cdot \nabla \psi \, \mathrm{d}x + \int_{B_1^+} \omega_1\Big(\frac{|\nabla u_j|}{\lambda_j \varrho_j}\Big) \nabla u_j \cdot \nabla \psi \, \mathrm{d}x = 0, \tag{3.3}$$

because u solves (1.5). Since

$$\frac{|\nabla u_j(x)|}{\lambda_j \varrho_j} = |\nabla u(\varrho_j x)| \leq C \varrho_j^\alpha \quad \text{for all } x \in B_1^+,$$

it follows that

$$\omega_1\Big(\frac{|\nabla u_j|}{\lambda_j \varrho_j}\Big) \to 0 \quad \text{as } j \to +\infty.$$

Thus, taking the limit in (3.3) and employing the convergence in C^1 of u_j to φ, we get

$$\int_{B_1^+} \nabla \varphi \cdot \nabla \psi \, \mathrm{d}x = 0,$$

which means that φ is harmonic in B_1^+. Next, let us discuss the limit of the conditions on B_1'. Since u solves (1.5), u_j solves

$$u_j \nabla_p f\Big(\frac{\nabla u_j}{\lambda_j \varrho_j}\Big) \cdot e_n = 0 \quad \text{on } B_1'.$$

Note that

$$\nabla_p f\Big(\frac{\nabla u_j}{\lambda_j \varrho_j}\Big) = h'\Big(\frac{|\nabla u_j|}{\lambda_j \varrho_j}\Big) \frac{\nabla u_j}{|\nabla u_j|} = \Big(1 + \omega_1\Big(\frac{|\nabla u_j|}{\lambda_j \varrho_j}\Big)\Big) \frac{\nabla u_j}{\lambda_j \varrho_j}.$$

Therefore,

$$\lambda_j \varrho_j \nabla_p f\Big(\frac{\nabla u_j}{\lambda_j \varrho_j}\Big) \cdot e_n = \Big(1 + \omega_1\Big(\frac{|\nabla u_j|}{\lambda_j \varrho_j}\Big)\Big) \nabla u_j \cdot e_n \to \nabla \varphi \cdot e_n,$$

and hence

$$\varphi \, \nabla \varphi \cdot e_n = 0 \quad \text{and} \quad \nabla \varphi \cdot e_n \le 0 \quad \text{on } B_1',$$

concluding that φ is a solution to the classical Signorini problem. For any radius $\sigma > 0$, we denote by $N_j(\sigma)$ the frequency for the sequence u_j and we have that

$$N_j(\sigma) = \frac{(\sigma \varrho_j)^{2-n} \int_{B^+_{\varrho_j \sigma}} |\nabla u(y)|^2 \, \mathrm{d}y}{(\sigma \varrho_j)^{1-n} \int_{\partial B^+_{\varrho_j \sigma}} |u(y)|^2 \, \mathrm{d}\sigma_y} = N_u(\varrho_j \sigma),$$

whence

$$N_j(\sigma) \to \lim_{j \to +\infty} N_u(\varrho_j \sigma) = N_u(0).$$

On the other hand, the convergence in C^1 ensures that

$$N_j(\sigma) \to N_\varphi(\sigma) \quad \text{for any radius } \sigma.$$

Combining the last two, we conclude that $N_\varphi(\sigma)$ is constant and by standard results we infer that φ is homogeneous of degree $N_u(0) \ge 3/2$ (see [4, 5]).

In order to establish (3.2), we start by noticing that as in (2.26) and (2.25), we know that

$$H'(\varrho) \ge C(1 - \varrho^\alpha) \, \frac{1}{\varrho} \, D(\varrho),$$

hence, employing the monotonicity of $N(\varrho)$,

$$(\ln H(\varrho))' = \frac{H'(\varrho)}{H(\varrho)} \ge C \frac{1}{\varrho} N(\varrho) \ge C \, \frac{1}{\varrho} \, N(0).$$

Integrating onto the interval (ϱ, σ) yields

$$\ln\Big(\frac{H(\sigma)}{H(\varrho)}\Big) \ge C \ln\Big(\frac{\sigma}{\varrho}\Big)^{3/2},$$

which means that

$$\Big(\frac{H(\sigma)}{H(\varrho)}\Big)^{1/2} \ge C\Big(\frac{\sigma}{\varrho}\Big)^{3/2}. \tag{3.4}$$

By virtue of (2.12),

$$H(\sigma) = \sigma^{1-n} \int_{\partial B^+_\sigma} u^2 \, \mathrm{d}\sigma_x \le C \sigma^{-n} \int_{B^+_\sigma} u^2 \, \mathrm{d}x,$$

while on the other side, from (2.16) and (2.10), we have

$$H(\varrho) \ge C \varrho^{-n} \int_{B^+_\varrho} u^2 \, \mathrm{d}x.$$

Therefore, (3.4) implies the conclusion

$$\Big(\varrho^{-n} \int_{B^+_\varrho} u^2 \, \mathrm{d}x\Big)^{1/2} \le C\Big(\frac{\varrho}{\sigma}\Big)^{3/2} \Big(\sigma^{-n} \int_{B^+_\sigma} u^2 \, \mathrm{d}x\Big)^{1/2}.$$

∎

4. Proof of the optimal regularity

Finally, in this section we show the optimal regularity $u \in C^{1,1/2}_{\rm loc}(B_1^+ \cup B_1')$. We fix the notation

$$\|u\|_{\varrho,x_0} := \Big(\varrho^{-n} \int_{B_\varrho^+(x_0)} u^2 \,\mathrm{d}x\Big)^{1/2}, \quad x_0 \in \Gamma_u.$$

We begin with the following preliminary lemma (cf. [39]).

Lemma 4.1. *Let u be solution to (1.1) and let $\delta \in (0,1/2)$. Then there exists $\varrho_0 \in (0,1/2]$, such that for every $x_0 \in \Gamma_u \cap B_{1/2}$ and every $\varrho \in (0,\varrho_0]$, the following implication holds:*

$$\|u\|_{\varrho/2,x_0} \geq 2^{3/2+\delta}\|u\|_{\varrho/4,x_0} \implies \|u\|_{\sigma,x_0} \geq \Big(\frac{2\sigma}{\varrho}\Big)^{3/2+\delta}\|u\|_{\varrho/2,x_0} \quad \textit{for all } \sigma \in \Big[\frac{3}{4}\varrho,\varrho\Big].$$

Proof. We argue by contradiction. Let us suppose that for any $l \in \mathbb{N}\setminus\{0,1\}$ there are $\varrho_0 = 1/l$, $\varrho_l \leq 1/l$, $\sigma_l \in [3\varrho_l/4, \varrho_l]$ and solutions $u_l = u$ to (1.1) with $x_l \in \Gamma_{u_l}$ such that the assertion does not hold. Then, introducing the rescaling

$$\tilde u_l(x) := \|u_l\|^{-1}_{2\varrho_l,x_l} u_l(2\varrho_l x + x_l),$$

by the estimates (2.1), we have, up to a not relabeled subsequence, that the functions $\tilde u_l$ converge in C^1 to φ on B_σ^+, with $\sigma = \lim_l \sigma_l/(2\varrho_l) \in [3/8, 1/2]$, that is a solution to the Signorini problem in B_1. Let us now consider the function $f(\varrho) = \varrho^{-3/2-\delta}\|\varphi\|_\varrho$, with $0 < \varrho \leq \sigma$ and $\delta > 0$. From the assumption, we have that the function f attains its maximum in $[1/8, \sigma]$ in an interior point $\tilde\varrho \in (1/8, \sigma)$. Indeed,

$$\begin{aligned}\|u_l\|_{\varrho_l/2} \geq 2^{3/2+\delta}\|u_l\|_{\varrho_l/4} &\implies \Big(\frac{1}{4}\Big)^{-3/2-\delta}\|\tilde u_l\|_{1/4} \geq \Big(\frac{1}{8}\Big)^{-3/2-\delta}\|\tilde u_l\|_{1/8}\\ &\implies f\Big(\frac14\Big) \geq f\Big(\frac18\Big)\end{aligned}$$

and, analogously,

$$\begin{aligned}\|u_l\|_{\sigma_l} < \Big(\frac{2\sigma_l}{\varrho_l}\Big)^{3/2+\delta}\|u_l\|_{\varrho_l/2} &\implies \Big(\frac{\sigma_l}{2\varrho_l}\Big)^{-3/2-\delta}\|\tilde u_l\|_{\sigma_l/(2\varrho_l)} < \Big(\frac{1}{4}\Big)^{-3/2-\delta}\|\tilde u_l\|_{1/4}\\ &\implies f(\sigma) \leq f\Big(\frac14\Big).\end{aligned}$$

Rather than $f'(\varrho)$, let us compute $(f^2(\varrho))'$:

$$\begin{aligned}\Big(\varrho^{-3-2\delta-n}\int_{B_\varrho^+}\varphi^2\,\mathrm{d}x\Big)' &= \Big(\varrho^{-3-2\delta}\int_{B_1^+}\varphi^2(\varrho x)\,\mathrm{d}x\Big)'\\ &= (-3-2\delta)\varrho^{-3-1-2\delta-n}\int_{B_\varrho^+}\varphi^2\,\mathrm{d}x + 2\varrho^{-3-2\delta-n}\int_{B_\varrho^+}\varphi_r\,\varphi\,\mathrm{d}x\\ &= (-3-2\delta)\varrho^{-3-1-2\delta-n}\int_0^\varrho\int_{\partial B_t^+}\varphi^2\,\mathrm{d}\sigma_x\,\mathrm{d}t + 2\varrho^{-3-2\delta-n}\int_0^\varrho\int_{\partial B_t^+}\varphi_r\,\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t\\ &= (-3-2\delta)\varrho^{-3-2\delta-2}\int_0^\varrho H_\varphi(t)\,\mathrm{d}t + 2\varrho^{-3-2\delta-2}\int_0^\varrho D_\varphi(t)\,\mathrm{d}t\\ &= 2\varrho^{-3-2\delta-2}\int_0^\varrho H_\varphi(t)\Big(N_\varphi(t) - \frac32 - \delta\Big)\,\mathrm{d}t.\end{aligned}$$

Since φ is a solution to the Signorini problem, the monotonicity of N and the fact that $N_\varphi \geq 3/2$ imply that f is monotone increasing. Moreover, considering that, under the contradiction assumption, f must have an interior maximum, we infer that f needs to be constant in the interval $I = [1/8, \sigma]$. Hence, we conclude that actually $N_\varphi(\varrho) = 3/2 + \delta$ for $\varrho \in I$. This implies that φ is a homogeneous solution of degree $3/2 + \delta < 2$, thus contradicting the fact that there are no solution with frequency $\lambda \in (3/2, 2)$ (see [5]). ∎

Remark 4.2. Let us notice that for $\sigma = \varrho/2 \in (0, \varrho_0/2)$ in Lemma 4.1, we have that

$$\|u\|_{\sigma,x_0} \geq 2^{3/2+\delta}\|u\|_{\sigma/2,x_0} \implies \|u\|_{2\sigma,x_0} \geq 2^{3/2+\delta}\|u\|_{\sigma,x_0}.$$

It follows that if $\|u\|_{\sigma,x_0} \geq 2^{3/2+\delta}\|u\|_{\sigma/2,x_0}$, then $\|u\|_{2^j\sigma,x_0} \geq 2^{3/2+\delta}\|u\|_{2^{j-1}\sigma,x_0}$ for every $j = 1, 2, \ldots$ such that $2^j\sigma \leq \varrho_0$, which implies

$$\|u\|_{\varrho,x_0} \leq C\varrho^{3/2+\delta}\|u\|_{\varrho_0,x_0} \quad \text{for any } \varrho \in (\sigma/2, \varrho_0/2],$$

and, by changing the constant C, also

$$\|u\|_{\varrho,x_0} \leq C\varrho^{3/2}\|u\|_{L^2(B_1)} \quad \text{for any } \varrho \in (\sigma/2, 1/2]. \tag{4.1}$$

Proposition 4.3. *Let $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$ be the solution of (1.1) and $x_0 \in \Gamma_u \cap B_{1/2}$. Then*

$$\sup_{B^+_{\varrho/2}(x_0)} |u| + \varrho \sup_{B^+_{\varrho/2}(x_0)} |\nabla u| \leq C\varrho^{3/2}\|u\|_{L^2(B_1^+)} \quad \textit{for all } \varrho \in (0, 1/2), \tag{4.2}$$

where $C = C(n, \alpha, \beta, k) > 0$.

Proof. Assume without loss of generality that $x_0 = 0 \in \Gamma_u$. Let us introduce

$$\sigma := \inf\left\{\frac{1}{2}, \{\varrho \in (0, 1/2] : \|u\|_\varrho \geq 2^{3/2+\delta}\|u\|_{\varrho/2}\}\right\},$$

where $\delta \in (0, 1/2)$. From Lemma 4.1 and (4.1), we have

$$\|u\|_\varrho \leq C\varrho^{3/2}\|u\|_{L^2(B_1^+)} \quad \text{for any } \varrho \in (\sigma/2, 1/2]. \tag{4.3}$$

On the other hand, if $\varrho \in (0, \sigma]$, then

$$\|u\|_\varrho < 2^{3/2+\delta}\|u\|_{\varrho/2}$$

and thus, by Proposition 3.1 and (3.2),

$$\|u\|_\varrho \leq C\left(\frac{\varrho}{\sigma}\right)^{3/2}\|u\|_\sigma \quad \text{for any } \varrho \in (0, \sigma]. \tag{4.4}$$

Combining (4.3) and (4.4), we deduce that there exists a constant $C > 0$ such that

$$\|u\|_\varrho \leq C\varrho^{3/2}\|u\|_{L^2(B_1)} \quad \text{for any } \varrho \in (0, 1/2).$$

By (2.1), we finally have

$$\sup_{B^+_{\varrho/2}}|u| + \varrho \sup_{B^+_{\varrho/2}}|\nabla u| \leq C\|u\|_\varrho \leq C\varrho^{3/2}\|u\|_{L^2(B_1^+)} \quad \text{for any } \varrho \in (0, 1/2).$$ ∎

Theorem 4.4. *Every solution u of* (1.1) *belongs to* $C^{1,1/2}_{\mathrm{loc}}(B_1^+ \cup B_1')$ *and for every* $r < 1$, *there exists a constant* $C > 0$ *depending on* r, $\mathrm{Lip}(u)$ *and the nonlinear energy density* f, *such that*

$$\|u\|_{C^{1,1/2}(B_r^+ \cup B_r')} \le C\|u\|_{L^2(B_1^+)}. \tag{4.5}$$

Proof. Recall that, by [12], every solution of (1.1) is $C^{1,\alpha}_{\mathrm{loc}}$. The local $C^{1,1/2}$ regularity of u is now an easy consequence of Proposition 4.3. We give the proof for readers' convenience.

We need to estimate the oscillation of the gradient. We distinguish between three cases.

Case 1. Let $z, y \in B^+_{1/2}$ be such that

$$\mathrm{dist}(z, \Gamma_u) \le 10^{-1}|z-y| \quad \text{and} \quad \mathrm{dist}(y, \Gamma_u) \le 10^{-1}|z-y|.$$

Then let $p_z, p_y \in \Gamma_u$ be such that

$$\mathrm{dist}(z, \Gamma_u) = |z - p_z|, \quad \mathrm{dist}(y, \Gamma_u) = |y - p_y|,$$

and note that

$$\begin{aligned} |\nabla u(z) - \nabla u(y)| &\le |\nabla u(z) - \nabla u(p_z)| + |\nabla u(p_z) - \nabla u(p_y)| + |\nabla u(p_y) - \nabla u(y)| \\ &= |\nabla u(z) - \nabla u(p_z)| + |\nabla u(p_y) - \nabla u(y)| \\ &\overset{(4.2)}{\le} C|z-p_z|^{1/2} + C|y-p_y|^{1/2} \\ &\le C|z-y|^{1/2}, \end{aligned}$$

where $C > 0$ is the constant in Proposition 4.3, which ultimately depends on $\mathrm{Lip}(u)$ and f.

Case 2. Let $z, y \in B^+_{1/2}$ be such that

$$10^{-1}|z-y| < \varrho_z := \mathrm{dist}(z, \Gamma_u), \quad B_{\varrho_z}(z) \cap \Lambda_u = \emptyset.$$

We consider the symmetrization matrix $S \in \mathbb{R}^{n \times n}$,

$$Sx := (x', -x_n) \quad \text{for } x = (x', x_n),$$

and set

$$v(x) := \begin{cases} u(Sx) & x_n \le 0, \\ u(x) & x_n > 0, \end{cases} \qquad x \in B_{\varrho_z}(z).$$

It is easy to verify that v satisfies

$$\mathrm{div}(\nabla_p f(\nabla v)) = 0 \quad \text{in } B_{\varrho_z}(z).$$

Indeed, if $x_n < 0$, then

$$\nabla v(x) = S\nabla u(Sx), \quad \nabla^2 v(x) = S\nabla^2 u(Sx)S.$$

Therefore,

$$
\begin{aligned}
&\operatorname{div}(\nabla_p f(\nabla v(x))) = \nabla_p^2 f(\nabla v(x)) : D^2 v(x) \\
&= \Big[(1+\omega_1(|\nabla v(x)|))I + \omega_1'(|\nabla v(x)|)\frac{\nabla v(x)\otimes\nabla v(x)}{|\nabla v(x)|}\Big] : D^2 v(x) \\
&= \Big[(1+\omega_1(|\nabla u(Sx)|))I + \omega_1'(|\nabla u(Sx)|)\frac{S\nabla u(Sx)\otimes S\nabla u(Sx)}{|\nabla u(Sx)|}\Big] : SD^2 u(Sx)S \\
&= (1+\omega_1(|\nabla u(Sx)|))I : D^2 u(Sx) + \omega_1'(|\nabla u(Sx)|)\frac{S\nabla u(Sx)\otimes S\nabla u(Sx)}{|\nabla u(Sx)|} : D^2 u(Sx) \\
&= \operatorname{div}(\nabla_p f(\nabla u(Sx))) = 0.
\end{aligned}
$$

Moreover, for every $\varphi \in C_c^1(B_{\varrho_z}(z))$, we have that

$$
\begin{aligned}
\int_{B_{\varrho_z}(z)} \nabla_p f(\nabla v(x))\cdot\nabla\varphi(x)\,\mathrm{d}x &= \int_{B^+_{\varrho_z}(z)} \nabla_p f(\nabla u(x))\cdot\nabla\varphi(x)\,\mathrm{d}x \\
&\quad + \int_{B_{\varrho_z}(z)\cap\{x_n<0\}} \nabla_p f(\nabla v(x))\cdot\nabla\varphi(x)\,\mathrm{d}x \\
&= -\int_{B'_{\varrho_z}(z)} \nabla_p f(\nabla u(x',0))\cdot e_n\,\varphi(x',0)\,\mathrm{d}x' \\
&\quad + \int_{B'_{\varrho_z}(z)} \nabla_p f(\nabla v(x',0))\cdot e_n\,\varphi(x',0)\,\mathrm{d}x' = 0,
\end{aligned}
$$

where we use that $\nabla_p f(p) = (1+\omega_1(|p|))p$ leads to

$$\nabla_p f(\nabla v(x',0))\cdot e_n \nabla_p f(\nabla u(x',0))\cdot e_n = 0 \quad \text{for all } (x',0)\in B_1'\setminus\Lambda_u.$$

Therefore, since v solves a uniformly elliptic equation in $B_{\varrho_z}(z)$, by standard estimates, we get

$$
\begin{aligned}
|\nabla u(z)-\nabla u(y)| &\le \|D^2 v\|_{L^\infty(B_{\varrho_z/2}(z))}|z-y| \\
&\le C\varrho_z^{-2}\|v\|_{L^\infty(B_{\varrho_z}(z))}|z-y| \le C\varrho_z^{-1/2}\,|z-y| \le C|z-y|^{1/2},
\end{aligned}
$$

where in the second inequality $C>0$ depends on $\operatorname{Lip}(u)$ and f, and in the third also on the constant in Proposition 4.3.

Case 3. Let $z, y \in B^+_{1/2}$ be such that

$$10^{-1}|z-y| < \varrho_z := \operatorname{dist}(z,\Gamma_u), \quad B_{\varrho_z}(z)\cap B_1' \subset \Lambda_u.$$

We consider the symmetrization matrix $S\in\mathbb{R}^{n\times n}$, $Sx := (x',-x_n)$ for $x=(x',x_n)$, and set

$$v(x) := \begin{cases} -u(Sx), & x_n \le 0, \\ u(x), & x_n > 0, \end{cases} \qquad x \in B_{\varrho_z}(z).$$

We need to verify as before that v solves the equation

$$\operatorname{div}(\nabla_p f(\nabla v)) = 0 \quad \text{in } B_{\varrho_z}(z).$$

Indeed, if $x_n < 0$, then, as above, the equation is pointwise verified and, for every $\varphi \in C^1_c(B_{\varrho_z}(z))$, we have that

$$\int_{B_{\varrho z}(z)} \nabla_p f(\nabla v(x)) \cdot \nabla\varphi(x)\,\mathrm{d}x = -\int_{B'_{\varrho z}(z)} \nabla_p f(\nabla u(x',0)) \cdot e_n\, \varphi(x'0)\,\mathrm{d}x' + \int_{B'_{\varrho z}(z)} \nabla_p f(\nabla v(x',0)) \cdot e_n\, \varphi(x'0)\,\mathrm{d}x' = 0,$$

where now we use that

$$\begin{aligned}\nabla_p f(\nabla v(x',0)) \cdot e_n &= (1+\omega_1(|\nabla v(x',0)|))\partial_n v(x',0)\\ &= (1+\omega_1(|\nabla u(x',0)|))\partial_n u(x',0)\\ &= \nabla_p f(\nabla u(x',0)) \cdot e_n \quad \text{for all } (x',0) \in B'_1 \cap \Lambda_u.\end{aligned}$$

We can then conclude exactly as in the previous case that

$$|\nabla u(z) - \nabla u(y)| \le C|z-y|^{1/2}.$$

By a covering argument one easily passes from $B^+_{1/2}$ to any B^+_r with $r < 1$, with the constant C in (4.5) depending on r too, thus finishing the proof. ■

A. Schauder estimates

In this section, we prove estimates (2.1) and (2.2). The proof follows very closely the arguments in [12].

A.1. Caccioppoli inequalities

We consider the solution u to (1.1) with $0 \in \Gamma_u$. When not explicitly specified, the constants of the statements below might depend on the Lipschitz constant of u.

Lemma A.1. *Let u be solution to (1.1). Then there exist $\bar{R}, C > 0$ such that*

$$\int_{A(k,\varrho)} |\nabla u|^2\,\mathrm{d}x \le \frac{C}{(R-\varrho)^2}\int_{A(k,R)} (u-k)^2\,\mathrm{d}x, \tag{A.1}$$

for ϱ such that $0 < \varrho < R < \bar{R}$ and for any $k \ge 0$, where $A(k,r) = \{x \in B^+_r : u(x) \ge k\}$.

Proof. Let $\varphi \in C^\infty(B^+_1)$ be such that $\varphi \equiv 1$ on B^+_ϱ, $\varphi \equiv 0$ outside B^+_R and $|\nabla\varphi| \le C/(R-\varrho)$ with $R < 1$. Then, testing the equation (1.5a) with $v := \varphi^2(u-k)_+$, we get

$$\begin{aligned}0 &= \int_{B^+_1} \operatorname{div}(\nabla_p f(\nabla u))v\,\mathrm{d}x\\ &= -\int_{B^+_1} \nabla_p f(\nabla u) \cdot \nabla(\varphi^2(u-k)_+)\,\mathrm{d}x - \int_{B'_1} \nabla_p f(\nabla u) \cdot e_n\, \varphi^2(u-k)_+\,\mathrm{d}x'\\ &\overset{(1.5b)}{\le} -\int_{B^+_1} \nabla_p f(\nabla u) \cdot \nabla(\varphi^2(u-k)_+)\,\mathrm{d}x\\ &= -\int_{A(k,1)} \varphi^2\, \nabla_p f(\nabla u) \cdot \nabla u\,\mathrm{d}x - 2\int_{B^+_1} \varphi(u-k)_+ \nabla_p f(\nabla u) \cdot \nabla\varphi\,\mathrm{d}x.\end{aligned}$$

Using the ellipticity condition (1.4), we get

$$\int_{A(k,1)} \varphi^2 \nabla_p f(\nabla u) \cdot \nabla u \,\mathrm{d}x = \int_{A(k,1)} \varphi^2 \int_0^1 \nabla_p^2 f(t\nabla u)\,\mathrm{d}t\, \nabla u \cdot \nabla u \,\mathrm{d}x$$
$$\geq \lambda \int_{A(k,1)} \varphi^2 |\nabla u|^2 \,\mathrm{d}x.$$

The conclusion now follows from a standard computation:

$$\lambda \int_{A(k,1)} \varphi^2 |\nabla u|^2 \,\mathrm{d}x \leq 2 \int_{A(k,1)} \varphi (u-k)_+ |\nabla_p f(\nabla u)| |\nabla \varphi| \,\mathrm{d}x$$
$$\leq \frac{\lambda}{2} \int_{A(k,1)} \varphi^2 |\nabla u|^2 \,\mathrm{d}x + \frac{C}{\lambda} \int_{A(k,1)} |u-k|^2 |\nabla \varphi|^2 \,\mathrm{d}x,$$

where we used that $|\nabla_p f(\nabla u)| \leq C|\nabla u|$, with the constant C depending on $\|\nabla u\|_\infty$. ■

For what concerns the Caccioppoli inequalities, for the derivatives of u, we argue as in Lemma 4.2 and Propositions 4.3 and 4.5 of [12].

Lemma A.2. *Let u be the solution of* (1.1). *Then there exist $\bar{R}, C > 0$ such that*

$$\int_{A(k,r)} |\nabla w|^2 \,\mathrm{d}x \leq \frac{C}{(R-r)^2} \int_{A(k,R)} (w-k)^2 \,\mathrm{d}x, \tag{A.2}$$

where $w = \pm\partial_i u$, with $i = 1, \dots, n$, ϱ is such that $0 < \varrho < R < \bar{R}$ and $k \geq 0$, with $A(k,r) = \{x \in B_r^+ : w(x) \geq k\}$.

Proof. We introduce the notation for the difference quotient, that is,

$$D_i^t w(x) := \frac{w(x+te_i) - w(x)}{t} \quad \text{for any } t \neq 0 \text{ and any } i = 1, \dots, n-1.$$

Let $\varphi \in C^\infty(B_1^+)$ be such that $\varphi \equiv 1$ on B_ϱ^+, $\varphi \equiv 0$ outside B_R^+ and $|\nabla \varphi| \leq C/(R-\varrho)$. As shown in Lemma 4.2 of [12], there exists $\varepsilon_0 > 0$ such that

$$v := u + \varepsilon D_i^{-t}(\varphi^2 (D_i^t u - k)_+) \in \mathcal{A}_g \quad \text{for all } k \geq 0, \text{ and all } \varepsilon \in (0, \varepsilon_0).$$

Employing v as a test function in (1.5a) and writing $z := (D_i^t u - k)_+$, we have

$$0 \leq \int_{B_1^+} \nabla_p f(\nabla u) \cdot \nabla (v-u) \,\mathrm{d}x = \int_{B_1^+} \nabla_p f(\nabla u) \cdot \nabla(\varepsilon D_i^{-t}(\varphi^2 z)) \,\mathrm{d}x$$
$$= -\varepsilon \int_{B_1^+} D_i^t(\nabla_p f(\nabla u)) \cdot \nabla(\varphi^2 z) \,\mathrm{d}x$$

Arguing as in the previous lemma, and using the difference quotient's properties, we obtain, for $t \to 0^+$,

$$\int_{B_R^+} \varphi^2 |\nabla(\partial_i u - k)_+|^2 \,\mathrm{d}x \leq C \int_{B_R^+} |\nabla \varphi|^2 |(\partial_i u - k)_+|^2 \,\mathrm{d}x.$$

Employing the properties of φ, we conclude (A.2).

For what concerns the normal derivative, we use equation (1.5):

$$0 = \operatorname{div}(\nabla_p f(\nabla u)) = \nabla_p^2 f(\nabla u) : D^2 u;$$

therefore, isolating the terms with $\partial^2_{nn} u$, we get

$$\partial^2_{nn} u \, \partial^2_{p_n p_n} f(\nabla u) = \sum_{(i,j)\neq(n,n)} \partial^2_{ij} u \, \partial^2_{p_i p_j} f(\nabla u).$$

By ellipticity (1.4),

$$\partial^2_{p_n p_n} f(\nabla u) \geq \lambda$$

(the constant λ depending on $\|\nabla u\|_\infty$), and hence

$$|\partial_{nn} u| \leq C \sum_i^{n-1} |\partial_i \nabla u|,$$

and employing (A.2) we conclude . The same arguments can be easily adapted for $-\partial_i u$ and the proof is complete. ∎

An immediate consequence of the previous lemma is the following corollary (whose simple proof is left to the readers).

Corollary A.3. *As consequence of* (A.1) *and* (A.2)*, it follows that*

$$\varrho^2 \int_{B^+_\varrho} |\nabla u|^2 \, dx + \varrho^4 \int_{B^+_\varrho} |\nabla^2 u|^2 \, dx \leq C \int_{B^+_{2\varrho}} |u|^2 \, dx.$$

A.2. De Giorgi's oscillation lemma

In this section we consider functions w satisfying the Caccioppoli inequalities:

$$\int_{A(k,r)} |\nabla w|^2 \, dx \leq \frac{C}{(R-r)^2} \int_{A(k,R)} (w-k)^2 \, dx, \tag{A.3}$$

for $k \geq 0$ and $0 < r < R$, with $A(h,s) = \{x \in B_s : w(x) \geq h\}$.

Proposition A.4. *Let w be a function in B_1^+ satisfying* (A.3)*. Then there exists $C > 0$ such that*

$$\sup_{B^+_{\varrho/2}} w \leq C \Big(\varrho^{-n} \int_{A(k_0,\varrho)} (w-k_0)^2 \, dx \Big)^{1/2} \Big(\frac{|A(k_0,\varrho)|}{\varrho^n} \Big)^{\gamma/2} + k_0$$

for every $k_0 \geq 0$, $0 < \varrho < 1$, where $\gamma^2 + \gamma = 2/n$, $\gamma > 0$.

Proof. We set $r' = (r+R)/2$, and let $\varphi \in C^\infty(B_R^+)$ such that $\operatorname{supp}\varphi \subset\subset B^+_{r'}$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in B_r^+ and $|\nabla\varphi| \leq C/(R-r)$. Then, applying the Sobolev and the Hölder inequalities, we have (consider for simplicity $n > 2$; the same holds for $n = 2$ by observing that $1^* = 2$ and then using the Jensen inequality)

$$\begin{aligned}\text{(A.4)}\quad \int_{A(k,r)} (w-k)^2\,\mathrm{d}x &\le \int_{A(k,r')} (\varphi(w-k))^2\,\mathrm{d}x \\ &\le |A(k,R)|^{2/n}\Big(\int_{A(k,r')} (\varphi(w-k))^{2^*}\,\mathrm{d}x\Big)^{2/2^*} \\ &\le |A(k,R)|^{2/n}\int_{A(k,r')} |\nabla(\varphi(w-k))|^2\,\mathrm{d}x \\ &\le C|A(k,R)|^{2/n}\Big(\int_{A(k,r')} |\nabla\varphi|^2(w-k)^2\,\mathrm{d}x + \int_{A(k,r')} |\nabla(w-k)|^2\varphi^2\,\mathrm{d}x\Big) \\ &\le C|A(k,R)|^{2/n}\Big(\frac{1}{(R-r)^2}\int_{A(k,R)} (w-k)^2\,\mathrm{d}x + \int_{A(k,r')} |\nabla w|^2\,\mathrm{d}x\Big) \\ &\overset{\text{(A.3)}}{\le} C|A(k,R)|^{2/n}\Big(\frac{1}{(R-r)^2}\int_{A(k,R)} (w-k)^2\,\mathrm{d}x\Big).\end{aligned}$$

Let us set

$$\phi(k,r) = |A(k,r)|^{\gamma}\int_{A(k,r)} (w-k)^2\,\mathrm{d}x.$$

Then observe that, for $0 \le h < k$ and $0 < r < R < 1$,

$$\text{(A.5)}\qquad |A(k,r)| \le \int_{A(k,r)} \Big(\frac{w-h}{k-h}\Big)^2\,\mathrm{d}x \le \frac{1}{(k-h)^2}\int_{A(h,R)} (w-h)^2\,\mathrm{d}x.$$

From (A.4), we get

$$\int_{A(k,r)} (w-k)^2\,\mathrm{d}x \le c|A(k,R)|^{\gamma^2+\gamma}\frac{1}{(R-r)^2}\int_{A(h,R)} (w-h)^2\,\mathrm{d}x.$$

Thus, combining the last two estimates, we obtain

$$\text{(A.6)}\qquad \phi(k,r) \le c\frac{1}{(k-h)^{2\gamma}}\,\frac{1}{(R-r)^2}\,\phi(h,R)^{1+\gamma}.$$

We now choose

$$r_j = \frac{\varrho}{2}\Big(1+\frac{1}{2^j}\Big) \quad\text{and}\quad k_j = k_0 + d\Big(1-\frac{1}{2^j}\Big),$$

where d is a positive number to be chosen later. Setting $R = r_j$, $r = r_{j+1}$, $k = k_{j+1}$ and $h = k_j$ in (A.6), we get

$$\phi(k_{j+1}, r_{j+1}) \le \frac{C(4^{\gamma+1})^j}{d^{2\gamma}\varrho^2}\phi(k_j,r_j)^{1+\gamma} = DB^j\phi(k_j,r_j)^{1+\gamma},$$

with $D = Cd^{-2\gamma}\varrho^{-2}$ and $B = 4^{\gamma+1}$. On the other side, choosing d such that

$$d \ge c\Big(\varrho^{-n}\int_{A(k_0,\varrho)} (w-k_0)^2\,\mathrm{d}x\Big)^{1/2}\Big(\frac{|A(k_0,\varrho)|}{\varrho^n}\Big)^{\gamma/2},$$

with c suitable positive constant, we have that $\phi(k_0,\varrho) \le D^{-1/\gamma} B^{-1/\gamma^2}$. Then employing Lemma 7.1 of [29], it follows that

$$\phi_j \le B^{-j/\gamma}\phi_0 = 4^{-j-j/\gamma}\phi_0 \quad \text{for all } j \ge 0, \text{ with } \phi_j = \phi(k_j, r_j),$$

which means

$$4^j |A(k_j,r_j)|^\gamma \int_{A(k_j,r_j)} (w-k_j)^2 \,\mathrm{d}x = 4^j \phi_j \le 4^{-j/\gamma}\phi_0.$$

Combining the last formula with (A.5) (adapting the indexes), we conclude that

$$|A(k_{j+1},r_{j+1})|^{\gamma+1} \le \frac{4^j}{d^2}|A(k_j,r_j)|^\gamma \int_{A(k_j,r_j)} (w-k_j)^2 \,\mathrm{d}x \le 4^{-j/\gamma}\frac{\phi_0}{d^2} \quad \text{for all } j \ge 0,$$

thus, taking the limit as $j \to +\infty$, we get $|A(d+k_0,\varrho/2)| = 0$, which gives the thesis. ∎

Remark A.5. From Lemma A.2, we have that $w = \pm\partial_i u$ satisfy (A.3). Therefore, we get

$$\|\nabla u\|_{L^\infty(B^+_{\varrho/2})} \le C\Big(\frac{1}{\varrho^n}\int_{B^+_\varrho} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}x\Big)^{1/2} \le C\varrho^{-1}\Big(\frac{1}{\varrho^n}\int_{B^+_{2\varrho}} |u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

Also Proposition A.4 implies

$$\sup_{B^+_{\varrho/2}} u \le C\Big(\frac{1}{\varrho^n}\int_{B^+_{2\varrho}} |u|^2 \,\mathrm{d}x\Big)^{1/2}.$$

By a simple covering, we conclude the estimates of the first two terms in (2.1).

In order to show the estimate on the Hölder seminorm $[\nabla u]_{\alpha,B_\varrho}$, we recall the following result.

Proposition A.6 (Proposition 4.7 in [12]). *Let u be the solution to the thin obstacle problem* (1.5)*. Then the co-normal derivative*

$$\Phi u := \partial_{p_n} f(\nabla u) = (1+\omega_1(|\nabla u|))\partial_n u$$

is $C^{0,\beta}(B_1^+ \cup B_1')$ for some $\beta \in (0,1)$ and

$$[\Phi u]_{\beta,B^+_\varrho} \le C r^{-\beta}\|\nabla u\|_{\infty,B^+_r} \quad \textit{for all } \varrho, r : 0 < \varrho < r < 1.$$

We can hence show the following proposition.

Proposition A.7. *Let u be the solution to the thin obstacle problem* (1.5)*. Then there exists $C > 0$ such that*

$$\varrho^{1+\alpha}[\nabla u]_{\alpha,B^+_\varrho} \le C\Big(\varrho^{-n}\int_{B^+_{2\varrho}} u^2 \,\mathrm{d}x\Big)^{1/2} \quad \textit{for some } \alpha \in (0,1). \tag{A.7}$$

Proof. Let $x_0 \in B'_r$ and $r \in (0, 1 - |x_0|)$. If $B'_r(x_0) \subset \Lambda(u)$, then we conclude by the classic regularity theory that there exists $\alpha > 0$ such that

$$\int_{B^+_\varrho(x_0)} |\nabla u - (\nabla u)_\varrho|^2 \,\mathrm{d}x \le C\left(\frac{\varrho}{r}\right)^{n+2\alpha} \int_{B^+_r(x_0)} |\nabla u - (\nabla u)_r|^2 \,\mathrm{d}x \quad \text{for } \varrho < r.$$

Let us now suppose that there exists $z \in B'_r(x_0) \setminus \Lambda(u)$. Let w be the solution of

$$\text{(A.8)} \qquad \begin{cases} \operatorname{div}(\nabla_p f(\nabla w)) = 0 & \text{in } B^+_\varrho(x_0), \\ \partial_n w = 0 & \text{in } B'_\varrho(x_0), \\ w = u & \text{on } (\partial B_\varrho(x_0))^+, \end{cases}$$

with $0 < \varrho < r$. By the triangular and the Jensen inequalities, we obtain

$$\text{(A.9)} \qquad \begin{aligned} &\int_{B^+_\varrho(x_0)} |\nabla u - (\nabla u)_\varrho|^2 \,\mathrm{d}x \\ &\quad \le C \int_{B^+_\varrho(x_0)} |\nabla u - \nabla w|^2 \,\mathrm{d}x + C \int_{B^+_\varrho(x_0)} |\nabla w - (\nabla w)_\varrho|^2 \,\mathrm{d}x. \end{aligned}$$

We begin by estimating the first addend, by testing (1.5) with $u - w \in H^1_0(B_r(x_0))$ and using the fact that w is solution of (A.8):

$$\begin{aligned} 0 &= \int_{B^+_\varrho(x_0)} (\operatorname{div}(\nabla_p f(\nabla u)) - \operatorname{div}(\nabla_p f(\nabla w)))(u - w) \,\mathrm{d}x \\ &= -\int_{B^+_\varrho(x_0)} (\nabla_p f(\nabla u) - \nabla_p f(\nabla w)) \cdot \nabla(u - w) \,\mathrm{d}x \\ &\quad - \int_{B'_\varrho(x_0)} (u - w)(\nabla_p f(\nabla u) - \nabla_p f(\nabla w)) \cdot e_n \,\mathrm{d}x \\ &= -\int_{B^+_\varrho(x_0)} (\nabla_p f(\nabla u) - \nabla_p f(\nabla w)) \cdot \nabla(u - w) \,\mathrm{d}x \\ &\quad - \int_{B'_\varrho(x_0)} (u - w)\nabla_p f(\nabla u) \cdot e_n \,\mathrm{d}x. \end{aligned}$$

Hence, we have that

$$\int_{B^+_\varrho(x_0)} (\nabla_p f(\nabla u) - \nabla_p f(\nabla w)) \cdot \nabla(u - w) \,\mathrm{d}x = -\int_{B'_\varrho(x_0)} (u - w)\nabla_p f(\nabla u) \cdot e_n \,\mathrm{d}x.$$

By ellipticity,

$$\lambda \int_{B^+_\varrho(x_0)} |\nabla u - \nabla w|^2 \,\mathrm{d}x \le \int_{B^+_\varrho(x_0)} (\nabla_p f(\nabla u) - \nabla_p f(\nabla w)) \cdot (\nabla u - \nabla w) \,\mathrm{d}x,$$

and therefore, using Proposition A.6,

$$\begin{aligned}
\lambda \int_{B_\varrho^+(x_0)} |\nabla u - \nabla w|^2 \,\mathrm{d}x &\le \int_{B'_\varrho(x_0)} |u-w||\nabla_p f(\nabla u)\cdot e_n| \,\mathrm{d}x' \\
&= \int_{B'_\varrho(x_0)} |u-w||\nabla_p f(\nabla u)(x')\cdot e_n - \nabla_p f(\nabla u)(z)\cdot e_n| \,\mathrm{d}x' \\
&\le C[\nabla_p f(\nabla u)\cdot e_n]_{C^{0,\beta}(B'_\varrho)}\, \varrho^\beta \int_{B'_\varrho(x_0)} |u-w| \,\mathrm{d}x' \\
&\le C[\nabla_p f(\nabla u)\cdot e_n]_{C^{0,\beta}(B'_\varrho)}\, \varrho^\beta \int_{B_\varrho^+(x_0)} \operatorname{div}(-|u-w|e_n) \,\mathrm{d}x \\
&\le C[\nabla_p f(\nabla u)\cdot e_n]_{C^{0,\beta}(B'_\varrho)}\, \varrho^\beta \int_{B_\varrho^+(x_0)} |\nabla(u-w)| \,\mathrm{d}x \\
&\le C[\nabla_p f(\nabla u)\cdot e_n]_{C^{0,\beta}(B'_\varrho)}\, \varrho^{\beta+n/2} \|\nabla(u-w)\|_{L^2(B_\varrho^+)},
\end{aligned}$$

where we used the Hölder inequality in the last step. Finally,

$$\int_{B_\varrho^+(x_0)} |\nabla u - \nabla w|^2 \,\mathrm{d}x \le C[\nabla_p f(\nabla u)\cdot e_n]^2_{C^{0,\beta}(B'_\varrho)}\, \varrho^{2\beta+n}. \tag{A.10}$$

We focus now on the second addend of (A.9). By standard regularity theory, there exists $\gamma > 0$ such that

$$\int_{B_\varrho^+(x_0)} |\nabla w - (\nabla w)_\varrho|^2 \,\mathrm{d}x \le C\left(\frac{\varrho}{r}\right)^{n+2\gamma} \int_{B_r^+(x_0)} |\nabla w|^2 \,\mathrm{d}x.$$

Then, using the triangular inequality and inequality (A.10), we deduce

$$\begin{aligned}
&\int_{B_\varrho^+(x_0)} |\nabla w - (\nabla w)_\varrho|^2 \,\mathrm{d}x \\
&\quad \le C\left(\frac{\varrho}{r}\right)^{n+2\gamma} \int_{B_r^+(x_0)} |\nabla u|^2 \,\mathrm{d}x + C[\nabla_p f(\nabla u)\cdot e_n]^2_{C^{0,\beta}(B'_\varrho)}\, \varrho^{2\beta+n}.
\end{aligned}$$

Therefore, for $\alpha := \min\{\beta,\gamma\}$, we get

$$\begin{aligned}
&\varrho^{-(n+2\alpha)} \int_{B_\varrho^+(x_0)} |\nabla u - (\nabla u)_\varrho|^2 \,\mathrm{d}x \\
&\quad \le C[\nabla_p f(\nabla u)\cdot e_n]^2_{\beta, B'_\varrho}\, \varrho^{2(\beta-\alpha)} + C\,\frac{\varrho^{2(\gamma-\alpha)}}{r^{n+2\gamma}} \int_{B_r^+(x_0)} |\nabla u|^2 \,\mathrm{d}x.
\end{aligned}$$

From Campanato's theorem, taking the supremum in $\varrho \le r/2$ and employing Proposition A.6, we get

$$[\nabla u]_{\alpha, B^+_{\varrho/2}} \le C r^{-\alpha} \int_{B_r^+(x_0)} |\nabla u|^2 \,\mathrm{d}x,$$

and, combining with Remark A.5, we infer that (A.7) follows. ■

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References

[1] Almgren, F. J., Jr.: *Almgren's big regularity paper. Q-Valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension* 2. World Sci. Monog. Ser. in Math. 1, World Scientific, River Edge, NJ, 2000. Zbl 0985.49001 MR 1777737

[2] Andreucci, G., Focardi, M. and Spadaro, E.: On the free boundary for thin obstacle problems with Sobolev variable coefficients. To appear in *Interfaces Free Bound.* (2024), DOI 10.4171/ifb/537.

[3] Athanasopoulos, I.: Coincidence set of minimal surfaces for the thin obstacle. *Manuscripta Math.* **42** (1983), no. 2-3, 199–209. Zbl 0517.49028 MR 701203

[4] Athanasopoulos, I. and Caffarelli, L. A.: Optimal regularity of lower dimensional obstacle problems. *J. Math. Sci., New York* **132** (2006), no. 3, 274–284 (2006); translation from *Zap. Nauchn. Semin. POMI* **310** (2004), 49–66, 226. Zbl 1108.35038 MR 2120184

[5] Athanasopoulos, I., Caffarelli, L. A. and Salsa, S.: The structure of the free boundary for lower dimensional obstacle problems. *Amer. J. Math.* **130** (2008), no. 2, 485–498. Zbl 1185.35339 MR 2405165

[6] Caffarelli, L. A.: Further regularity for the Signorini problem. *Comm. Partial Differential Equations* **4** (1979), no. 9, 1067–1075. Zbl 0427.35019 MR 542512

[7] Caffarelli, L. A., Salsa, S. and Silvestre, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.* **171** (2008), no. 2, 425–461. Zbl 1148.35097 MR 2367025

[8] Colombo, M., Fernández-Real, X. and Ros-Oton, X.: Optimal regularity for the fully nonlinear thin obstacle problem. To appear in *J. Eur. Math. Soc. (JEMS)* (2024), DOI 10.4171/jems/1445.

[9] De Lellis, C., Focardi, M. and Spadaro, E. N.: Lower semicontinuous functionals for Almgren's multiple valued functions. *Ann. Acad. Sci. Fenn. Math.* **36** (2011), no. 2, 393–410. Zbl 1243.49013 MR 2757522

[10] De Lellis, C. and Spadaro, E. N.: Q-valued functions revisited. *Mem. Amer. Math. Soc.* **211** (2011), no. 991, 79 pp. Zbl 1246.49001 MR 2663735

[11] De Philippis, G., Spolaor, L and Velichkov, B.: (Quasi-)conformal methods in two-dimensional free boundary problems. *J. Eur. Math. Soc. (JEMS)* **27** (2025), no. 8, 3369–3406.

[12] Di Fazio, L. and Spadaro, E.: Regularity of solutions to nonlinear thin and boundary obstacle problems. *Adv. Math.* **401** (2022), article no. 108263, 39 pp. Zbl 1494.35198 MR 4387851

[13] Duvaut, G. and Lions, J.-L.: *Inequalities in mechanics and physics*. Grundlehren Math. Wiss. 219, Springer, Berlin-New York, 1976. Zbl 0331.35002 MR 521262

[14] Fernández-Real, X.: $C^{1,\alpha}$ estimates for the fully nonlinear Signorini problem. *Calc. Var. Partial Differential Equations* **55** (2016), no. 4, article no. 94, 20 pp. Zbl 1357.35129 MR 3523661

[15] Fernández-Real, X.: The thin obstacle problem: a survey. *Publ. Mat.* **66** (2022), no. 1, 3–55. Zbl 1487.35002 MR 4366205

[16] Focardi, M. and Spadaro, E.: An epiperimetric inequality for the thin obstacle problem. *Adv. Differential Equations* **21** (2016), no. 1-2, 153–200. Zbl 1336.35370 MR 3449333

[17] Focardi, M. and Spadaro, E.: On the measure and the structure of the free boundary of the lower dimensional obstacle problem. *Arch. Ration. Mech. Anal.* **230** (2018), no. 1, 125–184. Zbl 1405.35257 MR 3840912

[18] Focardi, M. and Spadaro, E.: How a minimal surface leaves a thin obstacle. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **37** (2020), no. 4, 1017–1046. Zbl 1461.35250 MR 4104833

[19] Focardi, M. and Spadaro, E.: The local structure of the free boundary in the fractional obstacle problem. *Adv. Calc. Var.* **15** (2022), no. 3, 323–349. Zbl 1501.35463 MR 4451895

[20] Frehse, J.: Two dimensional variational problems with thin obstacles. *Math. Z.* **143** (1975), no. 3, 279–288. Zbl 0295.49003 MR 380550

[21] Frehse, J.: On Signorini's problem and variational problems with thin obstacles. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **4** (1977), no. 2, 343–362. Zbl 0353.49020 MR 509085

[22] Garofalo, N. and Petrosyan, A.: Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem. *Invent. Math.* **177** (2009), no. 2, 415–461. Zbl 1175.35154 MR 2511747

[23] Garofalo, N., Petrosyan, A. and Smit Vega Garcia, M.: The singular free boundary in the Signorini problem for variable coefficients. *Indiana Univ. Math. J.* **67** (2018), no. 5, 1893–1934. Zbl 1409.35239 MR 3875246

[24] Garofalo, N. and Smit Vega Garcia, M.: New monotonicity formulas and the optimal regularity in the Signorini problem with variable coefficients. *Adv. Math.* **262** (2014), 682–750. Zbl 1295.35155 MR 3228440

[25] Giaquinta, M. and Modica, G.: Regolarità Lipschitziana per la soluzione di alcuni problemi di minimo con vincolo. *Ann. Mat. Pura Appl. (4)* **106** (1975), 95–117. Zbl 0325.49009 MR 454852

[26] Gilbarg, D. and Trudinger, N. S.: *Elliptic partial differential equations of second order*. Second edition. Grundlehren Math. Wiss. 224, Springer, Berlin, 1983. Zbl 0562.35001 MR 737190

[27] Giusti, E.: Superfici minime cartesiane con ostacoli discontinui. *Arch. Rational Mech. Anal.* **40** (1971), 251–267. Zbl 0214.11201 MR 275251

[28] Giusti, E.: Non-parametric minimal surfaces with discontinuous and thin obstacles. *Arch. Rational Mech. Anal.* **49** (1972/73), 41–56. Zbl 0257.49015 MR 346640

[29] Giusti, E.: *Direct methods in the calculus of variations*. World Scientific, River Edge, NJ, 2003. Zbl 1028.49001 MR 1962933

[30] Kinderlehrer, D.: The smoothness of the solution of the boundary obstacle problem. *J. Math. Pures Appl. (9)* **60** (1981), no. 2, 193–212. Zbl 0459.35092 MR 620584

[31] Lewy, H.: On a variational problem with inequalities on the boundary. *J. Math. Mech.* **17** (1967/68), 861–884. Zbl 0153.43303 MR 222734

[32] Milakis, E. and Silvestre, L.: Regularity for the nonlinear Signorini problem. *Adv. Math.* **217** (2008), no. 3, 1301–1312. Zbl 1132.35025 MR 2383900

[33] Nitsche, J. C. C.: Variational problems with inequalities as boundary conditions or how to fashion a cheap hat for Giacometti's brother. *Arch. Rational Mech. Anal.* **35** (1969), 83–113. Zbl 0209.41601 MR 248585

[34] Richardson, D. J. A.: *Variational problems with thin obstacles*. Ph.D. Thesis, The University of British Columbia, 1978. MR 2628343

[35] Ros-Oton, X.: Obstacle problems and free boundaries: an overview. *SeMA J.* **75** (2018), no. 3, 399–419. Zbl 1411.35289 MR 3855748

[36] Ros-Oton, X. and Serra, J.: The structure of the free boundary in the fully nonlinear thin obstacle problem. *Adv. Math.* **316** (2017), 710–747. Zbl 1371.35365 MR 3672918

[37] Salsa, S.: Optimal regularity in lower dimensional obstacle problems. In *Subelliptic PDE's and applications to geometry and finance*, pp. 217–226. Lect. Notes Semin. Interdiscip. Mat. 6, Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2007. Zbl 1137.35083 MR 2384648

[38] Signorini, A.: Questioni di elasticità non linearizzata e semilinearizzata. *Rend. Mat. e Appl. (5)* **18** (1959), 95–139. Zbl 0091.38006 MR 118021

[39] Simon, L. and Wickramasekera, N.: A frequency function and singular set bounds for branched minimal immersions. *Comm. Pure Appl. Math.* **69** (2016), no. 7, 1213–1258. Zbl 1341.53097 MR 3503021

[40] Ural'tseva, N. N.: Hölder continuity of gradients of solutions of parabolic equations with boundary conditions of Signorini type. *Sov. Math., Dokl.* **31** (1985), 135–138; translation from *Dokl. Akad. Nauk SSSR* **280** (1985), 563–565. Zbl 0629.35047 MR 0775926

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