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Quasi-projective varieties whose fundamental group is a free product of cyclic groups

José I. Cogolludo-Agustín and Eva Elduque

Abstract. In the context of Serre's question, we study smooth complex quasi-projective varieties whose fundamental group is a free product of cyclic groups. In particular, we focus on the case of surfaces and prove the existence of an admissible map from such a quasi-projective surface to a smooth complex quasi-projective curve. Associated with this result, we prove addition-deletion lemmas which describe a natural operation correlating this family of quasi-projective surfaces and groups. Our methods also allow us to produce examples of curves in smooth projective surfaces whose complements have free products of cyclic groups as fundamental groups, generalizing classical results on $C_{p,q}$ curves and torus-type projective sextics, and describing the conditions under which this phenomenon occurs.

1. Introduction

Goals and motivation

This paper is devoted to the general problem of describing the topology of smooth complex quasi-projective varieties. From the point of view of first homotopy groups, using Lefschetz-type theorems, it is enough to focus on complements of curves in a smooth projective surface. In this context, we address Serre's question on the type of groups that can appear as fundamental groups of quasi-projective varieties. In particular, we consider the family of finite free products of cyclic groups. As it turns out, all of them can be realized as fundamental groups of quasi-projective surfaces, but only some as fundamental groups of complements of plane projective curves (see Section 3.3). Moreover, in Theorem 1.3 we describe the geometric structure of such quasi-projective surfaces as containing a natural fibration onto a projective curve where the finite order elements correspond with multiple fibers of the fibration.

Interest for this problem in the complex projective plane goes back to Zariski and his foundational paper [43]. He considered the space of plane projective curves of degree six having six simple cusps, and characterized whether these singular points are placed on a

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conic by the property $\pi_1(\mathbb{P}^2 \setminus D) = \mathbb{Z}_2 * \mathbb{Z}_3$ for any such sextic $D \subset \mathbb{P}^2$, or equivalently, by having a $(2,3)$-toric decomposition (i.e., given by an equation $f_3^2 + f_2^3 = 0$ where f_i is a generic form of degree i in $\mathbb{C}[x,y,z]$). This is equivalent to the existence of a morphism $F: \mathbb{P}^2 \setminus D \to \mathbb{P}^1 \setminus \{[1:-1]\}$, defined by $F(x,y,z) = [f_3^2 : f_2^3]$ with two multiple fibers. In the 70's, Oka proved his classical result on $C_{p,q}$ curves in Section 8 of [32] (see also Dimca [17], Proposition (4.16) in Chapter 4), exhibiting a family of irreducible curves

$$C_{p,q} = \{(x^p + y^p)^q + (y^q + z^q)^p = 0\} \tag{1.1}$$

for any $p, q > 1$ coprime such that $\pi_1(\mathbb{P}^2 \setminus C_{p,q}) \cong \mathbb{Z}_p * \mathbb{Z}_q$. This was generalized in the 80's by Némethi in [29].

This problem appeared in several other contexts, mainly focusing on a possible connection between Alexander polynomials of complements $\mathbb{P}^2 \setminus D$ and toric (or more generally quasi-toric) decompositions ([5,13,25,26,34–36,40,41]). For instance, Oka's conjecture [18] (discussed and proved by Degtyarev in [14, 15]) states that an irreducible plane sextic is of torus type if and only if its Alexander polynomial is non-trivial.

A classical tool to describe the topology of smooth quasi-projective surfaces is the existence of morphisms onto smooth curves (a complex quasi-projective manifold of dimension 1). This is described in the Castelnuovo–de Franchis theorem for the existence of morphisms onto smooth curves of genus $g \geq 2$, and in Arapura's structure theorem [3], as well as its twisted version [6]. If X denotes a smooth projective surface and $D \subset X$ a (non-empty) reduced curve, the latter paper uses certain properties of the fundamental group to describe the existence of a dominant morphism F from $X \setminus D$ to a smooth projective curve S, which induces an orbifold structure S^{orb} on S determined by the multiplicity of the fibers of F. Since the fibers of this morphism are generically connected, the authors show that the fundamental group $\pi_1(X \setminus D)$ surjects onto the orbifold fundamental group of S^{orb}, which, since $S \setminus F(X \setminus D) \neq \emptyset$, is a finitely generated free product of cyclic groups. The extreme case for this morphism induced by F occurs when the surjection becomes an isomorphism between $\pi_1(X \setminus D)$ and the orbifold fundamental group of S^{orb}. This motivates our use of these techniques for the study of quasi-projective varieties whose fundamental group is a free product of cyclic groups, and raises the question of the existence of such a morphism F realizing an isomorphism between (orbifold) fundamental groups in this setting. Indeed, similar connections between fundamental groups of smooth varieties and orbifold fundamental groups have been studied by Arapura [4] and Catanese [10] in the projective case ($D = \emptyset$), and by Bauer [7] and Catanese [9] in the quasi-projective case where $\pi_1(X \setminus D)$ is free (or more generally, it admits an epimorphism onto a free group with a finitely generated kernel).

In this paper, we provide both necessary and sufficient geometric conditions for a smooth quasi-projective variety to have a free (finite) product of cyclic groups as its fundamental group. However, before stating our results in full generality, let us provide some intuition by exhibiting some of their consequences in the case of plane curve complements. We show that if the fundamental group of the complement of a curve in $\mathbb{P}^2$ is a free product of cyclic groups, then it is isomorphic to $\mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q$ for some $r \geq 0$ and some $p, q \geq 1$ such that $\gcd(p,q) = 1$. We prove the following structure theorem, which provides necessary conditions and greatly constrains the type of polynomial equations that can give rise to curves in $\mathbb{P}^2$ whose fundamental group of their complement is a free product of two non-trivial cyclic groups. The following is an immediate consequence of Corollary 3.15.

Corollary 1.1. *If a curve complement in* $\mathbb{P}^2$ *has fundamental group* $\mathbb{Z}_p * \mathbb{Z}_q$, *with* $p, q \in \mathbb{Z}_{>1}$ *coprime integers, then the curve is given by a polynomial equation of the form* $f_p^q + f_q^p = 0$, *for some* f_p *and* f_q *homogeneous polynomials in* $\mathbb{C}[x, y, z]$ *of degrees* p *and* q *with no common factors, and such that neither of them is a* k*-th power of another polynomial for any* $k \geq 2$.

A partial converse is also proved by showing that the fundamental group of the complement of any curve defined by $f_p^q + f_q^p = 0$ is *generically* a free product of cyclic groups.

Theorem 1.2. *Let* f_p (*respectively,* f_q) *be a homogeneous polynomial of degree* p (*respectively,* q) *in* $\mathbb{C}[x, y, z]$ *with* $\gcd(p, q) = 1$. *Assume that*

- f_p *and* f_q *define an admissible map* $F = [f_p^q : f_q^p] : \mathbb{P}^2 \setminus \mathcal{B} \to \mathbb{P}^1$, *where* $\mathcal{B} = V(f_p) \cap V(f_q)$ *is a finite set.*
- *The multiple fibers of* F *lie over a subset of* $\{[0 : 1], [1 : 0]\}$ (*this always holds if* $p, q \geq 2$).

Let $r \geq 0$, *and let* $C_0, \ldots, C_r$ *be the closures in* $\mathbb{P}^2$ *of* $r + 1$ *distinct generic fibers of* F. *Let* $C = \bigcup_{i=0}^r C_i$. *Then,*

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q.$$

Moreover, assume that $V(f_p)$ *is irreducible and* $\pi_1(\mathbb{P}^2 \setminus V(f_p)) \cong \mathbb{Z}_p$. *Then*

$$\pi_1\big(\mathbb{P}^2 \setminus (C \cup V(f_p))\big) = \mathbb{F}_{r+1} * \mathbb{Z}_p.$$

In particular, this provides infinitely many examples of curves in $\mathbb{P}^2$ like the ones in the previously mentioned examples by Zariski, Oka and Némethi, that is, such that the fundamental group of their complement is a free product of two non-trivial cyclic groups. Zariski's and Oka's results were proved using braid monodromy computations (which can be very complicated) applied to very expertly chosen specific examples. The methods presented here are different and their scope is wider, as they rely on exploiting the extra structure on these varieties endowed by morphisms to smooth complex quasi-projective curves (such as F in Theorem 1.2). These morphisms are constructed using properties of the fundamental group itself, not invariants of it such as the Alexander polynomial.

Overview of the main results

The main objects of this paper are curves in a smooth projective surface. Whenever we refer to a divisor as a curve, the divisor is meant to have a reduced structure. One of the main theorems of this paper provides geometric necessary conditions for the complement of a curve D in a smooth projective surface X to have a free product of cyclic groups as its fundamental group. These conditions include the existence of an *admissible map* to a smooth curve S (see Section 2.1 for the definition of admissible map), and are stated using orbifold notation $S_{(n+1,\bar{m})}$, where $\Sigma_0 \subset S$ is the set of $\#\Sigma_0 = n + 1$ points removed and $\bar{m} = (m_1, \ldots, m_s)$ represents the orbifold structure on s points of $S \setminus \Sigma_0$ (see Section 2.4 for the relevant definitions).

Theorem 1.3. *Let X be a smooth connected projective surface, and let $D \subset X$ be a curve. Suppose that $\pi_1(X \setminus D) \cong \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ is infinite. Then, there exist a smooth projective curve S of genus g_S and an admissible map $F\colon X \dashrightarrow S$ such that:*

(i) *F induces an orbifold morphism*

$$F_| : X \setminus D \to S_{(n+1,\bar{m})},$$

where $S_{(n+1,\bar{m})}$ is maximal with respect to $F_|$, $n \geq 0$ and $\bar{m} = (m_1, \ldots, m_s)$.

(ii) *$F_*\colon \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$ is an isomorphism.*

(iii) *$D = D_f \cup D_t$, where*
- *$D_f = \overline{F^{-1}(\Sigma_0)}$ is a fiber-type curve which is the union of the $n+1$ fibers above the distinguished points $\Sigma_0 \subset S$, with $n = r - 2g_S$,*
- *and the meridians of D_t are trivial in $\pi_1(X \setminus D)$.*

In particular, the maximality condition of part (i) implies that $F_|\colon X \setminus D \to S \setminus \Sigma_0$ is a surjective algebraic map with exactly s multiple fibers of multiplicities $(m_1, \ldots, m_s)$, determined by the torsion of $\pi_1(X \setminus D)$. Theorem 1.3 is proved in Section 3.

The existence of the admissible map F is guaranteed by [3, 6] if $r \geq 1$ as long as $\pi_1(X \setminus D) \neq \mathbb{Z}$. The case $\pi_1(X \setminus D) = \mathbb{Z}$ has to be considered separately. If $r = 0$, the structure of the group $\pi_1(X \setminus D)$ is used to construct a finite covering with free fundamental group. We then show that the Albanese morphism of this covering produces an admissible map which, by the functoriality of the Albanese morphism, descends to the desired admissible map F (after normalization in the target curve and Stein factorization). Moreover, Corollary 3.9 discusses the role of the divisor D_t.

Theorem 1.3 is extended in Section 3.2 to the case where $\pi_1(X \setminus D)$ is finite as long as X is simply connected, under some extra assumptions that are always satisfied if $X = \mathbb{P}^2$. As a result, we prove a refinement of Theorem 1.3 for the case $X = \mathbb{P}^2$ in Section 3.3 (Corollary 3.15), and show that in this case, $\pi_1(X \setminus D) \cong \mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q$ for some $p, q \in \mathbb{Z}_{\geq 1}$ coprime and $r \in \mathbb{Z}_{\geq 0}$, and that $D = D_f$ is a union of irreducible fibers of a pencil $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. The already stated Corollary 1.1 is a particular case ($r = 0$ and $p, q > 1$) of Corollary 3.15.

The second type of main results are referred to as addition-deletion theorems in Section 4. Consider U a smooth quasi-projective surface and $F\colon U \to S$ an admissible map to a smooth projective curve S, and define $U_B = U \setminus F^{-1}(B)$ for any finite subset $B \subset S$. In this context, we prove a deletion Lemma 4.9 that describes the fundamental group of U_B if $F_|\colon U_{B \cup \{P\}} \to S$ induces an isomorphism between (orbifold) fundamental groups. This is done regardless of whether $P \in B_F$ or not, for B_F the set of atypical values of F, that is, whether $F^{-1}(P)$ is a typical fiber or not. We also prove the following generic addition-deletion lemma in Section 4.

Theorem 1.4 (Generic addition-deletion lemma). *Let U be a smooth quasi-projective surface and let $F\colon U \to S$ be an admissible map to a smooth projective curve S. Assume $B \subset S$, where $\#B = n \geq 1$, and let $P \in S \setminus (B_F \cup B)$. Consider $S_{(n+1,\bar{m})}$ (respectively, $S_{(n,\bar{m})}$) the maximal orbifold structure of S with respect to $F_|\colon U_{B \cup \{P\}} \to S \setminus (B \cup \{P\})$ (respectively, $F_|\colon U_B \to S \setminus B$).*

Then the following are equivalent.

- $F_*\colon \pi_1(U_B) \to \pi_1^{\text{orb}}(S_{(n,\bar{m})})$ *is an isomorphism,*
- $F_*\colon \pi_1(U_{B\cup\{P\}}) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$ *is an isomorphism.*

Moreover, in that case,

$$\pi_1(U_{B\cup\{P\}}) \cong \mathbb{Z} * \pi_1(U_B).$$

As a consequence of this, we prove that the fundamental group of the complement of r generic fibers of a primitive polynomial map from $\mathbb{C}^2$ onto $\mathbb{C}$ is free of order r in Corollary 4.11.

We finally devote Section 5 to a number of applications of these results to the calculation of fundamental groups of complements of projective curves, noting how the results in Section 4 provide sufficient geometric conditions for these groups to be a free product of cyclic groups. In particular, in Section 5.1 we prove the already stated Theorem 1.2. The key ingredients in the proof of this theorem are the addition-deletion results, which allow us to avoid any Zariski–Van Kampen/braid monodromy calculations.

Theorem 1.2 brings together several examples known in the literature. For instance, we apply it in Section 5.2 to provide a Zariski–Van Kampen-free proof of a result on the fundamental group of a union of lines and conics due to Amram–Teicher (see Theorems 2.2 and 2.5 in [2]). As a last application, in Section 5.3 we generalize a classical result due to Oka–Pho in [36] on fundamental groups of maximal tame torus-type sextics.

Let us remark that, after the initial preparation of this paper, more progress has been made in this topic, more concretely, about potential refinements of the sufficient conditions from Theorem 1.2 for the fundamental group of the complement in $\mathbb{P}^2$ of a plane curve to be a free product of cyclic groups. More concretely, the examples of [12] show that those sufficient conditions would need to involve more than local invariants.

Remark 1.5. The results in this paper could be generalized to the quasi-Kähler (complements of normal crossing divisors in a compact Kähler surface) case in the following sense. Section 2 can be extended to quasi-Kähler surfaces. These results would provide proofs of analogues to Theorems 1.3 and 1.4 in this context. However, our main interest in this paper is the pair (X, D), where X is a projective surface and D is a curve on X (not necessarily with normal crossings). The statements of Theorems 1.3 and 1.2 and Corollary 1.1 describe some properties of the curve D in X.

2. Preliminaries

A short exposition on the main concepts and tools of this paper will be given in this section in order to fix notation and for the sake of completeness.

2.1. Admissible maps on X

Following Arapura [3], we call a surjective morphism $F\colon U \to S'$ from a smooth quasi-projective surface U to a smooth quasi-projective curve S' *admissible* if it admits a surjective holomorphic enlargement $\hat{F}\colon \hat{U} \to S$ with connected fibers, where $\hat{U}$ and S are smooth compactifications of U and S', respectively. As a consequence, note that the admissible map F has connected generic fibers, and in fact, both conditions are equivalent (cf. Remark 2.2 in [12]).

If U is a Zariski dense subset of a smooth projective surface X, then F defines a rational map $F\colon X \dashrightarrow S$ which can be extended to an admissible map $F\colon X \setminus \mathcal{B} \to S$ on a maximal open set $X \setminus \mathcal{B}$, where $\mathcal{B}$ is the finite subset of base points. Note that F must be surjective and that $\mathcal{B} = \emptyset$ unless $S = \mathbb{P}^1$. For convenience, this rational map will also be referred to as *admissible*.

Remark 2.1. An admissible map $F\colon X \setminus \mathcal{B} \to S$ has connected generic fibers and hence it induces an epimorphism $F_*\colon \pi_1(X \setminus \mathcal{B}) = \pi_1(X) \to \pi_1(S)$. In particular, $H_1(X;\mathbb{C}) = 0$ implies $S = \mathbb{P}^1$.

Given $P \in S' \subset S$, the fiber $F^{-1}(P) \subset U$ defines an algebraic curve C_P. Assume $B \subset S'$ is a finite set. We will denote $C_B = \bigcup_{P\in B} C_P$. Any such curve will be referred to as a *fiber-type curve*. It is well known that the minimal set of values B_F for which $F\colon U \setminus C_{B_F} \to S' \setminus B_F$ is a locally trivial fibration is finite [39]. The points in B_F are called *atypical* values of $F\colon U \to S'$. We will distinguish between $F^*(P)$ as the pulled-back divisor and C_P as its reduced structure. Using this notation, one can describe the set of multiple fibers as

$$M_F = \{P \in S' \mid F^*(P) = mD, m > 1, \text{ for some effective divisor } D\} \subset B_F. \tag{2.1}$$

Note that, in general, the effective divisor D in (2.1) need not be reduced. If $P \in S$, the *multiplicity* of $F^*(P)$ is defined as $m \geq 1$ if $F^*(P) = mD$ for some D, and whenever $F^*(P) = m'D'$, then $m' \leq m$.

Remark 2.2. If X is a simply-connected surface and $F\colon X \dashrightarrow S$ is an admissible map, then $S = \mathbb{P}^1$ by Remark 2.1 and an analogous argument to the one given in the proof of Proposition 2.8 in [13] shows that the number of multiple fibers of F cannot exceed two.

From now on, we will use the following notation.

Notation 2.3. Let $F\colon U \to S'$ be an admissible map from a smooth quasi-projective surface U to a smooth quasi-projective curve S', and let $B \subset S'$ be a finite set. We denote by $U_B := U \setminus C_B$. Analogously, if $F\colon X \dashrightarrow S$ is an admissible rational map from a smooth projective surface X to a smooth projective curve S and $B \neq \emptyset$, one defines X_B as U_B for $U = X \setminus \mathcal{B}$. Note that $X_B = X \setminus \left(\bigcup_{P\in B} \overline{F^{-1}(P)}\right)$.

2.2. Fundamental groups of quasi-projective varieties

Let X be a smooth quasi-projective surface and let $D = \bigcup_{i\in I} D_i$ be a curve in X, where D_i are its irreducible components.

When studying $\pi_1(X \setminus D, p)$, one has the following generating homotopy classes of loops: take a regular point p_i on D_i and consider a disk $\mathbb{D}_i \subset X$ transversal to D_i at p_i and such that $\mathbb{D}_i \cap D = \{p_i\}$. Let $\hat{p}_i \in \partial\mathbb{D}_i$ and consider $\hat{\gamma}_i$ a loop based at $\hat{p}_i$ around $\partial\mathbb{D}_i$ travelled in the positive orientation. Define $\delta_i\colon [0,1] \to X \setminus D$ a path in $X \setminus D$ starting at the base point $\delta_i(0) = p$ and ending at $\delta_i(1) = \hat{p}_i$. Denote by $\bar{\delta}_i$ the reversed path defined as usual as $\bar{\delta}_i(t) := \delta_i(1-t)$, $t \in [0,1]$, starting at p_i and ending at p. The following loop $\gamma_i := \delta_i \star \hat{\gamma}_i \star \bar{\delta}_i$ is based at p and defines a homotopy class called a *meridian* around D_i. The following two results are well known.

Lemma 2.4. *Let γ be a meridian around D_i. A homotopy class γ' is a meridian around D_i if and only if γ' is in the conjugacy class of γ in $\pi_1(X \setminus D, p)$.*

Proof. See [31], and also Proposition 1.34 in [11], for a proof. ∎

Lemma 2.5. *Consider $X_i := X \setminus (\text{Þ}\bigcup_{j\in I\setminus\{i\}} D_j)$ and $(j_i)_* : \pi_1(X \setminus D, p) \to \pi_1(X_i, p)$ induced by the inclusion $X \setminus D \hookrightarrow X_i$. Then $(j_i)_*$ is surjective, and* $\ker(j_i)_*$ *is the normal closure of any meridian γ_i in $\pi_1(X \setminus D, p)$. In particular, if X is simply connected, then any set of the form $\{\gamma_i\}_{i\in I}$ normally generates $\pi_1(X \setminus D, p)$, where γ_i is a meridian around D_i for all $i \in I$.*

Proof. See [31]. Also, as a consequence of Lemma 2.3 in [38]. ∎

2.3. Homology of the complement

Consider X a smooth projective surface and let $D = D_0 \cup \cdots \cup D_r \subset X$ be the decomposition of a curve D into its irreducible components. Using excision and Lefschetz duality, the homology exact sequence of $(X, X \setminus D)$ becomes

$$H_2(X;\mathbb{Z}) \to H^2(D;\mathbb{Z}) \to H_1(X \setminus D;\mathbb{Z}) \to H_1(X;\mathbb{Z}) \to 0,$$

where $H^2(D;\mathbb{Z}) \cong \mathbb{Z}^{r+1}$ is generated by the cohomology classes of each D_i irreducible component of D. Hence, if $H_1(X;\mathbb{Z}) = 0$ (respectively, if $H_1(X;\mathbb{Q}) = 0$), one obtains

$$H_2(X;\mathbb{Z}) \xrightarrow{j} \mathbb{Z}^{r+1} \to H_1(X \setminus D;\mathbb{Z}) \to 0 \tag{2.2}$$

(respectively, with $\mathbb{Q}$-coefficients), where $j(C) = \sum_{i=0}^{r} (C, D_i)_X D_i$ (see for instance [8]). In particular,

$$H_1(X \setminus D;\mathbb{Z}) = \mathbb{Z}^{r+1}/\operatorname{Im} j \qquad (\text{respectively, } H_1(X \setminus D;\mathbb{Q}) = \mathbb{Q}^{r+1}/\operatorname{Im} j). \tag{2.3}$$

The following condition on the irreducible components of a curve allows for a particularly simple description of the first homology of $X \setminus D$.

Condition 2.6. *The curve D decomposes into irreducible components as $D = \bigcup_{i=0}^{r} D_i$, and the irreducible components are such that,*

$$m_i D_i \equiv m_j D_j \quad \textit{for some } m_0, \ldots, m_r \in \mathbb{Z}_{>0},$$

where $\equiv$ here means numerical equivalence.

Example 2.7. The following are typical sources of examples satisfying Condition 2.6: (i) For any D, if the surface X is such that $\operatorname{NS}(X) = \operatorname{Pic}(X)/\operatorname{Pic}^0(X) = \mathbb{Z}$. (ii) For any X, if there exists an admissible map F from X onto a curve and $m_i \geq 1$ such that $D = \bigcup_{i=0}^{r} D_i$ and $m_i D_i$ is a (multiple if $m_i > 1$) fiber of F. (iii) For any X, whenever D is irreducible, since $\operatorname{Im}(j) \subset \mathbb{Z}D$.

The following result is immediate from (2.3).

Lemma 2.8. *If X is a smooth projective surface such that $H_1(X;\mathbb{Z})$ is finite, and D is a curve satisfying Condition 2.6, then $H_1(X\setminus D;\mathbb{Q})\cong\mathbb{Q}^r$. Moreover, if $H_1(X;\mathbb{Z})=0$, then $H_1(X\setminus D;\mathbb{Z})\cong\mathbb{Z}^r\times\mathbb{Z}_d$, where $d\in\mathbb{Z}_{>0}$ is determined by the components D_i.*

Remark 2.9. In particular, if $D\subset X=\mathbb{P}^2$ is a curve with $r+1$ irreducible components, then $H_1(X\setminus D;\mathbb{Z})\cong\mathbb{Z}^r\times\mathbb{Z}_d$, where d is the greatest common divisor of the degrees of the irreducible components of D (see Proposition (1.3) in Chapter 4 of [17]).

2.4. Orbifold fundamental groups and orbifold admissible maps

In this section, we will define the concept of orbifold fundamental groups and orbifold morphisms induced by admissible maps. As a word of caution, the word orbifold might be misleading, since we do not need to develop any theory of orbifolds or V-manifolds in this context. This will become clear throughout the section. The first concept is a group theoretical object associated with a smooth projective curve (or in more generality with a projective manifold) and a divisor on it. The second concept is purely geometric and only reflects the existence of non-reduced fibers of an admissible map.

Consider a smooth projective curve S of genus g and choose a labeling map $\varphi\colon S\to\mathbb{Z}_{\geq 0}$ such that $\varphi(P)\neq 1$ only for a finite number of points, say $\Sigma=\Sigma_0\cup\Sigma_+\subset S$, for which $\varphi(P)=0$ if $P\in\Sigma_0$ and $\varphi(Q)=m_Q>1$ if $Q\in\Sigma_+$. In this context, we will refer to this as an *orbifold structure on* S. This structure will be denoted by $S_{(n+1,\bar m)}$, where $n+1=\#\Sigma_0$, and $\bar m$ is a $(\#\Sigma_+)$-tuple whose entries are the corresponding m_Q's. The *orbifold fundamental group* associated with $S_{(n+1,\bar m)}$, denoted by $\pi_1^{\mathrm{orb}}(S_{(n+1,\bar m)})$, is the quotient of $\pi_1(S\setminus\Sigma)$ by the normal closure of the subgroup $\langle\mu_P^{\varphi(P)},\,P\in\Sigma\rangle$, where μ_P is a meridian in $S\setminus\Sigma$ around $P\in\Sigma$. Note that $\pi_1^{\mathrm{orb}}(S_{(n+1,\bar m)})$ is hence generated by $\{a_i,b_i\}_{i=1,\dots,g}\cup\{\mu_P\}_{P\in\Sigma}$ and presented by the relations

$$\mu_P^{m_P}=1,\quad \text{for } P\in\Sigma_+,\quad\text{and}\quad \prod_{P\in\Sigma}\mu_P=\prod_{i=1,\dots,g}[a_i,b_i] \tag{2.4}$$

for appropriately chosen $\{a_i,b_i\}_{i=1,\dots,g}$ and meridians $\{\mu_P\}_{P\in\Sigma}$. In the particular case where $\Sigma_0\neq\emptyset$, (2.4) shows that $\pi_1^{\mathrm{orb}}(S_{(n+1,\bar m)})$ is a free product of cyclic groups as follows:

$$\pi_1^{\mathrm{orb}}(S_{(n+1,\bar m)})\cong\pi_1(S\setminus\Sigma_0)*\Big(\mathop{*}_{P\in\Sigma_+}\Big(\frac{\mathbb{Z}}{m_P\mathbb{Z}}\Big)\Big)\cong\mathbb{F}_r*\mathbb{Z}_{m_1}*\cdots*\mathbb{Z}_{m_s},$$

where $r=2g-1+\#\Sigma_0=2g+n$, $s=\#\Sigma_+$, and $\bar m=(m_1,\dots,m_s)$.

Definition 2.10. Let S be a smooth projective curve endowed with an orbifold structure. We refer to the orbifold fundamental group of S as an *open orbifold group* of S when the orbifold structure on S is such that $\Sigma_0\neq\emptyset$, or equivalently, $n\geq 0$.

Definition 2.11. The *orbifold Euler characteristic* of $S_{(n+1,\bar m)}$ is given as

$$\chi^{\mathrm{orb}}(S_{(n+1,\bar m)}):=2-2g-(n+1)-\sum_i\Big(1-\frac{1}{m_i}\Big)=1-(s+2g+n)+\sum_i\frac{1}{m_i}.$$

Definition 2.12. Let X be a smooth algebraic variety. A dominant algebraic morphism $F\colon X \to S_{(n+1,\bar{m})}$ is called an *orbifold morphism* if for all $P \in S$ such that $\varphi(P) > 0$, the divisor $F^*(P)$ is a $\varphi(P)$-multiple. The orbifold $S_{(n+1,\bar{m})}$ is said to be *maximal* (with respect to F) if $F(X) = S \setminus \Sigma_0$ and for all $P \in F(X)$, the divisor $F^*(P)$ is not an n-multiple for any $n > \varphi(P)$.

The following result is well known (see, for instance, Proposition 1.4 in [5]).

Remark 2.13. Let $F\colon X \to S_{(n+1,\bar{m})}$ be an orbifold morphism. Then F induces a morphism $F_*\colon \pi_1(X) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$. Moreover, if the generic fiber of F is connected, then F_* is surjective.

The following lemma extends [30].

Lemma 2.14. *Consider $G = \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, with $s \geq 2$, $m_i > 1$, and $m := \operatorname{lcm}(m_i, i \in I)$, for $I = \{1, \ldots, s\}$. Let $\pi_{\text{cl}}\colon G \to \mathbb{Z}_m$ be the natural epimorphism of G onto its maximal cyclic quotient. Then $\ker(\pi_{\text{cl}}) \cong \mathbb{F}_\rho$, a free group of rank*

$$\rho = 1 - m + m \sum_{i \in I} \Big(1 - \frac{1}{m_i}\Big) = 1 - m\,\chi^{\text{orb}}(\mathbb{P}^1_{(1,\bar{m})}).$$

Proof. It is straightforward using Reidemeister–Schreier techniques and induction over the number of distinct prime factors of m. ∎

The following well-known result is a generalization of Lemma 4 in [36].

Lemma 2.15. *Let G be a finitely generated free product of cyclic groups. Then G is a Hopfian group, i.e., every endomorphism of G which is an epimorphism is an isomorphism.*

Proof. Consider $G = \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, where $m_i \geq 2$ for any $i \in I = \{1, \ldots, s\}$. Let $J = \mathbb{F}_r$ and $H = \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$. Finitely generated free groups are Hopfian, so J is Hopfian. H is a free product of finitely many finite groups, so it is virtually free and thus residually finite. Finitely generated residually finite groups are Hopfian, so H is Hopfian. By [16], the free product of two finitely generated Hopfian groups is Hopfian. ∎

2.5. Fundamental groups of complements of fiber-type curves

In this section, $F\colon U \to S$ is going to be an admissible map from a smooth quasi-projective surface to a smooth projective curve S of genus g_S. Following [24], we say the generic fiber $F^{-1}(P)$ of an admissible map is of *type* (g_F, s_F) if $F^{-1}(P)$ is homeomorphic to a smooth projective curve of genus g_F with s_F points removed. We will denote by

$$\Omega_{(g,s)} = \Big\langle a_1, \ldots, a_g, b_1, \ldots, b_g, x_0, \ldots, x_{s-1} \;:\; \textstyle\prod_{i=1}^{g} [a_i, b_i] = \prod_{j=0}^{s-1} x_j \Big\rangle \tag{2.5}$$

the fundamental group of a smooth projective curve of genus g, where $x_0, \ldots, x_{s-1}$ are meridians around its s punctures.

As above, consider the admissible map $F\colon U \to S$, $B = \{P_0, P_1, \ldots, P_n\} \subset S$, $n \geq 0$. Let $\Gamma_S = \{\gamma_1, \ldots, \gamma_r\}$, $r = 2g_S + n$ be a set of loops in $\pi_1(U_B)$ such that:

(1) $\{F_*(\gamma_k)\}_{k=1}^r$ generates $\pi_1(S \setminus B) \cong \mathbb{F}_r$ for all $k = 1, \ldots, r$,
(2) the loops $\gamma_k \in \pi_1(U_B)$ result from lifting a meridian around $P_k \in S$, for $k = 1, \ldots, n$,
(3) the product $\tilde{\gamma} = \prod_{j=1}^{g_S}[\gamma_{n+2j-1}, \gamma_{n+2j}](\prod_{i=1}^n \gamma_i)^{-1}$ is such that $F_*(\tilde{\gamma})$ is a meridian around P_0.
(4) For every $P_k \in B$ such that $F^*(P_k)$ is not a multiple fiber, γ_k is a product of meridians (positively or negatively oriented) about irreducible components of $C_{P_k} \subset C_B$ for all $k = 1, \ldots, n$. In the particular case where C_{P_k} is irreducible, γ_k is a positively oriented meridian about C_{P_k}.

On the other hand, if $P \in S \setminus (B_F \cup B)$, then the typical fiber $F^{-1}(P)$ is a smooth curve of type (g_F, s_F) and $\pi_1(F^{-1}(P)) \cong \Omega_{(g_F, s_F)}$ as in (2.5). Let Γ_F denote the image of such a set of generators as in (2.5) by the homomorphism induced by the inclusion $\iota\colon F^{-1}(P) \hookrightarrow U_B$.

Definition 2.16. Any set of loops Γ_F (respectively, Γ_S) obtained as in the construction above will be referred to as an *adapted geometric set of fiber* (*respectively, base*) *loops* with respect to F and B.

The following shows that adapted geometric sets of fiber (respectively, base) loops exist for admissible maps.

Lemma 2.17. *Let $F\colon U \to S$ be an admissible map from a smooth quasi-projective surface U to a smooth projective curve S of genus g_S. Consider $B = \{P_0, P_1, \ldots, P_n\} \subset S$, $n \geq 0$. Then, there exists $\Gamma_S = \{\gamma_1, \ldots, \gamma_r\}$, $r = 2g_S + n$, (respectively, Γ_F) an adapted geometric set of base (respectively, fiber) loops with respect to F and B.*

Proof. The statement with respect to Γ_F follows by construction. As for Γ_S, let us choose a set of loops in $\pi_1(S \setminus B) = \Omega_{(g_S, n+1)}$ satisfying (2.5). Since F has connected fibers and is algebraic, $F_*\colon \pi_1(U_B) \to \pi_1(S \setminus B) \cong \mathbb{F}_r$ is surjective and one can choose liftings γ_k satisfying properties (1)–(3) above.

To see that we may choose Γ_S so that it also satisfies condition (4), note that there exists a meridian μ_C around each irreducible component C of $F^*(P_k)$ of multiplicity $m = \text{mult}\{C\}$, such that $F_*(\mu_C) = F_*(\gamma_k)^m$. Also note that

$$m_k = \gcd\{\text{mult}(C) \in \mathbb{Z}_{\geq 1} \mid C \text{ irreducible component in } F^*(P_k)\}.$$

Now, using Bézout's identity, one can obtain a product of meridians μ_k whose image is $F_*(\gamma_k)^{m_k}$. In particular, $\mu_k = \alpha\gamma_k^{m_k}$ for $\alpha \in \ker F_*$. If $F^*(P)$ is not multiple, then $m_k = 1$. Replacing γ_k by $\alpha\gamma_k$, condition (4) follows, and conditions (1)–(3) still hold. ∎

Lemma 2.18. *Let $F\colon U \to S$ be an admissible map from a smooth quasi-projective surface U to a smooth projective curve S of genus g_S. Consider $B = \{P_0, P_1, \ldots, P_n\} \subset S$, $n \geq 0$. Suppose, moreover, that $B \supset B_F$ contains the set B_F of atypical values of F, and let $P \in S \setminus B$. Then $\pi_1(U_B)$ is a semidirect product of the form*

$$\pi_1(U_B) \cong \pi_1(F^{-1}(P)) \rtimes \pi_1(S \setminus B).$$

Moreover, $\pi_1(U_B)$ has a presentation with generators $\Gamma_F \cup \Gamma_S$ for

$$\Gamma_F = \{a_i, b_i, x_j \mid i = 1, \ldots, g_F,\ j = 0, \ldots, s_F - 1\} \quad \textit{and} \quad \Gamma_S = \{\gamma_k \mid k = 1, \ldots, r\},$$

where (g_F, s_F) *is the type of* $F^{-1}(P)$*,* Γ_F (*respectively,* Γ_S) *is an adapted geometric set of fiber* (*respectively, base*) *loops with respect to* F *and* B*, and the following is a set of relations*:

$$\begin{aligned}&[\gamma_k, a_i] = \alpha_{i,k},\\ &[\gamma_k, b_i] = \beta_{i,k},\\ &[\gamma_k \delta_{j,k}, x_j] = 1,\\ &\textstyle\prod_j x_j = \prod_i [a_i, b_i],\end{aligned} \tag{2.6}$$

where $i \in \{1, \dots, g_F\}$*,* $j \in \{0, \dots, s_F - 1\}$*,* $k \in \{1, \dots, r = 2g_S + n\}$*, and* $\alpha_{i,k}$*,* $\beta_{i,k}$*, and* $\delta_{j,k}$ *are words in the elements of* Γ_F*.*

Proof. The condition $B \supset B_F$ implies that $F\colon U_B \to S \setminus B$ is a locally trivial fibration, with fiber $F^{-1}(P)$. Let $\iota\colon F^{-1}(P) \hookrightarrow U_B$ be the inclusion. The long exact sequence of a fibration for homotopy groups yields

$$\pi_2(S \setminus B) \to \pi_1(F^{-1}(P)) \xrightarrow{\iota_*} \pi_1(U_B) \xrightarrow{F_*} \pi_1(S \setminus B) \to 1.$$

Note that, since $\pi_1(S \setminus B) \cong \mathbb{F}_r$, the epimorphism F_* splits. Since $S \setminus B$ is homotopy equivalent to a wedge of circles, $\pi_2(S \setminus B) = 1$, which concludes the result.

The description of the semidirect product $\pi_1(F^{-1}(P)) \rtimes \pi_1(S \setminus B)$ is given by considering the action of γ_k on the group $\pi_1(F^{-1}(P))$ generated by Γ_F. For the generators a_i (respectively, b_i), one can write $\gamma_k^{-1} a_i \gamma_k = a_i \alpha_{i,k}$ ($\gamma_k^{-1} b_i \gamma_k = b_i \alpha_{i,k}$) for some $\alpha_{j,k}$ (respectively, $\beta_{j,k}$) in $\pi_1(F^{-1}(P))$. For the meridians x_j around the point p_j on the boundary of $F^{-1}(P)$, note that $\gamma_k^{-1} x_j \gamma_k$ must be also a meridian around p_j and hence $\gamma_k^{-1} x_j \gamma_k = \delta_{j,k} x_j \delta_{j,k}^{-1}$ for some $\delta_{j,k}$ in $\pi_1(F^{-1}(P))$. The last relation in (2.6) comes from the choice of the adapted geometric set of fiber loops Γ_F. ∎

Corollary 2.19. *Under the notation and assumptions of Lemma* 2.18*,* $F_*\colon \pi_1(U_B) \to \mathbb{F}_r$ *is an isomorphism if and only if* $F^{-1}(P)$ *is of type* $(0, 1)$ *or* $(0, 0)$*.*

Moreover, if $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ *and* $U = \mathbb{P}^2 \setminus \mathcal{B}$*, then* $M_F = B_F$ *and hence* $\#B_F \leq 2$*.*

Proof. The first statement is an immediate consequence of Lemma 2.18. If U is a Zariski open subset of $\mathbb{P}^2$, any pencil has at least a base point and thus any fiber $F^{-1}(P)$ of $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ must be an open curve, so the fibers of $F_{|U}$ will be open curves as well. The 'moreover' part follows from Theorem 6.1 in [24] by direct inspection, since all the pencils of type $(0, 1)$ are classified. ∎

Example 2.20. In particular, according to Corollary 2.19, the classification of all rational pencils on $\mathbb{P}^2$ of type $(0, 1)$ given in [24] provides a list of examples of curves whose complement have a free fundamental group.

In Lemma 2.18, $B_F \subset B$. However, one can understand $\pi_1(U_B)$ for any non-empty finite set $B \subset S$ as follows.

Corollary 2.21. *Assume* $F\colon U \to S$ *is an admissible map, and let* $B \subset S$*, with* $\#B = n + 1 \geq 1$*. Then,* $\pi_1(U_B) \cong \pi_1(U_{B \cup B_F})/N$*, where* N *is the normal closure of meridians* $\gamma \in \pi_1(U_{B \cup B_F})$ *of the components of* $C_{B_F \setminus B}$*.*

Proof. The result follows from Lemma 2.5. ∎

The following result is well known in different settings (cf. [27,31,38]), but we include it here for the sake of completeness.

Corollary 2.22. *Let $F\colon U \to S$ be an admissible map, let $B \subset S$ be a finite set with $\#B = n+1 \geq 1$, and let $P \in S \setminus (B \cup B_F)$. Let $S_{(n+1,\bar{m})}$ be the maximal orbifold structure on S with respect to $F_{|U_B}$. Then,*

$$\pi_1(F^{-1}(P)) \xrightarrow{\iota_*} \pi_1(U_B) \xrightarrow{F_*} \pi_1^{\text{orb}}(S_{(n+1,\bar{m})}) \to 1$$

is an exact sequence.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(F^{-1}(P)) & \longrightarrow & \pi_1(U_{B\cup B_F}) & \xrightarrow{F_*} & \pi_1(S \setminus (B \cup B_F)) & \longrightarrow & 1 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
 & & \pi_1(F^{-1}(P)) & \xrightarrow{\iota_*} & \pi_1(U_B) & \xrightarrow{F_*} & \pi_1^{\text{orb}}(S_{(n+1,\bar{m})}). & &
\end{array}$$

Here, the vertical arrows are all induced by inclusion, and the top row is exact, thanks to Lemma 2.18. The surjectivity of the map F_* in the bottom row follows from the diagram (but also from Remark 2.13). Also, $\operatorname{Im}(\iota_*) \subseteq \ker F_*$. Let us prove the other inclusion. Since $\operatorname{Im}(\iota_*)$ is a quotient of a normal subgroup of $\pi_1(U_{B\cup B_F})$, it is a normal subgroup. Using the exactness of the top row, the last paragraph of the proof of Lemma 2.17, Lemma 2.18 and Corollary 2.21, it is straightforward to see that the map

$$\pi_1(U_B)/\operatorname{Im}(\iota_*) \twoheadrightarrow \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$$

has a splitting σ taking $F_*(\gamma_k)$ to the class of γ_k for all $\gamma_k \in \Gamma_S$, where Γ_S is an adapted geometric set of base loops with respect to F and $B \cup B_F$. The commutativity of the diagram above implies that σ is surjective. ∎

Remark 2.23. Suppose that X is a projective surface, S a smooth projective curve, and $F\colon X \to S$ is a surjective holomorphic map with connected fibers. Let $S_{(0,\bar{m})}$ be the maximal orbifold structure of S with respect to $F\colon X \to S$. As mentioned in the proof of Lemma 4.2 in [9], one also has an exact sequence like the one in Corollary 2.22, namely $\pi_1(F^{-1}(P)) \to \pi_1(X) \to \pi_1^{\text{orb}}(S_{(0,\bar{m})}) \to 1$, where the first arrow is induced by the inclusion of a generic fiber $F^{-1}(P)$ over $P \in S$.

2.6. Characteristic varieties

Characteristic varieties are invariants of finitely presented groups G, and they can be computed using any connected topological space X (having the homotopy type of a finite CW-complex) such that $G = \pi_1(X, x_0)$, $x_0 \in X$ as follows. Let us denote $H := H_1(X;\mathbb{Z}) = G/G'$. Note that the space of characters on G is a complex torus $\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*) = \operatorname{Hom}(H, \mathbb{C}^*) = H^1(X;\mathbb{C}^*)$. This $\mathbb{T}_G$ can have multiple connected components, but it only contains one connected torus, which we denote by $\mathbb{T}_G^{\mathbf{1}}$.

Definition 2.24. The k-th *characteristic variety* of G is defined by

$$\mathcal{V}_k(G) := \{\xi \in \mathbb{T}_G \mid \dim H^1(X, \mathbb{C}_\xi) \geq k\},$$

where $H^1(X, \mathbb{C}_\xi)$ is classically called the *twisted cohomology of X with coefficients in the local system* $\xi \in \mathbb{T}_G$. It is also customary to use $\mathcal{V}_k(X)$ for $\mathcal{V}_k(G)$ whenever $\pi_1(X) = G$.

If $G = \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, the torus $\mathbb{T}_G$ is a disjoint union of $\mathbb{T}_G^{\mathbf{1}} \cong (\mathbb{C}^*)^r$ and translations $\mathbb{T}_G^\lambda$ of $\mathbb{T}_G^{\mathbf{1}}$ by every element λ of $C = C_{m_1} \times \cdots \times C_{m_s}$, where C_m is the multiplicative group of m-th roots of unity. Given a torsion element $\rho \in \mathbb{T}_G$, one can define

$$\operatorname{depth}(\rho) := \max\{k \in \mathbb{Z}_{\geq 0} \mid \mathbb{T}_G^\rho \subset \mathcal{V}_k(G)\}. \tag{2.7}$$

Remark 2.25. In Proposition 2.10 of [6], a complete description of $\mathcal{V}_k(G)$ is given for orbifold fundamental groups of smooth quasi-projective curves. If $S = S_{(n+1,\bar{m})}$, one can check that $\mathcal{V}_1(\pi_1^{\text{orb}}(S)) \neq \emptyset$ if and only if $\pi_1^{\text{orb}}(S)$ is not abelian, or equivalently, if $\chi^{\text{orb}}(S) < 0$, in which case S is called an *orbifold of general type*.

2.7. Iitaka's (quasi)-Albanese varieties

Let X be a smooth projective variety. The Albanese variety is defined as

$$\operatorname{Alb}(X) := H^0(X, \Omega_X^1)^\vee / \operatorname{Free} H_1(X; \mathbb{Z}),$$

where $\vee$ denotes the dual as a $\mathbb{C}$-vector space, and $\operatorname{Free} H_1(X;\mathbb{Z})$ denotes the torsion-free factor of $H_1(X;\mathbb{Z})$. It is an abelian variety. Moreover, fixing a base point $x_0 \in X$, the Albanese morphism $\alpha_X \colon X \to \operatorname{Alb}(X)$ defined by $x \mapsto \left(\omega \mapsto \int_{x_0}^x \omega\right)$ is an algebraic morphism. Iitaka [23] generalized this to smooth quasi-projective varieties U (for a detailed description, see [21]), $\operatorname{Alb}(U)$ being a semiabelian variety in this case. The Albanese map α_U satisfies that $(\alpha_U)_* \colon H_1(U;\mathbb{Z}) \to H_1(\operatorname{Alb}(U);\mathbb{Z})$ is surjective, whose kernel is $\operatorname{Tors}_{\mathbb{Z}} H_1(U;\mathbb{Z})$. Moreover, if X is a smooth compactification of U such that $D := X \setminus U$ is a simple normal crossing divisor, and $i : U \hookrightarrow X$ is the inclusion, then we have an exact sequence

$$1 \to (\mathbb{C}^*)^r \to \operatorname{Alb}(U) \xrightarrow{\operatorname{Alb}(i)} \operatorname{Alb}(X) \to 1, \tag{2.8}$$

where $r = \dim_{\mathbb{C}} H^0(X, \Omega_X^1(\log D)) - \dim_{\mathbb{C}} H^0(X, \Omega_X^1)$.

We include here two technical lemmas about Albanese varieties.

Lemma 2.26. *Let U be a smooth quasi-projective surface such that $\pi_1(U) \cong \mathbb{Z}$. Then,* $\operatorname{Alb}(U) \cong \mathbb{C}^*$*, and $\alpha_U \colon U \to \mathbb{C}^*$ is an admissible map with no multiple fibers inducing isomorphisms in fundamental groups.*

Proof. We have that $\operatorname{Alb}(U) \cong \mathbb{C}^*$ because the latter is the only semiabelian variety whose fundamental group is isomorphic to $\mathbb{Z}$. Note that α_U must be dominant.

Let us consider a holomorphic enlargement $F \colon X \to \mathbb{P}^1$ of α_U. Since $\pi_1(X)$ is a quotient of $\mathbb{Z}$ by Lemma 2.5, using Stein factorization on F and Remark 2.1, we obtain that α_U factors through an admissible map $f : U \to V \subset \mathbb{P}^1$. Thus, $\pi_1(V) \cong \mathbb{Z}$, and thus $V = \mathbb{C}^*$. By the universal property of the Albanese, $f : U \to \mathbb{C}^*$ coincides with α_U up to isomorphism of the target, so α_U is admissible. Finally, by Remark 2.13, α_U has no multiple fibers. ■

Lemma 2.27. *Let S' be a smooth quasi-projective curve such that $\pi_1(S') \cong \mathbb{F}_r$, for $r \geq 1$. Let U be a smooth quasi-projective surface and let $F: U \to S'$ be an admissible map such that $\overline{F}: X \to S$ is a holomorphic extension of F with connected fibers, where X is a smooth projective compactification of U, $X \setminus U$ is a simple normal crossing divisor, and S is a smooth projective curve of genus g_S. Let $i_U: U \hookrightarrow X$ be the inclusion.*

Suppose that $F_*: \pi_1(U) \to \pi_1(S')$ is an isomorphism.

(1) *If $g_S = 0$, then*
   (a) $\alpha_{S'} : S' \to \mathrm{Alb}(S')$ *is injective*;
   (b) $\mathrm{Alb}(U) \cong (\mathbb{C}^*)^r \cong \mathrm{Alb}(S')$;
   (c) $\mathrm{Alb}(F): \mathrm{Alb}(U) \to \mathrm{Alb}(S')$ *is an isomorphism*;
   (d) *up to isomorphism in the target, the map F coincides with the restriction of α_U to its image, namely $\alpha_U: U \to \alpha_U(U)$.*

(2) *If $g_S \geq 1$, then*
   (a) $\alpha_S: S \to \mathrm{Alb}(S)$ *is injective*;
   (b) $\mathrm{Alb}(\overline{F}): \mathrm{Alb}(X) \to \mathrm{Alb}(S)$ *is an isomorphism*;
   (c) *up to isomorphism in the target, the map F coincides with the restriction of $\alpha_X \circ i_U$ to its image, namely $\alpha_X \circ i_U: U \to \alpha_X(U)$.*

Proof. The variety $\mathrm{Alb}(S)$ has (complex) dimension g_S. Similarly, since X is a projective variety, the dimension of $\mathrm{Alb}(X)$ is half of the rank of $H_1(X, \mathbb{Z})$. Let us show that the rank of $H_1(X, \mathbb{Z})$ is $2g_S$. Applying Corollary 2.22 to $F: U \to S'$, the morphism $\pi_1(\overline{F}^{-1}(P)) \to \pi_1(X)$ induced by inclusion can be seen to be trivial, where $\overline{F}^{-1}(P)$ is a generic fiber of $\overline{F}$. Let $S_{(0,\bar{m})}$ be the maximal orbifold structure of S with respect to $\overline{F}: X \to S$. By Remark 2.23, $\overline{F}_*: \pi_1(X) \to \pi_1^{\mathrm{orb}}(S_{(0,\bar{m})})$ is an isomorphism. Abelianizing, we obtain that the rank of $H_1(X; \mathbb{Z})$ is $2g_S$. In particular, if $g_S = 0$, then $\mathrm{Alb}(X)$ (respectively, $\mathrm{Alb}(S = \mathbb{P}^1)$) is a point, and thus, using equation (2.8), $\mathrm{Alb}(U)$ (respectively, $\mathrm{Alb}(S')$) is a torus, which must necessarily have dimension r. This concludes the proof of part (1b).

Assume that $g_S = 0$. Part (1a) is well known. Note that $\mathrm{Alb}(F): \mathrm{Alb}(U) \to \mathrm{Alb}(S')$ is an algebraic map which is a homomorphism between $(\mathbb{C}^*)^r$ and itself and induces an isomorphism on fundamental groups. By Cartier duality (see [37]), $\mathrm{Alb}(F)$ is an isomorphism, and part (1c) is proved. Part (1d) now follows both from the functoriality of the Albanese map, and from parts (1a) and (1c).

Assume now that $g_S \geq 1$. Part (2a) is the well-known Abel–Jacobi theorem. Let us prove part (2b). Note that $\overline{F} : X \to S$ is surjective, so the classical Albanese map $\mathrm{Alb}(\overline{F}): \mathrm{Alb}(X) \to \mathrm{Alb}(S)$ is a surjective group homomorphism. Hence, $\mathrm{Alb}(\overline{F})$ must be a fibration, and the dimension of the fiber is 0 when $\mathrm{Alb}(X)$ and $\mathrm{Alb}(S)$ have the same dimension, which we know equals g_S in both cases. Thus $\mathrm{Alb}(\overline{F}): \mathrm{Alb}(X) \to \mathrm{Alb}(S)$ is a finite covering. Since the fibers of $\overline{F}: X \to S$ are connected, the functoriality of the Albanese and Remark 2.1 imply that $\mathrm{Alb}(\overline{F})$ is an isomorphism. In particular, $\alpha_X(X)$ is isomorphic to S. As in the $g_S = 0$ case, part (2c) now follows both from the functoriality of the Albanese map, and from parts (2a) and (2b). ∎

3. Main theorem

Our purpose in this section is to give a necessary geometric condition for a curve to have the fundamental group of its complement isomorphic to $\mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$. We will show that these curves come from admissible maps, with the only possible exceptions occurring when the fundamental group is finite and the compact surface is not simply connected (see Remark 3.14). We will prove the main theorem in two stages. If $r \geq 1$, this is done in Theorem 3.1. If $r = 0$ and the group is infinite, this is done in Theorem 3.2. The strategy to find the admissible map in Theorem 3.2 heavily relies on the structure of the fundamental group of $\pi_1(X \setminus D)$. The idea is to find a free finite order subgroup whose associated covering falls in the hypotheses of Theorem 3.1.

3.1. Proof of Theorem 1.3

Theorem 1.3 will be stated in two ways depending on whether or not the first Betti number of $X \setminus D$ vanishes.

Theorem 3.1 (Main theorem, $r \geq 1$). *Theorem 1.3 holds if $r \geq 1$.*

Proof. If $\pi_1(X \setminus D) \cong \mathbb{Z}$, then parts (i) and (ii) of Theorem 1.3 follow from Lemma 2.26 for $S_{(n+1,\bar{m})} = \mathbb{P}^1_{(2,-)} \cong \mathbb{C}^*$.

Let us now prove (i) and (ii) assuming that the group $G := \pi_1(X \setminus D) \cong \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ is non-abelian (i.e., $\not\cong \mathbb{Z}$). Using Remark 2.25 and Proposition 2.10 in [6], the 1-st characteristic variety of the group G has a positive dimensional irreducible component $\mathbb{T}^{\lambda}_G$ associated with the torsion character $\lambda = (\xi_1, \ldots, \xi_s) \in \mathbb{T}_G$, where $\xi_i \in \mathbb{C}$ is an m_i-th primitive root of 1. Since $\mathbb{T}^{\lambda}_G$ has dimension $r \geq 1$, by Theorem 1 in [6], there exist an orbifold structure $S_{(n',\bar{m}')}$, $n' \geq 0$ on a smooth projective curve S of genus g_S and an admissible map $F \colon X \dashrightarrow S$ which induces an orbifold morphism $F_| \colon X \setminus D \to S_{(n',\bar{m}')}$ such that $S_{(n',\bar{m}')}$ is maximal with respect to $F_|$ and $F_|^*(V_{\bar{m}'}) = \mathbb{T}^{\lambda}_G$ for some component $V_{\bar{m}'} = \mathbb{T}^{\lambda'}_{G_1}$ of the 1-st characteristic variety of the orbifold fundamental group $G_1 = \pi_1^{\text{orb}}(S_{(n',\bar{m}')})$. Since F has connected generic fibers, $F_|$ induces a surjection of (orbifold) fundamental groups $F_* \colon G \to G_1$ (cf. Proposition 2.6 in [6]), and hence an injection $F^* \colon \mathcal{V}_k(G_1) \hookrightarrow \mathcal{V}_k(G)$ for all k.

Since $\mathbb{T}^{\lambda}_G$ is positive dimensional, it contains a non-torsion character. By Lemma 6.4 in [6], the admissible map F is unique such that $F_|^*(\mathbb{T}^{\lambda'}_{G_1}) = \mathbb{T}^{\lambda}_G$ and $\operatorname{depth}(\lambda) = \operatorname{depth}(\lambda')$.

Assume $n' = 0$. According to the structure of its characteristic varieties (cf. Proposition 2.11 in [6]), one has $\dim \mathbb{T}_{G_1} = 2g_S - 2 = r - 1 = \dim \mathbb{T}_G$. Finally, $1 \in \mathcal{V}_{r+1}(G_1)$ but $1 \notin \mathcal{V}_{r+1}(G)$. This contradicts the inclusion of characteristic varieties for $k = r + 1$. Therefore $n' = n + 1$, $n \geq 0$, and hence

$$\pi_1^{\text{orb}}(S_{(n+1,\bar{m}')}) \cong \mathbb{F}_{r'} * \mathbb{Z}_{m'_1} * \cdots * \mathbb{Z}_{m'_{s'}},$$

where $n = r' - 2g_S$. For $k = 0$, $F_|^*(\mathbb{T}^{\lambda'}_{G_1}) = \mathbb{T}^{\lambda}_G$ implies $r = r'$.

To show (i), it remains to show that $s = s'$ and $m_i = m'_i$ for all $i = 1, \ldots, s$. Using Proposition 2.10 in [6] and (2.7), one obtains $s + r - 1 = \operatorname{depth}(\lambda) = \operatorname{depth}(\lambda') \leq s' + r - 1$, so $s \leq s'$. In addition, since $F_|^*$ induces injections between characteristic varieties, one obtains $s' \leq s$, which shows $s = s'$.

Let $G = \langle x_1, \ldots, x_{r+s} \mid x_1^{m_1} = x_2^{m_2} = \ldots = x_s^{m_s} = 1\rangle$ be the presentation of G that we have implicitly used to give coordinates to $\mathbb{T}_G$. Since F_* is a surjection and the only torsion elements of G and G_1 are conjugation of elements in the finite group free factors, Lemma 2.15 implies that, for some reordering of the m'_i, we can find a presentation $\langle y_1, \ldots, y_{r+s} \mid y_1^{m'_1} = y_2^{m'_2} = \cdots = y_s^{m'_s} = 1\rangle$ of G_1 such that $F_*(x_j) = y_j$ for all $j = 1, \ldots, r+s$. In particular, $m'_i \mid m_i$. We want to see that $m'_i = m_i$ for all $i = 1, \ldots, s$. We argue by contradiction. Without loss of generality, we may assume that $m'_1 < m_1$. Hence, $F_|^*\colon \mathbb{T}_{G_1} \to \mathbb{T}_G$ takes a generator of the $\mathbb{Z}_{m'_1}$ factor of $\mathbb{T}_{G_1} \cong \mathbb{Z}_{m'_1} \times \cdots \times \mathbb{Z}_{m'_s} \times (\mathbb{C}^*)^r$ to an element of the subgroup $\langle m_1/m'_1\rangle$ of the $\mathbb{Z}_{m_1}$ factor of $\mathbb{T}_G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_s} \times (\mathbb{C}^*)^r$. In particular, λ is not in the image of $F_|^*$, which yields a contradiction. This concludes the proof of (i) if G is non-abelian.

The fact that F_* is an isomorphism follows from Corollary 2.22 and Lemma 2.15. This concludes the proof of (ii) if G is non-abelian.

Let $B = \Sigma_0$ be the $n+1$ points of S of label 0 in the orbifold structure $S_{(n+1,\bar{m})}$. Note that, by the maximality of the orbifold structure of $S_{(n+1,\bar{m})}$ with respect to $F_|\colon X \setminus D \to S_{(n+1,\bar{m})}$, the extension $F\colon X \dashrightarrow S$ satisfies that $\overline{F^{-1}(B)} = D_f \subset D$, hence $D = D_f \cup D_t$, where D_t is the union of the irreducible components of D which are not in D_f. We can further decompose D_t as $D_v \cup D_h$, where D_v is the union of the vertical components (irreducible components C such that $F(C)$ is a point), and D_h is the union of the horizontal components (irreducible components C such that $F(C) = S$). Note that we can choose a meridian around any irreducible component C of D_h which is contained in a generic fiber of $F\colon X \setminus D \to S \setminus B$. Hence, by Corollary 2.22, any meridian about any irreducible component of D_h must be in the kernel of F_*, thus it must be trivial as a consequence of part (ii). Analogously, a meridian around an irreducible component of D_v must be also in $\ker F_*$ since its image is a power of a meridian around $P \in S \setminus B$ which is the boundary of a disk centered at P, and hence trivial. This concludes the proof of (iii). ■

Theorem 3.2 (Main theorem, $r = 0$). *Theorem 1.3 holds if $r = 0$ and $\pi_1(X \setminus D)$ is infinite.*

Proof. Denote
$$G := \pi_1(X \setminus D) = \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s},$$
where $m_i > 1$ for all $i \in I = \{1, \ldots, s\}$ and $s \geq 2$.

Consider the unramified covering associated with the projection onto the maximal cyclic quotient $\nu\colon G \to \mathbb{Z}_m$, where $m := \operatorname{lcm}(m_i, i \in I)$. For any irreducible component D_i of D, denote by $1 \leq e_i \leq m$ the order of $\nu(\gamma_i) \in \mathbb{Z}_m$, where γ_i is any meridian around D_i. By Theorem 1.3.8 in [28], the cyclic unbranched covering $\theta\colon \tilde{Y} \to X \setminus D$ associated with $\nu\colon G \to \mathbb{Z}_m$ extends to a finite Galois branched cover $Y \to X$ which branches at each component D_i with branching number e_i. By Theorem 1.1.7 in [28], the deck action of $\mathbb{Z}_m$ on $\tilde{Y}$ extends to the Galois action on Y, such that $Y/\mathbb{Z}_m \cong X$. Both of these actions are by algebraic automorphisms.

By Lemma 2.14, the kernel $\ker \nu \cong \pi_1(\tilde{Y})$ is the free group $\mathbb{F}_\rho$, where $\rho = 1 - m + m\sum_{i \in I}(1 - 1/m_i) \geq 1$.

Denote by $\bar{Y}$ a projective surface such that $\bar{Y} \setminus \bar{D} = \tilde{Y}$, where $\bar{D}$ is a normal-crossing divisor obtained by resolving the singularities of $\theta^{-1}(D) \subset Y$. We may assume that the action on Y (which on $\tilde{Y}$ is the action by Deck transformations) extends to $\bar{Y}$ [42]. By Theorem 3.1, $\tilde{Y}$ is induced by an admissible map $f'\colon \tilde{Y} \to C'$ onto an (open) smooth curve C' with no multiple fibers. Moreover, by Lemma 2.27(1d) and (2c), up to an isomorphism of the curve, f' is the restriction of the Albanese map to $\tilde{Y}$ and its image. By the functoriality of the Albanese map, the deck transformations can be carried over to C'.

Consider $\tilde{\theta}\colon C' \to C$ the quotient map by this action, where C is an open curve (not necessarily smooth). The map $f'\colon \tilde{Y} \to C'$ hence descends to a morphism $f\colon X \setminus D \to C \subset \overline{C}$, where $\overline{C}$ is a projective curve. Note that C (and $\overline{C}$) may not be smooth, but, by the universal property of the normalization, f lifts to $\widetilde{F}\colon X \setminus D \to \widetilde{S}$, where $\widetilde{S}$ is the normalization of $\overline{C}$ (a smooth projective curve). Applying Stein factorization, we know that $\widetilde{F} = \pi \circ F$, where $F\colon X \setminus D \to S$ is admissible when restricted to its image, and $\pi\colon S \to \widetilde{S}$ is a finite morphism.

Since the normalization $\widetilde{S} \to \overline{C}$ is a birational equivalence, a generic fiber of f is also a generic fiber of $\widetilde{F}$, which is a disjoint union of generic fibers of F. Moreover, since the generic fiber of $\tilde{\theta}$ is finite, the preimage through θ of a generic fiber of f is a disjoint union of generic fibers of f'. Restricting to connected components, one finds $P \in S$ and $P' \in C$ such that $\theta\colon (f')^{-1}(P') \to F^{-1}(P)$ is a finite covering map, where $(f')^{-1}(P')$ is a generic fiber of f' and $F^{-1}(P)$ is a generic fiber of F.

Let us check that the morphism $F_*\colon \pi_1(X \setminus D) \to \pi_1^{\mathrm{orb}}(S)$ is an isomorphism, where S is endowed with the maximal orbifold structure with respect to $F\colon X \setminus D \to S$. By Corollary 2.22, this is equivalent to showing that the image of $\pi_1(F^{-1}(P))$ in $\pi_1(X \setminus D)$ is trivial. Since $f'\colon U \to C'$ induces an isomorphism on fundamental groups, Corollary 2.22 tells us that the inclusion $(f')^{-1}(P') \hookrightarrow U$ induces the trivial map on fundamental groups. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1((f')^{-1}(P)) & \longrightarrow & \pi_1(U) \\ \big\downarrow \theta_* & & \big\downarrow \theta_* \\ \pi_1(F^{-1}(P)) & \xrightarrow{i_*} & \pi_1(X \setminus D) \cong \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}, \end{array}$$

where the horizontal arrows are induced by inclusion. Note that the arrow on the left is a finite covering space, so its image is a finite index normal subgroup of $\pi_1(F^{-1}(P))$. The commutativity of the diagram above implies that the morphism i_* factors through the quotient $\pi_1(F^{-1}(P))/\theta_*(\pi_1((f')^{-1}(P)))$. Hence, the image of i_* is a finite subgroup of $\mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$. Moreover, by Corollary 2.22, this subgroup is normal. By the Kurosh subgroup theorem, the only finite normal subgroup of $\mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ is the trivial subgroup. Hence, i_* is the trivial morphism, and thus $F_*\colon \pi_1(X \setminus D) \to \pi_1^{\mathrm{orb}}(S)$ is an isomorphism.

Note that if S is a smooth projective curve such that $\pi_1^{\mathrm{orb}}(S_{(n,\bar{m})}) \cong \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, then we will see that $S = \mathbb{P}^1$, and the image of F must be $\mathbb{P}^1$ with one point removed. Indeed, after abelianizing the presentation of $\pi_1^{\mathrm{orb}}(S)$ from Section 2.4, it follows that S must have genus 0 (so $S = \mathbb{P}^1$). Moreover, $F(X \setminus D)$ is either $\mathbb{P}^1$ or $\mathbb{C}$. Let us see that it is indeed the latter. Suppose that $F(X \setminus D) = \mathbb{P}^1$, so none of the irreducible components of D are fibers of $F\colon X \dashrightarrow \mathbb{P}^1$. Then, as in Theorem 3.1, the inclusion of $X \setminus D$ to X induces an isomorphism in fundamental groups, and thus $\pi_1(X)$ is isomorphic to a nontrivial free product. This is impossible by [22].

We have shown that, under the assumptions of this theorem, if $\pi_1(X \setminus D) \cong \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, for $s \geq 2$, with $m_i \geq 2$ for all $i = 1, \ldots, s$, then there exists an admissible map $F\colon X \setminus D \to \mathbb{P}^1 \setminus B$ that induces an isomorphism $F_*\colon \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(1,\bar{m})})$, where $\mathbb{P}^1$ is endowed with the maximal orbifold structure with respect to $F\colon X \setminus D \to \mathbb{P}^1$. Note that $\bar{m} = (m_1, \ldots, m_s)$, and those are the multiplicities of the multiple fibers. The remaining condition for $D = D_f \cup D_t$ can be proved as in 3.1. ∎

Remark 3.3. The mere existence of zero-dimensional components of the characteristic varieties of $\pi_1(X \setminus D)$ is not enough to ensure the existence of the admissible map F. For instance, in Theorem 4.5 of [5], an example of a quintic projective curve D considered by Degtyarev is presented, where the characteristic varieties of $\pi_1(\mathbb{P}^2 \setminus D)$ are the primitive 10th roots of unity, but no admissible map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ exists inducing a surjection from $\pi_1(X \setminus D)$ onto $\pi_1^{\text{orb}}(\mathbb{P}^1_{1,\bar{m}})$.

Example 3.4. Examples of smooth quasi-projective surfaces having infinite fundamental groups of the form $\mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ were found by Aguilar Aguilar in Theorem 1.2 of [1]. The author considers three concurrent lines $L + L_1 + L_2$ in $\mathbb{P}^2$ intersecting at P, and blows up P and the infinitely near points P_i on the strict transform $\tilde{L}_i$ of L_i, $i = 1, 2$. Hence $\pi^*(L_i) = E + 2E_i + \tilde{L}_i$, where E is the exceptional divisor of the first blow-up and E_i is the exceptional divisor that appears when blowing up the infinitely near points P_i. Denote by $\hat{\mathbb{P}}^2$ the resulting surface after the three blow-ups. The quasi-projective surface $X = \hat{\mathbb{P}}^2 \setminus (\tilde{L} \cup \tilde{L}_1 \cup \tilde{L}_2)$ has a well-defined morphism onto the orbifold $\mathbb{P}^1_{(1,(2,2))}$ whose generic fiber is the strict transform of a generic line through P, which is a rational smooth curve and hence simply connected. Hence $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_2$. The author shared with us how his method can be generalized by considering $r + s + 1$ concurrent lines and blowing up m_i times at infinitely near points of P in L_i, $i = 1, \ldots, s$, so that $\pi^*(L_i) = E + 2E_{i,2} + \cdots + m_i E_{i,m_i} + \tilde{L}_i$, to produce surfaces

$$X = \hat{\mathbb{P}}^2 \setminus \left(\bigcup_{i=1}^{r+s+1} \tilde{L}_i \cup \bigcup_{i=1,\ldots,s} \bigcup_{k_i=1}^{m_i-1} E_{i,k_i} \right),$$

with $\pi_1(X) = \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ for any $r \geq 0, m_1, \ldots, m_s \geq 0$. In particular, every finitely generated free product of cyclic groups can be realized as the fundamental group of a smooth quasi-projective surface.

We now make an observation about the type of free products of cyclic groups that can appear as fundamental groups of curve complements in *simply-connected* projective surfaces.

Remark 3.5. Under the conditions of Theorem 1.3, if X is simply connected, then $s \leq 2$ (see Remark 2.2), that is, $\pi_1^{\text{orb}}(\mathbb{P}^1_{(r+1,\bar{m})}) \cong \mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q$, where $p, q \geq 1$. Moreover, if D satisfies Condition 2.6, then $\gcd(p, q) = 1$ by Lemma 2.8.

Example 3.6. The simply-connectedness condition in Remark 3.5 is important. We will see an example satisfying the hypotheses of Theorem 3.2 with a non-simply-connected surface X, a curve D on X such that $\pi_1(X \setminus D) \cong \mathbb{Z}_3 * \mathbb{Z}_3$, and a rational map realizing the isomorphism with the orbifold fundamental group of $\mathbb{P}^1_{(1,(3,3))}$. Let X be the double cover of $\mathbb{P}^2$ ramified along a generic sextic $C = \{f_2^3 + f_3^2 = 0\}$, where f_i is a homogeneous polynomial in three variables of degree i. It is well known (cf. [43]) that $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 * \mathbb{Z}_3$ (also as a consequence of Theorem 1.2). A meridian around C can be given as xy, where $x^2 = y^3 = 1$. Using Lemma 2.14, one can show $\pi_1(X \setminus D) \cong \mathbb{Z}_3 * \mathbb{Z}_3$ generated by $y_0 := y$ and $y_1 := xyx$, where D is the preimage of C. Note that the six-fold cover ramifies fully along C and thus its preimage is irreducible. Since this cover factors through X, D must also be irreducible and thus it satisfies Condition 2.6. The preimage of $(yx)^2 = y_0 y_1$ becomes a meridian of D, and by Lemma 2.5, one has $\pi_1(X) = \pi_1(X \setminus D)/\operatorname{NCl}(y_0 y_1) = H_1(X;\mathbb{Z}) = \mathbb{Z}_3$. Theorem 3.2 ensures the existence of an admissible map $F\colon X \dashrightarrow \mathbb{P}^1$ with two multiple fibers of multiplicity 3 and such that D is the preimage of a point in $\mathbb{P}^1$.

Corollary 3.7. *Under the notation of Theorem 1.3, let us suppose that an admissible map $F\colon X \dashrightarrow S$ satisfies conditions (i)– (iii), so in particular, $\pi_1(X \setminus D)$ is isomorphic through the map induced by F to an open orbifold group of S (not necessarily infinite). If $H_1(X;\mathbb{Z})$ is torsion and D satisfies Condition 2.6, then $S = \mathbb{P}^1$ and $r + 1$ is in fact the number of irreducible components of D. In particular, $D = D_f$ is a fiber-type curve.*

Proof. The result follows from Remark 2.1 and Lemma 2.8. ∎

The following example will exhibit the importance of Condition 2.6 to establish that $r + 1$ is the number of irreducible components of D.

Example 3.8. Let $\mathcal{Q}$ be a smooth conic in $\mathbb{P}^2$ and let $P \in \mathcal{Q}$. Consider ℓ the tangent line to $\mathcal{Q}$ through P and D a union of ℓ and r lines through P. The quadric surface $X = \mathbb{P}^1 \times \mathbb{P}^1$ can be defined as the 2:1 cover $\sigma : X \to \mathbb{P}^2$ ramified along $\mathcal{Q}$. In particular, X is simply connected. Also note that $\sigma^{-1}(D)$ is a union of $r + 2$ irreducible components, namely r curves of bidegree $(1, 1)$ and the rulings $\sigma^{-1}(\ell) = \ell_1 \cup \ell_2$, for ℓ_1 (respectively, ℓ_2) of bidegree $(1, 0)$ (respectively, $(0, 1)$) all of them passing through $\sigma^{-1}(P)$. The curve $\sigma^{-1}(D)$ has $r + 2$ irreducible components and does not satisfy Condition 2.6. Note that $\pi_1(\mathbb{P}^2 \setminus D) = \mathbb{F}_r$, and one can check that also $\pi_1(X \setminus \sigma^{-1}(D)) = \mathbb{F}_r$.

Theorem 1.3 gives necessary geometric conditions for a quasi-projective surface $X \setminus D$ to have an infinite fundamental group which is a free product of cyclic groups. The curve D need not be of fiber-type, but D_f (which is a non-empty union of irreducible components of D) is a fiber-type curve coming from an admissible map $F\colon X \dashrightarrow S$. The following result illustrates that $X \setminus D_f$ behaves exactly like $X \setminus D$ in Theorem 1.3.

Corollary 3.9. *Under the conditions of Theorem 1.3 and using the notation therein, the inclusion induces an isomorphism $\pi_1(X \setminus D) \cong \pi_1(X \setminus D_f)$, and $F_*\colon \pi_1(X \setminus D_f) \to \pi_1^{\mathrm{orb}}(S_{(n+1,\bar{m})})$ is an isomorphism.*

Proof. The isomorphism $\pi_1(X \setminus D) \cong \pi_1(X \setminus D_f)$ follows from Lemma 2.5 and the fact that the meridians of D_t are trivial in $\pi_1(X \setminus D)$. Now, using Corollary 2.22 for $U = X \setminus (D_t \cup \mathcal{B})$ and $U = X \setminus \mathcal{B}$, we see that $F_|\colon X \setminus D_f \to S \setminus \Sigma_0$ must also induce isomorphisms in (orbifold) fundamental groups, and that the maximal orbifold structure on S with respect to $F_|\colon X \setminus D_f \to S \setminus \Sigma_0$ must coincide with the one with respect to $F_|\colon X \setminus D \to S \setminus \Sigma_0$. ∎

Example 3.10. Consider three concurrent lines L_1, L_2, and L_3 in $\mathbb{P}^2$. Blowing up $\mathbb{P}^2$ at the intersection point one obtains an exceptional divisor E and the strict transforms of the lines $\tilde{L}_1$, $\tilde{L}_2$, and $\tilde{L}_3$ in the blown-up surface $X = \hat{\mathbb{P}}^2$. Note that any meridian around E is trivial in $\pi_1(X \setminus D)$ since it is the inverse of a product of meridians around each line. Define $D = E \cup \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_3$ in X. Note that $\pi_1(X \setminus D) = \mathbb{F}_2$, $D_f = \tilde{L} \cup \tilde{L}_1 \cup \tilde{L}_2$ and $D_t = E$ in Theorem 1.3. As Corollary 3.9 states, $\pi_1(X \setminus D_f) = \mathbb{F}_2$.

3.2. Extensions of the main theorem to finite cyclic groups

Theorem 1.3 describes the geometry of a curve $D \subset X$ when $\pi_1(X \setminus D)$ is an *infinite* group of the form $\mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ (i.e., non-abelian or $\mathbb{Z}$), where X is a smooth projective surface. In this section, we give extra hypotheses under which similar results hold in the remaining abelian cases (the trivial group and finite cyclic groups).

Proposition 3.11 (Main theorem, trivial group case). *Let $D \subset X$ be a curve in a smooth projective surface X. Assume that $\pi_1(X \setminus D)$ is trivial. If D is an ample divisor, then there exists an admissible map $F\colon X \dashrightarrow \mathbb{P}^1$ as in Theorem 1.3 for $S_{(n+1,\bar{m})} = \mathbb{P}^1_{(1,-)}$ satisfying conditions (i)–(iii).*

Proof. Since D is ample, a multiple of D defines an embedding $X \setminus D \hookrightarrow \mathbb{C}^k$ for some k. Projecting to a generic 1-dimensional subspace inside $\mathbb{C}^k$, we get a dominant morphism $F\colon X \setminus D \to \mathbb{C}$ with connected fibers, which can be extended to a rational map $F\colon X \dashrightarrow \mathbb{P}^1$. Using Remark 2.13, we have that $\pi_1^{\mathrm{orb}}(\mathbb{P}^1_{(n+1,\bar{m})})$ is trivial, where $\mathbb{P}^1_{(n+1,\bar{m})}$ is the maximal orbifold structure with respect to $F\colon X \setminus D \to \mathbb{C}$. This implies that $n = 0$ (so in particular $F\colon X \setminus D \to \mathbb{C}$ is surjective) and that $\bar{m}$ is the trivial orbifold structure. ■

Remark 3.12. In order to clarify the hypothesis given in Proposition 3.11 we will exhibit an example where the result does not follow when $\pi_1(X \setminus D) = 1$ and D is not an ample divisor.

Consider a line L in a smooth cubic X. It is well known that $\pi_1(X) = \pi_1(X \setminus L) = 1$ and $L^2 = -1$. Thus L cannot be the fiber of an admissible map $X \dashrightarrow \mathbb{P}^1$.

Proposition 3.13 (Main theorem, case $\mathbb{Z}_m$, $m > 1$). *Let $D \subset X$ be a curve in a smooth projective surface X. Assume that $\pi_1(X \setminus D) \cong \mathbb{Z}_m$, for $m > 1$. If X is simply connected, then there exists an admissible map $F\colon X \dashrightarrow \mathbb{P}^1$ as in Theorem 1.3 for $S_{(n+1,\bar{m})} = \mathbb{P}^1_{(1,(m))}$ satisfying conditions (i)–(iii). Moreover, $D = D_f$ is a fiber-type curve.*

Proof. Let $D = \bigcup_{i=0}^r D_i$ be the decomposition of D into irreducible components. Any meridian γ_i around an irreducible component D_i of D has finite order e_i dividing m. Consider the divisor $E = \sum \frac{m}{e_i} D_i$. Since $\pi_1(X)$ is the result of factoring out $\pi_1(X \setminus D)$ by the normal closure of the meridians around all the D_i's, and X is simply connected, we note that E is not a positive multiple of an effective divisor. By Theorem 1.3.8 in [28], the unbranched universal covering $\tilde{Y} \to X \setminus D$ associated with G extends to a branched covering $Y \to X$. This implies there exists an effective divisor H in X such that $E \sim mH$. The linear equivalence provides a morphism $F\colon X \dashrightarrow \mathbb{P}^1$ such that $F_|\colon X \setminus D \to \mathbb{C}$ (so in particular, D is a fiber-type curve). After applying Stein factorization, we may assume that $F\colon X \dashrightarrow \mathbb{P}^1$ is the composition of $\tilde{F}\colon X \dashrightarrow \mathbb{P}^1$ and $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$, where $\tilde{F}$ has connected fibers and where β is generically $k:1$. In principle, D is a union of $n+1$ fibers of $\widetilde{F}$ above points in $\mathbb{P}^1$, although it must be only one fiber, or else $\widetilde{F}_*: \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(n+1,\bar{m})})$ would not be surjective, which would contradict Remark 2.13. Now, we have that $F^*([1:0]) = E$ and that $\beta^{-1}([1:0])$ is just a point, so E must be k times an effective divisor with support D, and hence $k = 1$. Thus, $F: X \dashrightarrow \mathbb{P}^1$ has connected fibers. We know that $F^*([0:1]) = mH$, so F induces a surjective morphism $\pi_1(X \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(1,(m))})$. Since $\mathbb{Z}_m$ is finite, this morphism is an isomorphism, and $\mathbb{P}^1_{(1,(m))}$ is the maximal orbifold structure with respect to $F_|: X \setminus D \to \mathbb{C}$. ■

Remark 3.14. To clarify the simply-connected hypothesis given in Proposition 3.13, we will exhibit an example where the result does not follow when $\pi_1(X) \neq 1$. Consider a surface $X \subset \mathbb{P}^n$ with finite cyclic fundamental group $\mathbb{Z}_m$ and containing a line $L \subset X$. Take a hyperplane $H \subset \mathbb{P}^n$ intersecting X transversally such that $L \not\subset H$. Note that $D = H \cdot X$ defines a reduced irreducible divisor in X and $L \cap D = L \cap H = \{P\}$. Hence, one can define a meridian γ of D around P such that $\gamma \subset L$. Since L is a rational curve, γ is trivial in X. If there was a map $F: X \dashrightarrow S$ such that D is a fiber of F, then $F_*: \pi_1(X \setminus D) \to \pi_1^{\text{orb}}(S \setminus \{p\})$ would be surjective. Since $\pi_1(X \setminus D) = \mathbb{Z}_m$, this implies $g_S = 0$ and the orbifold structure of S contains exactly one orbifold point of order $m > 1$. However, in that case $F_*(\gamma)$ would have order m, which is a contradiction since we have proved that γ is trivial.

3.3. Main theorem for curves in $\mathbb{P}^2$

We will pay a special attention to the case $X = \mathbb{P}^2$, in Theorem 1.3. Recall that every curve in $\mathbb{P}^2$ satisfies Condition 2.6 and is ample. Hence, the extra hypotheses needed in the relevant results of Sections 3.1 and 3.2 are always satisfied for curves in $\mathbb{P}^2$, and thus, the following stronger version of the Main Theorem 1.3 holds.

Corollary 3.15 (Main theorem, curves in $\mathbb{P}^2$). *Let D be a curve in $\mathbb{P}^2$. Suppose that $\pi_1(\mathbb{P}^2 \setminus D)$ is a free product of cyclic groups. Then, there exist $r \geq 0$ and $p \geq q \geq 1$ with* $\gcd(p,q) = 1$ *such that $\pi_1(\mathbb{P}^2 \setminus D) \cong \mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q$, and an admissible map $F: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that*:

(i) *F induces an orbifold morphism*

$$F_| : \mathbb{P}^2 \setminus D \to \mathbb{P}^1_{(r+1,\bar{m})},$$

where $\mathbb{P}^1_{(r+1,\bar{m})}$ is maximal with respect to $F_|$, and $\bar{m} = (p,q)$.

(ii) *$F_*: \pi_1(\mathbb{P}^2 \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(r+1,\bar{m})})$ is an isomorphism.*

(iii) *$D = \overline{F^{-1}(\Sigma_0)}$ is a fiber-type curve which is the union of $r+1$ irreducible fibers of F.*

Moreover, after possibly a change of coordinates in $\mathbb{P}^1$, one has $F = [f_{d_2}^{d_1} : f_{d_1}^{d_2}]$, where $d_1 \geq d_2 \geq 1$ satisfy that $\gcd(d_1, d_2) = 1$, *f_{d_i} is a homogeneous polynomial of degree d_i for $i = 1, 2$ which is not a k-th power of another polynomial for any $k \geq 2$, f_{d_1} and f_{d_2} do not have any components in common, and $M_F \subset \{[0:1], [1:0]\}$. More concretely,*

(1) *if $p > q > 1$, then $d_1 = p$, $d_2 = q$, and the pencil $F = [f_q^p : f_p^q]$ has exactly two multiple fibers corresponding to $[0:1], [1:0] \notin \Sigma_0$.*

(2) *If* $p > q = 1$, *then* $d_1 = p$, $[1:0] \in \Sigma_0$, *and the pencil* $F = [f_{d_2}^p : f_p^{d_2}]$ *has at least one multiple fiber corresponding to* $[0:1] \notin \Sigma_0$.

(3) *If* $q = p = 1$, *then* $F = [f_{d_2}^{d_1} : f_{d_1}^{d_2}]$ *has at most two multiple fibers corresponding to* $[0:1], [1:0] \in \Sigma_0$.

Proof. The existence of an admissible map $F: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that induces an isomorphism $F_*: \pi_1(\mathbb{P}^2 \setminus D) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(r+1,\bar{m})})$ follows from Theorem 1.3 and Remark 2.1, as well as from Propositions 3.11 and 3.13. By Remark 3.5, $\pi_1(X \setminus D) \cong \mathbb{F}_r * \mathbb{Z}_p * \mathbb{Z}_q$. This concludes the proof of parts (i) and (ii). Part (iii) follows from Corollary 3.7.

By Remark 2.2, after possibly a change of coordinates in $\mathbb{P}^1$, we may assume $M_F \subset \{[0:1],[1:0]\}$. F is of the form $F = [f_{k_2}^{d_1} : f_{k_1}^{d_2}]$ with where f_{k_i} is a homogeneous polynomial of degree k_i such that $d_1 k_2 = d_2 k_1$, f_{k_1} and f_{k_2} have no common factors, and f_{k_i} is not a l-th power of another polynomial for any $l \geq 2$, for $i = 1, 2$. The condition $\gcd(d_1, d_2) = 1$ is necessary for the fibers of F to be connected. In particular, there exists $l \in \mathbb{Z}_{>0}$ such that $k_i = d_i l$ for $i = 1, 2$. The rest of the proof is straightforward using Remark 2.9, including the fact that $l = 1$ (and hence $d_i = k_i$ for $i = 1, 2$) in each case (1)–(3). ∎

Remark 3.16. In a series of papers, Eyral–Oka calculated the fundamental group of the affine complement of a curve of type $g(x) = f(y)$ under certain conditions, as well as the fundamental group of the projective complement of the corresponding projectivized curve (see [33] for the case where the curve of that type is generic, see [19] for the semi-generic case, and see [20] for curves satisfying a certain arithmetic condition).

Under their conditions, the fundamental group of the complement in $\mathbb{P}^2$ of the projectivization of such a curve is a free product of cyclic groups if $d = \deg(f) = \deg(g)$. Note that, in this case, the projectivized polynomials $\bar{g}(x,z)$ and $\bar{f}(y,z)$ are of the form f_q^{rp} and f_p^{rq} for some p and q coprime and $r \geq 1$, where f_p and f_q are, respectively, homogeneous polynomials of degree p and q which are not a k-th power of any other polynomial for $k \geq 2$ (in fact, f_p and f_q are products of powers of degree 1 polynomials). Hence, in that case, the projective curve $\bar{g}(x,z) = \bar{f}(y,z)$ is a union of r fibers of an admissible map $F = [f_p^q : f_q^p] : \mathbb{P}^2 \setminus \mathcal{B} \to \mathbb{P}^1$. Moreover, the fundamental group of its complement in $\mathbb{P}^2$ is $\mathbb{Z}_p * \mathbb{Z}_q * \mathbb{F}_{r-1}$. This agrees with Corollary 3.15.

4. Addition-deletion lemmas

In Section 3, we have seen geometric conditions for a quasi-projective surface $X \setminus D$ to have a free product of cyclic groups as its fundamental group. Under the hypotheses and notation of Theorem 1.3, $D = D_f \cup D_t$. Hence, if we let $U = X \setminus (D_t \cup \mathcal{B})$, then $X \setminus D = U \setminus D_f$ is the complement of a fiber-type curve inside a smooth quasi-projective surface given by an admissible map $F: U \to S$ onto a smooth projective curve S.

The purpose of this section is to prove addition-deletion results of fibers for complements of fiber-type curves inside smooth quasi-projective surfaces U whose fundamental group is isomorphic by the map induced by F to an open orbifold fundamental group of S. Before we do that, we need some technical results regarding presentations of fundamental groups of complements of fiber-type curves.

4.1. Preparation lemmas

The results in this section give presentations of fundamental groups of fiber-type curve complements, but they do not make use of any Zariski–Van Kampen type computations. Instead, they follow from group-theoretical arguments and from Lemma 2.18, Corollaries 2.21 and 2.22, and the results in Section 2.2.

Assume $F: U \to S$ is an admissible map and consider $M_F \subset S$ (respectively, B_F) the set of multiple (respectively, atypical) values of F (see Section 2.1). In Lemma 2.18, we gave a presentation of $\pi_1(U_B)$ in the case where $B \supset B_F$. We now give a more explicit presentation for $\pi_1(U_B)$ in the general case when B does not necessarily contain B_F.

Throughout this section, $F: U \to S$ will be an admissible map from a smooth quasi-projective surface U to a smooth projective curve S. We will use the notation introduced in Section 2.5.

Lemma 4.1. *Assume that $F: U \to S$ is an admissible map. Let $B \subset S$, with $\#B \geq 1$. Consider Γ_F (respectively, $\Gamma_S = \Gamma_S(B \cup B_F)$) an adapted geometric set of fiber (respectively, base) loops with respect to F and B (respectively, $B \cup B_F$). Let $B' = M_F \setminus B$, and let $\Gamma'_S = \Gamma_S \setminus \{\gamma_k \mid P_k \in B_F \setminus (B \cup B')\}$.*

Then, Γ'_S is an adapted geometric set of base loops with respect to F and $B \cup B'$, and $\pi_1(U_B)$ has a finite presentation with generators $\Gamma_F \cup \Gamma'_S$ and relations of the form

- (R1) $[\gamma_k, w] = 1$, *for any $w \in \Gamma_F$ and any $\gamma_k \in \Gamma'_S$ a lift of a meridian of $P_k \in B \setminus B_F$,*
- (R2) $[\gamma_k, w] = z_{k,w}$, *for the remaining $\gamma_k \in \Gamma'_S$, for any $w \in \Gamma_F$, where $z_{k,w}$ is a word in Γ_F (that depends both on γ_k and w),*
- (R3) $y = 1$ *for a finite number of words y in Γ_F,*
- (R4) $z_k = \gamma_k^{m_k}$, *for any $\gamma_k \in \Gamma'_S$ a lift of a meridian of $P_k \in B'$, where m_k is the multiplicity of the fiber $F^*(P_k)$, and z_k is a word in Γ_F,*

Proof. Note that $B \cup B_F$ satisfies the conditions of Lemma 2.18, and hence $\pi_1(U_{B\cup B_F})$ admits a presentation generated by $\Gamma_F \cup \Gamma_S$, where Γ_F (respectively, Γ_S) is an adapted geometric set of fiber (respectively, base) loops with respect to F and $B \cup B_F$. Note that Γ_F is an adapted geometric set of fiber loops with respect to F and $B \cup B_F$ if and only if the same holds with respect to B.

By Corollary 2.21, a presentation of $\pi_1(U_B)$ can be obtained by factoring $\pi_1(U_{B\cup B_F})$ by the normal closure of all the meridians about irreducible components of fibers above points in $B_F \setminus B$. We abuse notation and see both the elements of Γ_F and Γ_S as elements of $\pi_1(U_B)$, $\pi_1(U_{B\cup B'})$ or $\pi_1(U_{B\cup B_F})$ when no ambiguity seems likely to arise.

Let γ_k be such that $F_*(\gamma_k)$ is a meridian around a point $P_k \in B_F \setminus (B \cup B')$. Since $F^*(P_k)$ is not a multiple fiber, γ_k is trivial in $\pi_1(U_B)$ by condition (4) in Definition 2.16. In particular, this proves that $\pi_1(U_B)$ can be generated by $\Gamma_F \cup \Gamma'_S$. Since Γ_S is an adapted geometric set of base loops with respect to F and $B \cup B_F$, then Γ'_S is an adapted geometric set of base loops with respect to F and $B \cup B'$. As in the proof of Lemma 2.17, there exists a meridian in $\pi_1(U_{B\cup B_F})$ about any given irreducible component of C_{P_k} of the form $w\gamma_k^m$ for some $m \geq 1$, and for $w \in \ker(F_{|U_{B\cup B_F}})_*$, which is a word in Γ_F by Corollary 2.22. Hence, we have shown that the normal closure of the subgroup generated by the meridians about each irreducible component of C_{P_k} is the normal closure of a subgroup of $\pi_1(U_{B\cup B_F})$ generated by γ_k and a finite number of words in Γ_F. This gives rise to relations of the form (R3).

Let γ_k be such that $F_*(\gamma_k)$ is a meridian around a point in B'. Similarly as in the previous paragraph, we can choose a meridian in $\pi_1(U_{B\cup B_F})$ about any of the irreducible components of C_{P_k} which is of the form $w(\gamma_k^{m_k})^m$ for some $m \geq 1$, and w a word in Γ_F. Let N_k be the normal closure of the subgroup generated by such choice of meridians about each of the irreducible components of C_{P_k}. Note that, by definition of $\gamma_k \in \pi_1(U_{B\cup B_F})$ and Corollary 2.22, the equality $\gamma_k^{m_k} = z_k$ holds in $\pi_1(U_B)$ for some word z_k in Γ_F (an element of $\ker F_*$). Hence, the relations given by $w(\gamma_k^{m_k})^m = 1$ coming from the chosen generators of N_k (as a normal closure) together with the relation $\gamma_k^{m_k} = z_k$ (which we know holds in $\pi_1(U_B)$) are equivalent to a finite number of relations of the form $y = 1$ for words y in the letters of Γ_F (type (R3)) and the relation $\gamma_k^{m_k} = z_k$ (all the relations of type (R4)).

Let $P_k \in B \setminus B_F$, hence $F_*(\gamma_k) \in \pi_1(\mathbb{P}^1 \setminus (B \cup B_F))$ induces the trivial monodromy morphism in the elements of $\pi_1(F^{-1}(P))$, since $P_k \notin B_F$. In other words, $\gamma_k \in \pi_1(U_B)$ commutes with any word in Γ_F.

The result now follows from adding the relations found in the previous paragraphs to the presentation of $\pi_1(U_{B\cup B_F})$ given in Lemma 2.18: the relation $\prod_j x_j = \prod_i [a_i, b_i]$ is of type (R3), the monodromy relations of $\pi_1(U_{B\cup B_F})$ corresponding to P_k are the ones of type (R2) if $P_k \in (B_F \cap B) \cup B'$), of type (R1) if $P_k \in B \setminus B_F$ (by the previous paragraph), or become of type (R3) after using that γ_k is trivial in $\pi_1(U_B)$ if $P_k \in B_F \setminus (B \cup B')$. ∎

Our next goal is to describe cases in which $\pi_1(U_B)$ has a presentation on generators Γ'_S. This will provide candidates for $B \subset S$ such that $F_*\colon \pi_1(U_B) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$ is an isomorphism (and, in particular, such that $\pi_1(U_B)$ is a free product of cyclic groups). This goal is achieved in Proposition 4.3, with the help of Lemma 4.2. We follow notation from Section 2.5 and Lemma 4.1.

Lemma 4.2. *Let $F\colon U \to S$ be an admissible map, $B = \{P_0, \dots, P_n\} \subset S$ a non-empty set, and assume $Q \in B \setminus (B_F \cap B)$. Let $S_{(n+1,\bar{m})}$ (respectively, $S_{(n,\bar{m})}$) be the maximal orbifold structure of S with respect to $F_|\colon U_B \to S \setminus B$ (respectively, with respect to $F_|\colon U_{B\setminus\{Q\}} \to S \setminus (B \setminus \{Q\})$). Consider $K := \ker F_*$, the kernel of $F_*\colon \pi_1(U_B) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$.*

Moreover, assume that $F_*\colon \pi_1(U_{B\setminus\{Q\}}) \to \pi_1^{\text{orb}}(S_{(n,\bar{m})})$ is an isomorphism, and furthermore, either

- *$n \geq 1$, or*
- *$n = 0$ and $B' \neq \emptyset$, where $B' := M_F \setminus B$.*

Then K is an abelian group. Furthermore, an adapted geometric set of base loops $\Gamma'_S = \Gamma_S(B \cup B')$ with respect to F and $B \cup B'$ can be chosen so that $K = N$, where N is the normal closure of the subgroup $\langle \gamma_k^{m_k} \mid P_k \in B' \rangle$.

Proof. We use the presentation of $\pi_1(U_B)$ given in Lemma 4.1 and the notation therein. Since $\gamma_k^{m_k} \in K$ for all $P_k \in B'$ and K is normal, $N \leq K$ for every choice of Γ'_S as in Lemma 4.1.

Assume now that F_* is an isomorphism and either $n \geq 1$ or $n = 0$ and $B' \neq \emptyset$. We will show that K is abelian and $K \leq N$ for some choice of Γ'_S. Let $\gamma \in \pi_1(U_B)$ be any positively oriented meridian about the irreducible fiber C_Q, such that $F_*(\gamma)$ is a positively oriented meridian around Q in $S \setminus (B \cup B_F)$. Note the following:

(1) Since C_Q is irreducible, $\mathrm{NCl}(\gamma)$ is independent of the choice of the meridian γ by Lemma 2.4.

(2) $K \trianglelefteq \mathrm{NCl}(\gamma) \trianglelefteq \pi_1(U_B)$ is a subgroup of the normal closure of $\langle\gamma\rangle$ in $\pi_1(U_B)$. To see this, consider $w \in K$. The projection of w in $\pi_1(U_{B\setminus\{Q\}})$ is trivial by Corollary 2.22, so w is an element of $\mathrm{NCl}(\gamma)$.

(3) $K \trianglelefteq \pi_1(U_B)$ is abelian. To see this, note that any meridian around the typical fiber C_Q commutes with $K = \iota_*(F^{-1}(P))$ by Lemma 4.1(R1). By Lemma 2.5, this means $\mathrm{NCl}(\gamma)$ is contained in the centralizer of K. By (2), $K \leq \mathrm{NCl}(\gamma)$, and in particular, K is an abelian subgroup.

(4) There exists an adapted geometric set of base loops Γ'_S with respect to F and $B \cup B'$ such that one such positively oriented meridian γ can be written as a word in Γ'_S.

If $n \geq 1$, we may assume $Q = P_1$ and $\gamma = \gamma_1$ in Γ'_S, which concludes (4) in the case $n \geq 1$.

Suppose that $n = 0$ and $b' = \#B' \geq 1$. In this case, $r = 2g_S$. Let $\Gamma'_S = \Gamma_S(B \cup B')$ be as in Lemma 4.1. By definition of Γ'_S, if

$$\tilde{\gamma} = \prod_{i=1}^{g_S} [\gamma_{b'+2i-1}, \gamma_{b'+2i}] \cdot \Big(\prod_{P_k \in B'} \gamma_k \Big)^{-1}$$

(see condition (3) in Definition 2.16), then one has that $F_*(\tilde{\gamma})$ is a meridian around Q in $\pi_1(S \setminus (B \cup B'))$. By Corollary 2.22, there exists a word z in the letters Γ_F such that $\gamma = \tilde{\gamma} z$ is a meridian around C_Q whose image by F_* is a meridian around Q. After replacing γ_1 by $z^{-1}\gamma_1$ in Γ'_S, one can assume $\gamma = \tilde{\gamma}$. This concludes (4) in the case $n = 0$ and $b' \geq 1$.

Finally, let us show $K \leq N$, where N is defined using the Γ'_S found in observation (4) above. By (4), $\gamma = w(\gamma_1, \ldots, \gamma_{r+b'})$ is a word in Γ'_S. Let $\varphi\colon \mathbb{F}_{r+b'} = \langle \delta_1, \ldots, \delta_{r+b'} \rangle \to \pi_1(U_B)$ be the group homomorphism given by $\delta_i \mapsto \gamma_i$ for all $i \in \{1, \ldots, r + b'\}$. Let α be an element in $\mathrm{NCl}(\gamma)$ and write α as a product $\prod_{i=1}^{u} g_i^{-1} \gamma^{d_i} g_i$, where g_i is an element of $\pi_1(U_B)$. By Lemma 2.18 applied to $\pi_1(U_{B \cup B_F})$, g_i in $\pi_1(U_B)$ can be written as $g_i = w_i h_i$, where w_i is a word in Γ_F (so $w_i \in K$) and h_i is a word in Γ'_S. Since γ commutes with the elements of $K \leq \pi_1(U_B)$, α can be written as the product $\prod_{i=1}^{u} h_i^{-1} \gamma^{d_i} h_i$. In other words, we have shown that $\mathrm{NCl}(\gamma) = \varphi(\mathrm{NCl}(a))$, where $\mathrm{NCl}(a)$ is the normal closure in $\mathbb{F}_{r+b'}$ of the subgroup generated by a, where $a = w(\delta_1, \ldots, \delta_{r+b'})$ (the word w for γ in the new letters $\delta_1, \ldots, \delta_{r+b'}$). In particular, one has $K \leq \varphi(\mathrm{NCl}(a))$.

A similar argument, this time using that K is abelian, shows that

$$N = \varphi(\mathrm{NCl}(\langle \delta_k^{m_k} \mid P_k \in B' \rangle)). \tag{4.1}$$

Now, the composition $F_* \circ \varphi$ induces an isomorphism in the quotients given by the composition of

$$\mathbb{F}_r * \Big(\mathop{*}_{P_k \in B'} \mathbb{Z}_{m_k} \Big) \cong \mathbb{F}_{r+b'} / \mathrm{NCl}\left(\langle \delta_j^{m_k} \mid P_k \in B' \rangle\right) \to \pi_1(U_B)/N$$

with

$$\pi_1(U_B)/N \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})}) \cong \mathbb{F}_r * \left(\mathop{*}_{P_k \in B'} \mathbb{Z}_{m_k} \right).$$

This implies that $\ker(F_*) \cap \operatorname{Im}(\varphi)/N$ is trivial. Recall that $K = \ker(F_*)$ and that, $K \leq \varphi(\operatorname{NCl}(a))$. Hence $K = \ker(F_*) \cap \operatorname{Im}(\varphi)$ and thus we arrive at the equality $K = N$. This concludes the proof. ∎

Proposition 4.3. *Under the same notation as that of Lemma 4.1 and the same hypotheses and choice of Γ'_S as in Lemma 4.2, every element of K can be written as a word in the letters Γ'_S. In particular, the presentation of $\pi_1(U_B)$ from Lemma 4.1 can be transformed to a presentation on generators Γ'_S.*

Proof. Let us consider the presentation of $\pi_1(U_B)$ given in Lemma 4.1. The generators of $\pi_1(U_B)$ in Γ_F are elements of K, which equals N by Lemma 4.2. Equation (4.1) in the proof of Lemma 4.2 implies that the elements of N are products of elements of the form $v\gamma_k^{m_k}v^{-1}$, where v is a word in Γ'_S, and $P_k \in B'$. Using this, one can eliminate the generators of $\pi_1(U_B)$ in Γ_F in the presentation given in Lemma 4.1. ∎

Remark 4.4. Suppose that $n \geq 1$, and assume the hypotheses, choice of Γ'_S and notation of Lemma 4.2. In the proof of Lemma 4.2, Γ'_S was chosen so that $\gamma = \gamma_1 \in \Gamma'_S$. Since $\operatorname{NCl}(\gamma)$ is contained in the centralizer of $N = K$ by (3) in the proof of Lemma 4.2, we can assume that the elements of N are products of elements of the form $v\gamma_k^{m_k}v^{-1}$, $P_k \in B'$, where v is a word in $\Gamma'_S \setminus \{\gamma_1\}$, i.e., the letter $\gamma_1 = \gamma$ does not appear in v.

Remark 4.5. Under the hypotheses, choice of Γ'_S and notation of Lemma 4.2, suppose moreover that U is simply connected, $B = \{Q\}$ ($Q \notin B_F$), and $M_F \neq \emptyset$. Note that in this case $r = n = 0$ (Remark 2.1) and $B' = M_F$. In this setting, Corollary 2.21 implies $\operatorname{NCl}(\gamma) = \pi_1(U_B)$. By (3) in the proof of Lemma 4.2, one has $K = N$ is contained in the center of $\pi_1(U_B)$. In particular, any subgroup generated by elements in K is normal, and thus

$$N = \langle \gamma_k^{m_k} \mid P_k \in M_F \rangle.$$

The following two corollaries pertain to the case $n \geq 1$ (Corollary 4.6) and $n = 0$, $B' = M_F \setminus B \neq \emptyset$ (Corollary 4.7) in Lemma 4.2, respectively, and give useful presentations of $\pi_1(U_B)$. Note that in Corollary 4.7, $S = \mathbb{P}^1$ and U are both assumed to be simply connected.

Corollary 4.6. *Assume $F\colon U \to S$ is an admissible map. Let $B = \{P_0, P_1, \ldots, P_n\} \subset S$ be such that $\#B = n + 1 \geq 2$. Let $S_{(n+1,\bar{m})}$ (respectively, $S_{(n,\bar{m})}$) be the maximal orbifold structure of S with respect to $F_|\colon U_B \to S \setminus B$ (respectively, $F_|\colon U_{B\setminus\{P_1\}} \to S \setminus (B \setminus \{P_1\})$).*

Suppose that $F_*\colon \pi_1(U_{B\setminus\{P_1\}}) \to \pi_1^{\text{orb}}(S_{(n,\bar{m})})$ is an isomorphism, and that $P_1 \notin B_F$. Let $B' = M_F \setminus B$, and let $\Gamma'_S = \Gamma'_S(B \cup B')$ be an adapted geometric set of base loops with respect to F and $B \cup B'$, as in Remark 4.4. Then $\pi_1(U_B)$ has a finite presentation

$$\pi_1(U_B) = \langle\, \Gamma'_S : \{R_j\}_{j\in J}, \{\widetilde{R}_i\}_{i\in I} \rangle,$$

where R_j is a word in $\Gamma'_S \setminus \{\gamma_1\}$ for all $j \in J$ and $\widetilde{R}_i = [\gamma_1, w_i]$ for all $i \in I$, where w_i is a word in $\Gamma'_S \setminus \{\gamma_1\}$.

Proof. Consider the presentation of $\pi_1(U_B)$ explained in the proof of Proposition 4.3 on generators Γ'_S, which arises from the presentation in Lemma 4.1. Using Remark 4.4, we see that all of the relations appearing in our presentation are either of type $\widetilde{R}_i$ (relations (R1) in Lemma 4.1, for $k = 1$) or of type R_j (rest of the relations in Lemma 4.1). ■

Corollary 4.7. *Assume $F: U \to \mathbb{P}^1$ is an admissible map, where U is a simply-connected quasi-projective surface. Let $Q \in \mathbb{P}^1 \setminus B_F$, and let $\mathbb{P}^1_{(1,\bar{m})}$ be the maximal orbifold structure of $\mathbb{P}^1$ with respect to $F_|: U_{\{Q\}} \to \mathbb{P}^1 \setminus \{Q\}$. Assume that $M_F \neq \emptyset$.*

Let Γ'_S be an adapted geometric set of base loops with respect to F and let $\{Q = P_0\} \cup M_F$ be given by Lemma 4.2. Then, $\pi_1(U_{\{Q\}})$ has a finite presentation

$$\pi_1(U_{\{Q\}}) = \langle\, \Gamma'_S : \{R_j\}_{j \in J}, \{[\gamma_k, \gamma_i^{m_i}]\}_{P_k, P_i \in M_F} \rangle,$$

where m_k is the multiplicity of the fiber $F^*(P_k)$ and $R_j = \prod_{P_k \in M_F} (\gamma_k^{m_k})^{n_{kj}}$ for some $n_{kj} \in \mathbb{Z}$, $j \in J$.

Remark 4.8. By Remark 2.2, note that $\#M_F = 1$ or 2 in Corollary 4.7.

Proof of Corollary 4.7. Since U is simply connected and F is admissible, $\pi_1^{\text{orb}}(\mathbb{P}^1_{(0,\bar{m})})$ must be trivial by Remark 2.13. Hence, $F_*: \pi_1(U) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(0,\bar{m})})$ is trivially an isomorphism, and the hypotheses of Lemma 4.2 are satisfied.

By Remark 4.5, the elements of $K = N$ (an abelian group) are all of the form

$$\prod_{P_k \in M_F} (\gamma_k^{m_k})^{n_k}, \quad \text{where } n_k \in \mathbb{Z}.$$

Since Remark 4.5 says that N is in the center of $\pi_1(U_{\{Q\}})$, we can add the relations $[\gamma_k, \gamma_i^{m_i}]$ for all $P_i, P_k \in M_F$ to the presentation of $\pi_1(U_{\{Q\}})$ of Proposition 4.3 without changing the group. After that, note that we can transform the relations already appearing in the presentation of Proposition 4.3 (coming from (R2)–(R4) in Lemma 4.1) to elements of $K = N$, and hence, as relations of type R_j. ■

4.2. Deletion lemma

Theorem 4.9 (Deletion lemma). *Let U be a smooth quasi-projective surface and let $F: U \to S$ be an admissible map to a smooth projective curve S. Assume $B \subset S$ is such that $\#B = n \geq 1$ and $r := 2g_S + n$. Consider $P \in S \setminus B$. Let $S_{(n+1,\bar{m})}$ (respectively, $S_{(n,\bar{m}')}$) be the maximal orbifold structure of S with respect to $F_|: U_{B \cup \{P\}} \to S \setminus (B \cup \{P\})$ (respectively, $F_|: U_B \to S \setminus B$).*

If $F_*: \pi_1(U_{B \cup \{P\}}) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$ is an isomorphism, then

$$F_* : \pi_1(U_B) \to \pi_1^{\text{orb}}(S_{(n,\bar{m}')})$$

is an isomorphism.

Moreover, if $\pi_1(U_{B \cup \{P\}}) \cong \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$, and $p \geq 1$ denotes the multiplicity of $F^*(P)$, then

$$\pi_1(U_B) \cong \mathbb{F}_{r-1} * \mathbb{Z}_p * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}.$$

Proof. By hypothesis, $\pi_1(U_{B\cup\{P\}}) \cong \pi_1(S_{(n+1,\bar{m})}) \cong \mathbb{F}_r * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_s}$ for integers $m_i \geq 2$, $i \in I = \{1,\ldots,s\}$. Also, by Corollary 2.22, the inclusion of the generic fiber (over a point $Q \in S \setminus (B \cup \{P\} \cup B_F)$) induces the trivial morphism $\pi_1(F^{-1}(Q)) \to \pi_1(U_{B\cup\{P\}})$. Since $U_{B\cup\{P\}} \subset U_B$, the inclusion of the generic fiber also induces the trivial morphism $\pi_1(F^{-1}(Q)) \to \pi_1(U_B)$. By Corollary 2.22,

$$F_* : \pi_1(U_B) \to \pi_1^{\text{orb}}(S_{(n,\bar{m}')})$$

is an isomorphism since $n > 0$.

For the *moreover* part, note that if $\bar{m} = (m_1,\ldots,m_s)$, then

$$\bar{m}' := \begin{cases} \bar{m} & \text{if } p = 1, \\ (p, m_1, \ldots, m_s) & \text{if } p > 1. \end{cases}$$ ∎

Remark 4.10. The deletion lemma also holds in the case $n = 0$ in the following way: if $F_*\colon \pi_1(U_{\{P\}}) \to \pi_1^{\text{orb}}(S_{(1,\bar{m})})$ is an isomorphism, then $F_*\colon \pi_1(U) \to \pi_1^{\text{orb}}(S_{(0,\bar{m}')})$ is also an isomorphism. A more subtle proof can be given using the presentation of $U_{\{P\}}$ from Lemma 4.1, taking into account that the elements of Γ_F are trivial in $\pi_1(U_{\{P\}})$. Since the result is not needed for the purpose of this paper, we omit it.

4.3. Proof of the generic addition-deletion lemma

Proof of Theorem 1.4. The 'if' as well as the 'moreover' parts of the statement are a particular case of the deletion Lemma 4.9.

Let us show the 'only if' part. Let $P_1 = P$, $B \cup \{P\} = \{P_0,\ldots,P_n\}$. By Corollary 4.6 (and using the notation therein) applied to B and $B \cup \{P\}$, we have that $\pi_1(U_{B\cup\{P\}})$ has a presentation of the form

$$\pi_1(U_{B\cup\{P\}}) = \langle\, \Gamma'_S : \{R_j\}_{j\in J}, \{\widetilde{R}_i\}_{i\in I} \rangle,$$

where R_j are words in $\Gamma'_S \setminus \{\gamma_1\}$, and $\widetilde{R}_i = [\gamma_1, w_i]$, where w_i is a word in $\Gamma'_S \setminus \{\gamma_1\}$.

Let $H = \langle\, \Gamma'_S \setminus \{\gamma_1\} : \{R_j\}_{j\in J} \rangle$, and let $\varphi : \mathbb{F}_{r+b'} = \langle \delta_1,\ldots,\delta_{r+b'} \rangle \to \pi_1(U_B)$ be the epimorphism sending δ_i to γ_i for all $i = 1,\ldots,r+b'$. We have that φ factors through $\widetilde{\varphi}\colon \mathbb{Z} * H \to \pi_1(U_{B\cup\{P\}})$, where the $\mathbb{Z}$ free factor is generated by the letter γ_1. In particular, $\widetilde{\varphi}$ is an epimorphism.

Moreover, according to the presentation of $\pi_1(U_{B\cup\{P\}})$ above,

$$\pi_1(U_B) \cong \pi_1(U_{B\cup\{P\}})/\operatorname{NCl}(\gamma_1) \cong H.$$

Hence, we have found an epimorphism

$$\widetilde{\varphi} : \mathbb{Z} * \pi_1(U_B) \to \pi_1(U_{B\cup\{P\}}).$$

Let $F_*\colon \pi_1(U_{B\cup\{P\}}) \to \pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$. Since $\widetilde{\varphi}$ is an epimorphism and F_* is also an epimorphism by Remark 2.13, $F_* \circ \widetilde{\varphi}$ is an epimorphism from $\mathbb{Z} * \pi_1(U_B)$ to $\pi_1^{\text{orb}}(S_{(n+1,\bar{m})})$, which is a free product of cyclic groups isomorphic to $\mathbb{Z} * \pi_1^{\text{orb}}(S_{(n,\bar{m})}) \cong \mathbb{Z} * \pi_1(U_B)$. Hence, $F_* \circ \widetilde{\varphi}$ is an isomorphism by Lemma 2.15. In particular, $\widetilde{\varphi}$ and F_* are both isomorphisms, and

$$\pi_1(U_{B\cup\{P\}}) \cong \mathbb{Z} * \pi_1(U_B).$$ ∎

Corollary 4.11. *The fundamental group of the complement of r generic fibers of a primitive polynomial map $f\colon \mathbb{C}^2 \to \mathbb{C}$ is free of order r.*

Proof. Consider $F\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $F(x,y,z) = [\overline{f}(x,y,z) : z^d]$, where $\overline{f}(x,y,z)$ is the homogenization of $f(x,y)$ and $d = \deg(f)$. Since f is primitive (it has connected generic fibers), F is an admissible map. Let $\mathcal{B}$ be the base points of F. Consider $U = \mathbb{P}^2 \setminus \mathcal{B}$ and $F_|\colon U \to \mathbb{P}^1$. The restriction $F_|\colon \mathbb{C}^2 \to \mathbb{C}$ induces an isomorphism of (trivial) fundamental groups. Note that $F_|\colon \mathbb{C}^2 \to \mathbb{C}$ does not have multiple fibers, or else F_* would factor through a surjection from $\pi_1(\mathbb{C}^2)$ to $\pi_1^{\text{orb}}(\mathbb{C})$ for an orbifold on $\mathbb{C}$, which is non-trivial. Consider r generic fibers of $F_|$. Then the result follows from the generic addition-deletion Lemma 1.4 to $F_|\colon U \to \mathbb{P}^1$ and $B = \{[1:0]\}$. ■

Example 4.12. Other examples of complements of curves with free fundamental groups include the following. Consider a polynomial $f(x,y)$ such that $\pi_1(\mathbb{C}^2 \setminus C_1) = \mathbb{Z}$ for $C_1 = V(f=0) \subset \mathbb{C}^2$ (for instance, if C_1 is irreducible and only has nodal singularities, including at infinity). In this case, C_1 is irreducible and hence the polynomial map $f(x,y)$ induces an admissible map. We have that $f_*\colon \pi_1(\mathbb{C}^2 \setminus C_1) \to \pi_1(\mathbb{C}^*)$ is an epimorphism from $\mathbb{Z}$ to itself, so it is an isomorphism. Note that f does not have multiple fibers, or else f_* would factor through a surjection from $\pi_1(\mathbb{C}^2 \setminus C_1) \cong \mathbb{Z}$ to $\pi_1^{\text{orb}}(\mathbb{C}^*)$ for an orbifold of general type on $\mathbb{C}^*$, which is a non-abelian group. Consider $C_2, \dots, C_r$ generic fibers of f. Then the generic addition-deletion Lemma 1.4 yields $\pi_1(\mathbb{C}^2 \setminus C) = \mathbb{F}_r$ for $C = C_1 \cup \dots \cup C_r$.

Analogously, if f_p (respectively, f_q) is a form of degree p (respectively, q) with $\gcd(p,q) = 1$ and $\pi_1(\mathbb{P}^2 \setminus C_1) = \mathbb{Z}$ for $C_1 = V(f_p) \cup V(f_q) \subset \mathbb{P}^2$. This implies that f_p and f_q are irreducible and in that case $F = [f_p^q : f_q^p]$ is also an admissible map (see, for instance, Lemma 2.6 in [12]). Consider $C_2, \dots, C_r$ generic fibers of $F = [f_p^q : f_q^p]$. Then we are under the hypotheses of the generic addition-deletion Lemma 1.4, and hence $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{F}_r$ for $C = C_1 \cup \dots \cup C_r$.

4.4. A base case for the addition lemma

Recall Notation 2.3. In light of Example 4.12, one might wonder if other pencils $F\colon X = \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ give rise to curves whose fundamental group of their complement is isomorphic to an open orbifold group of $\mathbb{P}^1$ through the morphism induced by F. The following result provides a base case for the addition lemma in the case $M_F \neq \emptyset$.

Theorem 4.13. *Let X be a simply-connected smooth projective surface, let $F\colon X \dashrightarrow \mathbb{P}^1$ be an admissible map, and let $P \in \mathbb{P}^1$ be such that $F^{-1}(P)$ is a typical fiber. Suppose that $M_F \neq \emptyset$.*

Let $\mathbb{P}^1_{(1,\bar{m})}$ be the maximal orbifold structure of $\mathbb{P}^1 \setminus \{P\}$ with respect to $F\colon X_{\{P\}} \to \mathbb{P}^1 \setminus \{P\}$, where $\bar{m} = (p,q)$, $p \geq q \geq 1$, and $\gcd(p,q) = 1$. Then the following statements are equivalent:

(1) $H_1(X_{\{P\}}) = \mathbb{Z}_{pq}$,

(2) $F_*\colon \pi_1(X_{\{P\}}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{(1,\bar{m})}) \cong \mathbb{Z}_p * \mathbb{Z}_q$ *is an isomorphism.*

Proof. After an isomorphism in $\mathbb{P}^1$, we may assume that $M_F \subset \{[0:1],[1:0]\}$ (see Remark 2.2), the fiber above $[0:1]$ has multiplicity p, the fiber above $[1:0]$ has multiplicity q, and $P \notin \{[0:1],[1:0]\}$.

(2) ⇒ (1) is trivial. Let us prove (1) ⇒ (2).

Assume that $H_1(X_{\{P\}}) = \mathbb{Z}_{pq}$. Note that

- $\#M_F = 2$ if and only if $q > 1$, and
- $\#M_F = 1$ if and only if $q = 1$.

Corollary 4.7 applied to $U = \mathbb{P}^2 \setminus \mathcal{B}$ and Remark 4.8 say that $\pi_1(X_{\{P\}})$ has a presentation of the form

$$\pi_1(X_{\{P\}}) = \langle \gamma_1, \gamma_2 \mid \{\gamma_1^{pk_j}\gamma_2^{ql_j}\}_{j\in J}, [\gamma_1, \gamma_2^q], [\gamma_2, \gamma_1^p]\rangle \tag{4.2}$$

where $k_j, l_j \in \mathbb{Z}$ for all $j \in J$. Indeed, this is clear if $\#M_F = b' = 2$ (i.e., $q > 1$), but it is also true if $b' = 1$ (i.e., $q = 1$), picking $k_1 = 0$, $l_1 = 1$. Hence, from now on we assume that $\pi_1(X_{\{P\}})$ has a finite presentation as in equation (4.2).

Let $k = \gcd_{j\in J}(k_j)$, and $l = \gcd_{j\in J}(l_j)$, with the convention that the greatest common divisor of various 0's is 0. Note that this group has a quotient

$$\langle \gamma_1, \gamma_2 \mid \gamma_1^{pk}, \gamma_2^{ql}, [\gamma_1, \gamma_2]\rangle,$$

so the quotient map induces an epimorphism on the abelianization

$$\mathbb{Z}_{pq} \twoheadrightarrow \mathbb{Z}_{pk} \times \mathbb{Z}_{ql}.$$

Hence, $k = l = 1$. Using that γ_1^p and γ_2^q commute, we can modify the presentation of $\pi_1(X_{\{P\}})$ in equation (4.2) to get

$$\langle \gamma_1, \gamma_2 \mid \gamma_1^p\gamma_2^{aq}, \gamma_2^{bq}, [\gamma_1, \gamma_2^q], [\gamma_2, \gamma_1^p]\rangle, \tag{4.3}$$

where $b \geq 1$, $a \in \{0, \dots, b-1\}$ and $\gcd(a, b) = 1$. Thus, a presentation matrix of the abelianization of $\pi_1(X_{\{P\}})$ as a $\mathbb{Z}$-module is given by

$$M = \begin{pmatrix} p & 0 \\ qa & qb \end{pmatrix}.$$

By hypothesis, we know that the matrix M is equivalent (over $\mathbb{Z}$) to the diagonal 2×2 matrix with diagonal (p, q), since both matrices present the same abelian group and have the same dimensions and rank. In particular, both matrices have the same determinant, so $b = 1$, and thus $a = 0$. Plugging that data back in for the presentation in equation (4.3), we get that

$$\pi_1(X_{\{P\}}) = \langle \gamma_1, \gamma_2 \mid \gamma_1^p, \gamma_2^q\rangle,$$

and the epimorphism (recall Remark 2.13)

$$F_* : \pi_1(X_{\{P\}}) \to \pi_1^{\text{orb}}(S_{(1,\bar{m})})$$

is in fact an isomorphism by Lemma 2.15. ∎

5. Applications

5.1. $C_{p,q}$-curves revisited

In this subsection, we prove a generalization of Oka's classical result on $C_{p,q}$ curves.

Proof of Theorem 1.2. If $p = q = 1$, the result is trivial. Suppose that $p \geq 1$ or $q > 1$, which implies that $M_F \neq \emptyset$. Let $P \in \mathbb{P}^1$ be such that $F^{-1}(P)$ is a typical fiber. In particular, $\overline{F^{-1}(P)}$ is given by the zeros of an irreducible degree pq polynomial, and by Remark 2.9, $H_1(X_{\{P\}}) \cong \mathbb{Z}_{pq}$. In other words, (2) in Theorem 4.13 holds, and the result for a finite union of generic fibers is proved for $r = 0$. The claim for $r > 0$ follows from the generic addition-deletion Lemma 1.4 applied to $U = \mathbb{P}^2 \setminus \mathcal{B}$.

Now consider $\mathbb{P}^1_{\bar{m}}$, the maximal orbifold structure of $\mathbb{P}^1 \setminus \{[0:1]\}$ with respect to $F_|: \mathbb{P}^2 \setminus V(f_p) = U_{\{[0:1]\}} \to \mathbb{P}^1 \setminus \{[0:1]\}$, and note that $\pi_1^{\mathrm{orb}}(\mathbb{P}^1_{\bar{m}}) \cong \mathbb{Z}_p$. Assume that $\pi_1(\mathbb{P}^2 \setminus V(f_p)) \cong \mathbb{Z}_p$. In that case, $F_*: \pi_1(\mathbb{P}^2 \setminus V(f_p)) \to \pi_1^{\mathrm{orb}}(\mathbb{P}^1_{\bar{m}})$ is an epimorphism between $\mathbb{Z}_p$ and itself, so it is an isomorphism. By the generic addition-deletion Lemma 1.4, one obtains $\pi_1(\mathbb{P}^2 \setminus (C \cup V(f_p))) = \mathbb{F}_{r+1} * \mathbb{Z}_p$. ■

Remark 5.1. Examples of these families also appear in Exercise (4.21) in Section 4 of Dimca's reference book [17]. Theorem 1.2 can serve as a proof for part (ii) of this exercise.

5.2. Fundamental group of a union of conics

Another instance where our results apply is given in a collection of conics in a pencil. We provide a new proof of Theorems 2.2 and 2.5 in [2] which does not depend on braid monodromy calculations.

Theorem 5.2. *Let $F = [f_2 : f_1^2]$ be a pencil generated by a smooth conic $C_0 = V(f_2)$ and a double line $\ell = V(f_1)$. Consider $C = C_0 \cup \cdots \cup C_r$ a union of $r + 1$ smooth conics of F. Then*

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{F}_r * \mathbb{Z}_2.$$

Proof. This result can be obtained from Theorem 1.2 for $p = 2, q = 1$. ■

5.3. Fundamental group of tame torus-type sextics

In a remarkable paper, Oka–Pho [36] describe the fundamental group of the complement of irreducible sextics in a pencil of type $F = [f_2^3 : f_3^2]$, where f_i is a homogeneous form of degree i, whose set of singular points are base points of the pencil, that is, $\operatorname{Sing} V(f) = V(f_2) \cap V(f_3)$. The term *torus type* refers to the former property, and the term *tame* refers to the latter. According to the authors, any such a sextic C satisfies $\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 * \mathbb{Z}_3$ except for the particular case where C has four singular points: one of type $C_{3,9}$ and three of type A_2. Type $C_{3,9}$ singularities have a local equation $f(x, y) = x^3 + y^9 + x^2y^2 \in \mathbb{C}\{x, y\}$, and A_2 singularities are ordinary cusps of local equation $f(x, y) = x^2 + y^3 \in \mathbb{C}\{x, y\}$.

It is enough to check the result on maximal irreducible tame torus-type sextics, that is, those with maximal total Milnor number (either 19 or 20). According to [36], there are seven types of such irreducible curves, which can be described by the configuration Σ of singularities (see Table 1), since their moduli space is connected for each configuration. Also, by the maximality of the total Milnor number, the multiple fibers $C_2 := V(f_2)$ and $C_3 := V(f_3)$ are uniquely determined by the moduli space of the curve $V(f_2) \cup V(f_3)$, that is, they depend only on the singularities of C_2 and C_3 and the topological type of their intersection. Such moduli spaces are connected in all cases.

	Σ	(μ, r, δ)	$f_2, f_3, C = \{f_2^3 + f_3^2 = 0\}$
(1)	$[C_{3,15}]$	$[(19, 2, 10)]$	$f_2 = yz - x^2$ $f_3 = 40y^3 + 21xyz - 21x^3$
(2)	$[C_{9,9}]$	$[(19, 2, 10)]$	$f_2 = y^2 - x^2$ $f_3 = 2y^2z - 2x^2z + \frac{32}{27}x^3$
(3)	$[C_{3,7}, A_8]$	$[(11, 2, 6), (8, 1, 4)]$	$f_2 = yz - x^2$ $f_3 = \frac{23}{27}x^3 - \frac{4}{9}x^2y + xyz + \frac{4}{9}xy^2 - \frac{4}{27}y^3$
(4)	$[Sp_1, A_2]$	$[(18, 1, 9), (2, 1, 1)]$	$f_2 = xy$ $f_3 = y^2z - y^3 - x^3$
(5)	$[B_{3,10}, A_2]$	$[(18, 1, 9), (2, 1, 1)]$	$f_2 = yz - y^2 - x^2$ $f_3 = y(yz - y^2 - x^2 + xy + \frac{18}{25}y^2)$
(6)	$[B_{3,8}, E_6]$	$[(14, 1, 7), (6, 1, 3)]$	$f_2 = y^2 - 2yz + z^2 + x^2 - z^2$ $f_3 = x^2y$
(7)	$[C_{3,9}, 3A_2]$	$[(13, 2, 7), (2, 1, 1)]$	$f_2 = yz - x^2$ $f_3 = y(yz + \frac{4}{3}y^2 + \frac{3\sqrt{3}}{2}x + \frac{2\sqrt{3}}{3}xy + y^2)$

Table 1. Configuration of singularities for maximal tame sextics of torus type $(2, 3)$.

The following recovers the well-known result by Oka–Pho [36] on irreducible maximal tame torus sextic of type $(2, 3)$ for families (1)–(6) in Table 1. For the sake of brevity, we will only show the details for family (1), but the same strategy can be followed to prove the remaining cases.

Theorem 5.3 ([36]). *Let $C = \{f_2^3 + f_3^2 = 0\}$ be an irreducible maximal tame torus sextic of type $(2, 3)$ whose configuration of singularities $\Sigma_C \neq \{[C_{3,9}, 3A_2]\}$ (see Table 1). Then*

$$\pi_1(\mathbb{P}^2 \setminus C) = \mathbb{Z}_2 * \mathbb{Z}_3.$$

Proof. The idea of the proof is to show that any irreducible maximal tame torus sextic of type $(2, 3)$ except for exceptional case $\Sigma_C \neq \{[C_{3,9}, 3A_2]\}$ (family (7) in Table 1) is a generic member of a primitive pencil satisfying the conditions in Theorem 1.2.

We will do it in detail for $C \in \mathcal{M}([C_{3,15}])$. Table 1 gives a possible equation $C = \{f_2^3 + f_3^2 = 0\}$ for such a curve as a member of a pencil generated by a smooth conic $C_2 = \{f_2 = 0\}$ and a nodal cubic $C_3 = \{f_3 = 0\}$ whose node $P \in C_2$ is such that $(C_2, C_3)_P = 6$, that is, $C_2 \cap C_3 = \{P\}$ (see Theorem 1 in [35]). To see that C is in fact a generic member, one can obtain the resolution of indeterminacies is shown in Figure 1, where

$$\hat{F}^*([0:1]) = 3C_2 + E_{2,1} + 2E_{2,2} + 3E_{2,3} + E_{2,4}, \quad \hat{F}^*([1:0]) = 2C_3 + E_{3,1} + E_{3,2}.$$

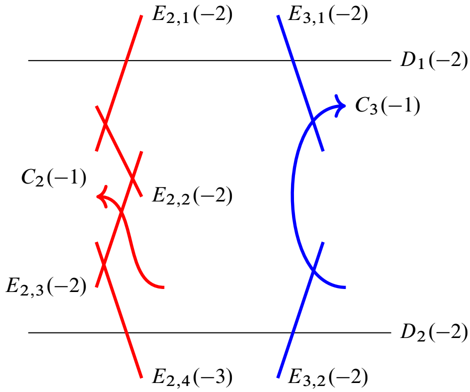

Figure 1. Resolution of indeterminacy.

The dicritical divisors D_1 and D_2 define 1:1 morphisms $\hat{F}_|: D_i \to \mathbb{P}^1$, so there is no degeneration of fibers on the dicritical divisors. One can check that C has no singularities outside the base point P. This implies that C is a generic fiber. ■

Remark 5.4. Let $C = \{f_2^3 + f_3^2 = 0\}$ be the curve in the moduli space $\mathcal{M}([C_{3,9}, 3A_2])$ given by the equation corresponding to family (7) in Table 1. The curve C is not a generic sextic in the pencil $[f_2^3 : f_3^2]$. If it were, Theorem 4.13 would contradict $\pi_1(\mathbb{P}^2 \setminus C) \not\cong \mathbb{Z}_2 * \mathbb{Z}_3$ (Oka–Pho). Nonetheless, one can check directly that C is not a typical fiber, since this pencil is of type $(1, 6)$, but C is a rational curve. This is a consequence of C being irreducible and $\delta(C_{3,9}) + 3\delta(A_2) = 7 + 3 = 10$ (see Table 1).

Theorem 5.5. *Let $F = [f_2^3 : f_3^2] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be a pencil such that $C = \{f_2^3 + f_3^2 = 0\}$ is an irreducible maximal tame torus sextic of type $(2, 3)$ and a generic member of the pencil F. Let Σ_C be its configuration of singularities. Consider $B \subset \mathbb{P}^1$ be a collection of $r + 1$ typical values, $C_B = \bigcup_{\lambda \in B} C_\lambda$, and $C_j = V(f_j)$ for $j = 2, 3$.*

(1) *If $\Sigma_C \neq \{B_{3,10}, A_2\}, \{B_{3,8}, E_6\}, \{C_{3,9}, 3A_2\}$, namely, if C is a curve in a family (1)–(4) (in Table 1), then*

$$\pi_1(\mathbb{P}^2 \setminus C_B \cup C_3) = \mathbb{F}_{r+1} * \mathbb{Z}_3.$$

(2) *If $\Sigma_C \neq \{Sp_1, A_2\}, \{C_{3,9}, 3A_2\}$, namely, if C is a curve in a family (1)–(3), (5) or (6), then*

$$\pi_1(\mathbb{P}^2 \setminus C_B \cup C_2) = \mathbb{F}_{r+1} * \mathbb{Z}_2.$$

Proof. For the proof of part (1), note that these are the only families where C_3 is irreducible. As mentioned before Table 1, the curves C_3 are well defined. In families (1)–(3), the curve C_3 is a nodal cubic, and in family (4), it is a cuspidal cubic transversal to the line at infinity. In both of these cases, $\pi_1(\mathbb{P}^2 \setminus C_3) \cong \mathbb{Z}_3$. The result now follows from the generic addition-deletion Lemma 1.4.

For the proof of part (2), note that these are the families where C_2 is irreducible (a smooth conic). Note that family (7) also has irreducible C_2, but it still does not satisfy the hypothesis, since C is not a typical fiber in that pencil (Remark 5.4). Since C_2 is smooth, $\pi_1(\mathbb{P}^2 \setminus C_2) \cong \mathbb{Z}_2$. The result follows from the generic addition-deletion Lemma 1.4. ■

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References

[1] Aguilar Aguilar, R.: The fundamental group of partial compactifications of the complement of a real line arrangement. *Topology Appl.* **283** (2020), article no. 107388, 19 pp. Zbl 1460.57023 MR 4155321

[2] Amram, M. and Teicher, M.: Erratum to "Fundamental groups of some special quadric arrangements". *Rev. Mat. Complut.* **22** (2009), no. 2, 517–550. Zbl 1172.14316 MR 2553947

[3] Arapura, D.: Geometry of cohomology support loci for local systems. I. *J. Algebraic Geom.* **6** (1997), no. 3, 563–597. Zbl 0923.14010 MR 1487227

[4] Arapura, D.: Toward the structure of fibered fundamental groups of projective varieties. *J. Éc. polytech. Math.* **4** (2017), 595–611. Zbl 1400.14078 MR 3665609

[5] Artal Bartolo, E. and Cogolludo-Agustín, J. I.: On the connection between fundamental groups and pencils with multiple fibers. *J. Singul.* **2** (2010), 1–18. Zbl 1292.14020 MR 2763015

[6] Artal Bartolo, E., Cogolludo-Agustín, J. I. and Matei, D.: Characteristic varieties of quasi-projective manifolds and orbifolds. *Geom. Topol.* **17** (2013), no. 1, 273–309. Zbl 1266.32035 MR 3035328

[7] Bauer, I.: Irrational pencils on non-compact algebraic manifolds. *Internat. J. Math.* **8** (1997), no. 4, 441–450. Zbl 0896.14008 MR 1460895

[8] Bredon, G. E.: *Topology and geometry*. Grad. Texts in Math. 139, Springer, New York, 1997. Zbl 0934.55001 MR 1700700

[9] Catanese, F.: Fibred Kähler and quasi-projective groups. *Adv. Geom.* **3** (2003), special issue, S13–S27. Zbl 1051.32013 MR 2028385

[10] Catanese, F.: Differentiable and deformation type of algebraic surfaces, real and symplectic structures. In *Symplectic 4-manifolds and algebraic surfaces*, pp. 55–167. Lecture Notes in Math. 1938, Springer, Berlin, 2008. Zbl 1145.14001 MR 2441412

[11] Cogolludo-Agustín, J. I.: Braid monodromy of algebraic curves. *Ann. Math. Blaise Pascal* **18** (2011), no. 1, 141–209. Zbl 1254.32043 MR 2830090

[12] Cogolludo-Agustín, J. I. and Elduque, E.: On the topology of fiber-type curves: a Zariski pair of affine nodal curves. Preprint 2025, arXiv:2306.07359v3.

[13] Cogolludo-Agustín, J. I. and Libgober, A.: Mordell–Weil groups of elliptic threefolds and the Alexander module of plane curves. *J. Reine Angew. Math.* **697** (2014), 15–55. Zbl 1326.14090 MR 3281651

[14] Degtyarev, A.: Oka's conjecture on irreducible plane sextics. *J. Lond. Math. Soc. (2)* **78** (2008), no. 2, 329–351. Zbl 1158.14026 MR 2439628

[15] Degtyarev, A.: Oka's conjecture on irreducible plane sextics. II. *J. Knot Theory Ramifications* **18** (2009), no. 8, 1065–1080. Zbl 1174.14027 MR 2554335

[16] Dey, I. M. S. and Neumann, H.: The Hopf property of free products. *Math. Z.* **117** (1970), 325–339. Zbl 0204.34205 MR 0276352

[17] Dimca, A.: *Singularities and topology of hypersurfaces*. Universitext, Springer, New York, 1992. Zbl 0753.57001 MR 1194180

[18] Eyral, C. and Oka, M.: On the fundamental groups of the complements of plane singular sextics. *J. Math. Soc. Japan* **57** (2005), no. 1, 37–54. Zbl 1070.14031 MR 2114719

[19] Eyral, C. and Oka, M.: On the fundamental groups of non-generic $\mathbb{R}$-join-type curves. In *Bridging algebra, geometry, and topology*, pp. 137–157. Springer Proc. Math. Stat. 96, Springer, Cham, 2014. Zbl 1328.14052 MR 3297113

[20] Eyral, C. and Oka, M.: On the fundamental groups of non-generic $\mathbb{R}$-join-type curves, II. *J. Math. Soc. Japan* **69** (2017), no. 1, 241–262. Zbl 1368.14041 MR 3597554

[21] Fujino, O.: On quasi-Albanese maps. To appear in *Boll. Unione Mat. Ital.* (2024), DOI 10.1007/s40574-024-00448-1.

[22] Gromov, M.: Sur le groupe fondamental d'une variété kählérienne. *C. R. Acad. Sci. Paris Sér. I Math.* **308** (1989), no. 3, 67–70. Zbl 0661.53049 MR 0983460

[23] Iitaka, S.: Logarithmic forms of algebraic varieties. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **23** (1976), no. 3, 525–544. Zbl 0342.14017 MR 0429884

[24] Kashiwara, H.: Fonctions rationnelles de type $(0, 1)$ sur le plan projectif complexe. *Osaka J. Math.* **24** (1987), no. 3, 521–577. Zbl 0668.14017 MR 0923877

[25] Kawashima, M. and Yoshizaki, K.: On $(2, 3)$ torus of decompositions of QL-configurations. *SUT J. Math.* **46** (2010), no. 1, 93–117. Zbl 1214.14023 MR 2732647

[26] Kulikov, V. S.: On plane algebraic curves of positive Albanese dimension. *Izv. Math.* **59** (1995), no. 6, 1173–1192; translation from *Izv. Ross. Akad. Nauk, Ser. Mat.* **59** (1995), no. 6, 75–94. Zbl 0939.14013 MR 1481615

[27] Matsuno, T.: On a theorem of Zariski–van Kampen type and its applications. *Osaka J. Math.* **32** (1995), no. 3, 645–658. Zbl 0879.14005 MR 1367896

[28] Namba, M.: *Branched coverings and algebraic functions*. Pitman Research Notes in Mathematics Series 161, Longman Scientific & Technical, Harlow; John Wiley, New York, 1987. Zbl 0706.14017 MR 0933557

[29] Némethi, A.: On the fundamental group of the complement of certain singular plane curves. *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 3, 453–457. Zbl 0679.14004 MR 0906619

[30] Nielsen, J.: The commutator group of the free product of cyclic groups. *Mat. Tidsskr. B* **1948** (1948), 49–56. MR 0029378

[31] Nori, M. V.: Zariski's conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)* **16** (1983), no. 2, 305–344. Zbl 0527.14016 MR 0732347

[32] Oka, M.: Some plane curves whose complements have non-abelian fundamental groups. *Math. Ann.* **218** (1975), no. 1, 55–65. Zbl 0335.14005 MR 0396556

[33] Oka, M.: On the fundamental group of the complement of certain plane curves. *J. Math. Soc. Japan* **30** (1978), no. 4, 579–597. Zbl 0387.14004 MR 0513071

[34] Oka, M.: Geometry of reduced sextics of torus type. *Tokyo J. Math.* **26** (2003), no. 2, 301–327. Zbl 1047.14002 MR 2020788

[35] Oka, M. and Pho, D. T.: Classification of sextics of torus type. *Tokyo J. Math.* **25** (2002), no. 2, 399–433. Zbl 1062.14036 MR 1948673

[36] Oka, M. and Pho, D. T.: Fundamental group of sextics of torus type. In *Trends in singularities*, pp. 151–180. Trends Math., Birkhäuser, Basel, 2002. Zbl 1054.14022 MR 1900785

[37] Russell, H.: Generalized Albanese and its dual. *J. Math. Kyoto Univ.* **48** (2008), no. 4, 907–949. Zbl 1170.14005 MR 2513591

[38] Shimada, I.: On the Zariski–van Kampen theorem. *Canad. J. Math.* **55** (2003), no. 1, 133–156. Zbl 1074.14517 MR 1952329

[39] Thom, R.: Ensembles et morphismes stratifiés. *Bull. Amer. Math. Soc.* **75** (1969), 240–284. Zbl 0197.20502 MR 0239613

[40] Tokunaga, H.-o.: $(2,3)$ torus sextics and the Albanese images of 6-fold cyclic multiple planes. *Kodai Math. J.* **22** (1999), no. 2, 222–242. Zbl 0990.14010 MR 1700594

[41] Tokunaga, H.-o.: Irreducible plane curves with the Albanese dimension 2. *Proc. Amer. Math. Soc.* **127** (1999), no. 7, 1935–1940. Zbl 0917.14015 MR 1637444

[42] Villamayor U., O. E.: Equimultiplicity, algebraic elimination, and blowing-up. *Adv. Math.* **262** (2014), 313–369. Zbl 1295.14015 MR 3228431

[43] Zariski, O.: On the problem of existence of algebraic functions of two variables possessing a given branch curve. *Amer. J. Math.* **51** (1929), no. 2, 305–328. Zbl 55.0806.01 MR 1506719

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José I. Cogolludo-Agustín
Departamento de Matemáticas, IUMA, Universidad de Zaragoza
Pedro Cerbuna 12, 50009 Zaragoza, Spain;
jicogo@unizar.es

Eva Elduque
Departamento de Matemáticas, ICMAT, Universidad Autónoma de Madrid
Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain;
eva.elduque@uam.es