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Solvability of concordance groups and Milnor invariants

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Abstract. Using Milnor invariants, we prove that the concordance group $\mathcal{C}(2)$ of 2-string links is not solvable. As a consequence, we prove that the equivariant concordance group of strongly invertible knots is also not solvable, and we answer a conjecture by Kuzbary (2023).

1. Introduction

Recall that a *strongly invertible knot* is given by a pair (K, ρ), where $K \hookrightarrow S^3$ is a knot and $\rho\colon S^3 \to S^3$ is a smooth orientation preserving involution such that $\rho(K) = K$ and ρ reverses the orientation on K. In [30], Sakuma introduced the definitions of equivariant connected sum and equivariant concordance for (directed) strongly invertible knots. Using these notions, he defines the group $\widetilde{\mathcal{C}}$ of equivariant concordance classes of strongly invertible knots. Following Sakuma, strongly invertible knots have been a classic object of study. Recently, there has been a strong new interest in this topic [5,8,17,22,23], and in particular towards the direction of equivariant concordance [2, 4, 9, 13, 26, 29]. However, very little is known concerning the group structure of $\widetilde{\mathcal{C}}$; even though the first author proved in [12] that $\widetilde{\mathcal{C}}$ is not abelian, a lot of questions remain open. For instance, Alfieri and Boyle [2] conjecture that the equivariant concordance group contains a non-abelian free group.

In this paper, we push the investigation of how far $\widetilde{\mathcal{C}}$ is from being abelian a step further. In Section 4, we prove the following.

Theorem 4.6. *The equivariant concordance group of strongly invertible knots $\widetilde{\mathcal{C}}$ is not solvable.*

The proof of Theorem 4.6 relies on the relations between strongly invertible knots, theta curves and string links, which we outline in Section 3. Recall that a *theta curve* is an embedding in S^3 of a graph with two vertices and three edges joining them. Given n distinct points $p_1, \dots, p_n$ in the interior of D^2, an *n-string link* is (the image of) a proper embedding σ of $\coprod_{k=1}^n I_i$, the disjoint union of n copies of the interval $I = [0, 1]$, in $D^2 \times I$ such that $\sigma|_{I_i}(j) = (p_i, j)$ for each i and for $j = 0, 1$. Analogously to the knot case, it is possible to give natural definitions of sum and cobordism/concordance for theta

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curves and string links. The set of cobordism classes of theta curves forms a group denoted by Θ, while n-string links up to concordance form a group denoted by $\mathcal{C}(n)$. One can easily check that $\mathcal{C}(1)$ coincides with the knot concordance group and hence is an abelian group. As shown by Le Dimet [20], the pure braid group $\mathcal{P}(n)$ naturally injects in $\mathcal{C}(n)$. Since for $n \geq 3$, $\mathcal{P}(n)$ contains a non-abelian free subgroup, we easily get that $\mathcal{C}(n)$ is not abelian and in particular not solvable for $n \geq 3$. The same argument does not work in the case $n = 2$, since $\mathcal{P}(2) \cong \mathbb{Z}$ is a central subgroup of $\mathcal{C}(2)$. De Campos [11] proved that Θ and $\mathcal{C}(2)$ are related by the following theorem.

Theorem 4.4 (Proposition 2 in [11]) *There is a split extension of groups*

$$1 \to \mathcal{P}(2) \to \mathcal{C}(2) \to \Theta \to 1.$$

Miyazaki in [28] provided a proof that the group Θ is not commutative (which would imply that $\mathcal{C}(2)$ is not abelian), appealing to a result of Gilmer [15]. However, Friedl [14] found gaps in the proof of the result in [15]. Another proof of the fact that $\mathcal{C}(2)$ is not abelian can be found in Theorem 1.8 of [24]. In Theorem 1.1 of [19], Kuzbary proves that for all $n \geq 2$, the group $\mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle$ is not abelian and conjectures that it is not solvable either (see Conjecture 1.3 in [19]).

Using Milnor invariants [16, 27], we prove in Theorem 4.1 that a certain subgroup $\mathcal{C}_u(2)$ (see Section 4) of $\mathcal{C}(2)$ is not solvable. Since the proof of this result is non-constructive, we would like to propose the following problem.

Problem 1.1. *For each $k \geq 1$, find a nontrivial explicit element $\sigma_k \in \mathcal{C}_u(2)$ lying in the k-th term of the derived series.*

From Theorem 4.1 we deduce Theorem 4.6. Moreover, the non-solvability of $\mathcal{C}_u(2)$ implies the following theorem, which answers the question posed by Kuzbary.

Theorem 4.7 *The quotient group $\mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle$ of the n-strand string links over the normal closure of the pure n-braids subgroup is not solvable for any $n \geq 2$. In particular, the cobordism group of theta curves $\Theta \cong \mathcal{C}(2)/\mathcal{P}(2)$ is not solvable.*

Content of the paper. In Section 2, we provide the basic notions and definitions regarding the algebraic tools we need in Section 5. Section 3 contains a recap on strongly invertible knots and string links. Moreover, we recall the definition of Milnor invariants. In Section 4, we show how Theorem 4.6 and Theorem 4.7 are implied by Theorem 4.1. Lastly, Section 5 is devoted to the proof of Theorem 4.1.

2. Groups and Lie algebras

In this section, we introduce the notation and the preliminary results that we will need in Section 5. For details on filtrations, see Section 1 of [10].

Definition 2.1. Let G be a group. A *strongly central filtration* $G_* = \{G_k\}_{k\geq 1}$ on G is a sequence of subgroups of G such that

- $G_1 = G$,

- G_{k+1} is a normal subgroup of G_k,
- $[G_i, G_j] \subset G_{i+j}$.

In the following, we will refer to a strongly central filtration just as a *filtration*.

Recall that the *lower central series* of a group G is defined inductively by setting $G_1 = G$ and $G_{n+1} = [G_n, G]$. It is not difficult to see that such a sequence of subgroups is, in particular, a filtration on G. The group G is said to be *nilpotent* if $G_n = 1$ for some n. Similarly, the *derived series* of G is defined by setting $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The group G is said to be *solvable* if $G^{(n)} = 1$ for some n. Observe that, in general, such a sequence of subgroups is not a strongly central filtration. Recall that the two series are related by $G^{(n)} \subseteq G_{n+1}$.

In the following, given a Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{g}^{(n)}$ the n-th term of its derived series, which is defined analogously as in the group case.

Definition 2.2. The *graded Lie ring* associated with a filtration G_* on a group G is defined as the graded group

$$\mathcal{L}(G_*) = \bigoplus_{k \geq 1} G_k / G_{k+1}$$

with the Lie bracket induced by the commutator operation on G.

Definition 2.3. Let G_* and H_* be filtrations on G and H, respectively. An *action* of G_* on H_* is a homomorphism $\phi\colon G \to \mathrm{Aut}(H)$ such that for all $i, j \geq 1$, $[G_i, H_j] \subset H_{i+j}$, where $[g, h] := \phi(g)(h) h^{-1}$, for $g \in G$, $h \in H$.

Given an action $\phi\colon G \to \mathrm{Aut}(H)$ of G_* on H_*, we have an induced homomorphism of Lie rings

$$J_\phi : \mathcal{L}(G_*) \to \mathrm{Der}(\mathcal{L}(H_*))$$

called the *Johnson homomorphism*, where $\mathrm{Der}(\mathcal{L}(H_*))$ is the Lie ring of derivations on $\mathcal{L}(H_*)$. Such homomorphism J_φ is defined on homogeneous elements $\bar{g} \in \mathcal{L}(G_*)_k = G_k / G_{k+1}$ by $J_\varphi(\bar{g}) = [g, -]$, where $g \in G$ is an arbitrary lift of $\bar{g}$. Observe that this is a generalization of the Johnson homomorphism defined in [18] on the Torelli subgroup of the mapping class group of a surface.

Lemma 2.4 (Proposition-Definition 1.8 in [10]). *Let $\phi\colon G \to \mathrm{Aut}(H)$ be a homomorphism and let H_* be a filtration on H. Then, the filtration G_* given by*

$$G_i = \{g \in G \mid \forall j \geq 1,\ [g, H_j] \subset H_{i+j}\},$$

is the unique maximal filtration on the subgroup $G_1 \subset G$ such that $\phi_{|G_1}\colon G_1 \to \mathrm{Aut}(H)$ is an action of G_* on H_*.

Remark 2.5. Notice that the subgroup G_1 in Lemma 2.4 is the largest subgroup of G which admits a filtration G_* such that $\phi_{|G_1}\colon G_1 \to \mathrm{Aut}(H)$ is an action of G_* on H_*.

Lemma 2.6. *Let G_* be a filtration on a group G. If G is a solvable group, then $\mathcal{L}(G_*)$ is a solvable Lie ring.*

Proof. We denote by $G^{(n)}$ the n-th term of the derived series of G, and we denote by $G^{(n)} \cap G_*$ the filtration on $G^{(n)}$ induced by the one on G. We begin by showing that

$\mathcal{L}(G_*)^{(1)} = [\mathcal{L}(G_*), \mathcal{L}(G_*)]$ is contained in $\mathcal{L}(G^{(1)} \cap G_*)$. Fix a degree $k \geq 1$ of the filtration. Then, by definition,

$$[\mathcal{L}(G_*), \mathcal{L}(G_*)]_k = \sum_{i+j=k} [G_i/G_{i+1}, G_j/G_{j+1}] \subseteq \frac{\langle [G_i, G_j] \mid i+j=k \rangle}{G_{k+1} \cap G^{(1)}}$$
$$\subseteq \frac{G_k \cap G^{(1)}}{G_{k+1} \cap G^{(1)}} = \mathcal{L}(G^{(1)} \cap G_*)_k.$$

By iteration, we obtain

$$\mathcal{L}(G_*)^{(n)} \subseteq \mathcal{L}(G^{(n)} \cap G_*).$$

Since G is solvable, for a large enough n the right-hand side will be zero, hence the thesis. ∎

Remark 2.7. Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ be a graded Lie algebra. Then the degree completion of $\mathfrak{g}$ is given by

$$\bar{\mathfrak{g}} = \prod_{k \geq 1} \mathfrak{g}_k.$$

Lemma 2.8. *Let $\mathfrak{g}$ be a graded Lie algebra. Then $\mathfrak{g}$ is solvable if and only if its degree completion $\bar{\mathfrak{g}}$ is solvable.*

Proof. We have that $\mathfrak{g} \subseteq \bar{\mathfrak{g}}$, so if $\mathfrak{g}$ is not solvable, $\bar{\mathfrak{g}}$ cannot be solvable. Vice versa, assume that $\bar{\mathfrak{g}}$ is not solvable and consider, for all $n \in \mathbb{N}$, the ideals

$$I_n = \bigoplus_{k \geq n} \mathfrak{g}_k \subset \mathfrak{g} \quad \text{and} \quad \bar{I}_n = \prod_{k \geq n} \mathfrak{g}_k \subset \bar{\mathfrak{g}}.$$

Denote by $\mathfrak{h}^{(n)}$ the n-th term of the derived series of a Lie algebra $\mathfrak{h}$. We want to prove that for every $k \in \mathbb{N}$, $\mathfrak{g}^{(k)} \neq 0$. This follows by showing that for every such k there exists n_k large enough such that $(\mathfrak{g}/I_{n_k})^{(k)} \neq 0$. Let $\pi_n \colon \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\bar{I}_n$ be the projection. Since π_n is a Lie ring homomorphism and hence commutes with taking the brackets, we have that for all k and n,

$$\pi_n(\bar{\mathfrak{g}}^{(k)}) = (\bar{\mathfrak{g}}/\bar{I}_n)^{(k)}.$$

Since $\bar{\mathfrak{g}}$ is not solvable, for all $k \geq 1$ there exist an $x_k \in \bar{\mathfrak{g}}^{(k)}$, $x_k \neq 0$. Let $n_k - 1$ be the degree of the first non-zero component of x_k. Since $\pi_{n_k}(x_k) \neq 0$, it follows that $(\bar{\mathfrak{g}}/\bar{I}_{n_k})^{(k)}$ is not trivial. Observe that $\bar{\mathfrak{g}}/\bar{I}_n = \mathfrak{g}/I_n$, hence $\mathfrak{g}^{(k)}$ is not trivial. Since this holds for every k, we deduce that $\mathfrak{g}$ is not solvable. ∎

Recall that given a Lie ring $\mathfrak{g}$, a *derivation* on $\mathfrak{g}$ is a linear map $D \colon \mathfrak{g} \to \mathfrak{g}$ such that for every $A, B \in \mathfrak{g}$,

$$D([A, B]) = [D(A), B] + [A, D(B)].$$

We denote by $\mathrm{Der}(\mathfrak{g})$ the set of derivations of $\mathfrak{g}$, which can be endowed with the structure of a Lie ring given by

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

In the following, we will denote by $\mathfrak{l}_2$ the free graded Lie ring on two generators X and Y in degree 1.

Definition 2.9 (Definitions 3.2 and 3.5 in [1]). We say that a derivation $D \in \operatorname{Der}(\mathfrak{l}_2)$ is *tangential* if there exist $U, V \in \mathfrak{l}_2$ such that

$$D(X) = [U, X] \quad \text{and} \quad D(Y) = [V, Y].$$

A tangential derivation D is called *special* (or *normalized*) if $D(X + Y) = 0$. We will denote by $\operatorname{tDer}(\mathfrak{l}_2)$ and $\operatorname{sDer}(\mathfrak{l}_2)$ the sets of tangential derivations and special derivations, respectively. It is not difficult to check that

$$\operatorname{sDer}(\mathfrak{l}_2) \subseteq \operatorname{tDer}(\mathfrak{l}_2) \subseteq \operatorname{Der}(\mathfrak{l}_2)$$

as Lie rings. Observe that since $\mathfrak{l}_2$ is graded, the Lie ring $\operatorname{sDer}(\mathfrak{l}_2)$ naturally inherits a graded structure, where $D \in \operatorname{sDer}(\mathfrak{l}_2)_n$ if and only if $D(X)$ and $D(Y)$ lie in degree $n + 1$. In a similar way, one can define the subalgebra of tangential and special derivations of the Lie algebra $\mathfrak{l}_2^{\mathbb{Q}} = \mathfrak{l}_2 \otimes \mathbb{Q}$ and of its degree completion $\overline{\mathfrak{l}_2^{\mathbb{Q}}}$ (see also Section 3 of [1]).

Lemma 2.10. *The following map induces an isomorphism of graded Lie rings*:

$$\begin{aligned} \operatorname{sDer}(\mathfrak{l}_2) \otimes \mathbb{Q} &\longrightarrow \operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}}) \\ D \otimes p/q &\longmapsto p/q \cdot D. \end{aligned}$$

Proof. It is not difficult to check that the inverse of the map above can be constructed as follows. Let $D \in \operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}})$ and let $U, V \in \mathfrak{l}_2^{\mathbb{Q}}$ such that $D(X) = [U, X]$ and $D(Y) = [V, Y]$. Then there exists $n \neq 0$ such that $nU, nV \in \mathfrak{l}_2 \subset \mathfrak{l}_2^{\mathbb{Q}}$. Then clearly nD restricts to a special derivation of $\mathfrak{l}_2$, and we get the inverse map as

$$\begin{aligned} \operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}}) &\longrightarrow \operatorname{sDer}(\mathfrak{l}_2) \otimes \mathbb{Q} \\ D &\longmapsto nD \otimes 1/n. \end{aligned}$$

∎

Lemma 2.11. *The degree completion* $\overline{\operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}})}$ *of the special derivations of* $\mathfrak{l}_2^{\mathbb{Q}}$ *is naturally isomorphic to* $\operatorname{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$.

Proof. Take $D \in \overline{\operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}})}$. We can see D as a formal sum $D = \sum_{i \geq 1} D_i$, where $D_i \in \operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}})_i$. Then D acts as a derivation on $\overline{\mathfrak{l}_2^{\mathbb{Q}}}$ as follows. Given $Z \in \overline{\mathfrak{l}_2^{\mathbb{Q}}}$, we can write it as $Z = \sum_{i \geq 1} Z_j$, with $Z_j \in (\mathfrak{l}_2^{\mathbb{Q}})_j$. Then we define $D(Z)$ as

$$D(Z) = \sum_{i,j \geq 1} D_i(Z_j),$$

which is a well-defined element of $\overline{\mathfrak{l}_2^{\mathbb{Q}}}$, since for every n the number of terms $D_i(Z_j)$ of degree $\leq n$ is finite. For every $i \geq 1$ take $U_i, V_i \in (\mathfrak{l}_2^{\mathbb{Q}})_i$ such that $D_i(X) = [U_i, X]$ and $D_i(Y) = [V_i, Y]$, and let $U = \sum_{i \geq 1} U_i,\ V = \sum_{i \geq 1} V_i \in \overline{\mathfrak{l}_2^{\mathbb{Q}}}$. Then it is immediate to see that $D(X) = [U, X]$ and $D(Y) = [V, Y]$, hence D is a tangential derivation of $\overline{\mathfrak{l}_2^{\mathbb{Q}}}$. Observe that since $D_i(X + Y) = 0$ for every i, we get that $D(X + Y) = 0$. Therefore we have a map $\overline{\operatorname{sDer}(\mathfrak{l}_2^{\mathbb{Q}})} \to \operatorname{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$. Its inverse is given as follows. Take $D \in \operatorname{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$. Since it is a tangential derivation there exist $U = \sum_{i\geq 1} U_i,\ V = \sum_{i\geq 1} V_i \in \overline{\mathfrak{l}_2^{\mathbb{Q}}}$, $U_i, V_i \in (\mathfrak{l}_2^{\mathbb{Q}})_i$ such that $D(X) = [U, X]$ and $D(Y) = [V, Y]$. Since D is special, one can easily see that $[U_i, X] + [V_i, Y] = 0$ for every i, by inspecting $D(X + Y)$ in each degree. Let D_i be the derivation of $\mathfrak{l}_2^{\mathbb{Q}}$ defined by $D_i(X) = [U_i, X]$ and $D_i(Y) = [V_i, Y]$, and observe that $D_i \in \mathrm{sDer}(\mathfrak{l}_2^{\mathbb{Q}})_i$. Hence we get an element

$$D = \sum_{i\geq 1} D_i \in \overline{\mathrm{sDer}(\mathfrak{l}_2^{\mathbb{Q}})}. \qquad \blacksquare$$

Proposition 2.12. *The Lie ring* $\mathrm{sDer}(\mathfrak{l}_2)$ *is not solvable.*

Proof. First of all, observe that $\mathrm{sDer}(\mathfrak{l}_2)$ is solvable if and only if $\mathrm{sDer}(\mathfrak{l}_2) \otimes \mathbb{Q}$ is solvable, where $\mathrm{sDer}(\mathfrak{l}_2) \otimes \mathbb{Q} \cong \mathrm{sDer}(\mathfrak{l}_2^{\mathbb{Q}})$ by Lemma 2.10. Therefore, using Lemmas 2.11 and 2.8, it is sufficient to show that $\mathrm{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$ is not solvable, By Theorem 4.1 in [1], we know that $\mathrm{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$ contains the Grothendieck–Teichmüller algebra $\mathfrak{grt}$ (see Section 4.2 of [1]) as a Lie subalgebra. In [6] Brown proved that $\mathfrak{grt}$ contains in turn a free Lie algebra on infinite generators (see also Section 7.2.3 of [25]). Therefore $\mathrm{sDer}(\overline{\mathfrak{l}_2^{\mathbb{Q}}})$ and hence $\mathrm{sDer}(\mathfrak{l}_2)$ are non-solvable. ■

3. String links and strongly invertible knots

The aim of this brief section is simply to recall and clarify the definitions concerning strongly invertible knots, theta curves, string links and Milnor invariants.

We start with the definition of a strongly invertible knot. Despite not being a fundamental definition in the following sections, we want to stress that all of this construction originates from the goal to investigate the non-abelianity of $\widetilde{\mathcal{C}}$.

Definition 3.1. A *strongly invertible knot* is a pair (K, ρ) where $K \subseteq S^3$ is an oriented knot and $\rho \in \mathrm{Diffeo}^+(S^3)$ is an involution such that $\rho(K) = K$ and ρ reverses the orientation on K.

Recall that by resolution of the Smith conjecture [3], the fixed point set $\mathrm{Fix}(\rho)$ of such an involution ρ is always an unknot.

Definition 3.2. A *direction* on a strongly invertible knot (K, ρ) is the choice of an oriented half-axis h, i.e., one of the two connected components of $\mathrm{Fix}(\rho) \setminus K$. We say that a triple (K, ρ, h) is a *directed strongly invertible knot*, or *DSI knot*.

Definition 3.3. We say that two DSI knots (K_i, ρ_i, h_i), $i = 0, 1$ are *equivariantly concordant* if there exists a smoothly and properly embedded annulus $C \cong S^1 \times I \subset S^3 \times I$, invariant with respect to some orientation-preserving smooth involution ρ of $S^3 \times I$ such that:

- $\partial(S^3 \times I, C) = (S^3, K_0) \sqcup -(S^3, K_1)$,
- ρ is in an extension of the strong inversion $\rho_0 \sqcup \rho_1$ on $S^3 \times 0 \sqcup S^3 \times 1$,
- the orientations of h_0 and $-h_1$ induce the same orientation on the annulus $\mathrm{Fix}(\rho)$, and h_0 and h_1 are contained in the same component of $\mathrm{Fix}(\rho) \setminus C$.

The *equivariant concordance group* is the set $\widetilde{\mathcal{C}}$ of classes of directed strongly invertible knots up to equivariant concordance, endowed with the operation induced by the *equivariant connected sum*, which we denote by $\widetilde{\#}$ (see [4, 30] for details).

Definition 3.4. A *labelled theta curve* is a graph G with two vertices v_1 and v_2 and three edges e_1, e_2 and e_3 joining v_1 and v_2, which are considered to be oriented from v_1 to v_2. A (*spatial*) *theta curve* is a piecewise linear locally flat embedding $\theta\colon G \to S^3$ (for details see [32]).

In [32], Taniyama introduced a notion of cobordism between theta curves and defined the so called *cobordism group of theta curves* Θ, which is a group with the operation of *vertex connected sum* of theta curves.

Remark 3.5. Observe that there exists a natural homomorphism

$$\pi : \widetilde{\mathcal{C}} \to \Theta.$$

In fact, we can associate with a directed strongly invertible knot (K, ρ, h) the theta curve θ given by the projection in $S^3/\rho \cong S^3$ of $K \cup \mathrm{Fix}(\rho)$. The theta curve is naturally labelled as follows:

- the vertex v_1 (respectively, v_2) is the projection of the initial (respectively, final) point of h,
- the edge e_1 is the projection of h,
- the edge e_2 is the projection of K,
- the edge e_3 is the projection of $\mathrm{Fix}(\rho) \setminus h$.

We now proceed with the definition of a *string link*, introduced originally in [20], which generalizes the notion of a braid.

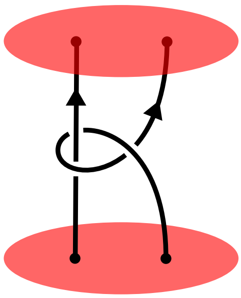

Figure 1. Example of a 2-string link.

Definition 3.6. Fix $p_1, \ldots, p_k$ distinct points in the interior of D^2. A k-*string link* is (the image of) a proper and piecewise linear locally flat embedding:

$$\sigma : \coprod_{i=1}^{k} I_i \hookrightarrow D^2 \times I$$

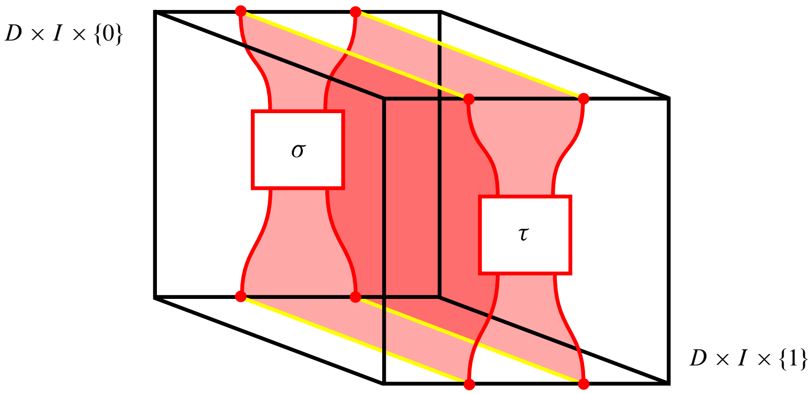

Figure 2. Concordance of two 2-string links σ and τ. In yellow, the two trivial string links lying in $D \times \{1\} \times I$ and $D \times \{0\} \times I$. The red disks are the components $(I \times I)_i$ of the concordance.

such that $\sigma|_{I_i}(0) = p_i \times 0$ and $\sigma|_{I_i}(1) = p_i \times 1$ for all $i = 1, \ldots, k$. The image of I_i is called the *i-th string* of the string link σ. We will also refer to the i-th string of σ by writing: $I_{\sigma,i}$. See Figure 1 for an example.

It is clear that each string of a string link inherits an orientation from the standard orientation on the interval.

In analogy with braids, given two k-string links σ and τ, it is possible to define their *sum* $\sigma \# \tau$, by stacking σ over τ.

The identity element of this sum is given by the trivial k-string link, which coincides with the trivial k-braid. Considering string links up to isotopies that fix the endpoints, it is easy to notice that the operation of sum is not commutative.

Now we want to define a suitable notion of *concordance* for string links (see Figure 2).

Definition 3.7. We say that two k-string links σ and τ are *concordant* if there exist k properly and piecewise linear locally flat embedded disks $\coprod_{i=1}^{k} (I \times I)_i \hookrightarrow D \times I \times I$ such that:

- $(I \times 0)_i = I_{\sigma,i}$ for all $i = 1, \ldots, k$.
- $(I \times 1)_i = I_{\tau,i}$ for all $i = 1, \ldots, k$.
- The string link $\coprod_{i=1}^{k} (0 \times I)_i \hookrightarrow D \times 0 \times I$ is the trivial one.
- The string link $\coprod_{i=1}^{k} (1 \times I)_i \hookrightarrow D \times 1 \times I$ is the trivial one.

Notice how the first two points above imply that, for each i, the disk $(I \times I)_i$ is a concordance disk from the i-th component of σ to the i-th component of τ. The concordance group $\mathcal{C}(k)$ of k-string links is defined as the set of k-string links up to concordance, together with the operation of sum # defined above.

As shown by Le Dimet in [20], the group of pure k-braids $\mathcal{P}(k)$ is naturally contained in $\mathcal{C}(k)$ as a subgroup.

We now recall briefly the definition of Milnor invariants for string links, following [16]. We will focus only on the construction for 2-string links, since it is the only case we will need in Section 5.

Let σ be a 2-string link and let $D_2 = D \setminus \{p_1, p_2\}$. Denote by j_0 and j_1 the inclusion of D_2 in $D \times I \setminus \sigma$ at time 0 and 1, respectively. Observe that, in general,

$$(j_0)_*, (j_1)_* : \pi_1(D_2) \to \pi_1(D \times I \setminus \sigma)$$

are not isomorphisms, but they induce isomorphisms on integral homology. Therefore, by Stallings' theorem [31], we have that j_0 and j_1 give isomorphisms

$$(j_0)_*, (j_1)_* \,:\, \frac{\pi_1(D_2)}{\pi_1(D_2)_n} \to \frac{\pi_1(D \times I \setminus \sigma)}{\pi_1(D \times I \setminus \sigma)_n}$$

for all n, where $\pi_1(X)_n$ is the n-th term of the lower central series of $\pi_1(X)$.

Identifying $\pi_1(D_2)$ with the free group F on two generators x and y, we get an automorphism $A_n(\sigma) = (j_1)_*^{-1}(j_0)_*$ of F/F_{n+1}. As in Theorem 1.1 of [16], one can prove that this actually defines a surjective homomorphism

$$A_n : \mathcal{C}(2) \to \mathrm{Aut}_0(F/F_{n+1}),$$

where $\mathrm{Aut}_0(F/F_{n+1})$ is the subgroup consisting of automorphisms which conjugate x and y and fix the product xy. In particular, A_n is an invariant of string link concordance.

Remark 3.8. In the following, we will denote by $\overline{F}$ the *algebraic closure* of F in its pronilpotent completion $\widehat{F} = \varprojlim F/F_n$ (see [21] for the definition), and by $\overline{F}_*$ the filtration on $\overline{F}$ given by the lower central series. As pointed out in Remark 1.2 of [16], Milnor invariants for string links can be gathered to define a *total Milnor invariant*

$$A : \mathcal{C}(2) \to \mathrm{Aut}_0(\overline{F}).$$

Then, for every n we can retrieve the homomorphism A_n by composition with the projection

$$\pi_n : \mathrm{Aut}_0(\overline{F}) \to \mathrm{Aut}_0(\overline{F}/\overline{F}_{n+1}),$$

since we can identify $F/F_{n+1} \cong \overline{F}/\overline{F}_{n+1}$.

4. Solvability of concordance groups

The aim of this section is to prove the non-solvability of multiple concordance groups. In particular, we prove that the equivariant concordance group is not solvable, and we confirm Conjecture 1.3 in [19] regarding the solvability of $\mathcal{C}(n)/\langle\langle \mathcal{P}(n)\rangle\rangle$.

First of all, let $\mathcal{C}_u(2)$ be the subgroup of $\mathcal{C}(2)$ generated by concordance classes of 2-string links that admit a representative whose first component is unknotted.

Theorem 4.1. *The group $\mathcal{C}_u(2)$ is not solvable.*

In Section 5, we provide a proof of this result. We now investigate the consequences of Theorem 4.1. It is straightforward that it implies the following.

Corollary 4.2. *The group $\mathcal{C}(2)$ of the concordance classes of 2-strand string links is not solvable.*

Let lk: $\mathcal{C}_u(2) \to \mathbb{Z}$ be the homomorphism mapping a 2-string link l to the linking number between its components, and denote by $\mathcal{C}_0(2)$ its kernel. Theorem 4.1 implies easily the following.

Corollary 4.3. *The group $\mathcal{C}_0(2)$ is not solvable.*

Proof. Observe that we have the following short exact sequence:

$$1 \to \mathcal{C}_0(2) \to \mathcal{C}_u(2) \xrightarrow{\text{lk}} \mathbb{Z} \to 1.$$

Since $\mathbb{Z}$ is abelian and $\mathcal{C}_u(2)$ is not solvable, it follows that $\mathcal{C}_0(2)$ cannot be solvable. ∎

We now need a way to deduce the non-solvability of the equivariant concordance group of strongly invertible knots $\widetilde{\mathcal{C}}$ from the results above. The key ingredient is the following result.

Theorem 4.4 (Proposition 2 in [11]). *There is a split extension of groups*

$$1 \to \mathcal{P}(2) \xrightarrow{i} \mathcal{C}(2) \xrightarrow{\alpha} \Theta \to 1,$$

where $\mathcal{P}(2)$ is the group of pure braids on 2 strands, $\mathcal{C}(2)$ are the concordance classes of 2-strand string links and Θ is the cobordism group of theta curves.

Remark 4.5. Define a homomorphism $\beta\colon \mathcal{C}(2) \to \mathcal{P}(2)$, where for a given 2-string link l, $\beta(l)$ is the unique pure braid on 2 strands whose linking number is the same as that of l. Then, according to the proof of Proposition 2 in [11], β is a retraction of i and the restriction of α to $\ker(\beta)$ is an isomorphism onto Θ. We denote by ϕ its inverse, which can be described as follows.

Given a theta curve $\theta \in \Theta$, we can always find a diagram such that edge e_3 is an arc without crossings (see [32]). This can be achieved by pulling a vertex along e_3 until it is a short segment with no crossings. Let $\mathcal{N}(e_3)$ be a regular neighbourhood small enough not to intersect any crossing on the diagram. Then, we define $\varphi(\theta)$ as the 2-string link in $D^2 \times I$ given by $\theta \setminus e_3 \subset S^3 \setminus \mathcal{N}(e_3)$ (see Figure 3). In order to obtain a well-defined map, we must choose appropriately the identification between $S^3 \setminus \mathcal{N}(e_3)$ and $D^2 \times I$, which is uniquely determined up to isotopy relative to the boundary by requiring the linking number between the two strands of $\theta \setminus e_3$ to be 0. In Figure 3, this corresponds to adding the blue compensatory twists.

Then we are able to define the following group homomorphism:

$$\widetilde{\mathcal{C}} \xrightarrow{\pi} \Theta \xrightarrow{\varphi} \mathcal{C}(2),$$

where the map π sends a directed strongly invertible knot (K, ρ, h) to its quotient theta curve as in Remark 3.5.

Observe that the image of the composition $\varphi \circ \pi$ is exactly $\mathcal{C}_0(2)$.

(1) $\operatorname{Im}(\varphi \circ \pi) \subseteq \mathcal{C}_0(2)$.

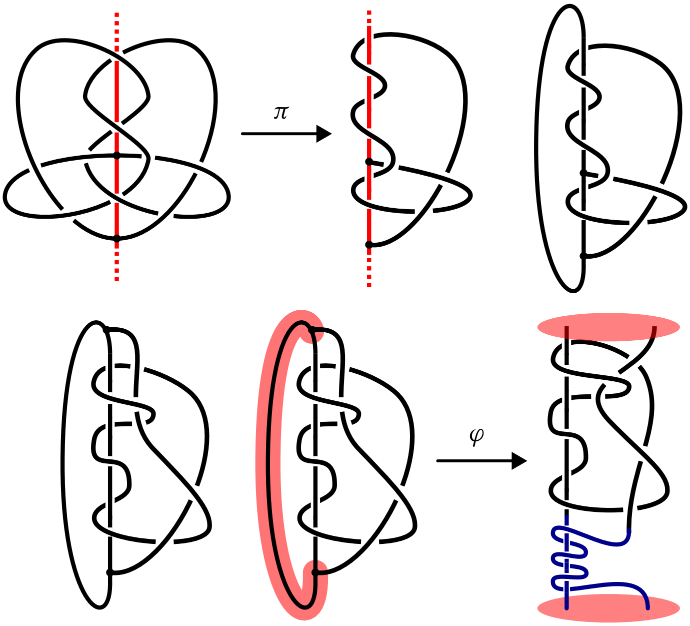

Figure 3. An example on the homomorphisms π and φ.

The image lies obviously in $\mathcal{C}(2)$. Since the first strand is the image of the chosen half axis of the fixed points locus of the strong inversion we also know that the image is contained in $\mathcal{C}_u(2)$. Moreover, by the definition of φ, the linking number between components is zero, hence $\mathrm{Im}(\varphi \circ \pi)$ is contained in $\mathcal{C}_0(2)$.

(2) $\mathcal{C}_0(2) \subseteq \mathrm{Im}(\varphi \circ \pi)$.

It is enough to glue back $\mathcal{N}(e_3)$ to obtain a theta curve. Observe that for such a theta curve, the union of e_1 and e_3 forms an unknot. Hence we get back a directed strongly invertible knot by considering the preimage of e_2 in the double cover branched over the unknot $e_1 \cup e_3$.

Remark 4.5 allow us to finally show our claims.

Theorem 4.6. *The equivariant concordance group of strongly invertible knots $\widetilde{\mathcal{C}}$ is not solvable.*

Proof. In Remark 4.5, we defined a surjective homomorphism $\widetilde{\mathcal{C}} \to \mathcal{C}_0(2)$. By Corollary 4.3, we know that $\mathcal{C}_0(2)$ is not solvable, therefore $\widetilde{\mathcal{C}}$ is not solvable since it has a non-solvable quotient. ∎

Theorem 4.7. *The quotient group $\mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle$ of the n-strand string links over the normal closure of the pure n-braids subgroup is not solvable for any $n \geq 2$. In particular, the cobordism group of theta curves $\Theta \cong \mathcal{C}(2)/\mathcal{P}(2)$ is not solvable.*

Proof. Notice that $\mathcal{C}_0(2) \cap \mathcal{P}(2) = 1$. Thus, by Corollary 4.3, we know that $\mathcal{C}(2)/\mathcal{P}(2)$, and hence Θ, are not solvable.

Observe that for $n > 2$, we have an injective homomorphism $i_n : \mathcal{C}(2) \to \mathcal{C}(n)$, which maps a 2-string link to an n-string link by adding $(n-2)$ trivial strands. The homomorphism i_n has a retraction $s_n : \mathcal{C}(n) \to \mathcal{C}(2)$ given by forgetting the last $(n-2)$ strands of the string link. It is not difficult to see that the maps above induce maps on the quotients

$$\mathcal{C}(2)/\mathcal{P}(2) \xrightarrow{\overline{i_n}} \mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle \xrightarrow{\overline{s_n}} \mathcal{C}(2)/\mathcal{P}(2)$$

which show that $\mathcal{C}(2)/\mathcal{P}(2)$ is a subgroup of $\mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle$ for every $n \geq 2$. Therefore $\mathcal{C}(n)/\langle\langle \mathcal{P}(n) \rangle\rangle$ is not solvable. ∎

5. Proof of Theorem 4.1

We start with three preliminary lemmas.

Lemma 5.1. *For every n, the restriction of the Artin representation on $\mathcal{C}_u(2)$,*

$$A_n : \mathcal{C}_u(2) \to \mathrm{Aut}_0(F/F_{n+1}),$$

is surjective.

Proof. The case $n = 1$ is trivial, since $F_2 = [F, F]$. For $n = 2$, $\mathrm{Aut}_0(F/F_3) \cong \mathbb{Z}$ and given $\sigma \in \mathcal{C}(2)$, we have that $A_2(\sigma)$ corresponds to the linking number between the components of σ. We proceed by induction on n. Consider the following commutative diagram (see Theorem 1.1 in [16] for the notation and details), where the bottom row is exact:

$$\begin{array}{ccccccccc} & & & & \mathcal{C}_u(2) & & & & \\ & & & & {\scriptstyle A_{n+1}}\downarrow & \searrow^{A_n} & & & \\ 1 & \longrightarrow & K_n & \longrightarrow & \mathrm{Aut}_0(F/F_{n+2}) & \longrightarrow & \mathrm{Aut}_0(F/F_{n+1}) & \longrightarrow & 1. \end{array}$$

Suppose by induction that the restriction of A_n on $\mathcal{C}_u(2)$ is surjective. In order to prove the lemma, it is sufficient to show that for every $\alpha \in K_n$, there exists $\sigma \in \mathcal{C}_u(2)$ such that $A_{n+1}(\sigma) = \alpha$. Given such an α, let $\tau \in \mathcal{C}(2)$ be a string link such that $A_{n+1}(\tau) = \alpha$, and denote by $\hat{\tau}$ its closure. Then by Lemma 3.7 in [16], we have that all Milnor invariants of $\hat{\tau}$ of length $\leq n$ vanish. By Theorem 3.3 in [7], we know that there exists a link L with unknotted components with the same Milnor invariants of $\hat{\tau}$ of length $\leq n+1$. Let now $\sigma \in \mathcal{C}_u(2)$ be any string link with closure L. It follows from Corollaries 3.6 and 3.8 in [16] that $A_{n+1}(\sigma) = A_{n+1}(\tau)$. Therefore $A_{n+1} : \mathcal{C}_u(2) \to \mathrm{Aut}_0(F/F_{n+2})$ is surjective. ∎

Let $\{\overline{F}_k\}_{k \geq 1}$ be the filtration on $\overline{F}$ (see Remark 3.8) given by the lower central series, and let $A : \mathcal{C}_u(2) \to \mathrm{Aut}_0(\overline{F})$ be the total Milnor invariant.

Recall from Lemma 2.4 that $\{\overline{F}_k\}_{k \geq 1}$ induces a filtration on $\mathcal{C}_u(2)$ via A, which is given by

$$\mathcal{C}_u(2)_i = \{\sigma \in \mathcal{C}_u(2) \mid \forall j \geq 0,\ [\sigma, \overline{F}_j] \subset \overline{F}_{i+j}\},$$

so that A gives an action of $C_u(2)_*$ on $\overline{F}_*$ (see Definition 2.3).

Lemma 5.2. *For all* $n \geq 0$*, we have that* $\mathcal{C}_u(2)_n = \ker(A_n)$.

Proof. Observe that since $A_n = \pi_n \circ A$, we have that

$$\ker(A_n) = \{\sigma \in \mathcal{C}_u(2) \mid [\sigma, \overline{F}_1] \subset \overline{F}_{n+1}\},$$

and hence $\mathcal{C}_u(2)_n \subset \ker(A_n)$.

Vice versa, given $\sigma \in \ker(A_n)$ we prove by induction on $j \geq 1$ that

$$[\sigma, \overline{F}_j] \subset \overline{F}_{n+j}.$$

First of all, observe that given $a, b \in \overline{F}_{j+1}$ such that $[\sigma, a], [\sigma, b] \in \overline{F}_{n+j+1}$, then $[\sigma, ab] \in \overline{F}_{n+j+1}$. In fact, we can write

$$[\sigma, ab] = \sigma(a)\sigma(b)b^{-1}a^{-1} = [\sigma, a][\sigma, b][[\sigma, b]^{-1}, a].$$

Since $\overline{F}_{j+1} = [\overline{F}_1, \overline{F}_j]$, it is sufficient to prove that given $a \in \overline{F}_1$ and $b \in \overline{F}_j$, we have $[\sigma, [a, b]] \in \overline{F}_{n+j+1}$. An easy computation shows that

$$[\sigma, [a, b]] = [[\sigma, a], b][a, [\sigma, b]],$$

up to elements in $\overline{F}_{n+j+2}$. ∎

Let $\mathcal{L}(\mathcal{C}_u(2)_*)$ be the graded Lie ring given by the filtration on $\mathcal{C}_u(2)$. Denote by $J_A \colon \mathcal{L}(\mathcal{C}_u(2)_*) \to \mathrm{Der}(\mathfrak{l}_2)$ the Johnson homomorphism, where we identify $\mathcal{L}(\overline{F}_*) \cong \mathcal{L}(F_*)$ with the free Lie ring $\mathfrak{l}_2$ with two generators of degree 1.

Since for every $\sigma \in \mathcal{C}_u(2)$, the automorphism $A(\sigma)$ acts on $\overline{F}$ by conjugating the generators x and y and preserving the product xy, we have that the image of J_A is actually contained in $\mathrm{sDer}(\mathfrak{l}_2)$ (see Definition 2.9).

Lemma 5.3. *Let* K_n *be the kernel of the map* $\mathrm{Aut}_0(F/F_{n+2}) \to \mathrm{Aut}_0(F/F_{n+1})$*. Then we have a natural identification between* K_n *and* $\mathrm{sDer}(\mathfrak{l}_2)_n$.

Proof. Let

$$p_n : (F_n/F_{n+1})^2 \to F_{n+1}/F_{n+2}$$

be the homomorphism

$$(\lambda_1, \lambda_2) \mapsto [\lambda_1, x][\lambda_2, y],$$

as defined in [16], and consider the map

$$\begin{aligned} \Phi_n : \ker p_n &\to K_n \\ (\lambda_1, \lambda_2) &\mapsto \varphi_{\lambda_1,\lambda_2}, \end{aligned}$$

where $\varphi_{\lambda_1,\lambda_2}$ is the automorphism defined by mapping x to $[\lambda_1, x]x = \lambda_1 x \lambda_1^{-1}$ and y to $[\lambda_2, y]y = \lambda_2 y \lambda_2^{-1}$. According to [16], Φ_n is an isomorphism for $n \geq 2$, and one can easily check that it is surjective for $n = 1$. Define an analogous map:

$$\begin{aligned} \Psi_n : \ker p_n &\to \mathrm{sDer}(\mathfrak{l}_2)_n \\ (\lambda_1, \lambda_2) &\mapsto D_{\lambda_1,\lambda_2}, \end{aligned}$$

where D_{λ_1,λ_2} is the derivation defined by mapping X to $[\lambda_1, X]$ and Y to $[\lambda_2, Y]$. By the definition of special derivation, we see that Ψ_n is surjective.

We conclude by noticing that $(\lambda_1, \lambda_2) \in \ker \Phi_n$ if and if $[\lambda_1, x] = [\lambda_2, y] = 1$ in F_{n+1}/F_{n+2} if and only if $(\lambda_1, \lambda_2) \in \ker \Phi_n$. ∎

Proof of Theorem 4.1. By Lemma 2.6, it is sufficient to show that $\mathcal{L}(\mathcal{C}_u(2)_*)$ is not solvable. In order to do so, we prove that the Johnson homomorphism

$$J_A : \mathcal{L}(\mathcal{C}_u(2)_*) \to \mathrm{sDer}(\mathfrak{l}_2)$$

is an isomorphism of Lie rings. Consider the following commutative diagram, with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{C}_u(2)_n/\mathcal{C}_u(2)_{n+1} & \longrightarrow & \mathcal{C}_u(2)/\mathcal{C}_u(2)_{n+1} & \longrightarrow & \mathcal{C}_u(2)/\mathcal{C}_u(2)_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_n & \longrightarrow & \mathrm{Aut}_0(F/F_{n+2}) & \longrightarrow & \mathrm{Aut}_0(F/F_{n+1}) & \longrightarrow & 1. \end{array}$$

According to Lemma 5.3, we can identify K_n with $\mathrm{sDer}(\mathfrak{l}_2)_n$. One can check that with this identification, the map $\mathcal{C}_u(2)_n/\mathcal{C}_u(2)_{n+1} \to K_n$ coincides with the restriction of the Johnson homomorphism in degree n. Finally, $\mathcal{C}_u(2)_n/\mathcal{C}_u(2)_{n+1} \to K_n$ is an isomorphism, since we know from Lemmas 5.1 and Lemma 5.2 that the central and right map are isomorphisms.

Finally, by Proposition 2.12, we know that $\mathrm{sDer}(\mathfrak{l}_2)$ is not solvable, therefore $\mathcal{C}_u(2)$ is not a solvable group. ■

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