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On the volume conjecture for hyperbolic Dehn-filled 3-manifolds along the figure-eight knot

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Abstract. Using Ohtsuki's method, we prove the asymptotic expansion conjecture and the volume conjecture of the Reshetikhin–Turaev and the Turaev–Viro invariants for all hyperbolic 3-manifolds obtained by doing a Dehn-surgery along the figure-8 knot.

1. Introduction

In [40], Witten interpreted values of the Jones polynomial using the Chern–Simons gauge theory, and constructed a sequence of complex valued 3-manifold invariants satisfying striking properties. This idea was mathematically rigorously formalized by Reshetikhin and Turaev [32, 33] though the representation theory of quantum groups and surgery descriptions [18] of 3-manifolds. In [39], Turaev, and Viro developed a different approach from triangulations, constructing a sequence of real valued invariants of 3-manifolds. These two invariants turned out to be closely related [34, 38, 40], and are expected to contain geometric and topological information of the manifold.

Kashaev's volume conjecture [16, 17] (see also Murakami–Murakami [23]) fulfilled such expectation by relating the colored Jones polynomials of a knot to the hyperbolic geometry of its complement. More precisely, the volume conjecture asserts that value of the nth normalized colored Jones polynomial of a hyperbolic knot evaluated at the nth primitive root of unit $t = e^{\frac{2\pi\sqrt{-1}}{n}}$ grows exponentially in n, and the growth rate is proportional to the hyperbolic volume of the complement of the knot. Recently, Chen and the second author [4] conjectured, now known as the Chen–Yang volume conjecture, that for odd r the values at the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ of the rth Reshetikhin–Turaev and Turaev–Viro invariants of a hyperbolic 3-manifold grow exponentially in r, with growth rate, respectively, proportional to the complex volume and the hyperbolic volume of the manifold. This conjecture was later refined

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independently by Ohtsuki [27] and Gang–Romo–Yamazaki [12] to include the adjoint twisted Reidemeister torsion [31] of the manifold in the asymptotic expansion of the invariants.

In [1], Belletti, Detcherry, Kalfagianni, and the second author proved the Chen–Yang volume conjecture for the family of fundamental shadow link complements. The fundamental shadow link complements were shown [5] to form a universal class of 3-manifolds in the sense that any orientable 3-manifold with empty or toroidal boundary is obtained from the complement of a fundamental shadow link by doing a Dehn-surgery along suitable components. Therefore, understanding the asymptotic behavior of the invariants under Dehn-surgeries becomes a necessary step toward the solution to the Chen–Yang volume conjecture.

An earlier work of Ohtsuki [27] was a result along this direction, where he obtained the asymptotic expansion of the Reshetikhin–Turaev invariants of all hyperbolic 3-manifolds obtained by doing an integral Dehn-surgery along the figure-8 knot. Together with a sequence of his works [25, 26, 28–30], Ohtsuki developed a method of attacking Kashaev's and Chen–Yang's volume conjectures consisting of a circle of creative ideas including the use of Faddeev's quantum dilogarithm functions, the Poisson summation formula, and the saddle point approximation.

The main result of this article is our first attempt to understand the asymptotic behavior of the Reshetikhin–Turaev and the Turaev–Viro invariants under Dehn-surgeries, which generalizes Ohtsuki's result from integral Dehn-surgeries to rational Dehn-surgeries along the figure-8 knot. We note that our approach also works for the integral Dehn-surgeries, and is up to details the same as Ohtsuki's. A new idea in our approach is a use of the reciprocity of generalized Gaussian sum in the simplification of our formula for the Reshetikhin–Turaev invariants. Another new feature is that we clarify the geometric meaning of the critical values of certain involved functions relating them to the desired geometric quantities (see Section 5), which previously could only be done by numerical computations. The argument in Section 3 can be directly applied to rationally Dehn-filled 3-manifold along any other knot, and together with Section 5 provides a reinforcement of Ohtsuki's method.

Theorem 1.1. *Let M be a closed oriented hyperbolic 3-manifold obtained by doing a Dehn-surgery along the figure-8 knot, and let $\mathrm{RT}_r(M)$ be its rth Reshetikhin–Turaev invariant evaluated at the root $q = e^{\frac{2\pi\sqrt{-1}}{r}}$. Then, as r varies along positive odd integers,*

$$\mathrm{RT}_r(M) = \frac{C_r}{2}\frac{1}{\sqrt{\mathrm{Tor}(M;\mathrm{Ad}_\rho)}}e^{\frac{r}{4\pi}(\mathrm{Vol}(M)+\sqrt{-1}\,\mathrm{CS}(M))}\left(1+O\left(\frac{1}{r}\right)\right),$$

where C_r is a constant of norm 1 independent of the geometric structure of M, $\mathrm{Tor}(M;\mathrm{Ad}_\rho)$ is the adjoint twisted Reidemeister torsion, $\mathrm{Vol}(M)$ is the hyperbolic *volume and* $\mathrm{CS}(M)$ *is the Chern–Simons invariant of* M*. As a consequence,*

$$\lim_{r\to\infty} \frac{4\pi}{r} \log \mathrm{RT}_r(M) = \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M) \quad (\mathrm{mod}\ \sqrt{-1}\pi^2\mathbb{Z}).$$

It is proved in [34, 38, 40], and at the root $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ in [7], that for a closed oriented 3-manifold, the Turaev–Viro invariant is up to a scalar the square of the norm of the Reshetikhin–Turaev invariant. As a consequence, we have the following theorem.

Theorem 1.2. *Let* M *be a closed oriented hyperbolic* 3*-manifold obtained by doing a rational Dehn-surgery along the figure-*8 *knot, and let* $\mathrm{TV}_r(M)$ *be its* r*th Turaev–Viro invariant evaluated at the root* $q = e^{\frac{2\pi\sqrt{-1}}{r}}$*. Then, as* r *varies along positive odd integers,*

$$\mathrm{TV}_r(M) = \frac{2^{b_2(M)-b_0(M)}}{|\mathrm{Tor}(M;\mathrm{Ad}_\rho)|} e^{\frac{r}{2\pi}\mathrm{Vol}(M)}\Big(1 + O\Big(\frac{1}{r}\Big)\Big),$$

where $b_0(M)$ *and* $b_2(M)$ *are, respectively, the zeroth and the second* $\mathbb{Z}_2$ *Betti-number of* M*. As a consequence,*

$$\lim_{r\to\infty} \frac{2\pi}{r} \log \mathrm{TV}_r(M) = \mathrm{Vol}(M).$$

Outline of the proof. The proof follows the guideline of Ohtsuki's method. In Proposition 3.4, we compute the Reshetikhin–Turaev invariants of M and write them as a sum of values of a holomorphic function f_r at integral points; and a key ingredient in the computation is Lemma 3.6 that iteratively using a reciprocity of generalized Gaussian sums, we can simplify a multi-sum into a single sum. The function f_r comes from Faddeev's quantum dilogarithm function. Using Poisson summation formula in Proposition 4.3, we write the invariants as a sum of the Fourier coefficients of f_r. In Proposition 5.4, we show that the critical values of the functions in the two leading Fourier coefficients $\hat{f}_r(s^+, m^+, 0)$ and $\hat{f}_r(s^-, m^-, 0)$ for $s^\pm$ and $m^\pm$ defined in Lemma 3.3 coincide with the complex volume of M and the determinant of the Hessian matrix gives the adjoint twisted Reidemeister torsion of M. The key observation is Lemmas 5.1 and 5.2 that the system of critical point equations is equivalent to the system of hyperbolic gluing equations (consisting of an edge equation and a Dehn-surgery equation) for a particular ideal triangulation of the figure-8 knot complement. In Proposition 6.4, we verify the conditions for applying the saddle point approximation showing that the growth rates of the leading Fourier coefficients are those critical values, i.e., the complex volume; and in Section 6.2, we estimate the other Fourier coefficients. Finally, we complete the proof by showing in Proposition 6.10 that the two leading Fourier coefficient do not cancel each other and the sum of all the other Fourier coefficients is neglectable.

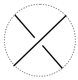
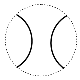
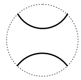
2. Preliminaries

2.1. Reshetikhin–Turaev invariants

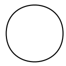
In this article, we will follow the skein theoretical approach of the Reshetikhin–Turaev invariants [2, 19] and focus on the case $q = e^{\frac{2\pi\sqrt{-1}}{r}}$, and equivalently $t = q^2 = e^{\frac{4\pi\sqrt{-1}}{r}}$, for odd integers $r \geqslant 3$.

A framed link in an oriented 3-manifold M is a smooth embedding L of a disjoint union of finitely many thickened circles $\mathrm{S}^1 \times [0, \varepsilon]$ for some $\varepsilon > 0$ into M. The Kauffman bracket skein module $\mathrm{K}_r(M)$ of M is the $\mathbb{C}$-module generated by the isotopy classes of framed links in M modulo the following two relations:

(1) *Kauffman bracket skein relation:*

 $= e^{\frac{\pi\sqrt{-1}}{r}}$  $+ e^{-\frac{\pi\sqrt{-1}}{r}}$ .

(2) *framing relation:*

$L \cup$  $= \Big(-e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}\Big)L.$

There is a canonical isomorphism

$$\langle\ \rangle : \mathrm{K}_r(\mathrm{S}^3) \to \mathbb{C}$$

defined by sending the empty link to 1. The image $\langle L\rangle$ of a framed link L is called the Kauffman bracket of L.

Let $\mathrm{K}_r(A \times [0, 1])$ be the skein module of the product of an annulus A with a closed interval. For any link diagram D in $\mathbb{R}^2$ with k ordered components and $b_1, \dots, b_k \in \mathrm{K}_r(A \times [0, 1])$, let

$$\langle b_1, \dots, b_k\rangle_D$$

be the complex number obtained by cabling $b_1, \dots, b_k$ along the components of D considered as an element of $K_r(\mathrm{S}^3)$ then taking the Kauffman bracket $\langle\ \rangle$.

On $\mathrm{K}_r(A \times [0, 1])$, there is a commutative multiplication induced by the juxtaposition of A, making it a $\mathbb{C}$-algebra; and as a $\mathbb{C}$-algebra, $\mathrm{K}_r(A \times [0, 1]) \cong \mathbb{C}[z]$, where z is the core curve of A. For an integer $n \geqslant 0$, let $e_n(z)$ be the nth Chebyshev polynomial defined by the recursive relations $e_0(z) = 1$, $e_1(z) = z$, and $e_n(z) = ze_{n-1}(z) - e_{n-2}(z)$. The Kirby coloring $\omega_r \in \mathrm{K}_r(A \times [0, 1])$ is then defined by

$$\omega_r = \sum_{n=0}^{r-2} (-1)^n [n+1] e_n,$$

where $[n]$ is the quantum integer defined by

$$[n] = \frac{e^{\frac{2n\pi\sqrt{-1}}{r}} - e^{-\frac{2n\pi\sqrt{-1}}{r}}}{e^{\frac{2\pi\sqrt{-1}}{r}} - e^{-\frac{2\pi\sqrt{-1}}{r}}}.$$

Suppose M is obtained from S^3 by doing a surgery along a framed link L, $D(L)$ is a standard diagram of L; i.e., the blackboard framing of $D(L)$ coincides with the framing of L, and $\sigma(L)$ is the signature of the linking matrix of L. Let U_+ be the diagram of the unknot with framing 1, and let

$$\mu_r = \frac{\sin\frac{2\pi}{r}}{\sqrt{r}}.$$

Then, the rth Reshetikhin–Turaev invariant of M is defined as

$$\mathrm{RT}_r(M) = \mu_r \langle \mu_r\omega_r, \dots, \mu_r\omega_r\rangle_{D(L)} \langle \mu_r\omega_r\rangle_{U_+}^{-\sigma(L)}. \tag{2.1}$$

2.2. Dilogarithm and Lobachevsky functions

Let $\log : \mathbb{C}\setminus(-\infty, 0] \to \mathbb{C}$ be the standard logarithm function defined by

$$\log z = \log|z| + \sqrt{-1}\arg z,$$

with $-\pi < \arg z < \pi$.

The dilogarithm function $\mathrm{Li}_2 : \mathbb{C}\setminus(1,\infty) \to \mathbb{C}$ is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\,du,$$

where the integral is along any path in $\mathbb{C}\setminus(1,\infty)$ connecting 0 and z, which is holomorphic in $\mathbb{C}\setminus[1,\infty)$ and continuous in $\mathbb{C}\setminus(1,\infty)$.

The dilogarithm function satisfies the following properties (see, e.g., Zagier [42]):

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\mathrm{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}(\log(-z))^2. \tag{2.2}$$

In the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$,

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \tag{2.3}$$

and on the unit circle $\{z = e^{2\sqrt{-1}\theta} \mid 0 \leqslant \theta \leqslant \pi\}$,

$$\mathrm{Li}_2(e^{2\sqrt{-1}\theta}) = \frac{\pi^2}{6} + \theta(\theta - \pi) + 2\sqrt{-1}\Lambda(\theta), \tag{2.4}$$

where $\Lambda : \mathbb{R} \to \mathbb{R}$ is the Lobachevsky function (see, e.g., Thurston's notes [37, Chapter 7]) defined by

$$\Lambda(\theta) = -\int_0^\theta \log|2\sin t|dt.$$

The Lobachevsky function is an odd function of period π. It achieves the absolute maximums at $k\pi + \frac{\pi}{6}$, $k \in \mathbb{Z}$, and the absolute minimums at $k\pi + \frac{5\pi}{6}$, $k \in \mathbb{Z}$. Moreover, it satisfies the functional equation

$$\frac{1}{2}\Lambda(2\theta) = \Lambda(\theta) + \Lambda\left(\theta + \frac{\pi}{2}\right).$$

2.3. Quantum dilogarithm functions

We will consider the following variant of Faddeev's quantum dilogarithm functions [9, 10]. All the results in this section are essentially due to Kashaev and some of them could also be found in [3]. For the readers' convenience, we also include a proof here.

Let $r \geqslant 3$ be an odd integer. Then, the following contour integral

$$\varphi_r(z) = \frac{4\pi\sqrt{-1}}{r}\int_\Omega \frac{e^{(2z-\pi)x}}{4x\sinh(\pi x)\sinh(\frac{2\pi x}{r})}dx \tag{2.5}$$

defines a holomorphic function on the domain

$$\left\{z \in \mathbb{C} \,\middle|\, -\frac{\pi}{r} < \operatorname{Re} z < \pi + \frac{\pi}{r}\right\},$$

where the contour is

$$\Omega = (-\infty, -\varepsilon] \cup \left\{z \in \mathbb{C} \mid |z| = \varepsilon, \operatorname{Im} z > 0\right\} \cup [\varepsilon, \infty)$$

for some $\varepsilon \in (0, 1)$. Note that the integrand has poles at $n\sqrt{-1}$, $n \in \mathbb{Z}$, and the choice of Ω is to avoid the pole at 0.

The function $\varphi_r(z)$ satisfies the following fundamental properties.

Lemma 2.1. *We have the following fundamental properties.*

(1) *For* $z \in \mathbb{C}$ *with* $0 < \operatorname{Re} z < \pi$,

$$1 - e^{2\sqrt{-1}z} = e^{\frac{r}{4\pi\sqrt{-1}}(\varphi_r(z-\frac{\pi}{r})-\varphi_r(z+\frac{\pi}{r}))}. \tag{2.6}$$

(2) *For* $z \in \mathbb{C}$ *with* $-\frac{\pi}{r} < \operatorname{Re} z < \frac{\pi}{r}$,

$$1 + e^{r\sqrt{-1}z} = e^{\frac{r}{4\pi\sqrt{-1}}(\varphi_r(z)-\varphi_r(z+\pi))}. \tag{2.7}$$

Proof. In the region enclosed by Ω in the upper half plane, the function $\frac{e^{(2z-\pi)x}}{2x\sinh(\pi x)}$ has simple poles at $x = n\sqrt{-1}$, $n \in \mathbb{Z}_+$. Hence, by the residue theorem,

$$\begin{aligned}
&\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(z-\frac{\pi}{r}\right)-\varphi_r\left(z+\frac{\pi}{r}\right)\right)\\
&\quad= -\int_\Omega \frac{e^{(2z-\pi)x}}{2x\sinh(\pi x)}dx\\
&\quad= -\sum_{n=1}^{\infty} 2\pi\sqrt{-1}\cdot \operatorname{Res}_{x=n\sqrt{-1}}\left(\frac{e^{(2z-\pi)x}}{2x\sinh(\pi x)}\right)\\
&\quad= -\sum_{n=1}^{\infty}\frac{(e^{2\sqrt{-1}z})^n}{n} = \log(1-e^{2\sqrt{-1}z}),
\end{aligned}$$

which proves (1).

In the same region, the function $\frac{e^{2zx}}{2x\sinh(\frac{2\pi x}{r})}$ has simple poles at $x = \frac{rn\sqrt{-1}}{2}$, $n \in \mathbb{Z}_+$. Hence, by the residue theorem, we have

$$\begin{aligned}
&\frac{r}{4\pi\sqrt{-1}}(\varphi_r(z)-\varphi_r(z+\pi))\\
&\quad= -\int_\Omega \frac{e^{2zx}}{2x\sinh(\frac{2\pi x}{r})}dx\\
&\quad= -\sum_{n=1}^{\infty} 2\pi\sqrt{-1}\cdot \operatorname{Res}_{x=\frac{rn\sqrt{-1}}{2}}\left(\frac{e^{2zx}}{2x\sinh(\frac{2\pi x}{r})}\right)\\
&\quad= -\sum_{n=1}^{\infty}\frac{(e^{r\sqrt{-1}z})^n}{(-1)^n n} = \log(1+e^{r\sqrt{-1}z}),
\end{aligned}$$

which proves (2). ∎

Using (2.6) and (2.7), for $z \in \mathbb{C}$ with $\pi + \frac{2(n-1)\pi}{r} < \operatorname{Re} z < \pi + \frac{2n\pi}{r}$, we can define $\varphi_r(z)$ inductively by the relation

$$\prod_{k=1}^{n}\left(1-e^{2\sqrt{-1}(z-\frac{(2k-1)\pi}{r})}\right) = e^{\frac{r}{4\pi\sqrt{-1}}(\varphi_r(z-\frac{2n\pi}{r})-\varphi_r(z))}, \tag{2.8}$$

extending $\varphi_r(z)$ to a meromorphic function on $\mathbb{C}$. The poles of $\varphi_r(z)$ have the form $(a+1)\pi + \frac{b\pi}{r}$ or $-a\pi - \frac{b\pi}{r}$ for all nonnegative integer a and positive odd integer b.

Let $t = e^{\frac{4\pi\sqrt{-1}}{r}}$, and let

$$(t)_n = \prod_{k=1}^{n}(1-t^k).$$

Lemma 2.2. *We have the following properties.*

(1) *For* $0 \leqslant n \leqslant r-1$,

$$(t)_n = e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{\pi}{r}\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}\right)\right)}. \tag{2.9}$$

(2) *For* $\frac{r-1}{2} \leqslant n \leqslant r-1$,

$$(t)_n = 2e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{\pi}{r}\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)\right)}. \tag{2.10}$$

Proof. Inductively, using (2.6), we have (1). To see (2), by (1), we have

$$(t)_n = e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{\pi}{r}\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)\right)} e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}\right)\right)}.$$

By analyticity, (2.7) holds for all z that is not a pole. In particular, it holds for $z = \frac{2\pi n}{r} + \frac{\pi}{r} - \pi$, and we have

$$e^{\frac{r}{4\pi\sqrt{-1}}\left(\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)-\varphi_r\left(\frac{2\pi n}{r}+\frac{\pi}{r}\right)\right)} = 1 + e^{r\sqrt{-1}\left(\frac{2\pi n}{r}+\frac{\pi}{r}-\pi\right)} = 2,$$

which proves (2). ∎

We consider (2.10) because there are poles in $(\pi, 2\pi)$, and we move everything into $(0, \pi)$ to avoid those poles.

The function $\varphi_r(z)$ and the dilogarithm function are closely related as follows.

Lemma 2.3. *We have the following properties.*

(1) *For every* z *with* $0 < \operatorname{Re} z < \pi$,

$$\varphi_r(z) = \operatorname{Li}_2\left(e^{2\sqrt{-1}z}\right) + \frac{2\pi^2 e^{2\sqrt{-1}z}}{3\left(1-e^{2\sqrt{-1}z}\right)} \frac{1}{r^2} + O\left(\frac{1}{r^4}\right). \tag{2.11}$$

(2) *For every* z *with* $0 < \operatorname{Re} z < \pi$,

$$\varphi_r'(z) = -2\sqrt{-1}\log\left(1-e^{2\sqrt{-1}z}\right) + O\left(\frac{1}{r^2}\right). \tag{2.12}$$

(3) *As* $r \to \infty$, $\varphi_r(z)$ *uniformly converges to* $\operatorname{Li}_2(e^{2\sqrt{-1}z})$ *and* $\varphi_r'(z)$ *uniformly converges to* $-2\sqrt{-1}\log(1-e^{2\sqrt{-1}z})$ *on a compact subset of* $\{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < \pi\}$.

Proof. For (1), since

$$\frac{1}{\sinh\left(\frac{2\pi x}{r}\right)} = \frac{r}{2\pi x} - \frac{\pi x}{3r} + O\left(\frac{1}{r^3}\right),$$

we have

$$\varphi_r(z)=\sqrt{-1}\int_\Omega \frac{e^{(2z-\pi)x}}{2x^2\sinh(\pi x)}dx-\frac{\pi^2\sqrt{-1}}{r^2}\int_\Omega \frac{e^{(2z-\pi)x}}{3\sinh(\pi x)}dx+O\Big(\frac{1}{r^4}\Big).$$

By the residue theorem, we have

$$\begin{aligned}\sqrt{-1}\int_\Omega \frac{e^{(2z-\pi)x}}{2x^2\sinh(\pi x)}dx&=-2\pi\sum_{n=1}^{\infty}\mathrm{Res}_{x=n\sqrt{-1}}\Big(\frac{e^{(2z-\pi)x}}{2x^2\sinh(\pi x)}\Big)\\&=\sum_{n=1}^{\infty}\frac{(e^{2\sqrt{-1}z})^n}{n^2}=\mathrm{Li}_2(e^{2\sqrt{-1}z}),\end{aligned}$$

where the last equality holds by (2.3) for z so that $e^{2\sqrt{-1}z}$ is in the unit disk, and it holds by analyticity for all z with $0<\mathrm{Re}\,z<\pi$.

By the residue theorem again, we have

$$\begin{aligned}-\frac{\pi^2\sqrt{-1}}{r^2}\int_\Omega \frac{e^{(2z-\pi)x}}{3\sinh(\pi x)}dx&=\frac{2\pi^3}{r^2}\sum_{n=1}^{\infty}\mathrm{Res}_{x=n\sqrt{-1}}\Big(\frac{e^{(2z-\pi)x}}{3\sinh(\pi x)}\Big)\\&=\frac{2\pi^2}{3r^2}\sum_{n=1}^{\infty}\big(e^{2\sqrt{-1}z}\big)^n=\frac{2\pi^2e^{2\sqrt{-1}z}}{3(1-e^{2\sqrt{-1}z})}\frac{1}{r^2}.\end{aligned}$$

This proves (2.11).

(2) follows from (1), and (3) follows from (1) and (2). ■

2.4. A geometric proposition

Proposition 2.4. *There is an* $\varepsilon>0$ *such that for any relatively prime pair* $(p,q)\neq(\pm 5,\pm 1)$ *so that the closed oriented* 3*-manifold* M *obtained by doing a* $\frac{p}{q}$ *Dehn-surgery along the figure-*8 *knot* K_{4_1} *is hyperbolic,*

$$\mathrm{Vol}(M)>\frac{1}{2}\mathrm{Vol}(S^3\setminus K_{4_1})+\varepsilon.$$

Proof. By Futer–Kalfagianni–Purcell [11, Theorem 1.1], if M is obtained from the complement of a hyperbolic knot K in S^3 by a Dehn-surgery along a boundary curve γ, then

$$\mathrm{Vol}(M)\geqslant\bigg(1-\Big(\frac{2\pi}{L(\gamma)}\Big)^2\bigg)^{\frac{3}{2}}\mathrm{Vol}(\mathrm{S}^3\setminus K),$$

where $L(\gamma)$ is the length of γ in the induced Euclidean metric on the boundary of the embedded horoball neighborhood of the cusp. For the K_{4_1} complement, the boundary of the maximum horoball neighborhood is a tiling of eight regular triangles of

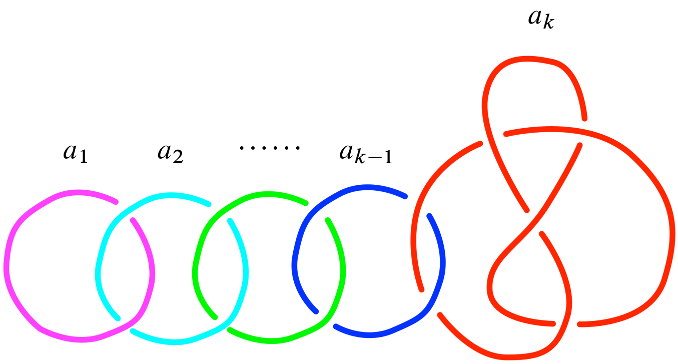


Figure 1. The link L.

side 1. Hence, as drawn in Figure 4 (see also Thurston's notes [37]), $L(x) = 4$ and $L(y) = 1$. As a consequence, the meridian $m = y$ and the longitude $l = x + 2y$ are perpendicular, $L(m) = 1$, $L(l) = 2\sqrt{3}$ and $L(pm + ql) = \sqrt{p^2 + 12q^2}$. As a consequence,

$$\mathrm{Vol}(M) \geqslant \left(1 - \frac{4\pi^2}{p^2 + 12q^2}\right)^{\frac{3}{2}} \mathrm{Vol}(S^3 \setminus K_{4_1}).$$

If $p^2 + 12q^2 > \frac{4\pi^2}{1-(\frac{1}{2})^{\frac{2}{3}}} \approx 106.67$, then $(1 - \frac{4\pi^2}{p^2+12q^2})^{\frac{3}{2}} > \frac{1}{2}$ and

$$\mathrm{Vol}(M) > \frac{1}{2} \mathrm{Vol}(S^3 \setminus K_{4_1}).$$

Therefore, by the symmetry of K_{4_1} complement, we only need to check for the pairs $(p, q) = (6, 1), (7, 1), (8, 1), (9, 1)$ $(1, 2), (3, 2), (5, 2)$, and $(7, 2)$, where $p^2 + 12q^2 < 107$, which could be numerically done by using SnapPy [8]. ■

Remark 2.5. The end of the proof of Proposition 2.4 is the only place in this article where we need a numerical computation.

3. Computation of the Reshetikhin–Turaev invariants

The main result of this section is Proposition 3.4 where we compute the Reshetikhin–Turaev invariants of the closed oriented 3-manifold obtained by doing a $\frac{p}{q}$ Dehn-surgery along the figure-8 knot K_{4_1}. Recall that if M is the 3-manifold obtained from S^3 by doing a $\frac{p}{q}$ Dehn-surgery along a knot K, then it can also be obtained by doing a surgery along a framed link L (see Figure 1) of k components with framings

$a_1, \dots, a_k$ coming from the continued fraction

$$\frac{p}{q} = a_k - \frac{1}{a_{k-1} - \frac{1}{\dots - \frac{1}{a_1}}}.$$

See, e.g., [35, page 273].

3.1. Continued fractions

We recall some notations related to the continued fraction of $\frac{p}{q}$, which will be used in the computation of the Reshetikhin–Turaev invariants. For a pair of relatively prime integers (p, q), let

$$\frac{p}{q} = a_k - \frac{1}{a_{k-1} - \frac{1}{\dots - \frac{1}{a_1}}}$$

be a continued fraction. For each $i \in \{1, \dots, k\}$, consider the matrix

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = T^{a_i} S \cdots T^{a_1} S,$$

where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

and as a convention, let

$$\begin{bmatrix} A_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.1}$$

Lemma 3.1. *We have the following properties.*

(1) *For* $i \in \{1, \dots, k\}$, $A_i = a_i A_{i-1} - C_{i-1}$ *and* $C_i = A_{i-1}$.

(2) *We have*

$$\frac{A_k}{C_k} = \frac{p}{q}.$$

Proof. (1) follows directly from induction. For (2), we show that

$$\frac{A_i}{C_i} = a_i - \frac{1}{a_{i-1} - \frac{1}{\dots - \frac{1}{a_1}}}$$

for each $i \in \{1, \dots, k\}$. For $i = 1$, we have

$$\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 1 \end{bmatrix}, \tag{3.2}$$

and $\frac{A_1}{C_1} = a_1$. Assume (2) holds for $i-1$. Then, by (1), we have

$$\frac{A_i}{C_i} = \frac{a_i A_{i-1} - C_{i-1}}{A_{i-1}} = a_i - \frac{C_{i-1}}{A_{i-1}} = a_i - \frac{1}{a_{i-1} - \frac{1}{\cdots - \frac{1}{a_1}}}. \qquad \blacksquare$$

Let us observe that A_k and C_k are relatively prime because $A_k D_k - B_k C_k = \det(T^{a_k} S \cdots T^{a_1} S) = 1$. Then, by Lemma 3.1 (2), $\left[\begin{smallmatrix} A_k \\ C_k \end{smallmatrix}\right] = \pm \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$. Since a (p,q) Dehn-surgery and a $(-p,-q)$ Dehn-surgery provide the same 3-manifold M, we may without loss of generality assume that

$$\begin{bmatrix} A_k \\ C_k \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{3.3}$$

As a consequence, we have

$$\begin{bmatrix} A_{k-1} \\ C_{k-1} \end{bmatrix} = \begin{bmatrix} q \\ -p + a_k q \end{bmatrix}. \tag{3.4}$$

We also let

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} D_k \\ -B_k \end{bmatrix} \tag{3.5}$$

so that $pp' + qq' = 1$.

For $i \in \{1, \dots, k\}$, we also consider the quantity

$$K_i = \frac{(-1)^{i+1} \sum_{j=1}^{i} a_j C_j}{C_i}. \tag{3.6}$$

The following Lemmas 3.2 and 3.3 are crucial in the computation of the Reshetikhin–Turaev invariants of M and the study of their asymptotics. The proofs are elementary, and the readers can skip them at the first time and come back later when needed.

Lemma 3.2. *$C_{k-1} K_{k-1} + C_{k-1} q$ is an even integer.*

Proof. By Lemma 3.1 and the definition of K_{k-1}, we have

$$\begin{aligned} K_{k-1} C_{k-1} &= (-1)^k \sum_{i=1}^{k-1} a_i C_i = (-1)^k \left(a_1 + a_2 A_1 + \sum_{i=3}^{k-1} a_i A_{i-1} \right) \\ &= (-1)^k \left(A_1 + (A_2 + C_1) + \sum_{i=3}^{k-1} (A_i + C_{i-1}) \right) \\ &= (-1)^k \left(A_1 + (A_2 + 1) + \sum_{i=3}^{k-1} (A_i + A_{i-2}) \right) \\ &= (-1)^k \left(1 + 2 \sum_{i=1}^{k-3} A_i + A_{k-1} + A_{k-2} \right) \equiv 1 + q + C_{k-1} \pmod 2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} K_{k-1}C_{k-1} + C_{k-1}q &\equiv 1 + q + C_{k-1} + C_{k-1}q \pmod 2 \\ &= (1+q)(1-p+a_k q). \end{aligned}$$

Now, if q is odd, then $1+q$ is even and the product is even. If q is even, then p must be odd since p and q are relatively prime. As a result, $1-p+a_k q$ is even and the product is even. ∎

Lemma 3.3. (1) *Let*

$$I : \{0, \dots, |q|-1\} \to \{0, \dots, 2|q|-1\}$$

be the map defined by

$$I(s) = -C_{k-1}(2s+1+K_{k-1}) \pmod{2|q|}.$$

Then, I is injective and its image is equal to the set of integers in $\{0, \dots, 2|q|-1\}$ that have the same parity as $1-q$.

In particular, there exist a unique $(s^+, m^+) \in \{0, \dots, |q|-1\} \times \mathbb{Z}$ such that

$$I(s^+) = 1 - q + 2m^+ q$$

and a unique $(s^-, m^-) \in \{0, \dots, |q|-1\} \times \mathbb{Z}$ such that

$$I(s^-) = -1 - q + 2m^- q.$$

Moreover,

$$s^+ - s^- \equiv p' \pmod q. \tag{3.8}$$

(2) *Let*

$$J : \{0, \dots, |q|-1\} \to \mathbb{Q}$$

be the map defined by

$$J(s) = \frac{2s+1}{q} + (-1)^k \sum_{i=1}^{k-1} \frac{(-1)^{i+1} K_i}{C_{i+1}}.$$

Then, for the s^+ and s^- in (1), *we have*

$$J(s^+) \equiv \frac{p'}{q} \pmod{\mathbb{Z}}$$

and

$$J(s^-) \equiv -\frac{p'}{q} \pmod{\mathbb{Z}}.$$

Moreover,

$$J(s^+) \equiv -J(s^-) \pmod{2\mathbb{Z}}.$$

(3) *Let*

$$K : \{0, \dots, |q|-1\} \to \mathbb{Q}$$

be the map defined by

$$K(s) = \frac{C_{k-1}(2s+1+K_{k-1})^2}{q} + \sum_{i=1}^{k-2} \frac{C_i K_i^2}{C_{i+1}}.$$

Then, for the s^+ and s^- in (1)*,*

$$K(s^+) \equiv -\frac{p'}{q} \pmod{\mathbb{Z}}$$

and

$$K(s^-) \equiv -\frac{p'}{q} \pmod{\mathbb{Z}}.$$

Proof. For (1), suppose otherwise that there exist distinct s and s' in $\{0, \dots, |q|-1\}$ such that $I(s) = I(s')$ modulo $2|q|$. Then, $2C_{k-1}(s-s') = 2hq$ for some integer h, and $\frac{C_{k-1}}{q} = \frac{h}{s-s'}$. Since $A_{k-1}D_{k-1} - B_{k-1}C_{k-1} = \det(T^{a_{k-1}}S \cdots T^{a_1}S) = 1$ and C_{k-1} and $q = A_{k-1}$ are relatively prime, $q \mid (s-s')$, which is a contradiction because $|s-s'| \leqslant |q|-1$. This proves that the map I is injective. To determine the image of I, for each $s \in \{0, \dots, |q|-1\}$, by (3.7), we have

$$\begin{aligned} I(s) &= -C_{k-1}(2s+1) - C_{k-1}K_{k-1} \\ &\equiv -C_{k-1}(2s+1) - 1 - q - C_{k-1} \pmod 2 \\ &= -2C_{k-1}(s+1) - 1 - q \\ &\equiv 1 - q \pmod 2. \end{aligned}$$

Since $\{0, \dots, 2|q|-1\}$ contains exactly $|q|$ even integers and $|q|$ odd integers and I is injective, the image of I consists of all integers with parity of $1-q$.

For (3.8), by computing $I(s^+) - I(s^-)$, we have

$$-2C_{k-1}(s^+ - s^-) = 2 + 2(m^+ - m^-)q,$$

which by the fact that $-C_{k-1} = p - a_k q$ implies

$$p(s^+ - s^-) - \big(a_k(s^+ - s^-) + (m^+ - m^-)\big)q = 1.$$

This completes the proof of (3.8).

For the first two identities of (2), it suffices to show that

$$pqJ(s^{\pm}) = \pm 1 \pmod q.$$

To this end, for each $i \in \{2, \ldots, k\}$, we consider the quantity

$$E_i = C_i \sum_{j=1}^{i-1} \frac{(-1)^{j+1} K_j}{C_{j+1}}.$$

Then, by that $p = a_k q - C_{k-1} = a_k C_k - C_{k-1}$, we have

$$\begin{aligned}
pqJ(s^{\pm}) &= (a_k q - C_{k-1})(2s^{\pm} + 1) + (-1)^k (a_k C_k - C_{k-1}) q \sum_{i=1}^{k-1} \frac{(-1)^{i+1} K_i}{C_{i+1}} \\
&= -C_{k-1}(2s^{\pm} + 1 + K_{k-1}) \\
&\quad + q\Bigg(a_k(2s^{\pm} + 1) + (-1)^k a_k \left(C_k \sum_{i=1}^{k-1} \frac{(-1)^{i+1} K_i}{C_{i+1}}\right) \\
&\quad - (-1)^k \left(C_{k-1} \sum_{i=1}^{k-2} \frac{(-1)^{i+1} K_i}{C_{i+1}}\right)\Bigg) \\
&= I(s^{\pm}) + q\big(a_k(2s^{\pm} + 1) + a_k(-1)^k E_k - (-1)^k E_{k-1}\big).
\end{aligned}$$

Since $I(s^{\pm}) \equiv \pm 1 \pmod q$ by (1), the result will follow if we can prove that both E_{k-1} and E_k are integers. For this, by a direct computation, we have that $E_2 = a_1$ and $E_3 = a_2 a_1 + a_2$ are integers. For $i \geqslant 4$, by Lemma 3.1, we have

$$C_i = A_{i-1} = a_{i-1} A_{i-2} - C_{i-2} = a_{i-1} C_{i-1} - C_{i-2}.$$

Then,

$$\begin{aligned}
E_i &= C_i \sum_{j=1}^{i-2} \frac{(-1)^{j+1} K_j}{C_{j+1}} + (-1)^i K_{i-1} \\
&= a_{i-1} C_{i-1} \sum_{j=1}^{i-2} \frac{(-1)^{j+1} K_j}{C_{j+1}} - C_{i-2} \sum_{j=1}^{i-2} \frac{(-1)^{j+1} K_j}{C_{j+1}} + (-1)^i K_{i-1} \\
&= a_{i-1} C_{i-1} \sum_{j=1}^{i-2} \frac{(-1)^{j+1} K_j}{C_{j+1}} - C_{i-2} \sum_{j=1}^{i-3} \frac{(-1)^{j+1} K_j}{C_{j+1}} \\
&\quad - C_{i-2} \frac{(-1)^{i-1} K_{i-2}}{C_{i-1}} + (-1)^i K_{i-1} \\
&= a_{i-1} E_{i-1} - E_{i-2} + a_{i-1},
\end{aligned}$$

where the last equality comes from that

$$\begin{aligned}
&- C_{i-2}\frac{(-1)^{i-1}K_{i-2}}{C_{i-1}} + (-1)^i K_{i-1} \\
&\quad = \frac{-(-1)^{i-1}C_{i-2}K_{i-2} + (-1)^i K_{i-1}C_{i-1}}{C_{i-1}} \\
&\quad = \frac{-\sum_{j=1}^{i-2} a_j C_j + \sum_{j=1}^{i-1} a_j C_j}{C_{i-1}} = \frac{a_{i-1}C_{i-1}}{C_{i-1}} = a_{i-1}.
\end{aligned}$$

By induction, all E_i's, in particular E_{k-1} and E_k, are integers; and the first two identities follow.

For the last identity of (2), by the first two identities, it suffices to show that

$$J(s^+) - J(s^-) \equiv \frac{2p'}{q} \pmod{2\mathbb{Z}}.$$

To see this, by (3.8), we have $s^+ - s^- = p' + nq$ for some integer n. Then, by the definition of J,

$$J(s^+) - J(s^-) = \frac{2(s^+ - s^-)}{q} = \frac{2p'}{q} + 2n,$$

which completes the proof.

For (3), by the definition of K_i and E_i, we first compute

$$\begin{aligned}
\sum_{i=1}^{k-1} \frac{C_i K_i^2}{C_{i+1}} &= \sum_{i=1}^{k-1}\sum_{j=1}^{i}(-1)^{i+1}\frac{a_j C_j K_i}{C_{i+1}} \\
&= \sum_{j=1}^{k-1}\sum_{i=j}^{k-1}(-1)^{i+1}\frac{a_j C_j K_i}{C_{i+1}} \\
&= \sum_{j=1}^{k-1}\sum_{i=1}^{k-1}(-1)^{i+1}\frac{a_j C_j K_i}{C_{i+1}} - \sum_{j=2}^{k-1}\sum_{i=1}^{j-1}(-1)^{i+1}\frac{a_j C_j K_i}{C_{i+1}} \\
&= \left(\sum_{j=1}^{k-1} a_j C_j\right)\left(\sum_{i=1}^{k-1}\frac{(-1)^{i+1}K_i}{C_{i+1}}\right) - \sum_{j=2}^{k-1} a_j\left(C_j\sum_{i=1}^{j-1}\frac{(-1)^{i+1}K_i}{C_{i+1}}\right) \\
&= (-1)^k C_{k-1}K_{k-1}\sum_{i=1}^{k-1}\frac{(-1)^{i+1}K_i}{C_{i+1}} \\
&\quad - \sum_{j=2}^{k-1} a_j E_j \equiv (-1)^k C_{k-1}K_{k-1}\sum_{i=1}^{k-1}\frac{(-1)^{i+1}K_i}{C_{i+1}} \pmod{\mathbb{Z}},
\end{aligned}$$

where the last equality uses that E_i's are integers from the proof of (3). Then, by that $C_k = q$,

$$\begin{aligned}
K(s^{\pm}) &= \frac{C_{k-1}((2s^{\pm}+1)^2 + 2(2s^{\pm}+1)K_{k-1})}{q} + \sum_{i=1}^{k-1} \frac{C_i K_i^2}{C_{i+1}} \\
&\equiv \frac{C_{k-1}((2s^{\pm}+1)^2 + 2(2s^{\pm}+1)K_{k-1})}{q} \\
&\quad + (-1)^k C_{k-1} K_{k-1} \sum_{i=1}^{k-1} \frac{(-1)^{i+1} K_i}{C_{i+1}} \pmod{\mathbb{Z}} \\
&= \frac{2s^{\pm}+1}{q} C_{k-1}(2s^{\pm}+1+K_{k-1}) \\
&\quad + C_{k-1}K_{k-1}\left(\frac{2s^{\pm}+1}{q} + (-1)^k \sum_{i=1}^{k-1} \frac{(-1)^{i+1}K_i}{C_{i+1}}\right) \\
&= -\frac{2s^{\pm}+1}{q} I(s^{\pm}) + C_{k-1}K_{k-1}J(s^{\pm}) \\
&\equiv \mp\frac{2s^{\pm}+1}{q} \pm \frac{C_{k-1}K_{k-1}p'}{q} \pmod{\mathbb{Z}},
\end{aligned}$$

where the last equality comes from (1) and (2). To prove the result, it suffices to show that

$$pq\left(\mp\frac{2s^{\pm}+1}{q} \pm \frac{C_{k-1}K_{k-1}p'}{q}\right) \equiv -1 \pmod{q}.$$

To this end, since $pp' + qq' = 1$ and $p = a_k q - C_{k-1}$, we have

$$\begin{aligned}
pq\left(-\frac{2s^{+}+1}{q} + \frac{C_{k-1}K_{k-1}p'}{q}\right) &= -p(2s^{+}+1) + C_{k-1}K_{k-1}pp' \\
&\equiv -p(2s^{+}+1) + C_{k-1}K_{k-1} \pmod{q} \\
&= -(a_k q - C_{k-1})(2s^{+}+1) + C_{k-1}K_{k-1} \\
&= -a_k q(2s^{+}+1) - I(s^{+}) \equiv -1 \pmod{q},
\end{aligned}$$

where the last equality comes from (2); and

$$\begin{aligned}
pq\left(\frac{2s^{-}+1}{q} - \frac{C_{k-1}K_{k-1}p'}{q}\right) &= p(2s^{-}+1) - C_{k-1}K_{k-1}pp' \\
&\equiv p(2s^{-}+1) - C_{k-1}K_{k-1} \pmod{q} \\
&= (a_k q - C_{k-1})(2s^{-}+1) - C_{k-1}K_{k-1} \\
&= a_k q(2s^{-}+1) + I(s^{-}) \equiv -1 \pmod{q},
\end{aligned}$$

where the last equality comes from (2). This completes the proof. ∎

3.2. The computation

Proposition 3.4. *For an odd integer $r \geqslant 3$ and at the root of unity $t = e^{\frac{4\pi\sqrt{-1}}{r}}$, the rth Reshetikin–Turaev invariant of the closed oriented 3-manifold M obtained by doing the $\frac{p}{q}$ Dehn-surgery along the figure-8 knot can be computed as*

$$\mathrm{RT}_r(M) = \kappa_r \sum_{s=0}^{|q|-1} \sum_{m=-\frac{r-2}{2}}^{\frac{r-2}{2}} \sum_{n=\max\{-m,m\}}^{\frac{r-2}{2}} g_r(s,m,n),$$

where

$$\kappa_r = \frac{(-1)^{\frac{3(k+1)}{4}+\sum_{i=1}^{k} a_i} e^{\frac{\pi\sqrt{-1}}{r}(3\sigma(L)-\sum_{i=1}^{k} a_i - \sum_{i=2}^{k}\frac{1}{C_{i-1}C_i}) + \frac{\pi\sqrt{-1}r}{4}(\sigma(L)+3a_k)}}{2r\sqrt{q}},$$

with $\sigma(L)$ the signature of the linking matrix of the link L in Figure 1, C_i and K_i as defined in Section 3.1, the first summation over integers s in between 0 and $|q|-1$, the second summation over half-integers m in between $-\frac{r-2}{2}$ and $\frac{r-2}{2}$ and the third summation over half-integers n in between $\max\{-m,m\}$ and $\frac{r-2}{2}$. For $s \in \{1,\dots,|q|-1\}$, let $I(s)$, $J(s)$ and $K(s)$ be as defined in Lemma 3.3. Then,

$$g_r(s,m,n) = \sin\left(\frac{2\pi m}{r}\frac{1}{q} - J(s)\pi\right)\varepsilon\left(\frac{2\pi m}{r}, \frac{2\pi n}{r}\right) e^{-\frac{2\pi m\sqrt{-1}}{r} + \frac{r}{4\pi\sqrt{-1}}V_r(s,\frac{2\pi m}{r},\frac{2\pi n}{r})},$$

where the functions $\varepsilon(x,y)$ and $V_r(s,x,y)$ are defined as follows: let φ_r be the quantum dilogarithm function as defined by (2.5).

(1) *If both $0 < y \pm x < \pi$, then $\varepsilon(x,y) = 2$ and*

$$V_r(s,x,y) = -\frac{px^2}{q} + I(s)\frac{2\pi x}{q} + 4xy - \varphi_r\left(\pi - y - x - \frac{\pi}{r}\right) + \varphi_r\left(y - x + \frac{\pi}{r}\right) + K(s)\pi^2.$$

(2) *If $0 < y + x < \pi$ and $\pi < y - x < 2\pi$, then $\varepsilon(x,y) = 1$ and*

$$V_r(s,x,y) = -\frac{px^2}{q} + I(s)\frac{2\pi x}{q} + 4xy - \varphi_r\left(\pi - y - x - \frac{\pi}{r}\right) + \varphi_r\left(y - x + \pi - \frac{\pi}{r}\right) + K(s)\pi^2.$$

(3) *If* $\pi < y + x < 2\pi$ *and* $0 < y - x < \pi$, *then* $\varepsilon(x, y) = 1$ *and*

$$V_r(s, x, y) = -\frac{px^2}{q} + I(s)\frac{2\pi x}{q} + 4xy - \varphi_r\left(2\pi - y - x - \frac{\pi}{r}\right) + \varphi_r\left(y - x + \frac{\pi}{r}\right) + K(s)\pi^2.$$

Remark 3.5. Here, by half-integers we mean rational numbers of the form $n + \frac{1}{2}$, $n \in \mathbb{Z}$.

Proof of Proposition 3.4. A direct computation shows that

$$\langle \mu_r \omega_r \rangle_{U_+} = e^{(-\frac{3}{r} - \frac{r+1}{4})\pi\sqrt{-1}}.$$

Let

$$\kappa'_r = \mu_r^{k+1} \langle \mu_r \omega_r \rangle_{U_+}^{-\sigma(L)} = \left(\frac{\sin\frac{2\pi}{r}}{\sqrt{r}}\right)^{k+1} e^{-\sigma(L)(-\frac{3}{r} - \frac{r+1}{4})\pi\sqrt{-1}}.$$

Then, by (2.1), we have

$$\begin{aligned}\mathrm{RT}_r(M) &= \kappa'_r \langle \omega_r, \dots, \omega_r \rangle_{D(L)} \\ &= \kappa'_r \sum_{m_1,\dots,m_k=0}^{r-2} (-1)^{m_k + \sum_{i=1}^{k} a_i m_i} t^{\sum_{i=1}^{k} \frac{a_i m_i (m_i+2)}{4}} [m_1 + 1] \\ &\quad \times \prod_{i=1}^{k-1} [(m_i + 1)(m_{i+1} + 1)] \langle e_{m_k} \rangle_{D(K_{4_1})},\end{aligned}$$

where the second equality comes from the fact that e_m is an eigenvector of the positive and the negative twist operator of eigenvalue $(-1)^m t^{\pm\frac{m(m+2)}{4}}$, and is also an eigenvector of the circle operator $c(e_n)$ (defined by enclosing e_m by e_n) of eigenvalue $(-1)^n \frac{[(m+1)(n+1)]}{[m+1]}$. By Habiro's formula [14] (see also [22] for a skein theoretical computation),

$$\begin{aligned}\langle e_m \rangle_{D(K_{4_1})} &= (-1)^m [m + 1] J'_{m+1}(K_{4_1}) \\ &= \frac{(-1)^m}{\{1\}} \sum_{n=0}^{\min\{m, r-2-m\}} \frac{\{m + 1 + n\}!}{\{m - n\}!} \\ &= \frac{(-1)^{m+1}}{\{1\}} \sum_{n=0}^{\min\{m, r-2-m\}} t^{-(m+1)(n+\frac{1}{2})} \frac{(t)_{m+1+n}}{(t)_{m-n}},\end{aligned}$$

where $J'_m(K)$ is the mth normalized colored Jones polynomial so that $J'_m(\text{unknot}) = 1$. Here, $\{m\} = t^{\frac{m}{2}} - t^{-\frac{m}{2}}$, $\{m\}! = \prod_{k=1}^{m} \{k\}$ and $(t)_m = \prod_{k=1}^{m} (1 - t^k)$.

Then,

$$
\begin{aligned}
&\mathrm{RT}_r(M)\\
&= -\frac{\kappa_r'}{\{1\}} \sum_{m_1,\dots,m_k=0}^{r-2} \sum_{n=0}^{\min\{m_k, r-2-m_k\}} (-1)^{\sum_{i=1}^k a_i m_i} t^{\sum_{i=1}^k \frac{a_i m_i (m_i+2)}{4} - (m_k+1)(n+\frac{1}{2})}\\
&\quad \times [m_1+1] \prod_{i=1}^{k-1} [(m_i+1)(m_{i+1}+1)] \frac{(t)_{m_k+1+n}}{(t)_{m_k-n}}\\
&= -\frac{\kappa_r'}{\{1\}} \sum_{m_1,\dots,m_{k-1}=0}^{r-1} \sum_{m_k=1}^{r-1} \sum_{n=0}^{\min\{m_k-1, r-1-m_k\}} (-1)^{\sum_{i=1}^k a_i (m_i-1)} t^{\sum_{i=1}^k \frac{a_i (m_i^2-1)}{4} - m_k(n+\frac{1}{2})}\\
&\quad \times [m_1] \prod_{i=1}^{k-1} [m_i m_{i+1}] \frac{(t)_{m_k+n}}{(t)_{m_k-n-1}},
\end{aligned}
$$

where the last equality is obtained by changing the variables m_i to $m_i - 1$ for $i \in \{1, \dots, k\}$ and the fact that $[0] = 0$. By reordering the summations, we have

$$
\begin{aligned}
\mathrm{RT}_r(M) &= -\frac{\kappa_r'}{\{1\}}\\
&\times \sum_{m_k=1}^{r-1} \sum_{n=0}^{\min\{m_k-1, r-1-m_k\}} \Bigg(\sum_{m_1,\dots,m_{k-1}=0}^{r-1} (-1)^{\sum_{i=1}^{k-1} a_i (m_i-1)} t^{\sum_{i=1}^{k-1} \frac{a_i (m_i^2-1)}{4}}\\
&\qquad\qquad \times [m_1] \prod_{i=1}^{k-1} [m_i m_{i+1}] \Bigg)\\
&\times (-1)^{a_k (m_k-1)} t^{\frac{a_k (m_k^2-1)}{4} - m_k(n+\frac{1}{2})} \frac{(t)_{m_k+n}}{(t)_{m_k-n-1}}\\
&= \kappa_r'' \sum_{m_k=1}^{r-1} \sum_{n=0}^{\min\{m_k-1, r-1-m_k\}} S(m_k) (-1)^{a_k m_k} t^{\frac{a_k m_k^2}{4} - m_k(n+\frac{1}{2})} \frac{(t)_{m_k+n}}{(t)_{m_k-n-1}},
\end{aligned}
$$

where

$$
\kappa_r'' = -\frac{(-1)^{\sum_{i=1}^k a_i} t^{-\sum_{i=1}^k \frac{a_i}{4}}}{\{1\}^{k+1}} \kappa_r'
$$

is a constant independent of $m_1, \dots, m_k$, and

$$
\begin{aligned}
S(m_k) = &\sum_{m_1,\dots,m_{k-1}=0}^{r-1} (-1)^{\sum_{i=1}^{k-1} a_i m_i} t^{\sum_{i=1}^{k-1} \frac{a_i m_i^2}{4}} \left(t^{\frac{m_1}{2}} - t^{-\frac{m_1}{2}} \right)\\
&\times \prod_{i=1}^{k-1} \left(t^{\frac{m_i m_{i+1}}{2}} - t^{-\frac{m_i m_{i+1}}{2}} \right)
\end{aligned}
$$

is a quantity depending only on m_k. By Lemma 3.6 in Section 3.3, the multi-sum $S(m_k)$ can be computed as the following single sum:

$$S(m_k) = \tau^+ \sum_{s=0}^{2|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}+\frac{(-1)^k}{C_{k-1}})^2} - \tau^- \sum_{s=0}^{2|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}-\frac{(-1)^k}{C_{k-1}})^2},$$

where

$$\tau^\pm = \frac{(-1)^{\frac{k-1}{4}}}{\sqrt{q}} 2^{k-2} r^{\frac{k-1}{2}} e^{-\frac{\pi\sqrt{-1}}{r}\sum_{i=1}^{k-2}\frac{1}{C_iC_{i+1}} - \frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_iK_i^2}{C_{i+1}} \mp \pi\sqrt{-1}\sum_{i=1}^{k-2}\frac{(-1)^{i+1}K_i}{C_{i+1}}}$$

are constants independent of m_k.

Then, we observe that, for each $s \in \{0, \ldots, |q|-1\}$,

$$e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+(s+q)r+\frac{K_{k-1}r}{2}\pm\frac{(-1)^k}{C_{k-1}})^2} = e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}\pm\frac{(-1)^k}{C_{k-1}})^2}.$$

Indeed, since all of C_{k-1}, m_k, s, q, and r are integers, a direct computation shows that

$$\frac{e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+(s+q)r+\frac{K_{k-1}r}{2}\pm\frac{(-1)^k}{C_{k-1}})^2}}{e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}\pm\frac{(-1)^k}{C_{k-1}})^2}} = e^{-\pi\sqrt{-1}r(K_{k-1}C_{k-1}+C_{k-1}q)} = 1,$$

where the last equality comes from Lemma 3.2 that $K_{k-1}C_{k-1} + C_{k-1}q$ is an even integer.

As a consequence, we have

$$\begin{aligned}
S(m_k) &= 2\tau^+ \sum_{s=0}^{|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}+\frac{(-1)^k}{C_{k-1}})^2} - 2\tau^- \sum_{s=0}^{|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2}-\frac{(-1)^k}{C_{k-1}})^2}\\
&= 2\tau\Big(\sum_{s=0}^{|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}((m_k+sr+\frac{K_{k-1}r}{2})^2+\frac{1}{C_{k-1}^2})-\pi\sqrt{-1}(\frac{(-1)^k}{rq}(2m_k+2sr+K_{k-1}r)+\sum_{i=1}^{k-2}\frac{(-1)^{i+1}K_i}{C_{i+1}})}\\
&\qquad - \sum_{s=0}^{|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}((m_k+sr+\frac{K_{k-1}r}{2})^2+\frac{1}{C_{k-1}^2})+\pi\sqrt{-1}(\frac{(-1)^k}{rq}(2m_k+2sr+K_{k-1}r)+\sum_{i=1}^{k-2}\frac{(-1)^{i+1}K_i}{C_{i+1}})}\Big)\\
&= \tau' \sum_{s=0}^{|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2})^2} \sin\Big(-\pi\Big(\frac{(-1)^k}{rq}(2m_k+2sr) + \sum_{i=1}^{k-1}\frac{(-1)^{i+1}K_i}{C_{i+1}}\Big)\Big),
\end{aligned}$$

where in the computation of the quantity in the sine we use that $C_k = q$,

$$\tau = \frac{(-1)^{\frac{k-1}{4}}}{\sqrt{q}} 2^{k-2} r^{\frac{k-1}{2}} e^{-\frac{\pi\sqrt{-1}}{r}\sum_{i=1}^{k-2}\frac{1}{C_iC_{i+1}} - \frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_iK_i^2}{C_{i+1}}}$$

and

$$\tau' = 4\sqrt{-1}\tau e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}\frac{1}{C_{k-1}^2}}$$
$$= \frac{(-1)^{\frac{k+1}{4}}}{\sqrt{q}} 2^k r^{\frac{k-1}{2}} e^{-\frac{\pi\sqrt{-1}}{r}\sum_{i=1}^{k-1}\frac{1}{C_i C_{i+1}} - \frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_i K_i^2}{C_{i+1}}}.$$

As a consequence,

$$\mathrm{RT}_r(M) = \kappa_r''\tau' \sum_{s=0}^{|q|-1}\sum_{m_k=1}^{r-1}\sum_{n=0}^{\min\{m_k-1,r-1-m_k\}} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k+sr+\frac{K_{k-1}r}{2})^2}$$
$$\times \sin\left(-\pi\left(\frac{(-1)^k}{rq}(2m_k+2sr) + \sum_{i=1}^{k-1}\frac{(-1)^{i+1}K_i}{C_{i+1}}\right)\right)$$
$$\times (-1)^{a_k m_k} t^{\frac{a_k m_k^2}{4} - m_k(n+\frac{1}{2})} \frac{(t)_{m_k+n}}{(t)_{m_k-n-1}}.$$

Let

$$m' = \frac{r}{2} - m_k, \quad n' = \frac{r-2}{2} - n$$

and

$$\kappa_r = (-1)^{\frac{3a_k r}{4}+k} e^{\frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_i K_i^2}{C_{i+1}}}\kappa_r''\tau',$$

we have

$$\mathrm{RT}_r(M)$$
$$= \kappa_r \sum_{s=0}^{|q|-1}\sum_{m'=-\frac{r-2}{2}}^{\frac{r-2}{2}}\sum_{n'=\max\{-m',m'\}}^{\frac{r-2}{2}} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(-m'+\frac{(2s+1+K_{k-1})r}{2})^2 - \frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_i K_i^2}{C_{i+1}}}$$
$$\times \sin\left(\frac{2\pi m'}{r}\frac{1}{q} - J(s)\pi\right) t^{\frac{a_k m'^2}{4} - m'(n'+\frac{1}{2})}\frac{(t)_{r-m'-n'-1}}{(t)_{n'-m'}}. \tag{3.9}$$

By a direct computation and that $C_k = q$, we have

$$\kappa_r =$$
$$\frac{(-1)^{\frac{3(k+1)}{4}+\sum_{i=1}^k a_i} e^{\frac{\pi\sqrt{-1}}{r}(3\sigma(L)-\sum_{i=1}^k a_i - \sum_{i=2}^k\frac{1}{C_{i-1}C_i}) + \frac{\pi\sqrt{-1}r}{4}(\sigma(L)+3a_k) + \frac{\pi\sqrt{-1}}{4}\sigma(L)}}{2r\sqrt{q}},$$

$$e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(-m'+\frac{(2s+1+K_{k-1})r}{2})^2 - \frac{\pi\sqrt{-1}r}{4}\sum_{i=1}^{k-2}\frac{C_i K_i^2}{C_{i+1}}}$$
$$= e^{\frac{r}{4\pi\sqrt{-1}}(\frac{C_{k-1}}{q}(\frac{2\pi m'}{r})^2 + \frac{2\pi I(s)}{q}(\frac{2\pi m'}{r}) + K(s)\pi^2)}$$

and

$$t^{\frac{a_k m'^2}{4} - m'(n'+\frac{1}{2})} = e^{-\frac{2\pi m'\sqrt{-1}}{r} + \frac{r}{4\pi\sqrt{-1}}(-a_k(\frac{2\pi m'}{r})^2 + 4(\frac{2\pi m'}{r})(\frac{2\pi n'}{r}))};$$

and by Lemma 2.2, we have the following:

(1) if $0 \leqslant n' \pm m' \leqslant \frac{r-1}{2}$, then

$$\frac{(t)_{r-m'-n'-1}}{(t)_{n'-m'}} = 2e^{\frac{4\pi\sqrt{-1}}{r}\left(-\varphi_r\left(\pi-\frac{2\pi n'}{r}-\frac{2\pi m'}{r}-\frac{\pi}{r}\right)+\varphi_r\left(\frac{2\pi n'}{r}-\frac{2\pi m'}{r}+\frac{\pi}{r}\right)\right)},$$

(2) if $0 \leqslant n' + m' \leqslant \frac{r-1}{2}$ and $\frac{r-1}{2} \leqslant n' - m' \leqslant r-1$, then

$$\frac{(t)_{r-m'-n'-1}}{(t)_{n'-m'}} = e^{\frac{4\pi\sqrt{-1}}{r}\left(-\varphi_r\left(\pi-\frac{2\pi n'}{r}-\frac{2\pi m'}{r}-\frac{\pi}{r}\right)+\varphi_r\left(\frac{2\pi n'}{r}-\frac{2\pi m'}{r}-\pi+\frac{\pi}{r}\right)\right)},$$

(3) if $\frac{r-1}{2} \leqslant n' + m' \leqslant r-1$ and $0 \leqslant n' - m' \leqslant \frac{r-1}{2}$, then

$$\frac{(t)_{r-m'-n'-1}}{(t)_{n'-m'}} = e^{\frac{4\pi\sqrt{-1}}{r}\left(-\varphi_r\left(2\pi-\frac{2\pi n'}{r}-\frac{2\pi m'}{r}-\frac{\pi}{r}\right)+\varphi_r\left(\frac{2\pi n'}{r}-\frac{2\pi m'}{r}+\frac{\pi}{r}\right)\right)}.$$

Putting all these together and using the fact that

$$C_{k-1} = -p + a_k q,$$

we complete the proof. ∎

3.3. Lemma 3.6

Lemma 3.6. *For each $i \in \{1,\dots,k\}$ and two non-zero integers m_0 and m_{i+1}, let*

$$S_i(m_0, m_{i+1}) = \sum_{m_1,\dots,m_i=0}^{r-1} (-1)^{\sum_{j=1}^i a_j m_j} t^{\sum_{j=1}^i \frac{a_j m_j^2}{4}} \prod_{j=0}^{i} \left(t^{\frac{m_j m_{j+1}}{2}} - t^{-\frac{m_j m_{j+1}}{2}}\right).$$

Let A_i, C_i, and K_i be the quantities introduced in Section 3.1. Then,

$$S_i(m_0, m_{i+1}) = \tau_i^+ \sum_{s=0}^{2|A_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_i}{A_i}\left(m_{i+1}+sr+\frac{K_i r}{2}+\frac{(-1)^{i+1}m_0}{C_i}\right)^2} - \tau_i^- \sum_{s=0}^{2|A_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_i}{A_i}\left(m_{i+1}+sr+\frac{K_i r}{2}-\frac{(-1)^{i+1}m_0}{C_i}\right)^2},$$

where

$$\tau_i^\pm = \frac{(-1)^{\frac{i}{4}}}{\sqrt{A_i}} 2^{i-1} r^{\frac{i}{2}} e^{-\frac{\pi\sqrt{-1}}{r} m_0^2 \sum_{j=1}^{i-1}\frac{1}{C_j C_{j+1}} - \frac{\pi\sqrt{-1}r}{4}\sum_{j=1}^{i-1}\frac{C_j K_j^2}{C_{j+1}} \mp \pi\sqrt{-1} m_0 \sum_{j=1}^{i-1}\frac{(-1)^{j+1}K_j}{C_{j+1}}}.$$

In particular, the quantity $S(m_k)$ *in the proof of Proposition* 3.4 *equals* $S_{k-1}(1, m_k)$, *and by* $A_{k-1} = q$, *we have*

$$\begin{aligned} S(m_k) = {} & \tau_{k-1}^{+} \sum_{s=0}^{2|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k + sr + \frac{K_{k-1} r}{2} + \frac{(-1)^k}{C_{k-1}})^2} \\ & - \tau_{k-1}^{-} \sum_{s=0}^{2|q|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{k-1}}{q}(m_k + sr + \frac{K_{k-1} r}{2} - \frac{(-1)^k}{C_{k-1}})^2}. \end{aligned}$$

The proof of Lemma 3.6 relies on the following reciprocity of generalized Gaussian sums.

Proposition 3.7 ([15, Proposition 2.3]). *For* $m, n \in \mathbb{Z}$, *if* mn *is even and* $n\psi \in \mathbb{Z}$, *then*

$$\sum_{\lambda=0}^{|n|-1} e^{\frac{m\lambda^2\pi\sqrt{-1}}{n}} e^{2\psi\lambda\pi\sqrt{-1}} = \left(\frac{\sqrt{-1}n}{m}\right)^{\frac{1}{2}} \sum_{s=0}^{|m|-1} e^{-\frac{n(s+\psi)^2\pi\sqrt{-1}}{m}}.$$

Proof of Lemma 3.6. We observe that, for any integer a, the quantities $(-1)^{am} t^{\frac{am^2}{4}}$ and $t^{\frac{am}{2}}$ are periodic in m with period r. As a consequence, we have

$$S_i(m_0, m_{i+1} + r) = S_i(m_0, m_{i+1}), \quad S_i(m_0, -m_{i+1}) = -S_i(m_0, m_{i+1}). \tag{3.10}$$

Now, we use induction. For $i = 1$ and non-zero integers m_0 and m_2, we have

$$\begin{aligned} S_1(m_0, m_2) &= \sum_{m_1=0}^{r-1} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}})(t^{\frac{m_1 m_2}{2}} - t^{-\frac{m_1 m_2}{2}}) \\ &= \sum_{m_1=0}^{r-1} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}}) t^{\frac{m_1 m_2}{2}} \\ &\quad - \sum_{m_1=0}^{r-1} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}}) t^{-\frac{m_1 m_2}{2}}. \end{aligned}$$

For the second sum, we have

$$\begin{aligned} & - \sum_{m_1=0}^{r-1} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}}) t^{-\frac{m_1 m_2}{2}} \\ &= \sum_{m_1=0}^{r-1} (-1)^{a_1 (-m_1)} t^{\frac{a_1 (-m_1)^2}{4}} (t^{\frac{m_0(-m_1)}{2}} - t^{-\frac{m_0(-m_1)}{2}}) t^{\frac{(-m_1) m_2}{2}} \\ &= \sum_{m_1=-r+1}^{0} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}}) t^{\frac{m_1 m_2}{2}} \\ &= \sum_{m_1=r}^{2r-1} (-1)^{a_1 m_1} t^{\frac{a_1 m_1^2}{4}} (t^{\frac{m_0 m_1}{2}} - t^{-\frac{m_0 m_1}{2}}) t^{\frac{m_1 m_2}{2}}, \end{aligned}$$

where the last equality comes from the periodicity of the summands in m_1 and that both $\{-r+1,\dots,0\}$ and $\{r,\dots,2r-1\}$ are a full period. Therefore,

$$\begin{aligned}
S_1(m_0,m_2) &= \sum_{m_1=0}^{2r-1}(-1)^{a_1m_1}t^{\frac{a_1m_1^2}{4}}(t^{\frac{m_0m_1}{2}}-t^{-\frac{m_0m_1}{2}})t^{\frac{m_1m_2}{2}}\\
&= \sum_{m_1=0}^{2r-1}(-1)^{a_1m_1}t^{\frac{a_1m_1^2}{4}}t^{\frac{m_0m_1}{2}}t^{\frac{m_1m_2}{2}}\\
&\quad -\sum_{m_1=0}^{2r-1}(-1)^{a_1m_1}t^{\frac{a_1m_1^2}{4}}t^{-\frac{m_0m_1}{2}}t^{\frac{m_1m_2}{2}}\\
&= \sum_{m_1=0}^{2r-1}e^{\frac{2a_1m_1^2\pi\sqrt{-1}}{2r}}e^{2(\frac{m_2+m_0}{r}+\frac{a_1}{2})m_1\pi\sqrt{-1}}\\
&\quad -\sum_{m_1=0}^{2r-1}e^{\frac{2a_1m_1^2\pi\sqrt{-1}}{2r}}e^{2(\frac{m_2-m_0}{r}+\frac{a_1}{2})m_1\pi\sqrt{-1}}.
\end{aligned}$$

Then, by Proposition 3.7 with $m=2a_1$, $n=2r$ and $\psi=\frac{m_2\pm m_0}{r}+\frac{a_1}{2}$, we have

$$\begin{aligned}
S_1(m_0,m_2) &= \left(\frac{\sqrt{-1}r}{a_1}\right)^{\frac{1}{2}}\sum_{s=0}^{2|a_1|-1}e^{-\frac{2r(s+\frac{m_2+m_0}{r}+\frac{a_1}{2})^2\pi\sqrt{-1}}{2a_1}}\\
&\quad -\left(\frac{\sqrt{-1}r}{a_1}\right)^{\frac{1}{2}}\sum_{s=0}^{2|a_1|-1}e^{-\frac{2r(s+\frac{m_2-m_0}{r}+\frac{a_1}{2})^2\pi\sqrt{-1}}{2a_1}}\\
&= \left(\frac{\sqrt{-1}r}{A_1}\right)^{\frac{1}{2}}\sum_{s=0}^{2|A_1|-1}e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_1}{A_1}(m_2+sr+\frac{K_1r}{2}+\frac{(-1)^2m_0}{C_1})^2}\\
&\quad -\left(\frac{\sqrt{-1}r}{A_1}\right)^{\frac{1}{2}}\sum_{s=0}^{2|A_1|-1}e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_1}{A_1}(m_2+sr+\frac{K_1r}{2}-\frac{(-1)^2m_0}{C_1})^2},
\end{aligned}$$

where the last equality uses the fact that $A_1=K_1=a_1$ and $C_1=1$. In this case, we have $\tau_1^{\pm}=(\frac{\sqrt{-1}r}{A_1})^{\frac{1}{2}}$.

Now, assume that

$$\begin{aligned}
S_{i-1}(m_0,m_i) &= \tau_{i-1}^{+}\sum_{s=0}^{2|A_{i-1}|-1}e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(m_i+sr+\frac{K_{i-1}r}{2}+\frac{(-1)^im_0}{C_{i-1}})^2}\\
&\quad -\tau_{i-1}^{-}\sum_{s=0}^{2|A_{i-1}|-1}e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(m_i+sr+\frac{K_{i-1}r}{2}-\frac{(-1)^im_0}{C_{i-1}})^2}.
\end{aligned}\tag{3.11}$$

By (3.10), we have

$$\begin{aligned}
S_i(m_0,m_{i+1}) &= \sum_{m_i=0}^{r-1}(-1)^{a_im_i}t^{\frac{a_im_i^2}{4}}(t^{\frac{m_im_{i+1}}{2}}-t^{-\frac{m_im_{i+1}}{2}})S_{i-1}(m_0,m_i)\\
&= \sum_{m_i=0}^{r-1}(-1)^{a_im_i}t^{\frac{a_im_i^2}{4}}t^{\frac{m_im_{i+1}}{2}}S_{i-1}(m_0,m_i)\\
&\quad+\sum_{m_i=0}^{r-1}(-1)^{a_i(-m_i)}t^{\frac{a_i(-m_i)^2}{4}}t^{\frac{(-m_i)m_{i+1}}{2}}S_{i-1}(m_0,-m_i)\\
&= \Big(\sum_{m_i=0}^{r-1}+\sum_{m_i=-r+1}^{0}\Big)(-1)^{a_im_i}t^{\frac{a_im_i^2}{4}}t^{\frac{m_im_{i+1}}{2}}S_{i-1}(m_0,m_i)\\
&= 2\sum_{m_i=0}^{r-1}(-1)^{a_im_i}t^{\frac{a_im_i^2}{4}}t^{\frac{m_im_{i+1}}{2}}S_{i-1}(m_0,m_i),
\end{aligned}$$

where the last equality comes from that both $\{-r+1,\dots,0\}$ and $\{0,\dots,r-1\}$ are a full period for m_i. Since the quantities $(-1)^{am}t^{\frac{am^2}{4}}$ and $t^{\frac{am}{2}}$ have period r in m, we have, for any integer s,

$$\begin{aligned}
(-1)^{a_im_i}t^{\frac{a_im_i^2}{4}}t^{\frac{m_im_{i+1}}{2}} &= (-1)^{a_i(m_i+sr)}t^{\frac{a_i(m_i+sr)^2}{4}}t^{\frac{(m_i+sr)m_{i+1}}{2}}\\
&= (-1)^{-(-1)^ia_i(m_i+sr)}t^{\frac{a_i(m_i+sr)^2}{4}}t^{\frac{(m_i+sr)m_{i+1}}{2}}\\
&= e^{\frac{\pi\sqrt{-1}}{r}a_i(m_i+sr)^2+2\pi\sqrt{-1}(\frac{m_{i+1}}{r}-\frac{(-1)^ia_i}{2})(m_i+sr)}.
\end{aligned}$$

Then, by this and (3.11), we have

$$\begin{aligned}
&S_i(m_0,m_{i+1})\\
&= 2\tau_{i-1}^{+}\sum_{m_i=0}^{r-1}\sum_{s=0}^{2|A_{i-1}|-1}\\
&\times e^{\frac{\pi\sqrt{-1}}{r}a_i(m_i+sr)^2+2\pi\sqrt{-1}(\frac{m_{i+1}}{r}-\frac{(-1)^ia_i}{2})(m_i+sr)-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(m_i+sr+\frac{K_{i-1}r}{2}+\frac{(-1)^im_0}{C_{i-1}})^2}\\
&- 2\tau_{i-1}^{-}\sum_{m_i=0}^{r-1}\sum_{s=0}^{2|A_{i-1}|-1}\\
&\times e^{\frac{\pi\sqrt{-1}}{r}a_i(m_i+sr)^2+2\pi\sqrt{-1}(\frac{m_{i+1}}{r}-\frac{(-1)^ia_i}{2})(m_i+sr)-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(m_i+sr+\frac{K_{i-1}r}{2}-\frac{(-1)^im_0}{C_{i-1}})^2}
\end{aligned}$$

We observe that, as s runs over $\{0,\dots,2|A_{i-1}|-1\}$ and m_i runs over $\{0,\dots,r-1\}$, the quantity m_i+sr runs over all integers in $\{0,\dots,2|A_{i-1}|r-1\}$. Thus, by letting

$\lambda = m_i + sr$, we can change the double sum above into the following single sum:

$$S_i(m_0, m_{i+1}) = 2\tau_{i-1}^{+} \sum_{\lambda=0}^{2|A_{i-1}|r-1} e^{\frac{\pi\sqrt{-1}}{r} a_i \lambda^2 + 2\pi\sqrt{-1}(\frac{m_{i+1}}{r} - \frac{(-1)^i a_i}{2})\lambda - \frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\lambda + \frac{K_{i-1}r}{2} + \frac{(-1)^i m_0}{C_{i-1}})^2}$$
$$- 2\tau_{i-1}^{-} \sum_{\lambda=0}^{2|A_{i-1}|r-1} e^{\frac{\pi\sqrt{-1}}{r} a_i \lambda^2 + 2\pi\sqrt{-1}(\frac{m_{i+1}}{r} - \frac{(-1)^i a_i}{2})\lambda - \frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\lambda + \frac{K_{i-1}r}{2} - \frac{(-1)^i m_0}{C_{i-1}})^2}$$
$$= 2\tau_{i-1}^{+} \sum_{\lambda=0}^{2|A_{i-1}|r-1} e^{\frac{\pi\sqrt{-1}}{r}(a_i - \frac{C_{i-1}}{A_{i-1}})\lambda^2 + 2\pi\sqrt{-1}(\frac{m_{i+1}}{r} - \frac{(-1)^i a_i}{2} - \frac{K_{i-1}C_{i-1}}{2A_{i-1}} - \frac{(-1)^i m_0}{A_{i-1}r})\lambda - \frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\frac{K_{i-1}r}{2} + \frac{(-1)^i m_0}{C_{i-1}})^2}$$
$$- 2\tau_{i-1}^{+} \sum_{\lambda=0}^{2|A_{i-1}|r-1} e^{\frac{\pi\sqrt{-1}}{r}(a_i - \frac{C_{i-1}}{A_{i-1}})\lambda^2 + 2\pi\sqrt{-1}(\frac{m_{i+1}}{r} - \frac{(-1)^i a_i}{2} - \frac{K_{i-1}C_{i-1}}{2A_{i-1}} + \frac{(-1)^i m_0}{A_{i-1}r})\lambda - \frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\frac{K_{i-1}r}{2} - \frac{(-1)^i m_0}{C_{i-1}})^2}.$$

By Lemma 3.1, we have $A_{i-1} = C_i$, $a_i - \frac{C_{i-1}}{A_{i-1}} = \frac{A_i}{C_i}$, and by the definition of K_i, we have $(-1)^i a_i + \frac{K_{i-1}C_{i-1}}{C_i} = -K_i$. As a consequence,

$$\begin{aligned} S_i(m_0, m_{i+1}) = {} & 2\tau_{i-1}^{+} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\frac{K_{i-1}r}{2} + \frac{(-1)^i m_0}{C_{i-1}})^2} \\ & \times \sum_{\lambda=0}^{2|C_i|r-1} e^{\frac{2A_i\lambda^2\pi\sqrt{-1}}{2C_i r}} e^{2(\frac{m_{i+1}}{r} + \frac{K_i}{2} + \frac{(-1)^{i+1}m_0}{C_i r})\lambda\pi\sqrt{-1}} \\ & - 2\tau_{i-1}^{-} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\frac{K_{i-1}r}{2} - \frac{(-1)^i m_0}{C_{i-1}})^2} \\ & \times \sum_{\lambda=0}^{2|C_i|r-1} e^{\frac{2A_i\lambda^2\pi\sqrt{-1}}{2C_i r}} e^{2(\frac{m_{i+1}}{r} + \frac{K_i}{2} - \frac{(-1)^{i+1}m_0}{C_i r})\lambda\pi\sqrt{-1}}. \end{aligned}$$

Finally, letting

$$\tau_i^{\pm} = 2e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{i-1}}{A_{i-1}}(\frac{K_{i-1}r}{2} \pm \frac{(-1)^i m_0}{C_{i-1}})^2} \left(\frac{\sqrt{-1}C_i r}{A_i}\right)^{\frac{1}{2}} \tau_{i-1}^{\pm}.$$

By Proposition 3.7 with $m = 2A_i$, $n = 2C_i r$ and $\psi = \frac{m_{i+1}}{r} + \frac{K_i}{2} \pm \frac{(-1)^i m_0}{C_i}$, we have

$$\begin{aligned} S_i(m_0, m_{i+1}) = {} & \tau_i^{+} \sum_{s=0}^{2|A_i|-1} e^{-\frac{2C_i r(s + \frac{m_{i+1}}{r} + \frac{K_i}{2} + \frac{(-1)^{i+1}m_0}{C_i r})^2 \pi\sqrt{-1}}{2A_i}} \\ & - \tau_i^{-} \sum_{s=0}^{2|A_i|-1} e^{-\frac{2C_i r(s + \frac{m_{i+1}}{r} + \frac{K_i}{2} - \frac{(-1)^{i+1}m_0}{C_i r})^2 \pi\sqrt{-1}}{2A_i}} \\ = {} & \tau_i^{+} \sum_{s=0}^{2|A_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_i}{A_i}(m_{i+1} + sr + \frac{K_i r}{2} + \frac{(-1)^{i+1}m_0}{C_i})^2} \\ & - \tau_i^{-} \sum_{s=0}^{2|A_i|-1} e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_i}{A_i}(m_{i+1} + sr + \frac{K_i r}{2} - \frac{(-1)^{i+1}m_0}{C_i})^2}. \end{aligned}$$

By induction and the fact that $A_{j-1} = C_j$ for each j,

$$\tau_i^{\pm} = \left(\frac{\sqrt{-1}r}{A_1}\right)^{\frac{1}{2}} \prod_{j=2}^{i}\left(2e^{-\frac{\pi\sqrt{-1}}{r}\frac{C_{j-1}}{A_{j-1}}(\frac{K_{j-1}r}{2}\pm\frac{(-1)^j m_0}{C_{j-1}})^2}\left(\frac{\sqrt{-1}C_j r}{A_j}\right)^{\frac{1}{2}}\right)$$

$$= \frac{(-1)^{\frac{i}{4}}}{\sqrt{A_i}} 2^{i-1} r^{\frac{i}{2}}$$

$$\times e^{-\frac{\pi\sqrt{-1}}{r}m_0^2\sum_{j=1}^{i-1}\frac{1}{C_jC_{j+1}} - \frac{\pi\sqrt{-1}r}{4}\sum_{j=1}^{i-1}\frac{C_jK_j^2}{C_{j+1}} \mp \pi\sqrt{-1}m_0\sum_{j=1}^{i-1}\frac{(-1)^{j+1}K_j}{C_{j+1}}}. \quad \blacksquare$$

4. Poisson summation formula

The main result of this section is Proposition 4.3 where, using the Poisson summation formula, we write the rth Reshetikhin–Turaev invariant of M as a sum of integrals.

Proposition 4.1. *For $\varepsilon > 0$ and $s \in \{1, \dots, |q| - 1\}$, we can choose a sufficiently small $\delta > 0$ so that if one of $\frac{2\pi n}{r} + \frac{2\pi m}{r}$ and $\frac{2\pi n}{r} - \frac{2\pi m}{r}$ is not in $(\delta, \frac{\pi}{2} - \delta) \cup (\pi + \delta, \frac{3\pi}{2} - \delta)$, then*

$$|g_r(s,m,n)| < O\left(e^{\frac{r}{4\pi}(\frac{1}{2}\mathrm{Vol}(S^3\smallsetminus K_{4_1})+\varepsilon)}\right).$$

To prove Proposition 4.1, we need the following estimate, which first appeared in [13, Proposition 8.2] for $t = e^{\frac{2\pi\sqrt{-1}}{r}}$, and in [6, Proposition 4.1] for $t = e^{\frac{4\pi\sqrt{-1}}{r}}$.

Lemma 4.2. *For any integer $0 < n < r$ and at $t = e^{\frac{4\pi\sqrt{-1}}{r}}$,*

$$\log|\{n\}!| = -\frac{r}{2\pi}\Lambda\left(\frac{2n\pi}{r}\right) + O(\log r).$$

Proof of Proposition 4.1. By (3.9), we have

$$|g_r(s,m,n)| = \left|\sin\left(\frac{2\pi m}{r}\frac{1}{q} - J(s)\right)\right|\left|\frac{\{r-m-n-1\}!}{\{n-m\}!}\right|,$$

and by Lemma 4.2, we have

$$\log|g_r(s,m,n)| = -\frac{r}{2\pi}\Lambda\left(\frac{2\pi(r-m-n-1)}{r}\right) + \frac{r}{2\pi}\Lambda\left(\frac{2\pi(n-m)}{r}\right) + O(\log r).$$

Choose $\delta > 0$ so that

$$\Lambda(\delta) < \frac{\varepsilon}{2}.$$

Now, if one of $\frac{2\pi n}{r} + \frac{2\pi m}{r}$ and $\frac{2\pi n}{r} - \frac{2\pi m}{r}$ is not in $(\delta, \frac{\pi}{2} - \delta) \cup (\pi + \delta, \frac{3\pi}{2} - \delta)$, then

$$\log|g_r(s,m,n)| < \frac{r}{2\pi}\left(\Lambda\left(\frac{\pi}{6}\right) + \frac{\varepsilon}{2}\right) = \frac{r}{4\pi}\left(\frac{1}{2}\mathrm{Vol}(S^3\smallsetminus K_{4_1}) + \varepsilon\right).$$

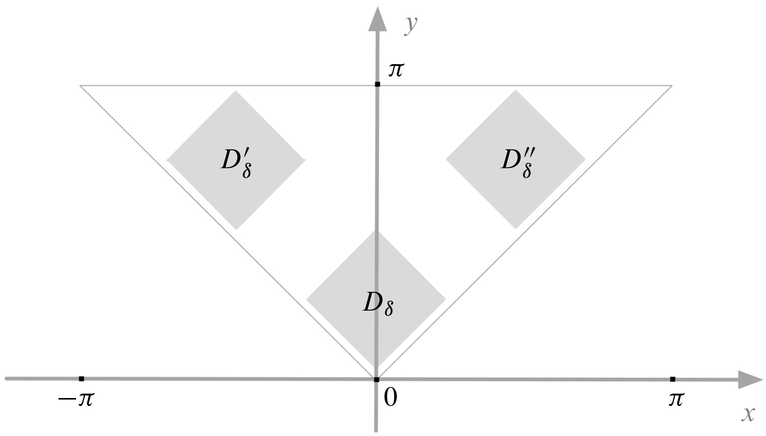


Figure 2. Regions D_δ, D'_δ, and D''_δ.

The last equality is true because by the properties of Lobachevsky function $\Lambda(\frac{\pi}{6}) = \frac{3}{2}\Lambda(\frac{\pi}{3})$, and the volume of $S^3 \setminus K_{4_1}$ equals $6\Lambda(\frac{\pi}{3})$. ■

For $\delta \geqslant 0$, let

$$D_\delta = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \delta < y + x < \frac{\pi}{2} - \delta, \delta < y - x < \frac{\pi}{2} - \delta\right\},$$
$$D'_\delta = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \delta < y + x < \frac{\pi}{2} - \delta, \pi + \delta < y - x < \frac{3\pi}{2} - \delta\right\}$$

and

$$D''_\delta = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, \pi + \delta < y + x < \frac{3\pi}{2} - \delta, \delta < y - x < \frac{\pi}{2} - \delta\right\},$$

and let $\mathcal{D}_\delta = D_\delta \cup D'_\delta \cup D''_\delta$. If $\delta = 0$, we omit the subscript and write $D = D_0$, $D' = D'_0$, $D'' = D''_0$ and $\mathcal{D} = D \cup D' \cup D''$. See Figure 2.

For a sufficiently small $\delta > 0$, we consider a C^∞-smooth bump function ψ on $\mathbb{R}^2$ such that

$$\begin{cases} \psi(x,y) = 1, & (x,y) \in \overline{\mathcal{D}_{\frac{\delta}{2}}}, \\ 0 < \psi(x,y) < 1, & (x,y) \in \mathcal{D} \setminus \overline{\mathcal{D}_{\frac{\delta}{2}}}, \\ \psi(x,y) = 0, & (x,y) \notin \mathcal{D}, \end{cases}$$

and let

$$f_r(s,m,n) = \psi\left(\frac{2\pi m}{r}, \frac{2\pi n}{r}\right) g_r(s,m,n).$$

Then, by Proposition 4.1, we have

$$\mathrm{RT}_r(M) = \kappa_r \sum_{s=0}^{|q|-1} \sum_{(m,n)\in(\mathbb{Z}+\frac{1}{2})^2} f_r(s,m,n) + O\big(e^{\frac{r}{4\pi}(\frac{1}{2}\mathrm{Vol}(S^3\smallsetminus K_{4_1})+\varepsilon)}\big). \tag{4.1}$$

Since f_r is C^∞-smooth and equals zero outside of $\mathcal{D}$, it is in the Schwartz space on $\mathbb{R}^2$. Recall that by the Poisson summation formula (see, e.g., [36, Theorem 3.1]), for any function f in the Schwartz space on $\mathbb{R}^k$,

$$\sum_{(m_1,\dots,m_k)\in\mathbb{Z}^k} f(m_1,\dots,m_k) = \sum_{(n_1,\dots,n_k)\in\mathbb{Z}^k} \hat{f}(n_1,\dots,n_k),$$

where $\hat{f}(n_1,\dots,n_k)$ is the $(n_1,\dots,n_k)$th Fourier coefficient of f defined by

$$\hat{f}(n_1,\dots,n_k) = \int_{\mathbb{R}^k} f(x_1,\dots,x_k) e^{\sum_{j=1}^k 2\pi\sqrt{-1}n_j x_j}\, dx_1\cdots dx_k.$$

As a consequence, we have the following proposition.

Proposition 4.3. *We have*

$$\mathrm{RT}_r(M) = \kappa_r \sum_{s=0}^{|q|-1} \sum_{(m,n)\in\mathbb{Z}^2} \hat{f}_r(s,m,n) + O\big(e^{\frac{r}{4\pi}(\frac{1}{2}\mathrm{Vol}(S^3\smallsetminus K_{4_1})+\varepsilon)}\big),$$

where

$$\hat{f}_r(s,m,n) = (-1)^{m+n}\Big(\frac{r}{2\pi}\Big)^2 \int_{\mathcal{D}} \psi(x,y)\sin\Big(\frac{x}{q} - J(s)\pi\Big)\varepsilon(x,y) \times e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s,x,y)-4\pi m x-4\pi n y)}\,dx\,dy.$$

Proof. To apply the Poisson summation formula, we need to make the summation in (4.1) over integers instead of half-integers. To do this, let $m' = m+\frac{1}{2}$ and $n' = n+\frac{1}{2}$. Then, for each $s\in\{0,\dots,|q|-1\}$,

$$\sum_{(m,n)\in(\mathbb{Z}+\frac{1}{2})^2} f_r(s,m,n) = \sum_{(m',n')\in\mathbb{Z}^2} f_r\Big(s, m'-\frac{1}{2}, n'-\frac{1}{2}\Big).$$

Now, by the Poisson summation formula, the right-hand side equals

$$\sum_{(a,b)\in\mathbb{Z}^2} \int_{\mathbb{R}^2} f_r\Big(s, m'-\frac{1}{2}, n'_k-\frac{1}{2}\Big) e^{2\pi\sqrt{-1}am'+2\pi\sqrt{-1}bn'}\,dm'dn'.$$

Using the change of variables $x = \frac{2\pi m}{r} = \frac{2\pi m'}{r} - \frac{\pi}{r}$ and $y = \frac{2\pi n}{r} = \frac{2\pi n'}{r} - \frac{\pi}{r}$, we get the result. ∎

In Section 6, we will show that among all the Fourier coefficients in this summation, the two $\hat{f}_r(s^+,m^+,0)$ and $\hat{f}_r(s^-,m^-,0)$ are the leading ones, where $s^\pm$ and $m^\pm$ are as defined in Lemma 3.3 (1). In the rest of the paper, let

$$V_r^+(x,y) = V_r(s^+,x,y) - 4\pi m^+ x,$$

and let

$$V_r^-(x,y) = V_r(s^-,x,y) - 4\pi m^- x.$$

Then, by Lemma 3.3 (1), we have

$$\begin{aligned} V_r^+(x,y) = {} & \frac{-px^2+2\pi x}{q} - 2\pi x + 4xy - \varphi_r\left(\pi - y - x - \frac{\pi}{r}\right) \\ & + \varphi_r\left(y - x + \frac{\pi}{r}\right) + K(s^+)\pi^2 \end{aligned}$$

and

$$\begin{aligned} V_r^-(x,y) = {} & \frac{-px^2-2\pi x}{q} - 2\pi x + 4xy - \varphi_r\left(\pi - y - x - \frac{\pi}{r}\right) \\ & + \varphi_r\left(y - x + \frac{\pi}{r}\right) + K(s^-)\pi^2 \end{aligned}$$

on the region D, and we have a similar formula on the regions D' and D''. As stated in Lemma 6.2 in Section 6, the functions $V_r^\pm$ are closely related to the functions

$$\begin{aligned} V^+(x,y) = {} & \frac{-px^2+2\pi x}{q} - 2\pi x + 4xy - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y+x)}\big) \\ & + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big) + K(s^+)\pi^2 \end{aligned}$$

and

$$\begin{aligned} V^-(x,y) = {} & \frac{-px^2-2\pi x}{q} - 2\pi x + 4xy - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y+x)}\big) \\ & + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big) + K(s^-)\pi^2 \end{aligned}$$

whose critical values will determine the exponential growth rate of the invariants.

We also notice that $V_r^\pm(x,y)$ and $V^\pm(x,y)$ define holomorphic functions on the regions $D_{\mathbb{C},\delta}$, $D'_{\mathbb{C},\delta}$, and $D''_{\mathbb{C},\delta}$ of $\mathbb{C}^2$, where for $\delta \geqslant 0$,

$$D_{\mathbb{C},\delta} = \left\{(x,y)\in\mathbb{C}^2 \,\middle|\, \delta < \mathrm{Re}(y)+\mathrm{Re}(x) < \frac{\pi}{2}-\delta,\ \delta < \mathrm{Re}(y)-\mathrm{Re}(x) < \frac{\pi}{2}-\delta\right\},$$

$$D'_{\mathbb{C},\delta} = \left\{(x,y)\in\mathbb{C}^2 \,\middle|\, \delta < \mathrm{Re}(y)+\mathrm{Re}(x) < \frac{\pi}{2}-\delta,\ \pi+\delta < \mathrm{Re}(y)-\mathrm{Re}(x) < \frac{3\pi}{2}-\delta\right\},$$

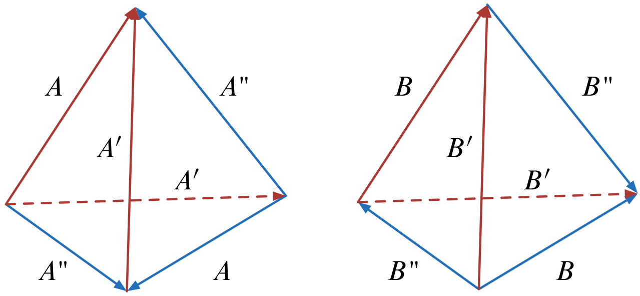


Figure 3. Thurston's ideal triangulation of the figure-8 knot complement.

and

$$D''_{\mathbb{C},\delta} = \left\{(x,y)\in\mathbb{C}^2 \,\middle|\, \pi+\delta<\mathrm{Re}(y)+\mathrm{Re}(x)<\frac{3\pi}{2}-\delta, \delta<\mathrm{Re}(y)-\mathrm{Re}(x)<\frac{\pi}{2}-\delta\right\}.$$

When $\delta = 0$, we denote the corresponding regions by $D_{\mathbb{C}}$, $D'_{\mathbb{C}}$ and $D''_{\mathbb{C}}$, and let $\mathcal{D}_{\mathbb{C}} = D_{\mathbb{C}} \cup D'_{\mathbb{C}} \cup D''_{\mathbb{C}}$.

5. Geometry of the critical points

The goal of this section is to understand the geometric meaning of the critical points and the critical values of the functions $V^{\pm}$ defined in the previous section. The main result is Proposition 5.4, which shows that the real and imaginary parts of the critical values of $V^{\pm}$ are the volume of M and modulo $\pi^2\mathbb{Z}$ the Chern–Simons invariant of M respectively, and the determinants of the Hessian matrices of $V^{\pm}$ at the critical points give the adjoint twisted Reidemeister torsion of M. The key observation is Lemmas 5.1 and 5.2 that the system of critical point equations of $V^{\pm}$ is equivalent to the system of hyperbolic gluing equations (consisting of an edge equation and a $\frac{p}{q}$ Dehn-surgery equation) for a particular ideal triangulation of the figure-8 knot complement.

According to Thurston's notes [37], the complement of the figure-8 knot has an ideal triangulation as drawn in Figure 3. Let A and B be the shape parameters of the two ideal tetrahedra, and let

$$A' = \frac{1}{1-A}, \quad A'' = 1-\frac{1}{A}, \quad B' = \frac{1}{1-B}, \quad B'' = 1-\frac{1}{B}.$$

In Figure 4 is a fundamental domain of the boundary of the complement of a tubular neighborhood of the figure-8 knot.

Recall that for $z \in \mathbb{C}\setminus(-\infty,0]$, the logarithmic function is defined by

$$\log z = \ln|z| + \sqrt{-1}\arg z,$$

with $-\pi < \arg z < \pi$.

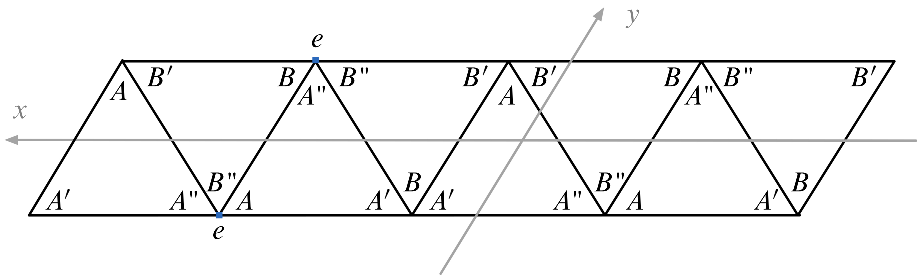


Figure 4. Combinatorics around the boundary.

Then, the holonomy $\mathrm{H}(e)$ around the edge e depicted in Figure 4 is

$$\mathrm{H}(e) = \log A + 2\log A'' + \log B + 2\log B'',$$

and the holonomies of the curves x and y depicted in Figure 4 are, respectively,

$$\mathrm{H}(x) = 2\log B + 2\log B'' - 2\log A - 2\log A''$$

and

$$\mathrm{H}(y) = \log B' - \log A''.$$

By [37], we can choose the meridian $m = y$ and the longitude $l = x + 2y$. Hence,

$$\mathrm{H}(m) = \log B' - \log A'',$$

and

$$\begin{aligned}\mathrm{H}(l) &= 2\log B + 2\log B' + 2\log B'' - 2\log A - 4\log A'' \\ &= 2\pi\sqrt{-1} - 2\log A - 4\log A''.\end{aligned}$$

Then, the system of hyperbolic gluing equations

$$\begin{cases}\mathrm{H}(e) = 2\pi\sqrt{-1}, \\ p\mathrm{H}(m) + q\mathrm{H}(l) = 2\pi\sqrt{-1}\end{cases}$$

can be written as

$$\begin{cases}\log A + 2\log A'' + \log B + 2\log B'' = 2\pi\sqrt{-1}, \\ p(\log B' - \log A'') + q(2\pi\sqrt{-1} - 2\log A - 4\log A'') = 2\pi\sqrt{-1}.\end{cases} \tag{5.1}$$

Now, for the critical point equations of $V^{\pm}$, by taking the partial derivatives, we have

$$\frac{\partial V^{\pm}}{\partial x} = \frac{-2px \pm 2\pi}{q} + 4y - 2\pi - 2\sqrt{-1}\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) + 2\sqrt{-1}\log\big(1 - e^{2\sqrt{-1}(y-x)}\big)$$

and

$$\frac{\partial V^{\pm}}{\partial y} = 4x - 2\sqrt{-1}\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) - 2\sqrt{-1}\log\big(1 - e^{2\sqrt{-1}(y-x)}\big).$$

Hence, the system of critical point equations of $V^{\pm}(x, y)$ is

$$\begin{cases} 4x - 2\sqrt{-1}\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) - 2\sqrt{-1}\log\big(1 - e^{2\sqrt{-1}(y-x)}\big) = 0, \\ \frac{-2px\pm 2\pi}{q} + 4y - 2\pi - 2\sqrt{-1}\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) \\ \quad +2\sqrt{-1}\log\big(1 - e^{2\sqrt{-1}(y-x)}\big) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1. *In $D_{\mathbb{C}}$, if we let $A = e^{2\sqrt{-1}(y+x)}$ and $B = e^{2\sqrt{-1}(y-x)}$, then the system of critical point equations (5.2) of V^{+} is equivalent to the system of hyperbolic glueing equations (5.1).*

Proof. In $D_{\mathbb{C}}$, we have

$$\begin{cases} \log A = 2\sqrt{-1}(y + x), \\ \log A' = \pi\sqrt{-1} - 2\sqrt{-1}(y + x) - \log\big(1 - e^{-2\sqrt{-1}(y+x)}\big), \\ \log A'' = \log\big(1 - e^{-2\sqrt{-1}(y+x)}\big), \\ \log B = 2\sqrt{-1}(y - x), \\ \log B' = -\log\big(1 - e^{2\sqrt{-1}(y-x)}\big), \\ \log B'' = \pi\sqrt{-1} - 2\sqrt{-1}(y - x) + \log\big(1 - e^{2\sqrt{-1}(y-x)}\big). \end{cases}$$

For one direction, we assume that $(x, y) \in D_{\mathbb{C}}$ is a solution of (5.2) with the "+" chosen. Then,

$$\begin{aligned} \mathrm{H}(e) &= \log A + 2\log A'' + \log B + 2\log B'' \\ &= 4x\sqrt{-1} + 2\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) + 2\log\big(1 - e^{2\sqrt{-1}(y-x)}\big) + 2\pi\sqrt{-1} \\ &= 2\pi\sqrt{-1}, \end{aligned}$$

where the last equality comes from the first equation of (5.2). Hence, the edge equation is satisfied.

Next, we compute $\mathrm{H}(m)$ and $\mathrm{H}(l)$. We have

$$\begin{aligned}\mathrm{H}(m) &= \log B' - \log A'' \\ &= -\log\big(1 - e^{2\sqrt{-1}(y-x)}\big) - \log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) \\ &= 2x\sqrt{-1},\end{aligned} \tag{5.3}$$

where the last equality comes from the first equation of (5.2); and

$$\begin{aligned}\mathrm{H}(l) &= 2\pi\sqrt{-1} - 2\log A - 4\log A'' \\ &= 2\pi\sqrt{-1} - 2\log A + \big(4x\sqrt{-1} - 2\log B'\big) - 2\log A'' \\ &= -4y\sqrt{-1} + 2\pi\sqrt{-1} - 2\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) + 2\log\big(1 - e^{2\sqrt{-1}(y-x)}\big),\end{aligned} \tag{5.4}$$

where the second equality comes from (5.3). Equations (5.3), (5.4) and the second equation of (5.2) then imply that

$$\frac{p\mathrm{H}(m)\sqrt{-1} + 2\pi}{q} + \mathrm{H}(l)\sqrt{-1} = 0,$$

which is equivalent to the $\frac{p}{q}$ Dehn-surgery equation

$$p\mathrm{H}(m) + q\mathrm{H}(l) = 2\pi\sqrt{-1}.$$

For the other direction, assume that (A, B) is a solution of (5.1). Then, the edge equation implies the first equation of (5.2); and (5.3), (5.4), and the Dehn-surgery equation imply the second equation of (5.2). ∎

Lemma 5.2. *In $D_{\mathbb{C}}$, if we let $A = e^{2\sqrt{-1}(y-x)}$ and $B = e^{2\sqrt{-1}(y+x)}$, then the system of critical point equations* (5.2) *of V^- is equivalent to the system of hyperbolic glueing equations* (5.1).

Proof. This time we have

$$\begin{cases}\log A = 2\sqrt{-1}(y - x), \\ \log A' = -\log\big(1 - e^{2\sqrt{-1}(y-x)}\big), \\ \log A'' = \pi\sqrt{-1} - 2\sqrt{-1}(y - x) + \log\big(1 - e^{2\sqrt{-1}(y-x)}\big), \\ \log B = 2\sqrt{-1}(y + x), \\ \log B' = \pi\sqrt{-1} - 2\sqrt{-1}(y + x) - \log\big(1 - e^{-2\sqrt{-1}(y+x)}\big), \\ \log B'' = \log\big(1 - e^{-2\sqrt{-1}(y+x)}\big).\end{cases}$$

The rest of the proof is similar to that of Lemma 5.1. For one direction, we assume that $(x, y) \in D_{\mathbb{C}}$ is a solution of (5.2) with the "–" chosen. Then,

$$\mathrm{H}(e) = \log A + 2\log A'' + \log B + 2\log B'' = 2\pi\sqrt{-1}.$$

Hence, the edge equation is satisfied.

For the computation of $\mathrm{H}(m)$ and $\mathrm{H}(l)$, we have

$$\begin{aligned}\mathrm{H}(m) &= \log B' - \log A'' \\ &= -4x\sqrt{-1} - \log\left(1 - e^{2\sqrt{-1}(y-x)}\right) - \log\left(1 - e^{-2\sqrt{-1}(y+x)}\right) \\ &= -2x\sqrt{-1},\end{aligned} \tag{5.5}$$

where the last equality comes from the first equation of (5.2); and

$$\begin{aligned}\mathrm{H}(l) &= 2\pi\sqrt{-1} - 2\log A - 4\log A'' \\ &= 2\pi\sqrt{-1} - 2\log A + \left(-4x\sqrt{-1} - 2\log B'\right) - 2\log A'' \\ &= 4y\sqrt{-1} - 2\pi\sqrt{-1} + 2\log\left(1 - e^{-2\sqrt{-1}(y+x)}\right) - 2\log\left(1 - e^{2\sqrt{-1}(y-x)}\right),\end{aligned} \tag{5.6}$$

where the second equality comes from (5.5). Equations (5.5), (5.6) and the second equation of (5.2) then imply that

$$\frac{-p\mathrm{H}(m)\sqrt{-1} - 2\pi}{q} - \mathrm{H}(l)\sqrt{-1} = 0,$$

which is equivalent to the $\frac{p}{q}$ Dehn-surgery equation

$$p\mathrm{H}(m) + q\mathrm{H}(l) = 2\pi\sqrt{-1}.$$

For the other direction, assume that (A, B) is a solution of (5.1). Then, the edge equation implies the first equation of (5.2); and (5.5), (5.6), and the Dehn-surgery equation imply the second equation of (5.2). ∎

By Thurston's notes [37], for each coprime of integers $(p,q) \neq (\pm 1, 1), (\pm 2, \pm 1), (\pm 3, \pm 1)$ and $(\pm 4, \pm 1)$, there is a unique solution A_0 and B_0 of (5.1) with $\operatorname{Im} A_0 > 0$ and $\operatorname{Im} B_0 > 0$. Then, by Lemmas 5.1 and 5.2, we have the following corollary.

Corollary 5.3. *The point*

$$(x_0, y_0) = \left(\frac{\log A_0 - \log B_0}{4\sqrt{-1}}, \frac{\log A_0 + \log B_0}{4\sqrt{-1}}\right)$$

is the unique critical point of V^+ *in* $D_{\mathbb{C}}$*, and* $(-x_0, y_0)$ *is the unique critical point of* V^- *in* $D_{\mathbb{C}}$*.*

Proposition 5.4. (1) *Let* Vol(M) *and* CS(M)*, respectively, be the hyperbolic volume and the Chern–Simons invariant of* M*. Then,*

$$V^+(x_0, y_0) - \left(K(s^+) + \frac{p'}{q}\right)\pi^2 = V^-(-x_0, y_0) - \left(K(s^-) + \frac{p'}{q}\right)\pi^2$$

for the integer p' *defined in* (3.5)*, and*

$$V^+(x_0, y_0) \equiv V^-(-x_0, y_0) \equiv \sqrt{-1}\big(\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)\big) \quad (\mathrm{mod}\ \pi^2\mathbb{Z}).$$

(2) *Let* $\mu = pm + ql$*, let* ρ *be the holonomy representation of the hyperbolic structure on* M *restricted to* $S^3 \setminus K_{4_1}$ *and let* $\mathrm{Tor}_\mu(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho)$ *be the Reidemeister torsion of* $S^3 \setminus K_{4_1}$ *twisted by the adjoint action of* ρ *with respect to the curve* μ [31]*. Then,*

$$\det(\mathrm{Hess}\, V^+)(x_0, y_0) = \det(\mathrm{Hess}\, V^-)(-x_0, y_0) = \frac{16}{q}\,\mathrm{Tor}_\mu(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho) \neq 0.$$

(3) *Let* ρ *be the holonomy representation of the hyperbolic structure on* M *and let* $\mathrm{Tor}(M; \mathrm{Ad}_\rho)$ *be the Reidemeister torsion of* M *twisted by the adjoint action of* ρ [31]*. Then,*

$$\frac{\sin(\frac{x_0}{q} - J(s^+)\pi)}{\sqrt{\det(\mathrm{Hess}\, V^+)(x_0, y_0)}} = -\frac{\sin(\frac{-x_0}{q} - J(s^-)\pi)}{\sqrt{\det(\mathrm{Hess}\, V^-)(-x_0, y_0)}} = \pm\frac{\sqrt{-q}}{8\sqrt{\mathrm{Tor}(M; \mathrm{Ad}_\rho)}},$$

with the sign $\pm$ *equal to* $(-1)^{J(s^+) - \frac{p'}{q}}$.

Proof. For (1), $(x, y) \in D_{\mathbb{C}}$, we have

$$\begin{aligned} -\mathrm{Li}_2\big(e^{-2\sqrt{-1}(y \pm x)}\big) &= \mathrm{Li}_2\big(e^{2\sqrt{-1}(y \pm x)}\big) + \frac{\pi^2}{6} + \frac{1}{2}\big(\log\big(-e^{2\sqrt{-1}(y \pm x)}\big)\big)^2 \\ &= \mathrm{Li}_2\big(e^{2\sqrt{-1}(y \pm x)}\big) + \frac{\pi^2}{6} \\ &\quad - 2y^2 - 2x^2 - \pi^2 \mp 4xy + 2\pi y \pm 2\pi x, \end{aligned}$$

where the first equality comes from (2.2), and the second equality comes from that $0 < \mathrm{Re}(y) \pm \mathrm{Re}(x) < \frac{\pi}{2}$, and hence,

$$\log\big(-e^{2\sqrt{-1}(y \pm x)}\big) = 2\sqrt{-1}(y \pm x) - \pi\sqrt{-1}.$$

From this, we have, for all (x, y) with $0 < \mathrm{Re}(y) \pm \mathrm{Re}(x) < \frac{\pi}{2}$,

$$\begin{aligned} &V^+(x, y) - \left(K(s^+) + \frac{p'}{q}\right)\pi^2 \\ &= \left(-\frac{p}{q} - 2\right)x^2 + \frac{2\pi x}{q} - 2y^2 + 2\pi y - \frac{5\pi^2}{6} \\ &\quad + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y+x)}\big) + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big) - \frac{p'\pi^2}{q} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
&V^-(-x, y) - \left(K(s^-) + \frac{p'}{q}\right)\pi^2 \\
&= \left(-\frac{p}{q} - 2\right)x^2 + \frac{2\pi x}{q} - 2y^2 + 2\pi y - \frac{5\pi^2}{6} \\
&\quad + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y+x)}\big) + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big) - \frac{p'\pi^2}{q},
\end{aligned} \tag{5.8}$$

which proves the first equality of (1).

For the second equality of (1), by (5.7) and (5.8) and Lemma 3.3 (3) that $K(s^\pm) + \frac{p'}{q}$ is an integer, we have

$$V^+(x_0, y_0) \equiv V^-(-x_0, y_0) \quad (\mathrm{mod}\ \pi^2\mathbb{Z}),$$

and it suffices to show that

$$V^+(x_0, y_0) \equiv \sqrt{-1}(\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)) \quad (\mathrm{mod}\ \pi^2\mathbb{Z}).$$

To this end, we need the following result of Yoshida [41, Theorem 2] that if the manifold M is obtained by doing a hyperbolic Dehn-filling from the complement of a hyperbolic knot K in S^3, m and l are, respectively, the meridian and longitude of the boundary of a tubular neighborhood of K, γ is isotopic to the core curve of the filled solid torus, and $\mathrm{H}(m)$, $\mathrm{H}(l)$ and $\mathrm{H}(\gamma)$ are, respectively, the holonomy of them, then

$$\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M) = \frac{\Phi(\mathrm{H}(m))}{\sqrt{-1}} - \frac{\mathrm{H}(m)\mathrm{H}(l)}{4\sqrt{-1}} + \frac{\pi\mathrm{H}(\gamma)}{2} \quad (\mathrm{mod}\ \sqrt{-1}\pi^2\mathbb{Z}),$$

where Φ is the function (see Neumann–Zagier [24]) defined on the deformation space of hyperbolic structures on $S^3\setminus K$ parametrized by the holonomy of the meridian $u = \mathrm{H}(m)$, characterized by

$$\begin{cases} \frac{\partial\Phi(u)}{\partial u} = \frac{\mathrm{H}(l)}{2}, \\ \Phi(0) = \sqrt{-1}\big(\mathrm{Vol}(\mathrm{S}^3\setminus K) + \sqrt{-1}\,\mathrm{CS}(\mathrm{S}^3\setminus K)\big). \end{cases} \tag{5.9}$$

We will show that

$$\Phi(\mathrm{H}(m)) = 4x_0y_0 - 2\pi x_0 - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y_0+x_0)}\big) + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y_0-x_0)}\big), \tag{5.10}$$

$$-\frac{\mathrm{H}(m)\mathrm{H}(l)}{4} = \frac{-px_0^2 + \pi x_0}{q} \tag{5.11}$$

and

$$\frac{\pi\sqrt{-1}}{2}\mathrm{H}(\gamma) = \frac{\pi x_0}{q} - \frac{p'\pi^2}{q} \tag{5.12}$$

so that

$$V^+(x_0, y_0) - \left(K(s^+) + \frac{p'}{q}\right)\pi^2 = \Phi(\mathrm{H}(m)) - \frac{\mathrm{H}(m)\mathrm{H}(l)}{4} + \frac{\pi\sqrt{-1}}{2}\mathrm{H}(\gamma),$$

from which the result follows.

For (5.10), let

$$U(x, y) = 4xy - 2\pi x - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y+x)}\big) + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big),$$

and define

$$\Psi(u) = U(x, y(x)),$$

where $u = 2x\sqrt{-1}$ and $y(x)$ is such that

$$\left.\frac{\partial V^+}{\partial y}\right|_{(x,y(x))} = 0.$$

Since

$$\frac{\partial U}{\partial y} = \frac{\partial V^+}{\partial y} \quad \text{and} \quad \left.\frac{\partial V^+}{\partial y}\right|_{(x,y(x))} = 0,$$

we have

$$\frac{\partial \Psi(u)}{\partial u} = \left(\frac{\partial U}{\partial x} + \left.\frac{\partial U}{\partial y}\right|_{(x,y(x))}\frac{\partial y}{\partial x}\right)\frac{\partial x}{\partial u} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial u} = \frac{\mathrm{H}(l)}{2},$$

where the last equality comes from (5.4). Also, a direct computation shows $y(0) = \frac{\pi}{6}$, and hence,

$$\Psi(0) = U\left(0, \frac{\pi}{6}\right) = 4\sqrt{-1}\Lambda\left(\frac{\pi}{6}\right) = \sqrt{-1}\big(\mathrm{Vol}(\mathrm{S}^3\setminus K_{4_1}) + \sqrt{-1}\,\mathrm{CS}(\mathrm{S}^3\setminus K_{4_1})\big).$$

Therefore, Ψ satisfies (5.9), and hence, $\Psi(u) = \Phi(u)$.

Since $y(x_0) = y_0$, and by (5.3) $\mathrm{H}(m) = 2x_0\sqrt{-1}$, we have

$$\Phi(\mathrm{H}(m)) = \Psi(2x_0\sqrt{-1}) = U(x_0, y_0),$$

which verifies (5.10).

For (5.11), by (5.3), we have that $\mathrm{H}(m) = 2x_0\sqrt{-1}$ and

$$\mathrm{H}(l) = \frac{2\pi\sqrt{-1} - p\mathrm{H}(m)}{q} = \frac{2\pi\sqrt{-1} - 2px_0\sqrt{-1}}{q}.$$

Then,

$$\mathrm{H}(m)\mathrm{H}(l) = 2x_0\sqrt{-1}\cdot\frac{2\pi\sqrt{-1} - 2px_0\sqrt{-1}}{q} = -4\cdot\frac{-px_0^2 + \pi x_0}{q},$$

from which (5.11) follows.

For (5.12), since $pp' + qq' = 1$, we can choose $\gamma = -q'm + p'l$ so that $\mu \cdot \gamma = (pm + ql) \cdot (-q'm + p'l) = 1$. Then,

$$\mathrm{H}(\gamma) = -q'\mathrm{H}(m) + p'\mathrm{H}(l) = -q' \cdot 2x_0\sqrt{-1} + p' \cdot \frac{2\pi\sqrt{-1} - 2px_0\sqrt{-1}}{q}$$
$$= \frac{2}{\pi\sqrt{-1}} \cdot \frac{\pi x_0 - p'\pi^2}{q}, \tag{5.13}$$

from which (5.12) follows.

This completes the proof of (1).

For (2), by (5.7), we have

$$\mathrm{Hess}\, V^+(x_0, y_0)$$
$$= \begin{bmatrix} -\frac{2p}{q} - 4 - \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} & -\frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} + \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} \\ -\frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} + \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} & -4 - \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} \end{bmatrix},$$

and by (5.8), we have

$$\mathrm{Hess}\, V^-(-x_0, y_0)$$
$$= \begin{bmatrix} -\frac{2p}{q} - 4 - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} - \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} & \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} \\ \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} & -4 - \frac{4e^{2\sqrt{-1}(y_0-x_0)}}{1-e^{2\sqrt{-1}(y_0-x_0)}} - \frac{4e^{2\sqrt{-1}(y_0+x_0)}}{1-e^{2\sqrt{-1}(y_0+x_0)}} \end{bmatrix}.$$

Hence,

$$\det(\mathrm{Hess}\, V^+)(x_0, y_0) = \det(\mathrm{Hess}\, V^-)(-x_0, y_0),$$

and it suffices to prove that

$$\det(\mathrm{Hess}\, V^+)(x_0, y_0) = \frac{16}{q}\mathrm{Tor}_\mu(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho).$$

To this end, for simplicity, let $X_0 = e^{2\sqrt{-1}x_0}$ and $Y_0 = e^{2\sqrt{-1}y_0}$. Then, the first equation of (5.2) implies

$$X_0 + X_0^{-1} = Y_0 + Y_0^{-1} + 1. \tag{5.14}$$

We have

$$\frac{\partial^2 V^+}{\partial x^2}(x_0, y_0) = -\frac{2p}{q} - 4 - \frac{4X_0Y_0}{1 - X_0Y_0} - \frac{4X_0^{-1}Y_0}{1 - X_0^{-1}Y_0}$$
$$= -\frac{2p}{q} + \frac{4(Y_0 - Y_0^{-1})}{Y_0 + Y_0^{-1} - X_0 - X_0^{-1}} = -\frac{2p}{q} - 4(Y_0 - Y_0^{-1}),$$

where the last equality comes from (5.14). Similarly, by (5.14), we have

$$\frac{\partial^2 V^+}{\partial y^2}(x_0, y_0) = -4 - \frac{4X_0Y_0}{1 - X_0Y_0} - \frac{4X_0^{-1}Y_0}{1 - X_0^{-1}Y_0} = -4(Y_0 - Y_0^{-1})$$

and

$$\frac{\partial^2 V^+}{\partial x \partial y}(x_0, y_0) = -\frac{4X_0Y_0}{1 - X_0Y_0} + \frac{4X_0^{-1}Y_0}{1 - X_0^{-1}Y_0} = -4(X_0 - X_0^{-1}).$$

Therefore,

$$\begin{aligned}
&\det(\operatorname{Hess} V^+)(x_0, y_0) \\
&= \Big(-\frac{2p}{q} - 4(Y_0 - Y_0^{-1})\Big)\big(-4(Y_0 - Y_0^{-1})\big) - \big(-4(X_0 - X_0^{-1})\big)^2 \\
&= \frac{8p}{q}(Y_0 - Y_0^{-1}) + 16\big((Y_0 - Y_0^{-1})^2 - (X_0 - X_0^{-1})^2\big) \\
&= \frac{8p}{q}(Y_0 - Y_0^{-1}) + 16\big((Y_0 + Y_0^{-1})^2 - (X_0 + X_0^{-1})^2\big) \\
&= \frac{8p}{q}(Y_0 - Y_0^{-1}) + 16(Y_0 + Y_0^{-1} + X_0 + X_0^{-1})(Y_0 + Y_0^{-1} - X_0 - X_0^{-1}) \\
&= \frac{8p}{q}(Y_0 - Y_0^{-1}) - 16(Y_0 + Y_0^{-1} + X_0 + X_0^{-1}),
\end{aligned} \tag{5.15}$$

where the last equality comes from (5.14).

Next, we will show that $\frac{16}{q}\operatorname{Tor}_\mu(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho)$ equals the same quantity. For this, by the change of curve formula [31, Theorem 4.1 (ii)], we have

$$\operatorname{Tor}_\mu(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho) = \frac{\partial \mathrm{H}(\mu)}{\partial \mathrm{H}(m)}\Big|_\rho \cdot \operatorname{Tor}_m(S^3 \setminus K_{4_1}; \mathrm{Ad}_\rho). \tag{5.16}$$

To compute the right-hand side, let $X = e^{2\sqrt{-1}x}$ and $Y = e^{2\sqrt{-1}y}$. Then, as the convention in Lemma 5.1, the shape parameters $A = XY$ and $B = X^{-1}Y$. Recall that the deformation space of the hyperbolic structures on M with this ideal triangulation consists of the shape parameters A and B that satisfy the edge equation $\mathrm{H}(e) = 2\pi\sqrt{-1}$, which can be written as the first equation of (5.1). Then, by Lemma 5.1, it is equivalent to the first equation of (5.2), which implies

$$X + X^{-1} = Y + Y^{-1} + 1. \tag{5.17}$$

As a consequence,

$$\frac{\partial Y}{\partial X} = \frac{Y^2(X^2 - 1)}{X^2(Y^2 - 1)}. \tag{5.18}$$

By (5.3) and (5.4), we have

$$\mathrm{H}(m) = 2x\sqrt{-1} = \log X \tag{5.19}$$

and

$$\begin{aligned}\mathrm{H}(l) &= 2\pi\sqrt{-1} - 2\log A - 4\log A'' \\ &= 2\pi\sqrt{-1} - 2\log(XY) - 4\log(1 - X^{-1}Y^{-1}).\end{aligned}$$

Then,

$$\frac{\partial \mathrm{H}(m)}{\partial X} = \frac{1}{X},$$

and by (5.18), we have

$$\begin{aligned}\frac{\partial \mathrm{H}(l)}{\partial X} &= -\frac{2}{X} - \frac{4}{X(XY-1)} - \frac{\partial Y}{\partial X}\left(\frac{2}{Y} + \frac{4}{Y(XY-1)}\right) \\ &= -\frac{2}{X} - \frac{4}{X(XY-1)} - \frac{Y^2(X^2-1)}{X^2(Y^2-1)}\left(\frac{2}{Y} + \frac{4}{Y(XY-1)}\right) \\ &= \frac{-2(X + X^{-1} + Y + Y^{-1})}{X(Y - Y^{-1})};\end{aligned}$$

and by the chain rule,

$$\frac{\partial \mathrm{H}(l)}{\partial \mathrm{H}(m)} = \frac{\frac{\partial \mathrm{H}(l)}{\partial X}}{\frac{\partial \mathrm{H}(m)}{\partial X}} = \frac{-2(X + X^{-1} + Y + Y^{-1})}{Y - Y^{-1}}.$$

As a consequence,

$$\frac{\partial \mathrm{H}(\mu)}{\partial \mathrm{H}(m)} = \frac{\partial(p\mathrm{H}(m) + q\mathrm{H}(l))}{\partial \mathrm{H}(m)} = p - \frac{2q(X + X^{-1} + Y + Y^{-1})}{Y - Y^{-1}}. \tag{5.20}$$

On the other hand, by [31, Example 1, page 113], for the holonomy representation $\rho_{\mathrm{H}(m)}$ of a (possibly incomplete) hyperbolic structure on $S^3\setminus K_{4_1}$ with $\mathrm{H}(m)$ the holonomy of the meridian m,

$$\begin{aligned}\mathrm{Tor}_m(S^3\setminus K_{4_1}; \mathrm{Ad}_{\rho_{\mathrm{H}(m)}}) &= \pm\frac{\sqrt{(e^{\mathrm{H}(m)} + e^{-\mathrm{H}(m)} - 3)(e^{\mathrm{H}(m)} + e^{-\mathrm{H}(m)} + 1)}}{2} \\ &= \pm\frac{\sqrt{(X + X^{-1} - 3)(X + X^{-1} + 1)}}{2} \\ &= \pm\frac{Y - Y^{-1}}{2},\end{aligned} \tag{5.21}$$

where the second equation comes from (5.19) and the last equation comes from (5.17). Notice that the Reidemeister torsion is a quantity defined up to $\pm$, and here we choose the "+" sign. Then, by (5.16), (5.20), and (5.21), we have

$$\begin{aligned}\frac{16}{q}\operatorname{Tor}_\mu(S^3\setminus K_{4_1};\mathrm{Ad}_\rho) &= \frac{16}{q}\frac{\partial \mathrm{H}(\mu)}{\partial \mathrm{H}(m)}\bigg|_\rho \cdot \operatorname{Tor}_m(S^3\setminus K_{4_1};\mathrm{Ad}_\rho)\\ &= \frac{16}{q}\frac{Y_0-Y_0^{-1}}{2}\left(p-\frac{2q(X_0+X_0^{-1}+Y_0+Y_0^{-1})}{Y_0-Y_0^{-1}}\right)\\ &= \frac{8p}{q}(Y_0-Y_0^{-1})-16(X_0+X_0^{-1}+Y_0+Y_0^{-1}).\end{aligned}$$

Comparing with (5.15), we have

$$\det(\operatorname{Hess} V^+)(x_0,y_0)=\frac{16}{q}\operatorname{Tor}_\mu(S^3\setminus K_{4_1};\mathrm{Ad}_\rho).$$

Finally, since the adjoint twisted Reidemeister torsion $\operatorname{Tor}_\mu(S^3\setminus K_{4_1};\mathrm{Ad}_\rho)$ is a non-zero quantity, $\det(\operatorname{Hess} V^+)(x_0,y_0)=\det(\operatorname{Hess} V^-)(-x_0,y_0)\neq 0$ and the Hessian matrices $\operatorname{Hess} V^+(x_0,y_0)$ and $\operatorname{Hess} V^-(-x_0,y_0)$ are non-singular.

For (3), by Lemma 3.3 (2) that $J(s^+)\equiv -J(s^-) \pmod{2\mathbb{Z}}$, we have

$$\sin\left(\frac{x_0}{q}-J(s^+)\pi\right)=-\sin\left(\frac{-x_0}{q}-J(s^-)\pi\right). \tag{5.22}$$

Together with (2), we have

$$\frac{\sin(\frac{x_0}{q}-J(s^+)\pi)}{\sqrt{\det(\operatorname{Hess} V^+)(x_0,y_0)}}=-\frac{\sin(\frac{-x_0}{q}-J(s^-)\pi)}{\sqrt{\det(\operatorname{Hess} V^-)(-x_0,y_0)}},$$

and it suffices to prove that

$$\frac{\sin(\frac{x_0}{q}-J(s^+)\pi)}{\sqrt{\det(\operatorname{Hess} V^+)(x_0,y_0)}}=\pm\frac{\sqrt{-q}}{8\sqrt{\operatorname{Tor}(M;\mathrm{Ad}_\rho)}}.$$

To this end, we will use the surgery formula [31, Theorem 4.1 (iii)] that

$$\operatorname{Tor}(M;\mathrm{Ad}_\rho)=\frac{1}{4\sinh^2\frac{\mathrm{H}(\gamma)}{2}}\operatorname{Tor}_\mu(S^3\setminus K_{4_1};\mathrm{Ad}_\rho). \tag{5.23}$$

By (5.13), we have

$$\frac{x_0}{q}-\frac{p'\pi}{q}=\frac{\sqrt{-1}}{2}\mathrm{H}(\gamma).$$

Then, by Lemma 3.3 (2) that $J(s^+)\equiv\frac{p'}{q} \pmod{\mathbb{Z}}$, we have

$$\begin{aligned}\sin\left(\frac{x_0}{q}-J(s^+)\pi\right)&=(-1)^{J(s^+)-\frac{p'}{q}}\sin\left(\frac{x_0}{q}-\frac{p'\pi}{q}\right)\\ &=(-1)^{J(s^+)-\frac{p'}{q}}\sqrt{-1}\sinh\frac{\mathrm{H}(\gamma)}{2}.\end{aligned} \tag{5.24}$$

Finally, by the surgery formula (5.23) and (2), we have

$$\frac{\sin(\frac{x_0}{q} - J(s^+)\pi)}{\sqrt{\det(\operatorname{Hess} V^+)(x_0, y_0)}} = (-1)^{J(s^+)-\frac{p'}{q}} \frac{\sqrt{-q}\sinh\frac{\mathrm{H}(\gamma)}{2}}{\sqrt{16\operatorname{Tor}_\mu(S^3\setminus K_{4_1}; \mathrm{Ad}_\rho)}}$$
$$= (-1)^{J(s^+)-\frac{p'}{q}} \frac{\sqrt{-q}}{8\sqrt{\operatorname{Tor}(M;\mathrm{Ad}_\rho)}}. \qquad \blacksquare$$

The following Lemmas 5.5 and 5.6 are crucial in the estimate of the Fourier coefficients in Section 6.

Lemma 5.5. *In* $\mathcal{D}_\mathbb{C} = D_\mathbb{C} \cup D'_\mathbb{C} \cup D''_\mathbb{C}$*,* $\operatorname{Im} V^\pm(x, y)$ *is strictly concave down in* $\operatorname{Re}(x)$ *and* $\operatorname{Re}(y)$*, and is strictly concave up in* $\operatorname{Im}(x)$ *and* $\operatorname{Im}(y)$*.*

Proof. Using (5.7), taking the second derivatives of $\operatorname{Im} V^\pm$ with respect to $\operatorname{Re}(x)$ and $\operatorname{Re}(y)$, we have

$$\operatorname{Hess}(\operatorname{Im} V^\pm)$$
$$= \begin{bmatrix} -\frac{4\operatorname{Im} e^{2\sqrt{-1}(y+x)}}{|1-e^{2\sqrt{-1}(y+x)}|^2} - \frac{4\operatorname{Im} e^{2\sqrt{-1}(y-x)}}{|1-e^{2\sqrt{-1}(y-x)}|^2} & -\frac{4\operatorname{Im} e^{2\sqrt{-1}(y+x)}}{|1-e^{2\sqrt{-1}(y+x)}|^2} + \frac{4\operatorname{Im} e^{2\sqrt{-1}(y-x)}}{|1-e^{2\sqrt{-1}(y-x)}|^2} \\ -\frac{4\operatorname{Im} e^{2\sqrt{-1}(y+x)}}{|1-e^{2\sqrt{-1}(y+x)}|^2} + \frac{4\operatorname{Im} e^{2\sqrt{-1}(y-x)}}{|1-e^{2\sqrt{-1}(y-x)}|^2} & -\frac{4\operatorname{Im} e^{2\sqrt{-1}(y+x)}}{|1-e^{2\sqrt{-1}(y+x)}|^2} - \frac{4\operatorname{Im} e^{2\sqrt{-1}(y-x)}}{|1-e^{2\sqrt{-1}(y-x)}|^2} \end{bmatrix}$$
$$= -\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{\operatorname{Im} e^{2\sqrt{-1}(y+x)}}{|1-e^{2\sqrt{-1}(y+x)}|^2} & 0 \\ 0 & \frac{\operatorname{Im} e^{2\sqrt{-1}(y-x)}}{|1-e^{2\sqrt{-1}(y-x)}|^2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}.$$

Since in $\mathcal{D}_\mathbb{C}$, $\operatorname{Im} e^{2\sqrt{-1}(y+x)} > 0$ and $\operatorname{Im} e^{2\sqrt{-1}(y-x)} > 0$, the diagonal matrix in the middle is positive definite, and hence, $\operatorname{Hess}(\operatorname{Im} V^\pm)$ is negative definite. Therefore, $\operatorname{Im} V$ is concave down in $\operatorname{Re}(x)$ and $\operatorname{Re}(y)$. Since $\operatorname{Im} V^\pm$ is harmonic, it is concave up in $\operatorname{Im}(x)$ and $\operatorname{Im}(y)$. ∎

Lemma 5.6. *We have* $\operatorname{Im}(x_0) \neq 0$.

Proof. By (5.3), the holonomy of the meridian $\mathrm{H}(m) = 2x_0\sqrt{-1}$. We prove by contradiction. Suppose $\operatorname{Im}(x_0) = 0$; then, $\mathrm{H}(m)$ is purely imaginary. As a consequence, $\mathrm{H}(l) = \frac{2\pi\sqrt{-1} - p\mathrm{H}(m)}{q}$ is also purely imaginary. This implies that the holonomy of the core curve of the filled solid torus $H(\gamma) = q'\mathrm{H}(m) - p'\mathrm{H}(l)$ is purely imaginary; i.e., γ has length zero, which is a contradiction. ∎

6. Asymptotics of the Reshetikhin–Turaev invariants

The goal of this section is to prove Theorem 1.1 by estimating each of the Fourier coefficients $\hat{f}_r(s, m, n)$ in Proposition 4.3. In Section 6.1, we estimate the two leading ones $\hat{f}_r(s^+, m^+, 0)$ and $\hat{f}_r(s^-, m^-, 0)$, and in Section 6.2, we estimate the others. Finally, in Section 6.3, we show that $\hat{f}_r(s^+, m^+, 0) + \hat{f}_r(s^-, m^-, 0)$ has the desired asymptotic behavior and the sum of all the other Fourier coefficients are neglectable, which completes the proof. We will also prove Theorem 1.2 at the end of Section 6.3.

6.1. Estimates of the leading Fourier coefficients

The main result of this section is Proposition 6.4, where we estimate the integrals in the Fourier coefficients $\hat{f}_r(s^+, m^+, 0)$ and $\hat{f}_r(s^-, m^-, 0)$ on the region $D_{\mathbb{C},\delta}$ as defined in Section 4, which turn out to be the leading terms in the summation of Proposition 4.3.

We need the following Proposition 6.1, which is a generalization of the standard steepest descent theorem (see, e.g., [21, Theorem 7.2.8]) and is also stated and proved by Ohtsuki [25, Proposition 3.5, Remarks 3.3, 3.6] in a slightly different form. For the readers' convenience, we include a proof in the appendix.

Proposition 6.1 ([25]). *Let D be a region in $\mathbb{C}^n$ and let $f(z_1, \dots, z_n)$, $g(z_1, \dots, z_n)$ be holomorphic functions on D independent of r. Let $f_r(z_1, \dots, z_n)$ be a holomorphic function of the form*

$$f_r(z_1, \dots, z_n) = f(z_1, \dots, z_n) + \frac{\upsilon_r(z_1, \dots, z_n)}{r^2}.$$

Let S be an embedded real n-dimensional closed disk in D and let $(c_1, \dots, c_n)$ be a point on S. If

(1) *$(c_1, \dots, c_n)$ is a critical point of f in D,*

(2) $\mathrm{Re}(f)(c_1, \dots, c_n) > \mathrm{Re}(f)(z_1, \dots, z_n)$ *for all* $(z_1, \dots, z_n) \in S \setminus \{(c_1, \dots, c_n)\}$,

(3) *the domain* $\{(z_1, \dots, z_n) \in D \mid \mathrm{Re}\, f(z_1, \dots, z_n) < \mathrm{Re}\, f(c_1, \dots, c_n)\}$ *deformation retracts to* $S \setminus \{(c_1, \dots, c_n)\}$,

(4) *the Hessian matrix* $\mathrm{Hess}(f)(c_1, \dots, c_n)$ *of f at $(c_1, \dots, c_n)$ is non-singular,*

(5) $g(c_1, \dots, c_n) \neq 0$, *and*

(6) *$|\upsilon_r(z_1, \dots, z_n)|$ is bounded from above by a constant independent of r in D,*

then

$$\int_S g(z_1, \dots, z_n) e^{r f_r(z_1, \dots, z_n)} dz_1 \cdots dz_n$$
$$= \left(\frac{2\pi}{r}\right)^{\frac{n}{2}} \frac{g(c_1, \dots, c_n)}{\sqrt{\det(-\mathrm{Hess}(f)(c_1, \dots, c_n))}} e^{r f(c_1, \dots, c_n)} \left(1 + O\left(\frac{1}{r}\right)\right).$$

To apply Proposition 6.1, we need the following Lemma 6.2. Recall from Section 4 that for $\delta \geqslant 0$, the region

$$D_{\mathbb{C},\delta} = \left\{ (x,y) \in \mathbb{C}^2 \mid \delta < \mathrm{Re}(y) + \mathrm{Re}(x) < \frac{\pi}{2} - \delta, \delta < \mathrm{Re}(y) - \mathrm{Re}(x) < \frac{\pi}{2} - \delta \right\},$$

and for (x,y) in $D_{\mathbb{C},\delta}$, the functions

$$\begin{aligned} V_r^{\pm}(x,y) &= V_r(s^{\pm}, m^{\pm}, 0) - 4\pi m^{\pm} x \\ &= \frac{-px^2 \pm 2\pi x}{q} - 2\pi x + 4xy - \varphi_r\left(\pi - y - x - \frac{\pi}{r}\right) \\ &\quad + \varphi_r\left(y - x + \frac{\pi}{r}\right) + K(s^{\pm})\pi^2, \end{aligned}$$

and

$$\begin{aligned} V^{\pm}(x,y) = \frac{-px^2 \pm 2\pi x}{q} &- 2\pi x + 4xy - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y+x)}\big) \\ &+ \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y-x)}\big) + K(s^{\pm})\pi^2. \end{aligned}$$

Lemma 6.2. *In* $\{(x,y) \in \overline{D_{\mathbb{C},\delta}} \mid |\mathrm{Im}\, x| < L, |\mathrm{Im}\, y| < L\}$ *for* $\delta > 0$ *and* $L > 0$,

$$\begin{aligned} V_r^{\pm}(x,y) &= V^{\pm}(x,y) \\ &\quad - \frac{2\pi\sqrt{-1}(\log(1 - e^{-2\sqrt{-1}(y+x)}) + \log(1 - e^{2\sqrt{-1}(y-x)}))}{r} \\ &\quad + \frac{\upsilon_r(x,y)}{r^2}, \end{aligned}$$

with $|\upsilon_r(x,y)|$ *bounded from above by a constant independent of* r.

Proof. Expanding in $\frac{1}{r}$, we have

$$\varphi_r\left(\pi - x - y - \frac{\pi}{r}\right) = \varphi_r(\pi - x - y) - \varphi_r'(\pi - x - y) \cdot \frac{\pi}{r} + O\left(\frac{1}{r^2}\right)$$

and

$$\varphi_r\left(y - x + \frac{\pi}{r}\right) = \varphi_r(y - x) + \varphi_r'(y - x) \cdot \frac{\pi}{r} + O\left(\frac{1}{r^2}\right).$$

By Lemma 2.3 (1), we have

$$\begin{aligned} &-\varphi_r(\pi - x - y) + \varphi_r(y - x) \\ &\quad = -\mathrm{Li}\big(e^{-2\sqrt{-1}(y+x)}\big) + \mathrm{Li}\big(e^{2\sqrt{-1}(y-x)}\big) \\ &\qquad + \left(-\frac{2\pi^2 e^{-2\sqrt{-1}(y+x)}}{3(1 - e^{-2\sqrt{-1}(y+x)})} + \frac{2\pi^2 e^{2\sqrt{-1}(y-x)}}{3(1 - e^{2\sqrt{-1}(y-x)})} \right) \frac{1}{r^2} + O\left(\frac{1}{r^4}\right), \end{aligned}$$

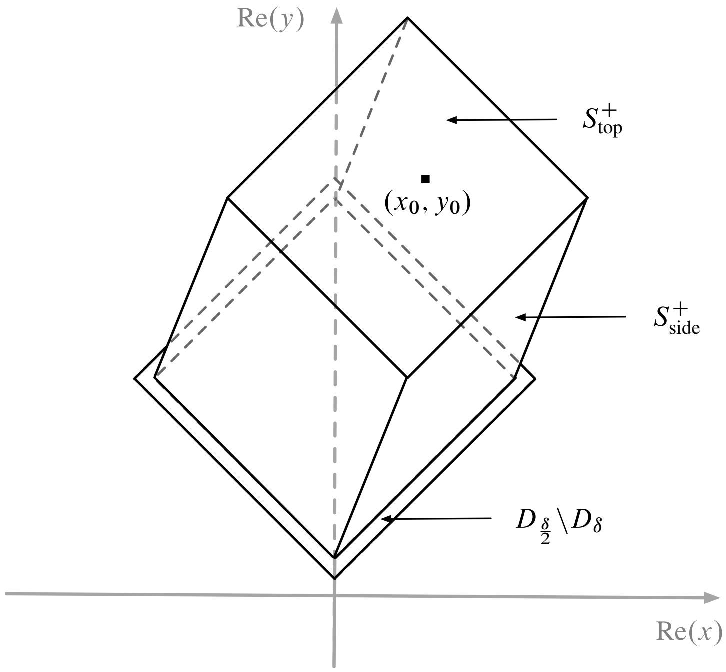


Figure 5. The deformed surface S^+.

and by Lemma 2.3 (2), we have

$$\begin{aligned}\varphi_r'(\pi - x - y)\cdot\frac{\pi}{r} + \varphi_r'(y-x)\cdot\frac{\pi}{r} \\ = -\frac{2\pi\sqrt{-1}}{r}\log\big(1 - e^{-2\sqrt{-1}(y+x)}\big) \\ -\frac{2\pi\sqrt{-1}}{r}\log\big(1 - e^{2\sqrt{-1}(y-x)}\big) + O\Big(\frac{1}{r^3}\Big).\end{aligned}$$

The result then follows with

$$\upsilon_r(x,y) = -\frac{2\pi^2 e^{-2\sqrt{-1}(y+x)}}{3(1-e^{-2\sqrt{-1}(y+x)})} + \frac{2\pi^2 e^{2\sqrt{-1}(y-x)}}{3(1-e^{2\sqrt{-1}(y-x)})} + O\Big(\frac{1}{r}\Big). \qquad \blacksquare$$

Let (x_0, y_0) be the unique critical point of V^+ in $D_{\mathbb{C}}$, and by Corollary 5.3, $(-x_0, y_0)$ is the unique critical point of V^- in $D_{\mathbb{C}}$. Let δ be as in Proposition 4.1, and as drawn in Figure 5, let $S^+ = S^+_{\text{top}} \cup S^+_{\text{side}} \cup (D_{\frac{\delta}{2}} \smallsetminus D_\delta)$ be the union of $D_{\frac{\delta}{2}} \smallsetminus D_\delta$ with the two surfaces

$$S^+_{\text{top}} = \big\{(x,y) \in D_{\mathbb{C},\delta} \mid (\mathrm{Im}(x), \mathrm{Im}(y)) = (\mathrm{Im}(x_0), \mathrm{Im}(y_0))\big\}$$

and

$$S^+_{\mathrm{side}} = \big\{(\theta_1 + \sqrt{-1}t\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}t\,\mathrm{Im}(y_0)) \mid (\theta_1,\theta_2) \in \partial D_\delta, t \in [0,1]\big\};$$

and let $S^- = S^-_{\mathrm{top}} \cup S^-_{\mathrm{side}} \cup (D_{\frac{\delta}{2}} \smallsetminus D_\delta)$ be the union of $D_{\frac{\delta}{2}} \smallsetminus D_\delta$ with the two surfaces

$$S^-_{\mathrm{top}} = \big\{(x,y) \in D_{\mathbb{C},\delta} \mid (\mathrm{Im}(x), \mathrm{Im}(y)) = (-\mathrm{Im}(x_0), \mathrm{Im}(y_0))\big\}$$

and

$$S^-_{\mathrm{side}} = \big\{(\theta_1 - \sqrt{-1}t\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}t\,\mathrm{Im}(y_0)) \mid (\theta_1,\theta_2) \in \partial D_\delta, t \in [0,1]\big\}.$$

Then, $S^\pm$ are homotopic to $D_{\frac{\delta}{2}}$; and since δ is sufficiently small, $S^\pm$, respectively, contain $(\pm x_0, y_0)$.

Proposition 6.3. $\mathrm{Im}\,V^+$ *achieves the only absolute maximum at* (x_0, y_0) *on* S^+*, and* $\mathrm{Im}\,V^-$ *achieves the only absolute maximum at* $(-x_0, y_0)$ *on* S^-*.*

Proof. First, by Propositions 4.1 and 2.4, on $D_{\frac{\delta}{2}} \smallsetminus D_\delta$, $\mathrm{Im}\,V(x,y) \leqslant \frac{1}{2}\mathrm{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \mathrm{Vol}(M) = \mathrm{Im}\,V^+(x_0,y_0)$.

By Lemma 5.5, $\mathrm{Im}\,V^+$ is concave down on S^+_{top}. Since (x_0,y_0) is the critical point of $\mathrm{Im}\,V^+$, it is the only absolute maximum on S^+_{top}.

On the side S^+_{side}, for each $(\theta_1,\theta_2) \in \partial D_\delta$, consider the function

$$g^+_{(\theta_1,\theta_2)}(t) \doteq \mathrm{Im}\,V^+\big(\theta_1 + \sqrt{-1}t\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}t\,\mathrm{Im}(y_0)\big)$$

on $[0,1]$. We show that $g^+_{(\theta_1,\theta_2)}(t) < \mathrm{Im}\,V^+(x_0,y_0)$ for each $(\theta_1,\theta_2) \in \partial D_\delta$ and $t \in [0,1]$. Indeed, since $(\theta_1,\theta_2) \in \partial D_\delta$, $g^+_{(\theta_1,\theta_2)}(0) = \mathrm{Im}\,V^+(\theta_1,\theta_2) < \frac{1}{2}\mathrm{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \mathrm{Vol}(M) = \mathrm{Im}\,V^+(x_0,y_0)$; and since $(\theta_1 + \sqrt{-1}\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}\,\mathrm{Im}(y_0)) \in S^+_{\mathrm{top}}$, by the previous step

$$g^+_{(\theta_1,\theta_2)}(1) = \mathrm{Im}\,V^+(\theta_1 + \sqrt{-1}\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}\,\mathrm{Im}(y_0)) < \mathrm{Im}\,V^+(x_0,y_0).$$

Now, by Lemma 5.5, $g^+_{(\theta_1,\theta_2)}$ is concave up; hence,

$$g^+_{(\theta_1,\theta_2)}(t) \leqslant \max\big\{g^+_{(\theta_1,\theta_2)}(0), g^+_{(\theta_1,\theta_2)}(1)\big\} < \mathrm{Im}\,V^+(x_0,y_0).$$

Putting all these together, we have $\mathrm{Im}\,V^+(x,y) \leqslant \mathrm{Im}\,V^+(x_0,y_0)$ on S^+.

The other case is similar. By Lemma 5.5, $\mathrm{Im}\,V^-$ is concave down on S^-_{top}. Since $(-x_0,y_0)$ is the critical point of $\mathrm{Im}\,V^-$, it is the only absolute maximum on S^-_{top}.

On the side S^-_{side}, for each $(\theta_1,\theta_2) \in \partial D_\delta$, consider the function

$$g^-_{(\theta_1,\theta_2)}(t) \doteq \mathrm{Im}\,V^-\big(\theta_1 - \sqrt{-1}t\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}t\,\mathrm{Im}(y_0)\big)$$

on [0, 1]. We show that $g^-_{(\theta_1,\theta_2)}(t) < \operatorname{Im} V^-(-x_0, y_0)$ for each $(\theta_1,\theta_2) \in \partial D_\delta$ and $t \in [0,1]$. Indeed, since $(\theta_1,\theta_2) \in \partial D_\delta$, $g^-_{(\theta_1,\theta_2)}(0) = \operatorname{Im} V^-(\theta_1,\theta_2) < \frac{1}{2}\operatorname{Vol}(\mathrm{S}^3\setminus K_{4_1}) + \varepsilon < \operatorname{Vol}(M) = \operatorname{Im} V^-(-x_0, y_0)$; and since $(\theta_1 - \sqrt{-1}\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}\operatorname{Im}(y_0)) \in S^-_{\mathrm{top}}$, by the previous step $g^-_{(\theta_1,\theta_2)}(1) = \operatorname{Im} V^-(\theta_1 - \sqrt{-1}\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}\operatorname{Im}(y_0)) < \operatorname{Im} V^\pm(-x_0, y_0)$. Now, by Lemma 5.5, $g^-_{(\theta_1,\theta_2)}$ is concave up; hence,

$$g^-_{(\theta_1,\theta_2)}(t) \leqslant \max\{g^-_{(\theta_1,\theta_2)}(0), g^-_{(\theta_1,\theta_2)}(1)\} < \operatorname{Im} V^-(\pm x_0, y_0).$$

Putting all these together, we have $\operatorname{Im} V^-(x,y) \leqslant \operatorname{Im} V^-(x_0,y_0)$ on S^-. ∎

Proposition 6.4. *We have the following estimations.*

(1) *For the integral in* $\hat{f}_r(s^+, m^+, 0)$*, we have*

$$\begin{aligned}\int_{D_{\frac{\delta}{2}}} &\psi(x,y)\sin\Big(\frac{x}{q} - J(s^+)\pi\Big)\varepsilon(x,y) \\ &\times e^{-x\sqrt{-1} + \frac{r}{4\pi\sqrt{-1}}(V_r(s^+,x,y) - 4\pi m^+ x)}\,dx\,dy \\ &= \frac{c_r^+}{\sqrt{\operatorname{Tor}(M;\operatorname{Ad}_\rho)}} e^{\frac{r}{4\pi}(\operatorname{Vol}(M) + \sqrt{-1}\operatorname{CS}(M))}\Big(1 + O\Big(\frac{1}{r}\Big)\Big),\end{aligned}$$

where

$$c_r^+ = -\frac{2\pi^2\sqrt{q}}{r}(-1)^{J(s^+) - \frac{p'}{q}} e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^+) + \frac{p'}{q})}.$$

(2) *For the integral in* $\hat{f}_r(s^-, m^-, 0)$*, we have*

$$\begin{aligned}\int_{D_{\frac{\delta}{2}}} &\psi(x,y)\sin\Big(\frac{x}{q} - J(s^-)\pi\Big)\varepsilon(x,y) \\ &\times e^{-x\sqrt{-1} + \frac{r}{4\pi\sqrt{-1}}(V_r(s^-,x,y) - 4\pi m^- x)}\,dx\,dy \\ &= \frac{c_r^-}{\sqrt{\operatorname{Tor}(M;\operatorname{Ad}_\rho)}} e^{\frac{r}{4\pi}(\operatorname{Vol}(M) + \sqrt{-1}\operatorname{CS}(M))}\Big(1 + O\Big(\frac{1}{r}\Big)\Big),\end{aligned}$$

where

$$c_r^- = \frac{2\pi^2\sqrt{q}}{r}(-1)^{J(s^+) - \frac{p'}{q}} e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^-) + \frac{p'}{q})}.$$

Proof. By Lemma 6.2, we have

$$\begin{aligned}&e^{-x\sqrt{-1} + \frac{r}{4\pi\sqrt{-1}}(V_r(s^\pm,x,y) - 4\pi m^\pm x)} \\ &= e^{-x\sqrt{-1} + \frac{r}{4\pi\sqrt{-1}}V_r^\pm(x,y)} \\ &= e^{-x\sqrt{-1} - \frac{\log(1 - e^{-2\sqrt{-1}(y+x)})}{2} - \frac{\log(1 - e^{2\sqrt{-1}(y-x)})}{2} + \frac{r}{4\pi\sqrt{-1}}(V^\pm(x,y) + \frac{\upsilon_r(x,y)}{r^2})}.\end{aligned}$$

By analyticity, the integrals remain the same if we deform the domains from $D_{\frac{\delta}{2}}$ to $S^{\pm}$.

Now, we apply Proposition 6.1, with the region

$$D = D_L \doteq \{(x,y) \in D_{\mathbb{C},\frac{\delta}{2}} \mid |\operatorname{Im} x| < L, |\operatorname{Im} y| < L\}$$

for a sufficiently large L, the embedded disk $S = S^{\pm}$, the functions $f(x,y) = \frac{1}{4\pi\sqrt{-1}} V^{\pm}(x,y)$, $f_r(x,y) = \frac{1}{4\pi\sqrt{-1}}(V^{\pm}(x,y) + \frac{\upsilon_r(x,y)}{r^2})$,

$$g(x,y) = \psi(x,y) \sin\left(\frac{x}{q} - J(s^{\pm})\pi\right)\varepsilon(x,y) \\ \times e^{-x\sqrt{-1} - \frac{\log(1-e^{-2\sqrt{-1}(y+x)})}{2} - \frac{\log(1-e^{2\sqrt{-1}(y-x)})}{2}},$$

and the point $(c_1, c_2) = (\pm x_0, y_0)$.

Then, by Corollary 5.3, $(\pm x_0, y_0)$ are, respectively, the critical points of $V^{\pm}$, and hence, of f, and (1) of Proposition 6.1 is satisfied. By Proposition 6.3, $\operatorname{Re}(f) = \frac{1}{4\pi} \operatorname{Im} V^{\pm}$ achieves the only absolute maximum on $S^{\pm}$ at $(\pm x_0, y_0)$, and (2) of Proposition 6.1 is satisfied.

To verify (3) of Proposition 6.1, for each $(\theta_1, \theta_2) \in D_{\frac{\delta}{2}}$, let

$$P_{(\theta_1,\theta_2)} = \{(x,y) \in D_L \mid \operatorname{Re}(x) = \theta_1, \operatorname{Re}(y) = \theta_2\},$$

and let

$$D^{\pm}_{(\theta_1,\theta_2)} = \{(x,y) \in P_{(\theta_1,\theta_2)} \mid \operatorname{Im} V^{\pm}(x,y) < \operatorname{Im} V^{\pm}(\pm x_0, y_0)\}.$$

Then, we claim that

$$D^{+}_{(\operatorname{Re}(x_0),\operatorname{Re}(y_0))} = D^{-}_{(\operatorname{Re}(-x_0),\operatorname{Re}(y_0))} = \emptyset,$$

$D^{+}_{(\theta_1,\theta_2)}$ is homeomorphic to a disk for $(\theta_1,\theta_2) \neq (\operatorname{Re}(x_0), \operatorname{Re}(y_0))$, and $D^{-}_{(\theta_1,\theta_2)}$ is homeomorphic to a disk for $(\theta_1,\theta_2) \neq (\operatorname{Re}(-x_0), \operatorname{Re}(y_0))$, from which we conclude that the domains

$$\{(x,y) \in D_L \mid \operatorname{Im} V^{\pm}(x,y) < \operatorname{Im} V^{\pm}(\pm x_0, y_0)\},$$

respectively, deformation retract to $S^{\pm} \setminus (\pm x_0, y_0)$ by shrinking each $D^{\pm}_{(\theta_1,\theta_2)}$, respectively, to $\{(\theta_1 \pm \sqrt{-1}\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}\operatorname{Im}(y_0))\}$, verifying (3) of Proposition 6.1.

To prove the claim, we by Lemma 5.5 have that, on $P_{(\operatorname{Re}(\pm x_0),\operatorname{Re}(y_0))}$, $\operatorname{Im} V^{\pm}$, respectively, achieve the absolute minimum at $(\pm x_0, y_0)$; hence, $D^{\pm}_{(\operatorname{Re}(\pm x_0),\operatorname{Re}(y_0))} = \emptyset$. For $(\theta_1,\theta_2) \neq (\operatorname{Re}(\pm x_0), \operatorname{Re}(y_0))$, we have

$$\min_{P_{(\theta_1,\theta_2)}} \operatorname{Im} V^{\pm} \leqslant \operatorname{Im} V^{\pm}\big(\theta_1 \pm \sqrt{-1}\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}\operatorname{Im}(y_0)\big) < \operatorname{Im} V^{\pm}(\pm x_0, y_0),$$

where the last inequality comes from that both $(\theta_1 \pm \sqrt{-1}\,\mathrm{Im}(x_0), \theta_2 + \sqrt{-1}\,\mathrm{Im}(y_0))$ and $(\pm x_0, y_0)$ are on $S^{\pm}_{\mathrm{top}}$ and $\mathrm{Im}\, V^{\pm}$ achieve the only maximum at $(\pm x_0, y_0)$. Then, by Lemma 5.5 that $\mathrm{Im}\, V^{\pm}$ is concave up on $P_{(\theta_1,\theta_2)}$, $D_{(\theta_1,\theta_2)}$ is a convex subset of $P_{(\theta_1,\theta_2)}$, which is homeomorphic to a disk. This proves the claim, and verifies (3) of Proposition 6.1.

By Proposition 5.4 (2), $\det(\mathrm{Hess}\, f)(c_1, c_2) = -\frac{1}{16\pi^2}\det(\mathrm{Hess}\, V^{\pm})(\pm x_0, y_0) \neq 0$, and (4) of Proposition 6.1 is satisfied. At the critical points $(\pm x_0, y_0)$, by the first equation of the system of critical equations (5.2), we have

$$\mp x_0\sqrt{-1} - \frac{\log(1 - e^{-2\sqrt{-1}(y_0 \pm x_0)})}{2} - \frac{\log(1 - e^{2\sqrt{-1}(y_0 \mp x_0)})}{2} = 0.$$

Together with (5.24) and (5.22), we have

$$g(c_1, c_2) = 2\sin\left(\frac{\pm x_0}{q} - J(s^{\pm})\pi\right) = \pm(-1)^{J(s^+) - \frac{p'}{q}} 2\sqrt{-1}\sinh\frac{\mathrm{H}(\gamma)}{2} \neq 0,$$

where the inequality comes from that $\mathrm{Re}(\mathrm{H}(\gamma))$ is the length of the core curve γ of the filled solid tori, which is non-zero. Hence, (5) of Proposition 6.1 is satisfied. (6) of Proposition 6.1 follows from Lemma 6.2. Therefore, all the conditions of Proposition 6.1 are satisfied.

By Proposition 5.4 (1), the critical values

$$\begin{aligned} f(c_1, c_2) &= \frac{V^{\pm}(\pm x_0, y_0)}{4\pi\sqrt{-1}} \\ &= \frac{1}{4\pi\sqrt{-1}}\left(\sqrt{-1}\big(\mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M)\big) + \left(K(s^{\pm}) + \frac{p'}{q}\right)\pi^2\right), \end{aligned}$$

and by Proposition 5.4 (3), we have

$$\begin{aligned} \frac{g(c_1, c_2)}{\sqrt{\det(-\mathrm{Hess}(f)(c_1, c_2))}} &= \frac{2\sin(\frac{\pm x_0}{q} - J(s^{\pm})\pi)}{\sqrt{-\frac{1}{16\pi^2}\det\mathrm{Hess}(V^{\pm})(\pm x_0, y_0)}} \\ &= \mp\frac{(-1)^{J(s^+) - \frac{p'}{q}}\pi\sqrt{q}}{\sqrt{\mathrm{Tor}(M; \mathrm{Ad}_\rho)}}, \end{aligned}$$

from which the result follows. ∎

6.2. Estimate of other Fourier coefficients

For $s \in \{0, \dots, |q| - 1\}$ and $(x, y) \in \mathcal{D}_{\mathbb{C}} = D_{\mathbb{C}} \cup D'_{\mathbb{C}} \cup D''_{\mathbb{C}}$, let

$$\begin{aligned} V(s, x, y) = \frac{-px^2}{q} + I(s)\frac{2\pi x}{q} + 4xy - \mathrm{Li}_2\big(e^{-2\sqrt{-1}(y+x)}\big) \\ + \mathrm{Li}_2\big(e^{2\sqrt{-1}(y-x)}\big) + K(s)\pi^2. \end{aligned}$$

Then, similar to Lemma 6.2, we have the following lemma.

Lemma 6.5. *In* $\{(x,y) \in \overline{D_{\mathbb{C},\delta}} \cup \overline{D'_{\mathbb{C},\delta}} \cup \overline{D''_{\mathbb{C},\delta}} \mid |\operatorname{Im} x| < L, |\operatorname{Im} y| < L\}$ *for* $\delta > 0$ *and* $L > 0$*,*

$$\begin{aligned} V_r(s,x,y) = V(s,x,y) \\ &- \frac{2\pi\sqrt{-1}(\log(1-e^{-2\sqrt{-1}(y+x)}) + \log(1-e^{2\sqrt{-1}(y-x)}))}{r} \\ &+ \frac{\upsilon_r(x,y)}{r^2} \end{aligned}$$

with $|\upsilon_r(x,y)|$ *bounded from above by a constant independent of* r*.*

6.2.1. Estimate on $D_{\frac{\delta}{2}}$.

Proposition 6.6. *There is an* $\varepsilon > 0$ *such that for each triple* $(s,m,0)$ *with* $(s,m) \neq (s^+,m^+)$ *and* $(s,m) \neq (s^-,m^-)$*,*

$$\begin{aligned} &\left|\int_{D_{\frac{\delta}{2}}} \psi(x,y)\sin\left(\frac{x}{q} - J(s)\pi\right)\varepsilon(x,y)e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s,x,y)-4\pi mx)}dx\,dy\right| \\ &\leqslant O\left(e^{\frac{r}{4\pi}(\operatorname{Vol}(M)-\varepsilon)}\right). \end{aligned}$$

Proof. We recall the surface $S^\pm = S^\pm_{\text{top}} \cup S^\pm_{\text{side}} \cup (D_{\frac{\delta}{2}} \smallsetminus D_\delta)$ from Section 6.1, where

$$S^\pm_{\text{top}} = \left\{(x,y) \in D_{\mathbb{C},\delta} \mid (\operatorname{Im}(x),\operatorname{Im}(y)) = (\pm\operatorname{Im}(x_0),\operatorname{Im}(y_0))\right\}$$

and

$$S^\pm_{\text{side}} = \left\{(\theta_1 \pm \sqrt{-1}t\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}t\operatorname{Im}(y_0)) \mid (\theta_1,\theta_2) \in \partial D_\delta, t \in [0,1]\right\},$$

with $(\pm x_0, y_0)$ the critical points of $V^\pm$ given in Corollary 5.3. We will prove that for some $\varepsilon > 0$,

$$\operatorname{Im}\big(V(s,x,y) - 4\pi mx\big) < \operatorname{Vol}(M) - \varepsilon$$

either on S^+ or on S^-. Then, the result follows form Lemma 6.5 and the analyticity of V_r that the domain of integral $D_{\frac{\delta}{2}}$ could be deformed to $S^\pm$.

By Lemma 5.6, $\operatorname{Im}(x_0) \neq 0$. Without loss of generality, we may assume that $\operatorname{Im}(x_0) > 0$, since otherwise, we can consider $\operatorname{Im}(-x_0)$.

Now, for $s \in \{0,\dots,|q|-1\}$ and $m \in \mathbb{Z}$, let

$$k^+(s,m) = -\frac{I(s)-1}{q} - 1 + 2m$$

and

$$k^-(s,m) = -\frac{I(s)+1}{q} - 1 + 2m.$$

Then, by a direct computation,

$$\begin{aligned} V(s,x,y) - 4\pi m x &= V^+(x,y) - 2\pi k^+(s,m)x + \big(K(s) - K(s^+)\big)\pi^2 \\ &= V^-(x,y) - 2\pi k^-(s,m)x + \big(K(s) - K(s^-)\big)\pi^2. \end{aligned}$$

By Lemma 3.3 (1), $(s,m) = (s^+, m^+)$ is the only pair such that $k^+(s,m) = 0$, and $(s,m) = (s^-, m^-)$ is the only pair such that $k^-(s,m) = 0$. By Lemma 3.3 (1), elements of

$$\big\{k^\pm(s,m) \mid s \in \{0,\dots,|q|-1\} \text{ and } m \in \mathbb{Z}\big\}$$

differ by a multiple of $\frac{2}{q}$; and for each (s,m), $k^+(s,m) - k^-(s,m) = \frac{2}{q}$. This implies that for each pair (s,m) other than $(s^\pm, m^\pm)$, $k^+(s,m)$ and $k^-(s,m)$ are either both strictly positive, or both strictly negative.

By Proposition 6.3, for any $(x,y) \in S^\pm_{\text{top}}$ we, respectively, have

$$\operatorname{Im} V^\pm(x,y) \leqslant \operatorname{Im} V^\pm(\pm x_0, y_0) = \operatorname{Vol}(M).$$

Then, for (s,m) with $k^\pm(s,m) > 0$, we have on S^+_{top} that

$$\operatorname{Im}\big(V(s,x,y) - 4\pi m x\big) = \operatorname{Im} V^+(x,y) - 2\pi k^+(s,m)\operatorname{Im}(x_0) < \operatorname{Vol}(M) - \varepsilon$$

for some $\varepsilon > 0$; and for (s,m) with $k^\pm(s,m) < 0$, we have on S^-_{top} that

$$\operatorname{Im}\big(V(s,x,y) - 4\pi m x\big) = \operatorname{Im} V^-(x,y) + 2\pi k^-(s,m)\operatorname{Im}(x_0) < \operatorname{Vol}(M) - \varepsilon$$

for some $\varepsilon > 0$.

By (2.4), Proposition 2.4, and the same computation as in the proof of Lemma 4.2 and the ε therein, we have on $D_{\frac{\delta}{2}} \smallsetminus D_\delta$ that

$$\begin{aligned} \operatorname{Im}\big(V(s,x,y) - 4\pi m x\big) &= \operatorname{Im} V(s,x,y) = 2\Lambda(y+x) + 2\Lambda(y-x) \\ &\leqslant 2\Lambda\left(\frac{\pi}{6}\right) + \varepsilon = \frac{1}{2}\operatorname{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \operatorname{Vol}(M) - \varepsilon. \end{aligned}$$

On $S^\pm_{\text{side}}$, we notice that $V(s,x,y) - 4\pi m x$ differs from $V^\pm$ by a linear function. Hence, by Lemma 5.5, for each $(\theta_1,\theta_2) \in \partial D_\delta$, the function

$$\begin{aligned} &g^\pm_{(\theta_1,\theta_2)}(t) \\ &\doteq \operatorname{Im}\Big(V\big(s, \theta_1 \pm \sqrt{-1}t\operatorname{Im}(x_0), \theta_2 + \sqrt{-1}t\operatorname{Im}(y_0)\big) - 4\pi m\big(\theta_1 \pm \sqrt{-1}t\operatorname{Im}(x_0)\big)\Big) \end{aligned}$$

is concave up on $[0,1]$. Therefore, for (s,m) with $k^\pm(s,m) > 0$, we have on S^+_{side} that

$$\operatorname{Im}\big(V(s,x,y) - 4\pi m x\big) = g^+_{(\theta_1,\theta_2)}(t) \leqslant \max\big\{g^+_{(\theta_1,\theta_2)}(0), g^+_{(\theta_1,\theta_2)}(1)\big\} < \operatorname{Vol}(M) - \varepsilon;$$

and for (s,m) with $k^{\pm}(s,m) < 0$, we have on S^-_{side} that

$$\operatorname{Im}\big(V(s,x,y) - 4\pi m x\big) = g^-_{(\theta_1,\theta_2)}(t) \leqslant \max\big\{g^-_{(\theta_1,\theta_2)}(0), g^-_{(\theta_1,\theta_2)}(1)\big\} < \operatorname{Vol}(M) - \varepsilon.$$

Putting all these together, we have for (s,m) with $k^{\pm}(s,m) > 0$, $\operatorname{Im}(V(s,x,y) - 4\pi m x) < \operatorname{Vol}(M) - \varepsilon$ on S^+, and for (s,m) with $k^{\pm}(s,m) < 0$, $\operatorname{Im}(V(s,x,y) - 4\pi m x) < \operatorname{Vol}(M) - \varepsilon$ on S^-. ∎

Proposition 6.7. *There is an $\varepsilon > 0$ such that for each triple (s,m,n) with $n \neq 0$,*

$$\left|\int_{D_{\frac{\delta}{2}}} \psi(x,y)\sin\left(\frac{x}{q} - J(s)\pi\right)\varepsilon(x,y)e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s,x,y)-4\pi m x-4\pi n y)}\,dx\,dy\right| \leqslant O\big(e^{\frac{r}{4\pi}(\operatorname{Vol}(M)-\varepsilon)}\big).$$

Proof. For $L > 0$, let

$$S^{\pm}_L = S^{\pm}_{L,\text{top}} \cup S^{\pm}_{L,\text{side}} \cup \big(D_{\frac{\delta}{2}} \smallsetminus D_\delta\big),$$

where

$$S^{\pm}_{L,\text{top}} = \big\{(x,y) \in D_{\mathbb{C},\delta} \mid \operatorname{Im}(x) = 0, \operatorname{Im}(y) = \pm L\big\}$$

and

$$S^{\pm}_{L,\text{side}} = \big\{(\theta_1, \theta_2 \pm \sqrt{-1}l) \mid (\theta_1,\theta_2) \in \partial D_\delta, l \in [0,L]\big\}.$$

Then, $S^{\pm}_L$ are homotopic to $D_{\frac{\delta}{2}}$.

We want to show a stronger statement: if L is sufficiently large, then there is an $\varepsilon > 0$ such that, for each triple (s,m,n) with $n \neq 0$,

$$\operatorname{Im}\big(V(s,x,y) - 4\pi m x - 4\pi n y\big) < \operatorname{Vol}(M) - \varepsilon$$

either on S^+_L or on S^-_L.

Then, the result follows form Lemma 6.5 and the analyticity of V_r.

To this end, first on $D_{\frac{\delta}{2}} \smallsetminus D_\delta$, by Propositions 4.1 and 2.4, we have

$$\operatorname{Im}\big(V(x,y) - 4\pi m x - 4\pi n y\big) = \operatorname{Im} V(x,y) \leqslant \frac{1}{2}\operatorname{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \operatorname{Vol}(M) - \varepsilon.$$

In $D_{\mathbb{C}}$, we have

$$0 < \arg(1 - e^{-2\sqrt{-1}(y+x)}) < \pi - 2(\operatorname{Re}(y) + \operatorname{Re}(x))$$

and

$$2(\operatorname{Re}(y) - \operatorname{Re}(x)) - \pi < \arg(1 - e^{2\sqrt{-1}(y-x)}) < 0.$$

For $n > 0$, let $y = \mathrm{Re}(y) + \sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial \, \mathrm{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)}{\partial l} \\
&\quad = 4\,\mathrm{Re}(x) + 2\arg\bigl(1 - e^{-2\sqrt{-1}(y+x)}\bigr) + 2\arg\bigl(1 - e^{2\sqrt{-1}(y-x)}\bigr) - 4n\pi \\
&\quad < 4\,\mathrm{Re}(x) + 2(\pi - 2(\mathrm{Re}(y) + \mathrm{Re}(x))) + 0 - 4n\pi \\
&\quad = 2\pi - 4\,\mathrm{Re}(y) - 4n\pi < -2\pi,
\end{aligned}$$

where the last inequality comes from that $0 < \mathrm{Re}(y) < \frac{\pi}{2}$ and $n > 0$. Therefore, pushing the domain D_δ along the $\sqrt{-1}l$ direction far enough (without changing $\mathrm{Im}(x)$), the imaginary part of $V(s,x,y) - 4\pi mx - 4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon > 0$ such that

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S^+_{L,\mathrm{top}}$.

Since $\mathrm{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)$ is already smaller than the volume of M on ∂D_δ and on $\partial S^+_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S^+_{L,\mathrm{side}}$.

Putting all these together, we have, for a sufficiently large L,

$$\mathrm{Im}\bigl(V(s,x,y) - 4\pi mx - 4\pi ny\bigr) < \mathrm{Vol}(M) - \varepsilon$$

on S^+_L for each triple (s,m,n) with $n > 0$.

For $n < 0$, let $y = \mathrm{Re}(y) - \sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial \, \mathrm{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)}{\partial l} \\
&\quad = -4\,\mathrm{Re}(x) - 2\arg\bigl(1 - e^{-2\sqrt{-1}(y+x)}\bigr) - 2\arg\bigl(1 - e^{2\sqrt{-1}(y-x)}\bigr) + 4n\pi \\
&\quad < -4\,\mathrm{Re}(x) - 0 - 2(2(\mathrm{Re}(y) - \mathrm{Re}(x)) - \pi) + 4n\pi \\
&\quad = 2\pi - 4\,\mathrm{Re}(y) + 4n\pi < -2\pi,
\end{aligned}$$

where the last inequality comes from that $0 < \mathrm{Re}(y) < \frac{\pi}{2}$ again and $n < 0$. Therefore, pushing the domain D_δ along the $-\sqrt{-1}l$ direction far enough (without changing $\mathrm{Im}(x)$), the imaginary part of $V(s,x,y) - 4\pi mx - 4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon > 0$ such that

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S^-_{L,\mathrm{top}}$.

Since $\operatorname{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)$ is already smaller than the volume of M on ∂D_δ and on $\partial S^-_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y) - 4\pi mx - 4\pi ny < \operatorname{Vol}(M) - \varepsilon$$

on $S^-_{L,\mathrm{side}}$.

Putting all these together, we have, for a sufficiently large L,

$$\operatorname{Im}\big(V(s,x,y) - 4\pi mx - 4\pi ny\big) < \operatorname{Vol}(M) - \varepsilon$$

on S^-_L for each triple (s,m,n), with $n < 0$. ∎

6.2.2. Estimate on $D'_{\frac{\delta}{2}}$.

Proposition 6.8. *There is an* $\varepsilon > 0$ *such that, for each triple* (s,m,n)*,*

$$\left|\int_{D'_{\frac{\delta}{2}}} \psi(x,y)\sin\left(\frac{x}{q} - J(s)\pi\right)\varepsilon(x,y)e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s,x,y)-4\pi mx-4\pi ny)}dx\,dy\right|$$
$$\leqslant O\big(e^{\frac{r}{4\pi}(\operatorname{Vol}(M)-\varepsilon)}\big).$$

Proof. For $L > 0$, let

$$S'^{\pm}_L = S'^{\pm}_{L,\mathrm{top}} \cup S'^{\pm}_{L,\mathrm{side}} \cup \big(D'_{\frac{\delta}{2}} \smallsetminus D'_\delta\big),$$

where

$$S'^{\pm}_{L,\mathrm{top}} = \big\{(x,y) \in D'_{\mathbb{C},\delta} \mid \operatorname{Im}(x) = 0, \operatorname{Im}(y) = \pm L\big\}$$

and

$$S'^{\pm}_{L,\mathrm{side}} = \big\{(\theta_1, \theta_2 \pm \sqrt{-1}l) \mid (\theta_1,\theta_2) \in \partial D'_\delta, l \in [0,L]\big\}.$$

Then, $S'^{\pm}_L$ are homotopic to $D'_{\frac{\delta}{2}}$.

We want to show a stronger statement: if L is sufficiently large, then there is an $\varepsilon > 0$ such that, for each triple (s,m,n),

$$\operatorname{Im}\big(V(s,x,y) - 4\pi mx - 4\pi ny\big) < \operatorname{Vol}(M) - \varepsilon$$

either on S'^+_L or on S'^-_L.

Then, the result follows form Lemma 6.5 and the analyticity of V_r.

To this end, first on $D'_{\frac{\delta}{2}} \smallsetminus D'_\delta$, by Propositions 4.1 and 2.4, we have

$$\operatorname{Im}\big(V(x,y) - 4\pi mx - 4\pi ny\big) = \operatorname{Im} V(x,y) \leqslant \frac{1}{2}\operatorname{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \operatorname{Vol}(M) - \varepsilon.$$

Then, in $D'_{\mathbb{C},\delta}$, we have

$$0 < \arg\big(1 - e^{-2\sqrt{-1}(y+x)}\big) < \pi - 2(\operatorname{Re}(y) + \operatorname{Re}(x))$$

and

$$2(\mathrm{Re}(y) - \mathrm{Re}(x)) - 3\pi < \arg\bigl(1 - e^{2\sqrt{-1}(y-x)}\bigr) < 0.$$

For $n \geqslant 0$, let $y = \mathrm{Re}(y) + \sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial \operatorname{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)}{\partial l} \\
&\quad = 4\,\mathrm{Re}(x) + 2\arg\bigl(1 - e^{-2\sqrt{-1}(y+x)}\bigr) + 2\arg\bigl(1 - e^{2\sqrt{-1}(y-x)}\bigr) - 4n\pi \\
&\quad < 4\,\mathrm{Re}(x) + 2(\pi - 2(\mathrm{Re}(y) + \mathrm{Re}(x))) + 0 - 4n\pi \\
&\quad = 2\pi - 4\,\mathrm{Re}(y) - 4n\pi < -2\delta,
\end{aligned}$$

where the last inequality comes from that $\frac{\pi}{2} + \frac{\delta}{2} < \mathrm{Re}(y) < \pi - \frac{\delta}{2}$ and $n \geqslant 0$. Therefore, pushing the domain D'_δ along the $\sqrt{-1}l$ direction far enough (without changing $\operatorname{Im}(x)$), the imaginary part of $V(s,x,y) - 4\pi mx - 4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon > 0$ such that

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S'^{+}_{L,\mathrm{top}}$.

Since $\operatorname{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)$ is already smaller than the volume of M on $\partial D'_\delta$ and on $\partial S'^{+}_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S'^{+}_{L,\mathrm{side}}$.

Putting all these together, we have, for a sufficiently large L,

$$\operatorname{Im}\bigl(V(s,x,y) - 4\pi mx - 4\pi ny\bigr) < \mathrm{Vol}(M) - \varepsilon$$

on S'^{+}_{L} for each triple (s,m,n) with $n \geqslant 0$.

For $n < 0$, let $y = \mathrm{Re}(y) - \sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial \operatorname{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)}{\partial l} \\
&\quad = -4\,\mathrm{Re}(x) - 2\arg\bigl(1 - e^{-2\sqrt{-1}(y+x)}\bigr) - 2\arg\bigl(1 - e^{2\sqrt{-1}(y-x)}\bigr) + 4n\pi \\
&\quad < -4\,\mathrm{Re}(x) - 0 - 2(2(\mathrm{Re}(y) - \mathrm{Re}(x)) - 3\pi) + 4n\pi \\
&\quad = 6\pi - 4\,\mathrm{Re}(y) + 4n\pi < -2\delta,
\end{aligned}$$

where the last inequality comes from that $\frac{\pi}{2} + \frac{\delta}{2} < \mathrm{Re}(y) < \pi - \frac{\delta}{2}$ again and $n < 0$. Therefore, pushing the domain D'_δ along the $-\sqrt{-1}l$ direction far enough (without changing $\operatorname{Im}(x)$), the imaginary part of $V(s,x,y) - 4\pi mx - 4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon > 0$ such that

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S'^{-}_{L,\mathrm{top}}$.

Since $\mathrm{Im}(V(s,x,y) - 4\pi mx - 4\pi ny)$ is already smaller than the volume of M on $\partial D'_\delta$ and on $\partial S'^{-}_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y) - 4\pi mx - 4\pi ny < \mathrm{Vol}(M) - \varepsilon$$

on $S'^{-}_{L,\mathrm{side}}$.

Putting all these together, we have, for a sufficiently large L,

$$\mathrm{Im}\big(V(s,x,y) - 4\pi mx - 4\pi ny\big) < \mathrm{Vol}(M) - \varepsilon$$

on S'^{-}_L for each triple (s,m,n) with $n < 0$. ■

6.2.3. Estimate on $D''_{\frac{\delta}{2}}$.

Proposition 6.9. *There is an $\varepsilon > 0$ such that, for each triple (s,m,n),*

$$\left|\int_{D''_{\frac{\delta}{2}}} \psi(x,y)\sin\left(\frac{x}{q} - J(s)\pi\right)\varepsilon(x,y)e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s,x,y)-4\pi mx-4\pi ny)}dx\,dy\right|$$
$$\leqslant O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon)}\big).$$

Proof. For $L > 0$, let

$$S''^{\pm}_L = S''^{\pm}_{L,\mathrm{top}} \cup S''^{\pm}_{L,\mathrm{side}} \cup \big(D''_{\frac{\delta}{2}} \smallsetminus D''_\delta\big),$$

where

$$S''^{\pm}_{L,\mathrm{top}} = \big\{(x,y) \in D''_{\mathbb{C},\delta} \mid \mathrm{Im}(x) = 0, \mathrm{Im}(y) = \pm L\big\}$$

and

$$S''^{\pm}_{L,\mathrm{side}} = \big\{(\theta_1, \theta_2 \pm \sqrt{-1}l) \mid (\theta_1,\theta_2) \in \partial D''_\delta, l \in [0,L]\big\}.$$

Then, $S''^{\pm}_L$ are homotopic to $D''_{\frac{\delta}{2}}$.

We want to show a stronger statement: if L is sufficiently large, then there is an $\varepsilon > 0$ such that for each triple (s,m,n),

$$\mathrm{Im}\big(V(s,x,y) - 4\pi mx - 4\pi ny\big) < \mathrm{Vol}(M) - \varepsilon$$

either on S''^{+}_L or on S''^{-}_L.

Then, the result follows form Lemma 6.5 and the analyticity of V_r.

To this end, first on $D''_{\frac{\delta}{2}} \smallsetminus D''_\delta$, by Propositions 4.1 and 2.4, we have

$$\mathrm{Im}\big(V(x,y) - 4\pi mx - 4\pi ny\big) = \mathrm{Im}\,V(x,y) \leqslant \frac{1}{2}\mathrm{Vol}(\mathrm{S}^3 \smallsetminus K_{4_1}) + \varepsilon < \mathrm{Vol}(M) - \varepsilon.$$

Then, in $D''_{\mathbb{C},\delta}$, we have

$$0 < \arg\big(1 - e^{-2\sqrt{-1}(y+x)}\big) < 3\pi - 2(\mathrm{Re}(y) + \mathrm{Re}(x))$$

and

$$2(\mathrm{Re}(y)-\mathrm{Re}(x))-\pi < \arg\bigl(1-e^{2\sqrt{-1}(y-x)}\bigr) < 0.$$

For $n>0$, let $y=\mathrm{Re}(y)+\sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial\,\mathrm{Im}(V(s,x,y)-4\pi mx-4\pi ny)}{\partial l}\\
&\quad= 4\,\mathrm{Re}(x)+2\arg\bigl(1-e^{-2\sqrt{-1}(y+x)}\bigr)+2\arg\bigl(1-e^{2\sqrt{-1}(y-x)}\bigr)-4n\pi\\
&\quad< 4\,\mathrm{Re}(x)+2(3\pi-2(\mathrm{Re}(y)+\mathrm{Re}(x)))+0-4n\pi\\
&\quad= 6\pi-4\,\mathrm{Re}(y)-4n\pi < -2\delta,
\end{aligned}$$

where the last inequality comes from that $\frac{\pi}{2}+\frac{\delta}{2}<\mathrm{Re}(y)<\pi-\frac{\delta}{2}$ and $n>0$. Therefore, pushing the domain D''_δ along the $\sqrt{-1}l$ direction far enough (without changing $\mathrm{Im}(x)$), the imaginary part of $V(s,x,y)-4\pi mx-4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon>0$ such that

$$V(s,x,y)-4\pi mx-4\pi ny < \mathrm{Vol}(M)-\varepsilon$$

on $S''^{+}_{L,\mathrm{top}}$.

Since $\mathrm{Im}(V(s,x,y)-4\pi mx-4\pi ny)$ is already smaller than the volume of M on $\partial D''_\delta$ and on $\partial S''^{+}_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y)-4\pi mx-4\pi ny < \mathrm{Vol}(M)-\varepsilon$$

on $S''^{+}_{L,\mathrm{side}}$.

Putting all these together, we have, for a sufficiently large L,

$$\mathrm{Im}\bigl(V(s,x,y)-4\pi mx-4\pi ny\bigr) < \mathrm{Vol}(M)-\varepsilon$$

on S''^{+}_{L} for each triple (s,m,n) with $n>0$.

For $n\leqslant 0$, let $y=\mathrm{Re}(y)-\sqrt{-1}l$. Then,

$$\begin{aligned}
&\frac{\partial\,\mathrm{Im}(V(s,x,y)-4\pi mx-4\pi ny)}{\partial l}\\
&\quad= -4\,\mathrm{Re}(x)-2\arg\bigl(1-e^{-2\sqrt{-1}(y+x)}\bigr)-2\arg\bigl(1-e^{2\sqrt{-1}(y-x)}\bigr)+4n\pi\\
&\quad< -4\,\mathrm{Re}(x)-0-2(2(\mathrm{Re}(y)-\mathrm{Re}(x))-\pi)+4n\pi\\
&\quad= 2\pi-4\,\mathrm{Re}(y)+4n\pi < -2\delta,
\end{aligned}$$

where the last inequality comes from that $\frac{\pi}{2}+\frac{\delta}{2}<\mathrm{Re}(y)<\pi-\frac{\delta}{2}$ again and $n\leqslant 0$. Therefore, pushing the domain D''_δ along the $-\sqrt{-1}l$ direction far enough (without changing $\mathrm{Im}(x)$), the imaginary part of $V(s,x,y)-4\pi mx-4\pi ny$ becomes as small as possible. In particular, for a sufficiently large L, there is an $\varepsilon>0$ such that

$$V(s,x,y)-4\pi mx-4\pi ny < \mathrm{Vol}(M)-\varepsilon$$

on $S''^{-}_{L,\mathrm{top}}$.

Since $\mathrm{Im}(V(s,x,y)-4\pi mx-4\pi ny)$ is already smaller than the volume of M on $\partial D''_\delta$ and on $\partial S''^{-}_{L,\mathrm{top}}$, by Lemma 5.5, it becomes even smaller on the side, i.e.,

$$V(s,x,y)-4\pi mx-4\pi ny < \mathrm{Vol}(M)-\varepsilon \quad \text{on } S''^{-}_{L,\mathrm{side}}.$$

Putting all these together, we have, for a sufficiently large L,

$$\mathrm{Im}\big(V(s,x,y)-4\pi mx-4\pi ny\big) < \mathrm{Vol}(M)-\varepsilon$$

on S''^{-}_{L} for each triple (s,m,n) with $n\leqslant 0$. ■

6.3. Proof of Theorems 1.1 and 1.2

Theorem 1.1 follows from the following proposition.

Proposition 6.10. (1) *The sum of the two leading Fourier coefficients*

$$\begin{aligned}&\hat{f}_r(s^+,m^+,0)+\hat{f}_r(s^-,m^-,0)\\ &\quad=\frac{c_r}{\sqrt{\mathrm{Tor}(M;\mathrm{Ad}_\rho)}}e^{\frac{r}{4\pi}(\mathrm{Vol}(M)+\sqrt{-1}\,\mathrm{CS}(M))}\Big(1+O\Big(\frac{1}{r}\Big)\Big),\end{aligned}$$

where

$$c_r=-(-1)^{m^++J(s^+)-\frac{p'}{q}}e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^+)+\frac{p'}{q})}r\sqrt{q}\neq 0.$$

(2) *The sum of all the other Fourier coefficients*

$$\sum_{(s,m,n)\neq(s^\pm,m^\pm,0)}\big|\hat{f}_r(s,m,n)\big|\leqslant O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon)}\big)\quad \textit{for some } \varepsilon>0.$$

Proof. For (1), recall that $\mathcal{D}_{\frac{\delta}{2}}=D_{\frac{\delta}{2}}\cup D'_{\frac{\delta}{2}}\cup D''_{\frac{\delta}{2}}$. Then, by Propositions 4.3, 6.4, 6.8, and 6.9, we have

$$\begin{aligned}&\hat{f}_r(s^+,m^+,0)+\hat{f}_r(s^-,m^-,0)\\
&\quad=(-1)^{m^+}\Big(\frac{r}{2\pi}\Big)^2\int_{\mathcal{D}_{\frac{\delta}{2}}}\psi(x,y)\sin\Big(\frac{x}{q}-J(s^+)\pi\Big)\varepsilon(x,y)\\
&\qquad\times e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s^+,x,y)-4\pi m^+x)}dx\,dy\\
&\qquad+(-1)^{m^-}\Big(\frac{r}{2\pi}\Big)^2\int_{\mathcal{D}_{\frac{\delta}{2}}}\psi(x,y)\sin\Big(\frac{x}{q}-J(s^-)\pi\Big)\varepsilon(x,y)\\
&\qquad\times e^{-x\sqrt{-1}+\frac{r}{4\pi\sqrt{-1}}(V_r(s^-,x,y)-4\pi m^-x)}dx\,dy\\
&\quad=\Big(\frac{r}{2\pi}\Big)^2\frac{((-1)^{m^+}c_r^++(-1)^{m^-}c_r^-)}{\sqrt{\mathrm{Tor}(M;\mathrm{Ad}_\rho)}}e^{\frac{r}{4\pi}(\mathrm{Vol}(M)+\sqrt{-1}\,\mathrm{CS}(M))}\Big(1+O\Big(\frac{1}{r}\Big)\Big)\\
&\qquad+O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon)}\big),\end{aligned}$$

where

$$c_r^+ = -\frac{2\pi^2\sqrt{q}}{r}(-1)^{J(s^+)-\frac{p'}{q}}e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^+)+\frac{p'}{q})}$$

and

$$c_r^- = \frac{2\pi^2\sqrt{q}}{r}(-1)^{J(s^+)-\frac{p'}{q}}e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^-)+\frac{p'}{q})}$$

are the constants in Proposition 6.4; and we are left to prove that

$$\big((-1)^{m^+}c_r^+ + (-1)^{m^-}c_r^-\big)\left(\frac{r}{2\pi}\right)^2 = c_r.$$

We claim that

$$-(-1)^{m^+}e^{-\frac{\pi\sqrt{-1}r}{4}K(s^+)} = (-1)^{m^-}e^{-\frac{\pi\sqrt{-1}r}{4}K(s^-)}, \tag{6.1}$$

from which the result follows. Indeed, by the definition of K and Lemma 3.3 (1), we have

$$K(s^+) - K(s^-) = -\frac{2}{q}(I(s^+)+I(s^-))(s^+-s^-) = -4(m^+ + m^- - 1)(s^+ - s^-).$$

Then,

$$\begin{aligned}\frac{-(-1)^{m^+}e^{-\frac{\pi\sqrt{-1}r}{4}K(s^+)}}{(-1)^{m^-}e^{-\frac{\pi\sqrt{-1}r}{4}K(s^-)}} &= (-1)^{m^+ + m^- - 1}e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^+)-K(s^-))}\\ &= (-1)^{(m^+ + m^- -1)(1+r(s^+-s^-))}.\end{aligned}$$

The result will follow if we can prove that $(m^+ + m^- - 1)(1 + r(s^+ - s^-))$ is even. For this, by Lemma 3.3 (1) and computing $I(s^+) - I(s^-)$, we have

$$C_{k-1}(s^+ - s^-) + (m^+ - m^-)q = -1.$$

Therefore, $s^+ - s^-$ and $m^+ - m^-$ cannot be both even. As a consequence, since r is odd, $m^+ + m^- - 1$ and $1 + r(s^+ - s^-)$ cannot be both odd. This completes the proof of (1).

For (2), let

$$\begin{aligned}\mathcal{S} = \big\{(s,m,n) \in \{0,\dots,|q|-1\}\times\mathbb{Z}^2 \mid (s,m,n) \neq (s^+,m^+,0), (s^-,m^-,0),\\ \text{and } (m,n)\neq(0,0)\big\}.\end{aligned}$$

Then, $\mathcal{S}$ contains all but finitely many triples in $\{0,\dots,|q|-1\}\times\mathbb{Z}^2$. By Propositions 6.6, 6.7, 6.8, and 6.9, for each $(s,m,n)\neq(s^\pm,m^\pm,0)$, especially for those finitely many that are not in $\mathcal{S}$,

$$\big|\hat{f}_r(s,m,n)\big| \leqslant O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon)}\big)$$

for some $\varepsilon > 0$. Therefore, it suffices to prove that

$$\sum_{(s,m,n)\in\mathcal{S}} \left|\hat{f}_r(s,m,n)\right| \leqslant O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon)}\big)$$

for some $\varepsilon > 0$.

Now, for each s, let

$$h_r(s,x,y) = \psi(x,y)\sin\left(\frac{x}{q} - J(s)\pi\right)\varepsilon(x,y)$$
$$\times\, e^{-x\sqrt{-1}-\frac{\log(1-e^{-2\sqrt{-1}(y+x)})}{2}-\frac{\log(1-e^{2\sqrt{-1}(y-x)})}{2}+\frac{\upsilon_r(x,y)}{4\pi\sqrt{-1}r}}.$$

Then, for each (s,m,n) in $\mathcal{S}$, since $\psi(x,y)$ vanishes outside of $\mathcal{D}$, by the integration by parts, we have

$$\begin{aligned}
&r^4(m^4+n^4)\int_{\mathcal{D}} h_r(s,x,y)e^{\frac{r}{4\pi\sqrt{-1}}(V(s,x,y)-4\pi mx-4\pi ny)}dx\,dy\\
&= \int_{\mathcal{D}} h_r(s,x,y)e^{\frac{r}{4\pi\sqrt{-1}}V(s,x,y)}\left(\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)e^{\frac{r}{4\pi\sqrt{-1}}(-4\pi mx-4\pi ny)}\right)dx\,dy\\
&= \int_{\mathcal{D}}\left(\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)\left(h_r(s,x,y)e^{\frac{r}{4\pi\sqrt{-1}}V(s,x,y)}\right)\right)e^{\frac{r}{4\pi\sqrt{-1}}(-4\pi mx-4\pi ny)}dx\,dy\\
&= \int_{\mathcal{D}} \tilde{h}_r(s,x,y)e^{\frac{r}{4\pi\sqrt{-1}}(V(s,x,y)-4\pi mx-4\pi ny)}dx\,dy,
\end{aligned}$$

where

$$\tilde{h}_r(s,x,y) = \frac{(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4})(h_r(s,x,y)e^{\frac{r}{4\pi\sqrt{-1}}V(s,x,y)})}{e^{\frac{r}{4\pi\sqrt{-1}}V(s,x,y)}}$$

is a smooth function independent of m and n, and has the form

$$\tilde{h}_r(s,x,y) = \tilde{h}(s,x,y)\cdot r^4 + O(r^3)$$

for a smooth function $\tilde{h}(s,x,y)$ independent of r. (Here, the remaining term being of order $O(r^3)$ uses the fact that as $r\to\infty$ all the partial derivatives $\frac{\partial^i \upsilon_r(x,y)}{\partial x^i}$ and $\frac{\partial^j \upsilon_r(x,y)}{\partial y^j}$ of $\upsilon_r(x,y)$ uniformly converge on a compact subset of $\{z\in\mathbb{C}\mid 0<\mathrm{Re}\,z<\pi\}$; hence, for $i,j\leqslant 4$ are bounded by a constant independent of r.) Therefore, on the compact subset $S^\pm\cup S_L^\pm\cup S_L'^\pm\cup S_L''^\pm$ of $\mathcal{D}_{\mathbb{C}}$, $\frac{\tilde{h}_r(s,x,y)}{r^4}$ is bounded from above by some $C>0$ independent of (s,m,n), where $S\pm$, $S_L^\pm$, $S_L'^\pm$ and $S_L''^\pm$ are the surfaces constructed in the proof of Propositions 6.6, 6.7, 6.8, and 6.9. Let A be the maximum of the areas of these surfaces, and let ε' be the minimum of those ε's in

Propositions 6.6, 6.7, 6.8 and 6.9. Then, we have

$$\begin{aligned}
&\sum_{(s,m,n)\in\mathcal{S}} \big|\hat{f}_r(s,m,n)\big| \\
&= \Big(\frac{r}{2\pi}\Big)^2 \sum_{(s,m,n)\in\mathcal{S}} \Big|\int_{\mathcal{D}} h_r(s,x,y) e^{\frac{r}{4\pi\sqrt{-1}}(V(s,x,y)-4\pi m x - 4\pi n y)} dx\,dy\Big| \\
&= \Big(\frac{r}{2\pi}\Big)^2 \sum_{(s,m,n)\in\mathcal{S}} \frac{1}{m^4+n^4}\Big|\int_{\mathcal{D}} \frac{\tilde{h}_r(s,x,y)}{r^4} e^{\frac{r}{4\pi\sqrt{-1}}(V(s,x,y)-4\pi m x - 4\pi n y)} dx\,dy\Big| \\
&\leqslant \Big(\frac{r}{2\pi}\Big)^2 \Big(\sum_{(s,m,n)\in\mathcal{S}} \frac{3AC}{m^4+n^4}\Big) O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\varepsilon')}\big) \leqslant O\big(e^{\frac{r}{4\pi}(\mathrm{Vol}(M)-\frac{\varepsilon'}{2})}\big).
\end{aligned}$$

The summation converges because $\sum_{(m,n)\neq(0,0)} \frac{1}{m^4+n^4}$ does; and the proof is completed with $\varepsilon = \frac{\varepsilon'}{2}$. ■

Proof of Theorem 1.1. By Propositions 4.3 and 6.10, we only need to compute the constant C_r. By a direct computation, we have

$$\begin{aligned}
\kappa_r c_r &= \frac{(-1)^{\frac{3(k+1)}{4}+\sum_{i=1}^k a_i} e^{\frac{\pi\sqrt{-1}}{r}(3\sigma(L)-\sum_{i=1}^k a_i - \sum_{i=2}^k \frac{1}{C_{i-1}C_i})+\frac{\pi\sqrt{-1}r}{4}(\sigma(L)+3a_k)}}{2r\sqrt{q}} \\
&\quad \cdot \big(-(-1)^{m^+ + J(s^+)-\frac{p'}{q}} e^{-\frac{\pi\sqrt{-1}r}{4}(K(s^+)+\frac{p'}{q})} r\sqrt{q}\big) \\
&= \frac{(-1)^{\frac{3k-1}{4}+\sum_{i=1}^k a_i + m^+ + J(s^+)-\frac{p'}{q}} e^{\frac{\pi\sqrt{-1}}{r}(3\sigma(L)-\sum_{i=1}^k a_i - \sum_{i=2}^k \frac{1}{C_{i-1}C_i})}}{2} \\
&\quad \cdot e^{\frac{\pi\sqrt{-1}r}{4}(\sigma(L)+3a_k-K(s^+)-\frac{p'}{q})}.
\end{aligned}$$

Therefore, $C_r = 2\kappa_r c_r$ has norm equal to 1.

To compute the exponential growth rate, by Lemma 3.3 (3), we have that $K(s^+)+\frac{p'}{q}$ is an integer, and

$$\begin{aligned}
\lim_{r\to\infty} \frac{4\pi}{r} \log \mathrm{RT}_r(M) &= \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M) \\
&\quad + \sqrt{-1}\Big(\sigma(L) + 3a_k - K(s^+) - \frac{p'}{q}\Big)\pi^2 \\
&\equiv \mathrm{Vol}(M) + \sqrt{-1}\,\mathrm{CS}(M) \pmod{\sqrt{-1}\pi^2\mathbb{Z}}.
\end{aligned}$$ ■

Proof of Theorem 1.2. At the root of unity $q = e^{\frac{2\pi\sqrt{-1}}{r}}$ for an odd integer r, let $\mathrm{TV}'_r(M)$ be the SO(3) Turaev–Viro invariants as defined in [7], namely, summing over even colors instead of all integral colors in the definition. Then, by [7, Theorem 2.9],

$$\mathrm{TV}_r(M) = 2^{b_2(M)-b_0(M)}\,\mathrm{TV}'_r(M);$$

and by the same argument as in [7, 34, 38, 40], we have

$$\mathrm{TV}'_r(M) = |\,\mathrm{RT}'(M)|^2,$$

where $\mathrm{RT}'_r(M)$ is the SO(3) Reshetikhin–Turaev invariants. (See [2, 20, 32], and for the exact definition used here, see [7, Definition 2.1].)

Next, we show that

$$\mathrm{RT}'_r(M) = 2\,\mathrm{RT}_r(M).$$

Indeed, in the SO(3) theory, the Kirby coloring

$$\omega'_r = \sum_{n=0}^{\frac{r-3}{2}} [2n+1] e_{2n}.$$

Then, by [2, Lemma 6.3 (iii)], we have

$$\omega_r = 2\omega'_r$$

in the Kauffman bracket skein module $\mathrm{K}_r(D^2 \times S^1)$ of the solid torus, and the constant

$$\mu'_r = \big(\langle \omega'_r \rangle_{U_+} \langle \omega'_r \rangle_{U_-}\big)^{-\frac{1}{2}} = \frac{2\sin\frac{2\pi}{r}}{\sqrt{r}} = 2\mu_r,$$

where $U_\pm$ are the diagram of the unknot with framing ± 1. As a consequence,

$$\mu'_r \omega'_r = \mu_r \omega_r$$

in $\mathrm{K}_r(D^2 \times S^1)$, and

$$\begin{aligned}\mathrm{RT}'_r(M) &= \mu'_r \langle \mu'_r \omega'_r, \dots, \mu'_r \omega'_r \rangle_{D(L)} \langle \mu'_r \omega'_r \rangle_{U_+}^{-\sigma(L)} \\ &= 2\mu_r \langle \mu_r \omega_r, \dots, \mu_r \omega_r \rangle_{D(L)} \langle \mu_r \omega_r \rangle_{U_+}^{-\sigma(L)} = 2\,\mathrm{RT}_r(M).\end{aligned}$$

Putting all these together, we have

$$\mathrm{TV}_r(M) = 2^{b_2(M)-b_0(M)+2} |\,\mathrm{RT}(M)|^2,$$

and by Theorem 1.1, we have

$$\mathrm{TV}_r(M) = \frac{2^{b_2(M)-b_0(M)}}{|\,\mathrm{Tor}(M;\mathrm{Ad}_\rho)|} e^{\frac{r}{2\pi}\mathrm{Vol}(M)} \Big(1 + O\Big(\frac{1}{r}\Big)\Big). \qquad \blacksquare$$

A. Proof of Proposition 6.1

To prove Proposition 6.1, we need the following lemmas, where the first one is the standard complex Morse lemma (see, e.g., [43, Lemma 1.6]).

Lemma A.1 (Complex Morse lemma). *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function with a non-degenerate critical point at $(c_1, \dots, c_n)$. Then, there exists a holomorphic change of variables $(z_1, \dots, z_n) = \psi(Z_1, \dots, Z_n)$ on a neighborhood V of $(c_1, \dots, c_n)$ such that $\psi(0, \dots, 0) = (c_1, \dots, c_n)$,*

$$f(\psi(Z_1, \dots, Z_n)) = f(c_1, \dots, c_n) - Z_1^2 - \cdots - Z_n^2$$

and

$$\det \mathrm{D}\psi(0, \dots, 0) = \frac{2^{\frac{n}{2}}}{\sqrt{\det(-\operatorname{Hess}(f)(c_1, \dots, c_n))}}.$$

Lemma A.2. *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

(1) *the asymptotics of the integral of e^{-rz^2} over $[-\varepsilon, \varepsilon]$ is given by*

$$\int_{-\varepsilon}^{\varepsilon} e^{-rz^2} dz = \sqrt{\frac{\pi}{r}} + O(e^{-\delta r})$$

and

(2) *the asymptotics of the integral of $z^2 e^{-rz^2}$ over $[-\varepsilon, \varepsilon]$ is given by*

$$\int_{-\varepsilon}^{\varepsilon} z^2 e^{-rz^2} dz = \frac{1}{2}\sqrt{\frac{\pi}{r^3}} + O(e^{-\delta r}).$$

Proof. For (1), we have

$$\int_{-\varepsilon}^{\varepsilon} e^{-rz^2} dz = \int_{-\infty}^{\infty} e^{-rz^2} dz - \int_{-\infty}^{-\varepsilon} e^{-rz^2} dz - \int_{\varepsilon}^{\infty} e^{-rz^2} dz,$$

where the first term

$$\int_{-\infty}^{\infty} e^{-rz^2} dz = \sqrt{\frac{\pi}{r}}$$

is a Gaussian integral, and the other two terms

$$\int_{-\infty}^{-\varepsilon} e^{-rz^2} dz = \int_{\varepsilon}^{\infty} e^{-rz^2} dz \leqslant \int_{\varepsilon}^{\infty} e^{-r\varepsilon z} dz = \frac{e^{-r\varepsilon^2}}{r\varepsilon} = O(e^{-\delta r}).$$

For (2), by integration by parts, we have

$$\int_{-\varepsilon}^{\varepsilon} e^{-rz^2} dz = z e^{-rz^2}\Big|_{-\varepsilon}^{\varepsilon} + 2r \int_{-\varepsilon}^{\varepsilon} z^2 e^{-rz^2} dz;$$

hence, by (1),

$$\int_{-\varepsilon}^{\varepsilon} z^2 e^{-rz^2} dz = \frac{1}{2r}\left(\int_{-\varepsilon}^{\varepsilon} e^{-rz^2} dz - 2\varepsilon e^{-r\varepsilon^2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{r^3}} + O(e^{-\delta r}). \qquad \blacksquare$$

Lemma A.3. *Let D be a region in $\mathbb{C}^n$ containing the origin $\mathbf{0}$ and g be a holomorphic function on D. Then, there exist functions $h_1, \dots, h_n$, and $k_1, \dots, k_n$ such that*

(1) *h_i has variables $z_{i+1}, \dots, z_n$ and is holomorphic in them,*

(2) *k_i has variables $z_i, \ldots, z_n$ and is holomorphic in them, and*

(3) *the holomorphic function g can be written as*

$$g(z_1, \ldots, z_n) = g(\mathbf{0}) + \sum_{i=1}^{n} h_i(z_{i+1}, \ldots, z_n) z_i + \sum_{i=1}^{n} k_i(z_i, \ldots, z_n) z_i^2.$$

Proof of Lemma A.3. We use induction on n. For $n = 1$, if $z_1 \neq 0$, then we can write

$$g(z_1) = g(0) + \frac{dg}{dz_1}(0) z_1 + \left(\frac{g(z_1) - g(0) - \frac{dg}{dz_1}(0) z_1}{z_1^2} \right) z_1^2,$$

and let

$$h_1 = g(0)$$

and

$$k_1(z_1) = \frac{g(z_1) - g(0) - \frac{dg}{dz_1}(0) z_1}{z_1^2}.$$

By computing the Laurent expansion of $k_1(z_1)$, one sees $z_1 = 0$ is a removable singularity, and $k_1(z_1)$ extends as a holomorphic function. This proves the case $n = 1$.

Now, assume that the result holds when $n = l$. For $n = l + 1$, if $z_1 \neq 0$, then we have

$$\begin{aligned} &g(z_1, \ldots, z_{l+1}) \\ &\quad = g(0, z_2, \ldots, z_{l+1}) + \frac{\partial g}{\partial z_1}(0, z_2, \ldots, z_{l+1}) z_1 \\ &\qquad + \left(\frac{g(z_1, \ldots, z_{l+1}) - g(0, z_2, \ldots, z_{l+1}) - \frac{\partial g}{\partial z_1}(0, z_2, \ldots, z_{l+1}) z_1}{z_1^2} \right) z_1^2, \end{aligned}$$

and let

$$h_1(z_2, \ldots, z_{l+1}) = \frac{\partial g}{\partial z_1}(0, z_2, \ldots, z_{l+1})$$

and

$$k_1(z_1, \ldots, z_{l+1}) = \frac{g(z_1, \ldots, z_{l+1}) - g(0, z_2, \ldots, z_{l+1}) - \frac{\partial g}{\partial z_1}(0, z_2, \ldots, z_{l+1}) z_1}{z_1^2}.$$

By computing the Laurent expansion again, one can see that k_1 holomorphically extends to $z_1 = 0$. Since $g(0, z_2, \ldots, z_{l+1})$ has l variables, by the induction assumption,

$$g(0, z_2, \ldots, z_{l+1}) = g(\mathbf{0}) + \sum_{i=2}^{l+1} h_i(z_{i+1}, \ldots, z_{l+1}) z_i + \sum_{i=2}^{l+1} k_i(z_i, \ldots, z_{l+1}) z_i^2.$$

As a consequence, we have

$$g(z_1, z_2, \dots, z_{l+1}) = g(\mathbf{0}) + \sum_{i=1}^{l+1} h_i(z_{i+1}, \dots, z_{l+1}) z_i + \sum_{i=1}^{l+1} k_i(z_i, \dots, z_{l+1}) z_i^2.$$

This completes the proof. ∎

Proof of Proposition 6.1. For simplicity, we use the bold letters $\mathbf{z} = (z_1, \dots, z_n)$, $d\mathbf{z} = dz_1 \cdots dz_n$, $\mathbf{c} = (c_1, \dots, c_n)$ and $\mathbf{0} = (0, \dots, 0)$.

We first consider a special case

$$\mathbf{c} = \mathbf{0}, \quad S = [-\varepsilon, \varepsilon]^n \subset \mathbb{R}^n \subset \mathbb{C}^n$$

and

$$f(\mathbf{z}) = -\sum_{i=1}^{n} z_i^2.$$

Let

$$\sigma_r(\mathbf{z}) = \upsilon_r(\mathbf{z}) \int_0^1 e^{\frac{\upsilon_r(\mathbf{z})}{r} s} ds.$$

Then, we can write

$$e^{\frac{\upsilon_r(\mathbf{z})}{r}} = 1 + \frac{\sigma_r(\mathbf{z})}{r},$$

and

$$g(\mathbf{z}) e^{r f_r(\mathbf{z})} = g(\mathbf{z}) e^{r f(\mathbf{z})} + \frac{1}{r} g(\mathbf{z}) \sigma_r(\mathbf{z}) e^{r f(\mathbf{z})}. \tag{A.1}$$

Since $|\upsilon_r(\mathbf{z})| < M$ for some $M > 0$ independent of r,

$$|\sigma_r(\mathbf{z})| < M \int_0^1 e^{\frac{M}{r} s} ds = M \left(\frac{e^{\frac{M}{r}} - 1}{\frac{M}{r}} \right) < 2M.$$

If M is big enough, then $|g(\mathbf{z})| < M$ on S, and by Lemma A.2 (1), we have

$$\begin{aligned} \left| \int_S \frac{1}{r} g(\mathbf{z}) \sigma_r(\mathbf{z}) e^{r f(\mathbf{z})} \right| &< \frac{2M^2}{r} \int_S e^{r f(\mathbf{z})} d\mathbf{z} \\ &= \frac{2M^2}{r} \left(\frac{\pi}{r} \right)^{\frac{n}{2}} + O(e^{-\delta r}) = O\left(\frac{1}{\sqrt{r^{n+2}}} \right). \end{aligned} \tag{A.2}$$

By Lemma A.3, we have

$$g(z_1, \dots, z_n) = g(\mathbf{0}) + \sum_{i=1}^{n} h_i(z_{i+1}, \dots, z_n) z_i + \sum_{i=1}^{n} k_i(z_i, \dots, z_n) z_i^2$$

for some holomorphic functions $\{h_i\}$ and $\{k_i\}$ on D, where $i = 1, \dots, n$. Then, by Lemma A.2 (1), we have

$$\int_S g(\mathbf{0}) e^{rf(\mathbf{z})} d\mathbf{z} = g(\mathbf{0}) \left(\frac{\pi}{r}\right)^{\frac{n}{2}} + O(e^{-\delta r}). \tag{A.3}$$

Since each $z_i e^{-rz_i^2}$ is odd, we have

$$\int_{-\varepsilon}^{\varepsilon} z_i e^{-rz_i^2} dz_i = 0.$$

As a consequence, for each i, we have

$$\begin{aligned} &\int_{[-\varepsilon,\varepsilon]^n} h_i(z_{i+1}, \dots, z_n) z_i e^{rf(\mathbf{z})} d\mathbf{z} \\ &\quad = \int_{[-\varepsilon,\varepsilon]^{n-1}} h_i(z_{i+1}, \dots, z_n) e^{-r \sum_{j \neq i} z_j^2} \prod_{j \neq i} dz_j \cdot \int_{-\varepsilon}^{\varepsilon} z_i e^{-rz_i^2} dz_i = 0. \end{aligned} \tag{A.4}$$

Besides, if M is big enough, $|k_i(\mathbf{z})| < M$ for all $\mathbf{z} \in S$, $i \in \{1, \dots, n\}$. By Lemma A.2, we have for each $i \in \{1, \dots, n\}$

$$\begin{aligned} \left| \int_S k_i(\mathbf{z}) z_i^2 e^{rf(\mathbf{z})} d\mathbf{z} \right| &< M \left(\int_{-\varepsilon}^{\varepsilon} z_i^2 e^{-rz_i^2} dz_i \right) \prod_{j \neq i} \left(\int_{-\varepsilon}^{\varepsilon} e^{-rz_j^2} dz_j \right) \\ &= O\left(\frac{1}{\sqrt{r^{n+2}}} \right). \end{aligned} \tag{A.5}$$

Putting (A.3), (A.4), and (A.5) together, we have the result for this special case.

For the general case, let (V, ψ) be the change of variable for f in the complex Morse lemma, and let $U \subset V$ such that

$$\psi^{-1}(U) = \prod_{i=1}^{n} \{ Z_i \in \mathbb{C} \mid -\varepsilon < \operatorname{Re}(Z_i) < \varepsilon, -\varepsilon < \operatorname{Im}(Z_i) < \varepsilon \}.$$

Let A be the volume of $S \smallsetminus U$. By the compactness and by conditions (2) and (5), there exist constants $M > 0$ and $\delta > 0$ such that

$$|g(\mathbf{z})| < M$$

and

$$\operatorname{Re} f_r(\mathbf{z}) < \operatorname{Re} f(\mathbf{c}) - \delta$$

on $S \smallsetminus U$. Then,

$$\left| \int_{S \smallsetminus U} g(\mathbf{z}) e^{rf_r(\mathbf{z})} d\mathbf{z} \right| < MAe^{r(\operatorname{Re} f(\mathbf{c}) - \delta)} = O\left(e^{r(\operatorname{Re} f(\mathbf{c}) - \delta)}\right). \tag{A.6}$$

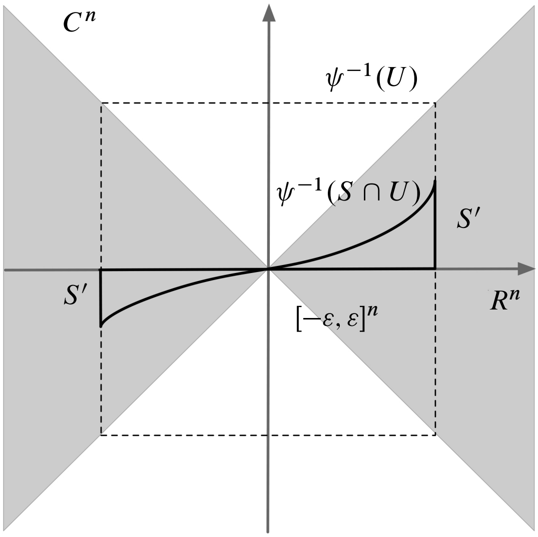


Figure 6. The image of ψ^{-1} around the non-degenerate critical point $\mathbf{c}$ of f.

In Figure 6, the shaded region is where

$$\mathrm{Re}\Big(-\sum_{i=1}^{n} Z_i^2\Big) < 0.$$

In $\overline{\psi^{-1}(U)}$, there is a homotopy H from $\overline{\psi^{-1}(S \cap U)}$ to $[-\varepsilon, \varepsilon]^n \subset \mathbb{R}^n$ defined by "pushing everything down" to the real part. This is where we need condition (3). Let

$$S' = H(\partial\psi^{-1}(S \cap U) \times [0,1]).$$

Then, $\overline{\psi^{-1}(S \cap U)}$ is homotopic to $S' \cup [-\varepsilon, \varepsilon]^n$. Denote $\mathbf{Z} = (Z_1, \ldots, Z_n)$. Then, by analyticity,

$$\begin{aligned}\int_{S\cap U} g(\mathbf{z}) e^{r f_r(\mathbf{z})} d\mathbf{z} &= \int_{\psi^{-1}(S\cap U)} g(\psi(\mathbf{Z})) \det \mathrm{D}(\psi(\mathbf{Z})) e^{r f_r(\psi(\mathbf{Z}))} d\mathbf{Z} \\ &= \int_{S'} g(\psi(\mathbf{Z})) \det \mathrm{D}(\psi(\mathbf{Z})) e^{r f_r(\psi(\mathbf{Z}))} d\mathbf{Z} \\ &\quad + \int_{[-\varepsilon,\varepsilon]^n} g(\psi(\mathbf{Z})) \det \mathrm{D}(\psi(\mathbf{Z})) e^{r f_r(\psi(\mathbf{Z}))} d\mathbf{Z}. \qquad \text{(A.7)}\end{aligned}$$

Since $\psi(S') \subset S \smallsetminus U$,

$$\int_{S'} g(\psi(\mathbf{Z})) \det \mathrm{D}(\psi(\mathbf{Z})) e^{r f_r(\psi(\mathbf{Z}))} d\mathbf{Z} = \int_{\psi(S')} g(\mathbf{z}) e^{r f_r(\mathbf{z})} d\mathbf{z} = O\big(e^{r(\mathrm{Re} f(\mathbf{c}) - \delta)}\big); \qquad \text{(A.8)}$$

and by the special case,

$$
\begin{aligned}
&\int_{[-\varepsilon,\varepsilon]^n} g(\psi(\mathbf{Z}))\det \mathrm{D}(\psi(\mathbf{Z}))e^{rf_r(\psi(\mathbf{Z}))}d\mathbf{Z} \\
&\quad = e^{rf(\mathbf{c})}\int_{[-\varepsilon,\varepsilon]^n} g(\psi(\mathbf{Z}))\det \mathrm{D}(\psi(\mathbf{Z}))e^{r(-\sum_{i=1}^n Z_i^2+\frac{\upsilon_r(\psi(\mathbf{Z}))}{r^2})}d\mathbf{Z} \\
&\quad = e^{rf(\mathbf{c})}g(\text{␣}(\mathbf{0}))\det \mathrm{D}(\psi(\mathbf{0}))\Big(\frac{\pi}{r}\Big)^{\frac{n}{2}}\Big(1+O\Big(\frac{1}{r}\Big)\Big) \\
&\quad = \Big(\frac{2\pi}{r}\Big)^{\frac{n}{2}}\frac{g(\mathbf{c})e^{rf(\mathbf{c})}}{\sqrt{\det(-\mathrm{Hess}(f)(\mathbf{c}))}}\Big(1+O\Big(\frac{1}{r}\Big)\Big).
\end{aligned}
$$

Together with (A.6), (A.7), and (A.8), we have the result. ■

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