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Classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories

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Abstract. A unitary fusion category is called $\mathbb{Z}/2\mathbb{Z}$-quadratic if it has a $\mathbb{Z}/2\mathbb{Z}$ group of invertible objects and one other orbit of simple objects under the action of this group. We give a complete classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories. The main tools for this classification are skein theory, a generalisation of Ostrik's results on formal codegrees to analyse the induction of the group elements to the centre, and a computation similar to Larson's rank-finiteness bound for $\mathbb{Z}/3\mathbb{Z}$-near group pseudounitary fusion categories. This last computation is contained in an appendix coauthored with attendees from the 2014 AMS MRC on Mathematics of Quantum Phases of Matter and Quantum Information.

1. Introduction

In the past several decades, unitary fusion categories (UFCs) have seen broad applications to many areas of mathematics and physics, including representation theory, operator algebras, topological quantum field theory (TQFT), theoretical condensed matter, and quantum information. Given the complete list of $6j$-symbols for a UFC, one can build unitary TQFTs which compute quantum invariants of links and 3-manifolds [4, 15], together with physical lattice models which realise these TQFTs [31, 32]. These computations are increasingly difficult in the presence of *multiplicity*, i.e., where there is a fusion channel with a dimension greater than 1, a.k.a. a fusion coefficient which is larger than 1.

While many classification techniques work well for multiplicity free fusion categories, more techniques are required to achieve classification in the multiplicity setting. We note that at the time of writing UFCs have only been classified up to rank 3 [46, 49]. For rank 4 fusion categories with a dual pair of simple objects, there is a classification of possible fusion rings in the pseudounitary setting [30]; our Theorem A (and Corollary B) below completes the classification of rank 4 UFCs with a dual pair of simples. The case of rank 4 with 4 self-dual objects still seems out

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of reach at this time. Multiplicity free fusion rings up to rank 6 admitting unitary categorifications have been classified [36].

Surprisingly, all currently known fusion categories fit into four families:

(1) those related to[1] groups,

(2) those related to quantum groups at roots of unity [1, 2, 18, 52, 54–56],

(3) the Haagerup–Izumi quadratic categories [3, 12, 13, 20, 24–26, 53],

(4) the extended Haagerup fusion categories [7, 19]. Given a finite group G, a G-*quadratic fusion category* is a fusion category $\mathcal{C}$ with a finite group G of simple objects and one other G-orbit $\{g\rho\}_{g\in G}$ of simple objects. (The collection of all G-quadratic fusion categories over all finite groups G is exactly the family (3) above.) The term "quadratic" comes from the existence of a quadratic relation for the self-fusion of an object ρ which generates the other G-orbit. For a family of fusion rings with a fixed rank, we say the family has rank-finiteness if only finitely many of these rings admit a categorification. Surprisingly, for a fixed group G beyond the trivial group, rank-finiteness is not known for G-quadratic fusion categories (for the trivial group, see [46]). The case

$$G = \mathbb{Z}/2\mathbb{Z}$$

is classified in the pivotal setting in [49], and rank-finiteness for $G = \mathbb{Z}/3\mathbb{Z}$ is achieved in the pseudounitary setting in [30].

In this article, we give a complete classification of $\mathbb{Z}/2\mathbb{Z}$-quadratic unitary fusion categories. While we do not find any new fusion categories in this article, we provide important techniques for finding $6j$-symbols for fusion categories with multiplicity. Our main theorem is as follows.

Theorem A. *The complete list of $\mathbb{Z}/2\mathbb{Z}$ quadratic UFCs is as follows.*

The 3 object categories are:

- *the Ising/Tambara–Yamagami categories of the form $\mathcal{TY}(\mathbb{Z}/2\mathbb{Z}, \chi, \pm)$ [53] with A_3 fusion rules, of which there are exactly 2: the case "+" is Temperley–Lieb–Jones at $q = \exp(2\pi i/8)$, and the case "−" is $SU(2)_2$,*
- *the three UFCs with* $\mathsf{Rep}(S_3)$ *fusion rules* [11, Remark 6.6] *and* [25, Theorem 5.1],
- *the two complex conjugate UFCs with* $\mathrm{Ad}(E_6)$ *fusion rules* [8, 24]*: these are exactly the adjoint subcategories of exceptional quantum subgroups of Temperley–Lieb–Jones at $q = \exp(2\pi i/24)$ and $SU(2)_{10}$* [29, 45].

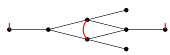
[1] Here, "related to" means obtained by iterating known constructions, such as equivariantisation, Morita equivalence, etc.

The 4 object categories are:

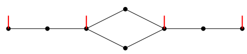
- *the pointed categories* $\mathsf{Hilb}(\mathbb{Z}/4\mathbb{Z}, \omega)$*, where* $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, U(1))$ *and* $\mathsf{Hilb}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \omega)$*, where*

$$\begin{aligned}\omega \in H^3(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, U(1))/\operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \\ \cong (\mathbb{Z}/2\mathbb{Z})^3/\operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\end{aligned}$$

  [10, Remark 4.10.4],
- *the Deligne products* $\mathsf{Fib} \boxtimes \mathsf{Hilb}(\mathbb{Z}/2\mathbb{Z}, \omega)$ *for* $\omega \in H^3(\mathbb{Z}/2\mathbb{Z}, U(1))$*, which have the* A_4 *fusion rules: these two categories are also Temperley–Lieb–Jones at* $q = \exp(2\pi i/10)$ *and* $SU(2)_3$*,*
- $\operatorname{Ad}(SU(2)_6)$*, which is also equivalent to the adjoint subcategory of* A_7 *Temperley–Lieb–Jones category with* $q = \exp(2\pi i/16)$*,*
- *the even parts of the two complex conjugate subfactor planar algebras with principal graphs*

  $\mathcal{S}' =$ 

  from [26, 34]*: these categories are also de-equivariantisations of* $2^{\mathbb{Z}/4\mathbb{Z}}1$ *near group fusion categories* [25, Example 9.5] [35, Example 2.2].
- *the even part of the 2D2 subfactor planar algebra with principal graph*

  from [39] *and* [26, Corollary 9.3]*: this category is also a de-equivariantisation of the even part of the* $3^{\mathbb{Z}/4\mathbb{Z}}$ *subfactor from* [26, 51].

All these UFCs are related to quantum groups at roots of unity or near group fusion categories [13, 25].

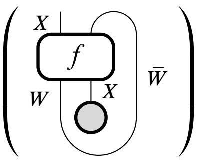
Remark 1.1. The results of Theorem A make no assumptions on the existence of a braiding on the category. The categories appearing in our classification which do not admit braidings are: the two UFC's with $\mathsf{Rep}(S_3)$ fusion rules which are not equivalent to $\mathsf{Rep}(S_3)$ [47, Section 4.4], the two UFC's with $\operatorname{Ad}(E_6)$ fusion rules [47, main theorem], the even parts of the two complex conjugate subfactor planar algebras with principal graphs $\mathcal{S}'$ [34], and the even part of the $2D2$ subfactor (which can be seen to not admit a braiding from the centre analysis in Section 3.2). It is interesting to note that the even parts of the subfactors with $\mathcal{S}'$ principal graphs admit σ-braidings as defined in [34, Definition 3.2].

The result [30, Theorem 1.1] gave a finite list of possible fusion rings for rank 4 pseudounitary fusion categories with a dual pair of simple objects, but included one fusion ring not previously known to be categorifiable (the case $c = 2$ from [30, Theorem 1.1 (6)]), and left open the classification of those fusion rings from [30, Theorem 1.1] which were previously known to be categorifiable.

Corollary B. *We have a complete classification of rank 4 unitary fusion categories with a dual pair of simple objects. In particular, there is no UFC with $c = 2$ from* [30, Theorem 1.1 (6)].

One tool to prove our classification is an adaptation of Larson's rank-finiteness bound for $\mathbb{Z}/3\mathbb{Z}$-near group pseudounitary fusion categories [30, Section 4]. This adaptation appears in Appendix A below, coauthored with attendees from the 2014 AMS MRC program on the Mathematics of Quantum Phases of Matter and Quantum Information.

Our main new technical tool to achieve Theorem A is a generalisation of Ostrik's results on formal codegrees of a spherical fusion category [48, 49]. We use the results of [41, Section 5], but we use the conventions of [23]. Suppose that $\mathcal{C}$ is a spherical fusion category, and denote by $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$ the forgetful functor and let $\mathcal{I} : \mathcal{C} \to Z(\mathcal{C})$ be its adjoint. Let A be the tube algebra of $\mathcal{C}$, and let $A_{X \leftarrow X}$ be the corner of A corresponding to $X \in \operatorname{Irr}(\mathcal{C})$. We pick a non-degenerate trace Tr_X on $A_{X \leftarrow X}$ given by

Tr_X 

$:= \delta_{W=1} \dim(X) \operatorname{tr}_{\mathcal{C}}(f).$

Given an irreducible representation (V, π_V) of $A_{X \leftarrow X}$, its *formal codegree* [38, 48] with respect to Tr_X is given by

$$f_V := \sum_b \operatorname{Tr}_V(\pi(b))\pi(b^\vee),$$

where $\{b\}$ is a basis of $A_{X \leftarrow X}$ and $\{b^\vee\}$ is the dual basis with respect to the non-degenerate pairing $(a, b) := \operatorname{Tr}_X(ab)$ on $A_{X \leftarrow X}$. Observe that f_V is independent of the choice of basis $\{b\}$, but depends on the choice of Tr_X. We refer the reader to Section 2.2 for more details.

Theorem C. *There is a bijective correspondence between irreducible representations (V, π_V) of $A_{X \leftarrow X}$ and simple subobjects $\Gamma_V \subset \mathcal{I}(X) \in Z(\mathcal{C})$. The formal codegree f_V of (V, π) with respect to Tr_X is a scalar, and the categorical dimension of Γ_V is given by $\frac{\dim(\mathcal{C})}{f_V \dim(X)}$. Moreover, if $Y \in \operatorname{Irr}(\mathcal{C})$ and ${}_X\pi_Y$ is the action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, then*

$$\dim \operatorname{Hom}_{\mathcal{C}}(\mathcal{F}(\Gamma_V) \to Y) = \dim \operatorname{Hom}(\pi_V \to {}_X\pi_Y).$$

In the case $X = \mathbf{1} \in \text{Irr}(\mathcal{C})$, this theorem recovers [49, Theorem 2.13], which allowed the computation of the simple decomposition of $\mathcal{I}(\mathbf{1})$ in terms of representations of the fusion algebra of $\mathcal{C}$. Our theorem generalises this result in several ways. The main improvement is that this result allows us to determine the simple decomposition of $\mathcal{I}(X)$ by studying the representations of the corner of the tube algebra $A_{X\leftarrow X}$. When $X = \mathbf{1}$, this algebra is isomorphic to the fusion algebra of $\mathcal{C}$. However, when X is non-trivial, this corner depends on certain $6j$-symbols of the category involving X. One immediate application of this theorem comes from the fact that the dimensions of simple objects in $Z(\mathcal{C})$ are highly restricted, which implies the representations of $A_{X\to X}$ (which depend on the $6j$-symbols) are also restricted. Hence, we obtain obstructions based on $6j$-symbols. We make use of this application in this article to determine several non-trivial $6j$-symbols involving the invertible object of a $\mathbb{Z}/2\mathbb{Z}$-quadratic UFC.

The other improvement Theorem C offers is that for each simple $\Gamma \subset \mathcal{I}(X)$, we can determine $\mathcal{F}(\Gamma) \in \mathcal{C}$. This information is encoded in the action of $A_{X\leftarrow X}$ acting on the entire tube algebra A. As these algebras are semisimple, it is easy to decompose this action into irreducibles and hence apply Theorem C. A surprising application of this theorem comes from the fact that if we know the action of $A_{X\leftarrow X}$ acting on the entire tube algebra A up to isomorphism, we can often determine the action on the nose. As this action is determined by the $6j$-symbols of $\mathcal{C}$, this allows us to find many linear equations involving the $6j$-symbols. We use this application later in this article to get our hands on many $6j$-symbols. In the general setting, this result allows the combinatorial data of the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$ to be leveraged into the categorical data of the $6j$-symbols of $\mathcal{C}$. As the forgetful functor can often be easily determined from the fusion ring of $\mathcal{C}$ [40], we expect this application to have many exciting future uses.

2. Preliminaries

We refer the reader to [10] for the basics of fusion categories. We refer the reader to [22, 50] for the basics of unitary fusion categories. In particular, we always assume a unitary fusion category is equipped with its unique unitary spherical structure where the daggers of cups are caps and the quantum dimensions are equal to the Frobenius–Perron dimensions [37].

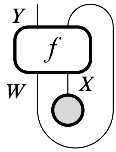
2.1. The tube algebra

One of the key tools in this paper is *Ocneanu's tube algebra* (or equivalently the annular category) of a fusion category. This algebra was first introduced by [44] and [14] in the context of subfactors and by [23, 24] and [41] in the context of fusion categories.

Definition 2.1. Let $\mathcal{C}$ be a spherical fusion category whose spherical trace is denoted by $\mathrm{tr}_{\mathcal{C}}$. The tube algebra A of $\mathcal{C}$ is the finite dimensional semisimple algebra

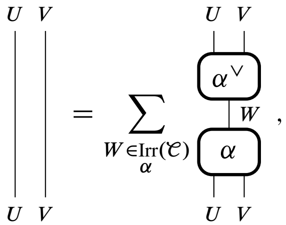
$$\bigoplus_{X,Y\in\mathrm{Irr}(\mathcal{C})} A_{Y\leftarrow X}, \quad \text{where } A_{Y\leftarrow X} := \bigoplus_{W\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(W\otimes X \to Y\otimes W).$$

We graphically represent a fixed basis element of A as

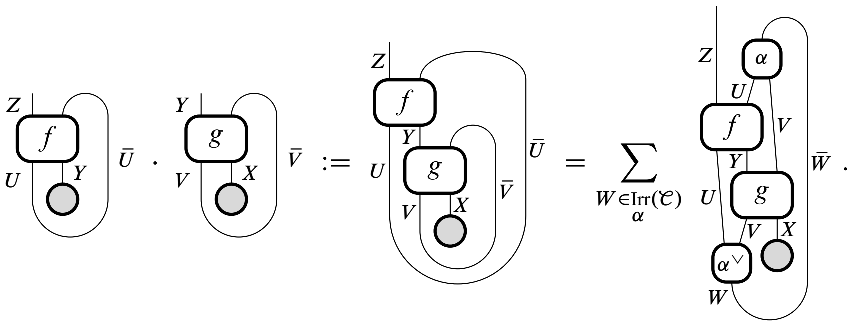


$$, \quad f \in \mathcal{C}(W\otimes X \to Y\otimes W).$$

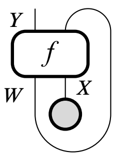
The multiplication on A is defined by composition of the tubes and applying the fusion relation obtained from semisimplicity to the strands around the outside. In more detail, we pick a basis $\{\alpha\} \subset \mathcal{C}(U\otimes V \to W)$ for all $U, V, W \in \mathrm{Irr}(\mathcal{C})$, and let $\{\alpha^\vee\} \subset \mathcal{C}(W \to U\otimes V)$ be the dual basis with respect to the non-degenerate pairing $(\cdot,\cdot) : \mathcal{C}(U\otimes V\to W)\times\mathcal{C}(W\to U\otimes V)\to\mathbb{C}$ determined by the formula $(h,k)\,\mathrm{id}_W = h\circ k \in \mathrm{End}_{\mathcal{C}}(W)$. We have the fusion relation



which gives the following formula for composition in the tube algebra, which is independent of the choice of basis $\{\alpha\}$:



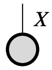
There is a non-degenerate linear functional ϕ on A given by



$$\longmapsto \delta_{X=Y}\delta_{W=\mathbf{1}} \dim(X)\,\mathrm{tr}_{\mathcal{C}}(f).$$

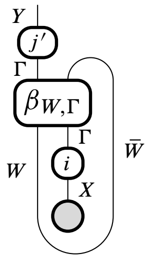
Its restriction to $A_{X\leftarrow X}$ is tracial for all $X \in \mathrm{Irr}(\mathcal{C})$, and we denote it by Tr_X.

Note that each of the spaces $A_{X\leftarrow X}$ is the corner $1_X A 1_X$ of A, where we cut down by the idempotent

$$1_X :=$$  ,

and $A_{X\leftarrow X}$ acts on the spaces $A_{X\leftarrow Y}$ by multiplication.

The tube algebra of $\mathcal{C}$ is intimately related to the Drinfeld centre $Z(\mathcal{C})$ of $\mathcal{C}$. From the data of $Z(\mathcal{C})$, we obtain a basis of matrix units for the spaces $A_{X\leftarrow Y}$ given by

$$e(\Gamma)_{(X,i),(Y,j)} := \frac{\dim(\Gamma)}{\dim(\mathcal{C})\sqrt{\dim(X)\dim(Y)}} \sum_{W\in \mathrm{Irr}\,\mathcal{C}} \dim(W)$$  ,

where $(\Gamma, \beta_\Gamma) \in \mathrm{Irr}(Z(\mathcal{C}))$, $\{i\}$ is a basis for $\mathcal{C}(X \to \mathcal{F}(\Gamma))$, and $\{j\}$ is a basis for $\mathcal{C}(Y \to \mathcal{F}(\Gamma))$, where $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor. Here, $\{j'\} \subset \mathcal{C}(\mathcal{F}(\Gamma) \to Y)$ denotes the dual basis of $\{j\}$ with respect to the pairing

$$k' \circ j = \delta_{j=k}\,\mathrm{id}_Y .$$

With respect to our functional ϕ on A, we have that

$$\phi(e(\Gamma)_{(X,i),(Y,j)}) = \delta_{X,Y}\delta_{i,j}\frac{\dim(X)\dim(\Gamma)}{\dim(\mathcal{C})},$$

and so, the dual basis with respect to the ϕ-pairing is given by

$$e(\Gamma)^{\vee}_{(X,i),(Y,j)} = \frac{\dim(\mathcal{C})}{\dim(X)\dim(\Gamma)} e(\Gamma)_{(Y,j),(X,i)}.$$

The construction above shows us that $Z(\mathcal{C})$ entirely determines the structure of the tube algebra of $\mathcal{C}$. The converse is also true. The tube algebra of $\mathcal{C}$ entirely determines the Drinfeld centre of $\mathcal{C}$. The following theorem gives a bijective correspondence between representations of the tube algebra and objects of $Z(\mathcal{C})$.

Theorem 2.2 ([23] and [41, Section 5]). *There is a bijective correspondence between equivalence classes of irreducible representations of the tube algebra of $\mathcal{C}$ and isomorphism classes of simple objects in $Z(\mathcal{C})$. This bijection sends*

$$(V,\pi) \mapsto \Gamma_V := \bigoplus_{X\in \mathrm{Irr}(\mathcal{C})} V|_{A_{X\leftarrow X}} \otimes X.$$

Further, we have that the minimal central projection $z_V \in A$ corresponding to (V, π) is given by

$$z_V = \sum_{\substack{X \in \mathrm{Irr}(\mathcal{C}), \\ \{i\} \subset \mathcal{C}(X \to \mathcal{F}(\Gamma_V))}} e(\Gamma_V)_{(X,i),(X,i)}.$$

2.2. A new result on formal codegrees

If one knows the full tube algebra of $\mathcal{C}$, then Theorem 2.2 essentially gives you the full data of $Z(\mathcal{C})$. However, in many situations, such as in this article, we only know information about some sub-algebra of the tube algebra, and we wish to leverage this information into partial information about $Z(\mathcal{C})$. Towards this goal, we introduce the *formal codegree* of a representation.

Definition 2.3 ([38, 48]). Let B be a finite dimensional semisimple algebra equipped with a non-degenerate trace Tr_B, and let (V, π) be a finite dimensional representation of B. We define the *formal codegree* of (V, π) as follows:

$$f_V := \sum_b \mathrm{Tr}_V(\pi(b))\pi(b^\vee) \in \pi(B) \subset \mathrm{End}(V),$$

where we sum over a basis $\{b\} \subset B$, and $\{b^\vee\}$ denotes the dual basis with respect to the Tr_B-pairing. Observe that f_V is independent of the choice of basis, but depends on the choice of trace Tr_B.

The following theorem allows us to determine the simple summands of $\mathcal{I}(X) \in Z(\mathcal{C})$ by classifying the representations of the subalgebra $A_{X \leftarrow X}$. Here, $\mathcal{I} : \mathcal{C} \to Z(\mathcal{C})$ is the induction functor which is adjoint to the forgetful functor $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$. Moreover, we can compute categorical dimensions in terms of formal codegrees of $A_{X \leftarrow X}$ with respect to Tr_X.

Theorem (C). *Let $\mathcal{C}$ be a spherical fusion category, and let A be the tube algebra of $\mathcal{C}$. Fix $X \in \mathrm{Irr}(\mathcal{C})$. There is a bijective correspondence between equivalence classes of irreducible representations (V, π) of $A_{X \leftarrow X}$ and isomorphism classes of simple subobjects*

$$\Gamma_V \subset \mathcal{I}(X) \in Z(\mathcal{C}).$$

The formal codegree f_V of (V, π) with respect to Tr_X is a scalar, and the categorical dimension of Γ_V is given by

$$\dim(\Gamma_V) = \frac{\dim(\mathcal{C})}{f_V \dim(X)}.$$

Moreover, if $Y \in \mathrm{Irr}(\mathcal{C})$ and ${}_X\pi_Y$ is the action of $A_{X \leftarrow X}$ on $A_{X \leftarrow Y}$, then

$$\dim(\mathcal{C}(Y \to \mathcal{F}(\Gamma_V))) = \dim(\mathrm{Hom}(\pi_V \to {}_X\pi_Y)).$$

Proof. Let (V, π) be an irreducible representation of $A_{X\leftarrow X}$. Since $A_{X\leftarrow X}$ is semisimple, (V, π) corresponds to a simple summand of $A_{X\leftarrow X}$. As $A_{X\leftarrow X}$ is a corner of A, each simple summand of $A_{X\leftarrow X}$ is of the form $A_{X\leftarrow X} z_\Gamma$ for a simple object $(\Gamma, \beta) \in \mathrm{Irr}(Z(\mathcal{C}))$. Hence, there is a simple $(\Gamma_V, \beta_{\Gamma_V})$ corresponding to (V, π), and by Theorem 2.2,

$$z_V 1_X = \sum_i e(\Gamma_V)_{(X,i),(X,i)}.$$

Moreover, for any other simple object $\Lambda \in \mathrm{Irr}(Z(\mathcal{C}))$, we have that

$$\pi(e(\Lambda)_{(X,i),(X,i)}) = 0$$

unless $\Lambda \cong \Gamma_V$. In particular,

$$\mathrm{Tr}_V(\pi(e(\Lambda)_{(X,i),(X,j)})) = 0$$

unless $\Lambda = \Gamma_V$ and $i = j$. We now compute

$$\begin{aligned} f_V &= \sum_{\substack{\Lambda \subseteq \mathcal{I}(X)\\ i,j}} \mathrm{Tr}_V(\pi(e(\Lambda)_{(X,i),(X,j)}))\pi(e(\Lambda)^\vee_{(X,i),(X,j)}) \\ &= \sum_i \frac{\dim(\mathcal{C})}{\dim(X)\dim(\Gamma_V)} \pi(e(\Gamma_V)_{(X,i),(X,i)}) \\ &= \frac{\dim(\mathcal{C})}{\dim(X)\dim(\Gamma_V)} \pi(z_V 1_X). \end{aligned}$$

Thus, the formal codegree of (V, π) is given by

$$f_V = \frac{\dim(\mathcal{C})}{\dim(X)\dim(\Gamma_V)} \quad \text{and} \quad \dim(\Gamma_V) = \frac{\dim(\mathcal{C})}{f_V \dim(X)}.$$

Finally, we observe that

$$\begin{aligned} \dim(\mathrm{Hom}(V \to {}_X\pi_Y)) &= \dim(\mathrm{Hom}(V \to 1_X A 1_Y)) \\ &= \dim(\mathrm{Hom}(V \to z_V 1_X A 1_Y)) \\ &= \dim(\mathrm{Hom}(V \to z_V 1_X A 1_Y z_V)) \\ &= \sum_j \underbrace{\dim(\mathrm{Hom}(V \to z_V 1_X A e(\Gamma_V)_{(Y,j),(Y,j)}))}_{=1} \\ &= \dim(\mathcal{C}(Y \to \mathcal{F}(\Gamma_V))). \end{aligned}$$ ∎

Note that if we just consider the subalgebra $A_{\mathbf{1}\leftarrow\mathbf{1}} \cong K_0(\mathcal{C})$, the fusion algebra of $\mathcal{C}$, then the above theorem recovers [49, Theorem 2.13], which shows that irreducible representations of $K_0(\mathcal{C})$ are in bijective correspondence with simple summands of $\mathcal{I}(\mathbf{1})$. Thus, our theorem generalises Ostrik's in two ways: (1) it gives us the simple summands of $\mathcal{I}(X)$ where X is any simple object of $\mathcal{C}$, and (2) it tells us the image under the forgetful functor of each of these summands.

2.3. $\mathbb{Z}/2\mathbb{Z}$-quadratic fusion categories

A $\mathbb{Z}/2\mathbb{Z}$-quadratic fusion category is a fusion category $\mathcal{C}$ whose invertible objects form the group $\mathbb{Z}/2\mathbb{Z}$, i.e., $\mathrm{Inv}(\mathcal{C}) = \{1, \alpha\}$ with $\alpha^2 \cong \mathbf{1}$, with one other orbit of simple objects under the $\mathbb{Z}/2\mathbb{Z}$-action. A simple associativity argument shows that we have three cases:

(Q1) simple objects: $\mathbf{1}, \alpha, \rho$; fusion rules determined by $\rho^2 \cong 1 \oplus m\rho \oplus \alpha$,

(Q2) simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ not self-dual; fusion rules determined by $\rho^2 \cong m\rho \oplus n\alpha\rho \oplus \alpha$,

(Q3) simple objects: $\mathbf{1}, \alpha, \rho, \alpha\rho$, ρ self-dual; fusion rules determined by $\rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho$.

Note that in all three cases we have

$$\alpha^2 \cong \mathbf{1} \quad \text{and} \quad \alpha\rho \cong \rho\alpha.$$

2.3.1. Multiplicity bounds and categorifiability. The case (Q1) was classified in the pivotal setting in [49, Theorem 4.1], where it was shown that $m \leq 2$. The complete classification of such unitary fusion categories which was known prior to this article is as follows:

($m = 0$) such a fusion category is a Tambara–Yamagami category of the form $\mathcal{TY}(\mathbb{Z}/2\mathbb{Z}, \chi, \pm)$ [53], of which there are exactly 2. Both such categories are unitarisable.

($m = 1$) such a fusion category has the fusion rules of $\mathsf{Rep}(S_3)$. There are exactly three such unitary fusion categories [25, Theorem 5.1].

($m = 2$) such a fusion category has the fusion rules of $\mathrm{Ad}(E_6)$, and there are exactly 4 such fusion categories [21], all within the same Galois orbit, and each admits a spherical structure. Two of these are unitary and complex conjugate to each other [24].

The case (Q2) was studied in the pseudounitary setting ($\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$) in [30], where it was shown that $m = n \leq 2$. The classification of such fusion categories prior to this article is as follows:

($m = 0$) such a fusion category is necessarily pointed with $\mathbb{Z}/4\mathbb{Z}$ fusion rules. It is thus of the form $\mathsf{Vect}(\mathbb{Z}/4\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, U(1)) = \mathbb{Z}/4\mathbb{Z}$, of which there are 4 categories [10, Remark 4.10.4].

($m = 1$) this case was open. Two such unitary fusion categories which are complex conjugate were known to exist from [26, 34].

($m = 2$) this case was open. No such examples were known to exist.

We finish this classification for unitary fusion categories in Theorem 4.1 below.

In Appendix A, we adapt the results of [30] in the pseudounitary setting to (Q3), where we prove the following theorem.

Theorem 2.4. *Suppose that $\mathcal{C}$ is a pseudounitary fusion category with the fusion rules (Q3). Then, (m, n) must be equal to one of* $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 2)$.

Proof. By Theorem A.12 in Appendix A, we must have $m + n \leq 5$. If either m or n is zero, then there is a fusion subcategory with 3 simple objects, so (m, n) must be one of $(0, 0)$, $(0, 1)$, $(1, 0)$ by [46]. If $0 \neq m \neq n \neq 0$, then $m + n \geq 11$ by Remark A.2 in Appendix A. The result follows. ∎

The proof that $m + n \leq 5$ that appears in Appendix A below was written by Ryan Johnson, Siu–Hung Ng, David Penneys, Jolie Roat, Matthew Titsworth, and Henry Tucker at the 2014 AMS MRC on The Mathematics of Quantum Phases of Matter and Quantum Information.

By [27, 33] (and applying Galois conjugation), any fusion category with fusion rules (Q3) with $(m, n) \in \{(0, 1), (1, 0)\}$ factorizes as a Deligne product of a fusion category with Fibonacci fusion rules, of which there are two, namely, Fib and YL, and a $\mathbb{Z}/2\mathbb{Z}$-pointed fusion category which must be of the form $\mathsf{Vect}(\mathbb{Z}/2\mathbb{Z}, \omega)$ for $\omega \in H^3(\mathbb{Z}/2\mathbb{Z}, U(1))$, of which there are two. Thus, there are exactly 4 such fusion categories, and 2 are unitarisable.

When $m = n \leq 2$, the complete classification of such unitary fusion categories as in Theorem 2.4 is given in Theorem 3.1 below.

3. The self-dual case

In this section, we will focus on the unitary categorification of the fusion ring with four simple objects $\mathbf{1}, \alpha, \rho, \alpha\rho$ and fusion rules

$$\alpha \otimes \alpha \cong \mathbf{1}, \quad \rho \otimes \rho \cong \mathbf{1} \oplus m\rho \oplus m\alpha\rho. \tag{$R(m)$}$$

Let us write ($R(m)$) for such a fusion ring. By Theorem 2.4 above, we know that $m \leq 2$. Our main result of this section is as follows.

Theorem 3.1. *The complete classification of unitary fusion categories $\mathcal{C}_m$ with K_0 $(\mathcal{C}_m) \cong$ ($R(m)$) for $m \leq 2$ is as follows:*

($m = 0$) *$\mathcal{C}_0$ is pointed and thus equivalent to one of the four monoidally distinct categories* $\mathsf{Vect}^\omega(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$, *where*

$$\omega \in H^3(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, U(1))/\operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$

[10, Remark 4.10.4].

($m = 1$) *$\mathcal{C}_1$ is equivalent to $\mathcal{C}(\mathfrak{sl}_2, 7)^{\text{ad}}$, which is also equivalent to the even part of the A_7 Temperley–Lieb–Jones category with*

$$q = \exp(2\pi i/16)$$

[26, Example 9.1].

($m = 2$) *$\mathcal{C}_2$ is equivalent to the even part of the $2D2$ subfactor from* [26, 39].

Proof. It suffices to restrict our attention to the cases of $m = 1$ and $m = 2$. The first step in our proof is to provide a set of numerical data which fully classifies a categorification of ($R(m)$); we do this in Section 3.1. By describing a sufficient list of local relations in our category, we are able to come up with such a set of numerical data. This data consists of $8m^4$ complex scalars and a collection of small roots of unity. This data is precisely a subset of the $6j + 4k$ symbols of such a categorification. Using techniques developed in the subfactor classification program, we prove that this subset of the $6j + 4k$ symbols is sufficient to describe the entire category.

In Section 3.2, in order to get a foothold on the numerical data of a categorification of ($R(m)$), we study the Drinfeld centre of such a category. By studying certain small representations of the tube algebra of the category (using basic combinatorial arguments), we are able to deduce a surprising amount of numerical data of the category. This centre analysis tells us nearly all of the small roots of unity in our numerical data and even gives us highly non-trivial linear equations involving the $8m^4$ complex scalars.

To reduce the $8m^4$ complex scalars down to a more manageable number, in Section 3.4, we apply the tetrahedral symmetries of the $6j + 4k$ symbols. These symmetries only apply in the unitary setting and give S_4 symmetries of these $8m^4$ complex scalars[2]. This essentially reduces the complexity of the problem by a factor of 24. For example, in the $m = 2$ case, we reduce from 128 complex scalars to roughly 10 (some of the S_4 symmetries are redundant). These symmetries turn an intractable amount of data into a set that can easily be dealt with by hand.

To finish off this section, we explicitly solve for the remaining numerical data which describes a categorification of ($R(m)$) in Section 3.5. The results of the previous sections essentially determine everything except the remaining complex scalars. By evaluating diagrams in our category in multiple ways, we are able to obtain equations of these complex scalars. In the $m = 2$ case, we find a single solution, which necessarily has to correspond to the even part of the $2D2$ subfactor. We prove this in Theorem 3.31 below. ■

[2] While writing this article, the article [17] was posted to the arXiv, which describes tetrahedral symmetries for general fusion categories. It would be interesting to use their work to extend our results to the non-unitary setting.

3.1. Numerical data

Let $\mathcal{C}_m$ be a unitary fusion category with $K(\mathcal{C}_m) \cong (R(m))$, $m \in \{1, 2\}$. The goal of this section is to pick nice basis for our morphisms spaces in $\mathcal{C}_m$ and to determine some local relations that these basis elements satisfy. These local relations will depend on the following data:

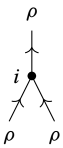
- two choices of signs $\lambda_\rho, \lambda_\alpha \in \{-1, 1\}$ which are the 2nd Frobenius–Schur indicators of α and ρ, respectively,
- a choice of sign $\mu \in \{-1, 1\}$,
- $2m$ choices of $\chi_{\mathbf{1},i} \in \{-1, 1\}$ and $\chi_{\alpha,i} \in \{-\sqrt{\lambda_\alpha}, \sqrt{\lambda_\alpha}\}$ for $0 \le i < m$,
- $2m$ choices of 3rd roots of unity $\omega_{\mathbf{1},i}, \omega_{\alpha,i} \in \{1, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}\}$ for $0 \le i < m$.

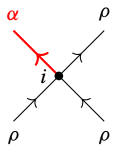
In the following section, we are able to pin down the data μ and χ by analysing the centre of $\mathcal{C}_m$.

To simplify notation, we define

$$d := \dim(\rho),$$

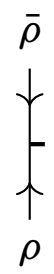
which is the largest solution to $d^2 = 1 + 2md$. If $m = 1$, then $d = 1 + \sqrt{2}$, and if $m = 2$, then $d = 2 + \sqrt{5}$. We choose orthonormal bases for our hom spaces

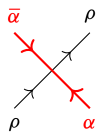


$\in \mathcal{C}_m(\rho \otimes \rho \to \rho)$,


$\in \mathcal{C}_m(\rho \otimes \rho \to \alpha\rho), \quad 0 \le i < m.$

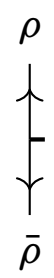
We also choose unitary isomorphisms[3]

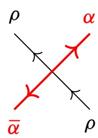
$\in \mathcal{C}_m(\alpha \to \bar{\alpha})$,


$\in \mathcal{C}_m(\rho \to \bar{\rho})$, and


$\in \mathcal{C}_m(\rho \otimes \alpha \to \bar{\alpha} \otimes \rho)$.

We can unambiguously write their inverses as

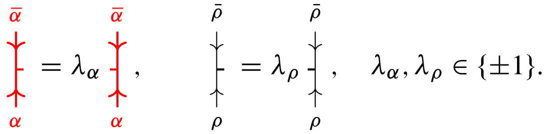
$\in \mathcal{C}_m(\bar{\alpha} \to \alpha)$,


$\in \mathcal{C}_m(\bar{\rho} \to \rho)$, and


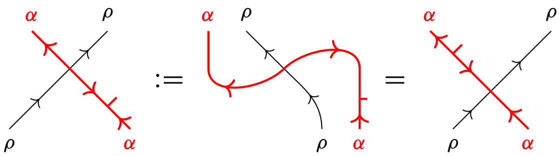
$\in \mathcal{C}_m(\bar{\alpha} \otimes \rho \to \rho \otimes \alpha)$.

[3] Using the convention of switching the orientation of the α-strand through the crossing works better for $\mathbb{Z}/2\mathbb{Z}$-equivariantisation, which is related to the $3^{\mathbb{Z}/4\mathbb{Z}}$-subfactor [26]. In the non-self-dual case in Section 4 below, we will use a more natural convention from a diagrammatic point of view which does not change the orientation of the α-strand.

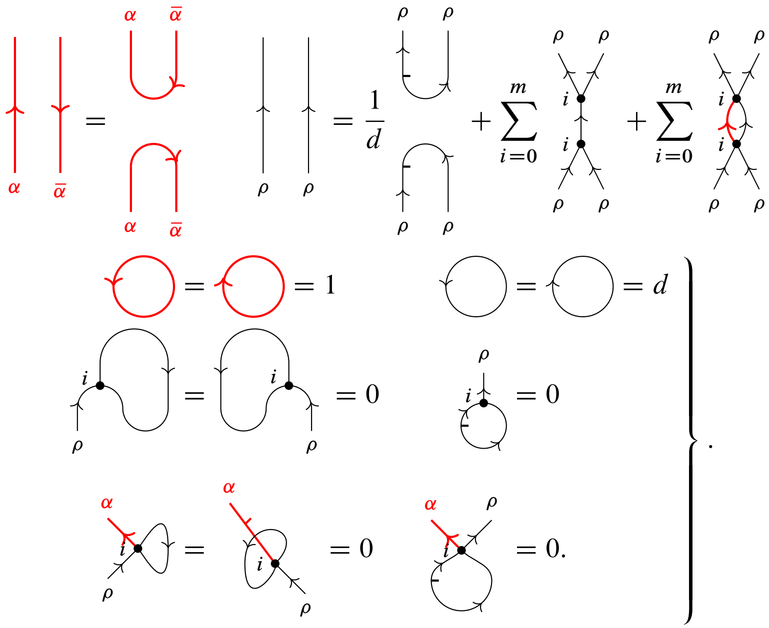
The duals of these first two isomorphisms are related to their inverses, respectively, by the Frobenius–Schur indicators of α and ρ via the following equations:



We can rescale the crossing so that

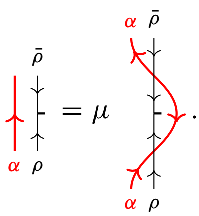


due to the implicit inverses on both sides. Semisimplicity gives us the local relations



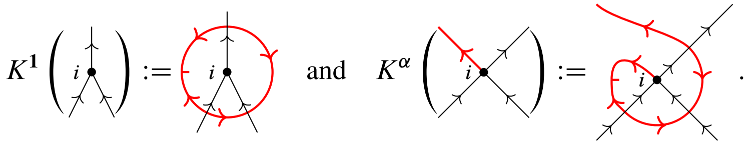
. (3.1)

Definition 3.2. Let $\mu \in \mathbb{C}^\times$ such that

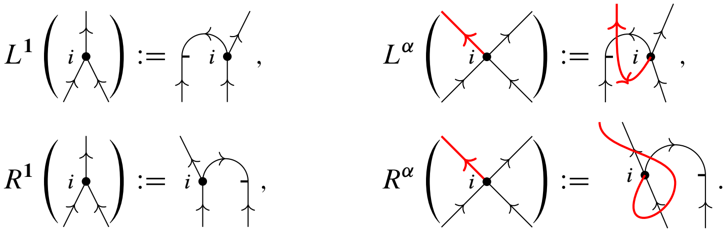


Clearly, $\mu^2 = 1$.

In order to choose a natural basis for the spaces $\mathcal{C}_m(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$, we introduce the following linear operators on these spaces. We often suppress the orientation on the α strands, as it may be inferred from the other orientations in the diagram:



We also define the anti-linear *Frobenius operators*

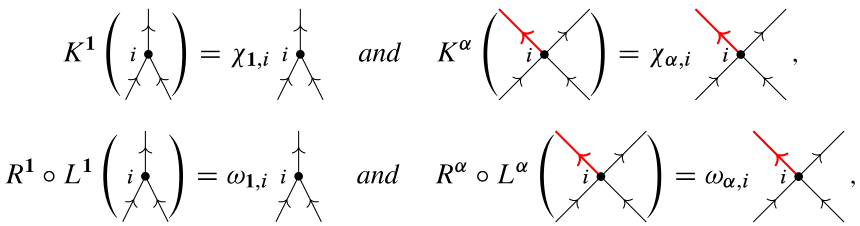


These operators are unitary with respect to the tracial inner product on hom spaces. By a straightforward but tedious calculation, one proves that these operators satisfy the following relations:

$$\begin{aligned}
K^{\mathbf{1}} \circ K^{\mathbf{1}} &= \lambda_\alpha \operatorname{Id}, & K^{\alpha} \circ K^{\alpha} &= \operatorname{Id},\\
L^{\mathbf{1}} \circ L^{\mathbf{1}} &= \lambda_\rho \operatorname{Id} = R^{\mathbf{1}} \circ R^{\mathbf{1}}, & L^{\alpha} \circ L^{\alpha} &= \lambda_\rho \mu \operatorname{Id} = R^{\alpha} \circ R^{\alpha},\\
K^{\mathbf{1}} \circ L^{\mathbf{1}} &= \mu(L^{\mathbf{1}} \circ K^{\mathbf{1}}), & K^{\alpha} \circ L^{\alpha} &= \mu\lambda_\alpha(L^{\alpha} \circ K^{\alpha}),\\
K^{\mathbf{1}} \circ R^{\mathbf{1}} &= \mu(R^{\mathbf{1}} \circ K^{\mathbf{1}}), & K^{\alpha} \circ R^{\alpha} &= \mu\lambda_\alpha(R^{\alpha} \circ K^{\alpha}),\\
(R^{\mathbf{1}} \circ L^{\mathbf{1}})^3 &= 1 = (L^{\mathbf{1}} \circ R^{\mathbf{1}})^3, & (R^{\alpha} \circ L^{\alpha})^3 &= 1 = (L^{\alpha} \circ R^{\alpha})^3.
\end{aligned}$$

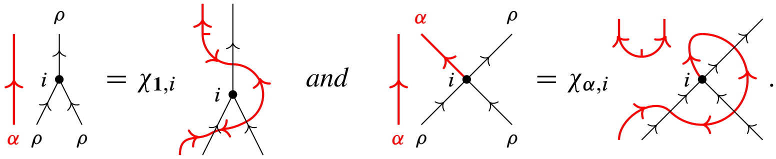
We can diagonalise our basis of $\mathcal{C}_m(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$ with respect to these operators to obtain the following lemma.

Lemma 3.3 (α-jellyfish). *We can choose bases for $\mathcal{C}_m(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$ such that*



where $\chi_{\mathbf{1},i}^2 = \lambda_a$, $\chi_{\alpha,i}^2 = 1$, and $\omega_{\mathbf{1},i}^3 = \omega_{\alpha,i}^3 = 1$.

In particular, we have the local relations



Proof. From the above relations, we have that K^1 and $R^1 \circ L^1$ commute. Furthermore, we have

$$(K^1)^4 = 1 \quad \text{and} \quad (R^1 \circ L^1)^3 = 1.$$

Hence, we can simultaneously diagonalise these operators to obtain the basis of $\mathcal{C}_m(\rho \otimes \rho \to \rho)$ claimed in the statement of the lemma. As $(K^1)^2 = \lambda_\alpha$, we have that $\chi_{1,i}^2 = \lambda_\alpha$, and as $(R^1 \circ L^1)^3 = 1$, we have that $\omega_{1,i}^3 = 1$. The same argument gives the claimed basis for $\mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$.

The local relations in the statement of the lemma follow by applying a local α relation to the operators K^1 and K^α. ∎

In the case that $m = 1$, we have that the spaces $\mathcal{C}_m(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_m(\rho \otimes \rho \to \alpha\rho)$ are 1-dimensional. Hence, the earlier operators are all scalars. In this special case, we determine the scalars λ_α, λ_ρ, and μ.

Corollary 3.4. *If $m = 1$, then $\lambda_\alpha = \lambda_\rho = \mu = 1$.*

Proof. As $m = 1$, we have that L^1 acts by a scalar $l^1 \in \mathbb{C}$. As L^1 is anti-linear, the relation $L^1 \circ L^1 = \lambda_\rho$ gives $l^1\overline{l^1} = \lambda_\rho$. Hence, $\lambda_\rho = 1$. The same analysis on the relation $L^\alpha \circ L^\alpha = \lambda_\rho\mu$ gives that $\lambda_\rho\mu = 1$, and so, $\mu = 1$. Finally, from Lemma 3.3, the linear operator K^α is a real scalar. The relation

$$K^\alpha \circ L^\alpha = \mu\lambda_\alpha(L^\alpha \circ K^\alpha)$$

then gives us that $\lambda_\alpha = 1$. ∎

Note that when $m = 1$ and $\lambda_\alpha = 1$, we have the classification of categories $\mathcal{C}_1$ from [26, Example 9.1]. Hence, we have the following corollary of the above lemma.

Corollary 3.5. *The statement of Theorem 3.1 is true when $m = 1$.*

Hence, for the remainder of this section, we may assume that $m = 2$.

Note that, at this point, we cannot fully determine the action of the operators L and R on our basis. However, we can make the following observation.

Lemma 3.6. *We have two cases depending on the value of $\lambda_\alpha\mu \in \{1, -1\}$:*

(1) *If $\lambda_\alpha\mu = 1$, then the operators L^1 and R^1 preserve the eigenspaces of K^1, and the operators L^α and R^α preserve the eigenspaces of K^α.*

(2) *If* $\lambda_\alpha \mu = -1$, *then the operators* $L^{\mathbf{1}}$ *and* $R^{\mathbf{1}}$ *exchange the eigenspaces of* $K^{\mathbf{1}}$, *and the operators* L^α *and* R^α *exchange the eigenspaces of* K^α. *In particular,* $\chi_{\mathbf{1},0} = -\chi_{\mathbf{1},1}$ *and* $\chi_{\alpha,0} = -\chi_{\alpha,1}$.

Proof. This follows from the commutation relations above, along with the fact that our L and R operators are anti-linear. Let us illustrate a few examples.

Suppose that $\lambda_\alpha \mu = 1$, and let v be an eigenvector for $K^{\mathbf{1}}$ (with eigenvalue χ). Then, the relation $K^{\mathbf{1}} \circ L^{\mathbf{1}} = \mu(L^{\mathbf{1}} \circ K^{\mathbf{1}})$ gives that

$$K^{\mathbf{1}} \circ L^{\mathbf{1}}(v) = \lambda_\alpha \overline{\chi} L^{\mathbf{1}}(v).$$

If $\lambda_\alpha = 1$, then χ is real by Lemma 3.3, and we get that $K^{\mathbf{1}} \circ L^{\mathbf{1}}(v) = \chi L^{\mathbf{1}}(v)$. If $\lambda_\alpha = -1$, then χ is imaginary by Lemma 3.3, and we get that $K^{\mathbf{1}} \circ L^{\mathbf{1}}(v) = -\overline{\chi} L^{\mathbf{1}}(v) = \chi L^{\mathbf{1}}(v)$. The argument for the eigenspaces of K^α is similar (and easier).

Suppose that $\lambda_\alpha \mu = -1$, and let v be an eigenvector for $K^{\mathbf{1}}$ (again with eigenvalue χ). Now, the relation $K^{\mathbf{1}} \circ L^{\mathbf{1}} = \mu(L^{\mathbf{1}} \circ K^{\mathbf{1}})$ gives that

$$K^{\mathbf{1}} \circ L^{\mathbf{1}}(v) = -\lambda_\alpha \overline{\chi} L^{\mathbf{1}}(v).$$

The same argument as above shows that $K^{\mathbf{1}} \circ L^{\mathbf{1}}(v) = -\chi L^{\mathbf{1}}(v)$. Thus, $L^{\mathbf{1}}$ exchanges the eigenspaces of $K^{\mathbf{1}}$, which then must be 1-dimensional. In particular, we must have that $\chi_{\mathbf{1},0} = -\chi_{\mathbf{1},1}$. The same argument holds for the eigenspaces of K^α.

The above arguments also work verbatim for the operators $R^{\mathbf{1}}$ and R^α. ■

3.2. Centre analysis

In this section, we will analyse the Drinfeld centre of the categories $\mathcal{C}_2$ in order to pin down the values of our data χ, μ and the operators L and R.

Our main tool in this section is Theorem 2.2. We remind the reader that, this result states that for an object $X \in \mathcal{C}$, the irreducible representations V of $A_{X \to X}$ are in bijective correspondence with simple summands $\Gamma_V \subset \mathcal{I}(X) \in Z(\mathcal{C})$, the dimension of Γ_V is given by $\frac{\dim(\mathcal{C})}{\dim(X) f_V}$, where f_V is the formal codegree of the representation V, and the multiplicity of $Y \in \mathcal{F}(\Gamma_V)$ is equal to the multiplicity of V in the left action of $A_{X \to X}$ on $A_{X \to Y}$.

With this tool in mind, we aim to study the tube algebra for $\mathcal{C}_2$:

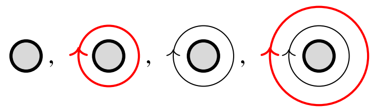
$$A = \begin{array}{|c|c|c|c|} A_{\mathbf{1}\leftarrow\mathbf{1}} & A_{\mathbf{1}\leftarrow\alpha} & A_{\mathbf{1}\leftarrow\rho} & A_{\mathbf{1}\leftarrow\alpha\rho} \\ A_{\alpha\leftarrow\mathbf{1}} & A_{\alpha\leftarrow\alpha} & A_{\alpha\leftarrow\rho} & A_{\alpha\leftarrow\alpha\rho} \\ A_{\rho\leftarrow\mathbf{1}} & A_{\rho\leftarrow\alpha} & A_{\rho\leftarrow\rho} & A_{\rho\leftarrow\alpha\rho} \\ A_{\alpha\rho\leftarrow\mathbf{1}} & A_{\alpha\rho\leftarrow\alpha} & A_{\alpha\rho\leftarrow\rho} & A_{\alpha\rho\leftarrow\alpha\rho} \end{array}$$

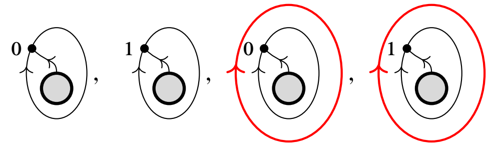
By determining the irreducible representations of the sub-algebras $A_{X\leftarrow X}$, and their multiplicities in the left action of $A_{X\leftarrow X}$ on $A_{X\leftarrow Y}$, we can determine the simple objects of $Z(\mathcal{C}_2)$ and their images under the forgetful functor.

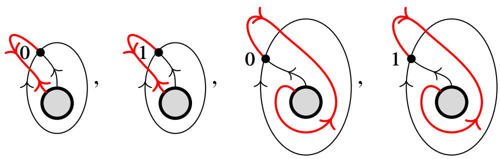
Performing this computation over all of the tube algebra is far too computationally taxing. Instead, we restrict our attention to the sub-algebra

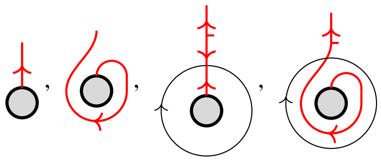
$$A_{\mathbf{1}\leftarrow\mathbf{1}} \oplus A_{\alpha\leftarrow\alpha} \oplus A_{\mathbf{1}\leftarrow\rho} \oplus A_{\alpha\leftarrow\rho} \oplus A_{\mathbf{1}\leftarrow\alpha\rho}.$$

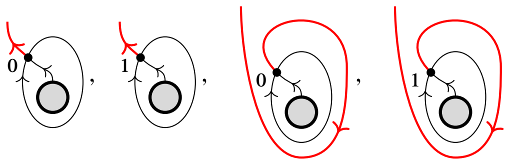
We pick the following bases for these spaces:

$A_{\mathbf{1}\leftarrow\mathbf{1}} = \text{span}$  ,

$A_{\mathbf{1}\leftarrow\rho} = \text{span}$  ,

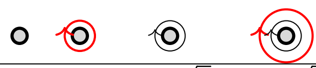
$A_{\mathbf{1}\leftarrow\alpha\rho} = \text{span}$  ,

$A_{\alpha\leftarrow\alpha} = \text{span}$  ,

$A_{\alpha\leftarrow\rho} = \text{span}$  .

By direct computation, we obtain the following:

- The algebra $A_{\mathbf{1}\leftarrow\mathbf{1}}$ has four 1-dimensional representations, which are



χ_0	1	1	$2+\sqrt{5}$	$2+\sqrt{5}$
χ_1	1	1	$2-\sqrt{5}$	$2-\sqrt{5}$
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

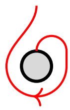
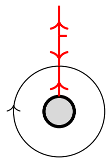
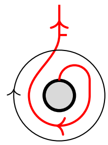
The formal codegrees of these representations are then $20+8\sqrt{5}$, $20-8\sqrt{5}$, 4, and 4, respectively. Hence, by Theorem C, the object $\mathcal{I}(\mathbf{1})$ is a direct sum of 4 simple objects X_i with dimensions

$$\dim(X_0) = 1, \quad \dim(X_1) = 9 + 4\sqrt{5}, \quad \text{and} \quad \dim(X_2) = \dim(X_3) = 5 + 2\sqrt{5}.$$

- Direct computation on the basis elements of $A_{\alpha\leftarrow\alpha}$ gives the following multiplications:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_\alpha \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_\alpha \\ \mu & 0 & \frac{\chi_1}{\sqrt{\lambda_\alpha}} & \chi_\alpha \\ 0 & \lambda_\alpha\mu & \lambda_\alpha\chi_\alpha & \frac{\chi_1}{\sqrt{\lambda_\alpha}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \lambda_\alpha\mu & \lambda_\alpha\chi_\alpha & \frac{\chi_1}{\sqrt{\lambda_\alpha}} \\ \lambda_\alpha\mu & 0 & \lambda_\alpha\frac{\chi_1}{\sqrt{\lambda_\alpha}} & \lambda_\alpha\chi_\alpha \end{bmatrix},$$

where $\chi_1 := \sqrt{\lambda_\alpha}(\chi_{1,0} + \chi_{1,1}) \in \{-2, 0, 2\}$ and $\chi_\alpha := \chi_{\alpha,0} + \chi_{\alpha,1} \in \{-2, 0, 2\}$. Here, we fix our choice of square roots so that $\sqrt{\lambda_\alpha} = 1$ if $\lambda_\alpha = 1$, and $\sqrt{\lambda_\alpha} = i$ if $\lambda_\alpha = -1$. From this, we determine that $A_{\alpha\leftarrow\alpha}$ has the four 1-dimensional representations:

				
τ_0	1	$\sqrt{\lambda_\alpha}$	$\frac{\chi_1+\chi_\alpha+\sqrt{4\mu\lambda_\alpha+(\chi_1+\chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1+\chi_\alpha+\sqrt{4\mu\lambda_\alpha+(\chi_1+\chi_\alpha)^2}}{2}$
τ_1	1	$\sqrt{\lambda_\alpha}$	$\frac{\chi_1+\chi_\alpha-\sqrt{4\mu\lambda_\alpha+(\chi_1+\chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1+\chi_\alpha-\sqrt{4\mu\lambda_\alpha+(\chi_1+\chi_\alpha)^2}}{2}$
τ_2	1	$-\sqrt{\lambda_\alpha}$	$\frac{\chi_1-\chi_\alpha+\sqrt{4\mu\lambda_\alpha+(\chi_1-\chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1-\chi_\alpha+\sqrt{4\mu\lambda_\alpha+(\chi_1-\chi_\alpha)^2}}{-2}$
τ_3	1	$-\sqrt{\lambda_\alpha}$	$\frac{\chi_1-\chi_\alpha-\sqrt{4\mu\lambda_\alpha+(\chi_1-\chi_\alpha)^2}}{2\sqrt{\lambda_\alpha}}$	$\frac{\chi_1-\chi_\alpha-\sqrt{4\mu\lambda_\alpha+(\chi_1-\chi_\alpha)^2}}{-2}$

Hence, by Theorem C, the object $\mathcal{I}(\alpha)$ is a direct sum of 4 simple objects Y_i with dimensions

$$\dim(Y_0) = \frac{\dim(\mathcal{C})}{2 + \frac{1}{2}|\chi_1 + \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}|^2},$$
$$\dim(Y_1) = \frac{\dim(\mathcal{C})}{2 + \frac{1}{2}|\chi_1 + \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 + \chi_\alpha)^2}|^2},$$
$$\dim(Y_2) = \frac{\dim(\mathcal{C})}{2 + \frac{1}{2}|\chi_1 - \chi_\alpha + \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}|^2},$$
$$\dim(Y_3) = \frac{\dim(\mathcal{C})}{2 + \frac{1}{2}|\chi_1 - \chi_\alpha - \sqrt{4\mu\lambda_\alpha + (\chi_1 - \chi_\alpha)^2}|^2}.$$

From [23, Lemma 5.4], we have that $t \cdot p_{\tau_i} = \theta_{Y_i} p_{\tau_i}$, where $p_{\tau_i} = \sum_b \tau(b) b^* \in A_{\alpha \leftarrow \alpha}$ is the minimal central idempotent corresponding to τ_i, and in our case, the operator t is simply right multiplication by the 2nd basis element. Hence, we have that

$$\theta_{Y_0} = \theta_{Y_1} = \lambda_\alpha \sqrt{\lambda_\alpha} \quad \text{and} \quad \theta_{Y_2} = \theta_{Y_3} = -\lambda_\alpha \sqrt{\lambda_\alpha}.$$

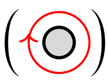
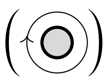
- Let Z_i be the remaining simple objects of $\mathcal{Z}(\mathcal{C}_2)$. Then

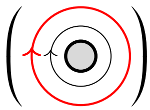
$$\mathcal{F}(Z_i) = p_i \rho \oplus q_i \alpha\rho$$

for some $p_i, q_i \in \mathbb{N}$. Further

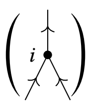
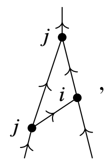
$$\dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 20 = \dim \operatorname{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\alpha\rho)),$$
$$\dim \operatorname{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\rho)) = 16.$$

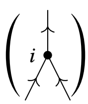
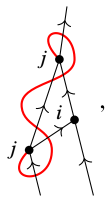
- Let ${}_1\pi_\rho$ be the action of $A_{1 \leftarrow 1}$ on $A_{1 \leftarrow \rho}$. Then

${}_1\pi_\rho$  $= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, ${}_1\pi_\rho$  $= \begin{bmatrix} \phi & \phi' \\ \phi' & \phi \end{bmatrix}$,

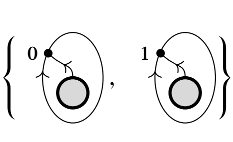
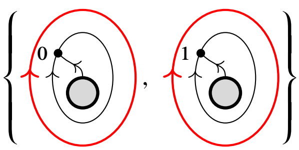
${}_1\pi_\rho$  $= \begin{bmatrix} \phi' & \phi \\ \phi & \phi' \end{bmatrix}$,

where ϕ and ϕ' are the operators on $\operatorname{Hom}(\rho \otimes \rho \leftarrow \rho)$ defined by

ϕ  $= \sum_j$  ,

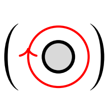
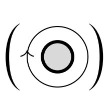
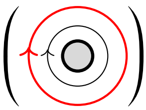
ϕ'  $= \sum_j$  ,

which we can identify as operators on the two spaces:

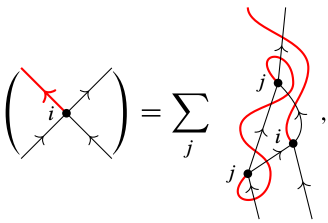
 and 

by local insertion, i.e., the elements of $A_{\mathbf{1}\leftarrow\mathbf{1}}$ which involve ϕ, ϕ' above act on $A_{\mathbf{1}\leftarrow\rho}$ by applying ϕ, ϕ' locally on the trivalent vertices in our standard basis of $A_{\mathbf{1}\leftarrow\rho}$.

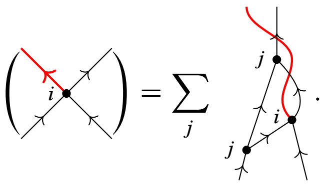
- Let ${}_{\mathbf{1}}\pi_{\alpha\rho}$ be the action of $A_{\mathbf{1}\leftarrow\mathbf{1}}$ on $A_{\mathbf{1}\leftarrow\alpha\rho}$. Then

${}_{\mathbf{1}}\pi_{\alpha\rho}$ 
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
${}_{\mathbf{1}}\pi_{\alpha\rho}$ 
$$= \begin{bmatrix} \psi' & \psi \\ \psi & \psi' \end{bmatrix},$$
${}_{\mathbf{1}}\pi_{\alpha\rho}$ 
$$= \begin{bmatrix} \psi & \psi' \\ \psi' & \psi \end{bmatrix},$$

where ψ and ψ' are the operators on $\mathrm{Hom}(\rho \otimes \rho \to \alpha\rho)$ defined by

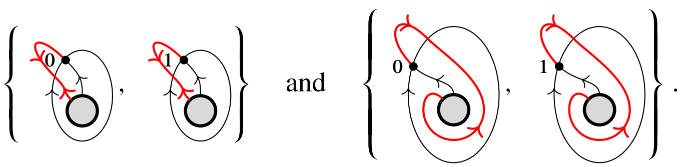
ψ 

,

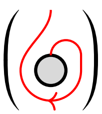
ψ' 

.

As before, we can naturally identify ψ, ψ' as operators on the following two spaces by local insertion:



- Denoting by ${}_{\alpha}\pi_{\rho}$ the action of $A_{\alpha\to\alpha}$ on $A_{\alpha\leftarrow\rho}$, we have

${}_{\alpha}\pi_{\rho}$ 
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_\alpha & 0 & 0 & 0 \\ 0 & \lambda_\alpha & 0 & 0 \end{bmatrix}.$$

We begin by analysing the corner of the tube algebra $A_{\mathbf{1}\leftarrow\mathbf{1}}$, and its actions on $A_{\mathbf{1}\leftarrow\rho}$ and $A_{\mathbf{1}\leftarrow\alpha\rho}$. This gives us the following result.

Lemma 3.7. *There exists* $b \in \{0, 1, 2\}$ *such that*

$$\begin{aligned}\mathcal{F}(X_0) &= \mathbf{1},\\ \mathcal{F}(X_1) &= \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho,\\ \mathcal{F}(X_2) &= \mathbf{1} \oplus b\rho \oplus (2-b)\alpha\rho,\\ \mathcal{F}(X_3) &= \mathbf{1} \oplus (2-b)\rho \oplus b\alpha\rho.\end{aligned}$$

Furthermore, if $b \in \{0, 2\}$*, then the operators* ϕ *and* ψ *are both the same scalar*

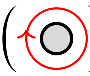
$$\phi = \psi = \frac{1+b-\sqrt{5}}{2},$$

and if $b = 1$*, the operators* ϕ *and* ψ *have the two eigenvalues*

$$\frac{1-\sqrt{5}}{2} \quad \text{and} \quad \frac{3-\sqrt{5}}{2}.$$

Proof. First, note that as X_0 is the tensor unit of $Z(\mathcal{C})$, we have that ${}_{\mathbf{1}}\pi_\rho$ and ${}_{\mathbf{1}}\pi_{\alpha\rho}$ contain no copies of χ_0. From the above computations, we have that

$$\mathrm{Tr}\left({}_{\mathbf{1}}\pi_\rho\left(\quad\right)\right) = 0.$$



As ${}_{\mathbf{1}}\pi_\rho$ is 4-dimensional, and χ_0 is not a sub-representation, we must have that

$${}_{\mathbf{1}}\pi_\rho \cong 2\chi_1 \oplus b\chi_2 \oplus (2-b)\chi_3, \quad \text{where } b \in \{0, 1, 2\}.$$

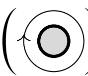
Thus, $\mathcal{F}(X_1)$ contains two copies of ρ, and a dimension count shows that

$$\mathcal{F}(X_1) = \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho.$$

From this, we can deduce three possibilities for the restrictions of X_2 and X_3.

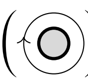
Case 1. $\mathcal{F}(X_2) = \mathcal{F}(X_3) = \mathbf{1} \oplus \rho \oplus \alpha\rho$, in which case, ${}_{\mathbf{1}}\pi_\rho \cong {}_{\mathbf{1}}\pi_{\alpha\rho} \cong 2\chi_1 \oplus \chi_2 \oplus \chi_3$, and, in particular,

$$\mathrm{Tr}\left({}_{\mathbf{1}}\pi_\rho\left(\quad\right)\right) = 4 - 2\sqrt{5}.$$

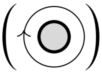


Case 2. $\mathcal{F}(X_2) = \mathbf{1} \oplus 2\rho$ and $\mathcal{F}(X_3) = \mathbf{1} \oplus 2\alpha\rho$, in which case, ${}_{\mathbf{1}}\pi_\rho \cong 2\chi_1 \oplus 2\chi_2$ and ${}_{\mathbf{1}}\pi_{\alpha\rho} = 2\chi_1 \oplus 2\chi_3$, and, in particular,

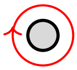
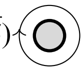
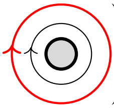
$$\mathrm{Tr}\left({}_{\mathbf{1}}\pi_\rho\left(\quad\right)\right) = 6 - 2\sqrt{5}.$$



Case 3. $\mathcal{F}(X_2) = \mathbf{1} \oplus 2\alpha\rho$ and $\mathcal{F}(X_3) = \mathbf{1} \oplus 2\rho$, in which case, ${}_{\mathbf{1}}\pi_\rho = 2\chi_1 + 2\chi_3$ and ${}_{\mathbf{1}}\pi_{\alpha\rho} = 2\chi_1 + 2\chi_2$, and in particular,

$\operatorname{Tr}\Big({}_{\mathbf{1}}\pi_\rho\Big($  $\Big)\Big) = 2 - 2\sqrt{5}.$

We now aim to deduce more information about the operator ψ. Note that

$z_1 := \frac{1}{\dim(\mathcal{C})}\Big($  $+$  $+ (2+\sqrt{5})$  $+ (2+\sqrt{5})$  $\Big)$

is the minimal central idempotent corresponding to the representation χ_0; i.e.,

$$z_1 \cdot x = \chi_0(x) \cdot z_1.$$

As ${}_{\mathbf{1}}\pi_\rho$ contains no copies of χ_0, we get

$${}_{\mathbf{1}}\pi_\rho(z_1) = 0,$$

and so,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \big(2+\sqrt{5}\big)(\phi + \phi') = 0 \Longrightarrow \phi' = -\left(\phi + \begin{bmatrix} \frac{1}{2+\sqrt{5}} & 0 \\ 0 & \frac{1}{2+\sqrt{5}} \end{bmatrix}\right).$$

To solve for ϕ, we use the fusion rule $\rho^2 = \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho$ to get

$$\phi^2 + \big(\sqrt{5} - 2\big)\phi = \begin{bmatrix} \frac{1}{2+\sqrt{5}} & 0 \\ 0 & \frac{1}{2+\sqrt{5}} \end{bmatrix}.$$

Together with knowing the trace of ${}_{\mathbf{1}}\pi_\rho(\rho)$ in each of the above cases, we can solve to get the statement of the lemma.

To obtain the statement about ψ, we repeat the above analysis with ${}_{\mathbf{1}}\pi_{\alpha\rho}$. ∎

This completes our analysis of $\mathcal{I}(\mathbf{1})$. We now analyse the object $\mathcal{I}(\alpha)$. Our first goal is to show that the object α never lifts to the centre. To begin, we prove the following lemma.

Lemma 3.8. *Suppose that α has a lift to the centre $\mathcal{Z}(\mathcal{C}_2)$. Then, $b = 1$.*

Proof. For a contradiction, suppose that $b = 2$.

By a relabeling, we can assume that Y_0 is a lift of α to $\mathcal{Z}(\mathcal{C}_2)$, and that $Y_i = Y_0 \otimes X_i$. As $\mathcal{F}$ is a $\otimes$-functor, this gives us $\mathcal{F}(Y_i)$ in terms of the b from Lemma 3.7. We then have

$$\mathcal{I}(\rho) = 2X_1 \oplus 2X_2 \oplus 2Y_1 \oplus 2Y_3 \oplus \bigoplus p_i Z_i,$$
$$\mathcal{I}(\alpha\rho) = 2X_1 \oplus 2X_3 \oplus 2Y_1 \oplus 2Y_2 \oplus \bigoplus q_i Z_i.$$

This gives us the following:

$$\begin{aligned}20 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 16 + \sum p_i^2,\\ 20 &= \dim \operatorname{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\alpha\rho)) = 16 + \sum q_i^2,\\ 16 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\alpha\rho)) = 8 + \sum p_i q_i.\end{aligned}$$

Hence,

$$\sum p_i^2 = 4 = \sum q_i^2 \quad \text{and} \quad \sum p_i q_i = 8,$$

which is impossible.

The contradiction when $b = 0$ is nearly identical. ■

With this lemma in hand, we can now show that α never lifts to the centre.

Lemma 3.9. *The object α does not have a lift to the centre.*

Proof. For a contradiction, suppose that α lifts to the centre. We have two cases depending on the value of λ_α.

First, suppose that $\lambda_\alpha = 1$. From Lemma 3.8, we have that $b = 1$, the same style of argument from the proof of this lemma shows that

$$\begin{aligned}\mathcal{I}(\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus p_i Z_i,\\ \mathcal{I}(\alpha\rho) &= 2X_1 \oplus X_2 \oplus X_3 \oplus 2Y_1 \oplus Y_2 \oplus Y_3 \oplus \bigoplus q_i Z_i,\end{aligned}$$

with

$$\sum p_i^2 = 8 = \sum q_i^2 \quad \text{and} \quad \sum p_i q_i = 4.$$

In particular,

$$\sum p_i(p_i + q_i) = 12.$$

We now use the Ng–Schauenburg formula for the 2nd Frobenius–Schur indicator of ρ to obtain

$$\begin{aligned}\lambda_\rho \dim(\mathcal{C}_2) &= 2(1+4d)(\theta_{X_1}^2 + \theta_{Y_1}^2) + (1+2d)(\theta_{X_2}^2 + \theta_{Y_2}^2 + \theta_{X_2}^2 + \theta_{Y_3}^2)\\ &\quad + \sum p_i(p_i + q_i) d\theta_{Z_i}^2\\ &= 8 + 24d + \sum p_i(p_i + q_i) d\theta_{Z_i}^2.\end{aligned}$$

For either case of λ_ρ, we have that $|\sum p_i(p_i + q_i)\theta_{Z_i}^2| \geq 4\sqrt{5} + 8 > 12$. Hence, α cannot have a lift in this case.

Now, suppose that $\lambda_\alpha = -1$. By analysing the dimensions formulas for Y_i over all the different cases of μ, $\chi_{\mathbf{1}}$, and χ_α, we see that α can only lift if $\mu = -1$, and $|\chi_{\mathbf{1}}| = |\chi_\alpha| = 2$. In this case, we have that the other Y_i have dimensions $9 + 4\sqrt{5}$ and

$5+2\sqrt{5}$ occurring twice. Furthermore, we see that the dimension 1 object and the dimension $9+4\sqrt{5}$ object have the same twist. Hence, by relabeling, we may assume

$$\begin{aligned} &\dim(Y_0)=1, &&\dim(Y_1)=9+4\sqrt{5}, &&\dim(Y_2)=\dim(Y_3)=5+2\sqrt{5},\\ &\theta_{Y_0}=\theta_{Y_1}=\pm\mathbf{i}, &&\theta_{Y_2}=\theta_{Y_3}=\mp\mathbf{i}, &&Y_i=Y_0\otimes X_i. \end{aligned}$$

As $\theta_{Y_0}=\pm\mathbf{i}$, we have that $\langle X_0, Y_0\rangle$ is a modular subcategory of $\mathcal{Z}(\mathcal{C}_2)$. Hence, $\mathcal{Z}(\mathcal{C}_2)$ factors as $\mathcal{Z}(\mathcal{C}_2)_0\boxtimes\langle X_0, Y_0\rangle$. Note that we have a simple $W\in\mathcal{Z}(\mathcal{C}_2)_0$ if and only if $\theta_{W\otimes Y_0}=\theta_W\theta_{Y_0}$. This implies that $\{X_0, X_1, Y_2, Y_3\}\in\mathcal{Z}(\mathcal{C}_2)_0$, and $\{X_2, X_3, Y_0, Y_1\}\in\mathcal{Z}(\mathcal{C}_2)_0\otimes Y_0$. Let us write $\{Z_i\}_{i\in\Lambda_0}$ for the remaining simple objects of $\mathcal{Z}(\mathcal{C}_2)$ which live in $\mathcal{Z}(\mathcal{C}_2)_0$. We then have

$$\begin{aligned} \mathcal{I}(\rho)&=2X_1\oplus X_2\oplus X_3\oplus 2Y_1\oplus Y_2\oplus Y_3\oplus\bigoplus_{i\in\Lambda_0}p_iZ_i\oplus\bigoplus_{i\in\Lambda_0}q_iY_0\otimes Z_i,\\ \mathcal{I}(\alpha\rho)&=2X_1\oplus X_2\oplus X_3\oplus 2Y_1\oplus Y_2\oplus Y_3\oplus\bigoplus_{i\in\Lambda_0}q_iZ_i\oplus\bigoplus_{i\in\Lambda_0}p_iY_0\otimes Z_i. \end{aligned}$$

From $20=\dim\operatorname{Hom}(\mathcal{I}(\rho),\mathcal{I}(\rho))$ and $16=\dim\operatorname{Hom}(\mathcal{I}(\alpha\rho),\mathcal{I}(\rho))$, we obtain

$$\sum_{i\in\Lambda_0}p_i^2+q_i^2=8\quad\text{and}\quad\sum_{i\in\Lambda_0}2p_iq_i=4.$$

Hence,

$$\sum_{i\in\Lambda_0}(p_i+q_i)^2=12\quad\text{and}\quad\sum_{i\in\Lambda_0}(p_i-q_i)^2=4.$$

By Cauchy–Schwarz applied to the vectors $(p_i+q_i)_i$ and $(|p_i-q_i|)_i$,

$$\sum_{i\in\Lambda_0}|p_i-q_i|^2=\sum_{i\in\Lambda_0}(p_i+q_i)|p_i-q_i|\le\sqrt{12\cdot 4}=4\sqrt{3}.$$

Again, we use the Ng–Schauenburg formula for the 2nd Frobenius–Schur indicator of ρ to obtain

$$\begin{aligned} \lambda_\rho\dim(\mathcal{C}_2)&=2(1+4d)(\theta_{X_1}^2+\theta_{Y_1}^2)+(1+2d)(\theta_{X_2}^2+\theta_{Y_2}^2+\theta_{X_2}^2+\theta_{Y_3}^2)\\ &\quad+\sum_{i\in\Lambda_0}p_i(p_i+q_i)d\theta_{Z_i}^2+\sum_{i\in\Lambda_0}q_i(p_i+q_i)d\theta_{Y_0\otimes Z_i}^2\\ &=\sum_{i\in\Lambda_0}(p_i^2-q_i^2)d\theta_{Z_i}^2. \end{aligned}$$

From this, we obtain

$$\sum_{i\in\Lambda_0}|p_i^2-q_i^2|\ge\frac{\dim(\mathcal{C}_2)}{d}=4\sqrt{5}.$$

Hence, α cannot have a lift in this case. ■

We can now deduce that both $\chi_{\mathbf{1}}$ and χ_α are 0. That is, $\chi_{\mathbf{1},1} = -\chi_{\mathbf{1},2}$ and $\chi_{\alpha,1} = -\chi_{\alpha,2}$.

Lemma 3.10. *We have that $\chi_{\mathbf{1},0} = -\chi_{\mathbf{1},1}$ and $\chi_{\alpha,0} = -\chi_{\alpha,1}$. In particular, we may assume that $\chi_{\mathbf{1},0} = \sqrt{\lambda_\alpha}$, $\chi_{\mathbf{1},1} = -\sqrt{\lambda_\alpha}$, $\chi_{\alpha,0} = 1$, and $\chi_{\alpha,1} = -1$.*

Proof. If $\lambda_\alpha\mu = -1$, then we have the first statement of the lemma from Lemma 3.6. Hence, we can assume that $\lambda_\alpha\mu = 1$.

First, consider the case that $|\chi_{\mathbf{1}}| = |\chi_\alpha| = 2$. Then, the earlier dimension formulas for $\dim(Y_i)$ show that one of these dimensions is 1, which implies that α lifts to the centre. But this contradicts Lemma 3.9.

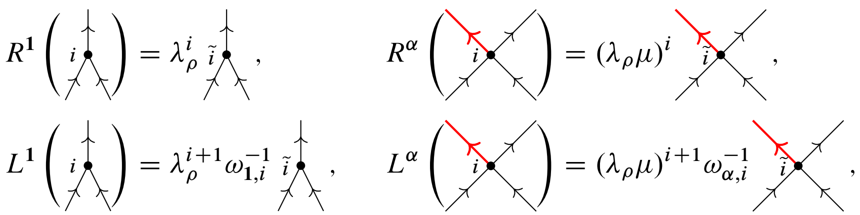
In the case that $\chi_{\mathbf{1}} = 0$ and $|\chi_\alpha| = 2$, or $|\chi_{\mathbf{1}}| = 2$ and $\chi_\alpha = 0$, then one of the Y_i's has dimension $\frac{5+2\sqrt{5}}{2+\sqrt{2}}$, which is impossible.

The only remaining case is that $\chi_{\mathbf{1}} = 0$ and $\chi_\alpha = 0$, which implies the first statement of the lemma.

As $\chi_{\mathbf{1},i}^2 = \lambda_\alpha$ and $\chi_{\alpha,i}^2 = 1$, we may reorder our basis to give the statement of the lemma. ■

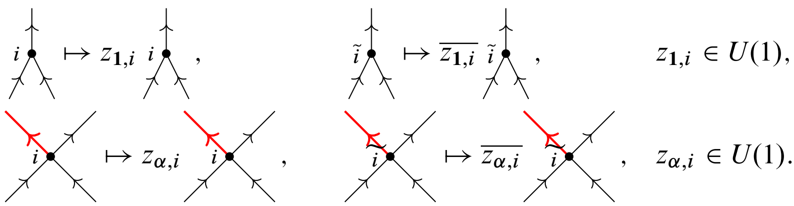
As a result of the above lemma, we know that the eigenspaces of $K^{\mathbf{1}}$ and K^α are 1-dimensional. We can pair this information with Lemma 3.6 to obtain the action of the L and R operators on our eigenbasis.

Lemma 3.11. *The basis of the spaces $\mathcal{C}_2(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_2(\rho \otimes \rho \to \alpha\rho)$ from Lemma 3.3 can be chosen so that*



where $i \mapsto \tilde{i}$ is an order two involution on the indexing set $\{0, 1\}$. If $\lambda_\alpha\mu = 1$, then $\tilde{0} = 0$ and $\tilde{1} = 1$. If $\lambda_\alpha\mu = -1$, then $\tilde{0} = 1$ and $\tilde{1} = 0$, and in this case, we have that $\omega_{\mathbf{1},0} = \omega_{\mathbf{1},1}$ and $\omega_{\alpha,0} = \omega_{\alpha,1}$. Furthermore, if $\lambda_\rho = -1$, then $\lambda_\alpha\mu = -1$, and if $\mu\lambda_\rho = -1$, then $\lambda_\alpha\mu = -1$.

We are free to exchange our distinguished basis elements and to rescale them by



Proof. Let us begin with the operator $R^{\mathbf{1}}$. In the case of $\lambda_\alpha \mu = 1$, we have from Lemma 3.6 that $R^{\mathbf{1}}$ preserves the eigenspaces of $K^{\mathbf{1}}$. As these eigenspaces are 1-dimensional, by Lemma 3.10, we have that $R^{\mathbf{1}}$ is of the form (using linear operator notation, even though $R^{\mathbf{1}}$ is anti-linear)

$$R^{\mathbf{1}} = \begin{bmatrix} R^{\mathbf{1}}_{0,0} & 0 \\ 0 & R^{\mathbf{1}}_{1,1} \end{bmatrix}.$$

As $(R^{\mathbf{1}})^4 = 1$, we have that these coefficients are elements of $U(1)$, and as $R^{\mathbf{1}}$ is anti-linear, we can rescale out two basis vectors by

$$\sqrt{R^{\mathbf{1}}_{0,0}} \in U(1) \quad \text{and} \quad \sqrt{R^{\mathbf{1}}_{1,1}} \in U(1)$$

to arrange that both these coefficients are 1. Note that this rescaling does not affect the relations of Lemma 3.3 as the operators $K^{\mathbf{1}}$ and $R^{\mathbf{1}} \circ L^{\mathbf{1}}$ are linear.

In the case of $\lambda_\alpha \mu = -1$, we have from Lemma 3.6 that $R^{\mathbf{1}}$ exchanges the eigenspaces of $K^{\mathbf{1}}$. We thus have that $R^{\mathbf{1}}$ is of the form

$$R^{\mathbf{1}} = \begin{bmatrix} 0 & R^{\mathbf{1}}_{0,1} \\ R^{\mathbf{1}}_{1,0} & 0 \end{bmatrix}.$$

By choosing our second basis vector as the image under $R^{\mathbf{1}}$ of the first, we arrange that $R^{\mathbf{1}}_{0,1} = 1$. Again, this does not affect the relations of Lemma 3.3 as $R^{\mathbf{1}}$ is unitary. We now use the relation $(R^{\mathbf{1}})^2 = \lambda_\rho$ to see that $R^{\mathbf{1}}_{1,0} = \lambda_\rho$.

Together, these give the action of $R^{\mathbf{1}}$ as in the statement of the lemma. The action of $L^{\mathbf{1}}$ follows from the action of $R^{\mathbf{1}} \circ L^{\mathbf{1}}$ from Lemma 3.3, along with the relation $(R^{\mathbf{1}})^2 = \lambda_\rho$. In the case of $\lambda_\alpha \mu = -1$, we can perform that same argument on $L^{\mathbf{1}} \circ R^{\mathbf{1}} = (R^{\mathbf{1}} \circ L^{\mathbf{1}})^{-1}$ to see that $\omega_{\mathbf{1},0} = \omega_{\mathbf{1},1}$.

Finally, from the relation $(R^{\mathbf{1}})^2 = \lambda_\rho$, we can see that if $\lambda_\rho = -1$, then only the case $\lambda_\alpha \mu = -1$ is possible.

The same analysis on the operators R^α and L^α gives the remaining statement of the lemma. ∎

Now that we have pinned down $\chi_{\mathbf{1}}$ and χ_α, we can describe the objects $Y_i \subset \mathcal{I}(\alpha)$ in more detail.

Lemma 3.12. *We have that*

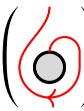
$$\begin{aligned} \mathcal{F}(Y_0) &= \alpha \oplus c_0\rho \oplus (2 - c_0)\alpha\rho, \\ \mathcal{F}(Y_1) &= \alpha \oplus (2 - c_0)\rho \oplus c_0\alpha\rho, \\ \mathcal{F}(Y_2) &= \alpha \oplus c_2\rho \oplus (2 - c_2)\alpha\rho, \\ \mathcal{F}(Y_3) &= \alpha \oplus (2 - c_2)\rho \oplus c_2\alpha\rho, \end{aligned} \qquad \textit{where } c_0, c_2 \in \{0, 1, 2\}.$$

Proof. As $\chi_{\mathbf{1},1} = -\chi_{\mathbf{1},2}$ and $\chi_{\alpha,1} = -\chi_{\alpha,2}$, we have that $\chi_{\mathbf{1}} = \chi_{\alpha} = 0$, and so each of the objects Y_i has dimension $5 + 2\sqrt{5}$. We thus have

$$\begin{aligned}
\mathcal{F}(Y_0) &= \alpha \oplus c_0\rho \oplus (2 - c_0)\alpha\rho, \\
\mathcal{F}(Y_1) &= \alpha \oplus c_1\rho \oplus (2 - c_1)\alpha\rho, \\
\mathcal{F}(Y_2) &= \alpha \oplus c_2\rho \oplus (2 - c_2)\alpha\rho, \\
\mathcal{F}(Y_3) &= \alpha \oplus c_3\rho \oplus (2 - c_3)\alpha\rho.
\end{aligned}$$

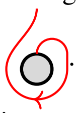
For some integers, $c_i \in \{0, 1, 2\}$.

From the computations in Section 3.2 determining the matrix for the operator ${}_\alpha\pi_\rho$, we see that

$\mathrm{Tr}\Big({}_\alpha\pi_\rho\Big($  $\Big)\Big) = 0.$

On the other hand, from Theorem C, we have that

$$ {}_\alpha\pi_\rho \cong c_0\tau_0 \oplus c_1\tau_1 \oplus c_2\tau_2 \oplus c_3\tau_3. $$

From the earlier tube algebra computations, we know the value of the representations τ_i on the element . In particular, we know the trace of this value. As traces are preserved under direct sums, we obtain that

$$ c_0 + c_1 - c_2 - c_3 = 0. $$

From the formula

$$ \dim \mathrm{Hom}(\alpha, \mathcal{R}\mathcal{I}(\rho)) = \dim \mathrm{Hom}(\mathcal{I}(\alpha), \mathcal{I}(\rho)) = 4, $$

we obtain $c_0 + c_1 + c_2 + c_3 = 4$. Together, we get the statement of the lemma. ∎

With the restrictions of the objects X_i and Y_i now understood, we can give a fairly explicit formula for the even Frobenius–Schur indicators of ρ. This formula will come in handy at several points later in this article.

Lemma 3.13. *We have that*

$$ \nu_{2n}(\rho)\dim(\mathcal{C}) = 28 + 12\sqrt{5} + \lambda_\alpha^n(20 + 8\sqrt{5}) + (2 + \sqrt{5})\sum p_i(p_i + q_i)\theta_{Z_i}^{2n}, $$

where p_i and q_i are integers satisfying $\sum p_i(p_i + q_i) = 16$, and the θ_{Z_i} are roots of unity.

Proof. From Lemmas 3.7 and 3.12, we know the image under the forgetful functor of each of the simple objects appearing in $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\alpha)$, up to some small integers

b, c_0, c_2. Then, we can write

$$\begin{aligned}\mathcal{I}(\rho) &= 2X_1 \oplus bX_2 \oplus (2-b)X_3 \oplus c_0 Y_0 \oplus (2-c_0)Y_1 \oplus c_2 Y_2 \oplus (2-c_2)Y_3 \\ &\quad \oplus \bigoplus p_i Z_i, \\ \mathcal{I}(\alpha\rho) &= 2X_1 \oplus (2-b)X_2 \oplus bX_3 \oplus (2-c_0)Y_0 \oplus c_0 Y_1 \oplus (2-c_2)Y_2 \oplus c_2 Y_3 \\ &\quad \oplus \bigoplus q_i Z_i.\end{aligned}$$

Using the fact that $\mathcal{F}(\mathcal{I}(\rho)) \cong \bigoplus_{X \in \mathrm{Irr}(\mathcal{C})} X\rho X^*$, we obtain

$$\begin{aligned}20 &= \dim \mathrm{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 4 + b^2 + (2-b)^2 + c_0^2 + (2-c_0)^2 \\ &\quad + c_2^2 + (2-c_2)^2 + \sum p_i^2, \\ 16 &= \dim \mathrm{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\rho)) = 4 + 2b(2-b) + 2c_0(2-c_0) + 2c_2(2-c_2) \\ &\quad + \sum p_i q_i,\end{aligned}$$

so, $36 = \dim \mathrm{Hom}(\mathcal{I}(\rho \oplus \alpha\rho), \mathcal{I}(\rho)) = 20 + \sum p_i^2 + \sum p_i q_i$, and thus, $\sum p_i(p_i + q_i) = 16$.

We have from the earlier computations that $\theta_{X_i}^2 = 1$ and $\theta_{Y_i}^2 = \lambda_\alpha$. We can use the Ng–Schauenburg formula for the $2n$th Frobenius–Schur indicator [42, Theorem 4.1] to obtain

$$\begin{aligned}\nu_{2n}(\rho)\dim(\mathcal{C}) &= \sum_{W \in Z(\mathcal{C})} \dim \mathrm{Hom}(\mathcal{F}(W) \to \rho) \dim(W) \theta_W^{2n} \\ &= 28 + 12\sqrt{5} + \lambda_\alpha^n(20 + 8\sqrt{5}) + (2+\sqrt{5}) \sum p_i(p_i + q_i)\theta_{Z_i}^{2n}. \qquad \blacksquare\end{aligned}$$

We finish this section by showing that $\mu = 1$ in all cases.

Lemma 3.14. *We have that $\mu = 1$.*

Proof. First, suppose that $\lambda_\alpha = \mu$, and for a contradiction, suppose that $\mu = -1$ so that $\lambda_\alpha = -1$. We thus have that one of λ_ρ or $\mu\lambda_\rho$ is -1. We thus get from Lemma 3.11 that $\lambda_\alpha \mu = -1$, which is our contradiction.

Now, suppose that $\lambda_\alpha = -\mu$, and for a contradiction, suppose that $\mu = -1$ so that $\lambda_\alpha = 1$. As $\mu = -1$, we can exchange ρ and $\alpha\rho$ if necessary to arrange $\lambda_\rho = -1$ (as a direct computation shows that $\nu_2(\rho) = \mu \cdot \nu_2(\alpha\rho)$).

We can now use Lemma 3.13, along with the fact that the 2nd Frobenius–Schur indicator of ρ is λ_ρ to get the equation

$$-20 - 8\sqrt{5} = 48 + 20\sqrt{5} + (2+\sqrt{5}) \sum p_i(p_i + q_i)\theta_{Z_i}^2,$$

where $\sum p_i(p_i + q_i) = 16$. Thus, $\sum p_i(p_i + q_i)\theta_{Z_i}^2 = -4 - 12\sqrt{5}$. However, Theorem A.9 implies that it takes at least $12\phi(10) = 48$ roots of unity to write $-4 - 12\sqrt{5}$, contradicting $\sum p_i(p_i + q_i) = 16$. Hence, $\mu = 1$. $\blacksquare$

3.3. Sufficient relations to evaluate closed diagrams

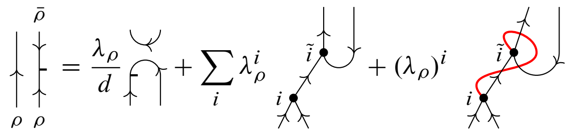
In this section, we will introduce several more 6-j style local relations in our category $\mathcal{C}_2$ and furthermore show that the full collection of relations described completely determine the category $\mathcal{C}_2$. We will do this via the standard technique of showing that our relations suffice to evaluate every endomorphism of the tensor unit to a scalar. These additional local relations will be determined by $8m^4$ complex scalars $A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell} \in \mathbb{C}$ for $0 \le i, j, k, \ell < 2$. These complex scalars are entries of the F-tensors $F^{\rho,\rho,\rho}_{\rho}$, $F^{\rho,\rho,\rho}_{\alpha\rho}$, $F^{\alpha\rho,\rho,\rho}_{\rho}$, and $F^{\alpha\rho,\rho,\rho}_{\alpha\rho}$.

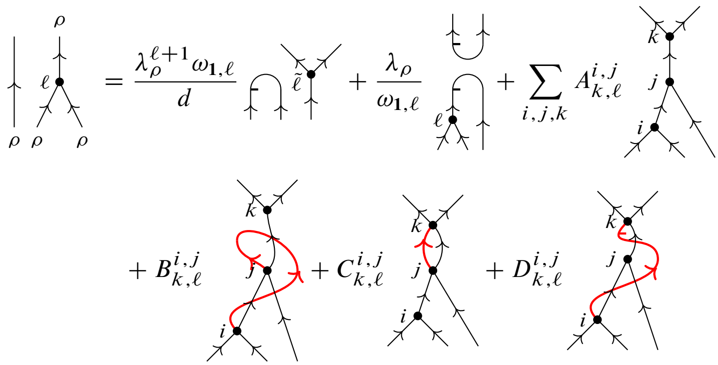
3.3.1. Jellyfish relations. In this section, we will introduce the following 128 complex scalars.

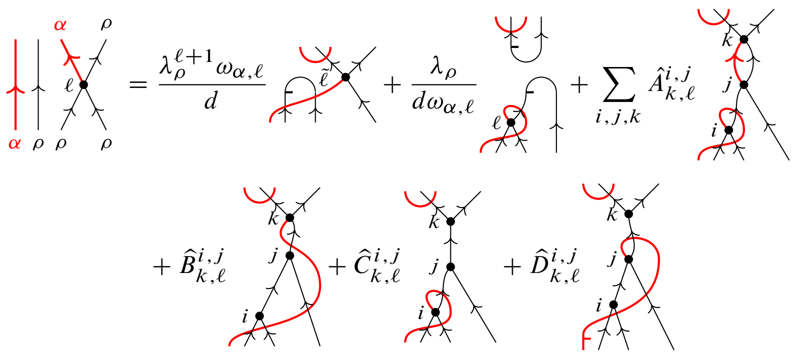
Lemma 3.15 (ρ-jellyfish). *There exist scalars*

$$A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell} \in \mathbb{C}, \quad 0 \le i, j, k, \ell < 2$$

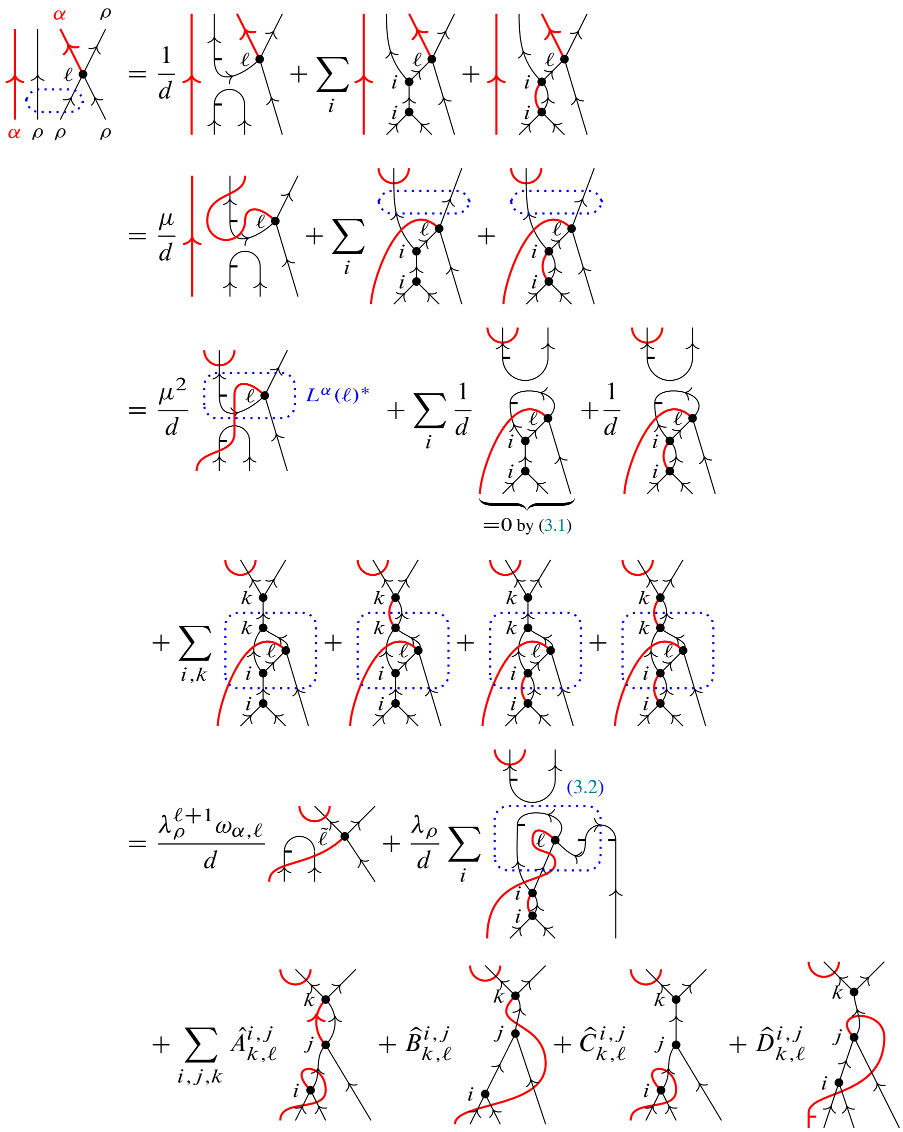
such that the following local relations hold in $\mathcal{C}_2$:





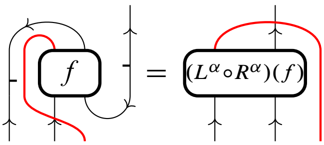


Proof. We provide the proof for the third relation, and the other two are left to the reader. We compute



For the final four sums of diagrams after the $\sum_{i,k}$ before the final equality above, we express each of the sub-diagrams in the dotted blue boxes in terms of our chosen basis for $\mathcal{C}_2(\rho^2 \to \rho)$ and $\mathcal{C}_2(\rho^2 \to \alpha\rho)$. For instance, for the diagram directly after the $\sum_{i,k}$, the sub-diagram in the blue box lives in $\mathcal{C}_2(\alpha\rho^2 \to \rho) \cong \mathcal{C}_2(\rho^2 \to \alpha\rho)$, so the sub-diagram can be expressed as a linear combination of 4-valent vertices. The coefficients $\widehat{A}^{i,j}_{k,\ell}$, $\widehat{B}^{i,j}_{k,\ell}$, $\widehat{C}^{i,j}_{k,\ell}$, $\widehat{D}^{i,j}_{k,\ell}$ then arise as arbitrary basis coefficients.

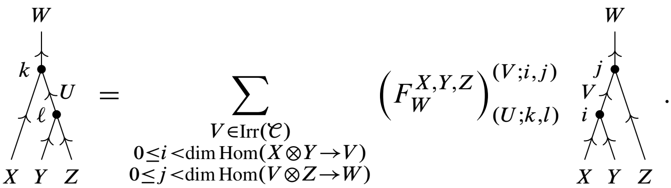
For the second diagram in the last line above, we used the relation that for $f \in \mathcal{C}_2(\rho^2 \to \alpha\rho)$,



(3.2)

which can be verified by a straightforward diagrammatic calculation. We then use the relation in Lemma 3.3 to obtain $(L^\alpha \circ R^\alpha)(f)$. We leave the final simplification of this second diagram using the α-jellyfish relation of Lemma 3.3 to the reader. ■

Remark 3.16. Recall that the associator F-tensors of a unitary fusion category are determined by the formula

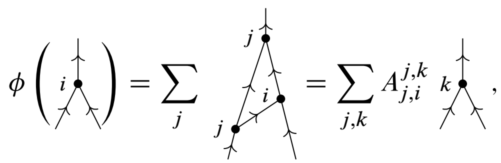


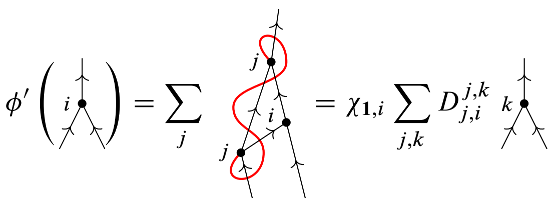
We have the following identification between the above 128 complex scalars and certain F-tensors of the category $\mathcal{C}_2$:

$$
\begin{aligned}
A_{k,\ell}^{i,j} &= \left(F_\rho^{\rho,\rho,\rho}\right)_{(\rho;k,\ell)}^{(\rho;i,j)}, & \hat{A}_{k,\ell}^{i,j} &= \left(F_{\alpha\rho}^{\alpha\rho,\rho,\rho}\right)_{(\alpha\rho;k,\ell)}^{(\rho;i,j)},\\
B_{k,\ell}^{i,j} &= \left(F_\rho^{\rho,\rho,\rho}\right)_{(\rho;k,\ell)}^{(\alpha\rho;i,j)}, & \hat{B}_{k,\ell}^{i,j} &= \left(F_{\alpha\rho}^{\alpha\rho,\rho,\rho}\right)_{(\alpha\rho;k,\ell)}^{(\alpha\rho;i,j)},\\
C_{k,\ell}^{i,j} &= \left(F_{\alpha\rho}^{\rho,\rho,\rho}\right)_{(\rho;k,\ell)}^{(\rho;i,j)}, & \hat{C}_{k,\ell}^{i,j} &= \left(F_{\rho}^{\alpha\rho,\rho,\rho}\right)_{(\alpha\rho;k,\ell)}^{(\rho;i,j)},\\
D_{k,\ell}^{i,j} &= \left(F_{\alpha\rho}^{\rho,\rho,\rho}\right)_{(\rho;k,\ell)}^{(\alpha\rho;i,j)}, & \hat{D}_{k,\ell}^{i,j} &= \left(F_{\rho}^{\alpha\rho,\rho,\rho}\right)_{(\alpha\rho;k,\ell)}^{(\alpha\rho;i,j)}.
\end{aligned}
$$

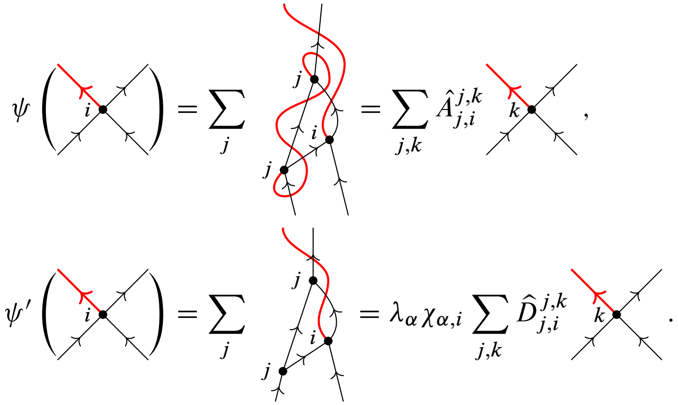
In the name of readability, we will not use this F-tensor notation in this article.

Remark 3.17. With the above jellyfish relations, we can describe the operators ϕ and ψ from Section 3.2 in terms of our free scalars. We have

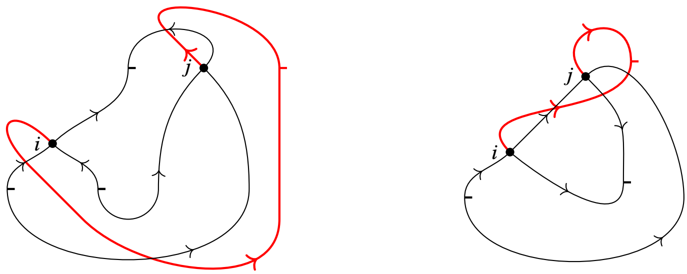




and

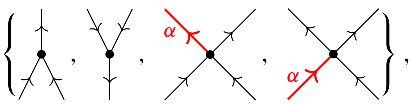


3.3.2. Absorption relations. Using the nomenclature from [7], a closed diagram in our generators is said to be in *jellyfish form* if all trivalent and tetravalent vertices and their labels appear on the external region of the closed diagram. By a slight abuse of nomenclature, we will say that a morphism in a hom space is in jellyfish form (or a *train* in the nomenclature of [6]) if all labels of trivalent and tetravalent vertices in the morphism meet the leftmost region of the morphism. In the examples below, the left diagram is not in jellyfish form, and the right diagram is in jellyfish form.

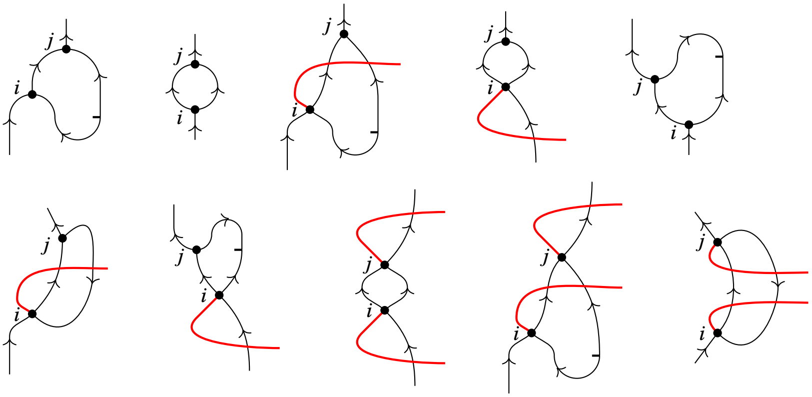


Lemma 3.18 (Absorption). *Using the relations from Sections* 3.1 *and* 3.3.1, *any two trivalent/tetravalent vertices in jellyfish form connected by two of their ρ strands so that the composite is still in jellyfish form may be simplified into a diagram with no trivalent/tetravalent vertices.*

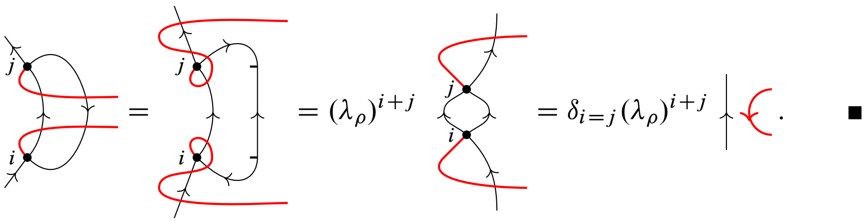
Proof. There are 16 words of length 2 on the symbols



and up to adjoints, 10 are distinct. Given any word of length 2, there is a unique composite in jellyfish form with two ρ strands connected, up to labels and moving tags through crossings. There are thus 10 cases to consider:



We give a full proof for the last case, and the others are similar and omitted:



3.3.3. Evaluation algorithm. With these local relations in hand, we can show that the numerical data we have described uniquely determines the category $\mathcal{C}_2$.

Proposition 3.19. *There is at most one unitary fusion category $\mathcal{C}_2$ realising each tuple of data*

$$\big(\lambda_\alpha, \lambda_\rho, \omega, A, B, C, D, \hat{A}, \widehat{B}, \widehat{C}, \widehat{D}\big).$$

Proof. The proof is an adaptation of Bigelow's jellyfish algorithm [5, 7]. Given any closed diagram in our generators, we show that it can be evaluated to a scalar using our relations. This immediately implies the stated result by [9, Lemma 2.4] which is the unshaded pivotal category version of [7, Proposition 3.5] for shaded planar algebras. Indeed, let $\mathcal{C}'$ be the quotient of the free category in our generators, modulo the relations corresponding to the data $(\lambda_\alpha, \lambda_\rho, \omega, A, B, C, D, \hat{A}, \widehat{B}, \widehat{C}, \widehat{D})$. If we can show that any closed diagram in our generators can be evaluated to a scalar using the given relations, then we have that every ideal of $\mathcal{C}'$ is contained in the negligible ideal. We then have an equivalence

$$\mathcal{C}' / \operatorname{Neg}(\mathcal{C}') \to \mathcal{C}_2$$

which shows that $\mathcal{C}_2$ is uniquely determined by the above tuple of data.

By the jellyfish relations from Lemmas 3.3 and 3.15, it suffices to show we can evaluate any closed diagram in *jellyfish form*, in which all trivalent and tetravalent vertices and their labels appear on the external boundary of the closed diagram. There are 3 cases for such a diagram.

Case 1. There are no vertices at all in the closed diagram. Then, we may use (3.1) to evaluate the closed diagram to a scalar.

Case 2. There is a trivalent/tetravalent vertex connected to itself. Then, we may use (3.1) to show that this closed diagram is equal to zero.

Case 3. There are two neighbouring trivalent/tetravalent vertices that are connected by at least 2 of their ρ strands. Then, using the absorption relations from Lemma 3.18, we can express our closed diagram in jellyfish form as a linear combination of diagrams with strictly fewer vertices, which are still in jellyfish form.

We are finished by a simple induction argument on the number of vertices in our closed diagram in jellyfish form. ∎

Remark 3.20. We wish to point out that we can also give an existence result for the categories $\mathcal{C}_m$ by realising them as actions by endomorphisms on the Cuntz algebras

$$O_{2m+1} \rtimes \mathbb{Z}_2.$$

To obtain existence, one needs to verify a finite list of polynomial equations that the above tuple needs to satisfy. As we can conclude existence of the examples in this article from the existing literature, we will not include the details of this existence result.

3.4. Symmetries

With the results of the last section in hand, the major task in front of us is to determine the 128 complex scalars

$$A^{i,j}_{k,\ell}, \quad B^{i,j}_{k,\ell}, \quad C^{i,j}_{k,\ell}, \quad D^{i,j}_{k,\ell},$$
$$\hat{A}^{i,j}_{k,\ell}, \quad \hat{B}^{i,j}_{k,\ell}, \quad \hat{C}^{i,j}_{k,\ell}, \quad \hat{D}^{i,j}_{k,\ell}.$$

In theory, we could begin evaluating diagrams in our category in multiple ways in order to obtain equations of these variables. However, in practice, this task is too complicated, given that we have 128 unknowns. To make our task of pinning down these scalars easier, we aim to find symmetries between them and to show that many of them must in fact vanish. The symmetries of these scalars come from the tetrahedral symmetries of the $6j$ symbols, which were rigorously studied in [16], and have been used in previous works of the second author [25, 26]. (See also footnote 2.)

The main result of this section is as follows.

Lemma 3.21. *The scalars $B^{i,j}_{k,\ell}$, $C^{i,j}_{k,\ell}$, $\widehat{B}^{i,j}_{k,\ell}$, $\widehat{C}^{i,j}_{k,\ell}$, $\widehat{D}^{i,j}_{k,\ell}$ can be expressed in terms of the $D^{i,j}_{k,\ell}$ as*

$$\begin{aligned}
B^{i,j}_{k,\ell} &= \lambda_\rho^{1+i+k}\lambda_\alpha\sqrt{\lambda_\alpha}(-1)^\ell\omega_{\mathbf{1},\ell}D^{j,\tilde{k}}_{\tilde{i},\ell},\\
\widehat{B}^{i,j}_{k,\ell} &= \lambda_\rho^{1+j+\ell}\lambda_\alpha\sqrt{\lambda_\alpha}(-1)^i\omega_{\mathbf{1},j}\omega^2_{\mathbf{1},i}\omega_{\alpha,\ell}D^{\tilde{\ell},i}_{k,\tilde{j}},\\
C^{i,j}_{k,\ell} &= \lambda_\rho^{1+j+k}\lambda_\alpha\sqrt{\lambda_\alpha}(-1)^\ell\omega^2_{\mathbf{1},\ell}D^{\tilde{k},i}_{\tilde{j},\ell},\\
\widehat{C}^{i,j}_{k,\ell} &= \lambda_\rho^{1+j+k}\sqrt{\lambda_\alpha}(-1)^k\omega_{\mathbf{1},k}\omega^2_{\mathbf{1},j}\omega_{\alpha,i}D^{i,\tilde{k}}_{\ell,\tilde{j}},\\
\widehat{D}^{i,j}_{k,\ell} &= \sqrt{\lambda_\alpha}(-1)^{k+j}\omega_{\mathbf{1},i}\omega^2_{\mathbf{1},k}\omega_{\alpha,\ell}\omega^2_{\alpha,j}D^{j,i}_{\ell,k}.
\end{aligned}$$

The scalars $A^{i,j}_{k,\ell}$ and $\hat{A}^{i,j}_{k,\ell}$ satisfy S_4 symmetries generated by the order three rotation

$$A^{i,j}_{k,\ell}=\lambda_\rho^{1+i+k}\omega_{\mathbf{1},\ell}A^{j,\tilde{k}}_{\tilde{i},\ell}=\lambda_\rho^{1+j+k}\omega^2_{\mathbf{1},\ell}A^{\tilde{k},i}_{\tilde{j},\ell}\quad\hat{A}^{i,j}_{k,\ell}=\lambda_\rho^{1+i+k}\omega_{\alpha,\ell}\hat{A}^{j,\tilde{k}}_{\tilde{i},\ell}=\lambda_\rho^{1+j+k}\omega^2_{\alpha,\ell}\hat{A}^{\tilde{k},i}_{\tilde{j},\ell}$$

and the order two flips

$$A^{i,j}_{k,\ell} = \omega_{\mathbf{1},k}\omega^2_{\mathbf{1},i}A^{\tilde{k},\tilde{\ell}}_{\tilde{i},\tilde{j}} = \lambda_\rho^{j+\ell}\overline{A^{k,\tilde{j}}_{i,\tilde{\ell}}},\qquad \hat{A}^{i,j}_{k,\ell} = \omega_{\alpha,k}\omega^2_{\alpha,i}\hat{A}^{\tilde{k},\tilde{\ell}}_{\tilde{i},\tilde{j}} = \lambda_\rho^{j+\ell}\overline{\hat{A}^{k,\tilde{j}}_{i,\tilde{\ell}}}.$$

The $D^{i,j}_{k,\ell}$ scalars satisfy the $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ symmetries generated by

$$D^{i,j}_{k,\ell} = \lambda_\rho^{j+l}\overline{D^{k,\tilde{j}}_{i,\tilde{\ell}}} = \lambda_\alpha(-1)^{j+\ell}\omega_{\alpha,k}\omega^2_{\alpha,i}D^{\tilde{k},\tilde{\ell}}_{\tilde{i},\tilde{j}} = \lambda_\rho^{i+k}\lambda_\alpha(-1)^{j+\ell}\omega_{\alpha,k}\omega^2_{\alpha,i}\overline{D^{\tilde{i},\ell}_{\tilde{k},j}}.$$

Finally, we have

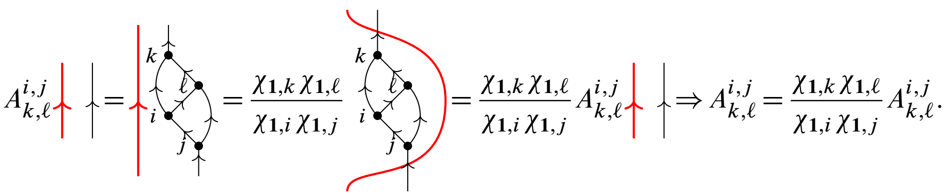
$$A^{i,j}_{k,\ell} = \hat{A}^{i,j}_{k,\ell} = D^{i,j}_{k,\ell} = 0 \quad \textit{if } i+j+k+\ell \not\equiv 0 \pmod 2.$$

This result reduces the number of complex scalars to solve for down to 11 in the $\lambda_\alpha = 1$ case and 7 in the $\lambda_\alpha = -1$ case. This simplification makes it feasible to solve for these scalars in the next section.

Lemma 3.22. *We have*

$$A^{i,j}_{k,\ell} = \hat{A}^{i,j}_{k,\ell} = D^{i,j}_{k,\ell} = 0 \quad \textit{if } i+j+k+\ell \not\equiv 0 \pmod 2.$$

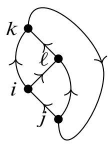
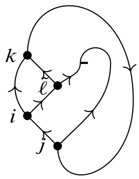
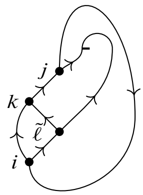
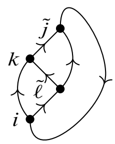
Proof. We will prove the statement of the lemma in the case of the $A^{i,j}_{k,\ell}$ coefficients, as the remaining two cases are nearly identical. We have



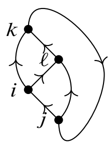
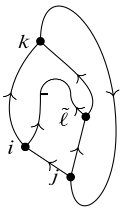
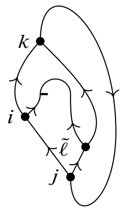
Recall from Lemma 3.10 that $\chi_{\mathbf{1},i} = (-1)^i \sqrt{\lambda_\alpha}$. Thus, if $i + j + k + \ell \not\equiv 0 \pmod 2$, then $\frac{\chi_{\mathbf{1},k}\chi_{\mathbf{1},\ell}}{\chi_{\mathbf{1},i}\chi_{\mathbf{1},j}} \neq 1$, which implies that $A^{i,j}_{k,\ell} = 0$. ■

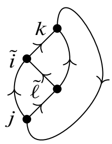
Now that we know that half of our coefficients vanish, we move on to describing the symmetries between them. As mentioned before, these symmetries are the standard tetrahedral symmetries of the $6j$-symbols. This completes the proof of the main result of this section.

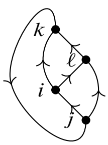
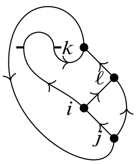
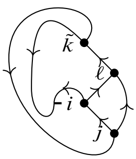
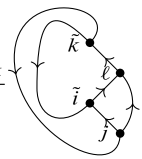
Proof of Lemma 3.21. We include enough examples to illuminate the necessary techniques, all of which involve using the Frobenius maps defined in Section 3.1. The symmetries of the $A^{i,j}_{k,\ell}$ coefficients are the easiest, as the diagrams only involve ρ strands. We compute the following symmetries:

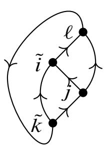
$A^{i,j}_{k,\ell} = \frac{1}{d}$  $= \frac{\lambda_\rho^{\ell+1}}{d}$  $= \frac{\lambda_\rho^{\ell}}{d}$  $= \frac{\lambda_\rho^{\ell+j}}{d}$ 

$= \lambda_\rho^{\ell+j}\,\overline{A^{k,\tilde{j}}_{i,\tilde{\ell}}}$,

$A^{i,j}_{k,\ell} = \frac{1}{d}$  $= \frac{\lambda_\rho^{1+\ell}}{d}\omega_{\mathbf{1},\ell}$  $= \frac{\lambda_\rho^{1+\ell}}{d}\omega_{\mathbf{1},\ell}$ 

$= \frac{\lambda_\rho^{1+i+\ell}}{d}\omega_{\mathbf{1},\ell}$  $= \lambda_\rho^{1+i+\ell}\omega_{\mathbf{1},\ell}\overline{A^{\tilde{i},k}_{j,\tilde{\ell}}}$,

$A^{i,j}_{k,\ell} = \frac{1}{d}$  $= \frac{1}{d}$  $= \frac{\lambda_\rho^{k}}{d}\omega_{\mathbf{1},k}$  $= \frac{\lambda_\rho^{i+k}}{d}\frac{\omega_{\mathbf{1},k}}{\omega_{\mathbf{1},i}}$ 

$= \frac{\lambda_\rho^{i+k}}{d}\frac{\omega_{\mathbf{1},k}}{\omega_{\mathbf{1},i}}$  $= \lambda_\rho^{i+k}\frac{\omega_{\mathbf{1},k}}{\omega_{\mathbf{1},i}}\overline{A^{\tilde{i},\ell}_{\tilde{k},j}}$.

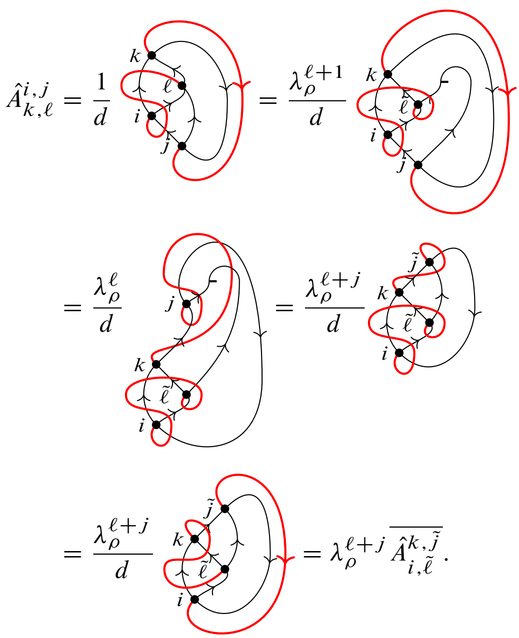
Together this shows that

$$A^{i,j}_{k,\ell} = \lambda_\rho^{i+k}\omega_{\mathbf{1},\ell} A^{j,\tilde{k}}_{\tilde{i},\ell} = \lambda_\rho^{1+j+k}\omega^2_{\mathbf{1},\ell} A^{\tilde{k},i}_{\tilde{j},\ell}$$

and

$$A^{i,j}_{k,\ell} = \lambda_\rho^{i+j+k+\ell}\frac{\omega_{\mathbf{1},i}}{\omega_{\mathbf{1},k}} A^{\tilde{k},\tilde{\ell}}_{\tilde{i},\tilde{j}} = \frac{\omega_{\mathbf{1},i}}{\omega_{\mathbf{1},k}} A^{\tilde{k},\tilde{\ell}}_{\tilde{i},\tilde{j}}$$

as claimed. These three tricks work to determine all of the symmetries in the statement of the lemma. In order to show how to deal with α strands, we include one final example as follows:



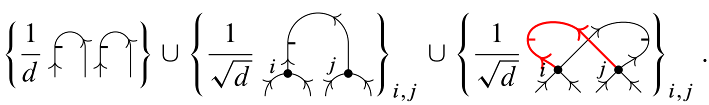
We leave the verification of the remaining identities to the reader. ■

To finish off this section, we explicitly compute the 4th Frobenius–Schur indicator of ρ in terms of our free variables. This formula will be useful in the next section.

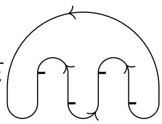
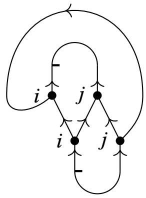
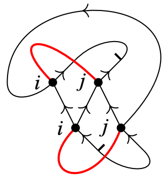
Lemma 3.23. *We have that*

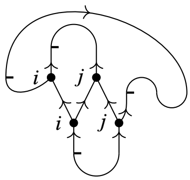
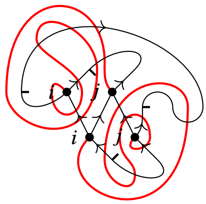
$$\nu_4(\rho) = \frac{1}{d} + \lambda_\rho \sum_{i,j} \omega_{\mathbf{1},i}\omega_{\mathbf{1},j} A^{i,j}_{i,j} + \lambda_\rho\lambda_\alpha \sum_{i,j} (-1)^{i+j}\omega_{\alpha,i}\omega_{\alpha,j}\hat{A}^{i,j}_{i,j}.$$

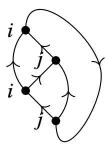
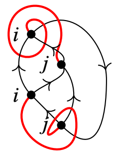
Proof. We pick the following orthonormal basis of $\mathcal{C}_2(\rho^{\otimes 4} \to \mathbf{1})$:

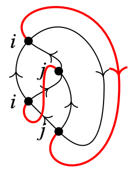


With this basis, we compute

$$\nu_4(\rho) = \frac{1}{d^2}$$

$$+ \frac{1}{d}\sum_{i,j}$$

$$+ \frac{1}{d}\sum_{i,j}$$


$$= \frac{1}{d} + \frac{\lambda_\rho}{d}\sum_{i,j}$$

$$+ \frac{\lambda_\rho}{d}\sum_{i,j}$$


$$= \frac{1}{d} + \frac{\lambda_\rho}{d}\sum_{i,j}\omega_{\mathbf{1},i}\omega_{\mathbf{1},j}$$

$$+ \frac{\lambda_\rho}{d}\sum_{i,j}\omega_{\alpha,i}\omega_{\alpha,j}$$


$$= \frac{1}{d} + \frac{\lambda_\rho}{d}\sum_{i,j}\omega_{\mathbf{1},i}\omega_{\mathbf{1},j}A^{i,j}_{i,j} + \frac{\lambda_\rho\lambda_\alpha}{d}\sum_{i,j}\chi_{\alpha,i}\chi_{\alpha,j}\omega_{\alpha,i}\omega_{\alpha,j}$$


$$= \frac{1}{d} + \lambda_\rho\sum_{i,j}\omega_{\mathbf{1},i}\omega_{\mathbf{1},j}A^{i,j}_{i,j} + \lambda_\rho\lambda_\alpha\sum_{i,j}(-1)^{i+j}\omega_{\alpha,i}\omega_{\alpha,j}\hat{A}^{i,j}_{i,j}. \qquad \blacksquare$$

3.5. Classification

In this final section, we complete the classification result in the self-dual case (Q3); i.e., we complete the proof of Theorem 3.1 and classify all categorifications of the rings ($R(m)$). We have two cases to consider depending on $\lambda_\alpha = \pm 1$.

3.5.1. The case $\lambda_\alpha = 1$. In the case of $\lambda_\alpha = 1$, we have determined that

$$\lambda_\rho = 1, \quad \mu = 1, \quad \tilde{i} = i, \quad \text{and} \quad \chi_{\mathbf{1},i} = \chi_{\alpha,i} = (-1)^i.$$

Thus, all that remains is to deduce the 3rd roots of unity $\omega_{\mathbf{1},0}, \omega_{\mathbf{1},1}, \omega_{\alpha,0}, \omega_{\alpha,1}$, along with the free variables $A^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}$. We express these free variables in the matrix form

$$\begin{bmatrix} X^{0,0}_{0,0} & X^{0,0}_{0,1} & X^{0,0}_{1,0} & X^{0,0}_{1,1} \\ X^{0,1}_{0,0} & X^{0,1}_{0,1} & X^{0,1}_{1,0} & X^{0,1}_{1,1} \\ X^{1,0}_{0,0} & X^{1,0}_{0,1} & X^{1,0}_{1,0} & X^{1,0}_{1,1} \\ X^{1,1}_{0,0} & X^{1,1}_{0,1} & X^{1,1}_{1,0} & X^{1,1}_{1,1} \end{bmatrix}, \qquad X = A, \hat{A}, D. \tag{3.3}$$

By applying the symmetries of Lemma 3.21, we have that our free variables are of the form

$$A = \begin{bmatrix} a_0 & 0 & 0 & \omega_{\mathbf{1},0} a_2 \\ 0 & a_2 & \omega_{\mathbf{1},0}^2 a_2 & 0 \\ 0 & \omega_{\mathbf{1},0}^2 a_2 & a_2 & 0 \\ \omega_{\mathbf{1},0} a_2 & 0 & 0 & a_1 \end{bmatrix},$$

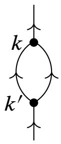
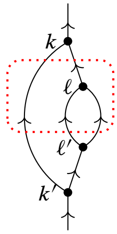
$$\hat{A} = \begin{bmatrix} \hat{a}_0 & 0 & 0 & \omega_{\alpha,0} \hat{a}_2 \\ 0 & \hat{a}_2 & \omega_{\alpha,0}^2 \hat{a}_2 & 0 \\ 0 & \omega_{\alpha,0}^2 \hat{a}_2 & \hat{a}_2 & 0 \\ \omega_{\alpha,0} \hat{a}_2 & 0 & 0 & \hat{a}_1 \end{bmatrix},$$

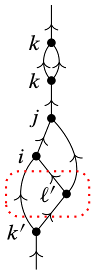
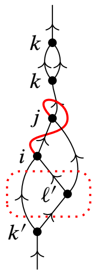
$$D = \begin{bmatrix} d_0 & 0 & 0 & -\frac{\omega_{\alpha,1}}{\omega_{\alpha,0}} \overline{d_4} \\ 0 & d_2 & d_4 & 0 \\ 0 & -\frac{\omega_{\alpha,0}}{\omega_{\alpha,1}} d_4 & d_3 & 0 \\ \overline{d_4} & 0 & 0 & d_1 \end{bmatrix},$$

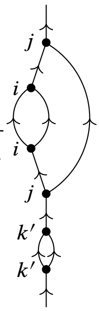
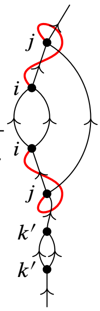
all of which are real apart from d_4. If these free coefficients are non-zero, then the tetrahedral symmetries imply conditions on our twists ω. We have

$$\begin{aligned} a_0 \neq 0 &\implies \omega_{\mathbf{1},0} = 1, \\ a_1 \neq 0 &\implies \omega_{\mathbf{1},1} = 1, \\ \hat{a}_0 \neq 0 &\implies \omega_{\alpha,0} = 1, \\ \hat{a}_1 \neq 0 &\implies \omega_{\alpha,1} = 1, \\ a_2 \neq 0 &\implies \omega_{\mathbf{1},0} = \omega_{\mathbf{1},1}, \\ \hat{a}_2 \neq 0 &\implies \omega_{\alpha,0} = \omega_{\alpha,1}. \end{aligned}$$

In order to solve for these complex variables, we evaluate certain morphisms in our categories in two ways to obtain equations of these variables. We compute

$\delta_{k,k'}\delta_{\ell,\ell'}$  $= \delta_{\ell,\ell'}$  $=$ 

$= \dfrac{\omega_{\mathbf{1},\ell}}{2+\sqrt{5}}$  $+ \sum_{i,j} A_{k,\ell}^{i,j}$  $+ B_{k,\ell}^{i,j}$ 

$= \delta_{k,\ell}\delta_{k',\ell'} \dfrac{\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2}{2+\sqrt{5}}$  $+ \sum_{i,j} A_{k,\ell}^{i,j}\overline{A_{k',\ell'}^{i,j}}$  $+ B_{k,\ell}^{i,j}\overline{B_{k',\ell'}^{i,j}}$ 

$= \left(\delta_{k,\ell}\delta_{k',\ell'} \dfrac{\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2}{2+\sqrt{5}} + \sum_{i',j'} A_{k,\ell}^{i,j}\overline{A_{k',\ell'}^{i,j}} + B_{k,\ell}^{i,j}\overline{B_{k',\ell'}^{i,j}}\right)$ 

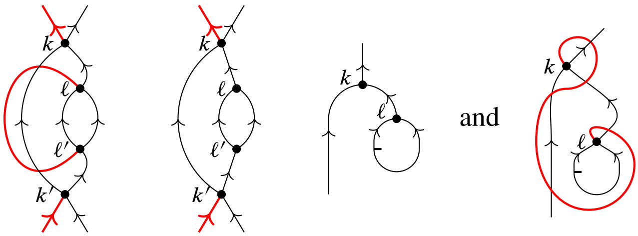
$= \left(\delta_{k,\ell}\delta_{k',\ell'} \dfrac{\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2}{2+\sqrt{5}} + \sum_{i',j'} A_{k,\ell}^{i,j}\overline{A_{k',\ell'}^{i,j}} + (-1)^{\ell+\ell'}\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2 D_{i,\ell}^{j,k}\overline{D_{i,\ell'}^{j,k'}}\right)$ 

$= \left(\delta_{k,\ell}\delta_{k',\ell'} \dfrac{\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2}{2+\sqrt{5}} + \omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2 \sum_{i,j} A_{i,\ell}^{j,k}\overline{A_{i,\ell'}^{j,k'}} + (-1)^{\ell+\ell'}\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2 D_{i,\ell}^{j,k}\overline{D_{i,\ell'}^{j,k'}}\right)$  .

Note that if $\ell \neq \ell'$, then the left-hand side vanishes, and we can cancel the $\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2$ terms. If $\ell = \ell'$, then $\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2 = 1$. In either case, we can remove the $\omega_{\mathbf{1},\ell}\omega_{\mathbf{1},\ell'}^2$ terms from the above equation. This leaves us with the equation

$$\sum_{i,j} A_{i,\ell}^{j,k}\overline{A_{i,\ell'}^{j,k'}} + (-1)^{\ell+\ell'} \sum_{i,j} D_{k,j}^{\ell,i}\overline{D_{k',j}^{\ell',i}} - +(2-\sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} = \delta_{k,k'}\delta_{\ell,\ell'}.$$

In a similar fashion, we can evaluate the diagrams



in two ways[4] to obtain

$$\sum_{i,j} \hat{A}^{i,k}_{j,\ell}\overline{\hat{A}^{i,k'}_{j,\ell'}} + \sum_{i,j} D^{j,k}_{i,\ell}\overline{D^{j,k'}_{i,\ell'}} - (2-\sqrt{5})\delta_{k,\ell}\delta_{k',\ell'} = \delta_{k,k'}\delta_{\ell,\ell'},$$

$$\sum_{i,j} \widehat{D}^{i,j}_{k,\ell}\overline{\widehat{D}^{i,j}_{k',\ell'}} + (-1)^{\ell+\ell'}\omega_{\mathbf{1},\ell'}\omega^2_{\mathbf{1},\ell}\sum_{i,j} D^{i,j}_{k',\ell'}\overline{D^{i,j}_{k,\ell}} = \delta_{k,k'}\delta_{\ell,\ell'},$$

$$\sum_i A^{i,k}_{i,\ell} + (-1)^\ell \sum_i D^{i,k}_{i,\ell} - (2-\sqrt{5})\delta_{k,\ell} = 0,$$

$$\sum_i \hat{A}^{i,k}_{i,\ell} + \sum_i (-1)^i D^{\ell,i}_{k,i} - (2-\sqrt{5})\delta_{k,\ell} = 0.$$

In terms of our free variables, this gives us the following equations:

$$\begin{aligned} 3-\sqrt{5} &= a_0^2 + a_2^2 + d_0^2 + d_3^2 = a_1^2 + a_2^2 + d_1^2 + d_2^2 = \hat{a}_0^2 + \hat{a}_2^2 + d_0^2 + d_2^2 \\ &= \hat{a}_1^2 + \hat{a}_2^2 + d_1^2 + d_3^2, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{1}{2} &= a_2^2 + |d_4|^2 = \hat{a}_2^2 + |d_4|^2 = d_0^2 + |d_4|^2 = d_1^2 + |d_4|^2 \\ &= d_2^2 + |d_4|^2 = d_3^2 + |d_4|^2, \end{aligned} \tag{3.5}$$

$$2-\sqrt{5} = (a_0 + a_1)a_2 - d_0 d_2 - d_1 d_3 = (\hat{a}_0 + \hat{a}_1)\hat{a}_2 + d_0 d_3 + d_1 d_2, \tag{3.6}$$

$$\begin{aligned} 2-\sqrt{5} &= a_0 + a_2 + d_0 + d_3 = a_1 + a_2 - d_1 - d_2 = \hat{a}_0 + \hat{a}_2 + d_0 - d_2 \\ &= \hat{a}_1 + \hat{a}_2 - d_1 + d_3, \end{aligned} \tag{3.7}$$

$$\begin{aligned} 0 &= (\omega_{\alpha,0} + \omega^2_{\alpha,0})a_2^2 + \omega_{\alpha,0}\omega^2_{\alpha,1}d_4^2 + \omega^2_{\alpha,0}\omega_{\alpha,1}\overline{d_4}^2 \\ &= (\omega_{\alpha,0} + \omega^2_{\alpha,0})\hat{a}_2^2 + \omega^2_{\alpha,0}\omega_{\alpha,1}d_4^2 + \overline{d_4}^2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 &= (1 - \omega_{\alpha,0}\omega^2_{\alpha,1})(d_2\overline{d_4} - d_3^2) \\ &= d_4(d_1 + d_0) - \overline{d_4}(\omega^2_{\alpha,0}\omega_{\alpha,1}d_0 + \omega_{\alpha,0}\omega^2_{\alpha,1}d_1). \end{aligned} \tag{3.9}$$

[4] When we say we evaluate a diagram in two ways to obtain a relation, one way is trivial, and the other uses the jellyfish relations from Lemmas 3.3 and 3.15. For the non self-dual case in Section 4 below, we use the jellyfish relations from (4.2), (4.3), and Lemma 4.4 instead.

While we could begin solving these equations directly, instead we opt for a more measured approach and use our previous centre analysis to simplify our solution.

Lemma 3.24. *There exists a* $\tau \in \{-1, 1\}$ *such that*

$$a_0 = a_1 = \hat{a}_0 = \hat{a}_1 = \frac{2 + 3\tau - \sqrt{5}}{4}$$

and

$$a_2 = \hat{a}_2 = d_0 = -d_1 = -d_2 = d_3 = \frac{2 - \tau - \sqrt{5}}{4}.$$

In particular, as $a_0, a_1, \hat{a}_0$*, and* $\hat{a}_1$ *are all non-zero, we have* $\omega_{\mathbf{1},0} = \omega_{\mathbf{1},1} = \omega_{\alpha,0} = \omega_{\alpha,1} = 1$.

Proof. We first observe from equation (3.5) that

$$a_2^2 = \hat{a}_2^2 = d_0^2 = d_1^2 = d_2^2 = d_3^2 = \frac{1}{2} - |d_4|^2,$$

and, in particular, we have that $a_2, \hat{a}_2, d_0, d_1, d_2$, and d_3 are real numbers which are equal up to sign. With this information in hand, we can now see from equation (3.4) that

$$a_0^2 = a_1^2 = \hat{a}_0^2 = \hat{a}_1^2 = 3 - \sqrt{5} - 3d_0^2.$$

To make additional progress on solving these equations, we recall the operators ϕ and ψ. In our case, via equation (3.7), we have that

$$\phi = \begin{bmatrix} a_0 + a_2 & 0 \\ 0 & a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{5} - d_0 - d_3 & 0 \\ 0 & 2 - \sqrt{5} + d_1 + d_2 \end{bmatrix},$$
$$\psi = \begin{bmatrix} \hat{a}_0 + \hat{a}_2 & 0 \\ 0 & \hat{a}_1 + \hat{a}_2 \end{bmatrix} = \begin{bmatrix} 2 - \sqrt{5} - d_0 + d_2 & 0 \\ 0 & 2 - \sqrt{5} + d_1 - d_3 \end{bmatrix}.$$

From Lemma 3.7, we know that ϕ and ψ have entries in $\{\frac{3-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$, so

$$d_0 + d_3, -d_1 - d_2, d_0 - d_2, d_3 - d_1 \in \left\{\frac{3 - \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right\}.$$

In particular, as the values d_0, d_1, d_2, and d_3 are real numbers which are the same up to sign, we have that

$$d_0 = -d_1 = -d_2 = d_3 = \frac{2 - \sqrt{5} - \tau}{4}$$

for some $\tau \in \{-1, 1\}$.

From equation (3.7), we can deduce that $a_0 = a_1$ and $\hat{a}_0 = \hat{a}_1$. We know that a_2 and c_0 are the same up to sign. If we have $a_2 = -c_0$, then equation (3.7) would imply that

$$a_0 = 2 - \sqrt{5} - d_0.$$

Plugging this value of a_0 into equation (3.4) gives a contradiction. Thus, $a_2 = c_0$, and so, equation (3.7) gives

$$a_0 = 2 - \sqrt{5} - 3d_0 = \frac{2 + 3\tau - \sqrt{5}}{4}.$$

A similar argument shows that $\hat{a}_2 = c_0$, and thus, $\hat{a}_0 = a_0$. ■

To pin down the value of τ, we return to our analysis of the centre of $\mathcal{C}_2$. By computing the 4th Frobenius–Schur indicator of ρ in two ways, we can show that $\tau = 1$.

Lemma 3.25. *We have that* $\tau = 1$.

Proof. From Lemma 3.23, we have that $\nu_4(\rho) = 3\tau$. On the other hand, we can use Lemma 3.13 to obtain

$$\nu_4(\rho)(20 + 8\sqrt{5}) = 48 + 20\sqrt{5} + (2 + \sqrt{5}) \sum p_i(p_i + q_i)\theta_i^4,$$

where $\sum p_i(p_i + q_i) = 16$ and the θ_i's are roots of unity. Thus,

$$\sum p_i(p_i + q_i)\theta_i^4 = 4\big(-1 - 2\sqrt{5} + 3\tau\sqrt{5}\big).$$

If $\tau = -1$, then Theorem A.9 implies that it would take at least 24 roots of unity to write $4(-1 - 2\sqrt{5} + 3\tau\sqrt{5})$, and hence, $\sum p_i(p_i + q_i) \geq 24$, giving a contradiction. Thus, we must have $\tau = 1$. ■

Now that we know all of our real free variables, we can solve for d_4, the one complex variable.

Lemma 3.26. *We have that*

$$d_4 = \eta_1 \frac{i}{2} + \eta_2 \frac{1}{2}\sqrt{\frac{-1 + \sqrt{5}}{2}}, \quad \textit{where } \eta_1, \eta_2 \in \{-1, 1\}.$$

Proof. From Lemmas 3.24 and 3.25, we have $a_2 = \frac{1-\sqrt{5}}{4}$. By equations (3.5) and (3.8), we have

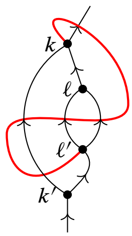
$$|d_4|^2 = \frac{1 + \sqrt{5}}{8} \quad \text{and} \quad d_4^2 + \overline{d_4}^2 = \frac{\sqrt{5} - 3}{4}.$$

The 4 intersection points of this hyperbola and circle yield the statement of the lemma. ■

Now that we have pinned down all of our variables, we can prove part of our main theorem which states that there is no fusion category when $\lambda_\alpha = 1$.

Theorem 3.27. *There is no unitary fusion category that categorifies $R(2)$ with $\lambda_\alpha = 1$.*

Proof. By evaluating the diagram



in two ways (see footnote 4), we obtain the equation

$$(-1)^{k+\ell}\sum_{i,j}\bar{D}^{k,\ell}_{i,j}\bar{D}^{i,j}_{\ell',k'}+\sum_{i,j}D^{\ell',k'}_{i,j}D^{i,j}_{k,\ell}=0.$$

Taking $k=\ell'=0$ and $\ell=k'=1$, we see that $\sum_{i,j}D^{0,1}_{i,j}D^{i,j}_{0,1}=d_2^2-d_4^2\in\mathbb{R}$. Since $d_2\in\mathbb{R}$ by Lemma 3.24, this means that $d_4^2\in\mathbb{R}$, which contradicts Lemma 3.26. ■

3.5.2. The case $\lambda_\alpha = -1$. In the case of $\lambda_\alpha=-1$, we have determined that

$$\mu=1,\qquad \tilde{i}=1-i,\qquad \omega_{\mathbf{1},0}=\omega_{\mathbf{1},1},$$
$$\omega_{\alpha,0}=\omega_{\alpha,1},\quad \chi_{\mathbf{1},i}=(-1)^i\mathbf{i},\quad \chi_{\alpha,i}=(-1)^i.$$

Thus, all that remains is to deduce λ_ρ, the 3rd roots of unity $\omega_{\mathbf{1},0}$ and $\omega_{\alpha,0}$, along with the free variables $A^{i,j}_{k,\ell}$, $\hat{A}^{i,j}_{k,\ell}$, and $D^{i,j}_{k,\ell}$. By studying the 4th Frobenius–Schur indicator of ρ, we are able to show that $\lambda_\rho=1$ and $\omega_{\alpha,0}=\omega_{\mathbf{1},0}^2$, along with the values of several of our free variables.

Lemma 3.28. *We have that $\lambda_\rho=1$, and $\omega_{\alpha,0}=\omega_{\mathbf{1},0}^2$. Further, we have that*

$$A^{0,0}_{0,0}=\frac{3-\sqrt{5}}{2(1+\omega_{\mathbf{1},0})}\quad\textit{and}\quad \hat{A}^{0,0}_{0,0}=\frac{3-\sqrt{5}}{2(1+\omega_{\mathbf{1},0}^2)}.$$

Proof. Recall the operators ϕ and ψ. By applying the symmetries of Lemma 3.21, we have that

$$\phi=\begin{bmatrix}A^{0,0}_{0,0}+A^{1,0}_{1,0} & 0\\ 0 & A^{0,1}_{0,1}+A^{1,1}_{1,1}\end{bmatrix}=\begin{bmatrix}A^{0,0}_{0,0}(1+\lambda_\rho\omega_{\mathbf{1},0}) & 0\\ 0 & A^{0,0}_{0,0}(1+\lambda_\rho\omega_{\mathbf{1},0})\end{bmatrix},$$
$$\psi=\begin{bmatrix}\hat{A}^{0,0}_{0,0}+\hat{A}^{1,0}_{1,0} & 0\\ 0 & \hat{A}^{0,1}_{0,1}+\hat{A}^{1,1}_{1,1}\end{bmatrix}=\begin{bmatrix}\hat{A}^{0,0}_{0,0}(1+\lambda_\rho\omega_{\alpha,0}) & 0\\ 0 & \hat{A}^{0,0}_{0,0}(1+\lambda_\rho\omega_{\alpha,0})\end{bmatrix}.$$

Thus, the operators ϕ and ψ are scalars, and Lemma 3.7 tells us that

$$A^{0,0}_{0,0} = \frac{2-\sqrt{5}+\tau}{2(1+\lambda_\rho\omega_{1,0})} \quad \text{and} \quad \hat{A}^{0,0}_{0,0} = \frac{2-\sqrt{5}+\tau}{2(1+\lambda_\rho\omega_{\alpha,0})}$$

for some $\tau \in \{-1, 1\}$.

From Lemma 3.23, we can write the 4th Frobenius–Schur indicator of ρ as

$$\begin{aligned}
\nu_4(\rho) &= \sqrt{5}-2+\lambda_\rho\omega_{1,0}^2(A^{0,0}_{0,0}+A^{0,1}_{0,1}+A^{1,0}_{1,0}+A^{1,1}_{1,1}) \\
&\quad -\lambda_\rho\omega_{\alpha,0}^2(\hat{A}^{0,0}_{0,0}-\hat{A}^{0,1}_{0,1}-\hat{A}^{1,0}_{1,0}+\hat{A}^{1,1}_{1,1}) \\
&= \sqrt{5}-2+\lambda_\rho\omega_{1,0}^2A^{0,0}_{0,0}(2+2\lambda_\rho\omega_{1,0})-\lambda_\rho\omega_{\alpha,0}^2\hat{A}^{0,0}_{0,0}(2-2\lambda_\rho\omega_{\alpha,0}) \\
&= \sqrt{5}-2+\lambda_\rho\omega_{1,0}^2(2-\sqrt{5}+\tau)-\lambda_\rho\omega_{\alpha,0}^2(2-\sqrt{5}+\tau)\frac{1-\lambda_\rho\omega_{\alpha,0}}{1+\lambda_\rho\omega_{\alpha,0}} \\
&= \left(-2+\lambda_\rho\omega_{1,0}^2(2+\tau)-\lambda_\rho\omega_{\alpha,0}^2(2+\tau)\frac{1-\lambda_\rho\omega_{\alpha,0}}{1+\lambda_\rho\omega_{\alpha,0}}\right) \\
&\quad +\sqrt{5}\left(1-\lambda_\rho\omega_{1,0}^2+\lambda_\rho\omega_{\alpha,0}^2\frac{1-\lambda_\rho\omega_{\alpha,0}}{1+\lambda_\rho\omega_{\alpha,0}}\right).
\end{aligned}$$

As $\nu_4(\rho) \in \mathbb{Z}[i]$, λ_ρ is a second root of unity, and $\omega_{1,0}$ and $\omega_{\alpha,0}$ are third roots of unity, we have

$$1-\lambda_\rho\omega_{1,0}^2+\lambda_\rho\omega_{\alpha,0}^2\frac{1-\lambda_\rho\omega_{\alpha,0}}{1+\lambda_\rho\omega_{\alpha,0}} = 0.$$

This implies that $\lambda_\rho = 1$, and that $\omega_{\alpha,0} = \omega_{1,0}^{-1}$. By simplifying the formula for $\nu_4(\rho)$ further, we find that $\nu_4(\rho) = \tau$.

To determine τ, we use Lemma 3.13 to write

$$\tau(20+8\sqrt{5}) = \nu_4(\rho) = 48+20\sqrt{5}+(2+\sqrt{5})\sum p_i(p_i+q_i)\theta_i^4,$$

where

$$\sum p_i(p_i+q_i) = 16,$$

and the θ_i's are roots of unity. If $\tau = -1$, then we have

$$\sum p_i(p_i+q_i)\theta_i^4 = -4-12\sqrt{5}.$$

However, Theorem A.9 implies that it takes at least 48 roots of unity to write $-4-12\sqrt{5}$, giving a contradiction. Thus, $\tau = 1$, which gives

$$A^{0,0}_{0,0} = \frac{3-\sqrt{5}}{2(1+\omega_{1,0})} \quad \text{and} \quad \hat{A}^{0,0}_{0,0} = \frac{3-\sqrt{5}}{2(1+\omega_{1,0}^2)}. \qquad \blacksquare$$

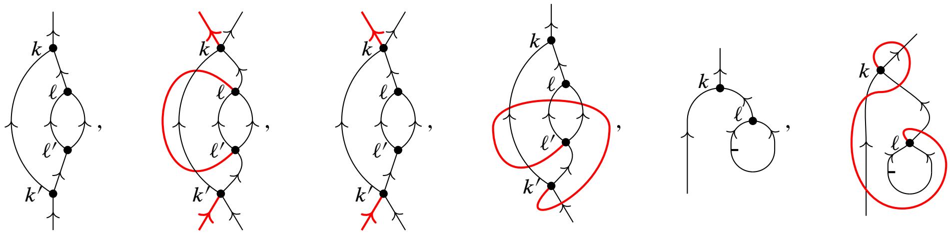
Now that we know $\lambda_\rho = 1$, the symmetries of Lemma 3.21 become much simpler. Using the same matrix notation as in the $\lambda_\alpha = 1$ case from (3.3), we can use these symmetries to express our free variables as

$$A = \begin{bmatrix} \omega_{1,0} r & 0 & 0 & \overline{a_1} \\ 0 & \omega_{1,0}^2 r & r & 0 \\ 0 & r & \omega_{1,0}^2 r & 0 \\ a_1 & 0 & 0 & \omega_{1,0} r \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} \omega_{1,0}^2 r & 0 & 0 & \overline{\hat{a}_1} \\ 0 & \omega_{1,0} r & r & 0 \\ 0 & r & \omega_{1,0} r & 0 \\ \hat{a}_1 & 0 & 0 & \omega_{1,0}^2 r \end{bmatrix},$$

$$D = \begin{bmatrix} d_0 & 0 & 0 & \overline{d_2} \\ 0 & \overline{d_0} & d_1 & 0 \\ 0 & \overline{d_1} & -\overline{d_0} & 0 \\ d_2 & 0 & 0 & -d_0 \end{bmatrix},$$

where $r = \frac{1}{\omega_{1,0}+\omega_{1,0}^2}\frac{3-\sqrt{5}}{2} \in \mathbb{R}$, and if either of a_1 or $\hat{a}_1$ are non-zero, then we have that $\omega_{1,0} = 1$.

Now that we have reduced our free variables down to 5 complex variables, all that remains is to solve for these variables and to determine the 3rd root of unity $\omega_{1,0}$. As in the $\lambda_\alpha = 1$ case, we get equations of these variables by evaluating the diagrams



in two ways (see footnote 4). This gives us the following equations:

$$\delta_{k,k'}\delta_{\ell,\ell'} = \sum_{i,j} A_{j,\ell}^{i,k}\overline{A_{j,\ell'}^{i,k'}} + (-1)^{\ell+\ell'}\sum_{i,j} D_{j,\ell}^{i,k}\overline{D_{j,\ell'}^{i,k'}} - (2-\sqrt{5})\delta_{k,\ell}\delta_{k',\ell'}$$

$$\delta_{k,k'}\delta_{\ell,\ell'} = \sum_{i,j} \hat{A}_{j,\ell}^{i,k}\overline{\hat{A}_{j,\ell'}^{i,k'}} + (-1)^{\ell+\ell'}\sum_{i,j} D_{1-k',j}^{1-\ell',i}\overline{D_{1-k,j}^{1-\ell,i}} - (2-\sqrt{5})\delta_{k,\ell}\delta_{k',\ell'}$$

$$\delta_{k,k'}\delta_{\ell,\ell'} = \sum_{i,j} D_{k,\ell}^{i,j}\overline{D_{k',\ell'}^{i,j}} + (-1)^{\ell+\ell'}\sum_{i,j} D_{1-k',1-\ell'}^{i,j}\overline{D_{1-k,1-\ell}^{i,j}}$$

$$0 = \sum_{i,j}(-1)^i A_{k,\ell}^{i,j}\overline{D_{k',1-j}^{1-\ell',i}} + (-1)^{\ell+1}\sum_{i,j} D_{1-i,\ell}^{j,1-k}\overline{\hat{A}_{k,\ell}^{i,j}}$$
$$- \omega_{1,0}^2(\sqrt{5}-2)\mathbf{i}\delta_{k,1-\ell}\delta_{k',1-\ell'}$$

$$0 = \sum_i A_{i,\ell}^{i,k} + (-1)^{\ell+1}\mathbf{i}\sum_i D_{i,\ell}^{i,k} - (2-\sqrt{5})\delta_{k,\ell}$$

$$0 = \sum_i \hat{A}_{i,\ell}^{i,k} + \mathbf{i}\sum_i (-1)^{i+1} D_{1-k,i}^{1-\ell,i} - (2-\sqrt{5})\delta_{k,\ell}.$$

In terms of our free variables, this gives us the following equations:

$$\operatorname{Im}(d_0) = \frac{1-\sqrt{5}}{4} = -\frac{\omega_{\mathbf{1},0}^2}{1+\omega_{\mathbf{1},0}^2}\frac{-1+\sqrt{5}}{2}, \tag{3.10}$$

$$r^2 + |d_0|^2 = \frac{3-\sqrt{5}}{2}, \tag{3.11}$$

$$r^2 + |a_1|^2 + |d_1|^2 + |d_2|^2 = r^2 + |\hat{a}_1|^2 + |d_1|^2 + |d_2|^2 = 1, \tag{3.12}$$

$$|d_1|^2 = |d_2|^2 = \frac{1}{2} - |d_0|^2, \tag{3.13}$$

$$(\omega_{\mathbf{1},0} + \omega_{\mathbf{1},0}^2)r^2 - (d_0^2 + \overline{d_0}^2) = 2 - \sqrt{5}, \tag{3.14}$$

$$ra_1 = d_1 d_2, \tag{3.15}$$

$$r\hat{a}_1 = -\bar{d}_1 d_2, \tag{3.16}$$

$$d_2\overline{a_1} + \hat{a}_1\overline{d_2} = d_1\overline{a_1} + \hat{q}\overline{d_1} = 0. \tag{3.17}$$

Remark 3.29. From equation (3.10), we see that $\frac{\omega_{1,0}^2}{1+\omega_{1,0}^2} = 1/2$, which implies that $\omega_{\mathbf{1},0} = 1$.

It is now straightforward to solve the above system of equations.

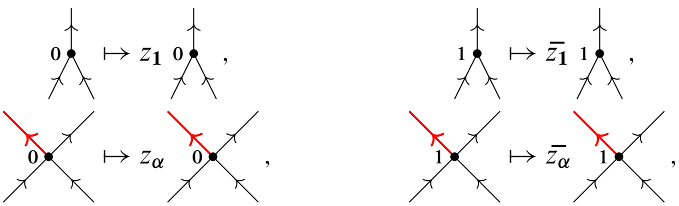
Lemma 3.30. *A general solution to equations* (3.10)–(3.17) *is given by*

$$a_0 = \hat{a}_0 = \frac{3-\sqrt{5}}{4}, \qquad a_1 = (3+\sqrt{5})d_1 d_2, \quad \hat{a}_1 = -(3+\sqrt{5})\bar{d}_1 d_2,$$
$$|d_1|^2 = |d_2|^2 = \frac{-1+\sqrt{5}}{8}, \quad d_0 = -\frac{1}{2} + \mathbf{i}\frac{1-\sqrt{5}}{4}.$$

With this lemma in hand, we can show the existence and uniqueness of the unitary fusion category with fusion ring $R(2)$.

Theorem 3.31. *There exists a unique fusion category categorifying the ring $R(2)$ with $\lambda_\alpha = -1$. This unitary fusion category can be realised as the even part of the $2D2$ subfactor.*

Proof. Note that, from Lemma 3.11, we are free to rescale our basis elements of $\mathcal{C}_2(\rho \otimes \rho \to \rho)$ and $\mathcal{C}_2(\rho \otimes \rho \to \alpha\rho)$ by



where $z_{\mathbf{1}}, z_\alpha \in U(1)$. This rescaling changes the phase of our free variables d_1 and d_2 by $z_{\mathbf{1}}^{-2} z_\alpha^2$ and $z_{\mathbf{1}}^{-2} z_\alpha^{-2}$, respectively. Thus, we can arrange so that

$$d_1 = d_2 = \mathbf{i}\frac{1}{2}\sqrt{\frac{1}{2}\big(-1+\sqrt{5}\big)}.$$

Hence, up to choice of our basis elements, we have a unique solution of all free parameters determining our category. Thus, Proposition 3.19 gives that we have at most one unitary fusion category with fusion ring $R(2)$, and $\lambda_\alpha = -1$.

We know that the even part of the $2D2$ subfactor is a unitary fusion category with fusion ring $R(2)$; hence, this must be the unique example. ■

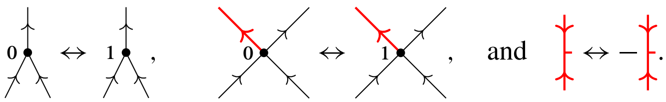
Let us write $\mathcal{C}_2$ for the categorification of $R(2)$ we have classified in this section.

Remark 3.32. We wish to point out the above solutions to our free variables can be used to construct a system of dualizable endomorphisms of the Cuntz algebra $O_5 \rtimes \mathbb{Z}/2\mathbb{Z}$. This gives an independent construction of the category $\mathcal{C}_2$.

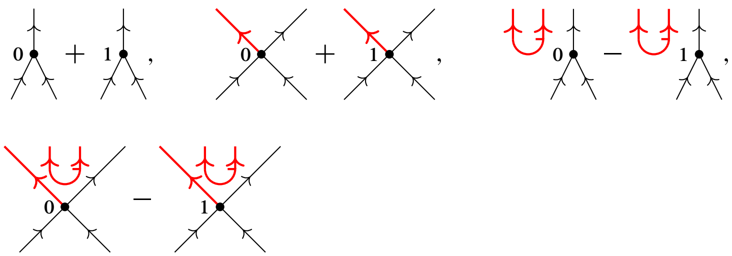
To finish up, we connect $\mathcal{C}_2$ to the even part of the $3^{\mathbb{Z}/4\mathbb{Z}}$ category.

Corollary 3.33. *There is a monoidal $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{C}_2$ such that equivariantisation by this action gives the $3^{\mathbb{Z}/4\mathbb{Z}}$ category of* [26, 51].

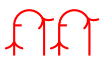
Proof. Using the same gauge choice as in the previous theorem, we can define an order two monoidal equivalence on $\mathcal{C}_2$ by



By equivariantising by this order two monoidal auto-equivalence, we obtain a unitary fusion category generated by the four morphisms



and the isomorphism

 $: \alpha^{\otimes 4} \xrightarrow{\sim} \mathbf{1}.$

This is the presentation of the $3^{\mathbb{Z}/4\mathbb{Z}}$ category from [26]. ■

4. The non-self-dual case

In this section, we focus on the unitary categorification of the fusion rings with four simple objects $\mathbf{1}, \alpha, \rho, \alpha\rho$ and fusion rules

$$\alpha \otimes \alpha \cong \mathbf{1}, \quad \rho \otimes \rho \cong \alpha \oplus m\rho \oplus m\alpha\rho. \tag{$S(m)$}$$

Let us write ($S(m)$) for such a fusion ring. By [30], we know that ($S(m)$) has a categorification only if $m = 0, 1, 2$.

Our main result of this section is as follows.

Theorem 4.1. *Let $\mathcal{D}_m$ be a unitary fusion category with $K_0(\mathcal{C}) \cong (S(m))$. Then, either*

- $m = 0$, *in which case $\mathcal{D}_0$ is equivalent to one of the four monoidally distinct categories* $\mathsf{Hilb}(\mathbb{Z}/4\mathbb{Z}, \omega)$, *where $\omega \in H^3(\mathbb{Z}/4\mathbb{Z}, \mathbb{C}^\times)$, or*
- $m = 1$, *in which case $\mathcal{D}_1$ is equivalent to the monoidally distinct even parts of the two complex conjugate subfactors with principal graphs $\mathcal{S}'$ from* [26, 34].

In particular, the case $m = 2$ from [30, Theorem 1.1 (6)] *is not categorifiable.*

Proof. The $m = 0$ case is easily seen to be pointed, and hence, the claim of the above theorem follows from [10, Remark 4.10.4]. Thus, it suffices to restrict our attention to the cases $m = 1$ and $m = 2$.

The general outline of this section follows for the most part as in the self-dual case. In Section 4.1, we begin by writing down a list of numerical data (essentially the 6-j symbols of the category) which fully describe a unitary fusion category with fusion ring ($S(m)$). In Section 4.2, by studying the Drinfeld centre via the tube algebra of the category, we are able to deduce the precise values of some of these numerical data. To reduce the complexity of our numerical data, in Section 4.3, we use tetrahedral symmetries to essentially cut down the number of free variables in our numerical data by a factor of 24. Finally, in Section 4.4, we solve for this numerical data by evaluating various morphisms in our categories in multiple ways to obtain equations.

In the case $m = 1$, we reduce our numerical data to two possible solutions, which shows that there are at most two distinct unitary fusion categories categorifying $S(1)$. From the subfactor classification literature [26, 34] we know that two such categories exist. We then show that there are no solutions to the numerical data in the case $m = 2$, and hence, there are no such unitary fusion categories. ∎

4.1. Numerical data

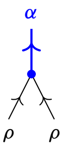
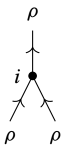
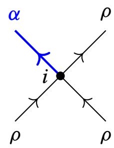
We now produce a set of numerical data which completely describes a categorification of the ring $S(m)$. Let us write $\mathcal{D}_m$ for such a unitary fusion category. We will show that the category $\mathcal{D}_m$ can be described by the following data:

- an 8th root of unity $\nu = e^{\pm i\pi\frac{1}{4}}$,

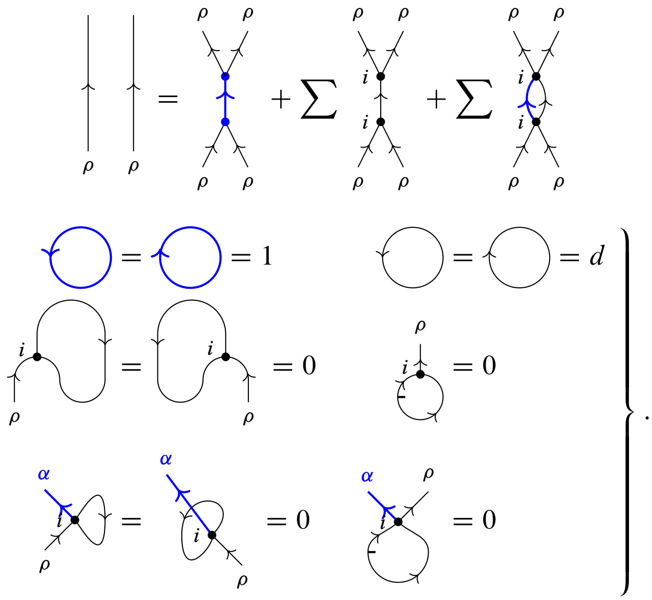
- m choices of signs $\chi_i \in \{-1, 1\}$,
- m choices of 3rd roots of unity $\omega_i \in \{1, e^{2i\pi\frac{1}{3}}, e^{2i\pi\frac{2}{3}}\}$,
- $8m^4$ complex scalars $A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell} \in \mathbb{C}$ for $0 \leq i, j, k, \ell < m$. These complex scalars are the entries of the F-tensors $F^{\rho,\rho,\rho}_{\rho}$ and $F^{\rho,\rho,\rho}_{\alpha\rho}$.

While the 128 complex scalars in the $m = 2$ case seem infeasible to deal with as is, we will use tetrahedral symmetries later on to reduce this 128 to a more workable number.

To simplify notation, we define $d := \dim(\rho)$, which is the largest solution to $d^2 = 1 + 2md$. If $m = 1$, then $d = 1 + \sqrt{2}$, and if $m = 2$, then $d = 2 + \sqrt{5}$. We pick orthonormal bases for the hom spaces

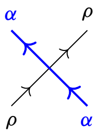
 $\in \mathcal{D}_m(\rho \otimes \rho \to \alpha)$,  $\in \mathcal{D}_m(\rho \otimes \rho \to \rho)$,  $\in \mathcal{D}_m(\rho \otimes \rho \to \alpha\rho)$,

where $0 \leq i \leq m$, so we have the local relations

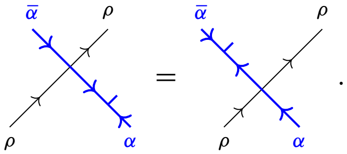


(4.1)

We also choose unitary isomorphisms

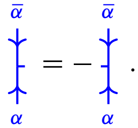
 $\in \mathcal{D}_m(\alpha \to \overline{\alpha})$ and  $\in \mathcal{D}_m(\rho \otimes \alpha \to \alpha \otimes \rho)$.

We normalise this last morphism so that

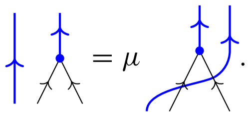


We are still free to rescale the crossing up to sign.

Note that, as pointed out in the proof of [30, Theorem 5.8], we may assume that α has second Frobenius–Schur indicator -1, so



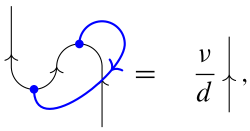
Let μ be the scalar defined by



(4.2)

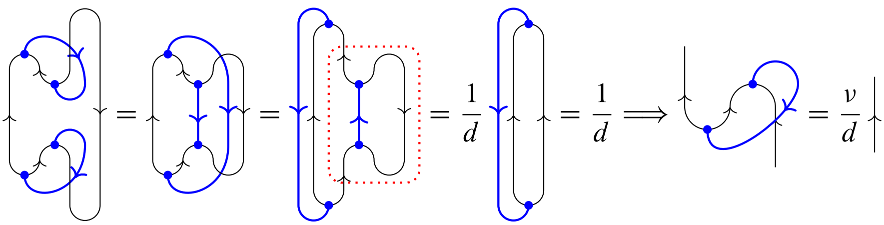
Note that from our normalisation we have that $\mu^2 = -1$.

Lemma 4.2. *Without loss of generality, we have the relation*

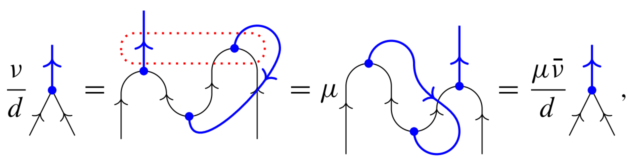


where $\nu = \exp(\pm\pi i/4)$.

Proof. First, by our normalisations for orthonormal bases of hom spaces, we observe that

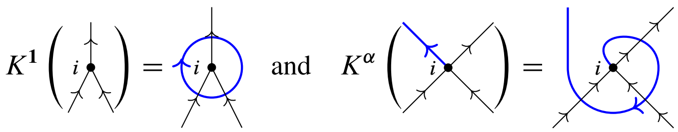


for some unimodular scalar ν. By computing

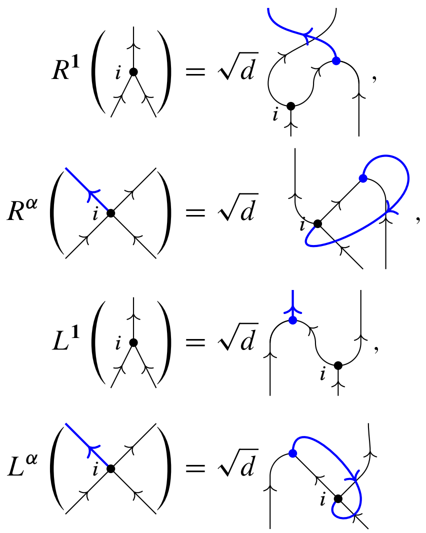


we find that $\nu^2 = \mu$. By rescaling the crossing by a sign, we may assume that $\nu = e^{\frac{\pm\pi i}{4}}$. ∎

In order to define natural orthonormal bases for the spaces $\mathcal{D}_m(\rho \otimes \rho \to \rho)$ and $\mathcal{D}_m(\rho \otimes \rho \to \alpha\rho)$, we define the operators



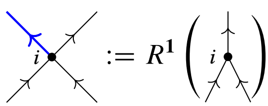
and the Frobenius operators



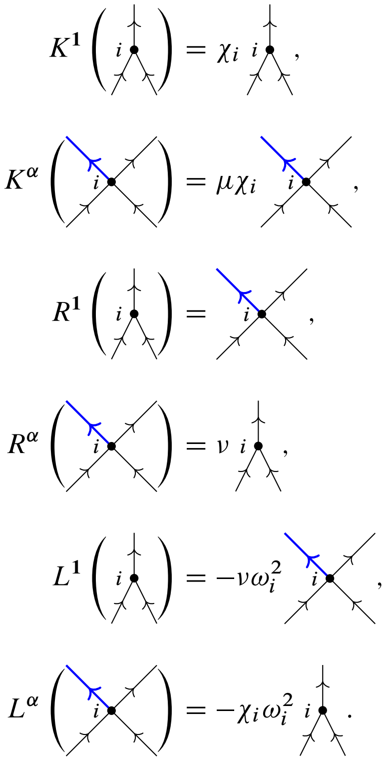
on these spaces. Direct computation shows that these operators satisfy the following relations:

$$\begin{aligned}
&K^1 \circ K^1 = 1, && K^\alpha \circ K^\alpha = -1,\\
&R^\alpha \circ R^1 = \nu, && R^1 \circ R^\alpha = \nu^{-1},\\
&L^\alpha \circ L^1 = \nu^{-1} K^1, && L^1 \circ L^\alpha = \nu^{-1} K^\alpha,\\
&K^\alpha \circ R^1 = \mu(R^1 \circ K^1), && K^\alpha \circ L^1 = \mu(L^1 \circ K^1),\\
&(R^\alpha \circ L^1)^3 = -1, && (R^1 \circ L^\alpha)^3 = \mu K^\alpha.
\end{aligned}$$

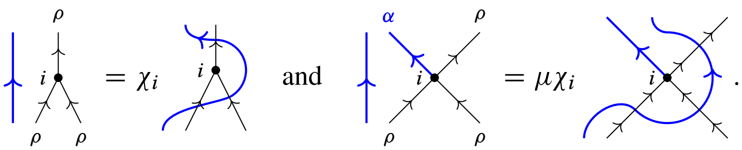
As a consequence of these relations, we can diagonalise the action of the operator K^1, and set



to obtain that there exist scalars $\chi_i \in \{-1, 1\}$ and $\omega_i \in \{1, e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{2}{3}}\}$ such that



In particular, this gives us the local relations



(4.3)

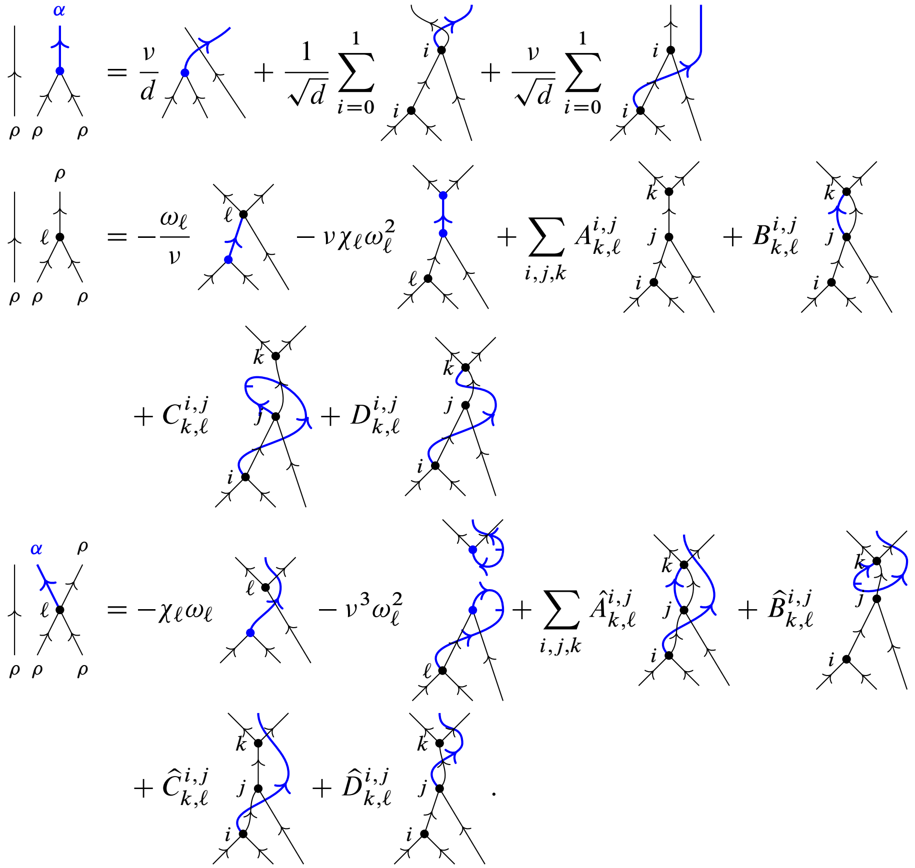
Remark 4.3. Note that we are free to change our basis of $\mathcal{D}_m(\rho \otimes \rho \to \rho)$ by a unitary which commutes with the operator $K^{\mathbf{1}}$. In particular, if $m = 2$ and $\chi_0 = \chi_1$, then we are free to pick any other orthonormal basis of $\mathcal{D}_m(\rho \otimes \rho \to \rho)$, and if $\chi_0 \neq \chi_1$, then we can only rescale each basis vector by an element of $U(1)$.

With this special choice of bases, we can determine the following local relations in $\mathcal{D}_m$.

Lemma 4.4. *There are scalars*

$$A^{i,j}_{k,\ell}, B^{i,j}_{k,\ell}, C^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, \widehat{B}^{i,j}_{k,\ell}, \widehat{C}^{i,j}_{k,\ell}, \widehat{D}^{i,j}_{k,\ell} \in \mathbb{C}, \quad 0 \leq i, j, k, \ell < m,$$

such that the following local relations hold in $\mathcal{D}_m$:



Proof. The proof is omitted as it is nearly identical to the proof of Lemma 3.15. ∎

Remark 4.5. As in the self-dual case described in Remark 3.16, the above complex scalars are precisely entries of certain F-tensors of $\mathcal{D}_m$.

With these local relations, we can show that our described numerical data fully determines the category $\mathcal{D}_m$.

Proposition 4.6. *There is at most one unitary fusion category $\mathcal{D}_m$ realising each tuple of data*

$$(\nu, \chi, \omega, A, B, C, D, \hat{A}, \widehat{B}, \widehat{C}, \widehat{D}).$$

Proof. We omit the proof which is nearly identical to the proof of Proposition 3.19 replacing (3.1) with (4.1), the jellyfish relations from Lemmas 3.3 and 3.15 with those from (4.3), (4.2), and Lemma 4.4, and using absorption relations similar to Lemma 3.18. ∎

4.2. Centre analysis

As in the self-dual case, we study the centre of $\mathcal{D}_m$ in order to determine information about our free variables. We restrict our attention to the case of $m = 2$, as this is the most difficult case, and we need as much information about our numerical data as possible in order to make progress on the classification. While we could repeat the analysis for $m = 1$, this is unnecessary as in this case the lack of multiplicity makes it easy to solve for our numerical data.

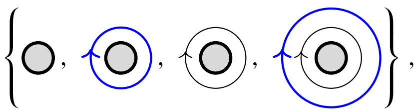
Our main result of this section is as follows.

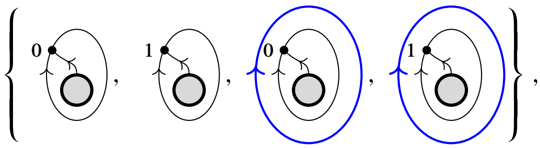
Lemma 4.7. *If* $m = 2$ *and* $\chi_0 = \chi_1$, *then*

$$\sum_i A_{i,0}^{i,0} = \frac{(2+i)-\sqrt{5}}{2}, \quad \sum_i A_{i,1}^{i,1} = \frac{(2-i)-\sqrt{5}}{2}, \quad \sum_i A_{i,0}^{i,1} = \sum_i A_{i,1}^{i,0} = 0.$$

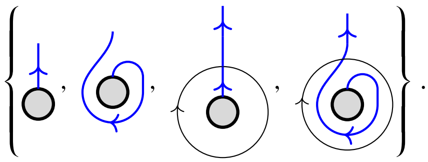
In the case of $\chi_1 = \chi_0$, knowing the above information about the free variables $A_{k,l}^{i,j}$ will be the key starting point in showing non-existence of the category $\mathcal{D}_2$ later on in this paper.

To show this result, we study the tube algebra of $\mathcal{D}_2$. As in the self-dual case, we only study a small sub-algebra. We choose the following bases:

$A_{\mathbf{1}\to\mathbf{1}} = \operatorname{span}$  ,

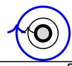
$A_{\mathbf{1}\to\rho} = \operatorname{span}$ 

,

$A_{\alpha\to\alpha} = \operatorname{span}$  .

By direct computation, we obtain the following:

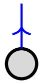
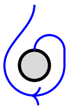
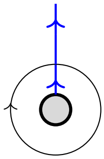
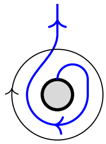
(1) The irreducible representations of $A_{\mathbf{1}\to\mathbf{1}}$ are

				
χ_0	1	1	$2+\sqrt{5}$	$2+\sqrt{5}$
χ_1	1	1	$2-\sqrt{5}$	$2-\sqrt{5}$
χ_2	1	-1	$\mathbf{i}$	$-\mathbf{i}$
χ_3	1	-1	$-\mathbf{i}$	$\mathbf{i}$

Hence, $\mathcal{I}(\mathbf{1})$ contains 4 simple objects X_i with dimensions

$$\dim(X_0) = 1, \quad \dim(X_1) = 9 + 4\sqrt{5}, \quad \dim(X_2) = \dim(X_3) = 5 + 2\sqrt{5}.$$

(2) The irreducible representations of $A_{\alpha\to\alpha}$ are

				
τ_0	1	$\mathbf{i}$	$\frac{a(1+\mathbf{i}\mu)+\sqrt{2a^2(1+\mathbf{i}\mu)+4\mathbf{i}\mu}}{2}$	$\mathbf{i}\frac{a(1+\mathbf{i}\mu)+\sqrt{2a^2(1+\mathbf{i}\mu)+4\mathbf{i}\mu}}{2}$
τ_1	1	$-\mathbf{i}$	$\frac{a(1-\mathbf{i}\mu)+\sqrt{2a^2(1-\mathbf{i}\mu)-4\mathbf{i}\mu}}{2}$	$-\mathbf{i}\frac{a(1-\mathbf{i}\mu)+\sqrt{2a^2(1-\mathbf{i}\mu)-4\mathbf{i}\mu}}{2}$
τ_2	1	$\mathbf{i}$	$\frac{a(1+\mathbf{i}\mu)-\sqrt{2a^2(1+\mathbf{i}\mu)+4\mathbf{i}\mu}}{2}$	$\mathbf{i}\frac{a(1+\mathbf{i}\mu)-\sqrt{2a^2(1+\mathbf{i}\mu)+4\mathbf{i}\mu}}{2}$
τ_3	1	$-\mathbf{i}$	$\frac{a(1-\mathbf{i}\mu)-\sqrt{2a^2(1-\mathbf{i}\mu)-4\mathbf{i}\mu}}{2}$	$-\mathbf{i}\frac{a(1-\mathbf{i}\mu)-\sqrt{2a^2(1-\mathbf{i}\mu)-4\mathbf{i}\mu}}{2}$

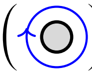
where $a := \chi_0 + \chi_1 \in \{0, \pm 2\}$. Hence, $\mathcal{I}(\alpha)$ contains 4 simple objects Y_i with dimensions

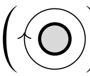
$$\frac{20 + 8\sqrt{5}}{2 + \frac{1}{2}|a(1 + \mathbf{i}\mu) \pm \sqrt{2a^2(1 + \mathbf{i}\mu) + 4\mathbf{i}\mu}|^2}$$

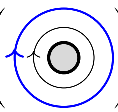
and

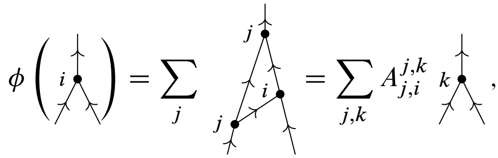
$$\frac{20 + 8\sqrt{5}}{2 + \frac{1}{2}|a(1 - \mathbf{i}\mu) \pm \sqrt{2a^2(1 - \mathbf{i}\mu) - 4\mathbf{i}\mu}|^2}.$$

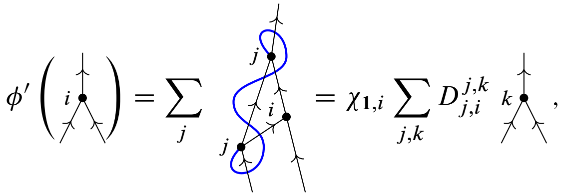
(3) Let ${}_{\mathbf{1}}\pi_\rho$ be the action of $A_{\mathbf{1}\to\mathbf{1}}$ on $A_{\mathbf{1}\to\rho}$. Then

${}_{\mathbf{1}}\pi_\rho\Big($  $\Big) =$ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

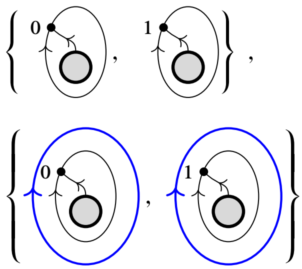
${}_{\mathbf{1}}\pi_\rho\Big($  $\Big) =$ $\begin{bmatrix} \phi & \phi' \\ \phi' & \phi \end{bmatrix}$,

${}_{\mathbf{1}}\pi_\rho\Big($  $\Big) =$ $\begin{bmatrix} \phi' & \phi \\ \phi & \phi' \end{bmatrix}$, where ϕ and ϕ' are the operators on $\mathrm{Hom}(\rho \otimes \rho \to \rho)$ defined by





which we can naturally identify as operators on the two spaces



by local insertion. That is, the elements of $A_{\mathbf{1}\leftarrow\mathbf{1}}$ which involve ϕ, ϕ' above act on $A_{\mathbf{1}\leftarrow\rho}$ by applying ϕ, ϕ' locally on the trivalent vertices in our standard basis of $A_{\mathbf{1}\leftarrow\rho}$.

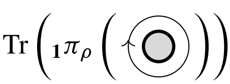
With these computations in hand, we either pin down the scalars χ_0 and χ_1 or determine the operator ϕ.

Proof of Lemma 4.7. Recall we have three possibilities for $a \in \{-2, 0, 2\}$. If $a = 0$, then we have $\chi_0 = -\chi_1$. Thus, we can restrict our attention to the case of $a = \pm 2$.

We begin by determining the decomposition of ${}_{\mathbf{1}}\pi_\rho$ into irreducible representations of $A_{\mathbf{1}\to\mathbf{1}}$. As X_0 is the tensor unit of $\mathcal{Z}(\mathcal{D}_m)$, its restriction contains no copies of ρ, and thus, ${}_{\mathbf{1}}\pi_\rho$ contains no copies of χ_0. We also know that $\mathrm{Tr}({}_{\mathbf{1}}\pi_\rho) = 0$, and so, from the character table of $A_{\mathbf{1}\to\mathbf{1}}$ above, we must have that

$$ {}_{\mathbf{1}}\pi_\rho \cong 2\chi_1 \oplus k\chi_2 \oplus (2-k)\chi_3 $$

with $k \in \{0, 1, 2\}$. In particular, we find that



$$ = 4 - 2\sqrt{5} + 2\mathbf{i}(k-1) \implies \mathrm{Tr}(\phi) = 2 - \sqrt{5} + \mathbf{i}(k-1). $$

To determine k, we study the restriction of the objects X_i and Y_i. By the above decomposition of ${}_{\mathbf{1}}\pi_\rho$ and from counting dimensions, we have

$$\begin{aligned}
\mathcal{F}(X_0) &= \mathbf{1},\\
\mathcal{F}(X_1) &= \mathbf{1} \oplus 2\rho \oplus 2\alpha\rho,\\
\mathcal{F}(X_2) &= \mathbf{1} \oplus k\rho \oplus (2-k)\alpha\rho,\\
\mathcal{F}(X_3) &= \mathbf{1} \oplus (2-k)\rho \oplus k\alpha\rho.
\end{aligned}$$

By our assumption that $a = \pm 2$, one of the objects Y_i must be invertible. Thus, we can label our Y_i so that

$$\begin{aligned}
\mathcal{F}(Y_0) &= \alpha,\\
\mathcal{F}(Y_1) &= \alpha \oplus 2\rho \oplus 2\alpha\rho,\\
\mathcal{F}(Y_2) &= \alpha \oplus (2-k)\rho \oplus k\alpha\rho,\\
\mathcal{F}(Y_3) &= \alpha \oplus k\rho \oplus (2-k)\alpha\rho.
\end{aligned}$$

Hence, we now know the restriction of all the objects in both $\mathcal{I}(\mathbf{1})$ and $\mathcal{I}(\alpha)$, up to the integer k. Denote by Z_i the remaining simple objects in $\mathcal{Z}(\mathcal{D})$, i.e., those simple objects such that

$$\mathcal{F}(Z_i) = p_i\rho \oplus q_i\alpha\rho,$$

where p_i, q_i are positive integers. This allows us to write

$$\begin{aligned}
\mathcal{I}(\rho) &= 2X_1 + kX_2 + (2-k)X_3 + 2Y_1 + (2-k)Y_2 + kY_3 + \sum p_i Z_i,\\
\mathcal{I}(\alpha\rho) &= 2X_1 + (2-k)X_2 + kX_3 + 2Y_1 + kY_2 + (2-k)Y_3 + \sum q_i Z_i.
\end{aligned}$$

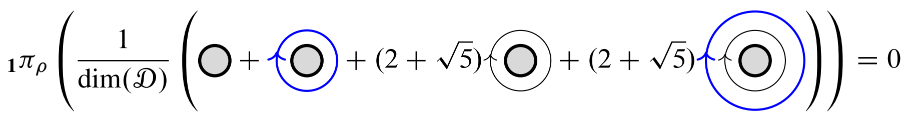
Therefore,

$$\begin{aligned}
20 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 4k^2 - 8k + 16 + \sum p_i^2,\\
20 &= \dim \operatorname{Hom}(\mathcal{I}(\alpha\rho), \mathcal{I}(\alpha\rho)) = 4k^2 - 8k + 16 + \sum q_i^2,\\
16 &= \dim \operatorname{Hom}(\mathcal{I}(\rho), \mathcal{I}(\alpha\rho)) = -4k^2 + 8k + 8 + \sum p_i q_i.
\end{aligned}$$

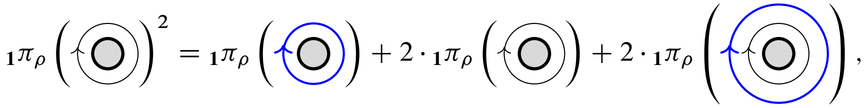
If $k \in \{0, 2\}$, then we get

$$\sum p_i^2 = \sum q_i^2 = 4 \quad \text{and} \quad \sum p_i q_i = 8,$$

which is impossible. Thus, we must have $k = 1$, and so, $\operatorname{Tr}(\phi) = 2 - \sqrt{5}$. From



and



we find that ϕ has the two distinct eigenvalues

$$\frac{(2+i)-\sqrt{5}}{2} \quad \text{and} \quad \frac{(2-i)-\sqrt{5}}{2}.$$

Finally, by Remark 4.3, we are free to unitarily change our basis of $\mathcal{D}_2(\rho \otimes \rho \to \rho)$ by any element of $U(2)$. In particular, we can choose this basis so that ϕ acts diagonally. This gives the statement of the lemma. ■

4.3. Symmetries

We now use the tetrahedral symmetries to determine relations between the 128 complex scalars:

$$A^{i,j}_{k,\ell}, \quad B^{i,j}_{k,\ell}, \quad C^{i,j}_{k,\ell}, \quad D^{i,j}_{k,\ell}, \quad \hat{A}^{i,j}_{k,\ell}, \quad \hat{B}^{i,j}_{k,\ell}, \quad \hat{C}^{i,j}_{k,\ell}, \quad \text{and} \quad \hat{D}^{i,j}_{k,\ell}.$$

Using the same techniques as in the self-dual case, we are able to show the following lemma.

Lemma 4.8. *The scalars $B^{i,j}_{k,\ell}, D^{i,j}_{k,\ell}, \hat{A}^{i,j}_{k,\ell}, \hat{C}^{i,j}_{k,\ell}, \hat{D}^{i,j}_{k,\ell}$ can be expressed in terms of the scalars $A^{i,j}_{k,\ell}$ as follows:*

$$B^{i,j}_{k,\ell} = -\nu\chi_\ell\omega_\ell^2 A^{k,i}_{j,\ell}, \quad D^{i,j}_{k,\ell} = -\nu^{-1}\omega_\ell A^{j,k}_{i,\ell},$$
$$\hat{A}^{i,j}_{k,\ell} = -\nu^3\omega_\ell^2 \overline{A^{j,i}_{k,\ell}}, \quad \hat{C}^{i,j}_{k,\ell} = -\chi_\ell\omega_\ell \overline{A^{i,k}_{j,\ell}}, \quad \hat{D}^{i,j}_{k,\ell} = \overline{A^{k,j}_{i,\ell}}.$$

The scalars $\hat{B}^{i,j}_{k,\ell}$ can be expressed in terms of the scalars $C^{i,j}_{k,\ell}$ as follows:

$$\hat{B}^{i,j}_{k,\ell} = \nu^3 \overline{C^{k,j}_{i,\ell}}.$$

The scalars $A^{i,j}_{k,\ell}$ satisfy $\mathbb{Z}/4\mathbb{Z}$ symmetries generated by the relations

$$A^{i,j}_{k,\ell} = -\nu\chi_i\omega_j\omega_k^2\omega_\ell^2 \overline{A^{k,\ell}_{j,i}} = \chi_i\chi_k\omega_i\omega_\ell\omega_j^2\omega_k^2 A^{j,i}_{\ell,k}.$$

The scalars $C^{i,j}_{k,\ell}$ satisfy S_3 symmetries generated by the order three rotation

$$C^{i,j}_{k,\ell} = \omega_\ell C^{j,k}_{i,\ell} = \omega_\ell^2 C^{k,i}_{j,\ell}$$

and the order two flip

$$C^{i,j}_{k,\ell} = \chi_j \chi_k \omega_i^2 \omega_k C^{k,\ell}_{i,j}.$$

Finally, we have that if $\chi_0 = -\chi_1$, *then*

$$A^{i,j}_{k,\ell} = C^{i,j}_{k,\ell} \quad \text{if } i + j + k + \ell = 0 \pmod 2.$$

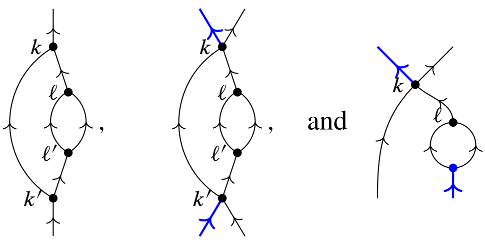
Proof. The proof of this lemma uses the exact same techniques as in the proof of Lemma 3.21. The only real difference is that we have different Frobenius operators in this case. ∎

4.4. Classification

We now complete the proof of Theorem 4.1 to complete the classification in the non self dual case. To prove this theorem, we break into three cases: (1) $m = 1$, (2) $m = 2$ and $\chi_0 = -\chi_1$, and (3) $m = 2$ and $\chi_0 = \chi_1$.

The case $m = 1$. If $m = 1$, then from our previous analysis we only have to determine the sign χ_0, the 3rd root of unity ω_0, the 8th root of unity ν, and the two complex scalars $a := A^{0,0}_{0,0}$ and $c := C^{0,0}_{0,0}$. Further, we have that if $\omega_0 \neq 1$, then $c = 0$.

By evaluating the diagrams



in two ways (see footnote 4), we obtain the following equations:

$$|a|^2 + |c|^2 = 1, \quad 2|a|^2 = 1 - \frac{1}{1+\sqrt{2}}, \quad \frac{\nu}{1+\sqrt{2}} = a(\chi_0\omega_0 - \nu).$$

With the first two of these equations, we can solve to find

$$|a|^2 = 1 - \frac{1}{\sqrt{2}} \quad \text{and} \quad |c|^2 = \frac{1}{\sqrt{2}},$$

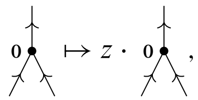
and thus, we have $\omega_0 = 1$. The general solution to these equations is then given by

$$\chi_0 = 1, \quad a = \frac{\nu}{(1-\nu)}(-1+\sqrt{2}), \quad \text{and} \quad c = e^{i\theta} 2^{\frac{-1}{4}},$$

where θ is any phase.

Lemma 4.9. *There are exactly two unitary fusion categories, up to monoidal equivalence, which categorify $S(1)$.*

Proof. By unitarily renormalising the basis element



$z \in U(1)$,

we change c to $z^{-2}c$. We can thus renormalise so that $c = 2^{\frac{-1}{4}}$. Hence, we have two solutions for our free variables, depending on the choice of $\nu = e^{\pm i\pi\frac{1}{4}}$. By Proposition 4.6, there are at most 2 unitary fusion categories with these fusion rules. These two unitary fusion categories are realised by the even parts of the two subfactors $\mathcal{S}'$ constructed in [34], which are monoidally non-equivalent and complex conjugate to each other. Indeed, they each admit a $\mathbb{Z}/2\mathbb{Z}$-equivariantisation, which produces monoidally non-equivalent $2^{\mathbb{Z}/4\mathbb{Z}}1$ near-group fusion categories which are complex conjugate [35, Example 2.2] and [25, Example 9.5]. ■

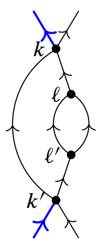
The case $m = 2$ and $\chi_0 = -\chi_1 = 1$. If $m = 2$ and $\chi_0 = -\chi_1 = 1$, then we have to determine the 3rd roots of unity ω_i, the 8th root of unity ν, and the free complex variables $A^{i,j}_{k,l}$ and $C^{i,j}_{k,l}$. We can represent these free complex variables in the same matrix notation as in (3.3) in the self-dual section. After applying the symmetries of Lemma 4.8, we obtain

$$A = \begin{bmatrix} a_0 & 0 & 0 & 0 \\ 0 & a_1 & -\nu\omega_0^2\overline{a_1} & 0 \\ 0 & \nu\omega_1^2\overline{a_1} & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 \end{bmatrix}.$$

Due to the large number of variables which are zero, it is fairly easy to derive a contradiction in this case.

Lemma 4.10. *There is no unitary fusion category that categorifies $S(2)$ with $\chi_0 = -\chi_1 = 1$.*

Proof. By evaluating



in two ways (see footnote 4), we obtain the equation

$$\chi_\ell \chi_{\ell'} \omega_\ell^2 \omega_{\ell'} \sum_{i,j} A_{j,\ell}^{k,i} \overline{A_{j,\ell'}^{k',i}} + \omega_\ell \omega_{\ell'}^2 \sum_{i,j} A_{i,\ell}^{j,k} \overline{A_{i,\ell'}^{j,k'}} - (2-\sqrt{5})\delta_{\ell,k}\delta_{\ell',k'} = \delta_{\ell,\ell'}\delta_{k,k'}.$$

Taking $k = k' = 0$ and $\ell = \ell' = 1$ gives $|a_1|^2 = \frac{1}{2}$, and taking $k = k' = \ell = l' = 0$ gives $2|a_0|^2 + 2|a_1|^2 = 3 - \sqrt{5}$. These two equations imply $2|a_0|^2 < 0$, a contradiction. ■

The case $m = 2$ and $\chi_0 = \chi_1$. Finally, we deal with the last case where $m = 2$ and $\chi_0 = \chi_1$. Let us again represent our free variables $A_{k,l}^{i,j}$ and $C_{k,l}^{i,j}$ in matrix form as in (3.3). After applying the symmetries of Lemma 4.8, we obtain the following:

$$A = \begin{bmatrix} a_0 & a_1 & \omega_0\omega_1^2 a_1 & a_2 \\ -\nu\chi_0\omega_1^2\overline{a_1} & a_3 & -\nu\chi_0\omega_0^2\overline{a_3} & a_4 \\ -\nu\chi_0\omega_0^2\overline{a_1} & -\nu\chi_0\omega_1^2\overline{a_3} & a_3 & \omega_0^2\omega_1 a_4 \\ -\nu\chi_0\omega_0\omega_1\overline{a_2} & -\nu\chi_0\omega_0^2\overline{a_4} & -\nu\chi_0\omega_1^2\overline{a_4} & a_5 \end{bmatrix},$$

$$C = \begin{bmatrix} c_0 & c_1 & \omega_0 c_1 & c_2 \\ c_1 & \omega_1^2 c_2 & \omega_0^2\omega_1^2 c_2 & \omega_1^2 c_3 \\ \omega_0^2 c_1 & \omega_1 c_2 & \omega_1^2 c_2 & c_3 \\ \omega_0\omega_1^2 c_2 & \omega_1 c_3 & c_3 & c_4 \end{bmatrix}.$$

Recall from Lemma 4.7 that in this case we have

$$\sum_i A_{i,0}^{i,0} = \frac{(2+i)-\sqrt{5}}{2}, \quad \sum_i A_{i,1}^{i,1} = \frac{(2-i)-\sqrt{5}}{2}, \quad \sum_i A_{i,0}^{i,1} = \sum_i A_{i,1}^{i,0} = 0,$$

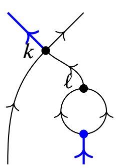
which implies that

$$a_0 + a_3 = \frac{(2+i)-\sqrt{5}}{2}, \quad a_3 + a_5 = \frac{(2-i)-\sqrt{5}}{2}, \quad a_1 = -a_4 = -\omega_0^2\omega_1 a_4.$$

With these linear equations in hand, it is straightforward to show non-existence in this case.

Theorem 4.11. *There is no unitary fusion category that categorifies $S(2)$ with $\chi_0 = \chi_1$.*

Proof. Evaluating the diagram



in two ways (see footnote 4) gives

$$\sum_i A_{i,\ell}^{i,k} + \nu^3 \chi_\ell \omega_\ell \sum_i A_{i,\ell}^{k,i} - \delta_{\ell,k}(2-\sqrt{5}) = 0.$$

In terms of our free variables, this gives us the following:

$$\begin{aligned} 2-\sqrt{5} &= a_0 + a_3 + 2\nu^3\chi_0\omega_0 a_0 + 2\overline{a_3} \\ &= \mathbf{i} - 2\nu^3\chi_0\omega_0\big(2\nu\chi_0\omega_0^2\overline{a_3} + 2a_3 + (-2-i) + \sqrt{5}\big), \\ 2-\sqrt{5} &= a_5 + a_3 + 2\nu^3\chi_0\omega_1 a_5 + 2\overline{a_3} \\ &= -\mathbf{i} - 2\nu^3\chi_0\omega_1\big(2\nu\chi_0\omega_1^2\overline{a_3} + 2a_3 + (-2+i) + \sqrt{5}\big). \end{aligned}$$

This system of equations of the complex variable a_3 does not hold for any values of our free variables $\chi_0 \in \{-1,1\}$, $\nu \in \{e^{\frac{i\pi}{4}}, e^{\frac{-i\pi}{4}}\}$, and $\omega_0, \omega_1 \in \{1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}\}$. ■

A. A multiplicity bound for $\mathbb{Z}/2\mathbb{Z}$-quadratic categories (by Ryan Johnson, Siu-Hung Ng, David Penneys, Jolie Roat, Matthew Titsworth, and Henry Tucker)

In this appendix, we prove Theorem 2.4. That is, given a pseudounitary $\mathbb{Z}/2\mathbb{Z}$ quadratic fusion category with simple objects $1, \alpha, \rho, \alpha\rho$ with ρ self-dual and fusion rules determined by

$$\alpha^2 \cong 1 \quad \text{and} \quad \rho^2 \cong 1 \oplus m\rho \oplus n\alpha\rho, \tag{Q3}$$

(m,n) must be one of $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(2,2)$.

A.1. Basic number theoretic constraints

Given a $\mathbb{Z}/2\mathbb{Z}$-quadratic category $\mathcal{C}$ with fusion rules (Q3), the fusion matrices are given in the ordering $1, \rho, \alpha\rho, \alpha$ by

$$L_\rho = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & m & n & 0 \\ 0 & n & m & 1 \\ 0 & 0 & 1 & 0 \end{array}\right), \quad L_{\alpha\rho} = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & n & m & 1 \\ 1 & m & n & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

$$L_\alpha = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right).$$

Setting $d := \dim(\rho)$, we have $\dim(\alpha) = 1$ and $d^2 = 1 + (m+n)d$ so that

$$d = \frac{1}{2}\big(m + n + \sqrt{4 + (m+n)^2}\big). \tag{A.1}$$

Then

$$\dim(\mathcal{C}) = 2 + 2d^2 = 2 + \frac{(m+n)^2 + 2(m+n)\sqrt{4+(m+n)^2} + 4 + (m+n)^2}{2}$$
$$= 4 + 2(m+n)d.$$

Since $K_0(\mathcal{C})$ is abelian of dimension 4, its irreducible representations are all 1-dimensional. Hence, by [49, Remark 2.11], the element

$$R := I + L_\rho^2 + L_{\alpha\rho}^2 + L_\alpha^2 = \begin{pmatrix} 4 & 2m & 2n & 0 \\ 2m & 2n^2+2m^2+4 & 4mn & 2n \\ 2n & 4mn & 2n^2+2m^2+4 & 2m \\ 0 & 2n & 2m & 4 \end{pmatrix}$$

is central in $K_0(\mathcal{C})$, and the roots of its characteristic polynomial are called the *formal codegrees* [48] of $\mathcal{C}$:

$$f_1 = 4 + (m+n)^2 + (m+n)\sqrt{4+(m+n)^2},$$
$$f_2 = 4 + (m+n)^2 - (m+n)\sqrt{4+(m+n)^2},$$
$$f_3 = 4 + (m-n)^2 + (m-n)\sqrt{4+(m-n)^2},$$
$$f_4 = 4 + (m-n)^2 - (m-n)\sqrt{4+(m-n)^2}.$$

A.2. Computing the induction and forgetful functor

We now assume that $\mathcal{C}$ is pseudounitary, and we analyse the centre $Z(\mathcal{C})$, the forgetful functor $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$, and the induction functor $\mathcal{I} : \mathcal{C} \to Z(\mathcal{C})$. Recall that

$$\mathcal{F}(\mathcal{I}(c)) \cong \bigoplus_{x \in \mathrm{Irr}(\mathcal{C})} x \otimes c \otimes x^* \quad \forall c \in \mathcal{C} \tag{A.2}$$

and that $\mathcal{F}$ is biadjoint to $\mathcal{I}$. We use the notation $(a,b) := \dim(\mathcal{C}(a \to b))$ and $(A,B) := \dim(Z(\mathcal{C})(A \to B))$.

Lemma A.1 ([49, Theorem 2.13]). *There are distinct simple objects* $1_{Z(\mathcal{C})}$, X_2, X_3, $X_4 \in \mathrm{Irr}(Z(\mathcal{C}))$ *such that*

$$\mathcal{I}(1_\mathcal{C}) = 1_{Z(\mathcal{C})} \oplus X_2 \oplus X_3 \oplus X_4 \quad \textit{and} \quad \dim(X_k) = \frac{f_1}{f_k}.$$

Setting

$$r := \sqrt{\frac{4+(m+n)^2}{4+(m-n)^2}},$$

it is straightforward to calculate that

$$\dim(X_2) = \frac{f_1}{f_2} = \frac{f_1^2}{f_1 f_2} = 1 + (m+n)d = \frac{\dim(\mathcal{C})}{2} - 1,$$
$$\dim(X_3) = \frac{f_1}{f_3} = \frac{f_1 f_4}{f_3 f_4} = 1 + \frac{1}{2}(m+n)d - \frac{r}{2}(m-n)d,$$
$$\dim(X_4) = \frac{f_1}{f_4} = \frac{f_1 f_3}{f_3 f_4} = 1 + \frac{1}{2}(m+n)d + \frac{r}{2}(m-n)d.$$

Remark A.2. Since $\dim(X_2), \dim(X_3), \dim(X_4) \in \mathbb{Z}(d)$, it must be the case that either $m = n$ or $r \in \mathbb{Q}(d)$. If $m = 0$ or $n = 0$, then $r = 1$. One may check that if $0 \neq m \neq n \neq 0$ and $r \in \mathbb{Q}(d)$, then $m + n \geq 11$. We will show below in Theorem A.12 that $m + n \leq 5$.

Proposition A.3. *The centre $Z(\mathcal{C})$ has 8 distinct simple objects $1_{Z(\mathcal{C})}$, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4 such that*

$$\mathcal{I}(1_{\mathcal{C}}) = 1_{Z(\mathcal{C})} \oplus X_2 \oplus X_3 \oplus X_4 \quad \text{and} \quad \mathcal{I}(\alpha) = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4.$$

Denote the rest of the simple objects of $Z(\mathcal{C})$ by $\{Z_s\}_{s\in S}$ where S is some finite set. The matrix F of the forgetful functor $\mathcal{F} : Z(\mathcal{C}) \to \mathcal{C}$ can then be represented as follows, where zero entries are omitted:

$F =$

	$1_{Z(\mathcal{C})}$	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	Z_s
$1_{\mathcal{C}}$	1	1	1	1					
α					1	1	1	1	
ρ		x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_s
$\alpha\rho$		x_2'	x_3'	x_4'	y_1'	y_2'	y_3'	y_4'	z_s'

and the induction matrix is given by F^T. Moreover,

$$\sum_{j=2}^{4} x_j = 2m \quad \text{and} \quad \sum_{j=2}^{4} x_j' = 2n, \tag{A.3}$$

$$x_2 + x_2' = m + n, \tag{A.4}$$

$$x_3 + x_3' = \frac{1}{2}(m+n) - \frac{r}{2}(m-n), \tag{A.5}$$

$$x_4 + x_4' = \frac{1}{2}(m+n) + \frac{r}{2}(m-n), \tag{A.6}$$

$$\sum_{j=1}^{4} y_j = 2n \quad \text{and} \quad \sum_{j=1}^{4} y_j' = 2m. \tag{A.7}$$

Proof. First, $1_{Z(\mathcal{C})}$ decomposes as desired by Lemma A.1. Next, observe that, by equation (A.2),

$$(\mathcal{I}(\alpha), \mathcal{I}(\alpha)) = 4,$$
$$(\mathcal{F}\mathcal{I}(1_{\mathcal{C}}), \alpha) = 0.$$

Since the first Frobenius–Schur indicator ν_1 satisfies $\mathrm{Tr}_{Z(\mathcal{C})}(\theta_{\mathcal{I}(\alpha)}) = 0$ [43, Remark 4.6] (see also [49, Theorem 2.4]), $\mathcal{I}(\alpha)$ decomposes as 4 distinct simples which are distinct from $1_{Z(\mathcal{C})}, X_2, X_3, X_4$. Equations (A.3) and (A.7) follow from calculating $\mathcal{F}\mathcal{I}(1_{\mathcal{C}})$ and $\mathcal{F}\mathcal{I}(\alpha)$. Equations (A.4), (A.5), and (A.6) now follow from the formulas for $\dim(X_k)$ for $k = 2, 3, 4$. ∎

We now compute the dimensions of all the hom spaces amongst $\mathcal{I}(\rho)$ and $\mathcal{I}(\alpha\rho)$ in two ways. The first way is by taking adjoints and using (A.2). The second way is by using the induction matrix F^T computed in Proposition A.3. This gives us the following:

$$(\mathcal{I}(\rho), \mathcal{I}(\rho)) = 4 + 2m^2 + 2n^2 = \sum_{j=2}^{4} x_j^2 + \sum_{j=1}^{4} y_j^2 + \sum_{s} z_s^2, \tag{A.8}$$

$$(\mathcal{I}(\rho), \mathcal{I}(\alpha\rho)) = 4mn = \sum_{j=2}^{4} x_j x_j' + \sum_{j=1}^{4} y_j y_j' + \sum_{s} z_s z_s', \tag{A.9}$$

$$(\mathcal{I}(\alpha\rho), \mathcal{I}(\alpha\rho)) = 4 + 2m^2 + 2n^2 = \sum_{j=2}^{4} (x_j')^2 + \sum_{j=1}^{4} (y_j')^2 + \sum_{s} (z_s')^2. \tag{A.10}$$

Lemma A.4. *The non-negative integers $x_j, x_j', y_j, y_j', z_s, z_s'$ satisfy*

$$8 + \frac{5}{2}(m+n)^2 - \frac{r^2}{2}(m-n)^2 = \sum_{j=1}^{4} (y_j + y_j')^2 + \sum_{s} (z_s + z_s')^2, \tag{A.11}$$

$$8 + 4(m-n)^2 = \sum_{j=2}^{4} (x_j - x_j')^2 + \sum_{j=1}^{4} (y_j - y_j')^+ \sum_{s} (z_s - z_s')^2. \tag{A.12}$$

Proof. To get the first equation, sum equations (A.8) and (A.10) and twice equation (A.9). Then, use equations (A.4), (A.5), and (A.6) and simplify. The second is similar. ∎

Proposition A.5. *We have the following upper bound:*

$$\sum_{s} (z_s + z_s')^2 \leq 8 + \frac{3}{2}(m+n)^2. \tag{A.13}$$

Proof. By equation (A.11), the desired inequality is implied by

$$\sum_{j=1}^{4}(y_j + y_j')^2 \geq \frac{1}{4}\left(\sum_{j=1}^{4} y_j + y_j'\right)^2 = (m+n)^2 \geq (m+n)^2 - \frac{r^2}{2}(m-n)^2,$$

which is true. The second equality above holds by equation (A.7), and the first inequality above follows from the fact that for any real numbers w, x, y, z, we have

$$\begin{aligned} &4(w^2 + x^2 + y^2 + z^2) - (w+x+y+z)^2 \\ &\quad = (w-x)^2 + (w-y)^2 + (w-z)^2 + (x-y)^2 + (x-z)^2 + (y-z)^2 \geq 0. \end{aligned}$$

The proof is complete. ■

Proposition A.6. *Denote the twists of $Y_1, \ldots, Y_4$ by $\theta_1, \ldots, \theta_4$. We have $\theta_1^2 = \theta_2^2 = \theta_3^2 = \theta_4^2 = \pm 1$. Setting $\theta = \theta_1$, without loss of generality, we have $\theta \in \{1, i\}$.*

Proof. We calculate the following Frobenius–Schur indicators [43] of α:

$$0 = \mathrm{Tr}_{Z(\mathcal{C})}(\theta_{\mathcal{I}(\alpha)}) = \sum_{j=1}^{4} \theta_j \dim(Y_j), \tag{A.14}$$

$$\pm \dim(\mathcal{C}) = \mathrm{Tr}_{Z(\mathcal{C})}(\theta^2_{\mathcal{I}(Z)}) = \sum_{j=1}^{4} \theta_j^2 \dim(Y_j). \tag{A.15}$$

Since $\dim(\mathcal{I}(\alpha)) = \dim(\mathcal{C}) = \sum_{j=1}^{4} \dim(Y_k)$, equation (A.15) implies that $\theta_1^2 = \theta_2^2 = \theta_3^2 = \theta_4^2 = \pm 1$. By equation (A.14), the θ_j's split up into two nonempty groups of opposite sign, so without loss of generality, $\theta_1 \in \{1, i\}$. ■

Definition A.7. For $j = 2, \ldots, 4$, we let $\varepsilon_j \in \{-1, +1\}$ such that $\varepsilon_j \theta = \theta_j$ where $\theta = \theta_1$ is the twist of Y_1. We also set $\gamma := \frac{1}{2}(m+n) + \frac{r}{2}$ and $\bar{\gamma} := \frac{1}{2}(m+n) - \frac{r}{2}$. Notice that $\gamma + \bar{\gamma} = m + n$ and that $\gamma^2 + \bar{\gamma}^2 = \frac{1}{2}(m+n)^2 + \frac{r^2}{2}(m-n)^2$.

Proposition A.8. *We have the following equalities:*

$$\begin{aligned} -\sum_s \theta_s (z_s + z_s')^2 d &= \frac{3}{2}(m+n)^2 d + \frac{r^2}{2}(m-n)^2 d + 2(m+n) \\ &\quad + \theta \sum_{j=1}^{4} \varepsilon_j (y_j + y_j')(1 + (y_j + y_j')d), \end{aligned} \tag{A.16}$$

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z_s')^2 d &= \Delta + \frac{3}{2}(m+n)^2 d + \frac{r^2}{2}(m-n)^2 d + 2(m+n) \\ &\quad + \theta^2 \sum_{j=1}^{4} (y_j + y_j')(1 + (y_j + y_j')d), \end{aligned} \tag{A.17}$$

where $\Delta := \pm(\mathrm{Tr}_{Z(\mathcal{C})}(\theta^2_{\mathcal{I}(\rho)}) + \mathrm{Tr}_{Z(\mathcal{C})}(\theta^2_{\mathcal{I}(\alpha\rho)})) \in \{0, \pm 2\dim(\mathsf{C}) = \pm(8 + 4(m + n)d)\}$.

Proof. To get (A.16), we add the following two equations for the first Frobenius–Schur indicators [43] of $\mathcal{I}(\rho)$ and $\mathcal{I}(\alpha\rho)$, and we use equations (A.4), (A.5), and (A.6) in conjunction with Definition A.7:

$$\begin{aligned}
0 = \mathrm{Tr}_{Z(\mathsf{C})}(\theta_{\mathcal{I}(\rho)}) &= x_2(1 + (m+n)d) + x_3(1+\gamma d) + x_4(1+\bar{\gamma} d)\\
&\quad + \theta \sum_{j=1}^{4} \varepsilon_j y_j (1 + (y_j + y_j')d) + \sum_s \theta_s z_s (z_s + z_s')d,\\
0 = \mathrm{Tr}_{Z(\mathsf{C})}(\theta_{\mathcal{I}(g\rho)}) &= x_2'(1 + (m+n)d) + x_3'(1+\gamma d) + x_4'(1+\bar{\gamma} d)\\
&\quad + \theta \sum_{j=1}^{4} \varepsilon_j y_j (1 + (y_j + y_j')d) + \sum_s \theta_s z_s' (z_s + z_s')d.
\end{aligned}$$

Obtaining equation (A.17) is similar using the second Frobenius–Schur indicators [43] of ρ and $\alpha\rho$:

$$\begin{aligned}
\pm\dim(\mathcal{C}) = \mathrm{Tr}_{Z(\mathcal{C})}(\theta^2_{\mathcal{I}(\rho)}) &= x_2(1 + (m+n)d) + x_3(1+\gamma d) + x_4(1+\bar{\gamma} d)\\
&\quad + \theta^2 \sum_{j=1}^{4} y_j (1 + (y_j + y_j')d) + \sum_s \theta_s^2 z_s (z_s + z_s')d,\\
\pm\dim(\mathcal{C}) = \mathrm{Tr}_{Z(\mathcal{C})}(\theta^2_{\mathcal{I}(\alpha\rho)}) &= x_2'(1 + (m+n)d) + x_3'(1+\gamma d) + x_4'(1+\bar{\gamma} d)\\
&\quad + \theta^2 \sum_{j=1}^{4} y_j' (1 + (y_j + y_j')d) + \sum_s \theta_s^2 z_s' (z_s + z_s')d.
\end{aligned}$$

Adding the above equations, applying (A.4), (A.5), (A.6), and Definition A.7, and rearranging give the result. ∎

Theorem A.9 ([30, Proposition 5.6 and Theorem 5.7]). *Suppose that* $u, v \in \mathbb{Z}$ *and* $t \in \mathbb{N}$ *is square free. We have the following:*

(1) *It requires at least* $|u| + 2|v|$ *roots of unity to write* $u + v\sqrt{2}$ *as a sum of roots of unity.*

(2) *It requires at least* $|v|\varphi(2t)$ *roots of unity to write* $u + v\sqrt{t}$ *as a sum of roots of unity.*

Corollary A.10. *Suppose that* $u \in \mathbb{Q}$, $v \in \mathbb{Z}$, *and* $t \in \mathbb{N}$ *is square free. We have the following:*

(1) *It requires at least* $|u| + 2|v|$ *roots of unity to write* $u + v\sqrt{2}$ *as a sum of roots of unity.*

(2) *It requires at least* $|v|\varphi(2t)$ *roots of unity to write* $u + v\sqrt{t}$ *as a sum of roots of unity.*

Proof. Suppose that $\sum_{i=1}^{N} \zeta_i = u + v\sqrt{2}$. Write $u = p/q$ in lowest terms with $q > 0$ so that

$$q \sum_{s=1}^{N} \zeta_s = p + qv\sqrt{2}.$$

By Theorem A.9(1), $qN \geq |p| + 2q|v|$, so $N \geq |u| + 2|v|$.

Now, suppose that $\sum_{i=1}^{N} \zeta_i = u + v\sqrt{t}$. Again, write $u = p/q$ in lowest terms with $q > 0$ so that $q\sum_{s=1}^{N} \zeta_s = p + qv\sqrt{t}$. By Theorem A.9(2), $qN \geq q|v|\varphi(2t)$, so $N \geq |v|\varphi(2t)$. ■

Lemma A.11. *For all* $t \in \mathbb{N}$ *with* $t \neq 1, 2, 3, 6$, $\varphi(2t) \geq \sqrt{\frac{16t}{5}}$.

Proof. By [28], for all $t \in \mathbb{N}$, $\varphi(2t) \geq 2(\frac{t}{3})^{2/3}$. It is straightforward to show that for $t > 42$, $2\left(\frac{t}{3}\right)^{2/3} \geq \sqrt{16t/5}$. One verifies directly that for $t = 4, 5$ and $7 \leq t \leq 42$, $\varphi(2t) \geq \sqrt{16t/5}$. The result follows. ■

Theorem A.12. *If there is a pseudounitary fusion category* $\mathcal{C}$ *with fusion rules* (Q3), *then* $(m + n) \leq 5$.

Proof. We consider the two cases for $\theta \in \{1, i\}$ afforded by Proposition A.6.

(1) Suppose that $\theta = i$. We add equation (A.16) to its complex conjugate, divide by d, and simplify to obtain

$$-\sum_{s}(\theta_s + \overline{\theta}_s)(z_s + z_s')^2 = (m+n)^2 + r^2(m-n)^2 + 2(m+n)\sqrt{4 + (m+n)^2}. \tag{A.18}$$

Case 1. Suppose that $(m+n)^2 + 4 = 2v_0^2$ for some integer $v_0 > 0$. Then, by Corollary A.10 (1) with $v = 2(m+n)v_0$, it requires at least

$$(m+n)^2 + r^2(m-n)^2 + \underbrace{4(m+n)\underbrace{\sqrt{\frac{4 + (m+n)^2}{2}}}_{=v_0}}_{=2v} \geq \left(1 + \frac{4}{\sqrt{2}}\right)(m+n)^2$$

roots of unity to write the right-hand side of equation (A.18). Together with inequality (A.13), we have the following:

$$16 + 3(m+n)^2 \geq 2\sum_{s}(z_s + z_s')^2 \geq \left(1 + \frac{4}{\sqrt{2}}\right)(m+n)^2,$$

which implies $m + n \leq 4$.

Case 2. If $4 + (m+n)^2 \neq 2v^2$, then we can write $4 + (m+n)^2 = v^2 t$, where v, t are integers with $v > 0$ and $t > 2$ is square free. Then, by Corollary A.10(2), it requires at least $2(m+n)v\varphi(2t)$ roots of unity to write the right-hand side of equation (A.18). Since $4 + (m+n)^2 \equiv \pm 1 \bmod 3$, we know $4 + (m+n)^2 \notin \{1, 2, 3, 6\}$. By Lemma A.11,

$$v^2 \varphi(2t)^2 \geq \frac{16 v^2 t}{5} \iff v\varphi(2t) \geq 4\sqrt{\frac{4 + (m+n)^2}{5}}.$$

Now, by inequality (A.13), we have the following:

$$\begin{aligned} 16 + 3(m+n)^2 &\geq 2\sum_s (z_s + z_s')^2 \geq 2(m+n)v\varphi(2t) \\ &\geq 8(m+n)\sqrt{\frac{4 + (m+n)^2}{5}}, \end{aligned}$$

which implies that $m + n \leq 4$.

(2) Suppose that $\theta = 1$. Then, dividing equation (A.17) by d and simplifying, we get

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z_s')^2 &= \frac{\Delta + 4(m+n)}{d} + \frac{3}{2}(m+n)^2 \\ &\quad + \frac{r^2}{2}(m-n)^2 + \sum_{j=1}^{4} (y_j + y_j')^2. \end{aligned} \tag{A.19}$$

There are now 2 cases depending on the value of Δ.

Case 1. Suppose that $\Delta = 0$. Then, equation (A.19) becomes

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z_s')^2 &= 2(a+b)\sqrt{4 + (m+n)^2} - \frac{1}{2}(m+n)^2 \\ &\quad + \frac{r^2}{2}(m-n)^2 + \sum_{j=1}^{4} (y_j + y_j')^2. \end{aligned} \tag{A.20}$$

Case 2. Suppose that $\Delta = \pm 2\dim(\mathsf{C}) = \pm(8 + 4(m+n)d)$. Then, equation (A.19) becomes

$$\begin{aligned} -\sum_s \theta_s^2 (z_s + z_s')^2 &= 2(m+n\pm 2)\sqrt{4 + (m+n)^2} - 2(m+n\pm 2)(a+b) \\ &\quad \pm 4(m+n) + \frac{3}{2}(m+n)^2 + \frac{r^2}{2}(m-n)^2 + \sum_{j=1}^{4} (y_j + y_j')^2. \end{aligned} \tag{A.21}$$

In either of the above cases, arguing as in (1) where $\theta = i$, we see that it takes at least

$$\min\left\{8(m+n-2)\sqrt{\frac{4+(m+n)^2}{5}}, 4(m+n-2)\sqrt{\frac{4+(m+n)^2}{2}}\right\}$$
$$\geq \frac{4}{\sqrt{2}}(m+n)(m+n-2)$$

roots of unity to write the right-hand sides of equations (A.20) and (A.21). Now, by inequality (A.13), we see that

$$8+\frac{3}{2}(m+n)^2 \geq \sum_s (z_s+z_s')^2 \geq \frac{4}{\sqrt{2}}(m+n)(m+n-2),$$

which implies $m+n \leq 5$. ∎

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References

[1] H. H. Andersen, Tensor products of quantized tilting modules. *Comm. Math. Phys.* **149** (1992), no. 1, 149–159 Zbl 0760.17004 MR 1182414

[2] H. H. Andersen and J. Paradowski, Fusion categories arising from semisimple Lie algebras. *Comm. Math. Phys.* **169** (1995), no. 3, 563–588 Zbl 0827.17010 MR 1328736

[3] M. Asaeda and U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$. *Comm. Math. Phys.* **202** (1999), no. 1, 1–63 Zbl 1014.46042 MR 1686551

[4] J. W. Barrett and B. W. Westbury, Invariants of piecewise-linear 3-manifolds. *Trans. Amer. Math. Soc.* **348** (1996), no. 10, 3997–4022 Zbl 0865.57013 MR 1357878

[5] S. Bigelow, Skein theory for the ADE planar algebras. *J. Pure Appl. Algebra* **214** (2010), no. 5, 658–666 Zbl 1192.46059 MR 2577673

[6] S. Bigelow and D. Penneys, Principal graph stability and the jellyfish algorithm. *Math. Ann.* **358** (2014), no. 1-2, 1–24 Zbl 1302.46049 MR 3157990

[7] S. Bigelow, E. Peters, S. Morrison, and N. Snyder, Constructing the extended Haagerup planar algebra. *Acta Math.* **209** (2012), no. 1, 29–82 Zbl 1270.46058 MR 2979509

[8] J. Bion-Nadal, An example of a subfactor of the hyperfinite II_1 factor whose principal graph invariant is the Coxeter graph E_6. In *Current topics in operator algebras (Nara, 1990)*, pp. 104–113, World Scientific, River Edge, NJ, 1991 Zbl 0816.46063 MR 1193933

[9] D. Copeland and C. Edie-Michell, Classification of pivotal tensor categories with fusion rules related to $SO(4)$. *J. Algebra* **619** (2023), 323–346 Zbl 1516.18015 MR 4530807

[10] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*. Math. Surveys Monogr. 205, American Mathematical Society, Providence, RI, 2015 Zbl 1365.18001 MR 3242743

[11] P. Etingof, S. Gelaki, and V. Ostrik, Classification of fusion categories of dimension pq. *Int. Math. Res. Not.* (2004), no. 57, 3041–3056 Zbl 1063.18005 MR 2098028

[12] D. E. Evans and T. Gannon, The exoticness and realisability of twisted Haagerup–Izumi modular data. *Comm. Math. Phys.* **307** (2011), no. 2, 463–512 Zbl 1236.46055 MR 2837122

[13] D. E. Evans and T. Gannon, Near-group fusion categories and their doubles. *Adv. Math.* **255** (2014), 586–640 Zbl 1304.18017 MR 3167494

[14] D. E. Evans and Y. Kawahigashi, On Ocneanu's theory of asymptotic inclusions for subfactors, topological quantum field theories and quantum doubles. *Internat. J. Math.* **6** (1995), no. 2, 205–228 Zbl 0820.46059 MR 1316301

[15] D. E. Evans and Y. Kawahigashi, *Quantum symmetries on operator algebras*. Oxford Math. Monogr., Oxford University Press, New York, 1998 Zbl 0924.46054 MR 1642584

[16] J. Fuchs, A. C. Ganchev, K. Szlachányi, and P. Vecsernyés, S_4 symmetry of $6j$ symbols and Frobenius–Schur indicators in rigid monoidal C^* categories. *J. Math. Phys.* **40** (1999), no. 1, 408–426 Zbl 0986.81044 MR 1657800

[17] J. Fuchs and T. Grøsfjeld, Tetrahedral symmetry of $6j$-symbols in fusion categories. 2021, arXiv:2106.16186

[18] S. Gelfand and D. Kazhdan, Examples of tensor categories. *Invent. Math.* **109** (1992), no. 3, 595–617 Zbl 0784.18003 MR 1176207

[19] P. Grossman, S. Morrison, D. Penneys, E. Peters, and N. Snyder, The extended Haagerup fusion categories. *Ann. Sci. Éc. Norm. Supér. (4)* **56** (2023), no. 2, 589–664 Zbl 07714642 MR 4598730

[20] U. Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$. In *Subfactors (Kyuzeso, 1993)*, pp. 1–38, World Scientific, River Edge, NJ, 1994 Zbl 0933.46058 MR 1317352

[21] T. J. Hagge and S.-M. Hong, Some non-braided fusion categories of rank three. *Commun. Contemp. Math.* **11** (2009), no. 4, 615–637 Zbl 1180.18003 MR 2559711

[22] A. Henriques and D. Penneys, Bicommutant categories from fusion categories. *Selecta Math. (N.S.)* **23** (2017), no. 3, 1669–1708 Zbl 1434.18007 MR 3663592

[23] M. Izumi, The structure of sectors associated with Longo–Rehren inclusions. I. General theory. *Comm. Math. Phys.* **213** (2000), no. 1, 127–179 Zbl 1032.46529 MR 1782145

[24] M. Izumi, The structure of sectors associated with Longo–Rehren inclusions. II. Examples. *Rev. Math. Phys.* **13** (2001), no. 5, 603–674 Zbl 1033.46506 MR 1832764

[25] M. Izumi, A Cuntz algebra approach to the classification of near-group categories. In *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday*, pp. 222–343, Proc. Centre Math. Appl. Austral. Nat. Univ. 46, Australian National University, Canberra, 2017 Zbl 1403.46049 MR 3635673

[26] M. Izumi, The classification of 3^n subfactors and related fusion categories. *Quantum Topol.* **9** (2018), no. 3, 473–562 Zbl 1403.46048 MR 3827808

[27] M. Izumi, S. Morrison, and D. Penneys, Quotients of $A_2 * T_2$. *Canad. J. Math.* **68** (2016), no. 5, 999–1022 Zbl 1364.46055 MR 3536926

[28] W. Jagy, Is the Euler phi function bounded below. Mathematics Stack Exchange, https://math.stackexchange.com/q/301856, visited on 6 February 2024

[29] A. Kirillov, Jr. and V. Ostrik, On a q-analogue of the McKay correspondence and the ADE classification of $\mathfrak{sl}_2$ conformal field theories. *Adv. Math.* **171** (2002), no. 2, 183–227 Zbl 1024.17013 MR 1936496

[30] H. K. Larson, Pseudo-unitary non-self-dual fusion categories of rank 4. *J. Algebra* **415** (2014), 184–213 Zbl 1304.18020 MR 3229513

[31] M. A. Levin and X.-G. Wen, String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B* **71** (2005), article no. 045110

[32] C.-H. Lin, M. Levin, and F. J. Burnell, Generalized string-net models: A thorough exposition. *Phys. Rev. B* **103** (2021), article no. 195155

[33] Z. Liu, Composed inclusions of A_3 and A_4 subfactors. *Adv. Math.* **279** (2015), 307–371 Zbl 1330.46061 MR 3345186

[34] Z. Liu, S. Morrison, and D. Penneys, 1-supertransitive subfactors with index at most $6\frac{1}{5}$. *Comm. Math. Phys.* **334** (2015), no. 2, 889–922 Zbl 1330.46062 MR 3306607

[35] Z. Liu, S. Morrison, and D. Penneys, Lifting shadings on symmetrically self-dual subfactor planar algebras. In *Topological phases of matter and quantum computation*, pp. 51–61, Contemp. Math. 747, American Mathematical Society, Providence, RI, 2020 Zbl 1455.46066 MR 4079744

[36] Z. Liu, S. Palcoux, and Y. Ren, Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six. *Lett. Math. Phys.* **112** (2022), no. 3, article no. 54, 37 pp. Zbl 1498.18024 MR 4434141

[37] R. Longo and J. E. Roberts, A theory of dimension. *K-Theory* **11** (1997), no. 2, 103–159 Zbl 0874.18005 MR 1444286

[38] G. Lusztig, *Hecke algebras with unequal parameters*. CRM Monogr. Ser. 18, American Mathematical Society, Providence, RI, 2003 Zbl 1051.20003 MR 1974442

[39] S. Morrison and D. Penneys, 2-supertransitive subfactors at index $3 + \sqrt{5}$. *J. Funct. Anal.* **269** (2015), no. 9, 2845–2870 Zbl 1341.46034 MR 3394622

[40] S. Morrison and K. Walker, The center of the extended Haagerup subfactor has 22 simple objects. *Internat. J. Math.* **28** (2017), no. 1, article no. 1750009, 11 pp. Zbl 1368.46052 MR 3611056

[41] M. Müger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. *J. Pure Appl. Algebra* **180** (2003), no. 1-2, 159–219 Zbl 1033.18003 MR 1966525

[42] S.-H. Ng and P. Schauenburg, Frobenius–Schur indicators and exponents of spherical categories. *Adv. Math.* **211** (2007), no. 1, 34–71 Zbl 1138.16017 MR 2313527

[43] S.-H. Ng and P. Schauenburg, Higher Frobenius–Schur indicators for pivotal categories. In *Hopf algebras and generalizations*, pp. 63–90, Contemp. Math. 441, American Mathematical Society, Providence, RI, 2007 Zbl 1153.18008 MR 2381536

[44] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras. In *Operator algebras and applications, Vol. 2*, pp. 119–172, London Math. Soc. Lecture Note Ser. 136, Cambridge University Press, Cambridge, 1988 Zbl 0696.46048 MR 996454

[45] A. Ocneanu, The classification of subgroups of quantum $\mathrm{SU}(N)$. In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, pp. 133–159, Contemp. Math. 294, American Mathematical Society, Providence, RI, 2002 Zbl 1193.81055 MR 1907188

[46] V. Ostrik, Fusion categories of rank 2. *Math. Res. Lett.* **10** (2003), no. 2-3, 177–183 Zbl 1040.18003 MR 1981895

[47] V. Ostrik, Pre-modular categories of rank 3. *Mosc. Math. J.* **8** (2008), no. 1, 111–118 Zbl 1161.18003 MR 2422269

[48] V. Ostrik, On formal codegrees of fusion categories. *Math. Res. Lett.* **16** (2009), no. 5, 895–901 Zbl 1204.18003 MR 2576705

[49] V. Ostrik, Pivotal fusion categories of rank 3. *Mosc. Math. J.* **15** (2015), no. 2, 373–396 Zbl 1354.18009 MR 3427429

[50] D. Penneys, Unitary dual functors for unitary multitensor categories. *High. Struct.* **4** (2020), no. 2, 22–56 Zbl 1457.18019 MR 4133163

[51] D. Penneys and E. Peters, Calculating two-strand jellyfish relations. *Pacific J. Math.* **277** (2015), no. 2, 463–510 Zbl 1339.46060 MR 3402358

[52] N. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103** (1991), no. 3, 547–597 Zbl 0725.57007 MR 1091619

[53] D. Tambara and S. Yamagami, Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Algebra* **209** (1998), no. 2, 692–707 Zbl 0923.46052 MR 1659954

[54] V. Turaev and H. Wenzl, Quantum invariants of 3-manifolds associated with classical simple Lie algebras. *Internat. J. Math.* **4** (1993), no. 2, 323–358 Zbl 0784.57007 MR 1217386

[55] H. Wenzl, Hecke algebras of type A_n and subfactors. *Invent. Math.* **92** (1988), no. 2, 349–383 Zbl 0663.46055 MR 936086

[56] H. Wenzl, Quantum groups and subfactors of type B, C, and D. *Comm. Math. Phys.* **133** (1990), no. 2, 383–432 Zbl 0744.17021 MR 1090432

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