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Generalizing Pauli spin matrices using cubic lattices

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Abstract. In quantum mechanics, the connection between the operator algebraic realization and the logical models of measurement of state observables has long been an open question. In the approach that is presented here, we introduce a new application of the cubic lattice. We claim that the cubic lattice may be faithfully realized as a subset of the self-adjoint space of a von Neumann algebra. Furthermore, we obtain a unitary representation of the symmetry group of the cubic lattice. In so doing, we re-derive the classic quantum gates and gain a description of how they govern a system of qubits of arbitrary cardinality.

1. Introduction

The cubic lattice [6] has long been thought of as an analogue of the standard Boolean lattice when adapted to the indeterminate setting of quantum logic. With this in mind, we see a substantial amount of literature that has been produced outlining the properties of a potential logic whose states are the cubic lattice in the finite case [7]. On the other hand, [8] introduces an axiomatic description of the cubic lattice without cardinality restrictions. We aim to combine these results. In so doing, we will obtain observables of an infinite quantum system and re-derive a universal set of quantum gates in the sense of the Solovay–Kitaev theorem. The key insight is that the reflection symmetries under consideration here can be represented as a subgroup of the unitary operator, which will be utilized to create a novel operator-based realization of a cubic lattice.

As none of the referenced approaches introduce an analytic structure, it is a natural starting point as we consider the infinite case. Therefore, we embed the cubic lattice into a specifically constructed Hilbert lattice.

Main result 1 (Theorem 2.1.11). *Let H be a Hilbert space constructed as a tensor product of 2-dimensional spaces over an index set I. For the given Hilbert lattice* HL *of H, there exists a cubic lattice* CL *such that* $\mathrm{CL} \subseteq \mathrm{HL}$ *and the atoms of* CL *are projections onto subspaces H forming an orthonormal basis of H.*

As the Hilbert lattice is much larger than our cubic lattice, we consider the minimal von Neumann algebra containing CL as well.

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Main result 2 (Theorem 3.2.8). *The atoms of* $W^*(\{s_i\}_{i\in I})$ *are the atoms of* CL.

We proceed to describe the algebra in our embedding of the cubic lattice, and in so doing, we generalize the standard result that the Pauli matrices span $M_2(\mathbb{C})$.

Main result 3 (Theorem 3.2.11). $B(H) = W^*(\{Us_iU^*\}_{i\in I}, \{s_i\}_{i\in I})$.

As a consequence, we generalize the Pauli matrices to infinite systems of qubits in our choice of matrix units when considered as a representation of $M_2(B)$ as opposed to $M_2(\mathbb{C})$, where $B \cong I_2 \otimes B(H_{I-i})$ for an indexing set I,

$$U_{\Delta_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i s_i U_{\Delta_i} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

1.1. Background and definitions

The standard approach for describing the spin states of n qubits is to consider a tensor product of the form $\bigotimes_{i=1}^n \mathbb{C}^2$, creating a vector space of dimension 2^n. In this setting, each pure state is represented by an orthonormal basis vector.

Definition 1.1.1. Let H be a Hilbert space. We define the lattice of the closed linear subspaces of H to be the *Hilbert lattice*, henceforth referred to as HL. In this context, $u \vee v = \text{span}\{u, v\}$, and $u \wedge v = \text{span}\{u\} \cap \text{span}\{v\}$.

In some literature, the Hilbert lattice is referred to as a "standard lattice." The term is used because this is the standard construction of lattice of projection operators of a Hilbert space; we refer the reader to [13] for an in-depth discussion.

The major issue with the above approach is that the geometry of the state space is not preserved because its dimension is too large. There are many unitary transformations that violate physical meaning, so we need a more restrictive symmetry group. With this in mind, we now move to the cubic lattice.

Definition 1.1.2 ([8]). A *cubic lattice* is a lattice C with 0 and 1 that satisfies the following axioms.

(1) For $x \in L$, there is an order-preserving map $\Delta_x \colon (x) \to (x)$, where (x) denotes the principal ideal generated by (x).

(2) If $0 < a, b < x$, then $a \vee \Delta_x(b) < x$ if and only if $a \wedge b = 0$.

(3) L is complete.

(4) L is atomistic.

(5) L is coatomistic.

In the finite case, the cubic lattice can be thought of as the lattice of the faces of an n-cube. For an arbitrary cardinal, the axiomatic description above relies on antipodal symmetry. We now tie together the geometric notion of the faces of the n-cube to the lattice of signed sets.

Definition 1.1.3. Let $S = \{1, 2, \ldots, n\}$. A *signed set* on S is a pair $x = (A^+, A^-)$ of subsets of S such that $A^+ \cap A^- = \emptyset$. The collection of signed sets, denoted by $L^+(S)$, is a poset with order relation $\leq$ defined by reverse inclusion $x = (A^+, A^-) \leq y = (B^+, B^-)$ if $B^+ \subseteq A^+$ and $B^- \subseteq A^-$. The pair (A^+, A^-) uniquely determines the face F if $A^+ \cap A^- = \emptyset$.

Now that we have considered a poset of the faces of a cube defined as a signed set, we can consider the lattice of signed sets. For some intuition from the finite case, the vertices of the cube are the atoms of the lattice, and its respective signed set (A^+, A^-) partitions the indexing set I. In contrast, the whole cube is represented by $(\emptyset, \emptyset)$. The ordering of the signed set can also be thought of as the inclusion of respective sub-faces of the cube.

Definition 1.1.4. If F, G are faces of I^n such that $F, G \neq \emptyset$, with $F = (A^+, A^-)$ and $G = (B^+, B^-)$, then $G \subseteq F$ if and only if $A^+ \subseteq B^+$ and $A^- \subseteq B^-$. Let $\mathcal{F}(I^n)$ be the set of all faces of I^n ordered by the above notion, so that $\mathcal{F}(I^n)$ forms a complete lattice, where $\vee$ is the union of faces, and $\wedge$ is the intersection of faces. With the addition of a 0 element, $L^+(S)$ becomes a lattice denoted by $L(S)$, where, for $x, y \in L(S)$, one has $x \vee y = (A^+ \cap B^+, A^- \cap B^-) \in L(S)$ and $x \wedge y = (A^+ \cup B^+, A^- \cup B^-) \in L(S)$ if $B^+ \cap A^- = \emptyset = B^- \cap A^+$, or $x \wedge y = 0 \in L(S)$ otherwise.

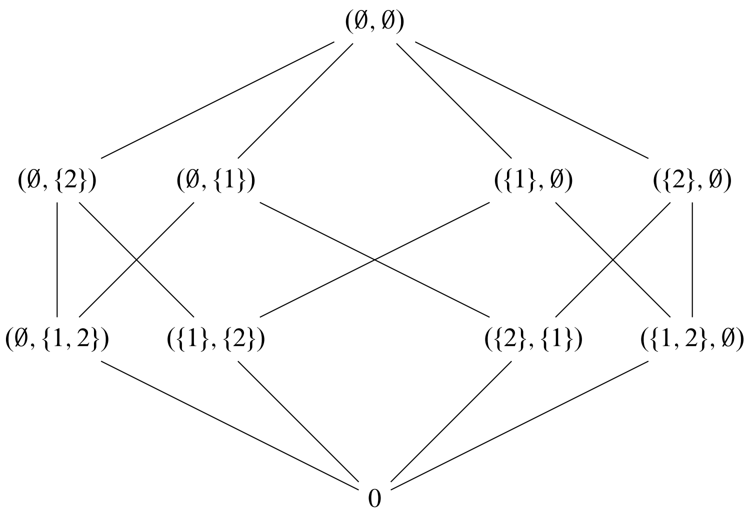
In addition to $\wedge$ and $\vee$, a cubic lattice has an additional operation.

Definition 1.1.5. Every cubic lattice $L(S)$ admits a partially-defined operation

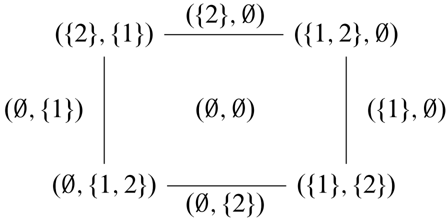
$$\Delta\colon L(S) \times L(S) \to L(S)$$

defined by $\Delta(x, 0) = 0$, and if $0 < x = (A^+, A^-)$, $0 < y = (B^+, B^-)$, $y \leq x$, then $\Delta(x, y) = (A^+ \cup (B^- - A^-), A^- \cup (B^+ - A^+))$.

Example 1.1.6. The face lattice of the 2-cube is given by



The signed sets of the 2-cube are given by



We have given a very terse description of cubic algebras, and we will now move towards creating a faithful realization of the cubic algebra as an operator algebra. A large amount of technology must be developed, as we do not yet even have a linear space of operators with which to begin.

2. Embeddings of the cubic lattice and octahedral lattice

Now that we have introduced the basic structures, we can build the necessary embedding to demonstrate that cubic lattices have a realization as a von Neumann algebra. In addition, we discuss the algebraic structure of the Hilbert lattice and compare it to the poset structure of the cubic lattice. Lastly, we compare the dual spaces with respect to both space categories. We show that there is a reasonable, direct relationship between the dual of the poset and the dual of the analytic structure.

2.1. Cubic lattice as a subset of a Hilbert lattice

We adapt the following definitions and proposition from [10] to our notation.

Proposition 2.1.1 ([10]). *The Hilbert lattice is an atomic, (completely) atomistic, complete, orthomodular lattice.*

For the following theorem, we will be constructing a Hilbert space from an infinite tensor product. We do so in an established but non-standard way. We outline the necessary definitions for expository purposes and use the results from [14]. Unless otherwise stated, when we refer to a Hilbert space formed by infinite tensor products, we mean the following construction, not the standard construction.

For the following, I is an index set of not necessarily countable cardinality, H_α is a finite-dimensional Hilbert space for all $\alpha \in I$, and the norm on $f_\alpha \in H_\alpha$ is the norm of the Hilbert space.

Definition 2.1.2 ([14]). $\prod_{\alpha \in I} z_\alpha$, $z_\alpha \in \mathbb{C}$, $\alpha \in I$, is *convergent,* and a is its *respective value,* if there exists, for every $\delta > 0$, a finite set $I_0 = I_0(\delta) \subseteq I$, such that, for every finite set $J = \{\alpha_1, \ldots, \alpha_n\}$ (mutually distinct α_i) with $I_0 \subset J \subset I$,

$$|z_{\alpha_1} \cdot \cdots \cdot z_{\alpha_n} - a| \leq \delta.$$

Definition 2.1.3. $\prod_{\alpha\in I} z_\alpha$ is *quasi-convergent* if $\prod_{\alpha\in I} |z_\alpha|$ is convergent. It value is

- the value of $\prod_{\alpha\in I} z_\alpha$ if it is convergent,
- 0 otherwise.

Now that we have a looser notion of convergence for infinite products, we adapt these definitions to functions in a normed space.

Definition 2.1.4. A sequence f_α, $\alpha \in I$, is a *C-sequence* if $f_\alpha \in H_\alpha$ for all $\alpha \in I$, and $\prod_{\alpha\in I} \|f_\alpha\|$ converges.

As we have an inner product for each H_α, we can consider the infinite product of the respective inner products.

Lemma 2.1.5 ([14]). *If f_α, $\alpha \in I$, and g_α, $\alpha \in I$, are two C-sequences, then $\prod_\alpha \langle f_\alpha, g_\alpha\rangle$ is quasi-convergent.*

Definition 2.1.6. Let $\Phi(f_\alpha; \alpha \in I)$ be the set of functionals on the product $\prod_{\alpha\in I} H_\alpha$ which is conjugate linear in each $f_\alpha \in I$ separately over C-sequences. The set of all such Φ for any C-sequence will be denoted by $\prod \bigodot_{\alpha\in I} H_\alpha$. We note that $\prod \bigodot_{\alpha\in I} H_\alpha$ is a linear space, but it is not an inner product space.

Although each functional f_α is conjugate linear for its respective H_α, we do not have an inner product on the entire space. We can form a conjugate linear inner product space by considering a fixed C-sequence.

Definition 2.1.7. Given a C-sequence f_α^0, $\alpha \in I$, we form the functional $\Phi(f_\alpha; \alpha) = \prod_{\alpha\in I}(f_\alpha^0, f_\alpha)$, where f_α, $\alpha \in I$, runs over all C-sequences. Denote such a functional by $\prod \bigotimes_{\alpha\in I} f_\alpha^0$.

We now turn the inner product space into a linear space.

Definition 2.1.8. Consider the set of all finite linear aggregates of the above elements:

$$\Phi = \sum_{v=1}^{p} \prod \bigotimes_{\alpha\in I} f_{\alpha,v}^0,$$

where $p = 0, 1, \ldots, p$ and $f_{\alpha,v}^0, \alpha \in I$, is a C-sequence for each $v = 1, 2, \ldots, p$. Denote the set of these Φ by $\prod' \bigotimes_{\alpha\in I} H_\alpha$. For $\Phi = \sum_{v=1}^{p} \prod \bigotimes_{\alpha\in I} f_{\alpha,v}^0$, $\Psi = \sum_{\mu=1}^{q} \prod \bigotimes_{\alpha\in I} g_{\alpha,\mu}^0 \in \prod' \bigotimes_{\alpha\in I} H_\alpha$, we define the inner product by

$$\langle \Phi, \Psi\rangle = \sum_{v=1}^{p} \sum_{\mu=1}^{q} \prod_{\alpha\in I} \langle f_{\alpha,v}^0, g_{\alpha,\mu}^0\rangle.$$

The Hilbert space of [14] has an inner product defined by a specific decomposition. For completeness, we highlight that the inner product is well defined.

Lemma 2.1.9 ([14]). *Let $\Phi, \Psi \in \prod' \bigotimes_{\alpha\in I} H_\alpha$. The value of $\langle \Phi, \Psi\rangle$ is independent of the choice of their respective decompositions.*

Lastly, [14] creates a Hilbert space by defining the completion with respect to our notion of convergence.

Definition 2.1.10. Consider the functions $\Phi \in \prod \bigotimes_{\alpha \in I} H_\alpha$ for which there exists a sequence

$$\Phi_1, \Phi_2, \ldots \in {\prod}' \bigotimes_{\alpha \in I} H_\alpha$$

such that

(1) $\Phi(f_\alpha; \alpha \in I) = \lim_{r \to \infty} \Phi_r(f_\alpha; \alpha \in I)$ for all C-sequences $f_\alpha, \alpha \in I$,

(2) $\lim_{r,s \to \infty} \|\Phi_r - \Phi_s\| = 0$

The set they form is the complete direct product of $H_\alpha, \alpha \in I$, to be denoted by $\prod \bigotimes_{\alpha \in I} H_\alpha$. Note that $\prod' \bigotimes_{\alpha \in I} H_\alpha \subseteq \prod \bigotimes_{\alpha \in I} H_\alpha \subseteq \prod \bigodot_{\alpha \in I} H_\alpha$.

For our application, the convergence criterion of Definition 2.1.2 is acceptable. We will only be concerned with forming the tensors of elementary basis elements of the respective H_α, so all of our elements are functionals derived from C-sequences as in Definition 2.1.7. We can then consider their span in the natural way.

Lastly, we want to highlight that the constructed Hilbert space is separable only if each H_α is finite dimensional and $|I|$ is finite. Therefore, the Hilbert spaces we are considering will in general be non-separable.

Theorem 2.1.11. *Let H be a Hilbert space constructed as a tensor product of 2-dimensional spaces over an index set I. For the given Hilbert lattice HL of H, there exists a cubic lattice CL such that CL $\subseteq$ HL, and the atoms of CL are projections onto subspaces of H forming an orthonormal basis of H.*

Proof. We begin with the standard construction of a basis over a tensor product of index I. Let e_i^+, e_i^- represent the 2 basis vectors for $i \in I$.

Now, each elementary tensor is a C-sequence, since $\|e_i\| = 1$ for all $i \in I$; so we have a linear functional of the form in H in Definition 2.1.7, and this linear functional can be represented by its respective projection operators. As these are projections onto 1-dimensional subspaces, they are atoms in HL, and in the cone $B(H)^+$.

For each atomic elementary tensor described above, we use the notation $v = \{A^+, A^-\}$, where $A^+ = \{i \in I : v_i = e_i^+\}$ and $A^- = \{i \in I : v_i = e_i^-\}$. By the construction of v, we have that $A^+ \cap A^- = \emptyset$ and $A^+ \cup A^- = I$. Now we observe that all such v form the atoms of a signed set over the indexing set I.

We define CL $= L(S_I)$, the lattice of signed sets generated by the closure of the above atoms under the operations of meet and join from the definition of cubic lattices. Recall, by Definition 1.1.5, that $\Delta\colon L(S) \times L(S) \to L(S)$ can be defined on any signed set.

As we have a description of the atoms of the cubic lattice in $\mathcal{M}^+$, we need to show that the atoms are closed under $\vee$. Consider $a, b \in$ CL $\cap$ HL, where $a = \{A^+, A^-\}$ and $b = \{B^+, B^-\}$. Then, $a \vee_{\mathrm{CL}} b = \{A^+ \cap B^+, A^- \cap B^-\}$. We now have that $a \vee_{\mathrm{CL}} b$ is the projection P_V onto the subspace $V = \bigotimes_{i \in I} V_i$, where $V_i = e_i^+$ for $i \in A^+ \cap B^+$,

$V_i = e_i^-$ for $i \in A^- \cap B^-$, and $V_i = \text{span}\{e_i^+, e_i^-\}$ otherwise, so that $a \vee_{\text{CL}} b \in \text{HL}$ and $P_V \in \mathcal{M}^+$. Therefore, $\text{CL} \subseteq \text{HL}$. In addition, as any element of CL is a join of its atoms by atomisticity, and $0 \in \text{HL}$ trivially, the result follows.

The atoms of CL form an orthonormal system in H. For any pair of distinct atoms $a, b \in \text{CL}$, there exists $i \in I$ such that $a_i \neq b_i$, so $\langle a_i, b_i \rangle_{H_\alpha} = 0$, which implies that $\langle a, b \rangle_H = 0$. Furthermore, these vectors span $\prod' \bigotimes_{\alpha \in I} H_\alpha$, and therefore are dense in H. ∎

Remark 2.1.12. The Hilbert lattice is not a cubic lattice. Suppose it were. Then, there exists a signed set realization of HL, $L(S)$, see [8]. Let $r(\cdot)$ denote the rank of a subspace. Consider the join of two linearly independent atoms $a = \{A^+, A^-\}$ and $b = \{B^+, B^-\}$ such that $|\{A^+ - B^+\}| > 1$, so $r(a \vee_C b) = |\{(A^+ \cap B^+) \cup (A^- \cap B^-)\}| > 2$. Then, $2 = r(a \vee_H b) < r((A^+ \cap B^+) \cup (A^- \cap B^-)) = r(a \vee_C b)$.

We now discuss relations of the distinct lattice structures of the cubic lattice and Hilbert lattice.

Definition 2.1.13. We write $\text{CL} \subseteq B(H)$ and say that H is *constructed as in Theorem* 2.1.11 to mean the set of orthogonal projections onto their respective closed subspaces of CL are in $B(H)$. We will use the notation $a \in \text{CL}$ and $p_a \in B(H)$.

It is worth discussing why we chose to construct a non-separable Hilbert space. The standard approach to model an n-qubit system is to embed these qubits into a 2^n-dimensional space. In order to keep our later results consistent with this property, we are forced for an $|I|$-qubit system to embed into an $2^{|I|}$-dimensional space, which again is countable if and only if $|I|$ is finite.

We now explore how some of the operations of the cubic lattice and Hilbert lattice relate.

Corollary 2.1.14. *The action of* $^\perp$ *on* HL *on the coatoms of* CL *is a symmetry that coincides element-wise with the unitary symmetry associated with* Δ.

Proof. The result follows as, for all $c \in \text{CL}$, $p_c^\perp = 1 - p_c = p_{\Delta(c)} = U_\Delta p_c U_\Delta$. ∎

Definition 2.1.15. Let $V = \bigotimes_{i \in I} V_i$ for some index set I over vector spaces $\{V_i\}_{i \in I}$. A *generalized simple tensor of* V is a subspace of V of the form $\bigotimes_{i \in I} U_i$, where U_i is a subspace of V_i.

Corollary 2.1.16. *The set* $\text{CL} \subseteq B(H)$ *is exactly the set of the operators represented by generalized simple tensors in the orthonormal basis.*

Lastly, although the join operation differs on the cubic lattice and the Hilbert lattice, the meet operation is the same.

Theorem 2.1.17. *For a proper principal lattice filter of the cubic lattice, $F \subseteq \mathrm{CL} \subseteq \mathrm{HL}$, $\wedge_H : F \times F \to \mathrm{HL} = \wedge_C : F \times F \to F$. Equivalently, the join of an ideal of* OL *agrees with the meet of the Hilbert lattice.*

Proof. Let $a, b \in F$. By definition, we can write a and b as the joins of atoms that are members of the orthonormal basis constructed as in Theorem 2.1.11, so we can write $a \wedge_H b$ in the same orthonormal basis as well. Therefore, we have the same relevant set of atoms for both HL and CL, and we reduce to this case implicitly for the remainder of the proof.

If α is an atom of $\mathrm{CL} \subseteq \mathrm{HL}$ such that $\alpha \leq a$ and $\alpha \leq b$, then $\alpha \leq a \wedge_C b$ and $\alpha \leq a \wedge_H b$. In addition, these are the only atoms in the commutative Boolean sub-lattice of HL that are less than or equal to $a \wedge_C b$ or $a \wedge_H b$. By atomisticity of the cubic lattice and the Boolean sub-lattice of the Hilbert lattice, $a \wedge_C b = \bigvee_C \{\alpha : \alpha \leq a \text{ and } \alpha \leq b\}$ and $a \wedge_H b = \bigvee_H \{\alpha : \alpha \leq a \text{ and } \alpha \leq b\}$.

As the ordering of CL is inherited from HL, $\alpha \vee_H \beta = \inf\{c \in \mathrm{HL} : c \geq \alpha, c \geq \beta\} \leq \inf\{c \in \mathrm{CL} : c \geq \alpha, c \geq \beta\}$. Therefore, $a \wedge_H b \leq a \wedge_C b$. Now, by reversing the above argument, $\alpha \wedge_H \beta = \sup\{c \in \mathrm{HL} : c \leq \alpha, c \leq \beta\} \geq \sup\{c \in \mathrm{CL} : c \leq \alpha, c \leq \beta\}$, and $a \wedge_H b \geq a \wedge_C b$. ∎

2.2. The lattice dual as an algebra anti-isomorphism

In order to expand our discussion of CL and HL as sets, we would benefit from compactness. Therefore, we consider the pre-dual space $\mathcal{M}_*$ and the dual space $\mathcal{M}^*$ of $\mathcal{M}$.

Definition 2.2.1 ([1, Definition 3.24]). Let $\sigma, \omega \in \mathcal{M}_*^+$, where $\mathcal{M}$ is a von Neuemann algebra. We say that σ is *absolutely continuous with respect to ω,* written as $\sigma \ll \omega$, if $\sigma(q) = 0$ for all projections $q \in M$ such that $\omega(q) = 0$.

Theorem 2.2.2 ([1, Theorem 3.27]). *If $\mathcal{M}$ is a von Neumann algebra and $\omega \in M_*^+$, then the norm closure of the face generated by $\omega \in M_*^+$ consists of all $\sigma \in M_*^+$ such that $\sigma \ll \omega$.*

Proposition 2.2.3. *For a base norm space X with generating hyperplane K, there is an order isomorphism from the non-zero faces of X to the faces of K.*

This is a standard fact, where the morphism is defined by a face F in X induces a face $F \cap K$ in K. One can also see this as a map from $0 \neq x \in X$ to $x/\|x\|$, assuming X is a normed space and observing the induced facial structure.

Proposition 2.2.4 ([1]). *The self-adjoint part M_*^+ of the pre-dual of a von Neumann algebra $\mathcal{M}$ is a base norm space whose distinguished base is the normal state space K_* of $\mathcal{M}$.*

Proposition 2.2.5. *If F is a face in $\mathcal{M}_*^+$, then there is an order isomorphism to faces in the normal state space K_*.*

Proof. A direct result of Proposition 2.2.4 and Proposition 2.2.3. ■

We use a direct application of [1] with slight abbreviation to avoid introducing notation that we will not use. For the full statement see the reference.

Proposition 2.2.6 ([1, Theorem 3.35]). *Let $\mathcal{M}$ be a von Neumann algebra with normal state space K_*, and denote by $\mathcal{F}$ the set of all norm closed faces of K_*, by $\mathcal{P}$ the set of all projections in $\mathcal{M}$, and by $\mathcal{J}$ the set of all σ-weakly closed left ideals in $\mathcal{M}$, each equipped with the natural ordering. Then there is an order-preserving bijection $\Phi\colon p \to F$ from $\mathcal{P}$ to $\mathcal{F}$, and an order reversing bijection $\Psi\colon p \to J$ from $\mathcal{P}$ to $\mathcal{J}$, and hence also an order reversing bijection $\Theta = \Psi \circ \Phi^{-1}$ from $\mathcal{F}$ to $\mathcal{J}$. The maps Φ, Ψ, and Θ and the final inverse are explicitly given by the equations*

(i) $F = \{\sigma \in K_* | \sigma(p) = 1\}$,

(ii) $J = \{a \in \mathcal{M} | ap = 0\}$,

(iii) $J = \{a \in \mathcal{M} | \sigma(a^*a) = 0 \text{ for all } \sigma \in F\}$, *where*

$$F = \{\sigma \in K_* | \sigma(a^*a) = 0 \text{ for all } a \in J\}.$$

We now want to show how our geometrically inspired Δ can be used somewhat synonymously with $\perp$ even across the dual space. We first have to embed the lattice pre-dual, the octahedron, into the pre-dual of our von Neumann algebra $B(H)$, where H is constructed as in Theorem 2.1.11. Note that the lattice dual is reflexive, so the dual and pre-dual are equivalent in this context. We first introduce some simplifying notation.

Definition 2.2.7. In the higher-dimensional embedding, we lose the $+1, -1$ directionality to gain orthogonality. Therefore, each $i \in S_+$ and $j \in S^-$ corresponds to a mutually linearly-independent linear functional for a total of $2|S|$ linear functionals. As an example, let $j \in A^+$, and $f_i \in \{e_i^+, e_i^-\}$, and $p_{\bigotimes_{i \in S} f_i}$ for all $i \in S$ be the projection onto $\bigotimes_{i \in S} f_i$; then

$$\varepsilon_j(p_{\bigotimes_{i \in S} f_i}) = \begin{cases} 1 & \text{if } f_j = e_j^+, \\ 0 & \text{if } f_j = e_j^-, \end{cases}$$

and extend linearly.

Definition 2.2.8. Define $\phi\colon \mathrm{CL} \to \mathcal{F}$ by $\phi((A^+, A^-))$ as the norm closed convex hull of the linear functionals $\{\varepsilon_i : i \in A^+\} \cup \{\varepsilon_j : j \in A^-\}$.

As we will show, the above ϕ will be the analytic equivalent of our ϕ defined as a lattice anti-isomorphism, and it will agree on the corresponding lattices, so the reuse of notation is intentional.

Definition 2.2.9. We define a unitary operator denoted U_Δ by linearly extending its action on the basis of H, and letting U_Δ act by inner automorphism on orthogonal projections of subspaces of HL.

As will be relevant later, $\Delta(a, b)$ is linearly extendable in this representation of CL if and only if $a = 1$. We now embed the octahedron into the pre-dual.

Lemma 2.2.10. *Let* CL $\subseteq$ HL *constructed as in Theorem* 2.1.11 *with the corresponding projections in* $B(H)$. *Then the restriction of the anti-isomorphism* $\Theta^{-1}\colon \mathcal{J} \to \mathcal{F}$ *of Proposition* 2.2.6 *to* CL *is equal to* $\phi \circ U_\Delta\colon \mathrm{CL} \to \mathcal{F}$.

Proof. Let J be the left ideal generated by a projection operator $p_{(A^+,A^-)}$ onto a subspace of $(A^+, A^-) \in \mathrm{CL} \subseteq \mathrm{HL}$. Note that $U_\Delta p_{(A^+,A^-)} U_\Delta = p_{\Delta(A^+,A^-)}$. For simplicity, we will assign $(B^+, B^-) = \Delta(A^+, A^-) = (A^-, A^+)$.

We claim that the face $\phi(U_\Delta p U_\Delta)$ in the normal state space

$$F_{U_\Delta(p)} = \overline{\{\sigma \in K_* : \sigma \ll \phi(U_\Delta p U_\Delta)\}}^{\|\cdot\|}$$

is equal to $\Theta^{-1}(J)$. Firstly, for any state $\omega \in \phi(U_\Delta p U_\Delta)$, we have $\omega(p) = 0$, as $\mathrm{supp}(\omega)$ is orthogonal to p. Therefore, $F_{U_\Delta(p)} \subseteq \Theta^{-1}(p)$.

Suppose that $F_{U_\Delta(p)} \subset \Theta^{-1}(p)$ and that there exists a state $\gamma \in \Theta^{-1}(J)$ such that γ is not absolutely continuous with respect to $F_{U_{\Delta(p)}}$. In particular, γ is not absolutely continuous with respect to a subset of $F_{U_{\Delta(p)}}$, namely the extreme points of $F_{U_{\Delta(p)}}$, consisting of $\{\varepsilon_i : i \in B^+\} \cup \{\varepsilon_j : j \in B^-\}$. So, there exists some projection $a \in \mathcal{M}$ such that $0 \neq a \subseteq (\bigcap_{i\in B^+} \mathrm{Ker}(\varepsilon_i)) \cap (\bigcap_{j\in B^-} \mathrm{Ker}(\varepsilon_j))$ and $\gamma(a) \neq 0$. By construction, any projection in $(\bigcap_{i\in B^+} \mathrm{Ker}(\varepsilon_i)) \cap (\bigcap_{j\in B^-} \mathrm{Ker}(\varepsilon_j))$ is less than or equal to p, so $\gamma(p) \neq 0$, which is a contradiction. ∎

Just as the atoms of the cubic lattice corresponded to atoms of the Hilbert lattice, the coatoms of the octahedral lattice, which are the image of the atoms of CL under the dual map, correspond to coatoms of the lattice of faces of the pre-dual.

Theorem 2.2.11. *For the normal state space* K_* *of* $B(H)$, *where* H *is constructed as in Theorem* 2.1.11, *there exists an* OL *such that the coatoms of* OL *are contained in the coatoms of* K_*.

Proof. By Theorem 2.1.11, the atoms of the cubic lattice form an orthonormal basis of H and the map $\phi\colon \mathrm{CL} \to OL$ as defined Lemma 2.2.10 is an order reversing map. As ϕ is the restriction of the map in M_*^+ whose facial structure is equivalent to K_*, we have our result. ∎

Example 2.2.12. The above results do not hold for the coatoms of CL. For a 2-cube, we see that the coatoms are rank-2 projection operators onto a given half space while the coatoms of the respective Hilbert lattice must be rank-3 operators.

3. The necessity of the cubic lattice

Throughout this document, we have created a sufficient structure to characterize the algebraic relations of an $|I|$-qubit system when considered in an analytic space. However, we now raise the question: what other structures suffice? Is there perhaps an entire set of such objects and what is the underlying characterizing feature? We now demonstrate that the symmetries required for an $|I|$-qubit system require a cubic lattice structure. Furthermore, we show that these algebraic relations are in fact measurable in the sense of [15]. We also show which von Neumann algebras contain a cubic lattice of a given cardinality up to $*$-isomorphism.

We will think of the commutant of U_Δ as being, in some sense, generated by the automorphism group of the lattice of signed sets. We will discuss this more in the following section.

3.1. The symmetry group of the cubic lattice and quantum relations

In the finite case, the automorphism (symmetry) group of the cubic lattice is the Coxeter group B_n, otherwise known as the *hyperoctahedral group* O_n.

Definition 3.1.1. Let $\text{Per}(C)$ be the group of permutations of coatoms of CL, $\text{Per}_\Delta(C)$ be the centralizer of Δ in $\text{Per}(C)$, and $L(S)$ be the lattice of signed sets over S.

Theorem 3.1.2 ([12]). *For a cubic lattice of cardinality* $\aleph$, $L(S)$, *one has* $\text{Aut}(L(S)) \cong \text{Per}_\Delta(C) \cong \mathbb{Z}_2 \wr S_\aleph$, *where* $\wr$ *denotes the unrestricted wreath product.*

In [12], their choice of embedding space is a Banach space of dimension equal to the indexing set S, as opposed to our exponentially larger Hilbert embedding. We now generalize these arguments to von Neumann algebras over the Hilbert space constructed as in Theorem 2.1.11.

Proposition 3.1.3. *The* C^**-algebra generated by* U_Δ *is a von Neumann algebra.*

Proof. Since U_Δ is a self-adjoint unitary operator, we have that $C^*(U_\Delta)$ is a finite-dimensional algebra, and so equal to its WOT closure. ∎

Lemma 3.1.4. *Let the prime sign* $'$ *denote the commutant. Then* $W^*(U_\Delta) = Z(W^*(U_\Delta)')$.

Proof. As $W^*(U_\Delta)$ is an abelian unital W^* algebra, $W^*(U_\Delta) \subseteq Z(W^*(U_\Delta)')$. Since $W^*(U_\Delta)$ is also a von Neumann algebra, we have that $Z(W^*(U_\Delta)') \subseteq W^*(U_\Delta)'' = W^*(U_\Delta)$, where the last equality follows from the double commutant theorem. ∎

We now have a large amount of insight into the structure $W^*(U_\Delta)$. There are three views to consider it: firstly, as a finite-dimensional abelian von Neumann algebra, $W^*(U_\Delta)$ is isomorphic to an $l^\infty(\{1, 2, \ldots, n\})$ for $n \in \mathbb{N}$; on the other hand, we know that $W^*(U_\Delta)$ is a unital commutative Banach algebra, so $W^*(U_\Delta)$ is also isomorphic to $C(K)$; and, lastly, $W^*(U_\Delta)$ is isomorphic to $p(U_\Delta)$, which, as Δ is an involution, is a two-dimensional $\mathbb{C}$ vector space. Of course, this all ultimately follows from the general principle that continuous maps over a finite space are a vacuous concept and devolve to maps from a finite set to the complex numbers. We describe this in general in the following statement.

Proposition 3.1.5. *Let $A \in B(H)$ be a normal operator such that $A^n = I$ for some $n \in \mathbb{N}$. Then, $C^*(A)$ is a von Neumann algebra and equals the center of its commutant.*

Before further discussing $W^*(U_\Delta)'$, we introduce some theory about the automorphism groups of the Hilbert lattice and the cubic lattice. To begin, the embedding of $L(S)$ in $H(L)$ constructed as in Theorem 2.1.11 is minimal in a fairly strict sense.

Theorem 3.1.6. *Let $f: L(S) \to \mathrm{HL}$, where the atoms of $L(S)$ are contained in the atoms of* HL, *and f is an injective order morphism. Then, there exists a unique injective order morphism $\psi: \widetilde{H}(L) \to \mathrm{HL}$, where $\widetilde{H}(L)$ is the embedding j of Theorem 2.1.11 such that $\psi \circ j = f$.*

Proof. First, we show existence of such an f. We only need to use that the Hilbert lattice is atomic and complete by Proposition 2.1.1. Therefore, if we have two Hilbert lattices, we have an injective order morphism if we have an injective mapping of orthonormal bases to the Hilbert spaces of the respective Hilbert lattices. Then, we have $\psi = f \circ j^{-1}$. Now we see that the uniqueness follows as j is a bijection between the orthonormal basis of $\widetilde{H}(L)$ and the atoms of $L(S)$, so if there exists another map ρ satisfying our criteria, then $\rho \circ j = f$ implies $\rho = f \circ j^{-1} = \psi$. ∎

In order to study a representation of the automorphisms of the cubic lattice, we first look at representing automorphisms of the Hilbert lattice.

Definition 3.1.7. A *conjugate linear operator* is a linear operator except for the fact that scalar multiplication is treated as conjugate scalar multiplication.

Definition 3.1.8 ([1]). Let H be a Hilbert space and consider $\Phi: B(H) \to B(H)$. Φ is said to be *implemented by a (conjugate) unitary* if there is a (conjugate) unitary map $U: H \to H$ such that $\Phi a = UaU^*$ for all $a \in B(H)$.

Lemma 3.1.9. *Let $g \in \mathrm{Aut}(H(L))$. There exists a unitary or conjugate linear unitary operator $U_g: H \to H$ such that g is implemented by U_g.*

Proof. If $g \in \mathrm{Aut}(H(L))$, then g is a unital order automorphism, so, by [1, Proposition 4.19], g is a Jordan automorphism. If g is a Jordan automorphism, then g is either a $*$-isomorphism or $*$-anti-isomorphism by [1, Proposition 5.69]. If g is a $*$-isomorphism, then g is implemented by a unitary, and if g is a $*$-anti-isomorphism, then g is implemented by a conjugate unitary by [1, Theorem 4.27]. ∎

Definition 3.1.10 ([1]). Let $A \in \mathcal{M}$ be invertible, then $\mathrm{Ad}_A: \mathcal{M} \to \mathcal{M}$ is defined by

$$\mathrm{Ad}_A(\cdot) = A(\cdot)A^{-1}.$$

Equivalently, one can view Ad_A as the inner automorphism induced by A on $\mathcal{M}$.

Definition 3.1.11. We say that the actions of two unitary operators *commute in a von Neumann algebra* $\mathcal{M}$ if their actions by inner automorphism commute.

Theorem 3.1.12. *Let* $g \in \mathrm{Aut}(H(L))$. *Then* $\mathrm{Ad}_g \in \mathrm{Aut}(L(S))$ *if and only if the action of* g *commutes with the action of* U_Δ *on* $W^*(L(S))$, *where* H *is constructed as in Theorem* 2.1.11.

Proof. By Lemma 3.1.9, we know that g can be implemented by a unitary or conjugate unitary operator U. Without loss of generality, we assume that U is a unitary operator, as this affects the associative multiplication consistent with the Jordan algebra of the Hilbert lattice, but it does not affect the action as an order automorphism.

Assume that the action U commutes with action of U_Δ on $L(S)$. It is sufficient to show that $\mathrm{Ad}_g \in \mathrm{Per}_\Delta(C)$. Both c and $\Delta(c)$ are coatoms in $L(S)$; therefore,

$$\Delta(g(p_c)) = U_\Delta U_g p_c U_g^* U_\Delta = U_g U_\Delta p_c U_\Delta U_g^*. = g(\Delta(p_c))$$

We have that g maps coatoms to coatoms in some lattice isomorphic to our original $L(S)$; in particular, it is an order isomorphism, so $\mathrm{Ad}_g \in \mathrm{Aut}(L(S))$.

Now, for the converse, if the inner automorphisms do not commute on $L(S)$, then there exists $c \in C$ such that $\Delta(g(p_c)) \neq g(\Delta(p_c))$. As $g \in \mathrm{Aut}(L(S))$, then $g(c)^\perp = \Delta(g(c))$ by Theorem 2.1.14. Therefore, $g(\Delta(p_c)) \neq g(p_c)^\perp$, but, by linearity,

$$g(\Delta(p_c)) + g(p_c) = g(\Delta(p_c) + p_c) = g(I) = I,$$

which leads to a contradiction. ∎

One can observe that there are many more unitary transformations, and, therefore, automorphims of the Hilbert lattice; therefore, there are many automorphisms of the cubic lattice. Now that we know that the elements of $\mathrm{Aut}(H(L))$ are either unitary or conjugate unitary transformations, we deduce exactly which unitary operators are automorphisms of the cubic lattice.

We see that we have a choice of equivalence class when we represent $\mathrm{Aut}(L(S))$ by its action on $L(S)$, as there are automorphisms acting as the identity on $L(S)$ that to do not act as the identity on HL. Namely, the abelian von Neumann algebra $W^*(\{p_c\}_{c\in C})$ – i.e., the symmetries associated $L(S)$ – is an example.

Due to this ambiguity, we choose to define a group representation of $\mathrm{Aut}(L(S))$ up to group isomorphism acting on H, rather than inner automorphisms acting on HL. From another perspective, we have, for each $U \in \mathrm{Aut}(L(S))$, the action $Uh_1 \mapsto h_2$ on the lattice of orthogonal projections HL, and we are instead considering the action $Uh_1 \mapsto h_2$ where $h_1, h_2 \in H$. One can see that any action in $\mathrm{Aut}(L(S))$ can be induced by the action on H and vice versa, but we have removed the ambiguity of the representation. We formalize this below.

Lemma 3.1.13. *There exists a unitary representation* $\rho\colon \mathrm{Aut}(L(S)) \to B(H)$ *such that* $U_g \in W^*(U_\Delta)'$ *for all* $g \in \mathrm{Aut}(L(S))$.

Proof. When considered as an automorphism group acting on the orthonormal basis constructed as in Theorem 2.1.11, we have a group representation of $\mathrm{Aut}(L(S)))$ contained in the permutation group over H. Therefore, we conclude that the group representation is a unitary representation.

Now we apply $\mathrm{Aut}(L(S)) \cong \mathrm{Per}_\Delta(C)$, the permutations of the coatoms that commute with Δ, to see that $\Delta \subseteq Z(\mathrm{Aut}(L(S)))$. As the commutativity of the group implies the commutativity of its representation, we conclude. ■

We want to decompose $W(U_\Delta)'$ into automorphisms of the cubic lattice and projections onto the cubic lattice. We first prove some facts about the maximality of an abelian algebra characterized by its projections.

Proposition 3.1.14 ([4]). *Every complete Boolean algebra, $\mathcal{B}$, corresponds to a unique Stonean completion $\mathcal{A}$ whose set of projections is equal to $\mathcal{B}$.*

Proposition 3.1.15. *If $\mathcal{A} \leq B(H)$ is an atomic abelian von Neumann algebra whose lattice of projections form an atomic complete Boolean algebra, which is maximal in the Hilbert lattice of H, then $\mathcal{A}$ is a maximal abelian algebra.*

Proof. From Proposition 3.1.14, we know that the Boolean lattice of projections correspond to abelian subalgebras of a von Neumann algebra. Let A be the atoms of $\mathcal{A}$, and $p \in \mathcal{A}' - \mathcal{A}$ be a projection. Furthermore, we can assume that p is orthogonal to every $a \in A$, otherwise let $p = p - (\bigvee_{a\in A} p \wedge a)$. Then p commutes the atoms $a \in \mathcal{A}$, so $a \geq a \wedge p \geq= ap = 0$, as a is an atom. The Hilbert lattice is atomic by Proposition 2.1.1, so let $b \leq p$ be an atom and, by the above, $a \wedge b \leq a \wedge p = 0$. Therefore, the lattice containing $\mathcal{B}$ and b is an atomic complete Boolean lattice strictly containing $\mathcal{B}$, contradicting the maximality. Therefore, $\mathcal{A}$ and $\mathcal{A}'$ contain the same projections and must be equal. The result follows as an abelian von Neumann algebra equal to its commutant is maximal. ■

Lemma 3.1.16. *Let* $\mathrm{CL} \subseteq \mathrm{HL}$ *be constructed as in Theorem 2.1.11 and $U \in W^*(\Delta)'$ be unitary. There exists a unitary $V \in \rho(\mathrm{Aut}(L(S)))$ such that* $\mathrm{Ad}_U = \mathrm{Ad}_V : \mathrm{CL} \to \mathrm{CL}$ *and $U = VS$ for $S \in W^*(\{p_c\}_{c\in C}) \cap W^*(U_\Delta)'$.*

Proof. If $U \in W^*(\Delta)'$, then $\mathrm{Ad}_U \in \mathrm{Aut}(L(S))$ by Theorem 3.1.12. Let $V = \rho(\mathrm{Ad}_U) \subseteq W^*(U_\Delta)'$. Then $\mathrm{Ad}_{V^*} = \mathrm{Ad}_V^{-1}$, so $\mathrm{Ad}_{UV^*}|_{\mathrm{CL}} = \mathrm{Ad}_I|_{\mathrm{CL}}$. As the action of the inner automorphism stabilizes CL, $UV^* \in W^*(\{p_c\}_{c\in C})'$, and $W^*(\{p_c\}_{c\in C})' = W^*(\{p_c\}_{c\in C})$ by Proposition 3.1.15.

Therefore, there exists $S \in W^*(\{p_c\}_{c\in C})$ such that $U = VS$. Furthermore, $S = UV^*$, so $S \in W^*(U_\Delta)'$ as well. ■

The above representation when considered as an action of inner automorphism on $B(H)$ can be seen to be identical to our previous notion when we fix $S = I$.

Theorem 3.1.17. $W^*(U_\Delta)' = W^*(\rho(\mathrm{Aut}(L(S))), W^*(\{p_c\}, U_\Delta)')$.

Proof. We use the above lemmas to show that both von Neumann algebras have the same set of unitaries for an appropriate representation of $\mathrm{Aut}(L(S))$. In particular, any unitary in $W^*(U_\Delta)'$ is a product of the two algebras $W^*(\rho(\mathrm{Aut}(L(S))))$ and $W^*(\{p_c\}, U_\Delta)'$. Now we recall that von Neumann algebras are generated by their unitaries (see [11, Proposition I.4.9]), so the result follows. ∎

Corollary 3.1.18. $W^*(U_\Delta) = Z(W^*(\rho(\mathrm{Aut}(L(S)))))$.

Proof. The statement follows immediately from the result $Z(\mathrm{Aut}(L(S))) = \{1, \Delta\}$ in [12], the definition of ρ, and the spectral theorem. ∎

We have extended the purely group theoretic ideas of [12] to the more general von Neumann algebra setting. Now that we have a legitimate and well-understood von Neumann algebra $W^*(\Delta)$, and some insight into its commutator, we can finally discuss the quantum relations that Δ induces. We demonstrate that the relations specified by the cubic lattice are natural and measurable in the sense of [15]. We first define a standard relation.

Definition 3.1.19. Let X be a set. A *binary relation* on X is a set of ordered pairs (a, b) where $a, b \in X$. In some literature, the notation aRb is used to indicate that the pair (a, b) is in the relation R on X. This relation is often denoted as (X, R).

The obvious issue with the classic notion of a relation is that, when one considers a non-atomic measure, these finite relations become vacuous. In [15], they generalize this notion to a measurable relation.

Definition 3.1.20. A measure space (X, μ) is *finitely decomposable* if X can be partitioned into a (possibly uncountable) family of finite measure subspaces X_λ such that a set $S \subseteq X$ is measurable if and only if its intersection with each X_λ is measurable, in which case $\mu(S) = \sum_\lambda \mu(S \cap X_\lambda)$.

As pointed out in [15], a measure space (X, μ) is finitely decomposable exactly when $L^\infty(X, \mu)$ is an abelian von Neumann algebra. A full explanation can be seen in [3].

Definition 3.1.21 ([15]). Let (X, μ) be a finitely decomposable measure space. A *measurable relation on* X is a family R of ordered pairs of non-zero projections in $L^\infty(X, \mu)$ such that $(\bigvee p_\lambda, \bigvee q_\kappa) \in R$ if and only, for any pair of families of non-zero projections $\{p_\lambda\}$ and $\{q_\kappa\}$, there are some p_λ and q_λ such that $(p_\lambda, q_\kappa) \in R$.

Equivalently, we can impose the two conditions

$$p_1 \leq p_2, q_1 \leq q_2, (p_1, q_1) \in R \implies (p_2, q_2) \in R$$

and

$$(\bigvee p_\lambda, \bigvee q_\kappa) \in R \implies \text{some } (p_\lambda, q_\kappa) \in R.$$

Of course, we are dealing with a more general, not necessarily abelian structure.

Definition 3.1.22 ([15]). A *quantum relation* on a von Neumann algebra $\mathcal{M} \subseteq B(H)$ is a W^*-bimodule over its commutant $\mathcal{M}'$, i.e., it is a weak* closed subspace $V \subseteq B(H)$ satisfying $\mathcal{M}' V \mathcal{M}' \subseteq V$.

Now we argue from the reverse perspective. If, *a priori,* we argued that a quantum logic must respect the symmetry group of a possibly infinite-dimensional cube, the infinite hyperoctahedral group $\operatorname{Aut}(L(S))$, then we could consider the von Neumann algebra generated by $\operatorname{Aut}(L(S))$.

Proposition 3.1.23. *Let $\mathcal{B}$ be the basis of atoms on $L(S)$ constructed as in Theorem 2.1.11. Then, $W^*(U_\Delta)$ acts transitively in $\mathcal{B}$.*

Proof. If $u, v \in \mathcal{B}$, then we consider the composition of $\mathbb{Z}_2$-actions on each disagreeing index, which is contained in $W^*(U_\Delta)$ by construction. ■

Geometrically, in the finite-dimensional case, we can observe that the Coxeter group B_n is the group of rigid motions of the cube and must be able to permute any two vertices.

Corollary 3.1.24. *The quantum relations associated with $W^*(U_\Delta)$ are weak* closed subspaces V satisfying $W^*(U_\Delta) V W^*(U_\Delta) \subseteq V$.*

The operator systems discussed above – i.e., the ideals of CL – have well-defined quantum relations. Furthermore, by presupposing the lattice, we have re-derived both Δ and the invariant subspaces of the cube.

From an experimental setting, this invariant subspace is a natural requirement, as one can rotate the axis for the detection of a spin $1/2$ particle. However, we still need cubic symmetry, as the experiment takes place in Euclidean space. Therefore, our notion Δ can be viewed as a necessary condition for the relations in the experiment. In addition, these symmetries can be verified by the single relation Δ, as opposed to the (infinite) hyperoctahedral group.

Furthermore, our original lattice-based definitions of the cube are now seen to be measurable in a much more general sense. As the principal ideals of a cubic lattice are again cubic lattices, we can further infer that these principal ideals form von Neumann subalgebras and therefore operator systems.

Definition 3.1.25. An *operator system* is a unital $*$-closed subspace contained in a unital C^*-algebra.

To be precise, we present the following theorem.

Theorem 3.1.26. *Let $p_a \in \mathrm{CL} \subseteq \mathrm{HL}$ constructed as in Theorem 2.1.11 and $\mathcal{M} = B(H)$. Then $p_a \mathcal{M} p_a$ forms a von Neumann subalgebra and $[a]_C$ forms a Boolean lattice.*

Proof. The fact that $p_a \mathcal{M} p_a \leq \mathcal{M}$ is a standard result. We refer to [8] for the construction of the complement ${}^c(\cdot) = a \vee_C \Delta\Delta(\cdot, a)$ making $[a]_C$ a Boolean algebra. ■

Therefore, an element in a cubic lattice can be seen as a dividing line between a Boolean algebra and a von Neumann algebra. This reflects our notion that the projection operators of a cubic lattice detect the minimum entangled state containing the respective atoms. Beyond that level of detection, elements become disentangled and therefore Boolean.

Corollary 3.1.27. *Let (p) be a principal ideal in* CL. *Then $C^*(\Delta_p)$ is a von Neumann algebra pDp.*

Proof. This follows from the fact that the principal ideals of cubic lattice are themselves cubic lattices. ∎

3.2. Operator algebras containing a cubic lattice

We can see that the above results can be generalized in a straightforward manner.

Definition 3.2.1. Let C be the coatoms of CL. Then, for each $c \in C$, we get a symmetry in the canonical form of $p_c - p_{\Delta c}$. We denote the set of these symmetries by $\{s_i\}_{i \in I}$.

Importantly, the i-th coordinate in the tensor product is equal to the matrix $s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Lemma 3.2.2. *With our previous choice of representation, $\rho\colon \mathrm{Aut}(L(S)) \to B(H)$, the mutual commutant of U_Δ and s_i is equal to*

$$W^*(W^*(\rho(\mathbb{Z}_2 \wr S_{I-i})), W^*(\{p_c\}_{c \in C}) \cap W^*(U_\Delta)');$$

again, by $\wr$ we mean the unrestricted wreath product.

Proof. By Lemma 3.1.16, we already have an explicit definition of the unitaries that commute with U_Δ, so we only need to consider the subset that also commute s_i.

We consider the elements of $\mathrm{Per}_\Delta(C)$ that fix p_{c_i} and $p_{\Delta c_i}$. These elements are the permutations fixing the i-th coordinate that commute with U_Δ, so we have again the infinite hyperoctahedral group, but on one less coordinate, or $\mathbb{Z}_2 \wr S_J$, where $|J| = |I| - 1$ or elements of $W^*(\{p_c\}_{c \in C}) \cap W^*(U_\Delta)'$.

Now the result follows by taking the WOT closure of the algebra generated by its unitary operators, which fully defines the von Neumann algebra again by [11, Proposition I.4.9]. ∎

Theorem 3.2.3. *Let H be constructed as in Theorem 2.1.11. Then, $B(H) \cong M_2(B)$, where $B \cong I_2 \otimes B(H_{I-i})$.*

Proof. Let U_{Δ_i} be the tensor product whose i-th index is equal to i-th index of U_Δ and I_2 elsewhere. We claim that the following form a system of matrix units of $B(H)$:

$$e_{11} = \frac{I + s_i}{2}, \qquad e_{12} = \frac{(I + s_i)U_{\Delta_i}}{2},$$
$$e_{21} = \frac{U_{\Delta_i}(I + s_i)}{2}, \quad e_{22} = \frac{I - s_i}{2}.$$

We can directly compute that

$$e_{11} + e_{22} = I, \quad e_{12} = e_{21}^*, \quad e_{ij}e_{kl} = \delta_{jk}e_{il}.$$

Therefore, $B(H) \cong M_2(B)$, where B commutes with all of the matrix units, see [2, Lemma 4.27].

Now we show thot $N = W^*(\{e_{ij}\}_{i,j\in\{1,2\}}) = W^*(U_{\Delta_i}, s_i)$. Firstly, $U_\Delta \in N$,

$$U_{\Delta_i} = \frac{U_{\Delta_i} + s_iU_{\Delta_i} - s_iU_{\Delta_i} + U_{\Delta_i}}{2} = \frac{U_{\Delta_i} + s_iU_{\Delta_i} + U_{\Delta_i}s_i + U_{\Delta_i}}{2}$$
$$= \frac{(I + s_i)U_{\Delta_i}}{2} + \frac{U_{\Delta_i}(I + s_i)}{2} = e_{12} + e_{21}.$$

Secondly, $s \in N$,

$$s_i = \frac{2s_i - I + I}{2} = \frac{I + s_i}{2} - \frac{I - s_i}{2} = e_{11} - e_{22}.$$

Therefore, $W^*(U_\Delta, s_i) \subseteq N$. For the reverse containment, the generators of N are in the algebra generated by U_Δ and s_i, so they are in the WOT closure of the algebra.

Now we apply $M_2(\mathbb{C})_i \otimes I_{I-i} = W^*(U_{\Delta_i}, s_i)$, so that $N' = I_2 \otimes B(H_{I-i})$, where $H_{I-i} = \bigotimes_{j\in(I-i)} \mathbb{C}^2$ is constructed as in Theorem 2.1.11. ■

Example 3.2.4. We see that, in our choice of matrix units, we again obtain that

$$U_{\Delta_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad is_iU_{\Delta_i} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

This is considered as a representation of $M_2(B)$, as opposed to $M_2(\mathbb{C})$. Of course, if we reduce to the single qubit case, then we have that $B \cong \mathbb{C}$ and only one choice of index for s_i, so our result is consistent.

We relate the above construction to a more familiar general object.

Definition 3.2.5 ([1]). A *Cartesian triple* is a set of operators r, s, t in a von Neumann algebra such that

(1) $r \circ s = s \circ t = t \circ r = 0$,

(2) $\mathrm{Ad}_r\,\mathrm{Ad}_s\,\mathrm{Ad}_t = I$.

Corollary 3.2.6. *For any $s_i \in S$, the set U_Δ, s_i, and $iU_\Delta s_i$ form a Cartesian triple in $B(H)$.*

Proof. Given our representation, the result follows from standard facts about Pauli matrices. ■

We can consider another von Neumann subalgebra of B(H). Namely, $W^*(\{s_i\}_{i\in I})$.

Lemma 3.2.7. *Given our representation of* Δ*, the set of coatoms* C *of* CL *is a generating set of projections of* $W^*(\{s_i\}_{i\in I})$.

Proof. We have that $C = \{\frac{1\pm s_i}{2}\}_{i\in I}$ generates $\{s_i\}_{i\in I}$ and vice versa. Therefore, $W^*(C)$ generates the unitaries of $W^*(\{s_i\}_{i\in I})$, and therefore generates all $W^*(\{s_i\}_{i\in I})$. ■

Theorem 3.2.8. *The atoms of* $W^*(\{s_i\}_{i\in I})$ *are the atoms of* CL.

Proof. We have shown that the coatoms of CL are in $W^*(\{s_i\}_{i\in I})$, by Lemma 3.2.7. By the coatomicity of CL, and by Lemma 2.1.17, we have that the atoms of CL are contained in $W^*(\{s_i\}_{i\in I})$.

Now, for the reverse direction, we consider the complete lattice of projections L generated by the canonical projections of $\{s_i\}_{i\in I}$. Here, we mean "complete" in the sense of lattice theory, not necessarily complete with respect to the norm; and generated in the sense of closure of meet and joins. As the canonical projections of $\{s_i\}_{i\in I}$ are exactly the coatoms of CL, we have that the atoms of L are exactly the set of atoms of CL by Lemma 2.1.17. Additionally, L is a complete lattice generated by an orthonormal basis and therefore Boolean.

In our specific application of Proposition 3.1.14, the atoms of L form a maximal set of mutually orthogonal projections, and the subalgebra of bounded operators of $\mathcal{A}$, $C^*(L)$, is abelian, so we have that $C^*(\{s_i\}_{i\in I}) = C^*(L)$ is a von Neumann algebra ([4, Remark 10.8]), whose atoms are the atoms of the cubic lattice. ■

Therefore, we now have a minimal von Neumann algebra containing $\mathrm{CL} = L(S)$ for a given $|S|$. Furthermore, we have shown that $W^*(\{s_i\}_{i\in I}) \leq B(H)$, where H is minimal as in Theorem 3.1.6.

Example 3.2.9. When reducing the one qubit case, we see that $W^*(U_\Delta, s)$ contains the Pauli matrices, which are a W^* algebra over $\mathbb{C}$ generating all of $M_2(\mathbb{C})$, which is a well-known result, as required. Furthermore, we have a unitary matrix $T \in M_2(\mathbb{C})$,

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

which is a unitary similarity sending s to U_Δ. We recognize this as the normalized Hadamard matrix.

We are now in a position to generalize the result that the Pauli spin matrices span $M_2(\mathbb{C})$.

Definition 3.2.10. We define $U = \bigotimes_{i\in I} T$.

Theorem 3.2.11. $B(H) = W^*(\{Us_iU^*\}_{i\in I}, \{s_i\}_{i\in I})$.

Proof. We only need to show that $W^*(\{Us_iU^*\}_{i\in I}, \{s_i\}_{i\in I})' = W^*(\{Us_iU^*\}_{i\in I})' \cap W^*(\{s_i\}_{i\in I})' = Z(B(H)) = \mathbb{C}I$.

Suppose that V is a unitary operator commuting with U_Δ, then, by Lemma 3.1.16, $\mathrm{Ad}_V \in \mathrm{Aut}(L(S))$, when considering its action by inner automorphism on $L(S)$. As V commutes with each coatom of $L(S)$, V acts trivially on the coatoms of $L(S)$, so by the coatomisticity of L(S), V acts trivially on $L(S)$. Then $V \in W^*(\{s_i\}_{i\in I})$. By symmetry, $V \in W^*(\{Us_iU^*\}_{i\in I})$.

Consider canonical projections p_i of Us_iU^* and $q_i \in s_i$ for some fixed index $i \in I$. Then $p_i \wedge q_i = \lim_{n\to\infty}(p_iq_ip_i)^n = \lim_{n\to\infty}(p_i/2)^n = 0$. By construction, any atom $a \in \{Us_iU^*\}$ is bounded by a canonical projection of Us_iU^*. So, we assume, without loss of generality, that $a \leq p$, and, by symmetry, we assume $b \leq q$. Then, $a \wedge b \leq p \wedge q = 0$. Therefore, the atomistic Boolean lattices of projections associated with $\{Us_iU^*\}_{i\in I}$ and $\{s_i\}_{i\in I}$, respectively, have distinct sets of atoms. By atomisticity, $W^*(\{s_i\}_{i\in I})$ and $W^*(\{Us_iU^*\}_{i\in I})$ are abelian von Neumann algebras whose only common projections are 0 and I, so their intersection is $\mathbb{C}I$, by Proposition 3.1.14. ∎

As we have demonstrated, the Hilbert lattice into which we originally embedded CL is not only minimal, but the von Neumann algebra as a whole is generated by two copies of CL with orthogonal atoms.

Corollary 3.2.12. *Let $L(C)$ be the meet semi-lattice generated by C, set of the coatoms of the cubic lattice adjoin* 1. *Then, the meets and joins of $L(S)$ are exactly* CL $\subseteq B(H)$.

We can see that our generation of $B(H)$ is a generalization of the single qubit case to arbitrary cardinals.

We have now shown that $B(H)$ is generated directly by Δ and CL. Additionally, $B(H)$ is a minimal structure containing both and is therefore a necessary structure if one considers an operator algebraic structure of the cubic lattice under the conditions detailed at the conclusion of [9, Section 4.1].

3.3. Phase rotations

So far, we have re-derived the Pauli and Hadamard gates, referred to as the X, Z, and H gates in the literature, and their respective roles in the underlying von Neumann algebra. As shown, this von Neumann algebra is over a Hilbert space constructed in the standard manner and generalized to arbitrary cardinals. The question now becomes: what types of observables can we obtain as functions of our already constructed observables? We will show that continuous functional calculus can be used to construct universal quantum gates in the sense of the Solovay–Kitaev theorem; see [5].

Definition 3.3.1. Let U_Δ, s be represented in $M_2(\mathbb{C})$, then

$$R_x(\theta) = e^{iU_\Delta\theta/2} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix},$$

$$R_y(\theta) = e^{iU_\Delta s\theta/2} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix},$$

$$R_z(\theta) = e^{is\theta/2} \quad = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

We will discuss group theoretic properties that can be shown directly from a computation in the case of $M_2(\mathbb{C})$, but we highlight a more general, standard technique to extend these results.

Proposition 3.3.2. *Let A be a normal operator in a C^*-algebra, $\mathcal{A}$. Then, for any $f \in C(\sigma(A))$ and unitary $U \in \mathcal{A}$, $Uf(A)U^* = f(UAU^*)$.*

Proof. Let $\rho(1) = I$, $\rho(z) = A$ be the standard continuous functional calculus on A. Let $\gamma = U\rho U^*$, and let $\tau(1) = I$, $\tau(z) = UAU^*$. As a transformation by unitary similarity does not change the spectrum of A, our mappings, γ and τ, both have the same domain $C(\sigma(A))$. We have $\gamma(1) = U\rho(1)U^* = UIU^* = I = \tau(1)$, and $\gamma(z) = U\rho(z)U^* = UAU^* = \tau(z)$, and the result follows for any continuous function by the uniqueness of the continuous functional calculus. ■

Lemma 3.3.3. *Let U be a unitary operator and A be a normal operator in a C^*-algebra such that $UA = -AU$. Then for any $t \in \mathbb{C}$, $Ue^{tA}U^* = e^{-tA}$.*

Proof. We apply Proposition 3.3.2 to see that $Ue^{tA}U^* = e^{tUAU^*} = e^{-tA}$. ■

Now we can use the above lemmas to immediately deduce that the action of unitary similarity of any member of a Cartesian triple acts as the inversion of the rotation of any other member of the same Cartesian triple. Explicitly, $e^x e^{-x} = 1$ when considered as standard continuous functions over $\mathbb{C}$, and we have an algebra homomorphism for the respective operator-valued functions. Furthermore, the action of unitary similarity of any normal element on its own exponent function is trivial.

Theorem 3.3.4. *Let $G = \langle U_\Delta, e^{2\pi\theta i s_i}\rangle$. Then $G \cong D_{2n}$ for some $n \in \mathbb{N}$, if θ is a rational or D_∞ if θ is an irrational.*

Proof. From the above discussion, we recognize that U_Δ embeds into the automorphism group generated by $e^{2\pi\theta i s_i}$ as an inversion. Therefore, we take the semidirect product. With the presentation $\langle U_\Delta, e^{2\pi\theta i s_i} : U_\Delta e^{2\pi\theta i s_i} U_\Delta = e^{-2\pi\theta i s_i}\rangle$, we see that the isomorphism type of the group follows from the order of $e^{2\pi\theta i s_i}$, which is finite if θ is rational and infinite otherwise. Thus, the result follows. ■

Corollary 3.3.5. *Let $G = \langle U_\Delta, e^{i\sum_{i\in I} 2\pi\theta_i s_i}\rangle$. Then, $G \cong D_{2n}$ for some $n \in \mathbb{N}$ if $\theta = 1$ for all but finitely $i \in I$ and $\theta_i \in \mathbb{Q}$ for finite i, or $G \cong D_\infty$ otherwise.*

Proof. We need only apply the previous theorem to each s_i and use the fact that continuous functions of commuting operators commute by functional calculus. If there are only finitely many rational θ not equal to one, then we can consider the least common multiple (lcm) of their respective orders to obtain a finite n satisfying the claim. ■

We now compare the above representation to the universal representation.

Corollary 3.3.6. *Let $\mathcal{A} = C^*(D_{2n})$, $3 \leq n \in \mathbb{N}$, in the representation $\pi: G \to B(H)$, where H is constructed as in Theorem 2.1.11, and let $\mathcal{B}$ be the reduced C^*-algebra of D_{2n} with left regular representation $\lambda: G \to B(l^2(G))$. Then, $\mathcal{A}$ is a nontrivial quotient of $\mathcal{B}$.*

Proof. We start with $n \geq 4$ and assume that $\theta_i = 1$ for all but exactly one $k \in I$. Without loss of generality, we assume $k = 1$. As D_{2n} is a group extension of discrete groups, D_{2n} is amenable. Therefore, the reduced C^*-algebra and the universal C^*-algebra are isomorphic. So we only need to show that $\|\pi(a)\| < \|\lambda(a)\|$ for some $a \in \mathcal{A}$.

Let us consider the group ring $\mathbb{C}[G]$ and restrict to elements over the cyclic subgroup $\mathbb{Z}_n \cong \langle r \rangle$ in each representation. Then $\lambda(\sum_{j=0}^{n-1} c_j r^j) \neq 0$ for any choice of $c_j \in \mathbb{C}$, as the r_j are linearly independent. However, $\pi(r) = R \otimes (\bigotimes_{i \in \{I-1\}} I_2)$, for an appropriate rotation matrix $R \in M_2(\mathbb{C})$, and, as a vector space, $C^*(R)$ has dimension at most 3 because $U_\Delta \notin C^*(R)$ as U_Δ does not commute with R and $C^*(R)$ is an abelian algebra. Therefore, $\pi(\sum_{j=0}^{n-1} c_j r^j) = 0$ for some choice of $c_j \in \mathbb{C}$.

Now, let $n = 3$. We can directly compute that $\pi(a) = I + R + R^2 = 0$, so $\pi(a) = 0 < \|\lambda(a)\|$, again using linear independence.

We have shown the result for a single coordinate of the tensor product, and if we extend to the multi-coordinate tensor case, then, for some element $a \in \mathcal{A}$, $\pi(a_i) < \lambda(a_i)$, so the same must be true for the product of the norms across the indexing set. ∎

Remark 3.3.7. We want to highlight that this behavior is quite different when considering the relation of anti-commutativity of the product of s_i. U_Δ and $\prod_{j \in J} s_i$ anti-commute exactly when J is odd, and the relationship is non-obvious when J is infinite. This is because -1 factors through the tensor product, and we get a term $(-1)^n$ as a leading coefficient. However, as described above, this does not occur when we consider exponentiation of the respective product.

As a summation of our results, we have demonstrated that many of the "classical" quantum gates are a direct consequence of our construction of the cubic lattice as an orthomodular lattice of orthogonal projections. Thanks to our construction, we have another natural choice of representation in a more geometric view, as a cube in dimension $|I|$ as opposed to the larger $2^{|I|}$, which may have interesting applications on its own. In addition, we have shown a number of group theoretic properties of their respective algebras, and that the remaining gates can be naturally constructed as functions of the already constructed gates, both in a direct sense via the exponential map and in a more general sense as observables constructed from a continuous or Borel function over the spectrum of a Cartesian triple using the spectral mapping theorems. From a physical perspective, we have given a mathematically formal description of the lattice of observables for a system of spin $-1/2$ of arbitrary cardinality. Furthermore, the gates I, Δ, $\sqrt{s}$, U, or more standardly I, X, $\sqrt{Z}$, and H, combined with classical circuits, generate a set of universal quantum gates.

Lastly, we want to conclude with a forward-looking note on the applications to real quantum computational systems. As a direct result of our work, any algorithm whose possible states are in a subset of a cubic lattice is immediately representable on a quantum system, thereby bringing a new set of algorithms into the quantum space. Furthermore, it is well known that the Pauli group is fundamental to most standard quantum error codes that are of a similar nature to the famous Shor code. In the finite case, U_Δ is a member of the Pauli group's typical representation in the context of quantum error correction. There is strong reason to believe this interplay has deep results pertaining to quantum algorithms living in a vector space stabilized by elements of the Pauli group. A full, rigorous exploration and explicit exemplification of such an algorithm would be a worthy area for future study.

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