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Torsors in super-symmetry

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Abstract. Torsors under affine groups are generalized in the super context by super-torsors under affine super-groups. We investigate those super-torsors by using Hopf-algebra language and techniques. It is shown in an explicit way that, under suitable assumptions, every super-torsor arises from an ordinary torsor. Especially, the objects with affinity restriction, or namely, the affine super-torsors and the affine ordinary torsors are proved to be precisely in one-to-one correspondence. The results play substantial roles in ongoing construction of super-symmetric Picard–Vessiot theory.

1. Introduction – basic notations and main theorems

Throughout this paper, we work over a fixed field k whose characteristic $\operatorname{char} k$ differs from 2, that is, in notation, $\operatorname{char} k \neq 2$.

This section, which is slightly long for an introduction, is devoted mainly to clarifying the notion of super-torsors and to formulating our two main theorems on super-torsors.

1.1. Very basic algebraic super-geometry

Such basics are here presented mostly by using the functor-of-points approach.

Recall that a *k-functor* is a set-valued functor defined on the category $\mathtt{Alg}_k$ of commutative algebras (over k). A representable k-functor is called an *affine scheme*, and it is called an *affine group* if it is group-valued. In addition, the category of k-functors includes faisceaux (or sheaves), faisceaux dur (or dur sheaves), schemes and affine schemes, as subcategories. These notions as well as relevant basic notions and constructions are directly generalized to the super context, in which everything is based on the tensor category $\mathtt{SMod}_k$ of vector spaces graded by the group $\mathbb{Z}_2$ of order 2, equipped with the so-called *super-symmetry*; see Section 2.1.

The generalized notions are called with "super" attached, in principle. For example, a *k-super-functor* is a set-valued functor $\mathtt{X}$ defined on the category $\mathtt{SAlg}_k$ of the super-commutative super-algebras (over k). It is important that, given such an $\mathtt{X}$, there is associated the k-functor $\mathtt{X}_{\mathsf{ev}}$ which is obtained from $\mathtt{X}$, restricting it to $\mathtt{Alg}_k$; ordinary algebras are regarded as *purely even* super-algebras, or namely, super-algebras consisting

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of even elements, only. We remark that the k-functor denoted here by X_{ev} is alternatively denoted by $\mathsf{X}_{\mathsf{res}}$ in [17], in which the symbol X_{ev} is used to denote the k-super-functor $R \mapsto \mathsf{X}(R_0)$.

The category of k-super-functors includes the following full subcategories:

$$\text{(faisceaux)} \supset \text{(faisceaux dur)} \supset \text{(super-schemes)}. \supset \text{(affine super-schemes)}$$

The definition of affine super-schemes, as well as that of affine super-groups, will be obvious. A *faisceaux dur* (resp., *faisceaux*) is a k-super-functor which preserves finite direct products and every equalizer diagram

$$R \to S \rightrightarrows S \otimes_R S$$

that naturally arises from an *fpqc* (resp., *fppf*) morphism $R \to S$ in $\mathtt{SAlg}_k$. Here, fpqc (resp., fppf) represents "faithfully flat" (resp., "faithfully flat and finitely presented"); see Section 2.5. A *super-scheme* is a k-super-functor X which is *local* (or roughly speaking, X behaves like a sheaf with respect to Zariski coverings $\{R \to R_{f_i}\}$, where $f_i \in R_0$ (finitely many) with $\sum_i f_i R_0 = R_0$), and which is covered by some open sub-functors that are isomorphic to affine super-schemes. By convention, every super-scheme X is supposed to be *non-trivial* in the sense that $\mathsf{X}(R) \neq \emptyset$ for some $0 \neq R \in \mathtt{SAlg}_k$.

If a k-super-functor X is contained in one of the categories noted above, then the associated k-functor X_{ev} is in the corresponding one defined in the ordinary (non-super) context. If G is an affine super-group, then G_{ev} is an affine group.

There is an alternative definition (see [1, Section 3.3], [8, Chapter 4, Section 1] or [17, Section 4], and rely on Leites [7] for historical background) of an (affine) super-scheme, which defines it as a super-ringed space satisfying some conditions that are naturally inferred from the ordinary setup. We will use in part (in fact, essentially only in Section 7) the word in this alternative, geometrical meaning. This is justified since we have a natural equivalence (see [17, Theorem 5.14]) between the two categories of (affine) super-schemes thus defined in the two distinct ways. The equivalence in one direction assigns to every super-scheme Z, in the geometrical sense, the k-super-functor $R \mapsto \mathrm{Mor}(\mathrm{Spec}\, R, \mathsf{Z})$ represented by it.

1.2. The first main theorem

Suppose that an affine super-group G acts on a super-scheme X. Here and in what follows, actions are supposed to be on the right. We say that the action is *free*, or G acts *freely* on X, if the morphism of k-super-functors

$$\mathsf{X} \times \mathsf{G} \to \mathsf{X} \times \mathsf{X}, \quad (x, g) \mapsto (x, xg) \tag{1.1}$$

is a monomorphism of super-schemes. Given a (free) G-action on X, there is a naturally-induced (free) action by the affine group G_{ev} associated with G on the scheme X_{ev} associated with X.

Suppose that an affine super-group G acts freely on a super-scheme X.

Definition 1.1. A morphism $\mathsf{X} \to \mathsf{Y}$ of super-schemes is called a *super* G*-torsor* (in the fpqc topology) if it constitutes the co-equalizer diagram of the faisceaux dur

$$\mathsf{X} \times \mathsf{G} \rightrightarrows \mathsf{X} \to \mathsf{Y}, \tag{1.2}$$

where the paired arrows denote the action and the projection. Alternatively, we say that X is a *super* G*-torsor over* Y.

A morphism of super G-torsors over Y is a G-equivariant morphism of super-schemes over Y; such morphism is necessarily an isomorphism, as will be seen in the paragraph following Proposition 1.7.

Remark 1.2. Demazure and Gabriel [2, Chapitre III, Section 4, Définition 1.3, p. 361] define the notion of torsors in the category of faisceaux, where the relevant objects are faisceaux and faisceaux en groupes. Our definition above, reduced to the ordinary situation, becomes equivalent to their definition, when the relevant objects are supposed to be schemes and affine algebraic groups; this fact is easily seen from [2, Chapitre III, Section 4, Corollaire 1.7, p. 362] and Corollary 1.8 below.

An affine super-group G is uniquely represented by a Hopf super-algebra. If the Hopf super-algebra is finitely generated, G is called an *affine algebraic super-group*.

Example 1.3. Let G be an affine algebraic super-group, and let H be a closed super-subgroup of G. Then H acts freely on G by right-multiplication. It is proved in [17, Theorem 0.1] (and re-proved by [16, Theorem 4.12]) that we have a Noetherian (in fact, algebraic) super-scheme G/H which fits into the co-equalizer diagram $\mathsf{G} \times \mathsf{H} \rightrightarrows \mathsf{G} \to \underset{\sim}{\mathsf{G}}/\mathsf{H}$ of faisceaux (dur); therefore, G/H may be presented alternatively by $\tilde{\mathsf{G}}/\mathsf{H}$ (and by $\tilde{\underset{\sim}{\mathsf{G}}}/\mathsf{H}$), by using the notation which will be introduced in the Section 1.3. One concludes that $\mathsf{G} \to \mathsf{G}/\mathsf{H}$ is a super H-torsor.

The first main theorem is the following: it shows, under some assumptions that include affinity, a remarkable one-to-one correspondence between super-torsors and ordinary torsors.

Theorem 1.4. *Let* G *be a smooth affine algebraic super-group, and let* Y *be a Noetherian smooth affine super-scheme. If* $\mathsf{X} \to \mathsf{Y}$ *is a super* G*-torsor, then the induced morphism* $\mathsf{X}_{\mathsf{ev}} \to \mathsf{Y}_{\mathsf{ev}}$ *of schemes is a* G_{ev}*-torsor. Moreover, the assignment* $\mathsf{X} \mapsto \mathsf{X}_{\mathsf{ev}}$ *gives rise to a bijection from*

- *the set of all isomorphism classes of super* G*-torsors over* Y

onto

- *the set of all isomorphism classes of* G_{ev}*-torsors over* Y_{ev}.

Remark 1.5. For those super G-torsors $\mathsf{X} \to \mathsf{Y}$ and G_{ev}-torsors $\mathsf{Z} \to \mathsf{Y}_{\mathsf{ev}}$ which constitute the two sets above, X and Z are necessarily affine, Noetherian, and smooth. The first two properties follow from Proposition 1.7 (2) below. The remaining smoothness will be proved by Proposition 5.1.

Since in Theorem 1.4, everything thus turns out to be affine, the theorem is re-formulated in terms of (Hopf) super-algebras, which represent the relevant functors. A super G-torsor $\mathsf{X} \to \mathsf{Y}$ with X, Y affine is translated into an *H-Galois extension A/B* (see Definition 3.1), where H is the Hopf super-algebra representing G, and A and B are the super-algebras representing X and Y, respectively. Theorem 1.4 will be obtained as an immediate consequence of Theorem 3.12, which explicitly describes such a Hopf–Galois extension in the super situation, in terms of the naturally associated extension in the ordinary situation; see also Corollary 3.14.

Remark 1.6. (1) It will be seen in Remark 3.15 that the bijection shown in Theorem 1.4 does not result from a category-equivalence from the category of super G-torsors over Y to the category of G_{ev}-torsors over Y_{ev}; these categories are in fact groupoids, as is seen from the paragraph following Definition 1.1.

(2) A super-scheme X, regarded as a super-ringed space, is said to be *split* (see [16, Section 2.6], for example) if the structure sheaf $\mathcal{O}_{\mathsf{X}}$ is isomorphic to the exterior algebra $\wedge_{\mathcal{O}_{\mathsf{X}_{\mathrm{ev}}}}(\mathcal{M})$ on some locally free $\mathcal{O}_{\mathsf{X}_{\mathrm{ev}}}$-module sheaf $\mathcal{M}$, where $\mathcal{O}_{\mathsf{X}_{\mathrm{ev}}}$ denotes the structure sheaf of the scheme associated with X; see the first paragraph of Section 7. Every affine algebraic super-group G is split (see (2.6)–(2.7) below), while G/H such as in Example 1.3 can be non-split; see [16, Remark 4.20 (2)]. In view of the cited example, it would be hopeless to generalize Theorem 1.4, removing the affinity assumption.

1.3. The second main theorem

Suppose that an affine super-group G acts freely on a super-scheme X. We have the k-super-functor which assigns to each $R \in \mathtt{SAlg}_k$ the set $\mathsf{X}(R)/\mathsf{G}(R)$ of $\mathsf{G}(R)$-orbits; it has the property

$$\mathsf{X}(R)/\mathsf{G}(R) \subset \mathsf{X}(S)/\mathsf{G}(S) \quad \text{whenever } R \subset S, \tag{1.3}$$

since the action is supposed to be free. We can construct in a simple manner (see [17, Remark 3.8], for example) the faisceau dur $\mathsf{X}\tilde{/}\mathsf{G}$ which includes the k-super-functor above as a sub-functor, and which is *associated with* that k-super-functor in the following sense: every morphism from the k-super-functor to any faisceau dur extends uniquely to a morphism from $\mathsf{X}\tilde{/}\mathsf{G}$. Clearly, this $\mathsf{X}\tilde{/}\mathsf{G}$ is characterized by the co-equalizer diagram of the faisceaux dur $\mathsf{X} \times \mathsf{G} \rightrightarrows \mathsf{X} \to \mathsf{X}\tilde{/}\mathsf{G}$ which is analogous to (1.2). An important consequence is the following.

Proposition 1.7. *Let G be an affine super-group.*

(1) *A super G-torsor refers precisely to the natural morphism $\mathsf{X} \to \mathsf{X}\tilde{/}\mathsf{G}$, where G acts freely on a super-scheme X and the associated faisceau dur $\mathsf{X}\tilde{/}\mathsf{G}$ is a super-scheme. (Here, strictly speaking, one identifies a super G-torsor $\mathsf{X} \to \mathsf{Y}$ with $\mathsf{X} \to \mathsf{X}\tilde{/}\mathsf{G}$ such as above, through a unique isomorphism $\mathsf{Y} \xrightarrow{\sim} \mathsf{X}\tilde{/}\mathsf{G}$ compatible with the morphisms from X.)*

(2) *Suppose that* $\mathsf{X} \to \mathsf{Y}$ *is a super* G*-torsor. Then it is an affine and faithfully-flat morphism of super-schemes. Moreover,*

$$\mathsf{X} \times \mathsf{G} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}, \quad (x, g) \mapsto (x, xg) \tag{1.4}$$

is an isomorphism of super-schemes. If G *is an affine algebraic super-group, then* $\mathsf{X} \to \mathsf{Y}$ *is a morphism of finite presentation.*

This is proved essentially by the same argument as [6, Part I, Section 5.7, (1)], which, in fact, uses property (1.3). The faithful flatness shown in Proposition 1.7 (2) implies that a morphism of super G-torsors over some Y is necessarily an isomorphism.

Note that the last assertion of Proposition 1.7 (2) above follows from the algebraicity assumption for G, by using (1.4), base-change, and faithfully-flat descent. As this assertion suggests, it would be more natural to consider the faisceau $\mathsf{X}\tilde{/}\mathsf{G}$ which is associated, now in an obvious sense, with the k-super-functor $R \mapsto \mathsf{X}(R)/\mathsf{G}(R)$, rather than the faisceau dur $\mathsf{X}\tilde{\tilde{/}}\mathsf{G}$ above, when we assume that G is algebraic. In general (without the algebraicity assumption), one sees from the constructions that

$$(\mathsf{X}\tilde{/}\mathsf{G})(R) \subset (\mathsf{X}\tilde{\tilde{/}}\mathsf{G})(R)$$

for every $R \in \mathtt{SAlg}_k$.

From the above-mentioned assertion one sees the following.

Corollary 1.8. *Suppose that an affine algebraic super-group* G *acts freely on a super-scheme* X*. Then the faisceau* $\mathsf{X}\tilde{/}\mathsf{G}$ *is a super-scheme if and only if the faisceau dur* $\mathsf{X}\tilde{\tilde{/}}\mathsf{G}$ *is. If these equivalent conditions are satisfied, we have*

$$\mathsf{X}\tilde{/}\mathsf{G} = \mathsf{X}\tilde{\tilde{/}}\mathsf{G},$$

and $\mathsf{X} \to \mathsf{X}\tilde{/}\mathsf{G}$ $(= \mathsf{X}\tilde{\tilde{/}}\mathsf{G})$ *is a super* G*-torsor.*

The second main theorem is the following: it shows, under some assumptions, that include (i) G is algebraic and (ii) the G-action is free, that $\mathsf{X}\tilde{/}\mathsf{G}$ is a super-scheme provided $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ is a scheme. Due to (i), we thus work with faisceaux, not with faisceaux dur.

Theorem 1.9. *Suppose that an affine algebraic super-group* G *acts freely on a locally Noetherian super-scheme* X*. Assume the following:*

(a) G *and* X *are both smooth;*

(b) *the faisceau* $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ *associated with the induced (necessarily, free) action by the affine algebraic group* G_{ev} *on the locally Noetherian scheme* X_{ev} *is a scheme; it then necessarily holds that* $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ *is locally Noetherian and smooth, and* $\mathsf{X}_{\mathsf{ev}} \to \mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ *is a* G_{ev}*-torsor.*

Then the faisceau $\mathsf{X}\tilde{/}\mathsf{G}$ *is a locally Noetherian, smooth super-scheme such that*

$$\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}} = (\mathsf{X}\tilde{/}\mathsf{G})_{\mathsf{ev}}. \tag{1.5}$$

In particular, $\mathsf{X} \to \mathsf{X}\tilde{/}\mathsf{G}$ *is a super* G*-torsor.*

By (1.5), we mean that the canonical morphism $\mathsf{X}_{\mathsf{ev}} \to (\mathsf{X}\tilde{/}\mathsf{G})_{\mathsf{ev}}$ induces an isomorphism such as noted. By convention, we thus present canonical isomorphisms by $=$, the equal sign.

To prove the theorem above, the essential case is when X is affine. In this situation, the theorem is re-formulated, again in terms of Hopf–Galois extensions, by Theorem 3.16.

Remark 1.10. One might think that the result [16, Theorem 4.12] referred to in Example 1.3 follows from the above Theorem 1.9 as the special case where H and G of the former are taken as G and X, respectively, of the latter, since the faisceau $\mathsf{G}_{\mathsf{ev}}\tilde{/}\mathsf{H}_{\mathsf{ev}}$ is known to be a scheme (see [6, Part I, Section 5.6, (8)]), and thus assumption (b) above is satisfied. But this is true only when assumption (a) (for H and G) as well is satisfied, which is always the case if $\operatorname{char} k = 0$; see Remark 3.13 (1).

1.4. Organization of the paper and additional remarks

As mentioned above, the two main theorems, Theorems 1.4 and 1.9, largely depend on Theorems 3.12 and 3.16, which are formulated in Section 3 in terms of Hopf–Galois extensions, the algebraic counterpart of super-torsors. Those Hopf-algebraic theorems are proved by spending the whole of Section 5 and of Section 6, respectively. Theorem 1.9 is proved in the final Section 7, while Theorem 1.4 is an immediate consequence of Theorem 3.12, as was already mentioned. Theorem 3.12 is accompanied with an analogous result, Proposition 3.11, which is used to prove Theorem 3.16; Section 4 is devoted to proving the proposition. Section 2 is devoted to preliminaries for the following Sections 3–6.

Sections 2–6 will be found to have a purely Hopf-algebraic flavor. There we use the Hopf-algebraic techniques developed so far to investigate algebraic super-groups, as well as several results already obtained by these techniques; see [9, 10, 14–18]. In addition, some important results proved by Schauenburg and Schneider [19] for Hopf–Galois extensions in the ordinary but non-commutative setting are applied; see Sections 3.1 and 5.1. A notable result of the application is the equivariant smoothness, Proposition 5.1, proved in the latter Section 5.1.

Remark 1.11. The two main theorems of ours both play substantial roles in the preprint [12] (continuing to [13]) by the first-named author. It generalizes Picard–Vessiot theory, which is the Galois theory of linear differential equations, to the super context.

2. Preliminaries

2.1. Super-linear algebra

We let $\otimes$ denote the tensor product $\otimes_k$ over our base field k; recall $\operatorname{char} k \neq 2$ by assumption. A vector space $V = V_0 \oplus V_1$ graded by the group $\mathbb{Z}_2 = \{0, 1\}$ is alternatively called a *super-vector space*; it is said to be *purely even* (resp., *purely odd*) if $V = V_0$ (resp., if

$V = V_1$). The super-vector spaces form a symmetric tensor category,

$$\mathtt{SMod}_k = (\mathtt{SMod}_k, \otimes, k),$$

with respect to the so-called super-symmetry

$$c = c_{V,W}: V \otimes W \xrightarrow{\simeq} W \otimes V, \quad c(v \otimes w) = (-1)^{|v||w|} w \otimes v,$$

where $V, W \in \mathtt{SMod}_k$, and $v(\in V)$, and $w(\in W)$ are homogeneous elements of degrees $|v|$ and $|w|$. A *super-algebra* is by definition an algebra in $\mathtt{SMod}_k$. Similarly, the notions of *super-coalgebra*, *Hopf super-algebra*, and *Lie super-algebra* are defined in an obvious manner.

2.2. Super-(co)algebras

Given a super-algebra R, we let

$$\mathtt{SMod}_R \quad (\text{resp., } {}_R\mathtt{SMod})$$

denote the category of right (resp., left) R-modules in $\mathtt{SMod}_k$; an object in the category is called a *right* (resp., *left*) *R-super-module.* As is shown in [9, Lemma 5.1 (2)], such a super-module is projective in the category if and only if it is projective regarded as an ordinary R-module.

Given a super-coalgebra C, we let

$$\mathtt{SMod}^C \quad (\text{resp., } {}^C\mathtt{SMod})$$

denote the category of right (resp., left) C-comodules in $\mathtt{SMod}_k$; an object in the category is called a *right* (resp., *left*) *C-super-comodule*. It follows by the dual argument of proving [9, Lemma 5.1 (2)] referred to above that such a super-comodule is injective in the category if and only if it is injective, regarded as an ordinary C-comodule. Suppose that $V = (V, \rho)$ is an object in $\mathtt{SMod}^C$ and $W = (W, \lambda)$ is in ${}^C\mathtt{SMod}$. The *co-tensor product* $V \square_C W$ is defined by the equalizer diagram

$$V \square_C W \to V \otimes W \rightrightarrows V \otimes C \otimes W \tag{2.1}$$

in $\mathtt{SMod}_k$, where the paired arrows indicate $\rho \otimes \mathrm{id}_W$ and $\mathrm{id}_V \otimes \lambda$. This gives rise to the functor $V\square_C : {}^C\mathtt{SMod} \to \mathtt{SMod}_k$, which is seen to be left exact. It is known that the functor $V\square_C$ is exact if and only if V is injective. An analogous result for the functor $\square_C W$ holds.

For $V = (V, \rho)$ as above, we define

$$V^{\mathrm{co}\,C} := \{v \in V \mid \rho(v) = v \otimes 1\}. \tag{2.2}$$

This is a super-subvector space of V, whose elements are called *C-co-invariants* in V. Analogously, for (W, λ) as above, we define

$${}^{\mathrm{co}\,C}W := \{w \in W \mid \lambda(w) = 1 \otimes w\}. \tag{2.3}$$

In what follows we assume that all super-algebras R are *super-commutative*, or namely, $ba = (-1)^{|a||b|}ab$ for all homogeneous elements a, b; the assumption is equivalent to saying that R includes the even component R_0 as a central subalgebra, and we have $ba = -ab$ for all $a, b \in R_1$. The two categories of super-modules are then naturally identified so that

$$ {}_R\mathtt{SMod} = \mathtt{SMod}_R. $$

2.3. Hopf super-algebras

Accordingly, all Hopf super-algebras H are assumed to be super-commutative, unless otherwise stated. The category

$$ \mathtt{SMod}^H = (\mathtt{SMod}^H, \otimes, k) $$

of right H-super-comodules then forms a symmetric tensor category with respect to the super-symmetry. We will mostly deal with these right H-super-comodules rather than the analogous left ones.

In general, given a super-algebra R, we let

$$ I_R := (R_1), \quad \bar{R} := R/I_R \tag{2.4} $$

denote the super-ideal of R generated by the odd component R_1, and the quotient super-algebra of R by I_R, respectively; the latter is in fact the largest purely-even quotient super-algebra of R.

Let H be a Hopf super-algebra. Then $\bar{H}$ is seen to be an ordinary Hopf algebra. Let H^+ be the augmentation super-ideal of H, or namely, the kernel $\mathrm{Ker}(\varepsilon\colon H \to k)$ of the co-unit of H. Let

$$ W := (H^+/(H^+)^2)_1 \tag{2.5} $$

denote the odd component of the super-vector space $H^+/(H^+)^2$. The exterior algebra $\wedge(W) = \bigoplus_{n\geq 0} \wedge^n(W)$ on W is regarded as a super-algebra whose even (resp., odd) component is the direct sum of those $\wedge^n(W)$ with n even (resp., odd). Since H is naturally an algebra in $\mathtt{SMod}^H$, it turns to be an algebra in $\mathtt{SMod}^{\bar{H}}$ along the quotient map $H \to \bar{H}$. The tensor-product decomposition theorem [9, Theorem 4.5] tells us that there is a co-unit-preserving isomorphism

$$ H \simeq \wedge(W) \otimes \bar{H} \tag{2.6} $$

of algebras in $\mathtt{SMod}^{\bar{H}}$, where the co-unit of the right-hand side is the co-unit of $\bar{H}$ tensored with the natural projection $\wedge(W) \to \wedge^0(W) = k$. As an exception from what we said above about the side, we will mainly use the opposite-sided version

$$ H \simeq \bar{H} \otimes \wedge(W) \tag{2.7} $$

of (2.6).

Remark 2.1. (1) As a typical example of affine algebraic super-groups, recall $\mathsf{GL}_{m|n}$; for a super-vector space V with $m = \dim V_0$, $n = \dim V_1$, the super-group assigns to every super-algebra R, the group of all automorphisms of the R-super-module $R \otimes V$. The Hopf super-algebra representing $\mathsf{GL}_{m|n}$ is presented, for example, in [11, Example 4.1]. [11, Example 4.2] explicitly gives a relevant isomorphism such as (2.7); see footnote 5 added to the arXiv version, in particular.

(2) We emphasize that the isomorphisms (2.6) and (2.7) preserve the $\bar{H}$-co-actions, in particular, and they hold without assuming that H (or $\bar{H}$) is finitely generated or smooth. They were first proved by [14, Proposition 2.4] under that assumption that $\operatorname{char} k = 0$ (whence $\bar{H}$ is smooth), and H is finitely generated; we remark that the cited proposition refers to a result by H. Boseck as a weaker version, but his proof was wrong.

2.4. Smooth super-algebras

A super-algebra R is said to be *Noetherian* if the super-ideals of R satisfy the ACC, or, equivalently, if the even component R_0 is a Noetherian algebra, and the odd component R_1, regarded as an R_0-module, is finitely generated; see [16, Section 2.4] for other equivalent conditions.

A super-algebra R is said to be *smooth* (over k) if, given a super-algebra surjection $A \to B$ with nilpotent kernel, every super-algebra map $R \to B$ factors through A by some super-algebra map $R \to A$. It is known that if $\operatorname{char} k = 0$, every Hopf super-algebra is smooth; see Remark 3.13 (1) below.

The following is part of [18, Theorem A.2].

Proposition 2.2. *For a Noetherian super-algebra R, the following conditions* (a) *and* (b) *are equivalent:*

(a) *R is smooth as a super-algebra;*

(b) (i) *$\bar{R}$ is smooth as an algebra,*

(ii) *I_R/I_R^2, which is seen to be a purely odd $\bar{R}$-super-module, is finitely generated projective as an $\bar{R}$-module, and*

(iii) *there is an isomorphism $\wedge_{\bar{R}}(I_R/I_R^2) \simeq R$ of super-algebras.*

In (iii) above, $\wedge_{\bar{R}}(I_R/I_R^2)$ denotes the exterior $\bar{R}$-algebra on I_R/I_R^2, which is regarded as a super-algebra just as the $\wedge(W)$ before. One can replace $\bar{R}$ and I_R/I_R^2, more generally, with a purely even super-algebra and a purely odd super-module over it, respectively.

Remark 2.3. (1) For a super-algebra R, one sees as in [18, p. 360] that

$$I_R/I_R^2 = R_1/R_1^3,$$

which is a purely odd $\bar{R}$-super-module. This is finitely generated, if R is Noetherian; recall that R_1 is then finitely generated as an R_0-module.

(2) Suppose that a Noetherian super-algebra R satisfies the equivalent conditions in the preceding proposition. By (b) (i), there is a section of the natural projection $R_0 \to \overline{R}$ ($= R_0/R_1^2$), through which we regard R as a super-algebra over $\overline{R}$. The natural projection $I_R \to I_R/I_R^2$ is seen to be a surjection in $\mathtt{SMod}_{\overline{R}}$, which has a section by (b) (ii). The proof of [18, Theorem A.2] shows that the $\overline{R}$-super-algebra map

$$\wedge_{\overline{R}}(I_R/I_R^2) \to R, \tag{2.8}$$

which uniquely extends the last section, is an isomorphism such as in (b) (iii).

2.5. Faithful flatness and finite presentation

A map $R \to S$ of super-algebras is said to be (*faithfully*) *flat* if the functor $S \otimes_R : {}_R\mathtt{SMod} \to \mathtt{SMod}_k$ is (faithfully) exact. This is equivalent to saying that S is (faithfully) flat, regarded as an ordinary right R-module; see [10, Lemma 5.1 (1)]. Each of the equivalent conditions is equivalent to the one with the side switched.

Given a map $R \to S$ of super-algebras, S is said to be *finitely presented* over R if S is presented so as $R[x_1, \dots, x_m; y_1, \dots, y_n]/I$, where $x_1, \dots, x_m$ are finitely many even variables, $y_1, \dots, y_n$ are finitely many odd variables, and I is a super-ideal which is generated by finitely many homogeneous elements.

Suppose that $\mathtt{X}$ is an affine super-scheme. We let $\mathcal{O}(\mathtt{X})$ denote the super-algebra representing $\mathtt{X}$; it is non-zero since $\mathtt{X}$ is non-trivial by convention; see Section 1.1. Note that the algebra $\mathcal{O}(\mathtt{X}_{\mathsf{ev}})$ which represents the associated affine scheme $\mathtt{X}_{\mathsf{ev}}$ equals $\overline{\mathcal{O}(\mathtt{X})}$ ($= \mathcal{O}(\mathtt{X})/(\mathcal{O}(\mathtt{X})_1)$). We say that $\mathtt{X}$ *has a property* (P) (e.g., Noetherian, smooth) if $\mathcal{O}(\mathtt{X})$ has the property (P).

3. Shifting to the Hopf-algebra world

3.1. Hopf–Galois extensions

Let $\mathtt{G}$ be an affine super-group, and set $H := \mathcal{O}(\mathtt{G})$, the Hopf super-algebra representing $\mathtt{G}$. A left $\mathtt{G}$-super-module may be understood to be a right H-super-comodule.

Let $\mathtt{X}$ is an affine super-scheme, and set $A := \mathcal{O}(\mathtt{X})(\neq 0)$. Suppose that $\mathtt{G}$ acts on $\mathtt{X}$ from the right, and the action $\mathtt{X} \times \mathtt{G} \to \mathtt{X}$ is represented by

$$\rho\colon A \to A \otimes H.$$

Thus, $A = (A, \rho)$ is an algebra in $\mathtt{SMod}^H$. Let

$$B := A^{\mathrm{co}\,H}\ (= \{a \in A \mid \rho(a) = a \otimes 1\});$$

see (2.2). This is a super-subalgebra of A, and is often alternatively presented as $B = A^G$, the G-invariants in A.

Recall that the $\mathtt{G}$-action on $\mathtt{X}$ is said to be *free* if the morphism (1.1) of k-super-functors is a monomorphism of super-schemes, or equivalently, if the natural maps

$$\mathtt{X}(R) \times \mathtt{G}(R) \to \mathtt{X}(R) \times \mathtt{X}(R), \quad (x, g) \mapsto (x, xg)$$

are injective for all $R \in \mathtt{SAlg}_k$. This is the case if the super-algebra map

$$\alpha\colon A \otimes A \to A \otimes H, \quad \alpha(a \otimes b) = a\rho(b), \tag{3.1}$$

which represents (1.1), is surjective. Notice that this super-algebra map induces

$$\beta\colon A \otimes_B A \to A \otimes H, \quad \beta(a \otimes_B b) = a\rho(b). \tag{3.2}$$

Definition 3.1. We say that A/B is (an) H-*Galois* (*extension*), or A is an H-*Galois extension over* B, if $B \hookrightarrow A$ is faithfully flat, and the map β above is bijective.

Remark 3.2. The definition above is generalized to the situation where H and A are not necessarily super-commutative. But it seems reasonable to assume then that the antipode of H is bijective, in which case, the condition that A is faithfully flat over B on the left is equivalent to the condition that it is so on the right, as is seen from [19, Theorem 4.10]; see also [17, Lemma 10.1] and Remark 3.10 below.

Recall that A is an algebra in $\mathtt{SMod}^H$. We let

$$\mathtt{SMod}_A^H = (\mathtt{SMod}^H)_A \tag{3.3}$$

denote the category of right A-modules in $\mathtt{SMod}^H$. Its objects may be called (H, A)-*Hopf super-modules* in view of [3, p. 244]. One sees $A \in \mathtt{SMod}_A^H$. More generally, $N \otimes_B A \in \mathtt{SMod}_A^H$, if $N \in \mathtt{SMod}_B$. In fact, we have the functor

$$\Psi\colon \mathtt{SMod}_B \to \mathtt{SMod}_A^H, \quad \Psi(N) = N \otimes_B A. \tag{3.4}$$

This is left adjoint to the functor $\Phi\colon \mathtt{SMod}_A^H \to \mathtt{SMod}_B$ which assigns to each M in $\mathtt{SMod}_A^H$, the H-co-invariants $M^{\mathrm{co}\,H}$ in M; see (2.2).

We reproduce the following from [17]. See also [23, Section 4] and [15, Remark 2.8].

Theorem 3.3 ([17, Theorem 7.1]). *Retain the situation as above.*

(1) *The following condition are equivalent:*

(i) *the* $\mathtt{G}$*-action on* $\mathtt{X}$ *is free, and the faisceau dur* $\mathtt{X}\tilde{/}\mathtt{G}$ *is an affine super-scheme;*

(ii) *the map* α *is surjective, and* A *is injective in* $\mathtt{SMod}^H$*;*

(iii) A/B *is an* H*-Galois extension;*

(iv) *the functor* Ψ *in* (3.4) *is an equivalence.*

If these equivalent conditions are satisfied, then the affine super-scheme $\mathtt{X}\tilde{/}\mathtt{G}$ *is represented by* B*, or, in notation,* $\mathcal{O}(\mathtt{X}\tilde{/}\mathtt{G}) = B$.

(2) *Suppose that* $\mathtt{G}$ *is algebraic and* $\mathtt{X}$ *is Noetherian (or, in other words,* H *is finitely generated and* A *is Noetherian). If the equivalent conditions above are satisfied, then* $\mathtt{X}/\!\!\!/\mathtt{G}$ *(or* B*) is Noetherian, and it coincides with the faisceau* $\mathtt{X}/\mathtt{G}$.

To prove part of this theorem, the same argument as the one we used before to see Proposition 1.7 has been used. Notice from Corollary 1.8 that the conclusion $\mathtt{X}/\!\!\!/\mathtt{G} = \mathtt{X}/\mathtt{G}$ in part (2) holds only assuming that $\mathtt{G}$ is algebraic. Part (1) of the theorem, combined with Proposition 1.7 (2), shows the following.

Corollary 3.4. *Let* $\mathtt{G}$ *and* H *be as above. Let* $\mathtt{Y}$ *be an affine super-scheme, and set* $B := \mathcal{O}(\mathtt{Y})$. *Every super* $\mathtt{G}$*-torsor* $\mathtt{X}$ *over* $\mathtt{Y}$ *is necessarily an affine super-scheme, and* $\mathcal{O}(\mathtt{X})$ *naturally turns into an* H*-Galois extension over* B. *Moreover, every* H*-Galois extension over* B *arises uniquely from a super* $\mathtt{G}$*-torsor over* $\mathtt{Y}$ *in this way.*

The following is well known at least in the ordinary situation, and is easily proved by using faithfully flatness of Hopf–Galois extension; see also the paragraph following Proposition 1.7.

Lemma 3.5. *Let* H *be a Hopf super-algebra, and let* B *be a non-zero super-algebra. Given* H*-Galois extensions* A/B *and* A'/B, *every* B*-algebra morphism* $A \to A'$ *in* $\mathtt{SMod}^H$ *is necessarily an isomorphism.*

We say that A/B and A'/B are *isomorphic* to each other, if there exists a morphism, necessarily an isomorphism, such as above.

The next Proposition 3.6 shows an interesting property of H-Galois extensions, which will be used to prove Lemmas 4.1 and 5.6. In fact, it directly generalizes an important result by Schauenburg and Schneider [19], which was proved in the non-super situation, to our super situation.

To prove the theorem above reproduced from [17, Section 10], there was used the so-called "bosonization technique," in order to reduce (the main part of) the theorem to a known result which was proved in the non-super, but non-commutative situation. The same technique is used, as will be seen soon, to prove the next proposition. We continue to assume the super-commutativity. But the proof works for proving the result without the assumption; see Remark 3.2.

Let A/B be an H-Galois extension. Since B is an algebra in $\mathtt{SMod}^H$ on which H co-acts trivially, the category

$$ {}_B\mathtt{SMod}^H \ (= {}_B(\mathtt{SMod}^H)) $$

of left B-modules in $\mathtt{SMod}^H$ consists precisely of those left B-super-modules M which are at the same time right H-super-comodules such that the B-action $B \otimes M \to M$ and the H-co-action $M \to M \otimes H$ commute in an obvious sense.

Proposition 3.6. *Let* A/B *be as above. Then the product map*

$$ B \otimes A \to A, \quad b \otimes a \mapsto ba \tag{3.5} $$

is clearly a surjective morphism in ${}_B\mathtt{SMod}^H$, and in fact, it splits as such a morphism. Hence, A is projective in ${}_B\mathtt{SMod}$, in particular.

Proof. The bosonization of H is the (ordinary) Hopf algebra

$$\widehat{H} = H \rtimes \mathbb{Z}_2$$

of smash product (resp., smash co-product) by $\mathbb{Z}_2$ as an algebra (resp., as a coalgebra). This Hopf algebra may not be commutative, but has a bijective antipode; see [17, Lemma 10.1]. In this proof, we present $\mathbb{Z}_2$ as a multiplicative group with generator σ. One sees from [17, Proposition 10.3] that the semi-direct product of A by $\mathbb{Z}_2$

$$\widehat{A} = A \rtimes \mathbb{Z}_2$$

naturally turns into an $\widehat{H}$-comodule algebra, and moreover, into an $\widehat{H}$-Galois extension over $B = B \times \{1\}$ (in the non-super situation), which is faithfully flat over B on both sides. It follows by [19, Theorem 5.6] that the product map $B \otimes \widehat{A} \to \widehat{A}$ has a left B-linear and right $\widehat{H}$-co-linear section,

$$s\colon \widehat{A} \to B \otimes \widehat{A}.$$

We suppose $\widehat{A} = A \otimes k\mathbb{Z}_2$, and define maps $r\colon B \otimes A \otimes k\mathbb{Z}_2 \to B \otimes A$ and $t\colon A \to A \otimes k\mathbb{Z}_2$ by

$$r(b \otimes a \otimes \sigma^i) = b \otimes a, \quad i = 0, 1,$$
$$t(a) = a \otimes e,$$

where $e = \frac{1}{2}(1 + \sigma)$, a primitive idempotent in $k\mathbb{Z}_2$. Then the composite $r \circ s \circ t\colon A \to B \otimes A$ is seen to be a left B-linear and right H-co-linear section of the product map (3.5), which may not preserve the parity. The section as well as the product map is naturally regarded as a map of left $B \otimes H^*$-modules, where H^* is the dual algebra of H. Since $B \otimes H^* \subset (B \otimes H^*) \rtimes \mathbb{Z}_2$ is a separable extension of rings by the assumption $\operatorname{char} k \neq 2$, one sees that the product map has a $(B \otimes H^*) \rtimes \mathbb{Z}_2$-linear section, which is indeed a desired section; cf. the proof of [9, Lemma 5.1 (2)]. ∎

3.2. Inflation of Hopf–Galois extensions

Günther [5] proves in the non-super situation an interesting result on the inflation of Hopf–Galois extensions, which is directly generalized to the super situation; see Remark 3.10 below. We are going to present the result in a restricted form as will be needed in the sequel.

Continue to suppose that $\mathtt{G}$ is an affine super-group represented by H. Recall that the affine group $\mathtt{G}_{\mathsf{ev}}$ is represented by the quotient Hopf algebra $\bar{H} = H/(H_1)$ of H. Let

$$\Delta\colon H \to H \otimes H, \quad \Delta(h) = h_{(1)} \otimes h_{(2)}$$

denote the co-product on H. Then H turns into an algebra in ${}^{\bar{H}}\mathtt{SMod}$ with respect to $\bar{\Delta}\colon H \to \bar{H} \otimes H$, $\bar{\Delta}(h) = \bar{h}_{(1)} \otimes h_{(2)}$, where $h \mapsto \bar{h}$ presents the quotient map. Let

$$R = {}^{\mathrm{co}\,\bar{H}}H \ (= \{h \in H \mid \bar{\Delta}(h) = \bar{1} \otimes h\}) \tag{3.6}$$

denote the super-subalgebra of *left* $\bar{H}$-co-invariants (see (2.3)), which is seen to satisfy $\Delta(R) \subset R \otimes H$, whence it is an algebra in $\mathtt{SMod}^H$. From (2.7), we see

$$R \simeq \wedge(W). \tag{3.7}$$

Suppose that $D = (D, \rho)$ is an algebra in $\mathtt{SMod}^{\bar{H}}$, which may not be purely even. The co-tensor product (see (2.1))

$$D\Box_{\bar{H}}H = \Big\{ \sum_i d_i \otimes h_i \in D \otimes H \ \Big|\ \sum_i \rho(d_i) \otimes h_i = \sum_i d_i \otimes \bar{\Delta}(h_i) \Big\}$$

is naturally an algebra in $\mathtt{SMod}^H$ such that

$$D^{\mathrm{co}\,\bar{H}} = (D\Box_{\bar{H}}H)^{\mathrm{co}\,H}. \tag{3.8}$$

This canonical isomorphism (3.8) is a restriction of $D \to D \otimes H$, $d \mapsto d \otimes 1$, and it has, indeed, an inverse, $\sum_i d_i \otimes h_i \mapsto \sum_i d_i \varepsilon(h_i)$, given by the co-unit ε of H.

Lemma 3.7. *Set $B := D^{\mathrm{co}\,\bar{H}}$. If D/B is $\bar{H}$-Galois, then $(D\Box_{\bar{H}}H)/B$ is H-Galois.*

Proof. One sees from (3.8) that $B = (D\Box_{\bar{H}}H)^{\mathrm{co}\,H}$. Notice from (2.7) and (3.7) that we have an isomorphism $\bar{H} \otimes R \simeq H$ of R-algebras in ${}^{\bar{H}}\mathtt{SMod}$, which, with $D\Box_{\bar{H}}$ applied, induces an isomorphism

$$D \otimes R \simeq D\Box_{\bar{H}}H \tag{3.9}$$

of super-algebras over $B \otimes R$. We conclude that $B \hookrightarrow D\Box_{\bar{H}}H$, $b \mapsto b \otimes 1$ is faithfully flat.

We claim that the inclusion $D\Box_{\bar{H}}H \hookrightarrow D \otimes H$ is faithfully flat. To prove this, let $\psi\colon \bar{H} \to H$ be the restriction of an isomorphism $\bar{H} \otimes R \simeq H$ such as above, to $\bar{H}$ $(= \bar{H} \otimes k)$. Then this ψ is an algebra morphism in ${}^{\bar{H}}\mathtt{SMod}$. Moreover, the isomorphism is given explicitly by $a \otimes x (\in \bar{H} \otimes R) \mapsto \psi(a)x$, and the induced one (3.9) is given by

$$d \otimes x (\in D \otimes R) \mapsto d_{(0)} \otimes \psi(d_{(1)})x,$$

where $d \mapsto d_{(0)} \otimes d_{(1)}$ presents the $\bar{H}$-co-action on D. Compose (3.9) with the inclusion in question. The resulting map is seen to be the composite of the inclusion $D \otimes R \hookrightarrow D \otimes H$ with the super-algebra map

$$D \otimes H \to D \otimes H, \quad d \otimes h \mapsto d_{(0)} \otimes \psi(d_{(1)})h,$$

which is indeed an isomorphism with inverse $d \otimes h \mapsto d_{(0)} \otimes \psi(\mathcal{S}(d_{(1)}))h$, where $\mathcal{S}$ denotes the antipode of $\bar{H}$. The faithful flatness of the last inclusion proves the claim.

It remains to prove that the beta map for $D\square_{\bar{H}}H$ is bijective. It suffices to show that the map turns to be so after the base extension along $D\square_{\bar{H}}H \hookrightarrow D \otimes H$, which is faithfully flat as was just proven. To the beta map for D, apply first $(D \otimes H)\otimes_D$, and then $\square_{\bar{H}}H$. The resulting bijection

$$(D \otimes H) \otimes_B (D\square_{\bar{H}}H) \xrightarrow{\simeq} (D \otimes H) \otimes H$$

is given explicitly by

$$(d \otimes h) \otimes_B \left(\sum_i d_i \otimes h_i\right) \mapsto \sum_i (-1)^{|d_i||h|} d\, d_i \otimes h \otimes h_i.$$

This, composed with the bijection

$$D \otimes H \otimes H \xrightarrow{\simeq} D \otimes H \otimes H, \quad d \otimes h \otimes h' \mapsto d \otimes hh'_{(1)} \otimes h'_{(2)},$$

coincides with the base extension which we wish to prove to be bijective; it is thus bijective, indeed. ∎

Definition 3.8. An H-Galois extension A over an arbitrary non-zero super-algebra B is said to *arise from* an $\bar{H}$-Galois extension D/B if it is isomorphic to the algebra $D\square_{\bar{H}}H$ in $\mathtt{SMod}^H$ over B. This $D\square_{\bar{H}}H$ is called the *inflation* of D along $H \to \bar{H}$.

For the R in (3.6), let

$$R^+ = R \cap \mathrm{Ker}(\varepsilon_H)$$

denote the augmentation super-ideal, where ε_H is the co-unit of H.

Proposition 3.9. *Let B be a non-zero super-algebra.*

(1) *An H-Galois extension A over B arises from some $\bar{H}$-Galois extension if and only if there exists an algebra morphism $R \to A$ in $\mathtt{SMod}^H$.*

(2) *Suppose that $A = (A, \rho)$ is an H-Galois extension over B, and $f: R \to A$ is an algebra morphism in $\mathtt{SMod}^H$. Then the quotient super-algebra of A*

$$D_f := A/(f(R^+)) \tag{3.10}$$

divided by the super-ideal $(f(R^+))$ generated by the image $f(R^+)$ of R^+ is a quotient algebra in $\mathtt{SMod}^{\bar{H}}$, and is in fact an $\bar{H}$-Galois extension over B. Moreover,

$$\iota: A \to D_f\square_{\bar{H}}H, \quad \iota(a) = \bar{a}_{(0)} \otimes a_{(1)} \tag{3.11}$$

is an isomorphism of B-algebras in $\mathtt{SMod}^H$, whence A/B arises from D_f/B. Here, $\rho(a) = a_{(0)} \otimes a_{(1)}$, and $A \to D_f$, $a \mapsto \bar{a}$ denotes the quotient map.

Proof. If $A = D\square_{\bar{H}}H$ is the inflation of D, then the map

$$f: R \to A = D\square_{\bar{H}}H, \quad f(x) = 1 \otimes x$$

is an algebra morphism in $\mathtt{SMod}^H$. This proves "only if" of part (1).

Suppose that A/B is H-Galois, and $f: R \to A$ is an algebra morphism in $\mathtt{SMod}^H$. Then A, regarded as an object in $\mathtt{SMod}_R$ through f, turns into an object in $\mathtt{SMod}_R^H$ $(= (\mathtt{SMod}^H)_R)$. [9, Proposition 1.1] proves a category equivalence,

$$\mathtt{SMod}^{\bar{H}} \xrightarrow{\approx} \mathtt{SMod}_R^H, \quad N \mapsto N\square_{\bar{H}} H,$$

and that ι is an isomorphism. It remains to prove that D_f is $\bar{H}$-Galois over B. Now, $B \to D_f \square_{\bar{H}} H$ $(= A)$ is faithfully flat. Notice from (2.7) that $N\square_{\bar{H}} H = N^{\oplus d}$ with $d = \dim(\wedge(W))$, whence the functor $\square_{\bar{H}} H$ is faithfully exact. Then it follows that $B \to D_f$ is faithfully flat. Apply $D_f \otimes_A$ to the beta map for A. Then there results an isomorphism in $\mathtt{SMod}_R^H$, which is seen to correspond, through the category equivalence above, to the beta map for D_f; it is, therefore, bijective. ∎

Remark 3.10. One can generalize to our super situation the result [5, Theorem 4] by Günther proved in the non-super situation, by modifying his proof. But the last proposition is formulated and proved under the restricted, super-commutativity assumption; it allows us to ignore the Miyashita–Ulbrich actions used in [5, p. 4391].

3.3. Main theorems re-formulated in the Hopf-algebra language

In this section, we let H be a Hopf super-algebra which is assumed to be finitely generated unless otherwise stated. We let $\mathtt{G}$ denote the affine super-group represented by H; it will be discussed only when H is assumed to be finitely generated, and will be, therefore, an affine algebraic super-group. Notice that the Hopf algebra $\bar{H}$ is finitely generated under the assumption.

Proposition 3.11. *Let* $\mathfrak{g} = \mathrm{Lie}(\mathtt{G})$ *be the Lie super-algebra of* $\mathtt{G}$*, and assume that odd component* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *satisfies*

$$[[\mathfrak{g}_1, \mathfrak{g}_1], \mathfrak{g}_1] = 0. \tag{3.12}$$

Suppose that B *is an arbitrary non-zero super-algebra. Then every* H*-Galois extension over* B *arises from an* $\bar{H}$*-Galois extension.*

The assumption above is clearly satisfied in case $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$, in which case the proposition will be applied to prove Lemma 6.4 contained in Step 2 of the proof of Theorem 3.16; the theorem is a key to the proof of the second main theorem, Theorem 1.9. The assumption is also satisfied if $\dim \mathfrak{g}_1 = 1$, since in general, a Lie super-algebra $\mathfrak{g}$ shall satisfy $[[x, x], x] = 0$ for all $x \in \mathfrak{g}_1$. Proposition 3.11 will be proved by spending the whole of Section 4.

Notice that the next theorem has a stronger conclusion than Proposition 3.11 above; see the sentence following the theorem.

Theorem 3.12. *Let* $B \neq 0$ *be a super-algebra. Assume*

(i) $\bar{H}$ *is smooth as an algebra,*

(ii) *the natural algebra surjection* $B_0 \to \bar{B}$ *splits, and*

(iii) *the super-ideal* (B_1) *of* B *is nilpotent.*

Choose arbitrarily a section $\overline{B} \to B_0$ of the surjection $B_0 \to \overline{B}$, and regard B as a super-algebra over $\overline{B}$ through it. Then every H-Galois extension A over B is isomorphic to

$$(B \otimes_{\overline{B}} \overline{A})\square_{\overline{H}} H. \tag{3.13}$$

Moreover, $\overline{A}$, which is regarded naturally as a purely even algebra in $\mathtt{SMod}^{\overline{H}}$, is a purely even $\overline{H}$-Galois extension over $\overline{B}$.

As a result, we can say that A/B arises from $B \otimes_{\overline{B}} \overline{A}$, which is clearly an $\overline{H}$-Galois extension over B.

Remark 3.13. (1) As for the assumption (i) above, it is proved by [18, Proposition A.3] without assuming H is finitely generated that the following are equivalent:

(a) $\overline{H}$ is smooth as an algebra;

(b) H is smooth as a super-algebra.

These equivalent conditions are necessarily satisfied if $\operatorname{char} k = 0$. Indeed, it is well known that $\overline{H}$ (in $\operatorname{char} k = 0$) is geometrically reduced; this is equivalent to saying that $\overline{H}$ is smooth in the restricted case where it is finitely generated. In the general case, $\overline{H}$ is a filtered union of finitely generated Hopf subalgebras which are smooth as was just seen, and it is, therefore, smooth; see [22, Exercise 9.3.2, Question 4].

(2) [15, Example 3.23] shows that Theorem 3.12 does not hold without assumption (i), even if the remaining (ii) and (iii) are satisfied.

(3) $(B \otimes_{\overline{B}} \overline{A})\square_{\overline{H}} H$ in (3.13) is canonically isomorphic to $B \otimes_{\overline{B}} (\overline{A}\square_{\overline{H}} H)$, as is easily seen by using (2.7).

Theorem 3.12 will be proved by spending the whole of Section 5. The theorem has the following corollary, which is seen to be translated into the first main theorem, Theorem 1.4, in view of Corollary 3.4 and Remark 3.13 (1) just above.

Corollary 3.14. *Assume that $\overline{H}$ is smooth. Let $B \neq 0$ be a Noetherian smooth algebra. If A/B is H-Galois, then $\overline{A}/\overline{B}$ is purely even $\overline{H}$-Galois. Moreover, the assignment $A \mapsto \overline{A}$ gives rise to a bijection from*

- *the set of all isomorphism classes of H-Galois extensions over B*

onto

- *the set of all isomorphism classes of purely even $\overline{H}$-Galois extensions over $\overline{B}$.*

Proof. To a purely even $\overline{H}$-Galois extension $E/\overline{B}$, one can assign an H-Galois extension, $(B \otimes_{\overline{B}} E)\square_{\overline{H}} H$, over B. This assignment gives rise to a desired inverse of $A \mapsto \overline{A}$, as is easily seen by using (2.7) and Theorem 3.12. ∎

Remark 3.15. We have an example of an H-Galois extension $(B \otimes_{\overline{B}} E)\square_{\overline{H}} H$ such as in the above proof, with the property that it has a B-algebra endomorphism (in fact, an automorphism by Lemma 3.5) in $\mathtt{SMod}^{H}$ which is not induced naturally from any

$\bar{H}$-comodule $\bar{B}$-algebra endomorphism of E. Indeed, let $H = \bar{H} = k[t, t^{-1}]$, the Laurent polynomial algebra regarded as the Hopf algebra which represents the multiplicative group. Let $B = \wedge(V)$ be the exterior algebra on a finite-dimensional vector space of dimension > 1. Choose as E the trivial $k[t, t^{-1}]$-Galois extension $k[t, t^{-1}]$ over k. Then one has $(B \otimes_{\bar{B}} E)\square_{\bar{H}} H = \wedge(V) \otimes k[t, t^{-1}]$. Every endomorphism in question uniquely arises from a super-algebra map $g\colon k[t, t^{-1}] \to \wedge(V)$ so that

$$\wedge(V) \otimes k[t, t^{-1}] \to \wedge(V) \otimes k[t, t^{-1}], \quad b \otimes h \mapsto b(g \otimes \mathrm{id})(\Delta(h)).$$

The endomorphism that arises from the g defined by $g(t^{\pm 1}) = 1 \pm x$, where $0 \neq x \in \wedge^2(V)$ such that $x^2 = 0$, is not induced from any endomorphism of $E = k[t, t^{-1}]$, as is easily seen. The example shows that the bijection proved in Corollary 3.14 does not result from a category-equivalence; this is translated into the fact stated in Remark 1.6 (1).

The second main theorem, Theorem 1.9, restricted to the affinity situation, is translated into the following.

Theorem 3.16. *Suppose that $A = (A, \rho)$ is a non-zero algebra in $\mathtt{SMod}^H$; $\bar{A}$ is then naturally a purely even algebra in $\mathtt{SMod}^{\bar{H}}$. Let $B = A^{\mathrm{co}\,H}$. Assume that*

(a) *the map $\alpha\colon A \otimes A \to A \otimes H$ in (3.1) is surjective,*

(b) *$\bar{A}/\bar{A}^{\mathrm{co}\,\bar{H}}$ is $\bar{H}$-Galois,*

(c) *$\bar{H}$ is smooth as an algebra, and*

(d) *A is Noetherian and smooth.*

Then A/B is H-Galois, and B is Noetherian, smooth and such that

$$\bar{B} = \bar{A}^{\mathrm{co}\,\bar{H}},$$

whence, with the assumptions of Theorem 3.12 all satisfied, A is of the form (3.13).

Remark 3.17. Under the situation of Theorem 3.16, assume that (a) is satisfied. It is proved by [15, Theorems 1.2 and 3.7] that A/B is H-Galois, if $\bar{H}$ is co-Frobenius as a coalgebra, or equivalently, if the affine algebraic group G_{ev}, which is now represented by $\bar{H}$, has an integral. When $\mathrm{char}\, k = 0$, the assumption is satisfied if and only if G_{ev} is linearly reductive; see Sullivan [20] also for characterizations in the positive characteristic case.

Theorem 3.16 will be proved in Section 6, and the result, regarded as the special, affinity case of Theorem 1.9, will be applied to prove that theorem in Section 7.

In what follows, until the end of Section 3.3, we let H be a Hopf super-algebra which may not be finitely generated.

The notion of smoothness for super-algebras (over k) is generalized to the notion for super-algebras over a fixed super-algebra and, moreover, for those super-algebras on which H-co-acts. For the latter we use the word "H-smooth," as follows.

Definition 3.18. Let us be given an algebra morphism $R \to S$ in $\mathtt{SMod}^H$. We say that S is *H-smooth over R* if, given a surjective R-algebra morphism $A \to B$ in $\mathtt{SMod}^H$ with nilpotent kernel, every R-algebra morphism $S \to B$ in $\mathtt{SMod}^H$ factors through A by some R-algebra morphism $S \to A$ in $\mathtt{SMod}^H$. The condition is equivalent to saying that every surjective R-algebra morphism in $\mathtt{SMod}^H$ that maps onto S and has nilpotent kernel necessarily splits, as is seen by a familiar augment using pull-backs; see the proof of [22, Proposition 9.3.3], for example. Here one should notice that the category of R-algebras in $\mathtt{SMod}^H$ has pull-backs.

We will use the word *H-smooth* when R such as above is trivial as an H-super-comodule (and, moreover, for applications in the sequel, when $R \to S$ is an embedding into $S^{\mathrm{co}\,H}$).

Proposition 3.19. *Let us be given an algebra morphism $R \to S$ in $\mathtt{SMod}^H$ such that R is trivial as an H-super-comodule. If S is H-smooth over R, then it is necessarily smooth over R.*

Proof. Let T be an R-super-algebra in general. Then $T \otimes H$ is naturally an R-algebra in $\mathtt{SMod}^H$. By the triviality assumption for R, the R-super-algebra maps $S \to T$ are in one-to-one correspondence with the R-algebra morphisms $S \to T \otimes H$ in $\mathtt{SMod}^H$, in which an R-super-algebra map $f\colon S \to T$ corresponds to

$$\tilde{f} := (f \otimes \mathrm{id}_H) \circ \rho\colon S \to T \otimes H, \tag{3.14}$$

where $\rho\colon S \to S \otimes H$ denotes the H-co-action on S.

Given an R-super-algebra surjection $A \to B$ with nilpotent kernel, we have a surjective R-algebra morphism $A \otimes H \to B \otimes H$ in $\mathtt{SMod}^H$ with nilpotent kernel. Assume that S is H-smooth over R. Given an R-super-algebra map $f\colon S \to B$, the R-algebra morphism $\tilde{f}\colon S \to B \otimes H$ in $\mathtt{SMod}^H$ factors through $A \otimes H$ by some $S \to A \otimes H$ that is necessarily of the form $\tilde{g}$ with $g\colon S \to A$ an R-super-algebra map. We see that f factors through A by g. This proves the desired smoothness. ∎

By the proposition just proven, H is smooth (over k), if it is H-smooth over k. The converse holds, as well, as is shown by the following.

Proposition 3.20. *H (or, equivalently, $\bar{H}$) is smooth if and only if it is H-smooth over k. These equivalent conditions are necessarily satisfied if* $\mathrm{char}\, k = 0$*; see Remark* 3.13 (1).

Proof. By modifying in an obvious manner the proof of the result [14, Proposition 1.10, (i) $\Longleftrightarrow$ (iii)] in the non-super situation, we see that the following are equivalent:

(a) for every trivial H-super-module M, the symmetric 2nd Hochschild cohomology $H^2_s(H, M)$ (constructed in $\mathtt{SMod}_k$) vanishes;

(b) every surjective algebra morphism in $\mathtt{SMod}^H$ that maps onto H and has nilpotent kernel necessarily splits.

We now have only to prove "only if." If H is smooth, then (a) and thus (b) are satisfied. But (b) is equivalent to saying that H is H-smooth over k, as was remarked at the last part of Definition 3.18. ∎

Remark 3.21. Let H' be a quotient Hopf algebra of H such that H is injective as a left or, equivalently, right H'-super-comodule; recall from the second paragraph of Section 2.2 that the condition is equivalent to saying that the co-tensor product functor $\square_{H'}H$ or $H\square_{H'}$ is exact. In the situation of Proposition 3.19, if S is H-smooth over R, then S is H'-smooth over R. This improves Proposition 3.19, generalizing k $(= H/\operatorname{Ker}\varepsilon)$ to H'. But the proof is a slight modification, which replaces (3.14) with $(f\square_{H'}\mathrm{id}_H)\circ\rho\colon S\to T\square_{H'}H$.

It follows that the conditions proved by Proposition 3.20 to be equivalent to each other are further equivalent to the condition that H is H'-smooth for some/any quotient Hopf super-algebra H' of H that satisfies the injectivity assumption.

4. Proof of Proposition 3.11

To prove the proposition (in 2 steps), let H be a finitely generated Hopf super-algebra, let $\mathtt{G}$ denote affine algebraic super-group represented by H, and let $\mathfrak{g} = \mathrm{Lie}(\mathtt{G})$ be the Lie super-algebra of $\mathtt{G}$. We assume $[[\mathfrak{g}_1,\mathfrak{g}_1],\mathfrak{g}_1] = 0$ as in (3.12).

4.1. Step 1

We claim the following.

Lemma 4.1. $\mathtt{G}$ *has a quotient affine super-group* $\mathtt{F}$ *which has the following two properties, where we let* $\mathfrak{f} = \mathrm{Lie}(\mathtt{F})$ *denote the Lie super-algebra of* $\mathtt{F}$.

(1) *The natural Lie super-algebra map* $\mathfrak{g}\to\mathfrak{f}$, *restricted to the odd components, is identical, or, in notation,* $\mathfrak{g}_1 = \mathfrak{f}_1$;

(2) *The closed embedding* $\mathtt{F}_{\mathrm{ev}}\hookrightarrow\mathtt{F}$ *splits (necessarily uniquely).*

Suppose that this lemma is proved. Let $J = \mathcal{O}(\mathtt{F})$. Then J is a Hopf super-subalgebra of H. By (2), this J is of the form of smash co-product

$$J = \bar{J}\blacktriangleright\!\!<\wedge(W). \tag{4.1}$$

Here W is a purely odd object in $\mathtt{SMod}^{\bar{J}}$; it constitutes the Hopf algebra $\wedge(W)$ in $\mathtt{SMod}^{\bar{J}}$, which contains elements of W as primitives. The Hopf algebra in turn constitutes the smash co-product above. By (1), we have $W = (\mathfrak{f}_1)^* = (\mathfrak{g}_1)^*$, whence

$${}^{\mathrm{co}\,\bar{H}}H = \wedge(W).$$

Note that the $k\oplus W$ in this ${}^{\mathrm{co}\,\bar{H}}H$ is a sub-object of H in $\mathtt{SMod}^H$, since it is, in fact, such of J in $\mathtt{SMod}^J$.

Given an H-Galois extension A/B, the inclusion $B \hookrightarrow A$ has a retraction $r\colon A \to B$ in $\mathtt{SMod}_B$. Indeed, by applying [19, Theorem 4.9] to the $\widehat{H}$-Galois extension $\widehat{A}/B$ in the non-super situation which was discussed in the proof of Proposition 3.6, one sees that the natural inclusion $B \hookrightarrow \widehat{A}$ $(= A \rtimes \mathbb{Z}_2)$ splits left (or right) B-linearly. The argument using separable extensions, which was used in the last-mentioned proof, ensures the existence of r. Now, the composite

$$k \oplus W \hookrightarrow H = k \otimes H \hookrightarrow A \otimes H \xrightarrow{\beta^{-1}} A \otimes_B A \xrightarrow{r \otimes_B \mathrm{id}_A} A$$

is a morphism in $\mathtt{SMod}^H$ which sends 1 in $k (\subset k \oplus W)$ to $1 \in A$. Therefore, this uniquely extends to an algebra morphism ${}^{\mathrm{co}\bar{H}}H = \wedge(W) \to A$ in $\mathtt{SMod}^H$. Proposition 3.9 (1) then proves Proposition 3.11 in question.

4.2. Step 2

It remains to prove Lemma 4.1 above. Let

$$C = H^\circ, \quad \underline{C} = \bar{H}^\circ$$

denote the dual Hopf super-algebra of H and the dual Hopf algebra of $\bar{H}$; see [10, Section 2.4]. The Hopf super-algebra C is not necessarily super-commutative, but is super-cocommutative. This C includes $\underline{C}$ as the largest purely even Hopf super-subalgebra. The spaces of all primitives in the Hopf (super-)algebras give the Lie (super-)algebras of the affine (super-)groups, so that

$$P(C) = \mathrm{Lie}(\mathtt{G})\ (= \mathfrak{g}), \quad P(\underline{C}) = \mathrm{Lie}(\mathtt{G}_{\mathrm{ev}})\ (= \mathfrak{g}_0).$$

Let $\langle\ ,\ \rangle\colon \underline{C} \times \bar{H} \to k$ denote the canonical pairing. Then $\bar{H}$ has the natural left $\underline{C}$-module structure given by

$$c \rightharpoonup a = a_{(1)} \langle c, a_{(2)} \rangle,$$

where $c \in \underline{C}$, $a \in \bar{H}$, and $\Delta(a) = a_{(1)} \otimes a_{(2)}$ denotes the co-product on $\bar{H}$. It is shown by [10, Lemma 27 and Theorem 29] that the set

$$\mathrm{Hom}_{\underline{C}}(C, \bar{H}) \tag{4.2}$$

of all left $\underline{C}$-linear maps $C \to \bar{H}$ has a natural structure of Hopf super-algebra, such that $H \simeq \mathrm{Hom}_{\underline{C}}(C, \bar{H})$ naturally as Hopf super-algebras. This Hom set is essentially the same as what is denoted by $\mathrm{Hom}_J(H(J, V), C)$ in [10, p. 1101], and the explicit Hopf-super-algebra structure is given by [10, Proposition 18 (2)–(3)].

Let

$$V = [\mathfrak{g}_1, \mathfrak{g}_1] (\subset \underline{C}).$$

One sees that, under the adjoint actions, $\underline{C}$ stabilizes V, and $\mathfrak{g}_1$ annihilates V by (3.12). It follows that the left and the right (super-)ideals of the Hopf (super-)algebra generated by V coincide, so that

$$\underline{C}V = V\underline{C}, \quad CV = VC.$$

Let

$$\underline{Q} = \underline{C}/V\underline{C}, \quad Q = C/VC$$

denote the resulting quotients. In view of [9, Theorem 3.6] one sees that Q is a super-cocommutative (but not necessarily, super-commutative) Hopf super-algebra, which includes $\underline{Q}$ as the largest purely even Hopf super-subalgebra, and the odd component of the primitives $P(Q)$ in Q remains to be $\mathfrak{g}_1$, or in notation,

$$P(Q)_1 = \mathfrak{g}_1. \tag{4.3}$$

Moreover, $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ in $P(Q)$, whence Q is of the form of smash product

$$Q = \underline{Q} \ltimes \wedge(\mathfrak{g}_1).$$

Define

$$\bar{H}' := \{a \in \bar{H} \mid V \rightharpoonup a = 0\}.$$

Then this is naturally a left $\underline{Q}$-module, whence we have the Hom set

$$J := \mathrm{Hom}_{\underline{Q}}(Q, \bar{H}') \tag{4.4}$$

as a subset of the one in (4.2). Since $\bar{H}$ is finitely generated, it is proper in the sense that the algebra map $\bar{H} \to (\underline{C})^*$ which arises from the canonical pairing is injective. Therefore, we have

$$\bar{H}' = \{a \in \bar{H} \mid \langle \underline{C}V, a\rangle = 0\}.$$

Moreover, since $\underline{C}V$ is a Hopf ideal of $\underline{C}$, it follows that $\bar{H}'$ is a Hopf subalgebra of $\bar{H}$. From the above-mentioned structure on $\mathrm{Hom}_{\underline{C}}(C, \bar{H})$, we see that the J in (4.4) is regarded as a Hopf super-subalgebra of H, such that

$$(\bar{J} =)J/(J_1) = \mathrm{Hom}_{\underline{Q}}(\underline{Q} \otimes k, \bar{H}') = \bar{H}'.$$

In addition, the natural embedding

$$\bar{J} = \mathrm{Hom}_{\underline{Q}}(Q/Q\mathfrak{g}_1, \bar{H}') \hookrightarrow \mathrm{Hom}_{\underline{Q}}(Q, \bar{H}')$$

is seen to be a Hopf super-algebra section of the projection $J \to \bar{J}$. It follows that J represents a quotient affine super-group, say F, of G which has the property (2) of Lemma 4.1. We see from (4.3) that it has the property (1), as well.

5. Proof of Theorem 3.12

Let H be a Hopf super-algebra, and suppose that A/B is an H-Galois extension. The proof is divided into three steps. In Steps 2 and 3 we will assume that H is finitely generated.

We start with only assuming (i), or namely, that $\bar{H}$ is smooth over k.

5.1. Step 1

This is a crucial step devoted to proving the following proposition, for which, we emphasize, H may not be finitely generated.

Proposition 5.1. *A is H-smooth over B, whence it is smooth over B by Proposition 3.19.*

We let $A^\natural$ denote the super-algebra A on which H co-acts trivially; thus we have $(A^\natural)^{\mathrm{co}\,H} = A^\natural$.

By Proposition 3.20, H is H-smooth. This implies the following.

Lemma 5.2. *$A^\natural \otimes H$ is H-smooth over $A^\natural$.*

Define

$$\mathbb{A} := A^\natural \otimes_B A.$$

Note that H co-acts on the second tensor factor A. Since this is isomorphic to $A^\natural \otimes H$ through the beta map in (3.2), the lemma above has the following corollary.

Corollary 5.3. *$\mathbb{A}$ is H-smooth over $A^\natural$.*

Recall from (3.3) the construction of the category $\mathtt{SMod}_A^H$, which is in fact k-linear abelian. By an analogous construction, we let

$$ {}_A\mathtt{SMod}_A^H \tag{5.1}$$

denote the k-linear abelian category of those (A, A)-bimodules M in $\mathtt{SMod}^H$ for which the left and the right B-actions coincide (in the super sense), or explicitly,

$$bm = (-1)^{|b||m|}mb, \quad b \in B, m \in M.$$

For example, $A \otimes_B A$, equipped with the co-diagonal H-co-action, is an object of this category. Clearly, A is as well. For simplicity, we will write ${}_A\mathtt{SMod}_A^H$ for (5.1). Notice that the category includes $\mathtt{SMod}_A^H$ as a full subcategory; it consists of all objects of ${}_A\mathtt{SMod}_A^H$ for which the left and the right A-actions coincide.

Recall from Theorem 3.3 that we have the (k-linear) category-equivalence

$$\Phi\colon \mathtt{SMod}_A^H \to \mathtt{SMod}_B, \quad \Phi(M) = M^{\mathrm{co}\,H}. \tag{5.2}$$

Lemma 5.4. *The direct sum*

$$\mathbb{P} := (A \otimes_B A)[0] \oplus (A \otimes_B A)[1]$$

of $A \otimes_B A$ ($= (A \otimes_B A)[0]$) and its degree shift $(A \otimes_B A)[1]$ is a projective generator in ${}_A\mathtt{SMod}_A^H$, whence the category has enough projectives, and for every $M \in {}_A\mathtt{SMod}_A^H$, the Ext *group*

$$\mathrm{Ext}^\bullet_{{}_A\mathtt{SMod}_A^H}(-, M)$$

is defined as the right derived functor of the Hom *functor* $\mathrm{Hom}_{{}_A\mathtt{SMod}_A^H}(-, M)$.

Proof. Let $M \in {}_A\mathtt{SMod}_A^H$. We have the natural identifications

$$\mathrm{Hom}_{{}_A\mathtt{SMod}_A^H}((A \otimes_B A)[i], M) = (M^{\mathrm{co}\,H})_i, \quad i = 0, 1$$

of k-vector spaces, which amount to

$$\mathrm{Hom}_{{}_A\mathtt{SMod}_A^H}(\mathbb{P}, M) = M^{\mathrm{co}\,H}. \tag{5.3}$$

With M regarded as an object in $\mathtt{SMod}_A^H$ through the obvious forgetful functor, the right-hand side of (5.3) coincides with $\Phi(M)$; here, we do not regard $\mathtt{SMod}_A^H \subset {}_A\mathtt{SMod}_A^H$. Since Φ preserves surjections, it follows that $\mathbb{P}$ as well as the two direct summands are projective. Since the inclusion $\Phi(M) = M^{\mathrm{co}\,H} \hookrightarrow M$ extends to a natural isomorphism $M^{\mathrm{co}\,H} \otimes_B A \xrightarrow{\simeq} M$ in $\mathtt{SMod}_A^H$, one sees that M is generated by $M^{\mathrm{co}\,H}$ as a right A-module, and so as an (A, A)-bimodule. Therefore, $\mathbb{P}$ is a generator. ∎

Remark 5.5. Since $\mathbb{A} = A^{\natural} \otimes_B A$ is an H-Galois extension over $A^{\natural}$, an analogous result of the lemma above holds, with $A \otimes_B A$ replaced by $\mathbb{A} \otimes_{A^{\natural}} \mathbb{A}$ $(= A^{\natural} \otimes_B (A \otimes_B A))$, in the k-linear abelian category

$${}_{\mathbb{A}}\mathtt{SMod}_{\mathbb{A}}^H,$$
$$A^{\natural}$$

which is defined in an obvious manner.

Note that B is a subalgebra of A in $\mathtt{SMod}^H$. Just as shown in [22, Section 9.3] in the ordinary situation (see also the last part of Definition 3.18), A is H-smooth over B if and only if an *extension of commutative B-algebras in* $\mathtt{SMod}^H$

$$0 \to M \to P \to A \to 0 \tag{5.4}$$

necessarily splits for every $M \in \mathtt{SMod}_A^H (\subset {}_A\mathtt{SMod}_A^H)$. By an *extension* such as the one above, we mean that $P \to A$ is a surjective morphism of commutative B-algebras in $\mathtt{SMod}^H$ with kernel M square-zero, $M^2 = 0$, in P. It is said to *split* if the last surjection splits as a B-algebra morphism in $\mathtt{SMod}^H$.

Lemma 5.6. *Every B-algebra extension of A, such as* (5.4)*, necessarily splits as a short exact sequence of B-modules in* $\mathtt{SMod}^H$.

Proof. By Proposition 3.6, the product map $B \otimes A \to A$ splits as a B-module morphism in $\mathtt{SMod}^H$.

It follows that every object in $\mathtt{SMod}_A^H$, and thus the kernel M, in particular, are injective in $\mathtt{SMod}^H$. Indeed, we see that

$$M = M^{\mathrm{co}\,H} \otimes_B A$$

is a direct summand of $M^{\mathrm{co}\,H} \otimes_B (B \otimes A) = M^{\mathrm{co}\,H} \otimes A$ in $\mathtt{SMod}^H$. The desired injectivity follows since A is injective in $\mathtt{SMod}^H$ by Theorem 3.3 (1).

We thus have a section $s\colon A \to P$ in $\mathtt{SMod}^H$ of the surjection $P \to A$. Choose arbitrarily a B-module section $t\colon A \to B \otimes A$ in $\mathtt{SMod}^H$ of the product map above. We see that the composite

$$A \xrightarrow{t} B \otimes A \xrightarrow{\mathrm{id}\otimes s} B \otimes P \to P,$$

where the last arrow indicates the product map, is a section of the B-module morphism $P \to A$ in $\mathtt{SMod}^H$. ∎

An important consequence of the lemma above is that B-algebra extensions of A such as above are classified by the symmetric 2nd Hochschild cohomology $H^2_s(A, M)$ (constructed in ${}_B\mathtt{SMod}^H$), which is naturally embedded into

$$\mathrm{Ext}^2_{{}_A\mathtt{SMod}^H_A}(A, M).$$

Let

$$J_{A/B} = \mathrm{Ker}(A \otimes_B A \to A)$$

be the kernel of the product map. Then one has the short exact sequence $0 \to J_{A/B} \to A \otimes_B A \to A \to 0$ in ${}_A\mathtt{SMod}^H_A$. The derived long exact sequence, combined with the projectivity result in Lemma 5.4, shows that the last 2nd Ext group is naturally isomorphic to

$$\mathrm{Ext}^1_{{}_A\mathtt{SMod}^H_A}(J_{A/B}, M).$$

Therefore, the algebra extension (5.4) splits if and only if the corresponding extension

$$0 \to M \to Q \to J_{A/B} \to 0 \tag{5.5}$$

in ${}_A\mathtt{SMod}^H_A$ splits; in other words, the cohomology class of (5.4) in $H^2_s(A, M)$ is zero if and only if its image in $\mathrm{Ext}^1_{{}_A\mathtt{SMod}^H_A}(J_{A/B}, M)$ through the injection

$$H^2_s(A, M) \hookrightarrow \mathrm{Ext}^2_{{}_A\mathtt{SMod}^H_A}(A, M) \xrightarrow{\simeq} \mathrm{Ext}^1_{{}_A\mathtt{SMod}^H_A}(J_{A/B}, M)$$

is zero. By the flat base-extension $A^\natural\otimes_B$, we obtain the mutually equivalent, analogous conditions, and see that they are both satisfied since $\mathbb{A} = A^\natural \otimes_B A$ is H-smooth over $A^\natural$ by Corollary 5.3. The second analogous condition tells us that the base extension

$$0 \to A^\natural \otimes_B M \to A^\natural \otimes_B Q \to A^\natural \otimes_B J_{A/B} = J_{\mathbb{A}/A^\natural} \to 0 \tag{5.6}$$

of (5.5) splits in ${}_{\mathbb{A}}\mathtt{SMod}^H_{A^\natural}$.

Now, we let $A^\natural$ recover the original structure as an algebra in $\mathtt{SMod}^H$, and suppose that the sequence (5.6) has an additional structure which arises from the tensor factors $A^\natural$ regarded as objects in

$${}_A\mathtt{SMod}^H \ (= {}_A(\mathtt{SMod}^H)).$$

To make this clearer, let A_i (resp., H_i), $i = 1, 2$, be copies of A (resp., H), regard A_i as an algebra in $\mathtt{SMod}^{H_i}$, and suppose that A_1 is the $A^\natural$ equipped with the recovered structure. One sees that

$$A_1 \otimes_B A_2 \quad \text{and} \quad A_2\ (= B \otimes_B A_2)$$

naturally form algebras in $\mathtt{SMod}^{H_1 \otimes H_2}$, which both include B as subalgebra. Therefore, we have the k-linear abelian category

$${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$$

of those $(A_1 \otimes_B A_2, A_2)$-bimodules in $\mathtt{SMod}^{H_1 \otimes H_2}$ for which the left and the right B-actions coincide. The short exact sequence (5.6) is naturally regarded as such a sequence in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$.

Lemma 5.7. *The short exact sequence* (5.6) *splits in* ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$.

Proof. Our argument below is essentially the same as the one which proves the Maschke-type Theorem [4, Theorem 1].

Let us consider the injection $A_1 \otimes_B M \to A_1 \otimes_B Q$ in (5.6). Since the category ${}_{\mathbb{A}}\mathtt{SMod}_{A^\natural}^{H}{}_{\mathbb{A}}$ is naturally identified with ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_2}_{A_2}$, the injection has a retraction, say r, in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_2}_{A_2}$. The tensor product

$$\tilde{r} := r \otimes \mathrm{id}_{H_1} \colon (A_1 \otimes H_1) \otimes_B Q \to (A_1 \otimes H_1) \otimes_B M.$$

is seen to be in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$, where A_1 acts on the $A_1 \otimes H_1$ in these objects through the H_1-co-action on A_1, and H_1 co-acts on the tensor factor H_1; to be precise, the latter co-action involves the super-symmetry relevant to Q or M. One sees that $\tilde{r}$ is a retraction, in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$, of the above injection tensored with id_{H_1}.

By Theorem 3.3 (1), A_1 is injective in $\mathtt{SMod}^{H_1}$. It follows that there is a morphism $\phi \colon H_1 \to A_1$ in $\mathtt{SMod}^{H_1}$ such that $\phi(1) = 1$. We see that the map $t \colon A_1 \otimes H_1 \to A_1$ defined by

$$t(a \otimes h) = a_{(0)}\phi(\mathcal{S}(a_{(1)})h),$$

where $\mathcal{S}$ denotes the antipode of H_1, is a retraction of the H_1-co-action $A_1 \to A_1 \otimes H_1$, $a \mapsto a_{(0)} \otimes a_{(1)}$ in ${}_{A_1}\mathtt{SMod}^{H_1}$; cf. [4, p. 100]. The tensor product

$$\tilde{t} := t \otimes_B \mathrm{id}_M \colon (A_1 \otimes H_1) \otimes_B M \to A_1 \otimes_B M$$

is a retraction in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$ of the H_1-co-action on $A_1 \otimes_B M$. One sees that the composite $\tilde{t} \circ \tilde{r}$, further composed with the H_1-co-action on $A_1 \otimes_B Q$, gives a retraction in ${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2}$ of the injection in question. This proves the lemma. ∎

Apply to (5.6) the equivalence of categories $(\ )^{\mathrm{co}(H_1)} \colon {}_{A_1}\mathtt{SMod}^{H_1} \approx {}_B\mathtt{SMod}$, which is essentially the same as the Φ in (5.2). This can be, in fact, replaced by

$${}_{A_1 \otimes_B A_2}\mathtt{SMod}_B^{H_1 \otimes H_2}_{A_2} \approx {}_{A_2}\mathtt{SMod}_{A_2}^{H_2}\ (= {}_{A_2}\mathtt{SMod}_B^{H_2}_{A_2}).$$

Then there results precisely the short exact sequence (5.5) in the category ${}_A\mathtt{SMod}_A^H$ on the right-hand side, which splits since, by Lemma 5.7, (5.6) splits in the category on the left-hand side. This implies that A is H-smooth over B, as was seen before.

5.2. Step 2

Let us be in the situation at the beginning of this Section 5. But here we assume that H is finitely generated (and continue to assume (i)). In addition, we assume that B is purely even, $B = B_0$, or equivalently, $B = \bar{B}$, and aim to prove Theorem 3.12, that is, the following.

Proposition 5.8. *Under the assumptions above, we have*

$$A \simeq \bar{A}\Box_{\bar{H}} H$$

(or, namely, A is of the form (3.13) with $B = \bar{B}$), and $\bar{A}/B$ is $\bar{H}$-Galois.

Lemma 5.9. *We have the following:*

(1) *$\bar{A}$ is smooth over B;*

(2) *$\bar{A}$ is faithfully flat over B.*

Proof. (1) By Proposition 5.1, A is smooth over the now purely even B. As a general fact, this implies that $\bar{A}$ is smooth over B. Indeed, given an algebra surjection $S \to T$ with nilpotent kernel, every algebra map $\bar{A} \to T$, identified with the naturally associated super-algebra map $A \to T$, factors through S by some $A \to S$ or by the naturally induced $\bar{A} \to S$.

(2) We claim that the super-ideal I_A $(= (A_1))$ of A is nilpotent. Indeed, since H is finitely generated by assumption, $A^\natural \otimes_B A (\simeq A^\natural \otimes H)$ is finitely generated over $A^\natural$. Since $B \to A^\natural$ is faithfully flat, A is finitely generated over B; this implies that the super-ideal I_A, being generated by finitely many odd elements, is nilpotent.

The claim above, combined with (1), shows that the B-super-algebra surjection $A \to \bar{A}$ (or in fact, $A_0 \to \bar{A}$) splits. The faithful flatness of A over B implies that $\bar{A}$ is faithful flat over B. ∎

Apply $\bar{A}\otimes_A$ to the isomorphism β in (3.2) to obtain

$$\bar{A} \otimes_B A \simeq \bar{A} \otimes H.$$

Then one sees that A is a twisted form of the B-algebra $B \otimes H$ in $\mathtt{SMod}^H$, which is split by $\bar{A}$; this $\bar{A}$ is faithfully flat over B by Lemma 5.9 (2). Such twisted forms are classified by the 1st Amitsur cohomology set

$$H^1(\bar{A}/B, \mathtt{G}) \tag{5.7}$$

for the affine super-group $\mathtt{G}$ represented by H. This cohomology set is in fact constructed in the super situation; the construction is analogous to the one, as given in [21, Part V], of

the ordinary cohomology set. One sees that super-algebra maps $H \to \bigotimes_B^n \bar{A}$, $1 \le n \le 3$, uniquely factor through $\bar{H}$. Moreover, the complex for computing (5.7) is naturally identified with the one for computing the ordinary 1st Amitsur cohomology set

$$H^1(\bar{A}/B, \mathsf{G}_{\mathrm{ev}})$$

for the affine group G_{ev} represented by $\bar{H}$. We see easily that the identification

$$H^1(\bar{A}/B, \mathsf{G}_{\mathrm{ev}}) \xrightarrow{\simeq} H^1(\bar{A}/B, \mathsf{G})$$

is realized by

$$D \mapsto D\square_{\bar{H}} H,$$

where D is a twisted form of $B \otimes \bar{H}$ which is split by $\bar{A}$. In particular, A is (uniquely) of the form $D\square_{\bar{H}} H$. To be more precise, A is isomorphic to $D\square_{\bar{H}} H$ through an isomorphism such as in (3.11), which is seen by (2.7) to induce $\bar{A} \simeq D$. Being a twisted form of $B \otimes \bar{H}$, $\bar{A}$ $(= D)$ is $\bar{H}$-Galois over B. This completes the proof of Proposition 5.8.

5.3. Step 3

Under the situation at the beginning of Section 5, we assume that H is finitely generated. In addition to (i), we assume that

(ii) the natural algebra surjection $B_0 \to \bar{B}$ splits, and

(iii) the super-ideal (B_1) of B is nilpotent,

as in Theorem 3.12. We are going to complete the proof of the theorem.

Let

$$F := A/B_1 A.$$

This is seen to be an algebra in $\mathtt{SMod}^H$. Given a purely even $\bar{B}$-super-module N, one sees that

$$N \otimes_B A = (N \otimes_B \bar{B}) \otimes_B A = N \otimes_{\bar{B}} F.$$

It follows that F is $\bar{B}$-faithfully flat, since A is B-faithfully flat.

By applying $\bar{B}\otimes_B$ to the isomorphism β in (3.2), we obtain

$$F \otimes_{\bar{B}} F \simeq F \otimes H.$$

Therefore, F is H-Galois over the purely even $\bar{B}$. By Proposition 5.1, F is H-smooth over $\bar{B}$. By (ii), the natural surjection $A \to F$ may be regarded as a $\bar{B}$-algebra morphism in $\mathtt{SMod}^H$, which splits since its kernel is nilpotent by (iii). Choose arbitrarily a section $F \to A$, through which we can regard A as an F-algebra in $\mathtt{SMod}^H$, and in particular, as an object in $\mathtt{SMod}_F^H$.

Notice that $\bar{F} = \bar{A}$. Then it follows by Proposition 5.8 that

$$F \simeq \bar{A}\square_{\bar{H}} H,$$

and $\bar{A}/\bar{B}$ is $\bar{H}$-Galois. The category equivalence $\mathtt{SMod}_F^H \approx \mathtt{SMod}_{\bar{B}}$ as in (5.2) shows

$$A \simeq B \otimes_{\bar{B}} F \simeq B \otimes_{\bar{B}} (\bar{A}\square_{\bar{H}} H) = (B \otimes_{\bar{B}} \bar{A})\square_{\bar{H}} H,$$

as desired; see Remark 3.13 (3) for the last canonical isomorphism.

6. Proof of Theorem 3.16

Let R be a super-algebra. Recall from (2.4) that we write $I_R = (R_1)$ and $\bar{R} = R/I_R$. Define

$$\mathrm{gr}\, R := \bigoplus_{n\geq 0} I_R^n/I_R^{n+1}.$$

Thus, we have

$$(\mathrm{gr}\, R)(0) = \bar{R}, \quad (\mathrm{gr}\, R)(1) = I_R/I_R^2,$$

in particular. This $\mathrm{gr}\, R$ is a *graded super-algebra*, by which we mean an $\mathbb{N}$-graded algebra which, regarded as $\mathbb{Z}_2$-graded by mod 2 reduction, is super-commutative. In other words, a *graded super-algebra* is a commutative algebra in the tensor category

$$\mathtt{GrMod}_k = (\mathtt{GrMod}_k, \otimes, k) \tag{6.1}$$

of the $\mathbb{N}$-graded vector spaces over k, equipped with the super-symmetry for which objects are regarded to be $\mathbb{Z}_2$-graded by mod 2 reduction; for example, the exterior algebras which have appeared so far are such algebras. We see

$$\overline{\mathrm{gr}\, R} = \bar{R}.$$

A *graded Hopf super-algebra* is a commutative Hopf algebra in the symmetric category above.

To prove Theorem 3.16 (in three steps), suppose that we are in the situation of the theorem. Thus, H is a finitely generated Hopf super-algebra, $A = (A, \rho)$ is a non-zero algebra in $\mathtt{SMod}^H$, and $B = A^{\mathrm{co}\, H}$.

6.1. Step 1

From the four assumptions (a)–(d) in Theorem 3.16, we here assume the following three conditions:

(b) $\bar{A}/\bar{A}^{\mathrm{co}\,\bar{H}}$ is $\bar{H}$-Galois;

(c) $\bar{H}$ is smooth as an algebra;

(d) A is Noetherian and smooth.

Lemma 6.1. *We have the following.*

(1) $\bar{A}^{\mathrm{co}\,\bar{H}}$ *is Noetherian and smooth.*

(2) $\bar{A}$ *is* $\bar{H}$*-smooth over* k.

Proof. (1) By (d), $\overline{A}$ is Noetherian and smooth; for the smoothness, see the proof of Lemma 5.9 (1).

In general, given a faithfully-flat homomorphism $R \to S$ of algebras over a field, we have that if S is Noetherian and smooth, then R is, as well. Indeed, R is then clearly Noetherian. To see that it is smooth or, equivalently, geometrically regular, it suffices, by base extension and localization, to prove that R is regular, assuming that $R \to S$ is a flat local homomorphism of Noetherian local algebras with S being regular. The desired regularity of R follows by using Serre's characterization of regularity as having finite global dimension.

Part (1) follows from this general fact applied to $\overline{A}^{\mathrm{co}\,\overline{H}} \hookrightarrow \overline{A}$, which is faithfully flat by (b).

(2) Everything here is purely even. We wish to prove that an extension

$$0 \to M \to T \to \overline{A} \to 0$$

of commutative algebras in $\mathtt{SMod}^{\overline{H}}$ necessarily splits. This splits as a short exact sequence in $\mathtt{SMod}^{\overline{H}}$, since M is in $\mathtt{SMod}^{\overline{H}}_{\overline{A}}$, and is, therefore, injective in $\mathtt{SMod}^{\overline{H}}$ by (b); see the second paragraph of the proof of Lemma 5.6. Therefore, the algebra map $T^{\mathrm{co}\,\overline{H}} \to \overline{A}^{\mathrm{co}\,\overline{H}}$ restricted to the $\overline{H}$-co-invariants is surjective, whence it splits by (1) above. The desired result follows, since $\overline{A}$ is $\overline{H}$-smooth over $\overline{A}^{\mathrm{co}\,\overline{H}}$, as is seen from Proposition 5.1 applied to $\overline{A}/\overline{A}^{\mathrm{co}\,\overline{H}}$; the last application is possible by (b) and (c). ∎

Proposition 6.2. *$A^{\mathrm{co}\,\overline{H}}$ is a Noetherian smooth super-algebra such that*

$$\overline{A^{\mathrm{co}\,\overline{H}}} = \overline{A}^{\mathrm{co}\,\overline{H}}.$$

Proof. By (d), I_A is nilpotent. By Lemma 6.1 (2), we have a section of the algebra projection $A \to \overline{A}$ in $\mathtt{SMod}^{\overline{H}}$, through which we can and we do regard A as an $\overline{A}$-algebra in $\mathtt{SMod}^{\overline{H}}$. Let

$$P_A := I_A/I_A^2 \ (= (\mathrm{gr}\,A)(1)). \tag{6.2}$$

Then this is an object in $\mathtt{SMod}^{\overline{H}}_{\overline{A}}$, which is a finitely generated projective $\overline{A}$-super-module by Proposition 2.2. The natural projection $I_A \to P_A$ is now a morphism in $\mathtt{SMod}^{\overline{H}}_{\overline{A}}$, and has a section essentially by the Maschke-type Theorem [4, Theorem 1]; see the proof of Lemma 5.7. The argument of Remark 2.3, slightly modified, shows that we have an isomorphism

$$\xi\colon \wedge_{\overline{A}}(P_A) \xrightarrow{\simeq} A \tag{6.3}$$

of $\overline{A}$-algebras in $\mathtt{SMod}^{\overline{H}}$. By the category equivalence $\mathtt{SMod}^{\overline{H}}_{\overline{A}} \approx \mathtt{SMod}_{\overline{A}^{\mathrm{co}\,\overline{H}}}$ ensured by the assumption (b), we see that P_A is of the form

$$P_A = Q_A \otimes_{\overline{A}^{\mathrm{co}\,\overline{H}}} \overline{A}, \quad \text{where } Q_A = P_A^{\mathrm{co}\,\overline{H}}.$$

This Q_A is a finitely generated projective $\overline{A}^{\mathrm{co}\,\overline{H}}$-module since $\overline{A}$ is faithfully flat over $\overline{A}^{\mathrm{co}\,\overline{H}}$. We thus have an isomorphism

$$\wedge_{\overline{A}^{\mathrm{co}\,\overline{H}}}(Q_A) \otimes_{\overline{A}^{\mathrm{co}\,\overline{H}}} \overline{A} \simeq A \tag{6.4}$$

of $\bar{A}$-algebras in $\mathtt{SMod}^{\bar{H}}$, which restricts to an isomorphism

$$\wedge_{\bar{A}^{\mathrm{co}\,\bar{H}}}(Q_A) \simeq A^{\mathrm{co}\,\bar{H}}$$

of super-algebras over $\bar{A}^{\mathrm{co}\,\bar{H}}$. This implies the desired result by Proposition 2.2. ∎

6.2. Step 2

In what follows we assume, in addition,

(a) the super-algebra map $A \otimes A \to A \otimes H$, $a \otimes b \mapsto a\rho(b)$ is surjective.

Applying gr to everything, we write

$$\mathcal{A} = \mathrm{gr}\, A, \quad \mathcal{H} = \mathrm{gr}\, H.$$

Then, $\mathcal{H}$ is a finitely generated graded Hopf super-algebra. Let $\mathtt{GrMod}^{\mathcal{H}}$ denote the symmetric category of the right $\mathcal{H}$-comodules in $\mathtt{GrMod}_k$; see (6.1). Then, $\mathcal{A} = (\mathcal{A}, \mathrm{gr}(\rho))$ is an algebra in $\mathtt{GrMod}^{\mathcal{H}}$. In particular, $\mathcal{H}$ is a Hopf super-algebra, and $\mathcal{A}$ is an algebra in $\mathtt{SMod}^{\mathcal{H}}$.

Lemma 6.3. *These $\mathcal{H}$ and $\mathcal{A}$ satisfy those conditions which corresponds to* (a)–(d) *for H and A; explicitly, they are the same* (b), (c), *and the following modified two:*

(a′) *the super-algebra map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$, $a \otimes b \mapsto a\,\mathrm{gr}(\rho)(b)$ is surjective;*

(d′) *$\mathcal{A}$ is Noetherian and smooth.*

Proof. By Proposition 2.2, (d′) follows from (d).

Since the super-algebra map in (a′) is induced from the one in (a) with gr applied, one sees that (a) implies (a′), in view of the following general fact. Given a surjection of super-algebras, $R \to S$, the induced map $\mathrm{gr}\, R \to \mathrm{gr}\, S$ of graded super-algebras is surjective. Indeed, one sees that R/I_R^2 (with $I_R = (R_1)$ as before) has the even component $(R/I_R^2)_0$ which is canonically isomorphic to $\bar{R} = R/I_R$, and has $(R/I_R^2)_1 = I_R/I_R^2$ as the odd component. It follows that the given super-algebra surjection induces surjections

$$\bar{R} \to \bar{S}, \quad I_R/I_R^2 \to I_S/I_S^2,$$

which are identified with the neutral and the 1st components of the induced, graded super-algebra map. The map is surjective since the graded super-algebras are generated by the described components. ∎

Thus, we may assume (a′), (b), (c), and (d′). We now claim that the conclusion of Theorem 3.16 holds for $\mathcal{H}$ and $\mathcal{A}$, as follows.

Lemma 6.4. *$\mathcal{A}/\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ is $\mathcal{H}$-Galois, and $\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ is Noetherian, smooth and such that*

$$\overline{\mathcal{A}^{\mathrm{co}\,\mathcal{H}}} = \bar{A}^{\mathrm{co}\,\bar{H}} \tag{6.5}$$

Proof. Let $W := (H^+/(H^+)^2)_1$ as in (2.5). Then $\mathcal{H}$ is (canonically) isomorphic to the smash co-product $\bar{H} \rtimes \wedge(W)$ just as in (4.1); see [16, (3.14)], for example. In other words, we have the split short exact sequence

$$\bar{H} \rightarrowtail \mathcal{H} \twoheadrightarrow \wedge(W)$$

of (graded) Hopf super-algebras, where $\wedge(W)$ is supposed to contain all elements of W as primitives. Note that $\mathcal{A}$ turns into an algebra in $\mathtt{SMod}^{\wedge(W)}$ along the last surjection $\mathcal{H} \twoheadrightarrow \wedge(W)$. The assumption (a′) implies that the alpha map (see (3.1)) for this $\mathcal{A}$ in $\mathtt{SMod}^{\wedge(W)}$ is surjective. Remark 3.17 (for H), now applied to the co-Frobenius $\wedge(W)$, shows that $\mathcal{A}/\mathcal{A}^{\mathrm{co}(\wedge(W))}$ is $\wedge(W)$-Galois, whence $\mathcal{A}^{\mathrm{co}(\wedge(W))} \to \mathcal{A}$ is faithfully flat, in particular.

Applying Proposition 3.11 (for H) to $\wedge(W)$, one can choose a morphism $\wedge(W) \to \mathcal{A}$ of algebras in $\mathtt{SMod}^{\wedge(W)}$. With gr applied, this morphism may be supposed to be graded. It uniquely extends to an isomorphism

$$\mathcal{A}^{\mathrm{co}(\wedge(W))} \otimes \wedge(W) \simeq \mathcal{A}$$

of $\mathcal{A}^{\mathrm{co}(\wedge(W))}$-algebras in $\mathtt{GrMod}^{\wedge(W)}$, as is seen by using the equivalence of categories

$$\mathtt{SMod}_{\mathcal{A}}^{\wedge(W)} \xrightarrow{\approx} \mathtt{SMod}_{\mathcal{A}^{\mathrm{co}(\wedge(W))}}, \quad M \mapsto M^{\mathrm{co}(\wedge(W))} \tag{6.6}$$

ensured by $\mathcal{A}/\mathcal{A}^{\mathrm{co}(\wedge(W))}$ being $\wedge(W)$-Galois. It follows that

$$\overline{\mathcal{A}^{\mathrm{co}(\wedge(W))}} = \bar{A}. \tag{6.7}$$

Moreover, by (d′), $\mathcal{A}^{\mathrm{co}(\wedge(W))}$ is Noetherian and smooth.

Notice that $\mathcal{A}^{\mathrm{co}(\wedge(W))}$ is an algebra in $\mathtt{GrMod}^{\bar{H}}$ such that

$$(\mathcal{A}^{\mathrm{co}(\wedge(W))})^{\mathrm{co}\,\bar{H}} = \mathcal{A}^{\mathrm{co}\,\mathcal{H}}.$$

Apply the equivalence of categories (6.6) to the surjection in (a′). The result is the faithfully-flat base extension $\mathcal{A} \otimes_{\mathcal{A}^{\mathrm{co}(\wedge(W))}}$ of the alpha map

$$\mathcal{A}^{\mathrm{co}(\wedge(W))} \otimes \mathcal{A}^{\mathrm{co}(\wedge(W))} \to \mathcal{A}^{\mathrm{co}(\wedge(W))} \otimes \bar{H}$$

for $\mathcal{A}^{\mathrm{co}(\wedge(W))}$ in $\mathtt{GrMod}^{\bar{H}}$. Therefore, this alpha map is surjective. By (6.7) we can apply (the argument of proving) Proposition 6.2 to $\mathcal{A}^{\mathrm{co}(\wedge(W))}$. It results that

$$\mathcal{A}^{\mathrm{co}\,\mathcal{H}}\ (= (\mathcal{A}^{\mathrm{co}(\wedge(W))})^{\mathrm{co}\,\bar{H}})$$

is Noetherian and smooth, and we have an isomorphism

$$\wedge_{\bar{A}^{\mathrm{co}\,\bar{H}}}(N_A) \otimes_{\bar{A}^{\mathrm{co}\,\bar{H}}} \bar{A} \simeq \mathcal{A}^{\mathrm{co}(\wedge(W))} \tag{6.8}$$

of $\bar{A}$-algebras in $\mathtt{GrMod}^{\bar{H}}$. This isomorphism is analogous to (6.4), but it is now canonical. Here, N_A denotes the $\bar{H}$-co-invariants in the 1st component of $\mathcal{A}^{\mathrm{co}(\wedge(W))}$, or, in other words,

$$N_A = P_A \cap \mathcal{A}^{\mathrm{co}\,\mathcal{H}};$$

see (6.2). This N_A is a finitely generated projective $\bar{A}^{\mathrm{co}\,\bar{H}}$-module. The isomorphism (6.8) shows equation (6.5). It also shows that $\mathcal{A}^{\mathrm{co}(\wedge(W))}$ is injective in $\mathtt{SMod}^{\bar{H}}$, whence by Theorem 3.3 (1), $\mathcal{A}^{\mathrm{co}(\wedge(W))}/\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ is $\bar{H}$-Galois. Therefore, $\mathcal{A}^{\mathrm{co}\,\mathcal{H}} \to \mathcal{A}$ is, as a composite of two faithfully-flat homomorphisms, faithfully flat.

It remains to show that the beta map $\mathcal{A} \otimes_{\mathcal{A}^{\mathrm{co}\,\mathcal{H}}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ (see (3.2)) for $\mathcal{A}$ in $\mathtt{GrMod}^{\mathcal{H}}$ is bijective. After applying (6.6), one has only to prove that

$$\mathcal{A} \otimes_{\mathcal{A}^{\mathrm{co}\,\mathcal{H}}} \mathcal{A}^{\mathrm{co}(\wedge(W))} \to \mathcal{A} \otimes \bar{H}$$

is bijective. But this is a base extension of the beta map

$$\mathcal{A}^{\mathrm{co}(\wedge(W))} \otimes_{\mathcal{A}^{\mathrm{co}\,\mathcal{H}}} \mathcal{A}^{\mathrm{co}(\wedge(W))} \xrightarrow{\simeq} \mathcal{A}^{\mathrm{co}(\wedge(W))} \otimes \bar{H}$$

for $\mathcal{A}^{\mathrm{co}(\wedge(W))}$ in $\mathtt{GrMod}^{\bar{H}}$, which is bijective since $\mathcal{A}^{\mathrm{co}(\wedge(W))}/\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ is $\bar{H}$-Galois. This completes the proof. ∎

6.3. Step 3

We have seen that $\mathcal{A}/\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ is an $\mathcal{H}$-Galois extension which satisfies the assumptions of Theorem 3.12. Let $\mathcal{R} := {}^{\mathrm{co}\,\bar{H}}\mathcal{H}$; this $\mathcal{R}$ is seen to be an algebra in $\mathtt{GrMod}^{\mathcal{H}}$. The theorem tells us that $\mathcal{A}/\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$ arises (uniquely) from an $\bar{H}$-Galois extension over $\mathcal{A}^{\mathrm{co}\,\mathcal{H}}$. By Proposition 3.9, we have an algebra morphism $f\colon \mathcal{R} \to \mathcal{A}$ in $\mathtt{SMod}^{\mathcal{H}}$, which can be chosen, with gr applied, so as to be in $\mathtt{GrMod}^{\mathcal{H}}$. Define $D_f := \mathcal{A}/(f(\mathcal{R}^+))$, just as in (3.10), and let $p\colon \mathcal{A} \to D_f$ denote the natural projection; these are in $\mathtt{GrMod}^{\bar{H}}$. Let

$$\iota\colon \mathcal{A} \xrightarrow{\simeq} D_f \square_{\bar{H}} \mathcal{H},$$

be the isomorphism of algebras in $\mathtt{GrMod}^{\mathcal{H}}$, such as given by (3.11). Recall that this ι is induced from the composite

$$\mathcal{A} \to \mathcal{A} \otimes \mathcal{H} \xrightarrow{p \otimes \mathrm{id}_{\mathcal{H}}} D_f \otimes \mathcal{H},$$

where the first arrow indicates the $\mathcal{H}$-co-action on $\mathcal{A}$. Using the isomorphism ξ in (6.3), define $q\colon A \to D_f$ to be the composite

$$q\colon A \xrightarrow{\xi^{-1}} \wedge_{\bar{A}}(P_A) \xrightarrow{\mathrm{gr}(\xi)} \mathcal{A} \xrightarrow{p} D_f. \tag{6.9}$$

This is an algebra morphism in $\mathtt{SMod}^{\bar{H}}$ such that $\mathrm{gr}(q) = p$. Note that the composite

$$A \to A \otimes H \xrightarrow{q \otimes \mathrm{id}_H} D_f \otimes H,$$

where the first arrow indicates the H-co-action on A, induces an algebra morphism in $\mathtt{SMod}^H$,

$$\eta\colon A \to D_f \square_{\bar{H}} H,$$

which is indeed an isomorphism, since we see $\mathrm{gr}(\eta) = \iota$.

One sees from the isomorphism ι that D_f is isomorphic, as an algebra in $\mathtt{SMod}^{\bar{H}}$, to $\mathcal{A}^{\mathrm{co}(\wedge W)}$. Therefore, D_f is injective in $\mathtt{SMod}^{\bar{H}}$, as is seen again from (6.8). The isomorphism η tells us that A is injective in $\mathtt{SMod}^H$. By Theorem 3.3 (1), assumption (a) implies that $A/A^{\mathrm{co}\,H}$ is H-Galois. Moreover, η and ι induce the isomorphisms

$$A^{\mathrm{co}\,H} \simeq (D_f)^{\mathrm{co}\,\bar{H}} \simeq \mathcal{A}^{\mathrm{co}\,\mathcal{H}} \tag{6.10}$$

of super-algebras. More precisely, the composite

$$A \xrightarrow{\xi^{-1}} \wedge_{\bar{A}}(P_A) \xrightarrow{\mathrm{gr}(\xi)} \mathcal{A}$$

of the first two isomorphisms in (6.9) induces an isomorphism $A^{\mathrm{co}\,H} \simeq \mathcal{A}^{\mathrm{co}\,\mathcal{H}}$.

By Lemma 6.4, $A^{\mathrm{co}\,H}$ is Noetherian, smooth and such that

$$\overline{A^{\mathrm{co}\,H}} = \bar{A}^{\mathrm{co}\,\bar{H}}.$$

This completes the proof of Theorem 3.16.

Remark 6.5. By applying gr to (6.10), we obtain an isomorphism

$$\mathrm{gr}(A^{\mathrm{co}\,H}) \simeq (\mathrm{gr}\,A)^{\mathrm{co}(\mathrm{gr}\,H)},$$

which is indeed canonical, arising from the inclusion $A^{\mathrm{co}\,H} \hookrightarrow A$ with gr applied. Let $B = A^{\mathrm{co}\,H}$, as above. Then it follows from (6.8) that the isomorphism in degree 1 gives

$$I_B/I_B^2 = N_A. \tag{6.11}$$

7. Proof of Theorem 1.9

Suppose that we are in the situation of Theorem 1.9. We work in the geometrical setting, in which super-schemes are regarded as super-ringed spaces; see [17, Section 4]. Let $\mathcal{O}_{\mathsf{X}}$ denote the structure sheaf of X, in particular. As a ringed space, the associated scheme X_{ev} has the same underlying topological space as X, and the structure sheaf assigns the algebra $\overline{\mathcal{O}_{\mathsf{X}}(Y)}$ to every open set Y.

Lemma 7.1. *Under the additional assumption that* X *and* $\mathsf{X}_{\mathsf{ev}}/\mathsf{G}_{\mathsf{ev}}$ *are both affine, Theorem 1.9 holds.*

Proof. Suppose that we are in the situation of the beginning of Section 3.1. We now assume that the Hopf super-algebra H is finitely generated and the algebra A in $\mathtt{SMod}^H$ is

Noetherian. Note that the freeness of the G-action on X is equivalent to saying that the map α in (3.1) is an epimorphism of super-algebras. Therefore, the lemma follows from Theorem 3.16 if we prove that α is necessarily surjective. But this follows from the following general fact for $R \to S$ applied to α. ∎

Lemma 7.2. *An epimorphism $R \to S$ of super-algebras is necessarily surjective, if the induced algebra map $\overline{R} \to \overline{S}$ is surjective and if S is Noetherian.*

Proof. The assumptions imply that S, regarded as an R-super-algebra through the map, is generated by finitely many nilpotent homogeneous elements in I_R, whence it is finitely generated as an R-super-module. The desired surjectivity now follows by a familiar argument which uses the fact that $R \to S$ is an epimorphism if and only if, tensored $\otimes_R \mathrm{id}_S$ with the identity map of S, it turns into a bijection, $S = R \otimes_R S \xrightarrow{\simeq} S \otimes_R S$; see the proof of [15, Proposition 2.6], for example. ∎

Let $\pi\colon \mathsf{X}_{\mathsf{ev}} \to \mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ denote the natural morphism. Choose arbitrarily an affine open subset $U \neq \emptyset$ in $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$, which is necessarily Noetherian and smooth. Then, as is shown in [6, Part I, Section 5.7, p. 83], $\pi^{-1}(U)$ is a G_{ev}-stable, open affine subscheme of X_{ev} such that $\pi^{-1}(U)\tilde{/}\mathsf{G}_{\mathsf{ev}} = U$.

Recall that X and X_{ev} have the same underlying topological space. One sees that $\pi^{-1}(U)$, regarded as an open subset of X, is an open Noetherian super-subscheme of X, which we denote by V. The associated scheme is the $\pi^{-1}(U)$ in X_{ev}, which is affine, as was seen above. Therefore, the V in X is affine by Zubkov's theorem [24, Theorem 3.1]. Clearly, the morphism of (affine) super-schemes $V \times \mathsf{G} \to V \times V$ restricted from (1.1) is a monomorphism; in other words, the G-action on V is free. By Lemma 7.1, we have $V\tilde{/}\mathsf{G} = \mathrm{Spec}(\mathcal{O}_{\mathsf{X}}(V)^{\mathsf{G}})$, and this is Noetherian and smooth. Moreover, the original U in $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$ coincides with $\mathrm{Spec}\,\overline{B}$, where we let $B := \mathcal{O}_{\mathsf{X}}(V)^{\mathsf{G}}$.

Let Z denote the underlying topological space of the scheme $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$.

Lemma 7.3. *There uniquely exists a sheaf $\mathcal{F}$ of super-algebras on Z, such that*

$$(U, \mathcal{F}|_U) = \mathrm{Spec}(\mathcal{O}_{\mathsf{X}}(V)^{\mathsf{G}})$$

for every U with $V = \pi^{-1}(U)(\subset \mathsf{X})$, as above.

Proof. Let $U \supset U'$ be a non-empty affine open subsets of $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$. Define $V := \pi^{-1}(U)$ and $V' := \pi^{-1}(U')$ in X. The commutative diagram

$$\begin{array}{ccc} V & \hookleftarrow & V' \\ {\scriptstyle \pi|_V}\downarrow & & \downarrow{\scriptstyle \pi|_{V'}} \\ U & \hookleftarrow & U' \end{array}$$

of affine super-schemes arises from the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ B & \longrightarrow & B' \end{array}$$

of super-algebras, where we have set

$$A = \mathcal{O}_{\mathsf{X}}(V), \quad B = A^{\mathrm{co}\,H}; \quad A' = \mathcal{O}_{\mathsf{X}}(V'), \quad B' = A'^{\mathrm{co}\,H}.$$

Choose arbitrarily a point $y \in U'$. There is a point $x \in V'$ such that $\pi(x) = y$. Let

$$\mathfrak{P}^{(\prime)} \in \mathrm{Spec}(A_0^{(\prime)}), \quad \mathfrak{p}^{(\prime)} \in \mathrm{Spec}(B_0^{(\prime)})$$

be the primes which correspond to the point x in $V^{(\prime)}$ and to y in $U^{(\prime)}$, respectively. To be precise, $\mathfrak{p}$ is first chosen from $\mathrm{Spec}(\overline{B})$, and is then regarded as being in $\mathrm{Spec}(B_0)$ through the canonical identification of the two sets of primes. This is also the case for $\mathfrak{p}'$. We have the canonical isomorphism

$$A_{\mathfrak{P}} = A'_{\mathfrak{P}'} \tag{7.1}$$

between stalks of $\mathcal{O}_{\mathsf{X}}$, and the analogous one

$$\overline{B}_{\mathfrak{p}} = \overline{B'}_{\mathfrak{p}'} \tag{7.2}$$

for the structure sheaf of $\mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}}$. To prove the lemma, it suffices to prove that the canonical map

$$\mathrm{cano}\colon B_{\mathfrak{p}} \to B'_{\mathfrak{p}'}$$

is an isomorphism, which shows $\mathcal{O}_{\mathrm{Spec}\,B}|_{U'} = \mathcal{O}_{\mathrm{Spec}\,B'}$. Notice that the super-algebras $B_{\mathfrak{p}}$ and $B'_{\mathfrak{p}'}$ both are Noetherian and smooth.

As is easily seen, localizations by the primes above are compatible with the relevant constructions which will appear below; for example, we have $(\overline{B})_{\mathfrak{p}} = \overline{(B_{\mathfrak{p}})}$. We will use freely such identifications.

Just as in (6.2), we write $P_R = I_R/I_R^2$ for $R = B_{\mathfrak{p}^{(\prime)}}^{(\prime)}$. Let us return to the isomorphism given in (6.8), where we wrote $\mathcal{A} = \mathrm{gr}\,A$. Localize both sides at $\mathfrak{P}$, and restrict to the first components. By using (6.11), there results

$$P_{B_{\mathfrak{p}}} \otimes_{\overline{B}_{\mathfrak{p}}} \overline{A}_{\mathfrak{P}} = \mathcal{A}_{\mathfrak{P}}^{\mathrm{co}(\wedge(W))}(1).$$

Here we remark that, since $\wedge(W)$ co-acts trivially on the $\overline{A}$ $(= \mathcal{A}(0))$ in $\mathcal{A}$, $\mathcal{A}_{\mathfrak{P}}^{\mathrm{co}(\wedge(W))}$ makes sense, and its first component has appeared on the right-hand side. The analogous result for A', B' and $\mathcal{A}' := \mathrm{gr}\,A'$, combined with (7.2) and the equation $\mathcal{A}_{\mathfrak{P}} = \mathcal{A}'_{\mathfrak{P}'}$ arising from (7.1), gives a canonical isomorphism

$$P_{B_{\mathfrak{p}}} = P_{B'_{\mathfrak{p}'}}. \tag{7.3}$$

Choose first a section s of the super-algebra projection $B_{\mathfrak{p}} \to \overline{B}_{\mathfrak{p}}$, and then a section t of the projection $I_{B_{\mathfrak{p}}} \to P_{B_{\mathfrak{p}}}$ in $\mathtt{SMod}_{\overline{B}_{\mathfrak{p}}}$, where the $B_{\mathfrak{p}}$-super-module $I_{B_{\mathfrak{p}}}$ is regarded to be in $\mathtt{SMod}_{\overline{B}_{\mathfrak{p}}}$ though the first chosen s. Define s' and t' to be the composites

$$\overline{B'}_{\mathfrak{p}'} = \overline{B}_{\mathfrak{p}} \xrightarrow{s} B_{\mathfrak{p}} \xrightarrow{\mathrm{cano}} B'_{\mathfrak{p}'}, \quad P_{B'_{\mathfrak{p}'}} = P_{B_{\mathfrak{p}}} \xrightarrow{t} I_{B_{\mathfrak{p}}} \xrightarrow{\mathrm{cano}\,|_{I_{B_{\mathfrak{p}}}}} I_{B'_{\mathfrak{p}'}},$$

respectively. Clearly, these are sections of $B'_{\mathfrak{p}'} \to \overline{B'}_{\mathfrak{p}'}$ and of $I_{B'_{\mathfrak{p}'}} \to P_{B'_{\mathfrak{p}'}}$, respectively. We have the following diagram:

$$\begin{array}{ccc} \wedge_{\overline{B}_{\mathfrak{p}}}(P_{B_{\mathfrak{p}}}) & \longrightarrow & B_{\mathfrak{p}} \\ \downarrow & & \downarrow \text{cano} \\ \wedge_{\overline{B'}_{\mathfrak{p}'}}(P_{B'_{\mathfrak{p}'}}) & \longrightarrow & B'_{\mathfrak{p}'} \end{array}$$

Here the horizontal arrow on the top (resp., on the bottom) indicates the isomorphism (see (2.8)) which arises from s and t (resp., from s' and t'), and the vertical arrow on the left-hand side indicates the isomorphism which arises from (7.2) and (7.3). It follows that the vertical cano on the right-hand side is an isomorphism, as desired, since the diagram above is easily seen to be commutative. ∎

We see that $\mathsf{Z} := (Z, \mathcal{F})$ is a locally Noetherian, smooth super-scheme. This represents $\mathsf{X}\tilde{/}\mathsf{G}$, and satisfies

$$\mathsf{Z}_{\mathsf{ev}} = \mathsf{X}_{\mathsf{ev}}\tilde{/}\mathsf{G}_{\mathsf{ev}},$$

since it has locally these properties. This proves Theorem 1.9.

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