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Spectral metric and Einstein functionals for the Hodge–Dirac operator

Ludwik Dąbrowski, Andrzej Sitarz, and Paweł Zalecki

Abstract. We examine the metric and Einstein bilinear functionals of differential forms introduced by Dąbrowski et al. (2023), for the Hodge–Dirac operator $d + \delta$ on an oriented, closed, even-dimensional Riemannian manifold. We show that they are equal (up to a numerical factor) to these functionals for the canonical Dirac operator on a spin manifold. Furthermore, we demonstrate that the spectral triple for the Hodge–Dirac operator is spectrally closed, which implies that it is torsion-free.

1. Introduction

Spectral geometry investigates relationships between geometric structures of manifolds and the spectra of certain differential operators. Its direct and inverse problems are inextricably linked to other areas of mathematics such as number theory and representation theory, and areas of mathematical physics such as quantum mechanics and general relativity. In this regard, starting with the Laplace–Beltrami operator on a closed Riemannian manifold, general Laplace-type operators have been extensively studied, and their spectra provide insights into the geometry and topology of the underlying space. The distribution of eigenvalues, for example, reveals information about the curvature or shape and global geometric properties such as diameter or volume, connectivity, or the presence of holes. In this vein, the Dirac-type operators have also been studied, beginning with the canonical Dirac operator on the spin manifold. When subsumed into Connes' concept of spectral triples [1, 2], they "can hear the shape of a drum" [14] in the sense that their equivalence (a suitably strengthened isospectrality) implies the isometricity of manifolds in virtue of the reconstruction theorem [4]. Furthermore, they allow for broad and captivating generalisations in noncommutative geometry.

Various (interrelated) spectral schemes that generate geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives are the small-time asymptotic expansion of the (localised) trace of the heat kernel [11, 12], certain values or residues of the (localised) zeta function of the Laplacian, the spectral action, and the Wodzicki residue $\mathcal{W}$ (also known as noncommutative residue).

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In this paper, we focus on the latter one, which is the unique (up to multiplication by a constant) tracial state on the algebra of pseudodifferential operators (ΨDO) on a complex vector bundle E over a compact manifold M of dimension $n \geq 2$ [13, 18]. For the oriented manifold M, it is given by an integral formula,

$$\mathcal{W}(P) := \int_M \mathcal{w}(P),$$

where the density $\mathcal{w}(P)$ is given in local coordinates by

$$\int_{|\xi|=1} \operatorname{tr} \sigma_{-n}(P)(x, \xi) \, \mathcal{V}_\xi \, d^n x.$$

Here, tr is the trace over endomorphisms of the bundle E at any given point of M, $\sigma_{-n}(P)$ is the symbol of order $-n$ of a pseudodifferential operator P and $\mathcal{V}_\xi$ denotes the volume form on the unit co-sphere.

When applied to the (scalar) Laplacian Δ on a Riemannian manifold M of dimension $n = 2m$ equipped with a metric tensor g, it yields, in a *localised* form, a functional of $f \in C^\infty(M)$,

$$\mathcal{v}(f) := \mathcal{W}(f\Delta^{-m}) = v_{n-1} \int_M f \operatorname{vol}_g,$$

where

$$v_{n-1} := \operatorname{vol}(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)}$$

is the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$.

A startling result regarding a higher power of the Laplacian presented by Connes [3] in the early 1990s (see [15, 16] for explicit computations) states that

$$\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M f R(g) \operatorname{vol}_g,$$

which for $n > 2$ and $f = 1$ is, up to a constant, a Riemannian analogue of the Einstein–Hilbert action functional of general relativity in vacuum. Here, $R = R(g)$ is the scalar curvature, that is, the g-trace $R = g^{jk}\mathrm{Ric}_{jk}$ of the Ricci tensor with components Ric_{jk} in local coordinates, where g^{jk} are the raised components of the metric g.

In the noncommutative realm, the spectral-theoretic approach to scalar curvature has been extended to quantum tori in the seminal work of Connes and Tretkoff [6] and extensively studied by many authors (see references in [8]).

In the recent paper [8], we accomplished the task of extracting two other important tensorial geometrical objects through spectral methods. These were the metric tensor g itself, its dual, and the Einstein tensor

$$G := \mathrm{Ric} - \frac{1}{2} R(g) \, g,$$

which directly enters the Einstein field equations with matter and its dual. In fact, for this purpose, we employed the Wodzicki residue of a suitable power of the Laplace-type operator or of the Dirac-type operator, multiplied by a pair of other differential operators. Notably, we have recovered the tensors g and G as the density of certain bilinear functionals of vector fields on a manifold M, while their dual tensors are the density of bilinear functionals of differential one-forms on M. The latter functionals (up to a numerical factor) we have obtained also for the canonical Dirac operators (in case M is a spin manifold). Then, using Connes' and Moscovici's [5] generalisation of pseudodifferential calculus for noncommutative spectral triples, we introduced their conspicuous quantum analogue and probed it on two- and four-dimensional noncommutative tori.

The aim of this paper is, employing methods of the Wodzicki residue, to analyse the metric and Einstein functionals for another natural Dirac-type operator, namely the Hodge–Dirac operator $d + \delta$ acting on (complex) differential forms $\Omega(M)$ of arbitrary order on an oriented even-dimensional Riemannian manifold M. It is worth mentioning that the associated Hodge–Dirac spectral triple is characterised [7] by the fact that its dense Hilbert subspace of continuous forms provides a Morita equivalence $Cl(M) - Cl(M)$ bimodule, where $Cl(M)$ is the C^*-algebra of continuous sections of the bundle of Clifford algebras on M. As is well known, the canonical spectral triple on a spin manifold is instead characterised by the fact that its dense Hilbert subspace of continuous Dirac spinors provides a Morita equivalence $Cl(M) - C(M)$ bimodule, where $C(M)$ is the algebra of continuous complex functions on M. As our first main result, we demonstrate that these two different pivotal cases yield in fact equal spectral metric and Einstein functionals (up to a numerical factor). Moreover, as our second main result, we prove that the associated spectral triple is spectrally closed, that is, for any operator T of zero-order,

$$\mathcal{W}(TD|D|^{-n}) \equiv 0.$$

A forthcoming result [9] demonstrates that, as a consequence, the Hodge–Dirac operator has no torsion.

2. Preliminaries

Let $n = 2m$ be the dimension of an oriented, closed, smooth Riemannian manifold M. We will use capital letters to denote increasing sequences of numbers between 1 and n, of fixed length $0 \leq \ell \leq n$. A differential ℓ-form $\omega = \sum_J \omega_J dx^J$ is determined by its coefficients ω_J, with respect to coordinates indicated by the multi-index J, where with a slight abuse of notation 0-forms (i.e., functions) will correspond to $J = \emptyset$.

We introduce the operators λ_+^j and λ_-^j, which respectively raise/lower the degree of forms, with components given by

$$(\lambda_+^p)_J^I = \varepsilon_{pJ}^I, \qquad (\lambda_-^p)_J^I = \varepsilon_J^{pI},$$

where $\varepsilon^I_{pJ} = (-)^{|\pi|}$ if the juxtaposed index pJ is a permutation π of I and $\varepsilon^I_{pJ} = 0$ otherwise, and similarly for ε^{pI}_J. They satisfy

$$\begin{aligned} \lambda^p_+\lambda^r_+ + \lambda^r_+\lambda^p_+ &= 0, \\ \lambda^p_-\lambda^r_- + \lambda^r_-\lambda^p_- &= 0, \\ \lambda^p_+\lambda^r_- + \lambda^r_-\lambda^p_+ &= \delta_{pr}\,\mathrm{id}, \end{aligned}$$

which follow from the relations (cf. [17])

$$\begin{aligned} &\sum_K \varepsilon^I_{pK}\varepsilon^K_{rJ} = \varepsilon^I_{prJ}, && \sum_K \varepsilon^{pI}_K\varepsilon^{rK}_J = \varepsilon^I_{rpJ}, \\ &\sum_K \varepsilon^I_{pK}\varepsilon^{rK}_J = \delta_{pr}\varepsilon^I_J - \varepsilon^{rI}_{pJ}, && \sum_K \varepsilon^{rI}_K\varepsilon^K_{pJ} = \varepsilon^{rI}_{pJ}, \end{aligned}$$

where the juxtaposed indices can be ordered using a signed permutation. We also introduce

$$\gamma^p = -i(\lambda^p_+ - \lambda^p_-),$$

which satisfy the following Clifford algebra relation

$$\{\gamma^p, \gamma^r\} = 2\delta_{pr}.$$

In the rest of the paper, we employ normal coordinates x centred around some fixed point on the manifold. Recall that then the components of the metric tensor g, its covariant (raised) components, and the square root of the determinant of the matrix of the components of g and the components of the Christoffel symbols of the Levi-Civitá connection have the following Taylor expansion around $x = 0$:

$$\begin{aligned} g_{ab} &= \delta_{ab} - \frac{1}{3}R_{acbd}x^c x^d + o(\mathbf{x}^2), \\ g^{ab} &= \delta_{ab} + \frac{1}{3}R_{acbd}x^c x^d + o(\mathbf{x}^2), \\ \sqrt{\det(g)} &= 1 - \frac{1}{6}\mathrm{Ric}_{ab}x^a x^b + o(\mathbf{x}^2), \\ \Gamma^a_{bc} &= -\frac{1}{3}(R_{abcd} + R_{acbd})x^d + o(\mathbf{x}^2). \end{aligned}$$

Here, R_{acbd} and Ric_{ab} are the components of the Riemann and Ricci tensors, respectively, at the point $x = 0$, and we use the notation $o(\mathbf{x}^k)$ to denote that we expand a function up to the polynomial of order k in the normal coordinates. The expansion in normal coordinates is more convenient notation to obtain the value of relevant quantities (symbols of operators) and their derivatives at a given point on the manifold, which we need to compute the products of symbols according to the rules of multiplication of symbols (A.1). The position of indices for $\lambda^p_\pm$, coordinates x and ξ, as well as Riemann and Ricci tensors at the chosen point is chosen for simplicity only (as we are using normal coordinates and the metric at this point is δ_{ab}). We use Einstein summation convention for repeated indices (independently of their position), by $\{\,,\}$ we denote anticommutators.

2.1. Hodge–Dirac operator

We focus on the Hodge–Dirac operator $D = d + d^*$, where d is the exterior derivative and d^* is its (formal) adjoint. Using our notation, we compute (locally) the symbol of D,

$$\sigma(D) = (i\lambda_+^p - ig^{pr}\lambda_-^r)\xi_p + \lambda_-^p\lambda_+^r\lambda_-^s\Gamma_{rt}^s g^{pt}, \tag{2.1}$$

which in normal coordinates takes the form

$$\sigma(D) = -\gamma^p\xi_p - \frac{1}{3}i\lambda_-^p R_{sapb}x^a x^b\xi_s - \frac{1}{3}\lambda_-^p\lambda_+^r\lambda_-^s(R_{srpa} + R_{spra})x^a + o(\mathbf{x}^2).$$

We compute then the symbols of the Hodge–Dirac Laplacian D^2 in normal coordinates up to orders relevant for our purposes.

Lemma 2.1. *The three homogeneous symbols of D^2 read*

$$\begin{aligned}
\mathfrak{a}_2 &= \left(\delta_{ab} + \frac{1}{3}R_{acbd}x^c x^d\right)\xi_a\xi_b + o(\mathbf{x}^2),\\
\mathfrak{a}_1 &= +\frac{2}{3}i\mathrm{Ric}_{ab}\xi_a x^b - \frac{2}{3}i\lambda_+^p\lambda_-^r(R_{rpab} + R_{rapb})x^b\xi_a + o(\mathbf{x}^1),\\
\mathfrak{a}_0 &= +\frac{2}{3}\lambda_+^p\lambda_-^r\mathrm{Ric}_{pr} + \frac{1}{3}\lambda_+^p\lambda_+^r\lambda_-^s\lambda_-^t(R_{tsrp} + R_{trsp}) + o(\mathbf{x}^0).
\end{aligned}$$

Proof. The computation of the principal symbol $\mathfrak{a}_2$ is obvious for the symbol of order 1:

$$\begin{aligned}
\mathfrak{a}_1 &= -\frac{1}{3}i\{\lambda_+^t, \lambda_-^p\lambda_+^r\lambda_-^s\}(R_{srpa} + R_{spra})x^a\xi_t\\
&\quad + \frac{1}{3}i\{\lambda_-^t, \lambda_-^p\lambda_+^r\lambda_-^s\}(R_{srpa} + R_{spra})x^a\xi_t + \frac{1}{3}\gamma^p\lambda_-^r(R_{aprb} + R_{abrp})x^b\xi_a\\
&= -\frac{1}{3}i\lambda_+^r\lambda_-^s(R_{srta} + R_{stra})x^a\xi_t - \frac{1}{3}i\lambda_-^p\lambda_+^r(R_{trpa} + R_{tpra})x^a\xi_t\\
&\quad - \frac{1}{3}i\lambda_-^p\lambda_-^s(R_{stpa} + R_{spta})x^d\xi_t - \frac{1}{3}i\lambda_+^p\lambda_-^s(R_{tpsa} + R_{tasp})x^a\xi_t\\
&\quad + \frac{1}{3}i\lambda_-^p\lambda_-^s(R_{tpsa} + R_{tasp})x^a\xi_t\\
&= \frac{2}{3}i\mathrm{Ric}_{sa}x^a\xi_s - \frac{1}{3}i\lambda_+^r\lambda_-^s(R_{srta} + R_{stra} - R_{trsa} - R_{tsra} + R_{trsa} + R_{tasr})x^a\xi_t\\
&\quad - \frac{1}{3}i\lambda_-^r\lambda_-^s(R_{stra} + R_{srta} - R_{trsa} - R_{tasr})x^a\xi_t\\
&= \frac{2}{3}i\mathrm{Ric}_{sa}x^a\xi_s - \frac{2}{3}i\lambda_+^r\lambda_-^s(R_{srta} + R_{stra})x^a\xi_t
\end{aligned}$$

and the order 0 symbol:

$$\begin{aligned}
\mathfrak{a}_0 &= -\frac{1}{3}i\gamma^p\lambda_-^q\lambda_+^r\lambda_-^s(R_{sqrp} + R_{srqp})\\
&= -\frac{1}{3}\lambda_+^p\lambda_-^q\lambda_+^r\lambda_-^s(R_{sqrp} + R_{srqp})\\
&= \frac{2}{3}\lambda_+^p\lambda_-^s\mathrm{Ric}_{ps} + \frac{1}{3}\lambda_+^p\lambda_+^r\lambda_-^q\lambda_-^s(R_{sqrp} + R_{srqp}).
\end{aligned}$$

■

2.2. The inverse of D^2 and its powers

In this section, we present the results that can be applied to a more general situation than the Hodge–Dirac operator. Note that since we work with pseudodifferential operators, we denote by the inverses of elliptic operator the corresponding parametrix. For this reason we can ignore the kernel of these operators. Let us start with the following lemma:

Lemma 2.2. *Let L be a Laplace-type operator with symbol*

$$\sigma(L) = \mathfrak{a}_2 + \mathfrak{a}_1 + \mathfrak{a}_0$$

expressed in normal coordinates as

$$\begin{aligned}
\mathfrak{a}_2 &= \left(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d\right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{a}_1 &= i P_{ab} \xi_a x^b + o(\mathbf{x}^1), \\
\mathfrak{a}_0 &= Q + o(\mathbf{x}^0).
\end{aligned}$$

Then the three leading symbols of $\sigma(L^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2}$ are:

$$\begin{aligned}
\mathfrak{c}_{2k} &= \|\xi\|^{-2k-2} \left(\delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d\right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{c}_{2k+1} &= -ik \|\xi\|^{-2k-2} P_{ab} \xi_a x^b + o(\mathbf{x}^1), \\
\mathfrak{c}_{2k+2} &= -k \|\xi\|^{-2k-2} Q + k(k+1) \|\xi\|^{-2k-4} \left(P_{ab} - \frac{1}{3} \mathrm{Ric}_{ab}\right) \xi_a \xi_b + o(\mathbf{x}^0).
\end{aligned}$$

Proof. First, observe that we follow the notation of most papers (see [6]), and to simplify the notation, indicate a negative order $-k$ by $k \geq 0$, so $\mathfrak{b}_2$ in the below equation denotes the symbol of order -2. We start with computing leading symbols of the inverse of L, i.e., $\sigma(L^{-1}) = \mathfrak{b}_2 + \mathfrak{b}_3 + \mathfrak{b}_4$ using the fact, that $\sigma(LL^{-1}) = \sigma(1) = 1$. We have

$$\begin{aligned}
\mathfrak{b}_2 &= (\mathfrak{a}_2)^{-1} = \|\xi\|^{-4} \left(\delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d\right) \xi_a \xi_b + o(\mathbf{x}^2), \\
\mathfrak{b}_3 &= \mathfrak{b}_2(-\mathfrak{a}_1 \mathfrak{b}_2 + i \partial_\xi^a \mathfrak{a}_2 \partial_a \mathfrak{b}_2) = -i \|\xi\|^{-4} P_{ab} \xi_a x^b + o(\mathbf{x}^1), \\
\mathfrak{b}_4 &= \mathfrak{b}_2 \left(-\mathfrak{a}_0 \mathfrak{b}_2 - \mathfrak{a}_1 \mathfrak{b}_3 + i \partial_\xi^a \mathfrak{a}_2 \partial_a \mathfrak{b}_3 + i \partial_\xi^a \mathfrak{a}_1 \partial_a \mathfrak{b}_2 + \frac{1}{2} \partial_\xi^{ab} \mathfrak{a}_2 \partial_{ab} \mathfrak{b}_2\right) \\
&= -\|\xi\|^{-4} Q + 2\|\xi\|^{-6} \left(P_{ab} - \frac{1}{3} \mathrm{Ric}_{ab}\right) \xi_a \xi_b + o(\mathbf{x}^0).
\end{aligned}$$

To finish, we apply [8, Lemma A1] and compute the three leading symbols of the powers of the pseudodifferential operator L^{-k}. ∎

Using the above lemma for $L = D^2$, with the Hodge–Dirac operator D (2.1), we get the following result.

Proposition 2.3. *The leading symbols of D^{-2k} are, up to the appropriate order in x,*

$$\begin{aligned}
\mathfrak{c}_{2k} &= \|\xi\|^{-2k-2}\Big(\delta_{ab} - \frac{k}{3}R_{acbc}x^c x^d\Big)\xi_a\xi_b + o(\mathbf{x}^2),\\
\mathfrak{c}_{2k+1} &= -\frac{2}{3}ki\,\|\xi\|^{-2k-2}\mathrm{Ric}_{ab}x^b\xi_a + \frac{2}{3}ki\,\|\xi\|^{-2k-2}\lambda_+^r\lambda_-^s(R_{srba}+R_{sbra})x^a\xi_b + o(\mathbf{x}^1)\\
\mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3}\|\xi\|^{-2k-4}\mathrm{Ric}_{ab}\xi_a\xi_b\\
&\quad - \frac{2}{3}k(k+1)\|\xi\|^{-2k-4}\,\lambda_+^r\lambda_-^s(R_{srab}+R_{sarb})\xi_a\xi_b\\
&\quad + \frac{1}{3}k\|\xi\|^{-2k-2}\lambda_+^p\lambda_-^q\lambda_+^r\lambda_-^s(R_{sqrp}+R_{srqp}) + o(\mathbf{x}^0).
\end{aligned}$$

Proof. For $L = D^2$, we substitute in Lemma 2.2

$$\begin{aligned}
P_{ab} &= \frac{2}{3}\mathrm{Ric}_{ab} - \frac{2}{3}\lambda_+^r\lambda_-^s(R_{srab}+R_{sarb}),\\
Q &= -\frac{1}{3}\lambda_+^p\lambda_-^q\lambda_+^r\lambda_-^s(R_{sqrp}+R_{srqp}).
\end{aligned}$$ ■

3. Spectral functionals

In [8], we defined two spectral functionals for finitely summable spectral triples, which for the canonical spectral triple over the spin manifold M allow to recover the metric and the Einstein tensors, viewed as bilinear functionals over a pair of one-forms. We recall the definition:

Definition 3.1 (cf. [8, Definition 5.4]). If $(\mathcal{A}, D, \mathcal{H})$ is an n-summable spectral triple, let Ω^1_D be the $\mathcal{A}$ bimodule of one-forms generated by $\mathcal{A}$ and $[D, \mathcal{A}]$. Moreover, assume that there exists a generalised algebra of pseudodifferential operators which contains $\mathcal{A}$, D, and $|D|^\ell$ for $\ell \in \mathbb{Z}$ with a tracial state $\mathcal{W}$ over this algebra (called a noncommutative residue), which identically vanishes on $T|D|^{-k}$ for any $k > n$ and a zero-order operator T (an operator in the algebra generated by $\mathcal{A}$ and $\Omega^1(\mathcal{A})$). Then, for $u, w \in \Omega^1_D(\mathcal{A})$, we call

$$g(u, w) := \mathcal{W}(uw|D|^{-n}),$$

the *metric functional*, and

$$\mathcal{G}(u, w) := \mathcal{W}(u\{D, w\}D|D|^{-n}).$$

the *Einstein functional*.

3.1. Hodge–de Rham spectral triple

We compute now these functionals for $\mathcal{A} = C^\infty(M)$ and $D = d + \delta$, identifying for $\dim M = n \geq 2$,

$$\Omega^1_D(A) \simeq \Omega^1(M)$$

via the correspondence of local components

$$u = \gamma^p u_p \quad \leftrightarrow \quad U = u_p e^p,$$

with respect to an orthonormal coframe e^p. First, let us compute the metric functional $\mathcal{g}$.

Proposition 3.2. *For $U, V \in \Omega^1(M)$, the metric spectral functional reads*

$$\mathcal{g}(U, W) = 2^n v_{n-1} \int_M g(U, W) \,\mathrm{vol}_g.$$

Proof. It can be seen that it is adequate to expand U, W, and so u, w, up to $o(\mathbf{x}^0)$ in normal coordinates:

$$u = \gamma^p u_p + o(\mathbf{x}^0), \qquad w = \gamma^r w_r + o(\mathbf{x}^0).$$

We compute (locally) the density of $\mathcal{W}(UW|D|^{-n})$ as

$$\int_{\|\xi\|=1} \mathrm{Tr}\,(\gamma^p \gamma^r u_p w_r) \mathfrak{c}_n(D)\, d^n x = v_{n-1}\sqrt{g}\,\mathrm{Tr}\,(\gamma^p \gamma^r u_p w_r)\, d^n x = 2^n v_{n-1}\sqrt{g}\, u_p w_p \, d^n x.$$

The factor 2^n comes from the trace of 1 over the space of differential forms. ∎

Proposition 3.3. *For $U, V \in \Omega^1(M)$, the Einstein functional reads*

$$\mathcal{G}(U, W) = \frac{2^n}{6} v_{n-1} \int_M G(U, W) \,\mathrm{vol}_g,$$

where G is the Einstein tensor for M.

Before we begin with the proof, let us demonstrate some useful lemmas. The first computes $\mathcal{W}(ED^{-2m+2})$ for two specific cases of endomorphism E.

Lemma 3.4. *If E is locally given by*

$$e^{(0)} + e^{(2)}_{pq} \gamma^p \gamma^q,$$

the density of the functional $\mathcal{W}(ED^{-2m+2})$ reads locally

$$\frac{n-2}{24} 2^n v_{n-1} \sqrt{g} R\big(-e^{(0)} - e^{(2)}_{pp}\big)\, d^n x.$$

On the other hand, if $\widetilde{E}$ is locally given by

$$\widetilde{e}^{(0)} + \widetilde{e}^{(2)}_{pq} \lambda^p_+ \lambda^q_-,$$

the local density of the functional $\mathcal{W}(\widetilde{E} D^{-2m+2})$ reads locally

$$-\frac{n-2}{24} 2^n v_{n-1} \sqrt{g} R \widetilde{e}^{(0)}\, d^n x.$$

The proof is based on direct calculations using Proposition 2.3:

$$\operatorname{Tr}\left(E\int_{\|\xi\|=1} \mathrm{c}_{2(m-1)+2}\right)\mathcal{V}_\xi\, d^n x$$
$$= \frac{n-2}{12} v_{n-1} \operatorname{Tr}\left(E\big(2(R_{srqp}+R_{sqrp})\lambda_+^p\lambda_-^q\lambda_+^r\lambda_-^s + R - 2\mathrm{Ric}_{qp}\lambda_+^p\lambda_-^q\big)\right)d^n x.$$

Lemma 3.5. *For P such that $\sigma(P) = F^{ab}\xi_a\xi_b + G^a\xi_a + H$, where locally*

$$F^{ab} = f^{(0)ab} + f_{pq}^{(2)ab}\gamma^p\gamma^q,$$

$\mathcal{W}(PD^{-n})$ reads locally

$$v_{n-1}\left(\operatorname{Tr} H + \frac{2^n}{24}R\big(-f^{(0)aa} - f_{pp}^{(2)aa}\big)\right)d^n x.$$

Also, for $\widetilde{P}$ such that $\sigma(\widetilde{P}) = \widetilde{F}^{ab}\xi_a\xi_b + G^a\xi_a + H$, where locally

$$\widetilde{F}^{ab} = \widetilde{f}^{(0)ab} + \widetilde{f}_{pq}^{(2)ab}\lambda_+^p\lambda_-^q,$$

$\mathcal{W}(\widetilde{P}D^{-n})$ reads locally

$$v_{n-1}\left(\operatorname{Tr} H + \frac{2^n}{48}\Big(-2R\widetilde{f}^{(0)aa} - \widetilde{f}_{pp}^{(2)aa}R + 2\big(\widetilde{f}_{pq}^{(2)ab} + \widetilde{f}_{pq}^{(2)ba}\big)R_{paqb}\Big)\right)d^n x.$$

The proof follows directly by computation using Proposition 2.3 and Lemma A.3 applied to the explicit expression:

$$\int_{\|\xi\|=1} \sigma_{-2m}(PD^{-2m})\mathcal{V}_\xi = \frac{1}{6}F^{aa}\left[(R_{srqp}+R_{bqrs})\lambda_+^p\lambda_-^q\lambda_+^r\lambda_-^s + \frac{1}{2}R - R_{pq}\lambda_+^p\lambda_-^q\right]$$
$$+ \frac{1}{6}F^{ab}[-R_{ab} + (R_{qapb}+R_{paqb})\lambda_+^p\lambda_-^q] + H.$$

Proof of Proposition 3.3. We begin with computing the symbol of $uDwD$ at $x=0$, where it suffices to expand u and w as:

$$u = \gamma^p u_p + o(\mathbf{x}^0), \qquad w = \gamma^s w_s + \gamma^s w_{sa}x^a + o(\mathbf{x}^1),$$

and thus locally:

$$uDwD = \gamma^p\gamma^q\gamma^r\gamma^s u_p w_r \xi_q\xi_s - i\gamma^p\gamma^q\gamma^r\gamma^s u_p w_{rq}\xi_s$$
$$- \frac{1}{3}i\gamma^p\gamma^q\gamma^r\lambda_-^s\lambda_+^t\lambda_-^z(R_{ztsq}+R_{zstq})u_p w_r + o(\mathbf{x}^0).$$

Then, we use Lemma 3.4 for $P = uDwD$ and $E = uw$. In this case, we have

$$\begin{aligned}
E &= u_p w_q \gamma^p \gamma^q, \\
H &= -\frac{1}{3} i \gamma^p \gamma^q \gamma^r \lambda_-^s \lambda_+^t \lambda_-^z (R_{ztsq} + R_{zstq}) u_p w_r \\
&= -\frac{2}{3} i \gamma^p \lambda_-^q \lambda_+^r \lambda_-^s (R_{srqt} + R_{sqrt}) u_p w_t \\
&\quad + \frac{1}{3} i \gamma^p \gamma^q \gamma^r \lambda_-^s \lambda_+^t \lambda_-^z (R_{ztsr} + R_{zstr}) u_p w_q, \\
F^{pq} \xi_p \xi_q &= \gamma^r \gamma^p \gamma^s \gamma^q u_r w_s \xi_p \xi_q = (2 u_r w_p \delta_{qs} \gamma^r \gamma^s - u_r w_s \gamma^r \gamma^s \delta_{pq}) \xi_p \xi_q,
\end{aligned}$$

where we used that $\gamma^p \gamma^q \xi_p \xi_q = \delta_{pq} \xi_p \xi_q$. Next, we see

$$\begin{aligned}
e^{(0)} &= 0, & e_{ab}^{(2)} &= u_a w_b, \\
f^{(0)} &= 0, & f_{cd}^{(2)ab} &= 2 u_c w_a \delta_{bd} - u_c w_d \delta_{ab}.
\end{aligned}$$

Finally, the contribution arising from E gives

$$-\frac{n-2}{24} 2^n v_{n-1} R\, u_a w_a,$$

whereas the part from F is

$$-\frac{2^n}{24} v_{n-1} R f_{ii}^{(2)aa} = \frac{n-2}{24} 2^n v_{n-1} R\, u_a w_a.$$

These two terms cancel each other, and we are left with terms that arise from H. The only possible terms in $\operatorname{Tr} H$ are linear combinations of $u_a w_a R$ and $u_a w_b \mathrm{Ric}_{ab}$; thus, we know that the result is symmetric in u_a, w_b. It allows us to simplify the second term in H:

$$\begin{aligned}
H &= -\frac{2}{3} i \gamma^p \lambda_-^q \lambda_+^s \lambda_-^t (R_{tsqb} + R_{tqsb}) u_p w_b \\
&\quad + \frac{1}{3} i \gamma^r \lambda_-^q \lambda_+^s \lambda_-^t (R_{tsqr} + R_{tqsr}) u_a w_a + \cdots,
\end{aligned}$$

where "$\cdots$" are terms antisymmetric in u_a, w_b, which we can neglect. We can also insert $-i\lambda_+$ instead of the remaining γ's because the part with λ_- will be traceless. Now, using (A.2), we get

$$\operatorname{Tr} H = \frac{1}{6} \mathrm{Ric}_{ab}\, u_a w_b - \frac{1}{12} R\, u_a w_a = \frac{1}{6} G_{ab}\, u_a w_b.$$

This proves the result. ∎

We deduce that for the spin$_c$ manifolds the spectral functionals for the Hodge–de Rham spectral triple are equal, up to the rank of the vector bundles, to those for the canonical spin$_c$ spectral triple.

3.2. Spectral closedness and torsion

In this section, we will prove that the Hodge–Dirac spectral triple has the property of being spectrally closed.

Theorem 3.6. *Let T be an operator of order 0 from the algebra generated by $a[D,b]$, $a,b \in C^\infty(M)$. Then,*

$$\mathcal{W}(TD|D|^n) = 0.$$

Proof. If we compute the symbol of TD at a chosen point on the manifold M in normal coordinates at $x = 0$, we obtain,

$$\sigma(TD) = T(-\gamma^p \xi_p).$$

Next, if we combine it with Proposition 2.3, we see that the symbol of order $-n$ of $TD|D|^n$ is

$$\sigma_{-n}(TD|D|^{-n}) = 0 + o(\mathbf{x^0}).$$

This ends the proof. ∎

As a consequence, we demonstrate in [9] that the Hodge–Dirac spectral triple is torsion-free. It is interesting to study generalised Hodge–Dirac operators, which are defined through an arbitrary linear connection, are not metric compatible, and have a torsion. Also, the extension to noncommutative Hodge–de Rham calculi and a comparison with the approach of [10] would be of great interest. Note that we focused here on even-dimensional manifolds only for the sake of simplicity and the result easily extends to the case of odd-dimensional manifolds (for technical details that include the computation of symbols, compare [9]).

A. Details of computations

We begin with a formula for the product of two pseudodifferential operators, P and Q, which have expansion in homogeneous symbols,

$$\sigma(P)(x,\xi) = \sum_\alpha \sigma(P)_\alpha(x,\xi), \qquad \sigma(Q)(x,\xi) = \sum_\beta \sigma(Q)_\beta(x,\xi),$$

respectively, where α and β are multi-indices (such that $|\alpha| = \sum_k \alpha^k$ and $|\beta|$ are bounded from above). The symbol $\sigma(P)_\alpha(x,\xi)$ is homogeneous of order $|\alpha|$ if, for $r > 0$, $\sigma(P)_\alpha(x,r\xi) = r^{|\alpha|}\sigma(P)_\alpha(x,\xi)$. The composition rule for the symbols of their product takes the form [11].

$$\sigma(PQ)(x,\xi) = \sum_\beta \frac{(-i)^{|\beta|}}{|\beta|!} \partial^\xi_\beta \sigma(P)(x,\xi) \partial_\beta \sigma(Q)(x,\xi), \tag{A.1}$$

where ∂^ξ_β denotes the partial derivative with respect to the coordinate of the cotangent bundle.

Lemma A.1. *A direct computation of traces of products of λ matrices is based on the following recursive formula:*

$$\begin{aligned}&\operatorname{Tr} \lambda_+^{p_1} \cdots \lambda_+^{p_k} \lambda_-^{q_1} \cdots \lambda_-^{q_k} \\ &\quad = \frac{1}{2} \sum_{j=1}^{k} (-1)^{k-j} \delta^{p_1 q_j} \operatorname{Tr} \left(\lambda_+^{p_2} \cdots \lambda_+^{p_k} \lambda_-^{q_1} \cdots \lambda_-^{q_{j-1}} \lambda_-^{q_{j+1}} \cdots \lambda_-^{q_k}\right).\end{aligned}$$

In particular, we have

$$\begin{aligned}\operatorname{Tr} (\lambda_+^{p} \lambda_-^{q}) &= 2^{n-1} \delta^{pq}, \\ \operatorname{Tr} (\lambda_+^{p_1} \lambda_-^{q_1} \lambda_+^{p_2} \lambda_-^{q_2}) &= 2^{n-2} (\delta^{p_1 q_1} \delta^{p_2 q_2} + \delta^{p_1 q_2} \delta^{p_2 q_1}),\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}&\operatorname{Tr} (\lambda_+^{p_1} \lambda_-^{q_1} \lambda_+^{p_2} \lambda_-^{q_2} \lambda_+^{p_3} \lambda_-^{q_3}) \\ &\quad = 2^n \left(\frac{1}{8} \sum_{\sigma \in S_3} \delta^{p_1 q_{\sigma(1)}} \delta^{p_2 q_{\sigma(2)}} \delta^{p_3 q_{\sigma(3)}} - \frac{1}{4} \delta^{p_1 q_2} \delta^{p_2 q_3} \delta^{p_3 q_1} \right).\end{aligned} \tag{A.3}$$

Next, we present the results on traces of products of γ and λ matrices.

Lemma A.2.

$$\operatorname{Tr} (\gamma^p \gamma^q \lambda_+^r \lambda_-^s) = 2^{n-2} \big(2\delta^{pq} \delta^{rs} + \delta^{ps} \delta^{qr} - \delta^{pr} \delta^{qs}\big) = 2^{n-1} \left(\delta^{pq} \delta^{rs} - \frac{1}{2} \varepsilon^{pq}_{rs} \right), \tag{A.4}$$

$$\begin{aligned}\operatorname{Tr} (\gamma^p \gamma^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z) &= 2^{n-2} \delta^{pq} (\delta^{rs} \delta^{tz} + \delta^{rz} \delta^{st}) \\ &\quad - 2^{n-3} (\delta^{rs} \varepsilon^{pq}_{tz} + \delta^{st} \varepsilon^{pq}_{rz} + \delta^{tz} \varepsilon^{pq}_{rs} + \delta^{rz} \varepsilon^{pq}_{st})\end{aligned}$$

We skip the computational proof, which is based on expressing γ-matrices in terms of λ-matrices,

$$\gamma^p \gamma^q = -\lambda_+^q \lambda_-^p + \lambda_+^p \lambda_-^q + \delta^{pq} + \cdots,$$

and using the results of Lemma A.1.

As a consequence, we obtain the following identities for the geometric quantities:

Lemma A.3. *In normal coordinates around $x = 0$, we have the following identities:*

$$\begin{aligned}\operatorname{Tr} (\lambda_+^p \lambda_-^q) \operatorname{Ric}_{pq} &= 2^{n-1} R, \\ \operatorname{Tr} (\lambda_+^p \lambda_-^q)(R_{paqb} + R_{qapb}) &= 2^n \operatorname{Ric}_{ab}, \\ \operatorname{Tr} (\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s)(R_{srqp} + R_{sqrp}) &= -2^{n-2} R, \\ \operatorname{Tr} (\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s) \operatorname{Ric}_{rs} &= 2^{n-2} (\delta_{pq} R + \operatorname{Ric}_{pq}), \\ \operatorname{Tr} (\gamma^p \gamma^q \lambda_+^r \lambda_-^s) \operatorname{Ric}_{rs} &= 2^{n-1} \delta_{pq} R, \\ \operatorname{Tr} (\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s)(R_{rasb} + R_{sarb}) &= 2^{n-2} (2\delta_{pq} \operatorname{Ric}_{ab} + R_{qapb} + R_{paqb}), \\ \operatorname{Tr} (\gamma^p \gamma^q \lambda_+^r \lambda_-^s)(R_{rasb} + R_{sarb}) &= 2^n \delta_{pq} \operatorname{Ric}_{ab}, \\ \operatorname{Tr} (\lambda_+^p \lambda_-^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z)(R_{ztsr} + R_{zstr}) &= 2^{n-3} (-R\delta_{pq} + 2\operatorname{Ric}_{pq}), \\ \operatorname{Tr} (\gamma^p \gamma^q \lambda_+^r \lambda_-^s \lambda_+^t \lambda_-^z)(R_{ztsr} + R_{zstr}) &= -2^{n-2} \delta_{pq} R.\end{aligned}$$

Proof. Direct computation using (A.2)–(A.4) and the properties of the Riemann and Ricci tensors. ■

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Ludwik Dąbrowski
SISSA (Scuola Internazionale Superiore di Studi Avanzati), Via Bonomea 265, 34136 Trieste, Italy; ludwik.dabrowski@sissa.it

Andrzej Sitarz
Institute of Theoretical Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland; andrzej.sitarz@uj.edu.pl

Paweł Zalecki
Institute of Theoretical Physics, Jagiellonian University, prof. Stanisława Łojasiewicza 11, 30-348 Kraków, Poland; pawel.zalecki@doctoral.uj.edu.pl