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# Yamabe systems, optimal partitions and nodal solutions to the Yamabe equation

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**Abstract.** We give conditions for the existence of regular optimal partitions, with an arbitrary number $\ell \geq 2$ of components, for the Yamabe equation on a closed Riemannian manifold $(M, g)$.

To this aim, we study a weakly coupled competitive elliptic system of $\ell$ equations, related to the Yamabe equation. We show that this system has a least energy solution with nontrivial components if $\dim M \geq 10$, $(M, g)$ is not locally conformally flat, and satisfies an additional geometric assumption whenever $\dim M = 10$. Moreover, we show that the limit profiles of the components of the solution separate spatially as the competition parameter goes to $-\infty$, giving rise to an optimal partition. We show that this partition exhausts the whole manifold, and we prove the regularity of both the interfaces and the limit profiles, together with a free boundary condition.

For $\ell = 2$ the optimal partition obtained yields a least energy sign-changing solution to the Yamabe equation with precisely two nodal domains.

*Keywords:* competitive elliptic system, Riemannian manifold, critical nonlinearity, optimal partition, free boundary problem, regularity, Yamabe equation, sign-changing solution.

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## 1. Introduction and statement of results

Consider the Yamabe equation

$$\mathscr{L}_g u := -\Delta_g u + \kappa_m S_g u = |u|^{2^*-2}u \quad \text{on } M, \tag{1.1}$$

where $(M, g)$ is a closed Riemannian manifold of dimension $m \geq 3$, $S_g$ is its scalar curvature, $\Delta_g := \mathrm{div}_g \nabla_g$ is the Laplace–Beltrami operator, $\kappa_m := \frac{m-2}{4(m-1)}$, and $2^* := \frac{2m}{m-2}$ is the critical Sobolev exponent. We assume that the quadratic form induced by the conformal Laplacian $\mathscr{L}_g$ is coercive.

If $\Omega$ is an open subset of $M$, we consider the Dirichlet problem

$$\begin{cases} \mathscr{L}_g u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Let $H_g^1(M)$ be the Sobolev space of square integrable functions on $M$ having square integrable first weak derivatives, and let $H_{g,0}^1(\Omega)$ be the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H_g^1(M)$. The (weak) solutions of (1.2) are the critical points of the $\mathcal{C}^2$-functional $J_\Omega : H_{g,0}^1(\Omega) \to \mathbb{R}$ given by

$$J_\Omega(u) := \frac{1}{2}\int_\Omega (|\nabla_g u|_g^2 + \kappa_m S_g u^2)\, \mathrm{d}\mu_g - \frac{1}{2^*}\int_\Omega |u|^{2^*}\, \mathrm{d}\mu_g.$$

The nontrivial ones belong to the Nehari manifold

$$\mathcal{N}_\Omega := \{u \in H_{g,0}^1(\Omega) : u \neq 0 \text{ and } J'_\Omega(u)u = 0\},$$

which is a natural constraint for $J_\Omega$. So, a minimizer for $J_\Omega$ over $\mathcal{N}_\Omega$ is a nontrivial solution of (1.2), called a *least energy solution*. Such a solution does not always exist. If $\Omega$ is the whole manifold $M$, it provides a solution to the celebrated Yamabe problem. In this case its existence was established thanks to the combined efforts of Yamabe [64], Trudinger [62], Aubin [4] and Schoen [46]. A detailed account is given in [41].

Set

$$c_\Omega := \inf_{u \in \mathcal{N}_\Omega} J_\Omega(u).$$

In this paper, given $\ell \geq 2$, we consider the optimal $\ell$-partition problem

$$\inf_{\{\Omega_1,\dots,\Omega_\ell\} \in \mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Omega_i}, \tag{1.3}$$

where $\mathcal{P}_\ell := \{\{\Omega_1, \dots, \Omega_\ell\} : \Omega_i \neq \emptyset$ is open in $M$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j\}$. A solution to (1.3) is an $\ell$-tuple $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell$ such that $c_{\Omega_i}$ is attained for every $i = 1, \dots, \ell$, and

$$\sum_{i=1}^{\ell} c_{\Omega_i} = \inf_{\{\Theta_1,\dots,\Theta_\ell\} \in \mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Theta_i}.$$

We call it an *optimal $\ell$-partition* for the Yamabe equation on $(M, g)$.

Optimal partitions do not always exist. In fact, there is no optimal $\ell$-partition for the Yamabe equation on the standard sphere $\mathbb{S}^m$ for any $\ell \geq 2$. This is because $c_\Omega$ is not attained in any open subset $\Omega$ of $\mathbb{S}^m$ whose complement has nonempty interior. Indeed, by means of the stereographic projection $\Sigma : \mathbb{S}^m \smallsetminus \{q\} \to \mathbb{R}^m$ from a point $q \in \mathbb{S}^m \smallsetminus \bar{\Omega}$, problem (1.2) translates into

$$-\Delta u = |u|^{2^*-2} u \quad \text{in } \Sigma(\Omega), \qquad u = 0 \quad \text{on } \partial[\Sigma(\Omega)].$$

It is well known that this problem does not have a least energy solution; see, e.g., [54, Theorem III.1.2].

Our aim is to give conditions on $(M, g)$ which guarantee the existence of an optimal $\ell$-partition for every $\ell$. To this end, we follow the approach introduced by Conti, Terracini and Verzini [21,22] and Chang, Lin, Lin and Lin [14] relating optimal partition problems to variational elliptic systems having large competitive interaction.

We consider the competitive elliptic system

$$\mathscr{L}_g u_i = |u_i|^{2^*-2} u_i + \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i \quad \text{on } M,\ i = 1, \dots, \ell, \tag{1.4}$$

where $\lambda_{ij} = \lambda_{ji} < 0$, $\alpha_{ij}, \beta_{ij} > 1$, $\alpha_{ij} = \beta_{ji}$, and $\alpha_{ij} + \beta_{ij} = 2^*$. Firstly, we provide sufficient conditions for (1.4) to have a least energy solution with nontrivial components; secondly, in the case $\alpha_{ij} = \beta_{ij}$ and $\lambda_{ij} \equiv \lambda$, we study the asymptotic profiles of such solutions as $\lambda \to -\infty$. As a byproduct, we obtain the existence of a regular optimal $\ell$-partition of (1.3), and the existence of a sign-changing solution of (1.1) with two nodal domains. Our results read as follows.

**Theorem 1.1.** *Assume that one of the following two conditions holds true:*

(A1) $\dim M = 3$, *$(M, g)$ is not conformal to the standard 3-sphere and* $2 < \alpha_{ij} < 4$ *for all* $i, j = 1, \dots, \ell$.

(A2) *$(M, g)$ is not locally conformally flat,* $\dim M \geq 9$, *and* $\frac{8}{m-2} < \alpha_{ij} < \frac{2(m-4)}{m-2}$ *for all* $i, j = 1, \dots, \ell$ *if* $m := \dim M = 9$.

*Then the system (1.4) has a least energy fully nontrivial solution $(u_1, \dots, u_\ell)$ such that $u_i \in \mathcal{C}^2(M)$ and $u_i \geq 0$ for every $i = 1, \dots, \ell$. If* $\dim M = 3$, *then $u_i > 0$ for every $i = 1, \dots, \ell$.*

Note that, as $\alpha_{ij} \in (1, 2^* - 1)$, it satisfies $\frac{8}{m-2} < \alpha_{ij} < \frac{2(m-4)}{m-2}$ when $m > 9$. By a least energy fully nontrivial solution we mean a minimizer of the variational functional for the system (1.4) on a suitable constraint that contains only solutions with nonzero components; see Section 2 below.

**Theorem 1.2.** *Assume that*

(A3) $(M, g)$ *is not locally conformally flat and* $\dim M \geq 10$. *If* $\dim M = 10$ *then*

$$|S_g(q)|^2 < \tfrac{5}{28}|W_g(q)|_g^2 \quad \forall q \in M,$$

*where* $W_g(q)$ *is the Weyl tensor of* $(M, g)$ *at* $q$.

*Let* $\lambda_n < 0$ *with* $\lambda_n \to -\infty$ *and set* $\beta := \frac{2^*}{2} = \frac{m}{m-2}$. *For each* $n \in \mathbb{N}$, *let* $(u_{n,1}, \ldots, u_{n,\ell})$ *be a least energy fully nontrivial solution to the system*

$$\mathscr{L}_g u_i = |u_i|^{2^*-2}u_i + \sum_{\substack{j=1\\ j\neq i}}^{\ell} \lambda_n \beta |u_j|^\beta |u_i|^{\beta-2}u_i \quad \text{on } M,\ i = 1, \ldots, \ell, \tag{1.5}$$

*such that* $u_{n,i} \in \mathcal{C}^2(M)$ *and* $u_{n,i} \geq 0$ *for all* $n \in \mathbb{N}$. *Then, after passing to a subsequence, we have:*

(i) $u_{n,i} \to u_{\infty,i}$ *strongly in* $H_g^1(M) \cap \mathcal{C}^{0,\alpha}(M)$ *for every* $\alpha \in (0, 1)$, *where* $u_{\infty,i} \geq 0$, $u_{\infty,i} \neq 0$, *and* $u_{\infty,i}|_{\Omega_i}$ *is a least energy solution to the problem* (1.2) *in* $\Omega_i := \{p \in M : u_{\infty,i}(p) > 0\}$ *for each* $i = 1, \ldots, \ell$. *Moreover,*

$$\int_M \lambda_n u_{n,i}^\beta u_{n,j}^\beta \to 0 \quad \text{as } n \to \infty \quad \text{whenever } i \neq j.$$

(ii) $u_{\infty,i} \in \mathcal{C}^{0,1}(M)$ *for each* $i = 1, \ldots, \ell$.

(iii) $\{\Omega_1, \ldots, \Omega_\ell\} \in \mathcal{P}_\ell$ *and it is an optimal* $\ell$*-partition for the Yamabe equation on* $(M, g)$. *In particular, each* $\Omega_i$ *is connected.*

(iv) $\Gamma := M \smallsetminus \bigcup_{i=1}^{\ell} \Omega_i = \mathscr{R} \cup \mathscr{S}$, *where* $\mathscr{R} \cap \mathscr{S} = \emptyset$, $\mathscr{R}$ *is an* $(m-1)$*-dimensional* $\mathcal{C}^{1,\alpha}$*-submanifold of* $M$ *and* $\mathscr{S}$ *is a closed subset of* $M$ *with Hausdorff measure* $\leq m-2$. *In particular,* $M = \bigcup_{i=1}^{\ell} \overline{\Omega}_i$. *Moreover,*

– *given* $p_0 \in \mathscr{R}$ *there exist* $i \neq j$ *such that*

$$\lim_{p \to p_0^+} |\nabla_g u_i(p)|^2 = \lim_{p \to p_0^-} |\nabla_g u_j(p)|^2 \neq 0,$$

*where* $p \to p_0^\pm$ *are the limits taken from opposite sides of* $\mathscr{R}$,

– *for* $p_0 \in \mathscr{S}$ *we have*

$$\lim_{p \to p_0} |\nabla_g u_i(p)|^2 = 0 \quad \text{for every } i = 1, \ldots, \ell.$$

(v) *If* $\ell = 2$, *then* $u_{\infty,1} - u_{\infty,2}$ *is a least energy sign-changing solution to the Yamabe equation* (1.1).

From Theorems 1.1 and 1.2 we immediately obtain the following results.

**Theorem 1.3.** *Assume* (A3). *Then for every $\ell \geq 2$ there exists an optimal $\ell$-partition $\{\Omega_1, \ldots, \Omega_\ell\}$ for the Yamabe equation on $(M, g)$ such that each $\Omega_i$ is connected and $M \smallsetminus \bigcup_{i=1}^{\ell} \Omega_i$ is the union of an $(m-1)$-dimensional $\mathcal{C}^{1,\alpha}$-submanifold of $M$ and a closed subset whose Hausdorff measure is at most $m-2$.*

**Theorem 1.4.** *Assume* (A3). *Then there exists a least energy sign-changing solution to the Yamabe equation* (1.1) *having precisely two nodal domains.*

The main difficulty in proving Theorem 1.1 lies in the lack of compactness of the variational functional for the system (1.4). Least energy fully nontrivial solutions are given by minimization on a suitable constraint, but minimizing sequences may blow up, as it happens for instance when $(M, g)$ is the standard sphere. To prove Theorem 1.1 we establish a compactness criterion (Proposition 2.8) that generalizes the condition given by Aubin for the Yamabe equation [3, Théorème 1]. To verify this criterion we introduce a test function and we make use of fine estimates established in [29] to show that, under assumptions (A1) and (A2), a minimizer exists.

The components of least energy fully nontrivial solutions to the system (1.4) may also blow up as the parameters $\lambda_{ij}$ go to $-\infty$. The standard sphere is again an example of this behavior. So, to prove Theorem 1.2, we establish a condition that prevents blow-up (see Lemma 4.2). To verify this condition we need to estimate the energy of suitable test functions. Rather delicate estimates are required, particularly in dimension 10 – where not only the exponents but also the coefficients of the energy expansion play a role – leading to the geometric inequality stated in assumption (A3). These estimates are derived in Appendix A.

But the occurrence of blow-up is not the only delicate issue in proving Theorem 1.2. To obtain an optimal $\ell$-partition we need the limit profiles of the components of the solutions to (1.5) to be continuous. To this end, we show that the components $(u_{n,i})$ are uniformly bounded in the $\alpha$-Hölder norm. This requires subtle regularity arguments which are well known in the flat case; see e.g. [11,43,49,54]. We adapt some of these arguments (for instance, a priori bounds, blow-up arguments and monotonicity formulas) to obtain uniform Hölder bounds for general systems involving an anisotropic differential operator. This result (Theorem B.2) is interesting in itself.

In order to prove the optimal regularity of the limiting profiles $u_i$, the regularity of the free boundaries $M \smallsetminus \bigcup_{i=1}^{\ell} \Omega_i$ and the free boundary condition, we use local coordinates. This reduces the problem to the study of segregated profiles satisfying a system involving divergence-type operators with variable coefficients. Using information arising from the variational system (1.5), we deduce limiting compatibility conditions between the $u_i$'s which allow us to prove an Almgren-type monotonicity formula and to perform a blow-up analysis, combining what is known in the case of the pure Laplacian [11, 49, 57] with some ideas from papers dealing with variable coefficient operators [32, 33, 39, 52]. This result (which we collect in a more general setting in Theorem C.1) is also interesting in its own right.

As mentioned before, optimal $\ell$-partitions on the standard sphere $\mathbb{S}^m$ do not exist. However, if one considers partitions with the additional property that every set $\Omega_i$ is invariant under the action of a suitable group of isometries, then optimal $\ell$-partitions of this kind do exist and they give rise to sign-changing solutions to the Yamabe equation (1.1) with precisely $\ell$-nodal domains for every $\ell \geq 2$, as shown in [19].

Already in 1986, W. Y. Ding [26] established the existence of infinitely many sign-changing solutions to (1.1) on $\mathbb{S}^m$, and quite recently Fernández and Petean [30] showed that there is a solution with precisely $\ell$ nodal domains for each $\ell \geq 2$. These results, like those in [19], make use of the fact that there are groups of isometries of $\mathbb{S}^m$ that do not have finite orbits. Looking for solutions which are invariant under such isometries allows avoiding blow-up. On the other hand, sign-changing solutions to (1.1) which blow-up along some special minimal submanifolds of the sphere $\mathbb{S}^m$ have been found by Del Pino, Musso, Pacard and Pistoia [23, 24]. The existence of a prescribed number of nodal solutions on some manifolds $(M, g)$ with symmetries having finite orbits is established in [18].

However, the existence of nodal solutions to the Yamabe equation (1.1) on an arbitrary manifold $(M, g)$ is largely an open problem. In [2] Ammann and Humbert established the existence of a least energy sign-changing solution when $(M, g)$ is not locally conformally flat and dim $M \geq 11$. Theorem 1.4 recovers and extends this result (see Remark 4.11). We also note that an optimal $\ell$-partition $\{\Omega_1, \dots, \Omega_\ell\}$ gives rise to what in [2] is called a *generalized metric* $\bar{g} := \bar{u}^{2^*-2} g$ *conformal to* $g$ by taking $\bar{u} := u_1 + \dots + u_\ell$ with $u_i$ a positive solution to (1.1) in $\Omega_i$. So Theorem 1.3 may be seen as an extension of the main result in [2].

We close this introduction with references to related problems. The study of elliptic systems like (1.4) with critical exponents in Euclidean spaces has been the subject of intensive research in the past two decades, starting from [15–17]; without being exhaustive, we refer to the recent contributions [27, 58, 59] for a state of the art and further references. For the use of Almgren's monotonicity formula in the classification of entire solutions to elliptic systems with competition terms, we refer for instance to [5,50]. On the other hand, optimal partition problems is another active field of research: see for instance the book [8] for an overview of a general theory using quasi-open sets and other relaxed formulations. Particular interest has been shown when the cost involves Dirichlet eigenvalues (leading to spectral optimal partitions) both in Euclidean spaces (see for instance the survey [7, 35] or the recent [1, 44, 60] and references therein), and in the context of metric graphs (see e.g. [36, 38] and references).

## 2. Compactness for the Yamabe system

We write $\langle \cdot , \cdot \rangle$ and $| \cdot |$ for the Riemannian metric and the norm in $(M, g)$ and for $v, w \in H^1_g(M)$ we define

$$\langle v, w \rangle_g := \int_M (\langle \nabla_g v, \nabla_g w \rangle + \kappa_m S_g v w) \, \mathrm{d}\mu_g \quad \text{and} \quad \|v\|_g := \sqrt{\langle v, v \rangle_g},$$

where $\nabla_g$ denotes the weak gradient. Since we are assuming that the conformal Laplacian $\mathscr{L}_g$ is coercive, $\| \cdot \|_g$ is a norm in $H^1_g(M)$, equivalent to the standard one, and the *Yamabe* *invariant*

$$Y_g := \inf_{u \in H_g^1(M) \smallsetminus \{0\}} \frac{\|u\|_g^2}{|u|_{g,2^*}^2}$$

of $(M, g)$ is positive. We write $|u|_{g,r} := (\int_M |u|^r \, \mathrm{d}\mu_g)^{1/r}$ for the norm in $L_g^r(M)$ with $r \in [1, \infty)$.

Set $\mathcal{H} := (H_g^1(M))^\ell$ and let $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ be given by

$$\mathcal{J}(u_1, \dots, u_\ell) := \frac{1}{2} \sum_{i=1}^{\ell} \|u_i\|_g^2 - \frac{1}{2^*} \sum_{i=1}^{\ell} |u_i|_{g,2^*}^{2^*} - \frac{1}{2^*} \sum_{\substack{i,j=1 \\ j \neq i}}^{\ell} \int_M \lambda_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}} \, \mathrm{d}\mu_g.$$

This functional is of class $\mathcal{C}^1$ and its partial derivatives are

$$\partial_i \mathcal{J}(u_1, \dots, u_\ell) v = \langle u_i, v \rangle_g - \int_M |u_i|^{2^*-2} u_i v \, \mathrm{d}\mu_g - \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \int_M \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i v \, \mathrm{d}\mu_g, \quad v \in H_g^1(M).$$

Hence, the critical points of $\mathcal{J}$ are the solutions to the system (1.4).

Note that every solution $u$ to the Yamabe equation (1.1) gives rise to a solution of the system (1.4) whose $i$-th component is $u$ and all other components are 0. We are interested in *fully nontrivial solutions*, i.e., solutions $(u_1, \dots, u_\ell)$ such that every $u_i$ is nontrivial. They belong to the Nehari-type set

$$\mathcal{N} := \{(u_1, \dots, u_\ell) \in \mathcal{H} : u_i \neq 0, \ \partial_i \mathcal{J}(u_1, \dots, u_\ell) u_i = 0, \ \forall i = 1, \dots, \ell\}.$$

Define

$$\hat{c} := \inf_{(u_1,\dots,u_\ell) \in \mathcal{N}} \mathcal{J}(u_1, \dots, u_\ell) = \inf_{(u_1,\dots,u_\ell) \in \mathcal{N}} \frac{1}{m} \sum_{i=1}^{\ell} \|u_i\|_g^2.$$

A fully nontrivial solution $u$ to (1.4) is called a *least energy fully nontrivial solution* if $\mathcal{J}(u) = \hat{c}$.

**Remark 2.1.** Since $\lambda_{ij} < 0$, it is not hard to check that minimization of $\mathcal{J}$ on the classical Nehari manifold $\{(u_1, \dots, u_\ell) \in \mathcal{H} \smallsetminus \{(0, \dots, 0)\} : \sum_i \partial_i \mathcal{J}(u_1, \dots, u_\ell) u_i = 0\}$ leads necessarily to solutions with only one nonzero component.

**Proposition 2.2.** *If* $(u_1, \dots, u_\ell) \in \mathcal{N}$, *then*

$$0 < Y_g^{m/2} \leq \|u_i\|_g^2 \leq |u_i|_{g,2^*}^{2^*} \quad \forall i = 1, \dots, \ell,$$

*where* $Y_g$ *is the Yamabe invariant of* $(M, g)$. *Hence,* $\mathcal{N}$ *is a closed subset of* $\mathcal{H}$.

*Proof.* Since $u_i \neq 0$, $\partial_i \mathcal{J}(u_1, \dots, u_\ell)u_i = 0$ and $\lambda_{ij} < 0$, we have

$$\|u_i\|_g^2 = |u_i|_{g,2^*}^{2^*} + \sum_{j \neq i} \int_M \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}}\,\mathrm{d}\mu_g \leq |u_i|_{g,2^*}^{2^*} \leq Y_g^{-m/(m-2)}\|u_i\|_g^{2^*}.$$

Hence, $Y_g^{m/2} \leq \|u_i\|_g^2 \leq |u_i|_{g,2^*}^{2^*}$, as claimed. ∎

For $u = (u_1, \dots, u_\ell) \in \mathcal{H}$ and $s = (s_1, \dots, s_\ell) \in (0, \infty)^\ell$, we write

$$su := (s_1u_1, \dots, s_\ell u_\ell).$$

**Proposition 2.3.** *Let $u = (u_1, \dots, u_\ell) \in \mathcal{H}$.*

(i) *If*

$$|u_i|_{g,2^*}^{2^*} > -\sum_{j \neq i} \int_M \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}}\,\mathrm{d}\mu_g \quad \forall i = 1, \dots, \ell,$$

*then there exists $s_u \in (0, \infty)^\ell$ such that $s_u u \in \mathcal{N}$.*

(ii) *If there exists $s_u \in (0, \infty)^\ell$ such that $s_u u \in \mathcal{N}$, then $s_u$ is unique and*

$$\mathcal{J}(s_u u) = \max_{s \in (0,\infty)^\ell} \mathcal{J}(su).$$

*Moreover, $s_u$ depends only on the values*

$$a_{u,i} := \|u_i\|_g^2, \quad b_{u,i} := |u_i|_{g,2^*}^{2^*}, \quad d_{u,ij} := \int_M \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}}\,\mathrm{d}\mu_g,$$

*$i = 1, \dots, \ell$, and it depends continuously on them.*

*Proof.* Define $J_u : (0, \infty)^\ell \to \mathbb{R}$ by

$$J_u(s) := \mathcal{J}(su) = \sum_{i=1}^{\ell} \frac{1}{2}a_{u,i}s_i^2 - \sum_{i=1}^{\ell} \frac{1}{2^*}b_{u,i}s_i^{2^*} - \sum_{i \neq j} \frac{1}{2}d_{u,ij}s_j^{\alpha_{ij}}s_i^{\beta_{ij}}.$$

If $u_i \neq 0$ for all $i = 1, \dots, \ell$, then, as

$$s_i\partial_i J_u(s) = \partial_i \mathcal{J}(su)[s_iu_i], \quad i = 1, \dots, \ell,$$

we see that $su \in \mathcal{N}$ iff $s$ is a critical point of $J_u$. Statements (i) and (ii) follow immediately from [20, Lemmas 2.1–2.3]. ∎

**Remark 2.4.** If $\ell = 1$, then $\mathcal{N} = \{u \in H_g^1(M) : u \neq 0,\ \|u\|_g^2 = |u|_{g,2^*}^{2^*}\}$ is the usual Nehari manifold for the Yamabe problem (1.1) and $s_u \in \mathbb{R}$ is explicitly given by $s_u^{2^*-2} = \|u\|_g^2/|u|_{g,2^*}^{2^*}$. Hence, for every $0 \neq u \in H_g^1(M)$,

$$\frac{1}{m}\left(\frac{\|u\|_g^2}{|u|_{g,2^*}^2}\right)^{m/2} = \mathcal{J}(s_u u) = \max_{s \in (0,\infty)} \mathcal{J}(su),$$

and $\widehat{c} = \frac{1}{m}Y_g^{m/2}$.

Set $\mathcal{T} := \{u \in \mathcal{H} : \|u_i\|_g = 1\ \forall i = 1, \dots, \ell\}$, and let

$$\mathcal{U} := \{u \in \mathcal{T} : su \in \mathcal{N} \text{ for some } s \in (0, \infty)^\ell\}.$$

Following [20, Proposition 3.1], it is easy to see that $\mathcal{U}$ is a nonempty open subset of $\mathcal{T}$. Define $\Psi : \mathcal{U} \to \mathbb{R}$ by

$$\Psi(u) := \mathcal{J}(s_u u),$$

with $s_u$ as in Proposition 2.3. This function has the following properties.

**Proposition 2.5.** (i) $\Psi \in \mathcal{C}^1(\mathcal{U}, \mathbb{R})$.

(ii) *Let $u_n \in \mathcal{U}$. If $(u_n)$ is a Palais–Smale sequence for $\Psi$, then $(s_{u_n} u_n)$ is a Palais–Smale sequence for $\mathcal{J}$. Conversely, if $(u_n)$ is a Palais–Smale sequence for $\mathcal{J}$ and $u_n \in \mathcal{N}$ for all $n \in \mathbb{N}$, then $(u_n/\|u_n\|_g)$ is a Palais–Smale sequence for $\Psi$.*

(iii) *Let $u \in \mathcal{U}$. Then $u$ is a critical point of $\Psi$ if and only if $s_u u$ is a fully nontrivial critical point of $\mathcal{J}$.*

(iv) *If $(u_n)$ is a sequence in $\mathcal{U}$ such that $u_n \to u \in \partial\mathcal{U}$, then $\Psi(u_n) \to \infty$.*

*Proof.* The proof is identical to that of [20, Theorem 3.3]. ∎

**Corollary 2.6.** *If $u \in \mathcal{N}$ and $\mathcal{J}(u) = \hat{c}$, then $u$ is a fully nontrivial solution to the system* (1.4).

*Proof.* Since $\Psi(u/\|u\|_g) = \inf_{\mathcal{U}} \Psi$ and $\mathcal{U}$ is an open subset of the smooth Hilbert manifold $\mathcal{T}$, we know that $u/\|u\|_g$ is a critical point of $\Psi$. By Proposition 2.5, $u$ is a critical point of $\mathcal{J}$. ∎

Recall that the operator $\mathcal{L}_g$ is *conformally invariant*, i.e., if $\tilde{g} = \varphi^{2^*-2} g$, $\varphi > 0$, is a metric conformal to $g$, then

$$\mathcal{L}_{\tilde{g}}(\varphi^{-1}u) = \varphi^{-(2^*-1)}\mathcal{L}_g(u) \quad \forall u \in H_g^1(M).$$

Since $\mathrm{d}\mu_{\tilde{g}} = \varphi^{2^*}\,\mathrm{d}\mu_g$, we have

$$\|\varphi^{-1}u\|_{\tilde{g}} = \|u\|_g \quad \text{and} \quad |\varphi^{-1}u|_{\tilde{g},2^*} = |u|_{g,2^*} \quad \forall u \in H_g^1(M).$$

So, changing the metric within the conformal class of $g$ does not affect our problem.

Let $\mathbb{S}^m$ be the standard $m$-sphere and $p \in \mathbb{S}^m$. Since the stereographic projection $\mathbb{S}^m \smallsetminus \{p\} \to \mathbb{R}^m$ is a conformal diffeomorphism, the Yamabe invariant of $\mathbb{S}^m$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$,

$$\sigma_m := \inf_{w \in D^{1,2}(\mathbb{R}^m) \smallsetminus \{0\}} \frac{\|w\|^2}{|w|_{2^*}^2},$$

where $D^{1,2}(\mathbb{R}^m) := \{w \in L^{2^*}(\mathbb{R}^m) : \nabla u \in [L^2(\mathbb{R}^m)]^m\}$ equipped with the norm $\|w\| := (\int_{\mathbb{R}^m} |\nabla w|^2)^{1/2}$, and $|w|_{2^*}$ is the norm of $w$ in $L^{2^*}(\mathbb{R}^m)$. It is well known that

$$\sigma_m = \frac{m(m-2)}{4}\omega_m^{2/m},$$

where $\omega_m$ denotes the volume of $\mathbb{S}^m$ [4,55].

It is shown in [20, Proposition 4.6] that $\hat{c} = \inf_{\mathcal{N}} \mathcal{J}$ is not attained if $M = \mathbb{R}^m$. Therefore, from the previous paragraph, $\hat{c}$ is also not attained if $M$ is the standard sphere $\mathbb{S}^m$. Our aim is to investigate whether this infimum is attained in some other cases, at least for some values of $\alpha_{ij}$ and $\beta_{ij}$.

To this end, we establish a compactness criterion for the system (1.4) that extends a similar well known criterion for the Yamabe equation [41, Theorem A]. The key ingredient is the following result of T. Aubin.

**Theorem 2.7** (Aubin, 1976). *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\sigma_m |u|^2_{g,2^*} \le (1+\varepsilon) \int_M |\nabla_g u|^2 \, \mathrm{d}\mu_g + C_\varepsilon \int_M u^2 \, \mathrm{d}\mu_g \quad \forall u \in H^1_g(M).$$

*Proof.* See [4, Théorème 9] or [41, Theorem 2.3]. ∎

For each $Z \subset \{1, \dots, \ell\}$, let $(\mathscr{S}_Z)$ be the system of $\ell - |Z|$ equations

$$(\mathscr{S}_Z) \qquad \begin{cases} \mathscr{L}_g u_i = |u_i|^{2^*-2} u_i + \sum\limits_{j \ne i} \lambda_{ij} \beta_{ij} |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}-2} u_i & \text{on } M, \\ i, j \in \{1, \dots, \ell\} \smallsetminus Z, \end{cases}$$

where $|Z|$ denotes the cardinality of $Z$. The *fully nontrivial solutions* of $(\mathscr{S}_Z)$ are the solutions $(u_1, \dots, u_\ell)$ of (1.4) which satisfy $u_i = 0$ iff $i \in Z$. We write $\mathcal{J}_Z$ and $\mathcal{N}_Z$ for the functional and the Nehari set associated to $(\mathscr{S}_Z)$, and define

$$\hat{c}_Z := \inf_{u \in \mathcal{N}_Z} \mathcal{J}_Z(u). \tag{2.1}$$

The following compactness criterion is inspired by [20, Lemma 4.10] (see also [58, Theorem 3.7] and [59, Theorem 2.6]).

**Proposition 2.8.** *Assume that*

$$\hat{c} < \min \left\{ \hat{c}_Z + \frac{|Z|}{m} \sigma_m^{m/2} : \emptyset \ne Z \subseteq \{1, \dots, \ell\} \right\}.$$

*Then $\hat{c}$ is attained by $\mathcal{J}$ on $\mathcal{N}$.*

*Proof.* By Ekeland's variational principle and Proposition 2.5 there is a sequence $(u_n)$ in $\mathcal{N}$ such that $\mathcal{J}(u_n) \to \hat{c}$ and $\mathcal{J}'(u_n) \to 0$. Then $(u_n)$ is bounded in $\mathcal{H}$ and, after passing to a subsequence, $u_{n,i} \rightharpoonup \bar{u}_i$ weakly in $H^1_g(M)$, $u_{n,i} \to \bar{u}_i$ strongly in $L^2_g(M)$ and $u_{n,i} \to \bar{u}_i$ a.e. on $M$, where $u_n = (u_{n,1}, \dots, u_{n,\ell})$. A standard argument shows that $\bar{u} = (\bar{u}_1, \dots, \bar{u}_\ell)$ is a solution of the system (1.4).

To prove that $\bar{u}$ is fully nontrivial, set $Z := \{i \in \{1, \dots, \ell\} : \bar{u}_i = 0\}$. As $u_n \in \mathcal{N}$ and $\lambda_{ij} < 0$, we have $\|u_{n,i}\|^2_g \le |u_{n,i}|^{2^*}_{g,2^*}$, and as $u_{n,i} \to 0$ strongly in $L^2_g(M)$ for each $i \in Z$, Theorem 2.7 and Proposition 2.2 yield, for each $\varepsilon > 0$,

$$\begin{aligned} \sigma_m &\le \frac{(1+\varepsilon) \int_M |\nabla_g u_{n,i}|^2 \, \mathrm{d}\mu_g + o(1)}{|u_{n,i}|^2_{g,2^*}} = \frac{(1+\varepsilon) \|u_{n,i}\|^2_g + o(1)}{|u_{n,i}|^2_{g,2^*}} \\ &\le (1+\varepsilon) (\|u_{n,i}\|^2_g)^{2/m} + o(1). \end{aligned}$$

So, after passing to a subsequence, $\lim_{n\to\infty}(1+\varepsilon)^{m/2}\|u_{n,i}\|_g^2 \geq \sigma_m^{m/2}$ for every $\varepsilon > 0$ and every $i \in Z$, and hence

$$\lim_{n\to\infty} \|u_{n,i}\|_g^2 \geq \sigma_m^{m/2} \quad \forall i \in Z. \tag{2.2}$$

Therefore,

$$\begin{aligned}\hat{c} &= \lim_{n\to\infty} \mathcal{J}(u_n) = \lim_{n\to\infty} \frac{1}{m}\sum_{i=1}^{\ell} \|u_{n,i}\|_g^2 = \lim_{n\to\infty} \frac{1}{m}\Big(\sum_{i\notin Z} \|u_{n,i}\|_g^2 + \sum_{i\in Z} \|u_{n,i}\|_g^2\Big) \\ &\geq \frac{1}{m}\sum_{i\notin Z} \|\bar{u}_i\|_g^2 + \frac{|Z|}{m}\sigma_m^{m/2} \geq \hat{c}_Z + \frac{|Z|}{m}\sigma_m^{m/2}.\end{aligned}$$

But then our assumption implies that $Z = \emptyset$, i.e., $\bar{u}$ is fully nontrivial. Hence, $\bar{u} \in \mathcal{N}$ and $\mathcal{J}(\bar{u}) = \hat{c}$. ∎

The proof of the following regularity result is standard; see, e.g., [54, Appendix B]. We include it here for the sake of completeness.

**Proposition 2.9.** *Let $u = (u_1, \dots, u_\ell) \in \mathcal{H}$ be a solution to the system (1.4). Then $u_i \in \mathcal{C}^{2,\gamma}(M)$ for any $\gamma \in (0,1)$ such that $\gamma < \beta_{ij} - 1$ for all $i, j = 1, \dots, \ell$.*

*Proof.* Let $s > 0$ and assume that $u_i \in L_g^{2(s+1)}(M)$. Note that this is true if $2(1+s) = 2^*$. Fix $\varepsilon > 0$. For each $L > 0$, define $\psi_{iL} := \min\{u_i^s, L\}$. Then $\nabla_g(u_i\psi_{iL}) = (1 + s1_{\{u_i^s\leq L\}})\,\psi_{iL}\nabla_g u_i$ and $\nabla_g(u_i\psi_{iL}^2) = (1 + 2s1_{\{u_i^s\leq L\}})\,\psi_{iL}^2\nabla_g u_i$. So, since $\partial_i\mathcal{J}(u)[u_i\psi_{iL}] = 0$ and $\lambda_{ij} < 0$, for any $K > 0$ we have

$$\begin{aligned}(1+s)^{-1}\int_M |\nabla_g(u_i\psi_{iL})|^2\,\mathrm{d}\mu_g &\leq \int_M \psi_{iL}^2|\nabla_g u_i|^2\,\mathrm{d}\mu_g \leq \int_M \langle\nabla_g u_i, \nabla_g(u_i\psi_{iL}^2)\rangle\,\mathrm{d}\mu_g \\ &\leq \int_M |u_i|^{2^*-2}(u_i\psi_{iL})^2\,\mathrm{d}\mu_g - \int_M \kappa_m S_g(u_i\psi_{iL})^2\,\mathrm{d}\mu_g \\ &\leq K\int_M |u_i\psi_{iL}|^2\,\mathrm{d}\mu_g + \int_{|u_i|^{2^*-2}\geq K} |u_i|^{2^*-2}|u_i\psi_{iL}|^2\,\mathrm{d}\mu_g + C\int_M |u_i\psi_{iL}|^2\,\mathrm{d}\mu_g \\ &\leq (K + C_1)\int_M u_i^{2(s+1)}\,\mathrm{d}\mu_g + \left(\int_{|u_i|^{2^*-2}\geq K} |u_i|^{2^*}\,\mathrm{d}\mu_g\right)^{2/m} |u_i\psi_{iL}|_{g,2^*}^2 \\ &\leq (K + C_2)\int_M u_i^{2(s+1)}\,\mathrm{d}\mu_g + \eta(K)(1+\varepsilon)\sigma_m^{-1}\int_M |\nabla_g(u_i\psi_{iL})|^2\,\mathrm{d}\mu_g,\end{aligned}$$

where the last inequality is given by Theorem 2.7 and $C_1, C_2$ are positive constants independent of $L$ and $K$. Since

$$\eta(K) := \left(\int_{|u_i|^{2^*-2}\geq K} |u_i|^{2^*}\,\mathrm{d}\mu_g\right)^{2/m} \to 0 \quad \text{as } K \to \infty,$$

we may fix $K$ such that $(1+s)\eta(K)(1+\varepsilon)\sigma_m^{-1} = \frac{1}{2}$. Then, as $u_i \in L_g^{2(s+1)}(M)$, the inequality above yields

$$\int_M |\nabla_g(u_i\psi_{iL})|^2 \, d\mu_g \le (1+s)\eta(K)(1+\varepsilon)\sigma_m^{-1}\int_M |\nabla_g(u_i\psi_{iL})|^2 + \widetilde{C}$$
$$\le \frac{1}{2}\int_M |\nabla_g(u_i\psi_{iL})|^2 + \widetilde{C},$$

with $\widetilde{C} > 0$ independent of $L$. Letting $L \to \infty$ we see that $\int_M |\nabla_g(u_i^{s+1})|^2 \, d\mu_g \le 2\widetilde{C}$. Hence, $u_i \in L_g^{2^*(s+1)}(M)$. Now, starting with $s$ such that $2(1+s) = 2^*$ and iterating this argument, we conclude that $u_i \in L_g^r(M)$ for every $r \ge 1$. Since $u_i$ is a weak solution of the equation

$$-\Delta_g u_i = -\kappa_m S_g u_i + |u_i|^{2^*-1} + \sum_{j\neq i} \lambda_{ij}\beta_{ij}|u_j|^{\alpha_{ij}}|u_i|^{\beta_{ij}-1} =: f_i,$$

from elliptic regularity [41, Theorem 2.5] and the Sobolev embedding theorem [41, Theorem 2.2] we get $u_i \in \mathcal{C}^{0,\gamma}(M)$ for any $\gamma \in (0,1)$. Then $f_i \in \mathcal{C}^{0,\gamma}(M)$ for any $\gamma \in (0,1)$ such that $\gamma < \beta_{ij} - 1$ for all $j \neq i$, and, by elliptic regularity again, we conclude that $u_i \in \mathcal{C}^{2,\gamma}(M)$ for any such $\gamma$. ■

## 3. The choice of the test function

To prove the strict inequality in Proposition 2.8 we need a suitable test function. We follow the approach of Lee and Parker [41].

Fix $N > m$. Given $p \in M$, there is a metric $\widetilde{g}$ on $M$ conformal to $g$ such that

$$\det \widetilde{g}_{ij} = 1 + O(|x|^N)$$

in $\widetilde{g}$-normal coordinates at $p$; see [41, Theorem 5.1]. These coordinates are called *conformal normal coordinates at* $p$.

Since the Yamabe invariant $Y_g$ is positive, the Green function $G_p$ for the conformal Laplacian $\mathscr{L}_g$ exists at every $p \in M$ and is strictly positive. Fix $p \in M$ and define the metric $\widehat{g} := G_p^{2^*-2} g$ on $\widehat{M} := M \smallsetminus \{p\}$. This metric is asymptotically flat of some order $\tau > 0$ which depends on $M$. If $m = 3, 4, 5$, or $(M, g)$ is locally conformally flat, the Green function has the asymptotic expansion

$$G_p(x) = b_m^{-1}|x|^{2-m} + A(p) + O(|x|)$$

in conformal normal coordinates $(x^i)$ at $p$, where $b_m = (m-2)\omega_{m-1}$ and $\omega_{m-1}$ is the volume of $\mathbb{S}^{m-1}$. The constant $A(p)$ is related to the mass of the manifold $(\widehat{M}, \widehat{g})$. It follows from the positive mass theorems of Schoen and Yau [47, 48] that $A(p) > 0$ if the manifold $(M, g)$ is not conformal to the standard sphere $\mathbb{S}^m$ and either $m < 6$ or $(M, g)$ is locally conformally flat. In the other cases the expansion of the Green function $G_p$ involves the Weyl tensor $W_g(p)$ of $(M, g)$ at $p$; see [41, Section 6] for details.

For $\delta > 0$, let

$$U_\delta(x) := [m(m-2)]^{(m-2)/4}\left(\frac{\delta}{\delta^2+|x|^2}\right)^{(m-2)/2},$$

written in conformal normal coordinates $(x^i)$ at $p$, and for suitably small $r > 0$ define

$$\widehat{V}_{\delta,p}(x) := \begin{cases} b_m|x|^{m-2}U_\delta(x) & \text{if } |x| \le r, \\ b_m r^{m-2}U_\delta(rx/|x|) & \text{otherwise.} \end{cases}$$

Note that $U_{1/\delta}(x/|x|^2) = |x|^{m-2}U_\delta(x)$. So, up to a constant, $\widehat{V}_{\delta,p}$ is the test function defined in [41, Section 7]. Now set

$$V_{\delta,p} := G_p\widehat{V}_{\delta,p}. \tag{3.1}$$

The following estimates were proved by Esposito, Pistoia and Vétois [29, proof of Lemma 1].

If $m = 3, 4, 5$, or $(M, g)$ is locally conformally flat, then

$$\begin{aligned} \|V_{\delta,p}\|_g^2 &= \sigma_m^{m/2} + (m-2)\,\overline{c}_m A(p)\delta^{m-2} + O(\delta^{m-1}), \\ |V_{\delta,p}|_{g,2^*}^{2^*} &= \sigma_m^{m/2} + m^2\,\overline{c}_m A(p)\delta^{m-2} + O(\delta^{m-1}). \end{aligned} \tag{3.2}$$

If $(M, g)$ is not locally conformally flat and $m = 6$, then

$$\begin{aligned} \|V_{\delta,p}\|_g^2 &= \sigma_6^3 + \overline{c}_6|W_g(p)|_g^2\,\delta^4|\ln\delta| + O(\delta^4), \\ |V_{\delta,p}|_{g,2^*}^{2^*} &= \sigma_6^3 + 9\,\overline{c}_6|W_g(p)|_g^2\,\delta^4|\ln\delta| + O(\delta^4). \end{aligned} \tag{3.3}$$

If $(M, g)$ is not locally conformally flat and $m \ge 7$, then

$$\begin{aligned} \|V_{\delta,p}\|_g^2 &= \sigma_m^{m/2} + \frac{(m-2)^2}{m+2}\,\overline{c}_m\omega_{m-1}|W_g(p)|_g^2\,\delta^4 + O(\delta^5), \\ |V_{\delta,p}|_{g,2^*}^{2^*} &= \sigma_m^{m/2} + \frac{m^2}{m-4}\,\overline{c}_m\omega_{m-1}|W_g(p)|_g^2\,\delta^4 + O(\delta^5). \end{aligned} \tag{3.4}$$

Here $\overline{c}_m$ is a positive constant depending only on $m$. In particular,

$$\overline{c}_m = \frac{1}{192}\,\frac{(m+2)[m(m-2)]^{(m-2)/2}}{2^{m-1}(m-6)(m-1)}\,\frac{\omega_m}{\omega_{m-1}} \quad \text{if } m \ge 7. \tag{3.5}$$

From these estimates we derive the following result.

**Lemma 3.1.** *Assume that $(M, g)$ is not conformal to the standard sphere $\mathbb{S}^m$. Then there exist $p \in M$ and $C_0 > 0$ such that*

$$\frac{1}{m}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,2^*}^2}\right)^{m/2} \le \frac{1}{m}\sigma_m^{m/2} - C_0R(\delta) + o(R(\delta))$$

*for all $\delta > 0$ sufficiently small, where*

$$R(\delta) = \begin{cases} \delta^{m-2} & \textit{if either } m < 6 \textit{ or } (M,g) \textit{ is l.c.f.}, \\ \delta^4|\ln\delta| & \textit{if } m = 6 \textit{ and } (M,g) \textit{ is not l.c.f.}, \\ \delta^4 & \textit{if } m > 6 \textit{ and } (M,g) \textit{ is not l.c.f.} \end{cases}$$

*Proof.* By Remark 2.4,

$$\frac{1}{m}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,2^*}^2}\right)^{m/2} = \frac{1}{2}\|s_\delta V_{\delta,p}\|_g^2 - \frac{1}{2^*}|s_\delta V_{\delta,p}|_{g,2^*}^{2^*},$$

where $s_\delta^{2^*-2} = \|V_{\delta,p}\|_g^2/|V_{\delta,p}|_{g,2^*}^{2^*}$.

If $m = 3, 4, 5$, or $(M, g)$ is locally conformally flat, the positive mass theorem ensures that $A(p) > 0$ for any $p \in M$, and from (3.2) we find

$$\begin{aligned}\frac{1}{m}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,2^*}^2}\right)^{m/2} &= \left(\frac{s_\delta^2}{2} - \frac{s_\delta^{2^*}}{2^*}\right)\sigma_m^{m/2} + C\left(\frac{s_\delta^2}{2}(m-2) - \frac{s_\delta^{2^*}}{2^*}m^2\right)\delta^{m-2} \\ &\quad + o(\delta^{m-2}) \\ &\le \frac{1}{m}\sigma_m^{m/2} + \frac{s_\delta^2}{2}C(m-2)(1 - m s_\delta^{2^*-2})\delta^{m-2} + o(\delta^{m-2}),\end{aligned}$$

where $C$ is a positive constant.

If $(M, g)$ is not locally conformally flat and $m \ge 6$, we choose $p \in M$ such that $|W_g(p)|_g^2 > 0$. Then if $m = 6$, estimates (3.3) yield

$$\frac{1}{6}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,3}^2}\right)^3 \le \frac{1}{6}\sigma_6^3 + \frac{s_\delta^2}{2}C(1 - 3s_\delta)\,\delta^4|\ln\delta| + o(\delta^4|\ln\delta|),$$

and if $m > 6$, from (3.4) we derive

$$\frac{1}{m}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,2^*}^2}\right)^{m/2} \le \frac{1}{m}\sigma_m^{m/2} + \frac{s_\delta^2}{2}(m-2)C\left(\frac{m-2}{m+2} - s_\delta^{2^*-2}\frac{m}{m-4}\right)\delta^4 + o(\delta^4),$$

for some positive constant $C$. Since $s_\delta \to 1$ as $\delta \to 0$, our claim is proved. ∎

**Lemma 3.2.** *Let $R(\delta)$ be as in* Lemma 3.1 *and $\alpha \in [1, \infty)$. Then*

$$\int_M |V_{\delta,p}|^\alpha \,\mathrm{d}\mu_g = o(R(\delta))$$

*if and only if either*

- *$m = 3$, $(M, g)$ is not conformal to $\mathbb{S}^3$ and $2 < \alpha < 4$, or*
- *$(M, g)$ is not locally conformally flat, $m \ge 9$, and $\frac{8}{m-2} < \alpha < \frac{2(m-4)}{m-2}$.*

*Proof.* Set $\gamma := \frac{m-2}{2}\alpha$. From (3.1) we deduce

$$I_\alpha := \int_M |V_{\delta,p}|^\alpha \,\mathrm{d}\mu_g = \begin{cases} O(\delta^\gamma) & \text{if } \gamma < m/2, \\ O(\delta^{m/2}|\ln\delta|) & \text{if } \gamma = m/2, \\ O(\delta^{m-\gamma}) & \text{if } \gamma > m/2. \end{cases}$$

Therefore,

$$\begin{aligned} I_\alpha = o(\delta^{m-2}) &\iff m-2 < \gamma < 2 \iff m = 3 \text{ and } 2 < \alpha < 4, \\ I_\alpha = o(\delta^4) &\iff 4 < \gamma < m-4 \iff m \ge 9 \text{ and } \tfrac{8}{m-2} < \alpha < \tfrac{2(m-4)}{m-2}, \end{aligned}$$

and our claim is proved. ∎

**Proposition 3.3.** *If either assumption* (A1) *or* (A2) *of Theorem* 1.1 *holds true, then*

$$\hat{c} < \min\left\{\hat{c}_Z + \frac{|Z|}{m}\sigma_m^{m/2} : \emptyset \neq Z \subseteq \{1,\dots,\ell\}\right\}. \tag{3.6}$$

*Proof.* We prove this statement by induction on $\ell$.

If $\ell = 1$ the system reduces to the Yamabe equation (1.1), and (3.6) is equivalent to $Y_g < \sigma_m$. This inequality follows from Lemma 3.1 if we take $\delta$ small enough.

Assume that the statement is true for every system $(\mathscr{S}_Z)$ with $|Z| \geq 1$ (i.e., for every system of less than $\ell$ equations). Then the proof of (3.6) reduces to showing that

$$\hat{c} < \min\left\{\hat{c}_Z + \frac{1}{m}\sigma_m^{m/2} : |Z| = 1\right\}.$$

Without loss of generality, we may assume that $Z = \{\ell\}$. By Proposition 2.8 and the induction hypothesis, there exists $(u_1,\dots,u_{\ell-1}) \in \mathcal{N}_Z$ such that $\mathcal{J}_Z(u_1,\dots,u_{\ell-1}) = \hat{c}_Z$. By Proposition 2.9, each $u_i$ is in $\mathcal{C}^0(M)$.

Let $V_{\delta,p}$ be as in Lemma 3.1. Since $\alpha_{ij} \in (1, \frac{m+2}{m-2})$, we have

$$\int_M |V_{\delta,p}|^{\alpha_{ij}} |u_i|^{\beta_{ij}} \,\mathrm{d}\mu_g \leq \max_{q\in M} |u_i(q)|^{\beta_{ij}} \int_M |V_{\delta,p}|^{\alpha_{ij}} \,\mathrm{d}\mu_g \to 0 \quad \text{as } \delta \to 0.$$

Hence, there exists $\delta_0 > 0$ such that, for every $\delta \in (0,\delta_0)$,

$$\begin{aligned}|u_i|_{g,2^*}^{2^*} + \sum_{j\neq i} \beta_{ij}\,\lambda_{ij} \int_M |u_j|^{\alpha_{ij}} |u_i|^{\beta_{ij}} \,\mathrm{d}\mu_g + \beta_{i\ell}\,\lambda_{i\ell} \int_M |V_{\delta,p}|^{\alpha_{i\ell}} |u_i|^{\beta_{i\ell}} \,\mathrm{d}\mu_g \\ = \|u_i\|_g^2 + \beta_{i\ell}\,\lambda_{i\ell} \int_M |V_{\delta,p}|^{\alpha_{i\ell}} |u_i|^{\beta_{i\ell}} \,\mathrm{d}\mu_g > 0, \quad i,j = 1,\dots,\ell-1,\end{aligned}$$

and

$$|V_{\delta,p}|_{g,2^*}^{2^*} + \sum_{j=1}^{\ell-1} \beta_{\ell j}\,\lambda_{\ell j} \int_M |u_j|^{\alpha_{\ell j}} |V_{\delta,p}|^{\beta_{\ell j}} \,\mathrm{d}\mu_g > 0.$$

Then, Proposition 2.3 asserts that there are $0 < r < R < \infty$ and $s_{\delta,1},\dots,s_{\delta,\ell} \in [r,R]$ such that

$$u_\delta = (s_{\delta,1}u_1,\dots,s_{\delta,\ell-1}u_{\ell-1}, s_{\delta,\ell}V_{\delta,p}) \in \mathcal{N} \quad \forall \delta \in (0,\delta_0).$$

By Proposition 2.3 and Lemmas 3.1 and 3.2,

$$\begin{aligned}\hat{c} \leq \mathcal{J}(u_\delta) &= \mathcal{J}_Z(s_{\delta,1}u_1,\dots,s_{\delta,\ell-1}u_{\ell-1}) + \frac{1}{2}s_{\delta,\ell}^2\|V_{\delta,p}\|_g^2 - \frac{1}{2^*}s_{\delta,\ell}^{2^*}|V_{\delta,p}|_{g,2^*}^{2^*} \\ &\quad - \sum_{i=1}^{\ell-1} s_{\delta,\ell}^{\alpha_{i\ell}} s_{\delta,i}^{\beta_{i\ell}} \lambda_{i\ell} \int_M |V_{\delta,p}|^{\alpha_{i\ell}} |u_i|^{\beta_{i\ell}} \,\mathrm{d}\mu_g \\ &\leq \hat{c}_Z + \frac{1}{m}\left(\frac{\|V_{\delta,p}\|_g^2}{|V_{\delta,p}|_{g,2^*}^2}\right)^{m/2} + \sum_{i=1}^{\ell-1} R^{2^*}|\lambda_{i\ell}| \max_{q\in M}|u_i(q)|^{\beta_{i\ell}} \int_M |V_{\delta,p}|^{\alpha_{i\ell}} \,\mathrm{d}\mu_g \\ &\leq \hat{c}_Z + \frac{1}{m}\sigma_m^{m/2} - C_0 R(\delta) + o(R(\delta)) < \hat{c}_Z + \frac{1}{m}\sigma_m^{m/2}\end{aligned}$$

for $\delta$ small enough, as claimed. ∎

*Proof of Theorem* 1.1. By Propositions 3.3 and 2.8, there is $u = (u_1, \dots, u_\ell) \in \mathcal{N}$ such that $\mathcal{J}(u) = \widehat{c}$. Then $\bar{u} = (|u_1|, \dots, |u_\ell|) \in \mathcal{N}$ and $\mathcal{J}(\bar{u}) = \widehat{c}$, and the result follows from Corollary 2.6 and Proposition 2.9. If dim $M = 3$, then since $\alpha_{ij} > 2$, each $|u_i|$ is positive by the strong maximum principle [41, Theorem 2.6]. ∎

## 4. Phase separation and optimal partitions

In this section we restrict to the case $\lambda_{ij} = \lambda$ and $\alpha_{ij} = \beta_{ji} = 2^*/2 =: \beta$. Our aim is to study the behavior of least energy fully nontrivial solutions to the system (1.5) as $\lambda \to -\infty$ and to derive the existence and regularity of an optimal partition.

Let $\Omega$ be an open subset of $M$. As mentioned in the introduction, the nontrivial solutions of (1.2) are the critical points of the restriction of the functional

$$J_\Omega(u) := \frac{1}{2}\|u\|_g^2 - \frac{1}{2^*}|u|_{g,2^*}^{2^*}$$

to the Nehari manifold

$$\mathcal{N}_\Omega := \{u \in H_{g,0}^1(\Omega) : u \neq 0 \text{ and } \|u\|_g^2 = |u|_{g,2^*}^{2^*}\}.$$

So, a minimizer for $J_\Omega$ on $\mathcal{N}_\Omega$ is a solution of (1.2). Note that $J_\Omega(u) := \frac{1}{m}\|u\|_g^2$ if $u \in \mathcal{N}_\Omega$. Therefore,

$$c_\Omega := \inf_{\mathcal{N}_\Omega} J_\Omega = \inf_{u \in \mathcal{N}_\Omega} \frac{1}{m}\|u\|_g^2.$$

Define

$$\mathcal{M}_\ell := \{(u_1, \dots, u_\ell) \in \mathcal{H} : u_i \neq 0,\ \|u_i\|_g^2 = |u_i|_{g,2^*}^{2^*},\ u_i u_j = 0 \text{ on } M \text{ if } i \neq j\},$$

$$c_\ell^* := \inf_{(u_1,\dots,u_\ell) \in \mathcal{M}_\ell} \frac{1}{m} \sum_{i=1}^{\ell} \|u_i\|_g^2.$$

**Lemma 4.1.** *Assume there exists $(u_1, \dots, u_\ell) \in \mathcal{M}_\ell$ such that $u_i \in \mathcal{C}^0(M)$ and*

$$\frac{1}{m} \sum_{i=1}^{\ell} \|u_i\|_g^2 = c_\ell^*.$$

*Set $\Omega_i := \{p \in M : u_i(p) \neq 0\}$. Then $\{\Omega_1, \dots, \Omega_\ell\}$ is an optimal $\ell$-partition for the Yamabe problem on $(M, g)$, each $\Omega_i$ is connected and $J_{\Omega_i}(u_i) = c_{\Omega_i}$ for all $i = 1, \dots, \ell$.*

*Proof.* As $u_i$ is continuous and nontrivial, we see that $\Omega_i$ is an open nonempty subset of $M$ and $u_i \in \mathcal{N}_{\Omega_i}$. Moreover, $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ because $u_i u_j = 0$. Therefore, $\{\Omega_1, \dots, \Omega_\ell\} \in \mathcal{P}_\ell$.

To prove the last two statements of the lemma we argue by contradiction. If, say, $\Omega_1$ were the disjoint union of two nonempty open sets $\Theta_1$ and $\Theta_2$, then, setting $\bar{u}_1(p) :=$

$u_1(p)$ if $p \in \Theta_1$ and $\bar{u}_1(p) := 0$ if $p \in \Theta_2$, we would have $(\bar{u}_1, \ldots, u_\ell) \in \mathcal{M}_\ell$ and

$$\frac{1}{m}\Big(\|\bar{u}_1\|_g^2 + \sum_{i=2}^{\ell} \|u_i\|_g^2\Big) < c_\ell^*,$$

contradicting the definition of $c_\ell^*$.

Similarly, if $J_{\Omega_i}(u_i) > c_{\Omega_i}$ for some $i$, then there would exist $v_i \in \mathcal{N}_{\Omega_i}$ with $c_{\Omega_i} < J_{\Omega_i}(v_i) < J_{\Omega_i}(u_i)$. But then

$$\frac{1}{m}\sum_{j\neq i} \|u_j\|_g^2 + \frac{1}{m}\|u_i\|_g^2 < c_\ell^*,$$

which is again a contradiction. Hence, $J_{\Omega_i}(u_i) = c_{\Omega_i}$ for all $i = 1, \ldots, \ell$ and

$$\inf_{\{\Theta_1,\ldots,\Theta_\ell\}\in\mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Theta_i} \le \frac{1}{m}\sum_{i=1}^{\ell} \|u_i\|_g^2 = c_\ell^* \le \inf_{\{\Theta_1,\ldots,\Theta_\ell\}\in\mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Theta_i}.$$

This shows that $\{\Omega_1, \ldots, \Omega_\ell\}$ is an optimal $\ell$-partition and concludes the proof. ∎

**Lemma 4.2.** *Let $\lambda_n < 0$ and $u_n = (u_{n,1}, \ldots, u_{n,\ell})$ be a least energy fully nontrivial solution to the system (1.5). Assume that $\lambda_n \to -\infty$ as $n \to \infty$ and $u_{n,i} \ge 0$ for all $n \in \mathbb{N}$. Assume further that*

$$c_\ell^* < \min\left\{c_k^* + \frac{\ell-k}{m}\,\sigma_m^{m/2} : 1 \le k < \ell\right\}. \tag{4.1}$$

*Then there exists $(u_{\infty,1}, \ldots, u_{\infty,\ell}) \in \mathcal{M}_\ell$ such that, after passing to a subsequence, $u_{n,i} \to u_{\infty,i}$ strongly in $H_g^1(M)$, $u_{\infty,i} \ge 0$, and*

$$c_\ell^* = \frac{1}{m}\sum_{i=1}^{\ell} \|u_{\infty,i}\|_g^2.$$

*Moreover,*

$$\int_M \lambda_n u_{n,j}^{\beta} u_{n,i}^{\beta} \to 0 \quad \text{as } n \to \infty \text{ whenever } i \neq j. \tag{4.2}$$

*Proof.* To highlight the role of $\lambda_n$, we write $\mathcal{J}_n$ and $\mathcal{N}_n$ for the functional and the Nehari set associated to the system (1.5) and we define

$$\hat{c}_n := \inf_{\mathcal{N}_n} \mathcal{J}_n.$$

Note that $\mathcal{M}_\ell \subset \mathcal{N}_n$ for each $n \in \mathbb{N}$. Therefore,

$$\frac{1}{m}\sum_{i=1}^{\ell} \|u_{n,i}\|_g^2 = \hat{c}_n \le c_\ell^* < \infty \quad \forall n \in \mathbb{N}.$$

So, after passing to a subsequence, we deduce that $u_{n,i} \rightharpoonup u_{\infty,i}$ weakly in $H^1_g(M)$, $u_{n,i} \to u_{\infty,i}$ strongly in $L^2_g(M)$ and $u_{n,i} \to u_{\infty,i}$ a.e. on $M$, for each $i = 1, \dots, \ell$. Hence, $u_{\infty,i} \geq 0$. Moreover, as $\partial_i \mathcal{J}_n(u_n)[u_{n,i}] = 0$, we have, for each $j \neq i$,

$$0 \leq \int_M \beta\, |u_{n,j}|^\beta |u_{n,i}|^\beta \, d\mu_g \leq \frac{|u_{n,i}|^{2^*}_{g,2^*}}{-\lambda_n} \leq \frac{C}{-\lambda_n}.$$

Fatou's lemma then yields

$$0 \leq \int_M |u_{\infty,j}|^\beta |u_{\infty,i}|^\beta \, d\mu_g \leq \liminf_{n\to\infty} \int_M |u_{n,j}|^\beta |u_{n,i}|^\beta \, d\mu_g = 0.$$

Hence, $u_{\infty,j} u_{\infty,i} = 0$ a.e. on $M$ whenever $i \neq j$. On the other hand, as $\partial_i \mathcal{J}_n(u_n)[u_{\infty,i}] = 0$ and $u_{n,i} \geq 0$, $u_{\infty,i} \geq 0$, we have

$$\langle u_{n,i}, u_{\infty,i} \rangle_g \leq \int_M u_{n,i}^{2^*-1} u_{\infty,i} \, d\mu_g.$$

So, passing to the limit as $n \to \infty$ we obtain

$$\|u_{\infty,i}\|^2_g \leq |u_{\infty,i}|^{2^*}_{g,2^*} \quad \forall i = 1, \dots, \ell. \tag{4.3}$$

We claim that

$$u_{\infty,i} \neq 0 \quad \forall i = 1, \dots, \ell. \tag{4.4}$$

To prove this claim, let $Z := \{i \in \{1, \dots, \ell\} : u_{\infty,i} = 0\}$. After reordering, we may assume that either $Z = \emptyset$ or $Z = \{k+1, \dots, \ell\}$ for some $0 \leq k < \ell$. Then, arguing as we did to prove (2.2), we get

$$\lim_{n\to\infty} \|u_{n,i}\|^2_g \geq \sigma_m^{m/2} \quad \forall i \in Z.$$

On the other hand, if $i \notin Z$, there exists $t_i \in (0, \infty)$ such that $\|t_i u_{\infty,i}\|^2_g = |t_i u_{\infty,i}|^{2^*}_{g,2^*}$. So $(t_1 u_{\infty,1}, \dots, t_k u_{\infty,k}) \in \mathcal{M}_k$. Inequality (4.3) implies that $t_i \in (0, 1]$. Therefore,

$$\begin{aligned} c_k^* + \frac{\ell - k}{m}\sigma_m^{m/2} &\leq \frac{1}{m}\sum_{i=1}^k \|t_i u_{\infty,i}\|^2_g + \frac{\ell-k}{m}\sigma_m^{m/2} \\ &\leq \frac{1}{m}\sum_{i=1}^k \|u_{\infty,i}\|^2_g + \frac{\ell-k}{m}\sigma_m^{m/2} \leq \frac{1}{m}\lim_{n\to\infty}\sum_{i=1}^\ell \|u_{n,i}\|^2_g = \lim_{n\to\infty}\hat{c}_n \leq c_\ell^*. \end{aligned} \tag{4.5}$$

But then assumption (4.1) implies that $k = \ell$, i.e., $Z = \emptyset$ and claim (4.4) is proved. Moreover, (4.5) becomes

$$c_\ell^* \leq \frac{1}{m}\sum_{i=1}^\ell \|t_i u_{\infty,i}\|^2_g \leq \frac{1}{m}\sum_{i=1}^\ell \|u_{\infty,i}\|^2_g \leq \frac{1}{m}\lim_{n\to\infty}\sum_{i=1}^\ell \|u_{n,i}\|^2_g \leq c_\ell^*.$$

Hence, $t_i = 1$, and so $(u_{\infty,1}, \dots, u_{\infty,\ell}) \in \mathcal{M}_\ell$, and

$$\frac{1}{m}\sum_{i=1}^\ell \|u_{\infty,i}\|^2_g = \lim_{n\to\infty}\frac{1}{m}\sum_{i=1}^\ell \|u_{n,i}\|^2_g = c_\ell^*. \tag{4.6}$$

Consequently, $u_{n,i} \to u_{\infty,i}$ strongly in $H^1_g(M)$. Finally, since

$$\sum_{i=1}^{\ell}\|u_{\infty,i}\|_g^2=\sum_{i=1}^{\ell}|u_{\infty,i}|_{g,2^*}^{2^*},$$
$$\sum_{i=1}^{\ell}\|u_{n,i}\|_g^2=\sum_{i=1}^{\ell}|u_{n,i}|_{g,2^*}^{2^*}+\sum_{\substack{i,j=1\\ j\neq i}}^{\ell}\int_M \lambda_n\beta|u_{n,j}|^{\alpha_{ij}}|u_{n,i}|^{\beta_{ij}},$$

and $u_{n,i}\to u_{\infty,i}$ strongly in $H_g^1(M)$ and $L_g^{2^*}(M)$, we obtain (4.2). ∎

**Lemma 4.3.** *Let $\lambda_n<0$ and $(u_{n,1},\dots,u_{n,\ell})$ be a solution to the system (1.5) such that $u_{n,i}\geq 0$ and $u_{n,i}\to u_{\infty,i}$ strongly in $H_g^1(M)$ as $n\to\infty$. Then $(u_{n,i})$ is uniformly bounded in $L^\infty(M)$ for all $i=1,\dots,\ell$.*

*Proof.* We write again $\mathcal{J}_n$ for the functional associated to the system (1.5). Note that, by Proposition 2.9, $u_{n,i}\in L^\infty(M)$ for all $n\in\mathbb{N}$, $i=1,\dots,\ell$. Fix $i\in\{1,\dots,\ell\}$.

Let $s\geq 0$ and set $w_{n,i}:=u_{n,i}^{1+s}$. Since $\partial_i\mathcal{J}_n(u_n)[u_{n,i}^{1+2s}]=0$ and $\lambda_{ij,n}<0$, we get

$$\begin{aligned}\int_M|\nabla_g w_{n,i}|^2\,\mathrm{d}\mu_g&=(1+s)^2\int_M u_{n,i}^{2s}|\nabla_g u_{n,i}|^2\,\mathrm{d}\mu_g\\&\leq(1+s)^2\int_M\langle\nabla_g u_{n,i},\nabla_g u_{n,i}^{1+2s}\rangle\,\mathrm{d}\mu_g\\&\leq(1+s)^2\int_M|u_{n,i}|^{2^*-2}w_{n,i}^2\,\mathrm{d}\mu_g-(1+s)^2\int_M\kappa_m S_g w_{n,i}^2\,\mathrm{d}\mu_g.\end{aligned}\tag{4.7}$$

Now, for any $K>0$, we have

$$\begin{aligned}\int_M|u_{n,i}|^{2^*-2}w_{n,i}^2\,\mathrm{d}\mu_g&\leq K^{2^*-2}\int_M w_{n,i}^2\,\mathrm{d}\mu_g\\&\quad+\int_{|u_{\infty,i}|\geq K}|u_{\infty,i}|^{2^*-2}w_{n,i}^2\,\mathrm{d}\mu_g+\int_M(|u_{n,i}|^{2^*-2}-|u_{\infty,i}|^{2^*-2})w_{n,i}^2\,\mathrm{d}\mu_g\\&\leq K^{2^*-2}|w_{n,i}|_{g,2}^2+\eta(K,n)|w_{n,i}|_{g,2^*}^2,\end{aligned}\tag{4.8}$$

where

$$\eta(K,n):=\left[\int_{|u_{\infty,i}|\geq K}|u_{\infty,i}|^{2^*}\,\mathrm{d}\mu_g\right]^{\frac{2^*-2}{2^*}}+\left||u_{n,i}|^{2^*-2}-|u_{\infty,i}|^{2^*-2}\right|_{g,\frac{2^*}{2^*-2}}.$$

Since $u_{n,i}\to u_{\infty,i}$ in $H_g^1(M)$, we have $|u_{n,i}|^{2^*-2}\to|u_{\infty,i}|^{2^*-2}$ in $L_g^{\frac{2^*}{2^*-2}}(M)$. Fix $\varepsilon>0$, and choose $K_s,n_s$ such that $\frac{1+\varepsilon}{\sigma_m}(1+s)^2\,\eta(K_s,n)<\frac{1}{2}$ for every $n\geq n_s$. From Theorem 2.7 and inequalities (4.7) and (4.8) we obtain

$$\begin{aligned}|w_{n,i}|_{g,2^*}^2&\leq\frac{1+\varepsilon}{\sigma_m}\int_M|\nabla_g w_{n,i}|^2\,\mathrm{d}\mu_g+C|w_{n,i}|_{g,2}^2\\&\leq\frac{1+\varepsilon}{\sigma_m}(1+s)^2\,\eta(K_s,n)|w_{n,i}|_{g,2^*}^2+C_s|w_{n,i}|_{g,2}^2\\&\leq\frac{1}{2}|w_{n,i}|_{g,2^*}^2+C_s|w_{n,i}|_{g,2}^2\quad\forall n\geq n_s.\end{aligned}$$

Therefore,

$$|u_{n,i}|_{g,2^*(1+s)}^{2(1+s)} = |w_{n,i}|_{g,2^*}^2 \leq \widetilde{C}_s |w_{n,i}|_{g,2}^2 = \widetilde{C}_s |u_{n,i}|_{g,2(1+s)}^{2(1+s)} \quad \forall n \in \mathbb{N},$$

whence

$$|u_{n,i}|_{g,2^*(1+s)} \leq \widetilde{C}_s' |u_{n,i}|_{g,2(1+s)} \quad \forall n \in \mathbb{N},$$

where $C_s$, $\widetilde{C}_s$ and $\widetilde{C}_s'$ are positive constants depending on $s$ but not on $n$. Iterating this inequality, starting with $s = 0$, we conclude that, for any $r \in [2, \infty)$,

$$|u_{n,i}|_{g,r}^2 \leq \bar{C}_r \quad \forall n \in \mathbb{N},$$

where $\bar{C}_r$ is a positive constant independent of $n$. Now, we fix $2R > 0$ smaller than the injectivity radius of $M$. Since $M$ is covered by a finite number of geodesic balls of radius $R$ and $u_{n,i}$ satisfies

$$\mathscr{L}_g u_{n,i} \leq |u_{n,i}|^{2^*-2} u_{n,i} \quad \text{on } M,$$

we deduce from [34, Theorem 8.17] that $(u_{n,i})$ is uniformly bounded in $L^\infty(M)$, as claimed. ∎

**Lemma 4.4.** *For $\lambda_n < 0$ such that $\lambda_n \to -\infty$ let $(u_{n,1}, \ldots, u_{n,\ell})$ be a solution to the system (1.5) such that $u_{n,i} \geq 0$ and $(u_{n,i})$ is uniformly bounded in $L^\infty(M)$ for each $i = 1, \ldots, \ell$. Then for any $\alpha \in (0, 1)$ there exists $C_\alpha > 0$ such that*

$$\|u_{n,i}\|_{\mathscr{C}^{0,\alpha}(M)} \leq C_\alpha \quad \forall n \in \mathbb{N},\ \forall i = 1, \ldots, \ell.$$

*Proof.* This is a particular case of Theorem B.1. ∎

**Lemma 4.5.** *Assume that $(M, g)$ is not locally conformally flat, $m \geq 10$, and there exists $(u_1, \ldots, u_{\ell-1}) \in \mathcal{M}_{\ell-1}$ such that $u_i \in \mathscr{C}^0(M)$, $u_i \geq 0$ and*

$$\frac{1}{m} \sum_{i=1}^{\ell-1} \|u_i\|_g^2 = c_{\ell-1}^*.$$

*If $m = 10$, assume further that there exists $p \in M$ such that*

$$0 < u_1(p) < \tfrac{5}{567} |W_g(p)|_g^2. \tag{4.9}$$

*Then*

$$c_\ell^* < \min \left\{ c_k^* + \frac{\ell - k}{m} \sigma_m^{m/2} : 1 \leq k < \ell \right\}.$$

*Proof.* It suffices to show that

$$c_\ell^* < c_{\ell-1}^* + \frac{1}{m} \sigma_m^{m/2}. \tag{4.10}$$

Set $\Omega_i := \{q \in M : u_i(q) > 0\}$. Then $\Omega_i$ is open and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Since $(M, g)$ is not locally conformally flat and $m \geq 4$, there exists $p \in M$ such that the Weyl tensor

$W_g(p)$ at $p$ does not vanish. After reordering, we may assume that either $p \in \Omega_1$, or $p \in M \smallsetminus \bigcup_{i=1}^{\ell-1} \overline{\Omega}_i$.

First, we consider the case where $p \in \Omega_1$. If $m = 10$ we take $p$ satisfying (4.9). Fix $r > 0$ suitably small so that the closed geodesic ball centered at $p$ is contained in $\Omega_1$ and let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth cut-off function such that $0 \le \chi \le 1$, $\chi \equiv 1$ in $[0, r/2]$ and $\chi \equiv 0$ in $[r, \infty)$. Define $\widetilde{V}_{\delta,p}$ on $M$ by

$$\widetilde{V}_{\delta,p}(x) = \chi(|x|)V_{\delta,p}(x) \quad \text{if } |x| \le r, \qquad \widetilde{V}_{\delta,p}(x) = 0 \quad \text{otherwise},$$

written in conformal normal coordinates around $p$, where $V_{\delta,p}$ is the function in (3.1). If $(M, g)$ is not locally conformally flat and $m \ge 7$, estimates (3.4) yield

$$\begin{aligned} \|\widetilde{V}_{\delta,p}\|_g^2 &= \sigma_m^{m/2} + \frac{(m-2)^2}{m+2}\,\bar{c}_m \omega_{m-1} |W_g(p)|_g^2\, \delta^4 + o(\delta^4), \\ |\widetilde{V}_{\delta,p}|_{g,2^*}^{2^*} &= \sigma_m^{m/2} + \frac{m^2}{m-4}\,\bar{c}_m \omega_{m-1} |W_g(p)|_g^2\, \delta^4 + o(\delta^4), \end{aligned} \tag{4.11}$$

with $\bar{c}_m$ as in (3.5). Now, set

$$v_1 := (u_1 - \widetilde{V}_{\delta,p})^+ \quad \text{and} \quad v_\ell := (u_1 - \widetilde{V}_{\delta,p})^-.$$

Note that $v_i \neq 0$ and $v_1 v_\ell = 0$ on $M$, and $v_1 = 0 = v_\ell$ in $M \smallsetminus \Omega_1$. Let $s_i > 0$ be such that $\|s_i v_i\|_g^2 = |s_i v_i|_{g,2^*}^{2^*}$. Then $(s_1 v_1, u_2, \ldots, u_{\ell-1}, s_\ell v_\ell) \in \mathcal{M}_\ell$ and

$$\|s_i v_i\|_g^2 = \left( \frac{\|v_i\|_g^2}{|v_i|_{g,2^*}^2} \right)^{m/2} \quad \text{for } i = 1, \ell. \tag{4.12}$$

For $m \ge 10$ from Remark 2.4 and Lemma A.1 we derive

$$\begin{aligned} \frac{1}{m}\left( \frac{\|v_1\|_g^2}{|v_1|_{g,2^*}^2} \right)^{m/2} &= \frac{1}{2}\|s_1 v_1\|_g^2 - \frac{1}{2^*}|s_1 v_1|_{g,2^*}^{2^*} \\ &= \frac{s_1^2}{2}\|u_1\|_g^2 - \frac{s_1^{2^*}}{2^*}|u_1|_{g,2^*}^{2^*} - (s_1^2 - s_1^{2^*})\mathfrak{a}_m u_1(p)\delta^{\frac{m-2}{2}} + o(\delta^4) \\ &\le \frac{1}{m}\|u_1\|_g^2 + o(\delta^4), \end{aligned} \tag{4.13}$$

because $\|u_1\|_g^2 = |u_1|_{g,2^*}^{2^*}$ and $s_1^{2^*-2} = \|v_1\|_g^2 / |v_1|_{g,2^*}^{2^*} \to 1$ as $\delta \to 0$. Similarly, using (4.11), we obtain

$$\begin{aligned} \frac{1}{m}\left( \frac{\|v_\ell\|_g^2}{|v_\ell|_{g,2^*}^2} \right)^{m/2} &= \frac{1}{2}\|s_\ell v_\ell\|_g^2 - \frac{1}{2^*}|s_\ell v_\ell|_{g,2^*}^{2^*} \\ &\le \frac{1}{m}\sigma_m^{m/2} + \left( \frac{s_\ell^2}{2}\,\frac{(m-2)^2}{m+2} - \frac{s_\ell^{2^*}}{2^*}\,\frac{m^2}{m-4} \right) \bar{c}_m \omega_{m-1} |W_g(p)|_g^2\, \delta^4 \\ &\quad + (s_\ell^2(\mathfrak{a}_m + \mathfrak{b}_m) - s_\ell^{2^*}\mathfrak{b}_m) u_1(p) \delta^{(m-2)/2} + o(\delta^4), \end{aligned}$$

where $\mathfrak{a}_m$ and $\mathfrak{b}_m$ are defined in (A.7). Since $s_\ell^{2^*-2} = \|v_\ell\|_g^2/|v_\ell|_{g,2^*}^{2^*} \to 1$ as $\delta \to 0$, and $\frac{1}{2}\frac{(m-2)^2}{m+2} < \frac{1}{2^*}\frac{m^2}{m-4}$ and $\frac{m-2}{2} > 4$ when $m \geq 11$, we find that, for $\delta$ small enough,

$$\frac{1}{m}\left(\frac{\|v_\ell\|_g^2}{|v_\ell|_{g,2^*}^2}\right)^{m/2} \leq \frac{1}{m}\sigma_m^{m/2} - C\delta^4 + o(\delta^4) \quad \text{if } m \geq 11, \tag{4.14}$$

with $C > 0$. On the other hand, if $m = 10$, then $\frac{m-2}{2} = 4$. Recalling that $\omega_m$ is the volume of the standard $m$-sphere $\mathbb{S}^m$ and using (3.5) we obtain

$$\begin{aligned}
&\mathfrak{a}_m u_1(p) + \left(\frac{1}{2}\frac{(m-2)^2}{m+2} - \frac{1}{2^*}\frac{m^2}{m-4}\right)\overline{c}_m\omega_{m-1}|W_g(p)|_g^2 \\
&= \mathfrak{a}_m\left[u_1(p) + \frac{1}{2}\left[\frac{m-2}{m+2} - \frac{m}{m-4}\right]\frac{1}{192}\,\frac{(m+2)[m(m-2)]^{\frac{m-2}{4}}}{2^{m-1}(m-6)(m-1)}\,\frac{\omega_m}{\omega_{m-1}}|W_g(p)|_g^2\right] \\
&= \mathfrak{a}_{10}\left[u_1(p) - \frac{5}{567}|W_g(p)|_g^2\right] < 0
\end{aligned}$$

by assumption (4.9). Hence, for $\delta$ small enough,

$$\frac{1}{m}\left(\frac{\|v_\ell\|_g^2}{|v_\ell|_{g,2^*}^2}\right)^{m/2} \leq \frac{1}{m}\sigma_m^{m/2} - C\delta^4 + o(\delta^4) \quad \text{if } m = 10, \tag{4.15}$$

with $C > 0$. From (4.12)–(4.15) we derive

$$\begin{aligned}
c_\ell^* &\leq \frac{1}{m}(\|t_1 v_1\|_g^2 + \|u_2\|_g^2 + \cdots + \|u_{\ell-1}\|_g^2 + \|t_\ell v_\ell\|_g^2) \\
&= \frac{1}{m}\left[\left(\frac{\|v_1\|_g^2}{|v_1|_{g,2^*}^2}\right)^{m/2} + \|u_2\|_g^2 + \cdots + \|u_{\ell-1}\|_g^2 + \left(\frac{\|v_\ell\|_g^2}{|v_\ell|_{g,2^*}^2}\right)^{m/2}\right] \\
&\leq \frac{1}{m}(\|u_1\|_g^2 + \|u_2\|_g^2 + \cdots + \|u_{\ell-1}\|_g^2) + \frac{1}{m}\sigma_m^{m/2} - C\delta^4 + o(\delta^4) \\
&< c_{\ell-1}^* + \frac{1}{m}\sigma_m^{m/2}
\end{aligned}$$

for $\delta$ small enough. This proves (4.10) when $p \in \Omega_1$.

If $p \in M \smallsetminus \bigcup_{i=1}^{\ell-1}\overline{\Omega}_i$, we fix $r > 0$ small enough that the closed geodesic ball of radius $r$ centered at $p$ is contained in $M \smallsetminus \bigcup_{i=1}^{\ell-1}\overline{\Omega}_i$ and define $u_\ell := t_\ell\widetilde{V}_{\delta,p}$ with $\widetilde{V}_{\delta,p}$ as above and $t_\ell > 0$ such that $\|u_\ell\|_g^2 = |u_\ell|_{g,2^*}^{2^*}$. Then $(u_1, \ldots, u_\ell) \in \mathcal{M}_\ell$ and estimates (4.11) yield

$$c_\ell^* \leq \frac{1}{m}\sum_{i=1}^{\ell}\|u_i\|_g^2 < c_{\ell-1}^* + \frac{1}{m}\sigma_m^{m/2},$$

as claimed. ■

**Remark 4.6.** The argument given above does not carry over to $m < 10$ or to the case where $(M, g)$ is locally conformally flat. Indeed, as can be seen from identities (3.2)–

(3.4) and Lemma A.1, in these cases

$$\frac{1}{m}\left(\frac{\|v_1\|_g^2}{|v_1|_{g,2^*}^2}\right)^{m/2} \leq \frac{1}{m}\|u_1\|_g^2 + o(\delta^{(m-2)/2}),$$
$$\frac{1}{m}\left(\frac{\|v_\ell\|_g^2}{|v_\ell|_{g,2^*}^2}\right)^{m/2} \leq \frac{1}{m}\sigma_m^{m/2} + Cu_1(p)\delta^{(m-2)/2} + o(\delta^{(m-2)/2}),$$

with $Cu_1(p) > 0$, for $\delta$ small enough.

**Remark 4.7.** If $m = 10$, then the following geometric conditions suffice to guarantee (4.9):

- For $\ell \geq 3$, inequality (4.9) holds true if $|W_g(q)|_g \neq 0$ for every $q \in M$, because for $p \in \Omega_1 := \{q \in M : u_1(q) > 0\}$ close enough to $\partial\Omega_1$ one has

$$u_1(p) < \tfrac{5}{567} \min_{q\in M} |W_g(q)|_g^2.$$

- For $\ell = 2$, inequality (4.9) holds true if

$$|S_g(q)|^2 < \tfrac{5}{28}|W_g(q)|_g^2 \quad \forall q \in M.$$

  Indeed, choosing $p$ to be a minimum point of $u_1$, since $u_1$ is a positive solution to the Yamabe equation (1.1) we have $\kappa_m S_g(p)u_1 = u_1^{2^*-1} + \Delta_g u_1 \geq u_1^{2^*-1}$. Setting $m = 10$ we get $u_1(p) \leq \frac{4}{81}|S_g(p)|^2 < \frac{5}{567}|W_g(p)|_g^2$.

**Lemma 4.8.** *Assume that $(M, g)$ satisfies the following conditions:*

(A4) *$(M, g)$ is not locally conformally flat and* $\dim M \geq 10$*. If* $\dim M = 10$*, then there exist a positive least energy fully nontrivial solution $\bar{u}$ to the Yamabe equation* (1.1) *and a point $p \in M$ such that $\bar{u}(p) < \frac{5}{567}|W_g(p)|_g^2$, and in addition $|W_g(q)|_g \neq 0$ for every $q \in M$ if $\ell \geq 3$.*

*Let $\lambda_n < 0$ and $u_n = (u_{n,1}, \ldots, u_{n,\ell})$ be a least energy fully nontrivial solution to the system* (1.5)*. Assume that $\lambda_n \to -\infty$ as $n \to \infty$ and $u_{n,i} \geq 0$ for all $n \in \mathbb{N}$. Then there exists $(u_{\infty,1}, \ldots, u_{\infty,\ell}) \in \mathcal{M}_\ell$ with $u_{\infty,i} \in \mathcal{C}^{0,\alpha}(M)$ for every $\alpha \in (0, 1)$ such that, after passing to a subsequence, $u_{n,i} \to u_{\infty,i}$ strongly in $H_g^1(M) \cap \mathcal{C}^{0,\alpha}(M)$, $u_{\infty,i} \geq 0$, and*

$$c_\ell^* = \frac{1}{m}\sum_{i=1}^{\ell} \|u_{\infty,i}\|_g^2.$$

*Moreover,*

$$\int_M \lambda_n u_{i,n}^\beta u_{j,n}^\beta \to 0 \quad \text{as } n \to \infty \quad \text{whenever } i \neq j.$$

*Proof.* The proof is by induction on $\ell$.

Let $\ell = 2$. Then we take $u_1$ to be a positive least energy solution to the Yamabe equation (if $m = 10$, take $u_1 := \bar{u}$ given by (A4)). It satisfies the hypotheses of Lemma 4.5. Therefore, inequality (4.1) holds true and Lemma 4.2 yields the existence of

$(u_{\infty,1},\dots,u_{\infty,\ell})\in\mathcal{M}_\ell$ such that, after passing to a subsequence, $u_{n,i}\to u_{\infty,i}$ strongly in $H^1_g(M)$, $u_{\infty,i}\geq 0$, and

$$c^*_\ell=\frac{1}{m}\sum_{i=1}^{\ell}\|u_{\infty,i}\|_g^2.$$

From Lemmas 4.3 and 4.4 we know that $(u_{n,i})$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(M)$. Therefore, the family $\{u_{n,i}\}$ is equicontinuous and, as $u_{n,i}\to u_{\infty,i}$ a.e. on $M$, the Arzelà–Ascoli theorem yields $u_{n,i}\to u_{\infty,i}$ in $\mathcal{C}^0(M)$.

Now, let $\ell\geq 3$ and assume that the statement holds true for $\ell-1$. Then, by Remark 4.7, the hypotheses of Lemma 4.5 are satisfied, and consequently (4.1) holds true for $\ell$. The same argument we gave for $\ell=2$ yields the result for $\ell$. ∎

**Remark 4.9.** Observe that to prove the previous lemma for $\ell$, we need it to be true for $\ell-1$, because inequality (4.1) must hold true in order to apply Lemma 4.2. Therefore, the inequality $\bar{u}(p)<\frac{5}{567}|W_g(p)|_g^2$ is required for every $\ell\geq 2$.

*Proof of Theorem* 1.2. As pointed out in Remark 4.7, assumption (A3) implies (A4). Statements (i) and (iii) follow immediately from Lemmas 4.8 and 4.1.

Proofs of (ii) and (iv) : These statements have a local nature. In local coordinates the system (1.5) becomes

$$-\operatorname{div}(A(x)\nabla u_i)=f_i(x,u_i)+a(x)\sum_{\substack{j=1\\ j\neq i}}^{\ell}\lambda_n|u_{n,j}|^\beta|u_{n,i}|^{\beta-2}u_{n,i},\quad x\in\Omega,$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^m$, $a(x)=\sqrt{|g(x)|}$, $A(x)=\sqrt{|g(x)|}(g^{kl}(x))$, and $f_i(x,s):=a(x)(|s|^{2^*-2}s-\kappa_m S_g(x)s)$. As usual, $(g_{kl})$ is the metric $g$ in local coordinates, $(g^{kl})$ is its inverse and $|g|$ its determinant. This system satisfies assumptions (H1′), (H2) and (H3) of Theorem C.1 in Appendix C. Statements (i) and (iii), which are already proved, yield assumptions (H4), (H5) and (H6). From Theorem C.1 we deduce that (ii) and (iv) hold true locally on $M$, hence also globally.

Proof of ($v$): If $u\in H^1_g(M)$ is a sign-changing solution of the Yamabe equation (1.1), then $u^+:=\max\{u,0\}\neq 0$, $u^-:=\min\{u,0\}\neq 0$ and $J_M'(u)[u^\pm]=0$. Hence, $u$ belongs to the set

$$\mathcal{E}_M:=\{u\in\mathcal{N}_M:u^+\in\mathcal{N}_M\text{ and }u^-\in\mathcal{N}_M\}.$$

Moreover, as shown in [13, Lemma 2.6], any minimizer of $J_M$ on $\mathcal{E}_M$ is a sign-changing solution of (1.1). For every $u\in\mathcal{E}_M$, we have $(u^+,u^-)\in\mathcal{M}_2$ and $J_M(u)=\frac{1}{m}(\|u^+\|_g^2+\|u^-\|_g^2)$. Therefore,

$$\inf_{\mathcal{E}_M}J_M\geq c_2^*=\frac{1}{m}(\|u_{\infty,1}\|_g^2+\|u_{\infty,2}\|_g^2).$$

As $u_{\infty,1}-u_{\infty,2}\in\mathcal{E}_M$, it is a minimizer of $J_M$ on $\mathcal{E}_M$. Hence, it is a sign-changing solution of (1.1), as claimed. ∎

**Remark 4.10.** As can be seen from its proof, Theorem 1.2 is true under assumption (A4), and consequently so are Theorems 1.3 and 1.4. As noted in Remark 4.7, (A4) is weaker than (A3), but it requires some knowledge on the least energy solution to the Yamabe equation (1.1) having precisely two nodal domains.

**Remark 4.11.** In [2], Ammann and Humbert defined the second Yamabe invariant of $(M, g)$ as

$$\mu_2(M,g) := \inf_{\widetilde{g}\in[g]} \lambda_2(\widetilde{g})\mathrm{Vol}(M,\widetilde{g})^{2/m},$$

where $\lambda_2(\widetilde{g})$ is the second eigenvalue of the operator $\kappa_m^{-1}\mathscr{L}_{\widetilde{g}}$ and $[g]$ is the conformal class of $g$. Using the variational characterization in [2, Proposition 2.1] one can easily verify that

$$\inf_{\mathcal{E}_M} J_M = \frac{1}{m}(\kappa_m\,\mu_2(M,g))^{m/2}.$$

The invariant $\mu_2(M, g)$ is not attained at a metric, but it is shown in [2] that if $(M, g)$ is not locally conformally flat and $m \geq 11$, this invariant is attained at the *generalized metric* conformal to $g$ which is given by a minimizer of $J_M$ in $\mathcal{E}_M$. So Theorem 1.4 recovers and extends this result.

**Remark 4.12.** It is interesting to compare our result with that proved by Robert and Vétois [45] under assumptions which are complementary to ours. In fact, they establish the existence of a sign-changing solution to the subcritical perturbation of the Yamabe equation

$$-\Delta_g u + \kappa_m S_g u = |u|^{2^*-2-\varepsilon}u \quad \text{on } M,$$

which looks like the difference between a positive solution $u_0$ to the Yamabe equation and a bubble. Their result holds true either in the locally conformally flat case, or in low dimensions $3 \leq m \leq 9$, or if $m = 10$ provided $u_0(p) > \frac{5}{567}|W_g(p)|_g^2$ for any $p \in M$.

An interesting open problem would be to show that under these assumptions a least energy sign-changing solution to the Yamabe problem (1.1) does not exist, as suggested by Remark 4.6.

## Appendix A. Some estimates

Fix $p \in M$ and $r > 0$ suitably small. Let $\chi : [0, \infty) \to \mathbb{R}$ be a smooth cut-off function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $[0, r/2]$ and $\chi \equiv 0$ in $[r, \infty)$, and let $\widetilde{V}_{\delta,p}$ be the function on $M$ given by

$$\widetilde{V}_{\delta,p}(x) = \chi(|x|)V_{\delta,p}(x) \quad \text{if } |x| \leq r, \qquad \widetilde{V}_{\delta,p}(x) = 0 \quad \text{otherwise}, \tag{A.1}$$

in conformal normal coordinates at $p$, where $V_{\delta,p}$ is the function defined in (3.1). Then, for some positive constant $c_0$,

$$0 < \widetilde{V}_{\delta,p}(x) \leq c_0\left(\frac{\delta}{\delta^2 + |x|^2}\right)^{(m-2)/2} \qquad \text{if } |x| \leq r. \tag{A.2}$$

Let $u \in H^1_g(M) \cap \mathcal{C}^0(M)$ be such that $u \geq 0$ and $u(x) > 0$ if $|x| \leq r$. Then there are positive constants $\mathfrak{c}_1, \mathfrak{c}_2$ such that

$$0 < \mathfrak{c}_1 \leq u(x) \leq \mathfrak{c}_2 \quad \text{if } |x| \leq r. \tag{A.3}$$

Set

$$v_1 := (u - \widetilde{V}_{\delta,p})^+ \quad \text{and} \quad v_\ell := (u - \widetilde{V}_{\delta,p})^-.$$

Observe that

$$v_1 = \begin{cases} u_1 & \text{if } |x| \geq r, \\ 0 & \text{if } |x| \leq r \text{ and } u_1 \leq \widetilde{V}_{\delta,p}, \\ u_1 - \widetilde{V}_{\delta,p} & \text{if } |x| \leq r \text{ and } u_1 \geq \widetilde{V}_{\delta,p}, \end{cases} \tag{A.4}$$

and

$$v_\ell = \begin{cases} 0 & \text{if } |x| \geq r, \\ \widetilde{V}_{\delta,p} - u_1 & \text{if } |x| \leq r \text{ and } u_1 \leq \widetilde{V}_{\delta,p}, \\ 0 & \text{if } |x| \leq r \text{ and } u_1 \geq \widetilde{V}_{\delta,p}. \end{cases} \tag{A.5}$$

By (A.1)–(A.3), there are positive constants $c_1, c_2, c_3$ such that

$$\begin{cases} |x| \leq r \text{ and } u_1(x) \leq \widetilde{V}_{\delta,p}(x) \Rightarrow |x| \leq c_1\sqrt{\delta}, \\ |x| \leq r/2 \text{ and } u_1(x) \geq \widetilde{V}_{\delta,p}(x) \Rightarrow |x| \geq c_2\sqrt{\delta}, \\ |x| \leq r/2 \Rightarrow \widetilde{V}_{\delta,p}(x) = V_{\delta,p}(x), \\ r/2 \leq |x| \leq r \Rightarrow |\widetilde{V}_{\delta,p}(x)|,\ |\nabla \widetilde{V}_{\delta,p}(x)| \leq c_3\delta^{(m-2)/2}. \end{cases} \tag{A.6}$$

**Lemma A.1.** *We have the following estimates:*

(i) $\|v_1\|_g^2 = \|u\|_g^2 - 2\mathfrak{a}_m u(p)\delta^{(m-2)/2} + o(\delta^{\nu(m)})$,

(ii) $|v_1|_{g,2^*}^{2^*} = |u|_{g,2^*}^{2^*} - 2^*\mathfrak{a}_m u(p)\delta^{(m-2)/2} + o(\delta^{(m-2)/2})$,

(iii) $\|v_\ell\|_g^2 = \|\widetilde{V}_{\delta,p}\|_g^2 + 2(\mathfrak{a}_m + \mathfrak{b}_m)u(p)\delta^{(m-2)/2} + o(\delta^{\nu(m)})$,

(iv) $|v_\ell|_{g,2^*}^{2^*} = |\widetilde{V}_{\delta,p}|_{g,2^*}^{2^*} + 2^*\mathfrak{b}_m u(p)\delta^{(m-2)/2} + o(\delta^{(m-2)/2})$,

*where*

$$\nu(m) := \begin{cases} (m-2)/2 & \textit{if either } m \leq 6 \textit{ or } (M,g) \textit{ is l.c.f.,} \\ 4 & \textit{if } m \geq 7 \textit{ and } (M,g) \textit{ is not l.c.f.,} \end{cases}$$

*and*

$$\mathfrak{a}_m := (m-2)(m(m-2))^{(m-2)/4}\omega_{m-1} \quad \textit{and} \quad \mathfrak{b}_m := \int_{\mathbb{R}^m} U_1^{2^*-1}. \tag{A.7}$$

*Proof.* (i) From (A.4) and (A.6) we obtain

$$\begin{aligned} &\|v_1\|_g^2 - \|u\|_g^2 \\ &= \int_{\{|x|\leq r\}\cap\{u\geq \widetilde{V}_{\delta,p}\}} [(|\nabla_g(u - \widetilde{V}_{\delta,p})|^2 + \kappa_m S_g (u - \widetilde{V}_{\delta,p})^2) - (|\nabla_g u|^2 + \kappa_m S_g u^2)]\, \mathrm{d}\mu_g \\ &\quad - \underbrace{\int_{\{|x|\leq r\}\cap\{u\leq \widetilde{V}_{\delta,p}\}} (|\nabla_g u|^2 + \kappa_m S_g u^2)\, \mathrm{d}\mu_g}_{=O(\delta^{m/2})} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\{|x|\le r\}\cap\{u\ge \widetilde{V}_{\delta,p}\}} (|\nabla_g \widetilde{V}_{\delta,p}|^2 + \kappa_m S_g \widetilde{V}_{\delta,p}^2)\, \mathrm{d}\mu_g \\
&\quad - 2\int_{\{|x|\le r\}\cap\{u\ge \widetilde{V}_{\delta,p}\}} (\langle \nabla_g u, \nabla_g \widetilde{V}_{\delta,p}\rangle_g + \kappa_m S_g u \widetilde{V}_{\delta,p})\, \mathrm{d}\mu_g + O(\delta^{m/2}) \\
&= \underbrace{O\Big(\int_{\{c_2\sqrt{\delta}\le |x|\le r\}} (|\nabla_g \widetilde{V}_{\delta,p}|^2 + \kappa_m S_g \widetilde{V}_{\delta,p}^2)\, \mathrm{d}\mu_g\Big)}_{=O(\delta^{m/2})} \\
&\quad - 2\underbrace{\int_{\{|x|\le r\}\cap\{u\ge \widetilde{V}_{\delta,p}\}} (\langle \nabla_g u, \nabla_g \widetilde{V}_{\delta,p}\rangle + \kappa_m S_g u \widetilde{V}_{\delta,p})\, \mathrm{d}\mu_g}_{\text{see (A.8)}} + O(\delta^{m/2}),
\end{aligned}
$$

and, using (A.6) again,

$$
\begin{aligned}
&\int_{\{|x|\le r\}\cap\{u\ge \widetilde{V}_{\delta,p}\}} (\langle \nabla_g u, \nabla_g \widetilde{V}_{\delta,p}\rangle + \kappa_m S_g u \widetilde{V}_{\delta,p})\, \mathrm{d}\mu_g \\
&\qquad = \int_{\{2c_1\sqrt{\delta}\le |x|\le r\}} (\langle \nabla_g u, \nabla_g \widetilde{V}_{\delta,p}\rangle + \kappa_m S_g u \widetilde{V}_{\delta,p})\, \mathrm{d}\mu_g \\
&\qquad = \int_{\{2c_1\sqrt{\delta}\le |x|\le r\}} (\langle \nabla_g u, \nabla_g \widetilde{V}_{\delta,p}\rangle + \kappa_m S_g u \widetilde{V}_{\delta,p})\, \mathrm{d}\mu_g + O(\delta^{m/2}) \\
&\qquad = \underbrace{\int_{\{2c_1\sqrt{\delta}\le |x|\le r\}} (-\Delta_g \widetilde{V}_{\delta,p} + \kappa_m S_g \widetilde{V}_{\delta,p} - \widetilde{V}_{\delta,p}^{2^*-1}) u\, \mathrm{d}\mu_g}_{\text{see (A.10)}} + O(\delta^{m/2}) \\
&\qquad\quad + \underbrace{\int_{\{2c_1\sqrt{\delta}\le |x|\le r\}} \widetilde{V}_{\delta,p}^{2^*-1} u\, \mathrm{d}\mu_g}_{=O(\delta^{m/2})} + \underbrace{\int_{\{2c_1\sqrt{\delta}=|x|\}} \partial_\nu \widetilde{V}_{\delta,p} u}_{\text{see (A.9)}} \\
&\qquad\quad + \underbrace{\int_{\{r=|x|\}} \partial_\nu \widetilde{V}_{\delta,p} u}_{=0}
\end{aligned} \tag{A.8}
$$

where $\partial_\nu$ is the exterior normal derivative,

$$
\int_{\{2c_1\sqrt{\delta}=|x|\}} \partial_\nu \widetilde{V}_{\delta,p} u = \underbrace{(m-2)(m(m-2))^{\frac{m-2}{4}} \omega_{m-1}}_{=\alpha_m} u(p)\delta^{\frac{m-2}{2}} + o(\delta^{\frac{m-2}{2}}) \tag{A.9}
$$

and

$$
\begin{aligned}
&\Big(\int_{\{2c_1\sqrt{\delta}\le |x|\le r\}} |-\Delta_g \widetilde{V}_{\delta,p} + \kappa_m S_g \widetilde{V}_{\delta,p} - \widetilde{V}_{\delta,p}^{2^*-1}|^{\frac{2m}{m+2}}\, \mathrm{d}\mu_g\Big)^{\frac{m+2}{2m}} \\
&\qquad = \begin{cases} O(\delta^{\frac{m-1}{2}}) & \text{if } m = 4, 5, \text{ or } M \text{ is l.c.f.}, \\ O(\delta^4 |\ln \delta|^{2/3}) & \text{if } m = 6 \text{ and } M \text{ is not l.c.f.}, \\ O(\delta^{\frac{m+10}{4}}) & \text{if } m \ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}
\end{aligned} \tag{A.10}
$$

Indeed, arguing as in [29] we obtain

$$\int_{\{2c_1\sqrt{\delta}\le|x|\le r\}} |-\Delta_g \widetilde{V}_{\delta,p} + \kappa_m S_g \widetilde{V}_{\delta,p} - \widetilde{V}_{\delta,p}^{2^*-1}|^{\frac{2m}{m+2}}\,\mathrm{d}\mu_g$$

$$= \begin{cases} O\Big(\displaystyle\int_{2c_1\sqrt{\delta}}^{r} \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^{\frac{2m^2}{m+2}-1+\frac{m(m-6)}{m+2}}\,\mathrm{d}s\Big) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,}\\ O\Big(\displaystyle\int_{2c_1\sqrt{\delta}}^{r} \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^{8}|\ln s|\,\mathrm{d}s\Big) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,}\\ O\Big(\displaystyle\int_{2c_1\sqrt{\delta}}^{r} \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^{\frac{2m^2}{m+2}-1+\frac{m(m-6)}{m+2}}\,\mathrm{d}s\Big) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}$$

$$= \begin{cases} O\Big(\delta^{\frac{2m(m-2)}{m+2}} \displaystyle\int_{1/\sqrt{\delta}}^{\infty} s^{-1+\frac{m(m-6)}{m+2}}\,\mathrm{d}s\Big) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,}\\ O\Big(\delta^{6} \displaystyle\int_{2c_1/\sqrt{\delta}}^{r/\sqrt{\delta}} \frac{|\ln \delta s|}{(1+s^2)^{\frac{9}{2}}} s^{8}\,\mathrm{d}s\Big) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,}\\ O\Big(\delta^{\frac{8m}{m+2}} \displaystyle\int_{1/\sqrt{\delta}}^{\infty} s^{-1-\frac{m(m-6)}{m+2}}\,\mathrm{d}s\Big) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}$$

$$= \begin{cases} O(\delta^{\frac{2m(m-2)}{m+2}-\frac{m(m-6)}{2(m+2)}}) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,}\\ O(\delta^8|\ln\delta|) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,}\\ O(\delta^{\frac{8m}{m+2}+\frac{m(m-6)}{2(m+2)}}) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}$$

This concludes the proof of statement (i).

(ii) Using the inequalities

$$\big||a+b|^{2^*}-|a|^{2^*}\big| \le c(|a|^{2^*-1}|b|+|b|^{2^*}) \quad \forall a,b\in\mathbb{R},$$
$$\big||a+b|^{2^*}-|a|^{2^*}-2^*a|a|^{2^*-2}|b|\big| \le c(|a|^{2^*-2}|b|^2+|b|^{2^*}) \quad \forall a,b\in\mathbb{R},$$

we obtain

$$\begin{aligned}|v_1|_{g,2^*}^{2^*} - |u|_{g,2^*}^{2^*} &= \int_{\{|x|\ge r\}} |u|^{2^*}\,\mathrm{d}\mu_g + \int_{\{|x|\le r\}} |(u-\widetilde{V}_{\delta,p})^+|^{2^*}\,\mathrm{d}\mu_g - \int_M |u|^{2^*}\,\mathrm{d}\mu_g\\ &= \int_{\{|x|\le r\}\cap\{u\ge\widetilde{V}_{\delta,p}\}} (|u-\widetilde{V}_{\delta,p}|^{2^*} - |u|^{2^*} + 2^* u^{2^*-1}\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g\\ &\quad - 2^*\int_{\{|x|\le r\}\cap\{u\ge\widetilde{V}_{\delta,p}\}} u^{2^*-1}\widetilde{V}_{\delta,p}\,\mathrm{d}\mu_g - \int_{\{|x|\le r\}\cap\{u\le\widetilde{V}_{\delta,p}\}} |u|^{2^*}\,\mathrm{d}\mu_g\end{aligned}$$

$$= \underbrace{O\Big(\int_{\{c_2\sqrt{\delta}\le|x|\le r\}}(u^{2^*-2}\widetilde{V}_{\delta,p}^2+\widetilde{V}_{\delta,p}^{2^*}\Big)\,d\mu_g)}_{O(\delta^{m/2})\text{ if } m\ge 5} - \underbrace{2^*\int_{\{2c_1\sqrt{\delta}\le|x|\le r\}}u^{2^*-1}\widetilde{V}_{\delta,p}\,d\mu_g}_{\text{see (A.11)}}$$

$$+ \underbrace{\int_{\{2c_1\sqrt{\delta}\le|x|\le r\}\cap\{u\le\widetilde{V}_{\delta,p}\}}u^{2^*-1}\widetilde{V}_{\delta,p}\,d\mu_g}_{=0\ (\text{see (A.6)})} - \underbrace{\int_{\{|x|\le 2c_1\sqrt{\delta}\}\cap\{u\ge\widetilde{V}_{\delta,p}\}}u^{2^*-1}\widetilde{V}_{\delta,p}\,d\mu_g}_{=O(\delta^{m/2})}$$

$$+ \underbrace{O\Big(\int_{\{|x|\le c_1\sqrt{\delta}\}}u^{2^*}\,d\mu_g\Big)}_{=O(\delta^{m/2})},$$

where

$$\int_{\{2c_1\sqrt{\delta}\le|x|\le r\}}u^{2^*-1}\widetilde{V}_{\delta,p}\,d\mu_g = \int_{\{2c_1\sqrt{\delta}\le|x|\le r\}}(-\Delta_g u+\kappa_m S_g u)\widetilde{V}_{\delta,p}\,d\mu_g$$
$$= \int_{\{2c_1\sqrt{\delta}\le|x|\le r\}}(-\Delta_g\widetilde{V}_{\delta,p}+\kappa_m S_g\widetilde{V}_{\delta,p})u\,d\mu_g - \underbrace{\int_{\{2c_1\sqrt{\delta}=|x|\}}\widetilde{V}_{\delta,p}\partial_\nu u}_{=O(\delta^{(m-1)/2})}$$
$$- \underbrace{\int_{\{r=|x|\}}\widetilde{V}_{\delta,p}\partial_\nu u}_{=0} + \underbrace{\int_{\{2c_1\sqrt{\delta}=|x|\}}\partial_\nu\widetilde{V}_{\delta,p}u}_{\text{see (A.9)}} + \underbrace{\int_{\{r=|x|\}}\partial_\nu\widetilde{V}_{\delta,p}u}_{=0}. \tag{A.11}$$

This concludes the proof of statement (ii).

(iii) Using (A.5) and (A.6) we obtain

$$\|v_\ell\|_g^2 - \|\widetilde{V}_{\delta,p}\|_g^2$$
$$= \int_{\{|x|\le r\}\cap\{u\le\widetilde{V}_{\delta,p}\}}[|\nabla_g(u-\widetilde{V}_{\delta,p})|^2+\kappa_m S_g(u-\widetilde{V}_{\delta,p})^2-|\nabla_g\widetilde{V}_{\delta,p}|^2+\kappa_m S_g\widetilde{V}_{\delta,p}^2]\,d\mu_g$$
$$- \int_{\{|x|\le r\}\cap\{u\ge\widetilde{V}_{\delta,p}\}}(|\nabla_g\widetilde{V}_{\delta,p}|^2+\kappa_m S_g\widetilde{V}_{\delta,p}^2)\,d\mu_g$$
$$= \int_{\{|x|\le r\}\cap\{u\le\widetilde{V}_{\delta,p}\}}(|\nabla_g u|^2+\kappa_m S_g u^2)\,d\mu_g$$
$$-2\int_{\{|x|\le r\}\cap\{u\le\widetilde{V}_{\delta,p}\}}(\langle\nabla_g u,\nabla_g\widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,d\mu_g$$
$$+ \underbrace{O\Big(\int_{\{c_2\sqrt{\delta}\le|x|\le r\}}(|\nabla_g\widetilde{V}_{\delta,p}|^2+\kappa_m S_g\widetilde{V}_{\delta,p}^2\Big)\,d\mu_g)}_{=O(\delta^{m/2})}$$

$$= \underbrace{O\Big(\int_{\{|x|\leq c_1\sqrt{\delta}\}}(|\nabla_g u|^2+\kappa_m S_g u^2\Big)\,\mathrm{d}\mu_g)}_{=O(\delta^{m/2})}$$

$$-2\int_{\{|x|\leq r\}\cap\{u\leq \widetilde{V}_{\delta,p}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g+O(\delta^{m/2})$$

$$= \underbrace{-2\int_{\{|x|\leq r\}\cap\{u\leq \widetilde{V}_{\delta,p}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g}_{\text{see (A.12)}}+O(\delta^{m/2})$$

and

$$\begin{aligned}
&\int_{\{|x|\leq r\}\cap\{u\leq \widetilde{V}_{\delta,p}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g\\
&\quad=\int_{\{|x|\leq \frac{c_2}{2}\sqrt{\delta}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g\\
&\qquad+\underbrace{\int_{\{\frac{c_2}{2}\sqrt{\delta}\leq|x|\leq r\}\cap\{u\leq \widetilde{V}_{\delta,p}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g}_{=O(\delta^{m/2})}\\
&\quad=\int_{\{|x|\leq \frac{c_2}{2}\sqrt{\delta}\}}(\langle\nabla_g u,\nabla_g \widetilde{V}_{\delta,p}\rangle+\kappa_m S_g u\widetilde{V}_{\delta,p})\,\mathrm{d}\mu_g+O(\delta^{m/2})\\
&\quad=\underbrace{\int_{\{|x|\leq \frac{c_2}{2}\sqrt{\delta}\}}(-\Delta_g\widetilde{V}_{\delta,p}+\kappa_m S_g\widetilde{V}_{\delta,p}-\widetilde{V}_{\delta,p}^{2^*-1})u\,\mathrm{d}\mu_g}_{\text{see (A.14)}}\\
&\qquad+\underbrace{\int_{\{|x|\leq \frac{c_2}{2}\sqrt{\delta}\}}\widetilde{V}_{\delta,p}^{2^*-1}u\,\mathrm{d}\mu_g}_{\text{see (A.13)}}\\
&\qquad+\underbrace{\int_{\{2c_1\sqrt{\delta}=|x|\}}\partial_\nu\widetilde{V}_{\delta,p}u}_{\text{see (A.9)}}+O(\delta^{m/2}),
\end{aligned}\tag{A.12}$$

where

$$\int_{\{|x|\leq \frac{c_2}{2}\sqrt{\delta}\}}\widetilde{V}_{\delta,p}^{2^*-1}u\,\mathrm{d}\mu_g=u(p)\underbrace{\Big(\int_{\mathbb{R}^m}U_1^{2^*-1}dx\Big)}_{=\mathfrak{b}_m}\delta^{(m-2)/2}+O(\delta^{m/2}),\tag{A.13}$$

and, arguing as in [29],

$$\int_{\{|x|\le \frac{c_2}{2}\sqrt{\delta}\}} |-\Delta_g \widetilde{V}_{\delta,p} + \kappa_m S_g \widetilde{V}_{\delta,p} - \widetilde{V}_{\delta,p}^{2^*-1}|^{\frac{2m}{m+2}} \,\mathrm{d}\mu_g$$

$$= \begin{cases} O\Big(\int_0^{\frac{c_2}{2}} \sqrt{\delta}\, \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^{\frac{2m^2}{m+2}-1+\frac{m(m-6)}{m+2}} \,\mathrm{d}s\Big) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,} \\ O\Big(\int_0^{\frac{c_2}{2}\sqrt{\delta}} \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^8 |\ln s| \,\mathrm{d}s\Big) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,} \\ O\Big(\int_0^{\frac{c_2}{2}\sqrt{\delta}} \frac{\delta^m}{(\delta^2+s^2)^{\frac{m^2}{m+2}}} s^{\frac{2m^2}{m+2}-1-\frac{m(m-6)}{m+2}} \,\mathrm{d}s\Big) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}$$

$$= \begin{cases} O\Big(\delta^m \int_0^{\frac{c_2}{2}\sqrt{\delta}} s^{-1+\frac{m(m-6)}{m+2}} \,\mathrm{d}s\Big) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,} \\ O\Big(\delta^6 \int_0^{\frac{c_2}{2}\sqrt{\delta}} \frac{|\ln s|}{s} \,\mathrm{d}s\Big) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,} \\ O\Big(\delta^m \int_0^{\frac{c_2}{2}\sqrt{\delta}} s^{-1-\frac{m(m-6)}{m+2}} \,\mathrm{d}s\Big) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases}$$

$$= \begin{cases} O(\delta^{\frac{m(3m-2)}{2(m+2)}}) & \text{if } m=4,5, \text{ or } M \text{ is l.c.f.,} \\ O(\delta^6 |\ln\delta|^2) & \text{if } m=6 \text{ and } M \text{ is not l.c.f.,} \\ O(\delta^{\frac{m(m+10)}{2(m+2)}}) & \text{if } m\ge 7 \text{ and } M \text{ is not l.c.f.} \end{cases} \tag{A.14}$$

This concludes the proof of statement (iii).

(iv) Using (A.5) and (A.6) we obtain

$$\begin{aligned} |v_\ell|_{g,2^*}^{2^*} - |\widetilde{V}_{\delta,p}|_{g,2^*}^{2^*} &= \int_{\{|x|\le r\}\cap\{u\le \widetilde{V}_{\delta,p}\}} (|\widetilde{V}_{\delta,p}-u|^{2^*} - |\widetilde{V}_{\delta,p}|^{2^*} + 2^* \widetilde{V}_{\delta,p}^{2^*-1} u) \,\mathrm{d}\mu_g \\ &\quad - 2^* \int_{\{|x|\le r\}\cap\{u\le \widetilde{V}_{\delta,p}\}} \widetilde{V}_{\delta,p}^{2^*-1} u \,\mathrm{d}\mu_g - \int_{\{|x|\le r\}\cap\{u\ge \widetilde{V}_{\delta,p}\}} \widetilde{V}_{\delta,p}^{2^*} \,\mathrm{d}\mu_g \\ &= \underbrace{O\Big(\int_{\{|x|\le c_1\sqrt{\delta}\}} (\widetilde{V}_{\delta,p}^{2^*-2} u^2 + u^{2^*}) \,\mathrm{d}\mu_g\Big)}_{=O(\delta^{m/2})} \underbrace{-2^* \int_{\{|x|\le r\}\cap\{u\le \widetilde{V}_{\delta,p}\}} \widetilde{V}_{\delta,p}^{2^*-1} u \,\mathrm{d}\mu_g}_{\text{see (A.15)}} \\ &\quad + \underbrace{O\Big(\int_{\{c_2\sqrt{\delta}\le|x|\le r\}} \widetilde{V}_{\delta,p}^{2^*} \,\mathrm{d}\mu_g\Big)}_{=O(\delta^{m/2})} \end{aligned}$$

and

$$\int_{\{|x|\le r\}\cap\{u\le\widetilde V_{\delta,p}\}} \widetilde V_{\delta,p}^{2^*-1} u \,\mathrm{d}\mu_g = \underbrace{\int_{\{|x|\le \frac{c_2}{2}\sqrt{\delta}\}} \widetilde V_{\delta,p}^{2^*-1} u\,\mathrm{d}\mu_g}_{\text{see (A.13)}}$$
$$- \underbrace{\int_{\{|x|\le\frac{c_2}{2}\sqrt{\delta}\}\cap\{u\ge\widetilde V_{\delta,p}\}} \widetilde V_{\delta,p}^{2^*-1} u\,\mathrm{d}\mu_g}_{=0 \text{ see (A.6)}} + \underbrace{\int_{\{\frac{c_2}{2}\sqrt{\delta}\le|x|\le r\}\cap\{u\le\widetilde V_{\delta,p}\}} \widetilde V_{\delta,p}^{2^*-1} u\,\mathrm{d}\mu_g}_{=O(\delta^{m/2})}$$
$$= \mathfrak{b}_m u(p)\delta^{(m-2)/2} + O(\delta^{m/2}). \qquad \text{(A.15)}$$

This concludes the proof of statement (iv). ■

## Appendix B. Uniform bounds in Hölder spaces

In this appendix we prove Lemma 4.4. Since it does not require additional effort, we consider the more general system

$$\mathscr{L}_g u_i = h_i(p,u_i) + \sum_{\substack{j=1\\ j\ne i}}^{\ell} \lambda|u_j|^{\gamma+1}|u_i|^{\gamma-1}u_i \quad \text{in } M,\ i=1,\dots,\ell, \qquad \text{(B.1)}$$

where $(M,g)$ is a closed Riemannian manifold of dimension $m\ge 1$, $\lambda<0$, $\gamma>0$, and $h_i : M\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $|h_i(p,s)|\le C|s|$ for every $p\in M$ and $|s|\le 1$.

Lemma 4.4 is a particular case of the following result.

**Theorem B.1.** *For each $\lambda<0$ let $(u_{\lambda,1},\dots,u_{\lambda,\ell})$ be a nonnegative solution to (B.1) such that $\{u_{\lambda,i} : \lambda<0\}$ is uniformly bounded in $L^\infty(M)$ for every $i=1,\dots,\ell$. Then, for any $\alpha\in(0,1)$, there exists $C_\alpha>0$ such that*

$$\|u_{\lambda,i}\|_{\mathscr{C}^{0,\alpha}(M)} \le C_\alpha \quad \textit{for every } \lambda<0,\ i=1,\dots,\ell.$$

In local coordinates, the system (B.1) becomes

$$\begin{cases} -\frac{1}{a(x)}\operatorname{div}(A(x)\nabla u_i) = -\kappa_m S_g(x)u_i + h_i(x,u_i) + \sum\limits_{\substack{j=1\\ j\ne i}}^{\ell} \lambda|u_j|^{\gamma+1}|u_i|^{\gamma-1}u_i \quad \text{in } \Omega,\\ i=1,\dots,\ell, \end{cases}$$

where $\Omega$ is an open subset of $\mathbb{R}^m$, $a(x) := \sqrt{|g(x)|}$, $A(x) := \sqrt{|g(x)|}(g^{kl}(x))$, $(g_{kl})$ is the metric written in local coordinates, $(g^{kl})$ is its inverse and $|g|$ is the determinant of $(g_{kl})$. Observe that the second order differential operator is uniformly elliptic, and since $M$ is compact, $a$ is bounded away from 0. Therefore, we end up with a system of the form

$$-\operatorname{div}(A(x)\nabla u_i) = f_i(x,u_i) + a(x)\sum_{\substack{j=1\\ j\ne i}}^{\ell} \lambda|u_j|^{\gamma+1}|u_i|^{\gamma-1}u_i \quad \text{in } \Omega,\ i=1,\dots,\ell. \qquad \text{(B.2)}$$

Let $\mathrm{Sym}_m \equiv \mathbb{R}^{m(m+1)/2}$ be the space of real symmetric $m \times m$ matrices. For the system (B.2) we prove the following result.

**Theorem B.2.** *Let $\Omega$ be an open subset of $\mathbb{R}^m$, and $\gamma > 0$. Assume that*

(H1) *$a \in \mathcal{C}^0(\Omega)$ and $a > 0$ in $\Omega$,*

(H2) *$A \in \mathcal{C}^1(\Omega, \mathrm{Sym}_m)$ and there exists $\theta > 0$ such that*

$$\langle A(x)\xi, \xi\rangle \geq \theta|\xi|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^m,$$

(H3) *$f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\bar{c} > 0$ such that*

$$|f_i(x, s)| \leq \bar{c}|s| \quad \text{for all } x \in \Omega,\ |s| \leq 1,\ i = 1, \dots, \ell.$$

*For each $\lambda < 0$ let $(u_{\lambda,1}, \dots, u_{\lambda,\ell})$ be a nonnegative solution to the system (B.2) such that $\{u_{\lambda,i} : \lambda < 0\}$ is uniformly bounded in $L^\infty(\Omega)$ for every $i = 1, \dots, \ell$. Then, given a compact subset $\mathcal{K}$ of $\Omega$ and $\alpha \in (0, 1)$, there exists $C > 0$ such that*

$$\|u_{\lambda,i}\|_{\mathcal{C}^{0,\alpha}(\mathcal{K})} \leq C \quad \text{for every } \lambda < 0,\ i = 1, \dots, \ell.$$

We now show that Theorem 1.4 follows from Theorem B.2.

*Proof of Theorem* B.1. Arguing by contradiction, assume that $\{u_{\lambda,i} : \lambda < 0\}$ is unbounded in $\mathcal{C}^{0,\alpha}(M)$ for some $\alpha \in (0, 1)$ and some $i = 1, \dots, \ell$. Since, by assumption, this set is uniformly bounded in $L^\infty(M)$, there exist $\lambda_n \to -\infty$ and $p_n \neq q_n$ in $M$ such that $u_{n,i} := u_{\lambda_n,i}$ satisfies

$$\frac{|u_{n,i}(p_n) - u_{n,i}(q_n)|}{[d_g(p_n, q_n)]^\alpha} \to \infty,$$

where $d_g$ is the geodesic distance in $(M, g)$. As $(u_{n,i})$ is uniformly bounded in $L^\infty(M)$, this implies that $d_g(p_n, q_n) \to 0$. Moreover, since $M$ is compact, a subsequence satisfies $p_n \to \bar{p}$ in $M$. Hence, $q_n \to \bar{p}$. Now, in local coordinates around $\bar{p}$ the system (B.1) becomes (B.2) with $f_i(x, s) := a(x)(-\kappa_m S_g(x)s + h_i(x, s))$. This contradicts Theorem B.2. ∎

**Remark B.3.** We point out that proving local uniform Hölder bounds for solutions of (B.2) (and hence (B.1)) is the starting point to prove local Lipschitz uniform bounds, which are optimal in this context. This has been done recently in [25, Theorem 1.1 and Corollary 1.4].

For the remainder of the appendix, our goal is to prove Theorem B.2. We follow very closely the proof of [49, Theorem 1.2], where the case $A(x) = \mathrm{Id}$ was treated, mainly highlighting the differences that arise from having a divergence-type operator instead of the Laplacian.

We use the following notation for the seminorm in Hölder spaces:

$$[u]_{\mathcal{C}^{0,\alpha}(\Omega)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

*B.1. Auxiliary lemmas*

We present the following generalization of [9, Lemma 5.2], [51, Lemma 4.1] and [53, Lemma 2.2] to our setting. The first part of the lemma is required to treat the case $\gamma \le 1$, while the second part is needed for $\gamma > 1$ (see the upcoming proof of Lemma B.7 for more details).

**Lemma B.4** (Decay estimates). *Let $\widetilde{\Omega}$ be an open subset of $\mathbb{R}^m$ and $\widetilde{A} \in \mathcal{C}^1(\widetilde{\Omega}, \mathrm{Sym}_m)$ be bounded in the $\mathcal{C}^1$-norm and such that there are $0 < \theta < \Theta$ with $\theta|\xi|^2 \le \langle \widetilde{A}(x)\xi, \xi\rangle \le \Theta|\xi|^2$ for all $x \in \widetilde{\Omega}$ and $\xi \in \mathbb{R}^m$. Let $a_0 \ge \|\widetilde{A}\|_{\mathcal{C}^1(\widetilde{\Omega},\mathrm{Sym}_m)}$. For any $R > 0$ satisfying $\overline{B_{2R}(0)} \subset \widetilde{\Omega}$ we have the following results:*

(1) *Take $C \ge 1$ and let $u \in H^1(B_{2R}(0)) \cap \mathcal{C}^0(\overline{B_{2R}(0)})$ be a nonnegative solution of*

$$-\operatorname{div}(\widetilde{A}(x)\nabla u) \le -Cu \quad \text{in } B_{2R}(0).$$

*Then there exist constants $c_1, c_2 > 0$, depending only on $m$, $\Theta$ and $a_0$, such that*

$$\|u\|_{L^\infty(B_R(0))} \le c_1 \|u\|_{L^\infty(B_{2R}(0))}\, e^{-c_2 R\sqrt{C}}.$$

(2) *Take $\delta > 0$, $\gamma \ge 1$, $C \ge 1$ and let $u \in H^1(B_{2R}(0)) \cap \mathcal{C}^0(\overline{B_{2R}(0)})$ be a nonnegative solution of*

$$-\operatorname{div}(\widetilde{A}(x)\nabla u) \le -Cu^\gamma + \delta \quad \text{in } B_{2R}(0).$$

*Then there exists a constant $c > 0$, depending only on $m$, $\Theta$ and $a_0$, such that*

$$C\|u\|^\gamma_{L^\infty(B_R(0))} \le \frac{c}{R + R^2}\|u\|_{L^\infty(B_{2R}(0))} + \delta.$$

*Proof.* (1) For the first statement we follow closely the proof of [51, Lemma 4.1], which concerns the case of a constant matrix. Define

$$z(x) := \sum_{i=1}^m \cosh(\sqrt{C/L}x_i) \quad \text{with} \quad L := \max\{1, (a_0 m + \Theta)^2\}.$$

Observe that, for $x \in B_{2R}(0)$,

$$\begin{aligned}
\operatorname{div}(\widetilde{A}(x)\nabla z) &= \sum_{i,j=1}^m \left( \frac{\partial \widetilde{A}_{ij}}{\partial x_i}(x)\frac{\partial z}{\partial x_j} + \widetilde{A}_{ij}(x)\frac{\partial^2 z}{\partial x_i \partial x_j}\right) \\
&= \sqrt{\frac{C}{L}} \sum_{i,j}^m \frac{\partial \widetilde{A}_{ij}}{\partial x_i}(x)\sinh\left(\sqrt{\frac{C}{L}}x_j\right) + \frac{C}{L}\sum_{i=1}^m \widetilde{A}_{ii}(x)\cosh\left(\sqrt{\frac{C}{L}}x_i\right) \\
&\le \sqrt{\frac{C}{L}} a_0 \sum_{j=1}^m \left|\sinh\left(\sqrt{\frac{C}{L}}x_j\right)\right| + \frac{C\Theta}{L} z(x) \\
&\le Cz(x)\left(\frac{a_0}{\sqrt{CL}} + \frac{\Theta}{L}\right) \le Cz(x),
\end{aligned}$$

where in the last inequality we have used $C \geq 1$ and the definition of $L$. Moreover, observe that there exist $c_1, c_2 > 0$, depending on $L$ (that is, on $m$, $a_0$ and $\Theta$), such that

$$z(x) \geq c_1 e^{c_2|x|\sqrt{C}} \quad \text{for every } x \in \mathbb{R}^m.$$

Therefore, given $x_0 \in B_R(0)$, by the comparison principle (which we can apply because $C > 0$) we have

$$\frac{u(x)}{\|u\|_{L^\infty(B_{2R}(0))}} \leq \frac{z(x-x_0)}{c_1 e^{c_2 R\sqrt{C}}} \quad \text{in } B_R(x_0).$$

Evaluating the previous inequality at $x = x_0$ yields

$$u(x_0) \leq \frac{m}{c_1} e^{-c_2 R\sqrt{C}} \|u\|_{L^\infty(B_{2R}(0))},$$

and the conclusion follows.

(2) We follow the proof of [53, Lemma 2.2] (which deals with the Laplace operator). Our main addition is the use of the mean value theorem for divergence operators, which reads as follows: Given $\Omega \subset \mathbb{R}^m$ there exist $k, K > 0$, only depending on $\theta, \Theta > 0$, such that for $y \in \Omega$ there exists an increasing family of sets $D_r(y) \subset \Omega$ such that $B_{kr}(y) \subset D_r(y) \subset B_{Kr}(y)$ and, for every solution $w$ of $-\operatorname{div}(A(x)\nabla w) \leq 0$ in $\Omega$,

$$r \mapsto \frac{1}{|D_r(y)|}\int_{D_r(y)} w \quad \text{is increasing,} \quad \text{and} \quad w(y) \leq \frac{1}{|D_r(x_0)|}\int_{D_r(y)} w.$$

(See [6, Theorem 6.3] for the proof of this result, which was previously stated in [10,12].) Now take a nonnegative solution $v \in H^1(B_{2R}(0))$ of

$$-\operatorname{div}(A(x)\nabla w) + Cw^\gamma = 0 \quad \text{in } B_{2R}(0), \qquad v = \|u\|_{L^\infty(B_{2R}(0))} \quad \text{on } \partial B_{2R}(0).$$

Using the uniform ellipticity and since $C > 0$, we can apply the maximum principle to deduce that $v \leq \|u\|_{L^\infty(B_{2R}(0))}$ in $B_{2R}(0)$. Let $\eta \in \mathcal{C}_c^\infty(B_{2R}(0))$ with $0 \leq \eta \leq 1$ be a cut-off function such that $\eta = 1$ in $B_{3R/2}(0)$, and take $\eta_R(x) := \eta(x/R)$. Then

$$\begin{aligned}
\int_{B_{3R/2}(0)} Cv^\gamma &\leq \int_{B_{2R}(0)} Cv^\gamma \eta_R = \int_{B_{2R}(0)} \operatorname{div}(\widetilde{A}(x)\nabla v)\eta_R \\
&= \sum_{i,j=1}^{\ell} \int_{B_{2R}(0)} v\left(\widetilde{A}_{ij}(x)\frac{\partial^2\eta}{\partial x_j \partial x_i}\left(\frac{x}{R}\right)\frac{1}{R^2} + \frac{\partial \widetilde{A}_{ij}}{\partial x_j}(x)\frac{\partial \eta}{\partial x_i}\left(\frac{x}{R}\right)\frac{1}{R}\right) \\
&\leq a_0(R^{m-1} + R^{m-2})\|v\|_{L^\infty(B_{2R}(0))}.
\end{aligned}$$

Now let $y \in B_R(0)$. Since $\gamma \geq 1$, by the mean value theorem presented above and Jensen's inequality we have

$$\begin{aligned}
Cv(y)^\gamma &\leq C\left(\frac{1}{|D_{R/(2K)}(y)|}\int_{D_{R/(2K)}(y)} v\right)^\gamma \leq \frac{1}{|D_{R/(2K)}(y)|}\int_{|D_{R/(2K)}(y)|} Cv^\gamma \\
&\leq \frac{1}{|B_{\frac{kR}{2K}}(y)|}\int_{B_{R/2}(y)} Cv^\gamma \leq \left(\frac{2K}{kR}\right)^m \frac{1}{|B_1(0)|}\int_{B_{3R/2}(0)} Cv^\gamma \\
&\leq \frac{a_0}{R+R^2}\|v\|_{L^\infty(B_{2R}(0))}.
\end{aligned}$$

By the maximum principle we have $u \le v + (\delta/C)^{1/\gamma}$, from which the conclusion follows. ■

**Lemma B.5** (Liouville-type results). *Let $A \in \mathrm{Sym}_m$ be a constant matrix and $\alpha \in (0,1)$.*

(1) *Let $u, v \in H^1_{\mathrm{loc}}(\mathbb{R}^m) \cap \mathcal{C}^0(\mathbb{R}^m)$ be nonnegative functions satisfying $uv \equiv 0$ and*

$$-\operatorname{div}(A\nabla u) \le 0, \quad -\operatorname{div}(A\nabla v) \le 0 \quad \text{in } \mathbb{R}^m.$$

*If $[u]_{\mathcal{C}^{0,\alpha}(\mathbb{R}^m)} < \infty$ and $[v]_{\mathcal{C}^{0,\alpha}(\mathbb{R}^m)} < \infty$ then either $u \equiv 0$ or $v \equiv 0$.*

(2) *Let $u, v \in H^1_{\mathrm{loc}}(\mathbb{R}^m) \cap \mathcal{C}^0(\mathbb{R}^m)$ be nonnegative solutions of*

$$-\operatorname{div}(A\nabla u) \le -au^\gamma v^{\gamma+1}, \quad -\operatorname{div}(A\nabla u) \le -av^\gamma u^{\gamma+1} \quad \text{in } \mathbb{R}^m,$$

*with $a > 0$ and $\gamma > 0$. If $[u]_{\mathcal{C}^{0,\alpha}(\mathbb{R}^m)} < \infty$ and $[v]_{\mathcal{C}^{0,\alpha}(\mathbb{R}^m)} < \infty$ then either $u \equiv 0$ or $v \equiv 0$.*

(3) *Let $u$ be a solution of $-\operatorname{div}(A\nabla u) = 0$ in $\mathbb{R}^m$ such that $[u]_{C^{0,\alpha}(\mathbb{R}^m)} < \infty$. Then $u$ is constant.*

*Proof.* Inspired by the proof of [51, Theorem 3.1] we find that since $A$ is symmetric and positive definite, there exist an orthogonal matrix $O$ and a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_\ell)$ with $d_i > 0$ such that $O^t AO = D$. Then, for $\sqrt{D} := \operatorname{diag}(\sqrt{d_1}, \dots, \sqrt{d_\ell})$ and $\bar{u}(x) := u(O\sqrt{D}x)$, we have

$$\Delta\bar{u}(x) = \operatorname{div}(A\nabla u)(O\sqrt{D}x).$$

Under this change of variables, we reduce the proof to the case of the Laplace operator. Therefore, parts (1) and (3) follow from [43, Proposition 2.2 and Corollary 2.3], while (2) follows from [49, Lemma A.3] (see also [50, Corollary 1.14 (ii)] for the case $\gamma \ge 1$). ■

### *B.2. A contradiction argument and a blow-up analysis*

Fix $\alpha \in (0,1)$. Without loss of generality, we assume that $\overline{B_3(0)} \subset \Omega$. Under the assumptions of Theorem B.2, we aim at proving the uniform Hölder bound in $B_1(0)$. Fix $\Lambda > 0$ such that

$$\|u_{\lambda,i}\|_{L^\infty(B_3(0))} \le \Lambda \quad \forall \lambda < 0,\ i = 1, \dots, \ell. \tag{B.3}$$

Let $\eta \in \mathcal{C}^1_c(\mathbb{R}^m)$ be a radially decreasing cut-off function such that

$$\begin{cases} \eta(x) = 1 & \text{for } x \in B_1(0), \\ \eta(x) = 0 & \text{for } x \in \mathbb{R}^m \smallsetminus B_2(0), \\ \eta(x) = (2 - |x|)^2 & \text{for } x \in B_2(0) \smallsetminus B_{3/2}(0). \end{cases}$$

For $x \in B_2(0)$, let $d_x := \operatorname{dist}(x, \partial B_2(0))$. It is shown in [49, Remark 2.1] that

$$\sup_{x \in B_2(0)} \sup_{\rho \in (0, d_x/2)} \frac{\sup_{B_\rho(x)} \eta}{\inf_{B_\rho(x)} \eta} \le 16. \tag{B.4}$$

Our goal is to prove that there exists $C > 0$ such that

$$\sup_{\substack{x \neq y \\ x,y \in \overline{B_2(0)}}} \frac{|(\eta u_{n,i})(x) - (\eta u_{n,i})(y)|}{|x-y|^\alpha} \leq C \quad \forall \lambda < 0,\ i = 1, \dots, \ell. \tag{B.5}$$

Since $\eta = 1$ in $B_1(0)$, Theorem B.2 follows readily from (B.5).

To prove (B.5), assume that it is false. Then there exist $\lambda_n \to -\infty$ such that $u_{n,i} := u_{\lambda_n,i}$ satisfies

$$L_n := \max_{i=1,\dots,\ell} \sup_{\substack{x \neq y \\ x,y \in \overline{B_2(0)}}} \frac{|(\eta u_{n,i})(x) - (\eta u_{n,i})(y)|}{|x-y|^\alpha} \to \infty \quad \text{as } n \to \infty. \tag{B.6}$$

We may assume that the maximum is attained at $i = 1$. Then, for each $n$, we fix a pair of points $x_n, y_n \in B_2(0)$ with $x_n \neq y_n$ such that

$$L_n = \frac{|(\eta u_{n,1})(x_n) - (\eta u_{n,1})(y_n)|}{|x_n - y_n|^\alpha}.$$

As $(u_n)$ is uniformly bounded in $L^\infty(B_2(0))$, this implies that $|x_n - y_n| \to 0$. So $(x_n)$ and $(y_n)$ converge to the same point. We denote

$$x_\infty := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n, \quad A_\infty := A(x_\infty), \quad a_\infty := a(x_\infty). \tag{B.7}$$

The contradiction argument is based on two blow-up sequences

$$v_{n,i}(x) := \eta(x_n)\frac{u_{n,i}(x_n + r_n x)}{L_n r_n^\alpha} \quad \text{and} \quad \bar{v}_{n,i}(x) := \frac{(\eta u_{n,i})(x_n + r_n x)}{L_n r_n^\alpha},$$

both defined in the scaled domain $\Omega_n := (B_3(0) - x_n)/r_n$; see [49,53,61,63]. Here $r_n \in (0,1)$, $r_n \to 0$, will be conveniently chosen later. Observe that $B_{1/r_n}(0) \subset \Omega_n$, therefore $\Omega_n$ approaches $\mathbb{R}^m$ as $n \to \infty$. Since $\eta$ is positive in $B_2(0)$, the functions $v_{n,i}$ and $\bar{v}_{n,i}$ are nonnegative and nontrivial in $\Omega_n' := (B_2(0) - x_n)/r_n$. Note that as $x_n \in B_2(0)$, $\Omega_n'$ approaches a limit domain $\Omega_\infty$ which is either a half-space or the whole $\mathbb{R}^m$, as $n \to \infty$.

**Lemma B.6.** *Under the assumptions of* Theorem B.2, *$v_{n,i}$ and $\bar{v}_{n,i}$ have the following properties:*

(1) *The sequence $([\bar{v}_{n,i}]_{\mathcal{C}^{0,\alpha}(\overline{\Omega_n'})})$ of $\alpha$-Hölder seminorms is uniformly bounded. Furthermore, for every $n \in \mathbb{N}$,*

$$\max_{i=1,\dots,\ell} \sup_{\substack{x \neq y \\ x,y \in \overline{\Omega_n'}}} \frac{|\bar{v}_{n,i}(x) - \bar{v}_{n,i}(y)|}{|x-y|^\alpha} = \frac{\left|\bar{v}_{n,1}(0) - \bar{v}_{n,1}\left(\frac{y_n - x_n}{r_n}\right)\right|}{\left|\frac{y_n - x_n}{r_n}\right|^\alpha} = 1.$$

(2) *$v_{n,i}$ solves*

$$-\operatorname{div}(A_n(x)\nabla v_{n,i}) = g_{n,i}(x) + a_n(x)\sum_{j \neq i} \Lambda_n |v_{n,j}|^{\gamma+1}|v_{n,i}|^{\gamma-1}v_{n,i} \quad \text{in } \Omega_n,$$

*where*

$$a_n(x) := a(x_n + r_n x), \quad A_n(x) := A(x_n + r_n x),$$
$$g_{n,i}(x) := \frac{\eta(x_n) r_n^{2-\alpha}}{L_n} f_{n,i}(x_n + r_n x, u_{n,i}(x_n + r_n x)),$$
$$\Lambda_n := \lambda_n r_n^{2(\alpha\gamma+1)} \left(\frac{L_n}{\eta(x_n)}\right)^{2\gamma}.$$

(3) *There exist* $a_0, a_1, a_2, \theta, \Theta > 0$ *such that, for every* $n \in \mathbb{N}$,

$$a_1 \le a_n(x) \le a_2, \quad \theta|\xi|^2 \le \langle A_n(x)\xi, \xi\rangle \le \Theta|\xi|^2 \quad \forall x \in \Omega_n,\ \xi \in \mathbb{R}^m,$$
$$\|A_n\|_{\mathcal{C}^1(\Omega_n, \mathrm{Sym}_m)} \le a_0.$$

(4) $\|g_{n,i}\|_{L^\infty(\Omega_n)} \to 0$, *and there is* $c_0 > 0$ *such that* $|g_{n,i}(x)| \le c_0 r_n^2 |v_{n,i}(x)|$ *for all* $x \in \Omega_n'$, $n \in \mathbb{N}$.

(5) $\|v_{n,i} - \bar{v}_{n,i}\|_{L^\infty(\mathcal{K})} \to 0$ *for any compact set* $\mathcal{K} \subset \mathbb{R}^m$ *and every* $i = 1, \ldots, \ell$.

(6) *For any compact set* $\mathcal{K} \subset \mathbb{R}^m$ *there exists* $C > 0$ *such that*

$$|v_{n,i}(x) - v_{n,i}(y)| \le C + |x - y|^\alpha \quad \forall x, y \in \mathcal{K},\ i = 1, \ldots, \ell.$$

*In particular,* $(v_{n,i})$ *has uniformly bounded oscillation in any compact set.*

*Proof.* The first two statements are proved by direct computation. The third one follows from (H1)–($H2$) with $a_1 := \min_{x \in \overline{B_3(0)}} a(x)$, $a_2 := \max_{x \in \overline{B_3(0)}} a(x)$, $\Theta := \|A\|_{\mathcal{C}^0(\overline{B_3(0)}, \mathrm{Sym}_m)}$ and $a_0 := \|A\|_{\mathcal{C}^1(\overline{B_3(0)}, \mathrm{Sym}_m)}$, while the fourth one is a consequence of (H3), (B.3), (B.6) and $r_n \to 0$. The last two statements are proved exactly as those in [49, Lemma 2.2 (4)–(5)]. ∎

**Lemma B.7.** *Take* $r_n \to 0^+$ *such that*

$$\liminf_{n\to\infty} |\Lambda_n| > 0 \quad \textit{and} \quad \limsup_{n\to\infty} \frac{|x_n - y_n|}{r_n} < \infty. \tag{B.8}$$

*Then the sequence* $(v_n(0))$ *is bounded in* $\mathbb{R}^\ell$, *where* $v_n := (v_{n,1}, \ldots, v_{n,\ell})$.

*Proof.* We follow [49, Lemma 2.3], to which we refer for further details.

Assume towards a contradiction that $|v_{n,\bar{i}}(0)| \to \infty$ for some $\bar{i} \in \{1, \ldots, \ell\}$. Take $R \ge |y_n - x_n|/r_n$ for all $n \in \mathbb{N}$. From Lemma B.6 (1) we get $|v_{n,\bar{i}}(0)| = |\bar{v}_{n,\bar{i}}(0)| \le |\bar{v}_{n,\bar{i}}(x)| + (2R)^\alpha$ if $x \in \Omega_n' \cap B_{2R}(0)$. So $\inf_{\Omega_n' \cap B_{2R}(0)} |\bar{v}_{n,\bar{i}}(x)| \to \infty$ and, as $\bar{v}_{n,\bar{i}}|_{\mathbb{R}^m \smallsetminus \Omega_n'} = 0$, we conclude that $B_{2R}(0) \subset \Omega_n'$ for $n$ large enough. Since $R$ is arbitrary, it follows that $\Omega_n'$ approaches $\mathbb{R}^m$ as $n \to \infty$.

Let $\varphi \in \mathcal{C}_c^\infty(B_{2R}(0))$ be a nonnegative cut-off function such that $\varphi = 1$ in $B_R(0)$. Take $i = 1, \ldots, \ell$. Testing the equation for $v_{i,n}$ in Lemma B.6 (2) against $v_{i,n}\varphi^2$, we obtain

$$\begin{aligned}
\theta \int_{B_{2R}(0)} |\nabla v_{n,i}|^2\varphi^2 &+ \int_{B_{2R}(0)} a_n \sum_{j\neq i} |\Lambda_n|\,|v_{n,j}|^{\gamma+1}|v_{n,i}|^{\gamma+1}\varphi^2 \\
&\leq \int_{B_{2R}(0)} \langle A_n\nabla v_{n,i}, \nabla v_{n,i}\rangle\varphi^2 + \int_{B_{2R}(0)} a_n \sum_{j\neq i} |\Lambda_n|\,|v_{n,j}|^{\gamma+1}|v_{n,i}|^{\gamma+1}\varphi^2 \\
&= -2\int_{B_{2R}(0)} \langle A_n\nabla v_{n,i}, \nabla\varphi\rangle v_{n,i}\varphi + \int_{B_{2R}(0)} g_{n,i}v_{n,i}\varphi^2 \\
&\leq \frac{\theta}{2}\int_{B_{2R}(0)} |\nabla v_{n,i}|^2\varphi^2 + C\int_{B_{2R}(0)} (v_{n,i}^2+1),
\end{aligned}$$

where in the last inequality we have used Lemma B.6 (4), (H2) and Young's inequality. By Lemma B.6 (3) we have

$$\sum_{j\neq i}\int_{B_R(0)} |\Lambda_n|\,|v_{n,j}|^{\gamma+1}|v_{n,i}|^{\gamma+1} \leq C\int_{B_{2R}(0)} (v_{n,i}^2+1).$$

Combining this inequality with $\liminf|\Lambda_n| > 0$ and Lemma B.6 (6) we deduce that

$$|v_{n,j}(x)|^{2(p+1)}|v_{n,i}(x)|^{2(p+1)} \leq C(|v_{n,i}(x)|^2+1)(|v_{n,j}(x)|^2+1) \quad \forall x \in B_R(0),$$

for any $i \neq j$. Using again Lemma B.6 (6) and our assumption that $|v_{n,\bar{i}}(0)| \to \infty$ we derive

$$\inf_{B_{2R}(0)} |v_{n,\bar{i}}| \to \infty, \qquad \sup_{B_{2R}(0)} |v_{n,i}| \to 0 \quad \forall i \neq \bar{i}.$$

Now we consider two cases.

Assume first that $\gamma \in (0,1]$ (as when $2(\gamma+1) = 2^*$ and $m \geq 5$). There are again two possibilities:

*Case 1.* If $\bar{i} = 1$, take $I_n := a_1|\Lambda_n|\inf_{B_{2R}(0)}|v_{n,1}|^{\gamma+1} \to \infty$. Since $v_{n,i} \to 0$ in $B_{2R}(0)$ and $\gamma \leq 1$, from Lemma B.6 we get

$$-\operatorname{div}(A_n\nabla v_{n,i}) \leq Cr_n^2 v_{n,i} - I_n v_{n,i}^\gamma \leq -\frac{I_n}{2}v_{n,i} \quad \text{in } B_{2R}(0),\ \forall i > 1,$$

Since $\|A_n\|_{\mathcal{C}^1(\Omega_n,\mathrm{Sym}_m)} \leq a_0$ for all $n \in \mathbb{N}$ (by Lemma B.6 (5)), Lemma B.4 (1) yields

$$0 \leq v_{n,i} \leq c_1 e^{-c_2\sqrt{I_n}} \quad \text{in } B_R(0),$$

and therefore

$$|\lambda_n|M_n v_{n,i}^{\gamma+1}v_{n,1}^{\gamma} \leq 2I_n c_1 e^{-c_2(\gamma+1)\sqrt{I_n}} \to 0 \quad \text{in } B_R(0)$$

for $n$ large, and

$$\operatorname{div}(A_n\nabla v_{n,i}) \to 0 \quad \text{in } L^\infty(B_R(0)).$$

Then, setting $w_n(x) := v_{n,1}(x) - v_{n,1}(0)$, by the Arzelà–Ascoli theorem we deduce that $w_n \to w_\infty$ in $L^\infty(B_R(0))$. Moreover, $A_n(x) \to A(x_\infty) =: A_\infty$ with $x_\infty$ as in (B.7). Observing that $R$ may be taken arbitrarily large, we conclude that

$$\operatorname{div}(A_\infty \nabla w_\infty) = 0 \quad \text{in } \mathbb{R}^m.$$

Arguing as in [49, pp. 401–402] and using (B.4), we see that $[w_\infty]_{C^{0,\alpha}(\mathbb{R}^m)} = 1$, which contradicts Lemma B.5 (3).

*Case 2.* If $\bar{i} > 1$, take $I_n := \sum_{j>1} |\Lambda_n| \inf_{B_{2R}(0)} |v_{n,j}|^{\gamma+1} \to \infty$. Then

$$-\operatorname{div}(A_n \nabla v_{n,1}) \le -\frac{I_n}{2}|v_{1,n}|^\gamma \le -\frac{I_n}{2}|v_{n,1}| \quad \text{in } B_{2R}(0).$$

Therefore, again by Lemma B.4 (1), $v_{n,1} \le c_1 e^{-c_2\sqrt{I_n}}$ in $B_R(0)$, which gives again

$$\operatorname{div}(A_n(x)\nabla v_{n,1}(x)) \to 0 \quad \text{uniformly in } B_R(0),$$

a contradiction.

Finally, if $\gamma > 1$, one may argue exactly as in Case 1 of the proof of [49, Lemma 2.3], using this time the decay estimate Lemma B.4 (2). In both cases, $\bar{i} = 1$ and $\bar{i} > 1$, we end up with $\operatorname{div}(A_n(x)\nabla v_{n,1}(x)) \to 0$ locally uniformly in $\mathbb{R}^m$, leading as before to a contradiction. ∎

**Lemma B.8.** *Up to a subsequence, we have*

$$|\lambda_n| \left(\frac{L_n}{\eta(x_n)}\right)^{2\gamma} |x_n - y_n|^{2(\alpha\gamma+1)} \to \infty.$$

*Proof.* We follow [49, proof of Lemma 2.5]. To reach a contradiction, assume that the sequence considered in the statement is bounded and take

$$r_n := \left(|\lambda_n|\left(\frac{L_n}{\eta(x_n)}\right)^{2\gamma}\right)^{-1/(2(\alpha\gamma+1))} \to 0.$$

With this choice, (B.8) is satisfied and from Lemma B.7 we deduce that $(\bar{v}_n(0))$ is bounded in $\mathbb{R}^\ell$. Combining this fact with Lemma B.6 (1), we deduce the existence of $(v_1, \dots, v_\ell) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that $\bar{v}_{n,i} \to v_i$ in the $\alpha$-Hölder norm as $n \to \infty$. Under the previous choice of $r_n$ one has $\Lambda_n = -1$. Hence, by elliptic regularity, the convergence of $\bar{v}_{n,i}$ to $v_i$ is actually in $\mathcal{C}^{1,\alpha}$, and

$$-\operatorname{div}(A_\infty \nabla v_i) = -a_\infty v_i^\gamma \sum_{j \ne i} v_j^{\gamma+1} \quad \text{in } \Omega_\infty,$$

where $\Omega_\infty$ is the limit domain of $\Omega_n'$ and $A_\infty$, $a_\infty$ are defined in (B.7). In particular, for any $i \ne j$, the pair $(v_i, v_j)$ is a nonnegative solution of

$$-\operatorname{div}(A_\infty \nabla v_i) \le -a_\infty v_i^\gamma v_j^{\gamma+1}, \quad -\operatorname{div}(A_\infty \nabla v_j) \le -a_\infty v_j^\gamma v_i^{\gamma+1} \quad \text{in } \Omega_\infty,$$

with bounded $\alpha$-Hölder seminorm. Using Lemma B.5 (2), and reasoning from this point on word for word as in [49, proof of Lemma 2.5], we obtain a contradiction. ∎

**Lemma B.9.** *Let $r_n := |x_n - y_n|$. Then there exists $(v_1, \dots, v_\ell) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that, up to a subsequence,*

(i) *$v_{n,i} \to v_i$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^m) \cap H^1_{\mathrm{loc}}(\mathbb{R}^m)$ for all $i = 1, \dots, \ell$;*

(ii) *for any $r > 0$,*

$$\lim_{n\to\infty} \int_{B_r(0)} |\Lambda_n|\,|v_{n,i}|^{\gamma+1}|v_{n,j}|^{\gamma+1} = 0 \quad \forall i, j = 1, \dots, \ell \text{ with } i \neq j.$$

*In particular, $v_i v_j \equiv 0$ in $\mathbb{R}^m$ for any $i \neq j$.*

*Proof.* Using Lemmas B.6–B.8, in particular the smoothness, boundedness and uniform ellipticity of $A_n$, the proof is obtained from a straightforward adaptation of that of [49, Lemma 2.6] (which, in turn, is based on [43, proof of Lemma 3.6]). Observe that $v_i v_j \equiv 0$ is a direct consequence consequence of the strong convergence of $v_n$, the convergence in (ii) and the fact that

$$|\Lambda_n| = |\lambda_n| \left(\frac{L_n}{\eta(x_n)}\right)^{2\gamma} |x_n - y_n|^{2(\alpha\gamma+1)} \to \infty. \qquad \blacksquare$$

**Lemma B.10.** *Let $(v_1, \dots, v_\ell)$ be as in Lemma B.9 and $A_\infty$ be as in (B.7). Then*

(i) *$\max_{x\in\partial B_1} |v_1(x) - v_1(0)| = 1$ and*

$$\operatorname{div}(A_\infty \nabla v_1) = 0 \quad \text{in } \Omega_1 := \{x \in \mathbb{R}^m : v_1(x) > 0\},$$

*where $\Omega_1$ is open and connected, and $\Omega_1 \neq \mathbb{R}^m$;*

(ii) *$v_i \equiv 0$ in $\mathbb{R}^m$ for every $i > 1$.*

*Proof.* Using the previous lemma together with Lemma B.5, the proof goes exactly as the one of [49, Lemma 2.7]. ■

### *B.3. The domain variation formula: end of the proof*

**Lemma B.11.** *Let $(v_1, \dots, v_\ell)$ be as in Lemma B.9 and $A_\infty \in \mathrm{Sym}_m$ as in (B.7). Then, for any vector field $Y \in \mathcal{C}^1_c(\mathbb{R}^m, \mathbb{R}^m)$, we have*

$$\int_{\mathbb{R}^m} \big(\langle \mathrm{d}Y A_\infty \nabla v_1, \nabla v_1\rangle - \tfrac{1}{2}\langle A_\infty \nabla v_1, \nabla v_1\rangle \operatorname{div} Y\big) = 0. \tag{B.9}$$

*Proof.* We test the $i$-th equation in Lemma B.6 (2) against $\langle \nabla v_{n,i}, Y\rangle$, integrate by parts and take the sum for all $i = 1, \dots, \ell$ to obtain

$$\sum_{i=1}^{\ell} \int_{\Omega_n} \langle A_n \nabla v_{n,i}, \nabla\langle \nabla v_{n,i}, Y\rangle\rangle + \sum_{\substack{i,j=1\\ j\neq i}}^{\ell} \int_{\Omega_n} |\Lambda_n| a_n v_{n,j}^{\gamma+1} v_{n,i}^{\gamma} \langle \nabla v_{n,i}, Y\rangle$$
$$= \sum_{i=1}^{\ell} \int_{\Omega_n} g_{n,i}(x) \langle \nabla v_{n,i}, Y\rangle.$$

Observe that

$$\begin{aligned}\int_{\Omega_n}\langle A_n\nabla v_{n,i},\nabla\langle\nabla v_{n,i},Y\rangle\rangle &= \int_{\Omega_n}\langle A_n\nabla v_{n,i},\, D^2v_{n,i}Y+(\mathrm{d}Y)^t\nabla v_{n,i}\rangle\\ &=\int_{\Omega_n}(\langle D^2v_{n,i}A_n\nabla v_{n,i},Y\rangle+\langle \mathrm{d}YA_n\nabla v_{n,i},\nabla v_{n,i}\rangle)\\ &=\int_{\Omega_n}\Big(-\tfrac{1}{2}\langle A_n\nabla v_{n,i},\nabla v_{n,i}\rangle\operatorname{div}Y\\ &\quad -\tfrac{1}{2}\sum_{j,k,l}\frac{\partial(A_n)_{jk}}{\partial x_l}Y_l\frac{\partial v_{n,i}}{\partial x_k}\frac{\partial v_{n,i}}{\partial x_j}+\langle \mathrm{d}YA_n\nabla v_{n,i},\nabla v_{n,i}\rangle\Big)\\ &\to\int_{\mathbb{R}^m}\big(-\tfrac{1}{2}\langle A_\infty\nabla v_1,\nabla v_1\rangle\operatorname{div}Y+\langle \mathrm{d}YA_\infty\nabla v_1,\nabla v_1\rangle\big)\end{aligned}$$

because $Y$ has compact support, $v_{n,i}\to v_i$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^m)$ and

$$\frac{\partial(A_n)_{jk}}{\partial x_l}=r_n^2\frac{\partial A_{jk}}{\partial x_l}(x_n+r_n\,\cdot)\to 0\quad\text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^m).$$

Moreover,

$$\begin{aligned}\sum_{\substack{i,j=1\\ j\neq i}}^{\ell}\int_{\Omega_n}|\Lambda_n|a_nv_{n,j}^{\gamma+1}v_{n,i}^{\gamma}\langle\nabla v_{n,i},Y\rangle &=\frac{1}{\gamma+1}\sum_{\substack{i,j=1\\ i<j}}^{\ell}\int_{\Omega_n}|\Lambda_n|a_n\langle\nabla(v_{n,j}^{\gamma+1}v_{n,i}^{\gamma+1}),Y\rangle\\ &=-\sum_{\substack{i,j=1\\ i<j}}^{\ell}\int_{\Omega_n}|\Lambda_n|a_n(x)v_{n,j}^{\gamma+1}v_{n,i}^{\gamma+1}\operatorname{div}Y\\ &\quad-\sum_{\substack{i,j=1\\ i<j}}^{\ell}\int_{\Omega_n}|\Lambda_n|r_n^2v_{n,j}^{\gamma+1}v_{n,i}^{\gamma+1}\langle\nabla a(x_n+r_nx),Y\rangle\to 0\end{aligned}$$

by Lemma B.9 (ii). The statement follows from this facts. ∎

*End of the proof of Theorem* B.2. Since $A_\infty\in\mathrm{Sym}_m$ is positive definite, there exist an orthogonal matrix $O$ and a diagonal matrix $D=\operatorname{diag}(d_1,\dots,d_\ell)$ with $d_i>0$ such that $O^tA_\infty O=D$. Let

$$u_1(x):=v_1(O\sqrt{D}x),\qquad\text{so that}\quad \nabla u_1(x)=\sqrt{D}O^tv_1(O\sqrt{D}x).$$

Then, from Lemma B.10 we get

- $\max_{|\sqrt{D}x|=1}|u_1(x)-u_1(0)|=1$;
- $\Delta u_1=0$ in $\{u_1>0\}$, which is an open connected set that does not coincide with $\mathbb{R}^m$;

while (B.9) turns into

$$\int_{\mathbb{R}^m}\big(\langle \mathrm{d}Z\nabla u_1,\nabla u_1\rangle-\tfrac{1}{2}|\nabla u_1|^2\operatorname{div}Z\big)=0 \tag{B.10}$$

for $Z(x) := \sqrt{D}O^tY(O\sqrt{D}x)$. Since $Y$ is an arbitrary vector field with compact support, then (B.10) holds true for every $Z \in \mathcal{C}^1_c(\mathbb{R}^m, \mathbb{R}^m)$. Given $x_0 \in \mathbb{R}^m$ and $r > 0$, let $\eta_\delta \in \mathcal{C}^\infty_c(B_{r+\delta}(x_0))$ be a cut-off function such that $0 \le \eta_\delta \le 1$ and $\eta_\delta = 1$ in $B_r(x_0)$. Then taking $Z(x) = Z_\delta(x) := (x - x_0)\eta_\delta$ in (B.10) and letting $\delta \to 0$, we derive the local Pohozaev identity

$$(2-m)\int_{B_r(x_0)} |\nabla u_1|^2 = \int_{\partial B_r(x_0)} r(2(\partial_\nu u_1)^2 - |\nabla u_1|^2)$$

(see for instance [44, Corollary 3.16] for the details). From this, it is now classical to deduce an Almgren monotonicity formula, namely, if we set

$$E(x_0,r) := \frac{1}{r^{m-2}}\int_{B_r(x_0)} |\nabla u_1|^2, \quad H(x_0,r) := \frac{1}{r^{m-1}}\int_{\partial B_r(x_0)} u_1^2,$$
$$N(x_0,r) := \frac{E(x_0,r)}{H(x_0,r)},$$

then $H(x_0,r) \neq 0$ for every $r > 0$, the function $r \mapsto N(x_0,r)$ is absolutely continuous and nondecreasing, and

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(x_0,r) = \frac{2}{r}N(x_0,r)$$

(see for instance [44, Theorem 3.21] for a proof). Moreover, if $N(x_0,r) = \varrho$ for every $r \in [r_1, r_2]$, then $u_1 = r^\varrho \hat{u}_1(\vartheta)$ in $\{r_1 < r < r_2\}$, where $(r,\vartheta)$ denotes a system of polar coordinates centered at $x_0$. Therefore we have obtained precisely the statements contained in [49, Lemma 2.7 and Proposition 2.9]. From this point on we argue *exactly* as in [49, Section 2.3] to obtain a contradiction. ■

## Appendix C. Lipschitz continuity of the limiting profiles and regularity of the free boundaries

Staying within the framework of Appendix B we continue our study of the system (B.2). Our aim now is to prove the following result.

**Theorem C.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^m$ and $\gamma > 0$. Assume that*

(H1′) $a \in \mathcal{C}^1(\Omega)$ *and* $a > 0$ *in* $\Omega$,

*and that $A$ and $f_i$ satisfy assumptions* (H2) *and* (H3) *of* Theorem B.2. *For each $\lambda < 0$, let $(u_{\lambda,1}, \dots, u_{\lambda,\ell})$ be a nonnegative solution to the system* (B.2) *satisfying*

(H4) $u_{\lambda,i} \to u_i$ *strongly in* $H^1(\Omega) \cap \mathcal{C}^{0,\alpha}(\Omega)$ *for every* $\alpha \in (0,1)$, *as* $\lambda \to -\infty$, *where* $u_i \not\equiv 0$;

(H5) $\int_\Omega \lambda u_{\lambda,i}^\gamma u_{\lambda,j}^\gamma \to 0$ *whenever* $i \neq j$; *in particular,* $u_iu_j \equiv 0$ *if* $i \neq j$;

(H6) $-\operatorname{div}(A(x)\nabla u_i) = f_i(x,u_i)$ *in the open set* $\{x \in \Omega : u_i(x) > 0\}$.

*Then the following statements hold true:*

(a) *$u_i$ is Lipschitz continuous for every $i = 1, \dots, \ell$.*

(b) *The nodal set $\Gamma := \{x \in \Omega : u_i(x) = 0 \ \forall i = 1, \dots, \ell\}$ is the disjoint union $\mathscr{R} \cup \mathscr{S}$, where $\mathscr{R}$ is an $(m-1)$-dimensional $\mathcal{C}^{1,\alpha}$-submanifold of $\Omega$ and $\mathscr{S}$ is a relatively closed subset of $\Omega$ whose Hausdorff measure is smaller than or equal to $m-2$. Moreover,*

– *given $x_0 \in \mathscr{R}$, there exist $i, j$ such that*

$$\lim_{x \to x_0^+} \langle A(x)\nabla u_i(x), \nabla u_i(x)\rangle = \lim_{x \to x_0^-} \langle A(x)\nabla u_j(x), \nabla u_j(x)\rangle \neq 0,$$

*where $x \to x_0^\pm$ are the limits taken from opposite sides of $\mathscr{R}$;*

– *if $x_0 \in \mathscr{S}$, then*

$$\lim_{x \to x_0} \langle A(x)\nabla u_i(x), \nabla u_i(x)\rangle = 0 \quad \text{for every } i = 1, \dots, \ell.$$

The proof of the Lipschitz continuity of the limiting profiles goes along the lines of [43, Theorem 1.2 and Section 4] (see also [49, Theorem 1.5 (4)], while the regularity of the nodal set follows [57, Theorem 1.1] (see also [49, Theorem 1.7]), where the differential operator is the Laplacian. The proof requires a careful blow-up analysis and is mainly based on the use of an Almgren-type monotonicity formula. Adapting it to divergence-form operators with nonconstant matrices is not completely straghtforward, and for that we use ideas from [32, 33, 39, 52].

### *C.1. Almgren's monotonicity formula: the case $A(0) = \mathrm{Id}$*

Assume that $0 \in \Omega$ and, for now, that $A(0) = \mathrm{Id}$. Our goal is to prove an Almgren monotonicity formula centered at the origin. Later on we shall see how to reduce the case where $A(0)$ is any matrix to the one where $A(0) = \mathrm{Id}$, and in which way this affects the formulas. The advantage of making this assumption stems from the fact that, near the origin, the problem looks like the one for the Laplacian, for which formulas are easier to derive. As in [33, 39, 52], we define

$$\mu(x) := \left\langle A(x)\frac{x}{|x|}, \frac{x}{|x|}\right\rangle, \quad x \in \mathbb{R}^m \setminus \{0\}.$$

The next lemma quantifies the behavior of various functions involving $A$ as $x \to 0$, in terms of

$$\|DA\|_\infty := \max\{\|\partial_{x_k} a_{ij}(x)\|_{L^\infty(\Omega)} : i, j = 1, \dots, \ell,\ k = 1, \dots, m\}$$

(which we assume to be finite, possibly by taking a smaller $\Omega$ from the start). The proof follows computations in [33]. Here, however, we need to keep track of the dependencies of the constants involved in the monotonicity formula. This is a key factor in passing from a general $A$ to one with $A(0) = \mathrm{Id}$ (see the proof of Theorem C.9 below, and its relation to Theorem C.3).

**Lemma C.2.** *There exists a constant $C$, depending only on the dimension $m$ and on an upper bound for $\|DA\|_\infty$, such that, as $|x| \to 0$,*

(1) $\|A(x) - \mathrm{Id}\| \le C|x|$,

(2) $|\mu(x) - 1| \le C|x|$,

(3) $\left|\frac{1}{\mu(x)} - 1\right| \le \frac{C}{1-C|x|}|x|$,

(4) $\left|\frac{1}{\mu^2(x)} - 1\right| \le \frac{C}{(1-C|x|)^2}|x|$,

(5) $|\nabla\mu(x)| \le C$,

(6) $\left|\mathrm{div}(A(x)\nabla|x|) - \frac{m-1}{|x|}\right| \le C$,

(7) $\left|\mathrm{div}\big(\frac{A(x)x}{\mu(x)}\big) - m\right| \le C|x|$.

*Proof.* The first statement is a direct consequence of the mean value theorem and the fact that the coefficients of $A$ are of class $\mathcal{C}^1$, which yields

$$\|A(x) - \mathrm{Id}\| \le \sqrt{m}\|DA\|_\infty|x|.$$

The second one follows from the identity

$$\mu(x) = \left\langle \frac{x}{|x|}, \frac{x}{|x|} \right\rangle + \left\langle (A(x) - \mathrm{Id})\frac{x}{|x|}, \frac{x}{|x|} \right\rangle,$$

combined with the Cauchy–Schwarz inequality and item *1.*, which allows us to conclude that $|\mu(x) - 1| \le \sqrt{m}\|DA\|_\infty|x|$. Items (3) and (4) are direct consequences of (2), namely,

$$\left|\frac{1}{\mu(x)} - 1\right| \le \frac{\sqrt{m}\|DA\|_\infty}{1 - \sqrt{m}\|DA\|_\infty|x|}|x|,$$
$$\left|\frac{1}{\mu^2(x)} - 1\right| \le \frac{\sqrt{m}\|DA\|_\infty(2 + \sqrt{m}\|DA\|_\infty)}{(1 - \|DA\|_\infty|x|)^2}|x|.$$

Regarding (5), from the proof of [33, Lemma 4.2] we get

$$\begin{aligned}
&|\partial_{x_k}\mu(x)| \\
&\quad \le \left|\sum_{i,j=1}^m \frac{\partial_{x_k}a_{ij}(x)x_ix_j}{|x|^2} + \sum_{j=1}^m \frac{2(a_{kj}(x) - \delta_{kj})x_j}{|x|^2} - \sum_{i,j=1}^m \frac{2(a_{ij}(x) - \delta_{ij})x_ix_jx_k}{|x|^4}\right| \\
&\quad \le \|DA\|_\infty\left(\sum_{i,j=1}^m \frac{|x_i||x_j|}{|x|^2} + \sum_{j=1}^m \frac{2|x_j||x|}{|x|^2} + \sum_{i,j=1}^m \frac{2|x_i||x_j||x_k||x|}{|x|^4}\right) \\
&\quad \le \|DA\|_\infty(3m^2 + 2m).
\end{aligned}$$

As for item (6), following the computations in [33, Lemma 4.1], we see that

$$\mathrm{div}(A(x)\nabla|x|) = \frac{m-1}{|x|} + \mathrm{div}((A(x) - \mathrm{Id})\nabla|x|),$$

and

$$|\mathrm{div}(A(x) - \mathrm{Id})\nabla|x|)| = \left|\sum_{i,j=1}^{m} \partial_{x_i} a_{ij}(x)\frac{x_j}{|x|} + \left(\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right)(a_{ij}(x) - \delta_{ij})\right|$$
$$\leq \|DA\|_\infty \sum_{i,j=1}^{m} \left(\frac{|x_j|}{|x|} + \left(\frac{1}{|x|} + \frac{|x_i x_j|}{|x|}\right)|x|\right) \leq 3m^2\|DA\|_\infty.$$

Finally, following [33, Lemma A.5], we have

$$\mathrm{div}\left(\frac{|x|A(x)\nabla|x|}{\mu(x)} - m\right) = \frac{|x|\,\mathrm{div}(A(x)\nabla|x|)}{\mu(x)} + 1 - m - \frac{|x|\langle A(x)\nabla|x|, \nabla\mu(x)\rangle}{\mu^2(x)}$$
$$= \frac{|x|}{\mu(x)}\left(\mathrm{div}(A(x)\nabla|x|) - \frac{m-1}{|x|}\right) + (m-1)\left(\frac{1}{\mu(x)} - 1\right) - \frac{|x|\langle A(x)\nabla|x|, \nabla\mu(x)\rangle}{\mu^2(x)}.$$

Since

$$|\langle A(x)\nabla|x|, \nabla\mu(x)\rangle| \leq (|A(x) - \mathrm{Id}| + 1)|\nabla\mu(x)| \leq C$$

(by item (5)), we see that (7) follows from (3), (4) and (6). ■

Set $B_r := B_r(0)$, $u = (u_1, \ldots, u_\ell)$, $|u|^2 := \sum_{i=1}^{\ell} u_i^2$, and

$$\langle A(x)\nabla u, \nabla u\rangle := \sum_{i=1}^{\ell}\langle A(x)\nabla u_i, \nabla u_i\rangle,$$
$$f(x,u) := (f_1(x,u_1), \ldots, f_\ell(x,u_\ell)).$$

Define

$$E(r) := \frac{1}{r^{m-2}}\int_{B_r}(\langle A(x)\nabla u, \nabla u\rangle - \langle f(x,u), u\rangle)\,\mathrm{d}x$$
$$= \frac{1}{r^{m-2}}\sum_{i=1}^{\ell}\int_{B_r}(\langle A(x)\nabla u_i, \nabla u_i\rangle - f_i(x,u_i)u_i)\,\mathrm{d}x,$$
$$H(r) := \frac{1}{r^{m-1}}\int_{\partial B_r}\mu(x)|u|^2\,\mathrm{d}\sigma = \frac{1}{r^{m-1}}\sum_{i=1}^{\ell}\int_{\partial B_r}\mu(x)u_i^2\,\mathrm{d}\sigma,$$

and the Almgren quotient

$$N(r) := \frac{E(r)}{H(r)} \quad \text{whenever } H(r) \neq 0.$$

The main purpose of this section is to prove the following result.

**Theorem C.3** (Monotonicity formula, case $A(0) = \mathrm{Id}$). *Take $\omega \Subset \Omega$ such that $0 \in \omega$. There exist $C, \bar{r} > 0$ (depending only on the dimension $m$, the ellipticity constant $\theta$, the domain $\omega$, and on an upper bound for $\|DA\|_\infty$ and $\|u\|_\infty$) such that, whenever $r \in (0, \bar{r})$, we have $H(r) \neq 0$, the function $N$ is absolutely continuous, and*

$$N'(r) \geq -C(N(r) + 1).$$

*In particular,* $e^{Cr}(N(r)+1)$ *is a nondecreasing function and the limit*

$$N(0^+) := \lim_{r\to 0^+} N(r)$$

*exists and is finite. Moreover,*

$$\left|(\log H(r))' - \frac{2}{r}N(r)\right| \leq C \quad \textit{for every } r \in (0,\bar{r}). \tag{C.1}$$

We present here a sketch of the proof of this result, which is based in ideas from [43, 49, 57], with adaptations obtained in [32, 33, 39, 52] allowing us to deal with variable coefficients. Our main goal in carrying out this proof and in repeating some computations is to focus on the dependence of the constants $C$ and $\bar{r}$, specially on the matrix $A$, something that was not needed in previous papers.

**Lemma C.4.** *We have*

$$E(r) = \frac{1}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i \langle A(x)\nabla u_i, \nu(x)\rangle \, \mathrm{d}\sigma.$$

*Proof.* For $\lambda < 0$, we have

$$\begin{aligned} E_\lambda(r) &:= \frac{1}{r^{m-2}} \int_{B_r} (\langle A(x)\nabla u_\lambda, \nabla u_\lambda\rangle - \langle f(x,u_\lambda), u_\lambda\rangle)\, \mathrm{d}x \\ &\quad - \frac{1}{r^{m-2}} \int_{B_r} a(x) \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \lambda |u_{\lambda,j}|^{\gamma+1} |u_{\lambda,i}|^{\gamma+1}\, \mathrm{d}x \\ &= \frac{1}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_{\lambda,i} \langle A(x)\nabla u_{\lambda,i}, \nu(x)\rangle\, \mathrm{d}\sigma, \end{aligned}$$

where the last identity is a consequence of testing the $i$-th equation in (B.2) by $u_{\lambda,i}$, integrating by parts, and taking the sum over $i$. Passing to the limit as $\lambda \to -\infty$ and using assumption (H4) yields the claim. ∎

**Lemma C.5.** *Let* $\omega \Subset \Omega$ *be such that* $0 \in \omega$*. There exist constants* $C, \bar{r} > 0$*, depending only on the dimension* $m$*, on* $\omega$ *and on an upper bound for* $\|DA\|_\infty$*, such that*

$$\left|H'(r) - \frac{2}{r}E(r)\right| \leq CH(r) \quad \textit{for every } r \in (0,\bar{r}).$$

*In particular, we get* (C.1)*.*

*Proof.* We combine the proof of [52, Lemma 3.3] with the estimates from Lemma C.2. By [52, (3.7)],

$$\frac{d}{dr}\int_{\partial B_r} \mu(x) u_i^2 = 2\int_{\partial B_r} u_i \langle A(x)\nabla u_i, \nabla|x|\rangle + \int_{\partial B_r} u_i^2 \,\mathrm{div}(A(x)\nabla|x|),$$

which, together with Lemma C.4, yields

$$H'(r) = \frac{1-m}{r} H(r) + \frac{2}{r^{m-1}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i \langle A(x)\nabla u_i, \nu(x)\rangle + \frac{1}{r^{m-1}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i^2 \operatorname{div}(A(x)\nabla|x|)$$
$$= \frac{1-m}{r} H(r) + \frac{2}{r} E(r) + \frac{1}{r^{m-1}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i^2 \operatorname{div}(A(x)\nabla|x|).$$

The conclusion now follows from the estimate

$$\left| \frac{1-m}{r} H(r) + \frac{1}{r^{m-1}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i^2 \operatorname{div}(A(x)\nabla|x|) \right| \le \frac{1}{r^{m-1}} \sum_{i=1}^{\ell} \int_{\partial B_r} \mu(x) u_i^2 \left( \frac{\operatorname{div}(A(x)\nabla|x|)}{\mu(x)} - \frac{1-m}{r} \right) \le CH(r),$$

where the constant $C > 0$ arises from items (3) and (6) of Lemma C.2. ■

Define

$$Z(x) := \frac{A(x)x}{\mu(x)} \sim x \quad \text{as } x \to 0.$$

From now on we use the summation convention for repeated indices, unless stated otherwise.

**Lemma C.6** (Local Pohozaev-type identities). *For every $r > 0$ such that $B_r \subset \Omega$, we have the following identity* (*where $A = (a_{ij})$*)

$$r \int_{\partial B_r} \langle A\nabla u_i, \nabla u_i\rangle = \int_{B_r} \operatorname{div} Z \langle A\nabla u_i, \nabla u_i\rangle + 2\int_{B_r} f_i(x, u_i)\langle \nabla u_i, Z\rangle + 2\int_{\partial B_r} \langle Z, \nabla u_i\rangle\langle A\nabla u_i, \nu\rangle + \int_{B_r} \langle Z, \nabla a_{hl}\rangle \frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_l} - 2\int_{B_r} a_{hl} \frac{\partial Z_j}{\partial x_h}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l}. \tag{C.2}$$

*Proof.* From system (B.2) we derive an identity for the $u_{\lambda,i}$'s, and then pass to the limit as $\lambda \to -\infty$. For each $i$, from the divergence theorem and the definition of $\mu(x)$ and $Z(x)$, we derive

$$r \int_{\partial B_r} \langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle = \int_{\partial B_r} \langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle\langle Z, \nu\rangle = \int_{B_r} \operatorname{div}(\langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle Z)$$
$$= \int_{B_r} \operatorname{div} Z \langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle + \int_{B_r} \langle Z, \nabla(\langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle)\rangle.$$

Following now [52, Lemma A.1] (see also [33, Lemma A.9]), we obtain

$$\int_{B_r} \langle Z, \nabla(\langle A\nabla u_{\lambda,i}, \nabla u_{\lambda,i}\rangle)\rangle = \int_{B_r} \langle Z, \nabla a_{hl}\rangle \frac{\partial u_{\lambda,i}}{\partial x_h}\frac{\partial u_{\lambda,i}}{\partial x_l} + 2\int_{\partial B_r} \langle Z, \nabla u_{\lambda,i}\rangle\langle A\nabla u_{\lambda,i}, \nu\rangle - 2\int_{B_r} a_{hl}\frac{\partial Z_j}{\partial x_h}\frac{\partial u_{\lambda,i}}{\partial x_j}\frac{\partial u_{\lambda,i}}{\partial x_l} + 2\int_{B_r} \langle Z, \nabla u_i\rangle\Big(f_i(x, u_{\lambda,i}) + a(x)\lambda \sum_{\substack{j=1\\ j\neq i}}^{\ell} |u_{\lambda,j}|^{\gamma+1}|u_{\lambda,i}|^{\gamma-1}u_{\lambda,i}\Big).$$

The conclusion will follow once we prove that

$$\sum_{i=1}^{\ell}\int_{B_r} \langle Z, \nabla u_{\lambda,i}\rangle a(x)\lambda \sum_{\substack{j=1\\ j\neq i}}^{\ell} |u_{\lambda,j}|^{\gamma+1}|u_{\lambda,i}|^{\gamma-1}u_{\lambda,i} \to 0 \quad \text{as } \lambda\to-\infty. \tag{C.3}$$

This is true due to the variational character of the coupling term; in fact, as $\lambda\to-\infty$,

$$\begin{aligned}&\sum_{i=1}^{\ell}\int_{B_r} \langle Z, \nabla u_{\lambda,i}\rangle a(x)\lambda \sum_{\substack{j=1\\ j\neq i}}^{\ell} |u_{\lambda,j}|^{\gamma+1}|u_{\lambda,i}|^{\gamma-1}u_{\lambda,i}\\ &= \sum_{i<j}\int_{B_r} \langle Z, \nabla(|u_{\lambda,i}|^{\gamma+1}|u_{\lambda,j}|^{\gamma+1})\rangle \frac{a(x)\lambda}{\gamma+1}\\ &= -\sum_{i<j}\int_{B_r} \operatorname{div} Z\frac{a(x)\lambda}{\gamma+1}|u_{\lambda,i}|^{\gamma+1}|u_{\lambda,j}|^{\gamma+1} - \sum_{i<j}\int_{B_r}\frac{\langle Z, \nabla a(x)\rangle}{\gamma+1}\lambda|u_{\lambda,i}|^{\gamma+1}|u_{\lambda,j}|^{\gamma+1}\\ &\quad + \sum_{i<j}\int_{\partial B_r}\frac{\langle Z,\nu\rangle a(x)}{\gamma+1}\lambda|u_{\lambda,i}|^{\gamma+1}|u_{\lambda,j}|^{\gamma+1} \to 0\end{aligned}$$

by assumption (H5) and because $\operatorname{div} Z = m + O(r)$ (see Lemma C.2 (7)). This proves (C.3) and completes the proof of Lemma C.6. ∎

Let

$$\widetilde{E}(r) := \frac{1}{r^{m-2}}\int_{B_r}\langle A(x)\nabla u, \nabla u\rangle = \frac{1}{r^{m-2}}\int_{B_r}\langle A(x)\nabla u_i, \nabla u_i\rangle.$$

**Lemma C.7.** *We have*

$$\widetilde{E}'(r) = \sum_{i=1}^{\ell}\Big(\frac{2}{r^{m-2}}\int_{\partial B_r}\frac{\langle A\nabla u_i, \nu\rangle^2}{\mu} + \frac{2}{r^{m-1}}\int_{B_r} f_i(x,u_i)\langle Z, \nabla u_i\rangle + \frac{1}{r^{m-1}}\int_{B_r}\langle Z, \nabla a_{hl}\rangle\frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_l} + \frac{1}{r^{m-1}}\int_{B_r}\operatorname{div}(Z-x)\langle A\nabla u_i, \nabla u_i\rangle - \frac{2}{r^{m-1}}\int_{B_r} a_{hl}\frac{\partial(Z_j - x_j)}{\partial x_h}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l}\Big). \tag{C.4}$$

*In particular, given $\omega \Subset \Omega$ with $0 \in \omega$, there exist constants $C, \bar{r} > 0$ (depending only on $m$, $\theta$, $\omega$ and on an upper bound for $\|DA\|_\infty$) such that, for every $r \in (0, \bar{r})$,*

$$\left| \widetilde{E}'(r) - \frac{2}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} \frac{\langle A\nabla u_i, \nu\rangle^2}{\mu} - \frac{2}{r^{m-1}} \int_{B_r} f_i(x, u_i)\langle Z, \nabla u_i\rangle \right| \leq C\widetilde{E}(r).$$

*Proof.* From Lemma C.6 and since

$$\frac{2}{r} \int_{\partial B_r} \langle Z, \nabla u_i\rangle\langle A\nabla u_i, \nu\rangle = 2\int_{\partial B_r} \frac{\langle A\nabla u_i, \nu\rangle^2}{\mu},$$

we have

$$\begin{aligned}\left(\int_{B_r} \langle A(x)\nabla u_i, \nabla u_i\rangle\right)' &= \frac{m-2}{r} \int_{B_r} \langle A\nabla u_i, \nabla u_i\rangle + \frac{2}{r}\int_{B_r} f_i(x, u_i)\langle \nabla u_i, Z\rangle \\ &\quad + 2\int_{\partial B_r} \frac{\langle A\nabla u_i, \nu\rangle^2}{\mu} + \frac{1}{r}\int_{B_r} \operatorname{div}(Z - x)\langle A\nabla u_i, \nabla u_i\rangle \\ &\quad + \frac{1}{r}\int_{B_r} \langle Z, \nabla a_{hl}\rangle \frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_l} - \frac{2}{r}\int_{B_r} a_{hl}\frac{\partial (Z_j - x_j)}{\partial x_h}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l}.\end{aligned}$$

As

$$\widetilde{E}'(r) = \frac{2-m}{r}\widetilde{E}(r) + \frac{1}{r^{m-2}}\left(\int_{B_r} \langle A\nabla u_i, \nabla u_i\rangle\right)',$$

we conclude that identity (C.4) is true.

Now, by Lemma C.2 (3, 4, 7), we have

$$\left\|\frac{1}{\mu} - 1\right\|_{L^\infty(B_r)}, \left\|\frac{1}{\mu^2} - 1\right\|_{L^\infty(B_r)}, \|\operatorname{div}(Z - x)\|_{L^\infty(B_r)} \leq Cr$$

for some constant $C > 0$ depending only on the dimension $m$, on $\omega$ and on an upper bound for $\|DA\|_\infty$. Then, using also (H2), we obtain

$$\begin{aligned}&\left|\frac{1}{r^{m-1}}\int_{B_r} \langle Z, \nabla a_{hl}\rangle \frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_l} + \frac{1}{r^{m-1}}\int_{B_r} \operatorname{div}(Z - x)\langle A\nabla u_i, \nabla u_i\rangle\right. \\ &\qquad\qquad \left. - \frac{2}{r^{m-1}}\int_{B_r} a_{hl}\frac{\partial (Z_j - x_j)}{\partial x_h}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l}\right| \\ &\qquad\qquad \leq C\frac{1}{r^{m-2}}\int_{B_r} \langle A\nabla u_i, \nabla u_i\rangle = C\widetilde{E}(r)\end{aligned}$$

(see [52, equations (A.3)–(A.12)] for more details). This completes the proof. ∎

**Remark C.8.** Observe that identities (C.2) and (C.7), which can be seen as local Pohozaev-type identities, are equivalent. They correspond to condition (G3) for the Laplacian stated in [49] and [57] respectively.

*Proof of Theorem* C.3. This result now follows from standard arguments. Here, as before, we mainly verify the dependence of the constants. Within this proof, $O(1)$ will represent a bounded function of $r$ depending only on $m$, $\theta$, $\omega$ and on an upper bound for $\|DA\|_\infty$ (but which is independent of $\|u\|_\infty$). We have, by Lemma C.7,

$$E'(r) = \widetilde{E}'(r) - \frac{2-m}{r^{m-1}} \int_{B_r} f_i(x,u_i)u_i - \frac{1}{r^{m-2}} \int_{\partial B_r} f_i(x,u_i)u_i$$

$$= \frac{2}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} \frac{\langle A\nabla u_i, \nu\rangle^2}{\mu} + O(1)E(r) + R(r), \tag{C.5}$$

where

$$R(r) := \frac{2}{r^{m-1}} \int_{B_r} f_i(x,u_i)\langle Z, \nabla u_i\rangle + \frac{O(1)}{r^{m-1}} \int_{B_r} f_i(x,u_i)u_i - \frac{1}{r^{m-2}} \int_{\partial B_r} f_i(x,u_i)u_i.$$

By (H3), there exists $\bar{d}$ depending on an upper bound for $\|u\|_\infty$ such that $|f_i(x,u_i)| \le \bar{d}u_i$. This together with assumption (H2) and Lemma C.2 (2, 3), yields

$$Z(x) = A(x)x/\mu(x) = O(1)|x| \quad \text{as } x \to 0$$

and

$$|R(r)| \le O(1)\bar{d} \sum_{i=1}^{\ell} \left( \frac{1}{r^{m-2}} \int_{B_r} |u_i|\,|\nabla u_i| + \frac{1}{r^{m-1}} \int_{B_r} u_i^2 + \frac{1}{r^{m-2}} \int_{\partial B_r} u_i^2 \right)$$

$$\le O(1)\bar{d} \sum_{i=1}^{\ell} \left( \frac{1}{r^{m-2}} \int_{B_r} \langle A\nabla u_i, \nabla u_i\rangle + \frac{1}{r^{m}} \int_{B_r} u_i^2 + \frac{1}{r^{m-1}} \int_{\partial B_r} u_i^2 \right)$$

$$\le O(1)\bar{d} \left( E(r) + H(r) + \frac{1}{r^m} \sum_{i=1}^{\ell} \int_{B_r} u_i^2 \right). \tag{C.6}$$

Using Poincaré's inequality (see [57, pp. 279–280] for the details), we conclude that

$$\frac{1}{r^m} \sum_{i=1}^{\ell} \int_{B_r} u_i^2 \le O(1)\bar{d}(E(r) + H(r)) \tag{C.7}$$

for every $r \in (0,\bar{r})$ sufficiently small. Combining (C.5)–(C.7), we arrive at

$$E'(r) = \frac{2}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} \frac{\langle A\nabla u_i, \nu\rangle^2}{\mu} + O(1)\bar{d}(E(r) + H(r)).$$

Recalling from Lemmas C.4 and C.5 that

$$E(r) = \frac{1}{r^{m-2}} \sum_{i=1}^{\ell} \int_{\partial B_r} u_i \langle A(x)\nabla u_i, \nu(x)\rangle \quad \text{and} \quad H'(r) = \frac{2}{r}E(r) + O(1)H(r),$$

we finally deduce the existence of a constant $C$ with the required properties such that

$$
\begin{aligned}
N'(r) &= \frac{E'(r)H(r) - E(r)H'(r)}{H^2(r)} \\
&= \frac{2}{H^2(r) r^{2m-3}} \Bigg( \sum_{i=1}^{\ell} \int_{B_r} \frac{\langle A\nabla u_i, \nu\rangle}{\mu} \sum_{j=1}^{\ell} \int_{\partial B_r} \mu u_j^2 - \Big( \sum_{i=1}^{\ell} \int_{\partial B_r} u_i \langle A\nabla u_i, \nu\rangle \Big)^2 \Bigg) \\
&\quad + \frac{1}{H^2(r)} \big( O(1)\bar{d}\, H(r)(E(r) + H(r)) + O(1)E(r)H(r) \big) \\
&\geq -C(N(r)+1),
\end{aligned}
$$

and $e^{Cr}(N(r)+1)$ is nondecreasing whenever $H(r) \neq 0$. Now observe that $H$ solves $H'(r) = a(r)H(r)$ with $a(r) = \frac{2}{r}N(r) + O(1)r$, and by the existence and uniqueness theorem for this ODE we find that $H > 0$ for sufficiently small $r > 0$. Finally, the validity of (C.1) is given by Lemma C.5. ∎

### *C.2. Almgren's monotonicity formula: the general case*

We have proved a monotonicity formula under the assumption that $A(0) = \mathrm{Id}$. The general case can be reduced to this case in the following way: let $A(x_0)^{1/2}$ be the square root of the (positive definite) matrix $A(x_0)$, that is, the unique positive definite matrix whose square is $A(x_0)$. We recall that $A(x_0)^{1/2}$ is also symmetric, it commutes with $A(x_0)$, it has real entries and the map $x_0 \mapsto A(x_0)^{1/2}$ is continuous (see for instance [37]). Following [32,52], we set

$$
\begin{aligned}
T_{x_0}x &:= x_0 + A(x_0)^{1/2}x, \\
A_{x_0}(x) &:= A(x_0)^{-1/2} A(T_{x_0}x) A(x_0)^{-1/2}, \\
\mu_{x_0}(x) &:= \Big\langle A_{x_0}\frac{x}{|x|}, \frac{x}{|x|} \Big\rangle, \\
f_{x_0}(x,s) &:= f(T_{x_0}x, s), \\
v_{i,x_0} &:= u_i(T_{x_0}x).
\end{aligned}
$$

Observe that $A_{x_0}(0) = \mathrm{Id}$. Let now

$$
N(x_0, u, r) := \frac{E(x_0, u, r)}{H(x_0, u, r)},
$$

where

$$
\begin{aligned}
E(x_0, u, r) &:= \frac{1}{r^{N-2}} \int_{B_r(0)} \big( \langle A_{x_0}\nabla v_{x_0}, \nabla v_{x_0}\rangle - \langle f_{x_0}(x, v_{x_0}), v_{x_0}\rangle \big)\, \mathrm{d}x, \\
H(x_0, u, r) &:= \frac{1}{r^{N-1}} \int_{\partial B_r(0)} \mu_{x_0}(x) |v_{x_0}|^2.
\end{aligned}
$$

These quantities can be expresed in terms of the original function $u$ in the ellipsoidal set

$$\mathcal{E}_r(x_0) := \{x \in \mathbb{R}^m : |A(x_0)^{-1/2}(x - x_0)| < r\}.$$

Namely, by a change of variables one has

$$\int_{B_r(0)} \langle A_{x_0} \nabla v_{x_0}, \nabla v_{x_0} \rangle = \det(A(x_0)^{-1/2}) \int_{\mathcal{E}_r(x_0)} \langle A \nabla u, \nabla u \rangle,$$
$$\int_{B_r(0)} \langle f_{x_0}(x, v_{x_0}), v_{x_0} \rangle = \det(A(x_0)^{-1/2}) \int_{\mathcal{E}_r(x_0)} \langle f(x, u), u \rangle,$$
$$\int_{\partial B_r(0)} \mu_{x_0}(x) |v_{x_0}(x)|^2 \, d\sigma(x) = \int_{\partial \mathcal{E}_r(x_0)} b_{x_0}(y) |u(y)|^2 \, d\sigma(y),$$

where $b_{x_0}(y) := c(x_0, y) |A(x_0)^{-1/2}(y - x_0)|^{-2} \langle A(x_0)^{-1} A(y) A(x_0)^{-1} y, y \rangle$, $c(x_0, y)$ being the dilation coefficient/tangential Jacobian (see for instance [42, Chapter 11]), which is continuous and positive.

**Theorem C.9** (Monotonicity formula, general case). *Take $\omega \Subset \Omega$ and let $u$ be as before. Then there exist $C, \bar{r} > 0$ (depending on the dimension $m$, the ellipticity constant $\theta$ and the domain $\omega$, but* independent of $x_0$*) such that, whenever $r \in (0, \bar{r})$ and $x_0 \in \omega$, the function $v_{x_0}$ satisfies identities (C.2) and (C.4), $H(x_0, u, r) \neq 0$, the function $r \mapsto N(x_0, u, r)$ is absolutely continuous, and*

$$\frac{\partial}{\partial r} N(x_0, u, r) \geq -C(N(x_0, u, r) + 1).$$

*In particular, $e^{Cr}(N(x_0, u, r) + 1)$ is nondecreasing and the limit*

$$N(x_0, u, 0^+) := \lim_{r \to 0^+} N(x_0, u, r)$$

*exists and is finite. Moreover,*

$$\left| \frac{\partial}{\partial r} \log H(x_0, u, r) - \frac{2}{r} N(r) \right| \leq C \quad \text{for every } r \in (0, \bar{r}). \tag{C.8}$$

*Proof.* This is basically a direct consequence of Theorem C.3. The only thing left to check is the dependence of the constants. But this is straightforward by observing that $\|v_{i,x_0}\|_\infty = \|u_i\|_\infty$ for every $i = 1, \ldots, \ell$, and that

$$\|DA_{x_0}(x)\|_\infty = \|DA(T_{x_0} x) A(x_0)^{1/2}\|_\infty,$$

which is uniformly bounded for $x \in \omega$, because of (H2) and the continuity of the map $x_0 \mapsto A(x_0)^{1/2}$. This allows us to take $C$ and $\bar{r}$ which are independent of $x_0$. ∎

Now that we have shown an Almgren monotonicity formula with constants independent of $x_0$ in any compactly contained subset of $\Omega$, we have all the tools required to conclude the proof of the main result of this appendix.

### C.3. Proof of the regularity result

*End of the proof of Theorem C.1.* (a) To prove that the functions $u_i$ are Lipschitz continuous for any $i = 1, \dots, \ell$ we argue as in the proof of [49, Proposition 3.4], with minimal adaptations at this point:

- we have an elliptic divergence-type operator instead of the pure Laplacian operator, therefore the estimates will depend on the ellipticity constant $\theta$;
- the identity (C.2) plays the role of the identity in the last assumption of [49, Proposition 3.4], while the monotonicity formula (Theorem C.9) plays the role of the monotonicity formula in [49, Theorem 3.3].

For related proofs of Lipschitz continuity in similar contexts, see also [43, Section 4.1] or [56, Section 2.4], the latter being a more detailed version of the former.

(b) Regarding the regularity properties of $\Gamma := \{x \in \Omega : u_i(x) = 0 \ \forall i = 1, \dots, \ell\}$, we argue as in the proof of [57, Theorem 1.1]:

- again, here we have an elliptic divergence-type operator instead of the Laplacian;
- formula of (C.4) plays the role of the expression for the derivative of $\widetilde{E}(x_0, U, r)$ in the statement of [57, Theorem 1.1] (see condition (G3) therein), while our Theorem C.9 plays the role of the monotonicity formula of [57, Theorem 2.2].

At a regular point $x_0 \in \Gamma$, identity (C.4) (or, equivalently, the local Pohozaev identities (C.2)) together with the equations

$$-\operatorname{div}(A(x)\nabla u_i) = f_i(x, s) \quad \text{in the open set } \{x \in \Omega : u_i(x) > 0\},\ i = 1, \dots, \ell,$$

given by assumption (H6), provide the free boundary condition

$$\lim_{x \to x_0^+} \langle A(x)\nabla u_i, \nabla u_i \rangle = \lim_{x \to x_0^-} \langle A(x)\nabla u_j, \nabla u_j \rangle \neq 0,$$

where $x \to x_0^{\pm}$ are the limits taken from opposite sides of $\Gamma$; see [57, Section 2] for the details.

For related proofs of regularity in similar contexts, see also [49, Theorem 1.7] or [56, Chapter 3]. ∎

**Remark C.10.** We remark that Theorem C.1 can be seen as a direct consequence of [57, Theorem 7.1]. However, since the latter result is presented without proof, we have decided to write this appendix and give all the necessary details.

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